Curve Sketching Techniques

Illustration 7

$f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$

- Asymptote/s:

  VA: none, since f is continuos everywhere
  HA: none

$$\lim_{x\to+\infty} x^{\frac{2}{3}}(6-x)^{\frac{1}{3}} = (+\infty)(-\infty) = -\infty$$

$$\lim_{x\to-\infty} x^{\frac{2}{3}}(6-x)^{\frac{1}{3}} = (+\infty)(+\infty) = +\infty$$

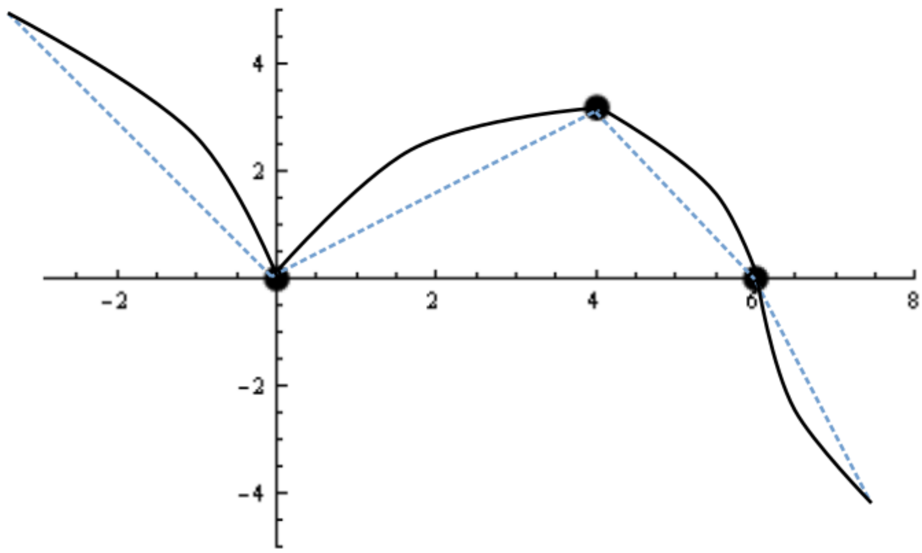


Illustration 8

$f(x) = x^3 - 8x^2 + 15x$

- **Domain:** Dom $f = \mathbb{R}$

- **Intercept/s:**

  x-intercept/s: $f(x) = x^3 - 8x^2 + 15x = 0$

$$x(x^2 - 8x + 15) = 0$$

$$x(x-5)(x-3) = 0$$

$$x = 0, 3, 5$$

y-intercept: $f(0) = 0$

- **Asymptote/s:**

VA: none
HA: none

- **Critical Number/s:**

$f'(x) = 3x^2 - 16x + 15$
$f'(x) = 0 \Rightarrow x = \frac{8-\sqrt{19}}{3}, \frac{8+\sqrt{19}}{3}$
$f'(x)$ dne $\Rightarrow$ no solution

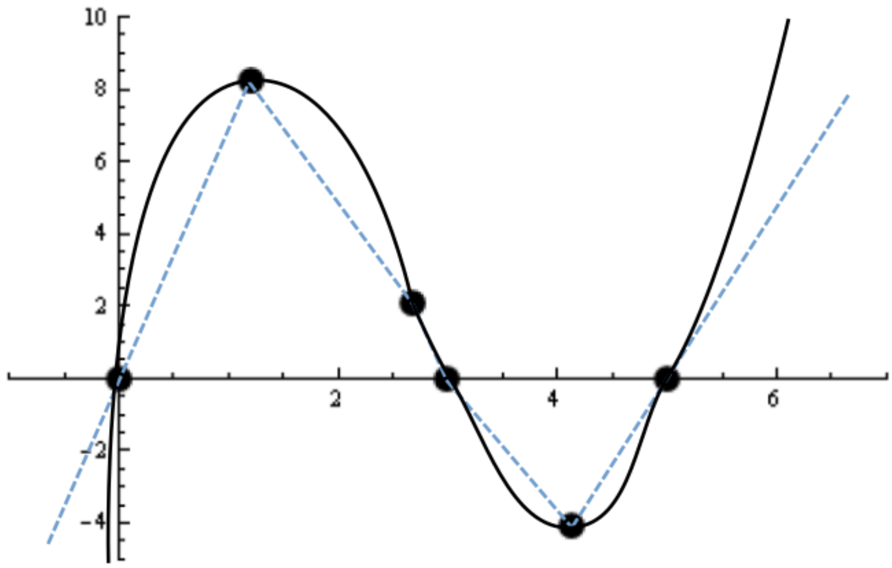
$f''(x) = 6x - 16$
$f''(x) = 0 \Rightarrow x = \frac{8}{3}$
$f''(x)$ dne $\Rightarrow$ no solution

