Measure Theory

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1. **Exercise 1.3**
   - If $A \in \mathcal{G}_1$ and is different from empty set and the whole set, then A^c is closed and and doe not belong to $\mathcal{G}_1$. Thus, $\mathcal{G}_1$ is not an algebra.
   - If $a = b$ then $\emptyset \in \mathcal{G}_2$. If $b \leq a$, then $\mathbb{R} \in \mathcal{G}_2$. Thus by definition of G_2, it is an algebra.
   - If $a = b$ then $\emptyset \in \mathcal{G}_2$. If $b \leq a$, then $\mathbb{R} \in \mathcal{G}_2$. Thus by definition of G_2, it is an $\sigma-$algebra.
2. **Exercise 1.7**
   - $\mathcal{A}$ is a $\sigma-$algebra. Thus, by definition it contains $\{\emptyset, X\}$
   - $\mathcal{A}$ is a $\sigma-$algebra. By definition it is a family of subsets of X, thus, $\mathcal{A} \subset \mathcal{P}(X)$.
3. **Exercise 1.10**
   - $\emptyset \in \mathcal{S}_\alpha \ \ \forall \alpha \ \ \Rightarrow \emptyset \in \cap_\alpha \mathcal{S}_\alpha$
   - If $A \in \cap_\alpha \mathcal{S}_\alpha$, then $A \in \mathcal{S}_\alpha$ and $A^c \in \mathcal{S}_\alpha$ $\forall \alpha$, hence, $A^c \in \cap_\alpha \mathcal{S}_\alpha$
   - If $A_n \in \cap_\alpha \mathcal{S}_\alpha$, then then $A_n \in \mathcal{S}_\alpha$ and $\cup_n A_n \in \mathcal{S}_\alpha$ $\forall \alpha$, hence, $\cap_n A_n \in \cup_\alpha \mathcal{S}_\alpha$
4. **Exercise 1.22**
   - $A \subset B \ \Rightarrow B = A \cup (B \cap A^c)$. Since A and $B \cap A^c$ are disjoint, $\mu(B) = \mu(A) + \mu(B \cap A^c)$. Thus, $\mu(A) \leq \mu(B)$.
   - $A = (A \cap B^c) \cup (A \cap B)$ and $B = (B \cap A^c) \cup (B \cap A)$. $\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cap B) + \mu(A \cap B^c) + \mu(A^c \cap B) = \mu(A \cap B) + \mu(A \cup B)$. Thus, $\mu(A) + \mu(B) \geq \mu(A \cup B)$.
5. **Exercise 1.23**
   - $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.
   - $\lambda(\cup_n \mathcal{A}_n) = \mu((\cup_n \mathcal{A}_n) \cap B) = \mu(\cup_n(\mathcal{A}_n \cap B)) = \sum_n \mu(\mathcal{A}_n \cap B) = \sum_n \lambda(\mathcal{A}_n)$. ($\{\mathcal{A}_n\}$ disjoint $\rightarrow \{\mathcal{A}_n \cap B\}$ disjoint.)
6. **Exercise 1.26**

- $\mu(\cap_n \mathcal{A}_n) = \mu((\cup_n \mathcal{A}_n^c)^c) = \mu(X) - \mu(\cup_n \mathcal{A}_n^c) = \mu(X) - \lim_{n\to\infty} \mu(\mathcal{A}_n^c) = \mu(X) - \lim_{n\to\infty}(\mu(X) - \mu(\mathcal{A}_n)) = \lim_{n\to\infty} \mu(\mathcal{A}_n)$. By (i).

7. **Exercise 2.10**

8. **Exercise 2.14**

   - By Theorem 2.12, $\sigma(\mathcal{A}) \subset \mathcal{M}$. In order to show that $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}$, it is sufficient to prove that $\sigma(\mathcal{O}) \in \sigma(\mathcal{A})$. $(a, b) = \cup_n (a, b - 1/n]$.

9. **Exercise 3.1**

   - Let B be a countable subset of $\mathbb{R}$. Then the Lebesgue outer measure given by $\mu^*(B) := \inf \sum_{n=1}^{\infty}(b_n - a_n) : \ B \subset \cup_{i=1}^{\infty}(a_i, b_i]$, achieves the infimum for the partition given by the set B, with $a_i = b_i = B_i$, where $B_i$is the $i-th$ element of the set B. Hence, the result.

10. **Exercise 3.7**

    - $\sigma(\{(-\infty, a), \ \forall a \in \mathbb{R}\}) \ = \sigma(\{(-\infty, a], \ \forall a \in \mathbb{R}\}) \ = \ \sigma(\{(a, \infty), \ \forall a \in \mathbb{R}\}) \ = \sigma(\{[a, \infty), \ \forall a \in \mathbb{R}\}) \ = \mathcal{B}(\mathbb{R})$.

11. **Exercise 3.10**

    - $\max(f, g)$, $\min(f, g)$ and $|f| = max(f, -f)$ are measurable by (2).
    - $f + g$ and $f - g$ are measurable by (4).

12. **Exercise 3.17**

    Simillarly to the proof for (1), we define N such that $N \geq M$ and $\frac{1}{2^N} < \epsilon$. Since f is bounded M doesnot depend on x, hence we have uniform convergence.e

13. **Exercise 4.13**

    - $0 <= |f| < M$, by proposition 4.5, $\int_E |f| d\mu < M\mu(E) < \infty$.

14. **Exercise 4.14**

    - Let $\mathcal{A} \subset E$ denote the subset where f is not finite. By contrast, let's assume that E $\forall B : \mathcal{A} \subset B \subset E$, we have that $\mu(B) > 0$. by the properties of the monotonocity of μ and by the definition of Lebesgue integral, $\int_E |f| d\mu > \int_{\mathcal{B}} |f| d\mu = \infty$. This would mean that $f \notin \mathcal{L}^1(\mu, E)$.

15. **Exercise 4.15**

    - If $f \leq g$, then $0 \leq f_+ \leq g_+$ and $0 \leq -f_- \leq -g_-$. By Proposition 4.7, $\int_E f_+ d\mu \leq \int_E g_+ d\mu$ and $-\int_E f_- d\mu \leq -\int_E g_- d\mu$. Hence,$\int_E f_+ d\mu - \int_E g_- d\mu \leq \int_E g_+ - \int_E g_-$. Then by definition $\int_E f d\mu \leq \int_E g d\mu$.

16. **Exercise 4.16**

    $A \subset E$, by definition of Lebesgue integral, $\int_A |f| d\mu \leq \int_E |f| d\mu < \infty$. Hence, $f \in \mathcal{L}^1(\mu, A)$.

17. **Exercise 4.21**

    $\tilde{\mu}(A) := \int_A f d\mu$. $\tilde{\mu}(A) = \tilde{\mu}(B) + \tilde{\mu}(A - B)$. $\int_A f d\mu = \int_{A-B} f d\mu + \int_B f d\mu$. Since $\mu(A - B) = 0$, $\int_A f d\mu = \int_B f d\mu$.