

Week 16.

- ① Show that the set of all integers is countable
List all of the integers correctly:

0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, ... \rightarrow in this way we can predictably list all integers, therefore, the set of all integers is countable.

- ② Prove that the set of all rational numbers is countable.

The key is to list all rational numbers correctly:

The image displays three rows of handwritten mathematical sequences, each consisting of two parts separated by ellipses. Arrows indicate relationships between terms across the two parts.

- Row 1:** The first part shows a sequence of fractions where both numerator and denominator increase by 1 from left to right: $\left(\frac{1}{1}\right), \left(-\frac{1}{1}\right), \left(\frac{2}{1}\right), \left(-\frac{2}{1}\right), \left(\frac{3}{1}\right), \left(-\frac{3}{1}\right), \left(\frac{4}{1}\right), \left(-\frac{4}{1}\right), \left(\frac{5}{1}\right), \left(-\frac{5}{1}\right), \dots$. The second part shows a similar sequence but with denominators fixed at 2: $\left(\frac{1}{2}\right), \left(-\frac{1}{2}\right), \frac{2}{2}, -\frac{2}{2}, \left(\frac{3}{2}\right), \left(-\frac{3}{2}\right), \frac{4}{2}, -\frac{4}{2}, \left(\frac{5}{2}\right), \left(-\frac{5}{2}\right), \dots$. Arrows point from the first part's terms to the second part's terms, specifically from the positive terms.
- Row 2:** The first part has denominators fixed at 3: $\left(\frac{1}{3}\right), \left(-\frac{1}{3}\right), \left(\frac{2}{3}\right), \left(-\frac{2}{3}\right), \frac{3}{3}, -\frac{3}{3}, \left(\frac{4}{3}\right), \left(-\frac{4}{3}\right), \left(\frac{5}{3}\right), \left(-\frac{5}{3}\right), \dots$. The second part has denominators increasing by 1: $\frac{1}{4}, -\frac{1}{4}, \frac{2}{4}, -\frac{2}{4}, \frac{3}{4}, -\frac{3}{4}, \frac{4}{4}, -\frac{4}{4}, \frac{5}{4}, -\frac{5}{4}, \dots$. Arrows point from the first part's terms to the second part's terms, specifically from the positive terms.
- Row 3:** This row follows the same pattern as Row 2, with the first part having denominators fixed at 4 and the second part having denominators increasing by 1.

- ③ Prove that the power set of a set of all natural numbers (\mathbb{Z}^+) is uncountable.

Let's assume that it is countable

$$\{1, 2, 3, 4, 5, 6, 7, \dots\}$$

1 → 2 3 4 5 6 7 ...

2 → Y (Y) Y Y Y Y Y

3 → N N (N) N N N N ...

4 → N Y N (Y) N Y N

...

Using Cantor's diagonal argument we can build a subset that will be different in at least one way from every other subset on \aleph_0 sized list.

$$|P(z^*)| > |z^*|$$

- ④ Hilbert's Hotel - infinite buses w/ infinite number of people

Hotel	R_1	R_2	R_3	R_4	R_5	\dots
B_1	B_1S_1	B_1S_2	B_1S_3	B_1S_4	B_1S_5	\dots
B_2	B_2S_1	B_2S_2	B_2S_3	B_2S_4	B_2S_5	\dots
B_3	B_3S_1	B_3S_2	B_3S_3	B_3S_4	B_3S_5	\dots
\vdots						

- ⑤ Hilbert's Hotel - single bus w/ infinite number of people w/ infinitely long names consist. of A+B
No room BAAA,...

Room 1 ~~AB~~BAAAA,...

Room 2 A~~BB~~BBAA,...

Room 3 BA~~B~~ABAAA,...

Room 4 BBB~~B~~AAA,...

\vdots

← Cantor's
diagonalization

We have uncountable infinity of people w/ infinitely long names consisting of A&B.