- **Tables of Signs:**

x **Values**	**Sign of** $f'(x)$	**Conclusion for** $f(x)$
$x < \frac{8-\sqrt{19}}{3}$	$+$	Increasing
$x = \frac{8-\sqrt{19}}{3}$	0	Relative Maximum
$\frac{8-\sqrt{19}}{3} < x < \frac{8+\sqrt{19}}{3}$	$-$	Decreasing
$x = \frac{8+\sqrt{19}}{3}$	0	Relative Minimum
$x > \frac{8+\sqrt{19}}{3}$	$+$	Increasing

x **Values**	**Sign of** $f''(x)$	**Conclusion for** $f(x)$
$x < \frac{8}{3}$	$-$	Concave Downward
$x = \frac{8}{3}$	0	Inflection Point
$x > \frac{8}{3}$	$+$	Concave Upward

- **Curve:**



Exercises D

1. $f(x) = x^{\frac{5}{4}} + 10x^{\frac{1}{4}}$

- **Domain:** Dom $f = [0, +\infty)$

- **Intercept/s:**

x-intercept/s: $f(x) = x^{\frac{5}{4}} + 10x^{\frac{1}{4}} = 0$

$$x^{\frac{1}{4}}(x + 10) = 0$$

$$x = 0 \text{ or } x = -10 \notin \text{Dom } f$$

y-intercept: $f(0) = 0$

- **Asymptote/s:**

VA: none, since f is continuous on $[0, +\infty)$
HA: none

$$\lim_{x\to+\infty} \left(x^{\frac{5}{4}} + 10x^{\frac{1}{4}}\right)$$

$$= \lim_{x\to+\infty} x^{\frac{1}{4}}(x + 10)$$

$$= (+\infty)(+\infty)$$

$$= +\infty$$

$$\lim_{x\to-\infty}\left(x^{\frac{5}{4}}+10x^{\frac{1}{4}}\right) \text{ dne}$$

- **Critical Number/s:**

$$f'(x)=\frac{5}{4}x^{\frac{1}{4}}+\frac{5}{2}x^{-\frac{3}{4}}$$
$$f'(x)=0 \Rightarrow \text{no solution}$$
$$f'(x) \text{ dne} \Rightarrow x=0$$

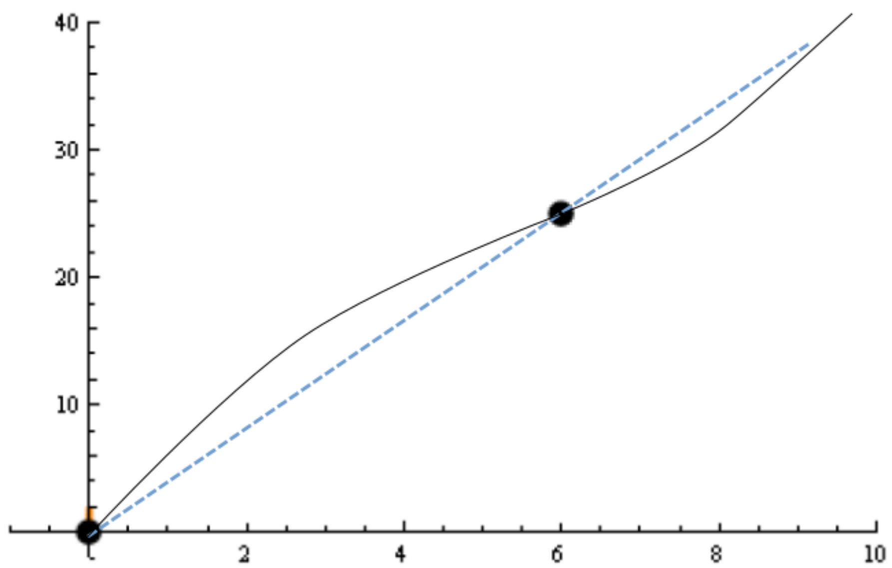
$$f''(x)=\frac{5}{16}x^{-\frac{3}{4}}-\frac{15}{8}x^{-\frac{7}{4}}$$
$$f''(x)=0 \Rightarrow x=6$$
$$f''(x) \text{ dne} \Rightarrow x=0$$

- **Tables of Signs:**

x Values	Sign of $f'(x)$	Conclusion for $f(x)$
$x=0$	d.n.e	
$x>0$	$+$	Increasing

x Values	Sign of $f''(x)$	Conclusion for $f(x)$
$x=0$	d.n.e.	
$0<x<6$	$-$	Concave Downward
$x=6$	0	Inflection Point
$x>6$	$+$	Concave Upward

- **Curve:**



3. $f(x) = \dfrac{\ln x}{\sqrt{x}}$

- **Domain:** Dom $f = (0, +\infty)$
- **Intercept/s:**

x-intercept/s: $f(x) = \frac{\ln x}{\sqrt{x}} = 0 \Rightarrow \ln x = 0 \Rightarrow x = 1$

y-intercept: none

- **Asymptote/s:**

VA:

Candidate/s:

$\sqrt{x} = 0 \Rightarrow x = 0$

Checking:

$$\lim_{x\to 0^+} \frac{\ln x}{\sqrt{x}} \quad \left(\frac{-\infty}{0^+}\right)$$

$$= \lim_{x\to 0^+} \ln x \left(\frac{1}{\sqrt{x}}\right)$$

$$= (-\infty)(+\infty)$$

$$= -\infty$$

$$\Rightarrow x = 0 \text{ is a VA}$$

HA:

$$\lim_{x\to +\infty} \frac{\ln x}{\sqrt{x}} \quad \left(\frac{+\infty}{+\infty}\right)$$

$$=_{LH} \lim_{x\to +\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}}$$

$$= \lim_{x\to +\infty} 2 \cdot \frac{1}{x} \cdot x^{\frac{1}{2}}$$

$$= \lim_{x \to +\infty} \frac{2}{x^{\frac{1}{2}}}$$

$$= 0$$

$$\Rightarrow y = 0 \text{ is a HA}$$

$$\lim_{x \to -\infty} \frac{\ln x}{\sqrt{x}} \text{ dne}$$

- **Critical Number/s:**

$$f'(x) = \frac{1 - \frac{1}{2}\ln x}{x^{\frac{3}{2}}}$$

$$f'(x) = 0 \Rightarrow x = e^2$$

$$f'(x) \text{ dne} \Rightarrow x = 0 \notin \text{Dom } f$$

$$f''(x) = \frac{-2 + \frac{3}{4}\ln x}{x^{\frac{5}{2}}}$$

$$f''(x) = 0 \Rightarrow x = e^{\frac{8}{3}}$$

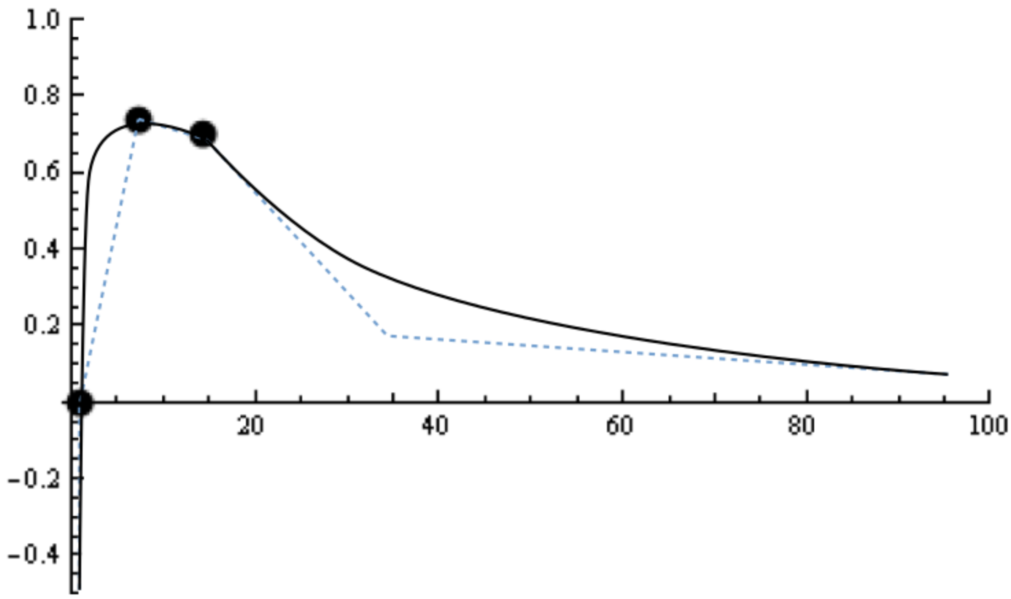
$$f''(x) \text{ dne} \Rightarrow x = 0 \notin \text{Dom } f$$

- **Tables of Signs:**

x **Values**	**Sign of** $f'(x)$	**Conclusion for** $f(x)$
$0 < x < e^2$	$+$	Increasing
$x = e^2$	0	Relative Maximum
$x > e^2$	$-$	Decreasing

x **Values**	**Sign of** $f''(x)$	**Conclusion for** $f(x)$
$0 < x < e^{8/3}$	$-$	Concave Downward
$x = e^{8/3}$	0	Inflection Point
$x > e^{8/3}$	$+$	Concave Upward

- **Curve:**



5. $f(x) = \cos^2 x - 2\sin x, x \in [0, 2\pi]$

- **Domain:**

  $\text{Dom } f = [0, 2\pi]$

- **Intercept/s:**

  x-intercept/s:

$$f(x) = 0 \Rightarrow \qquad \cos^2 x - 2\sin x = 0$$

$$(1 - \sin^2 x) - 2\sin x = 0$$

$$\sin^2 x + 2\sin x - 1 = 0$$

$$\sin x = -1 + \sqrt{2} \approx 0.41$$

$$x = \sin^{-1}(-1 + \sqrt{2}), \pi - \sin^{-1}(-1 + \sqrt{2})$$

  y-intercept: $y = f(0) = \cos^2 0 - 2\sin 0 = 1$

- **Asymptote/s:**

  VA: none, since f is continuous on $[0, 2\pi]$

  HA: none, since $\text{Dom } f = [0, 2\pi]$

- **Critical Number/s:**

  $f'(x) = -2\cos x(\sin x + 1)$

$$f'(x) = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \ or \ x = \frac{3\pi}{2}$$

$f'(x)$ dne $\Rightarrow$ no solution

$$f''(x) = 2(2\sin x - 1)(\sin x + 1)$$

$$f''(x) = 0 \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6} \ or \ x = \frac{3\pi}{2}$$

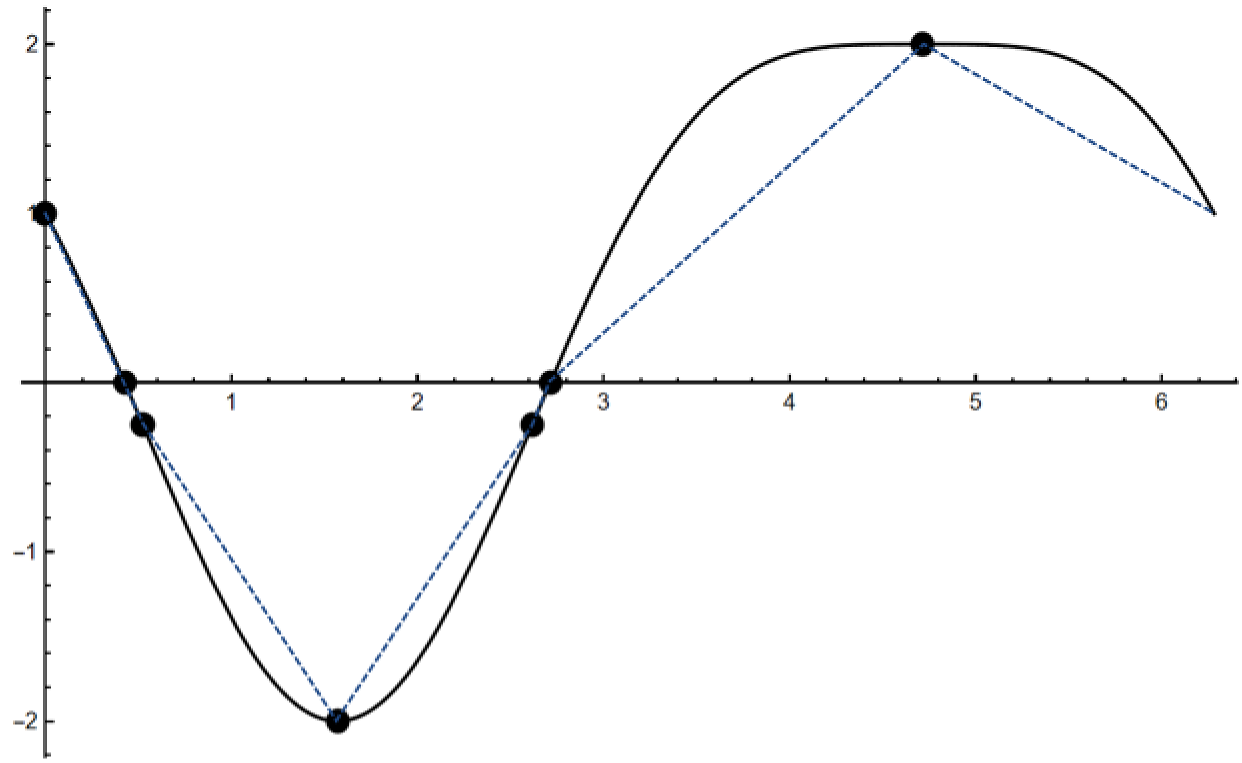
$f''(x)$ dne $\Rightarrow$ no solution

- **Tables of Signs:**

x Values	Sign of $f'(x)$	Conclusion for $f(x)$
$0 \leq x < \frac{\pi}{2}$	$-$	Decreasing
$x = \frac{\pi}{2}$	0	Relative Minimum
$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$+$	Increasing
$x = \frac{3\pi}{2}$	0	Relative Maximum
$\frac{3\pi}{2} < x \leq 2\pi$	$-$	Decreasing

x Values	Sign of $f'(x)$	Conclusion for $f(x)$
$0 \leq x < \frac{\pi}{6}$	$-$	Concave Downward
$x = \frac{\pi}{6}$	0	Inflection Point
$\frac{\pi}{6} < x < \frac{5\pi}{6}$	$+$	Concave Upward
$x = \frac{5\pi}{6}$	0	Inflection Point
$\frac{5\pi}{6} < x < \frac{3\pi}{2}$	$-$	Concave Downward
$x = \frac{3\pi}{2}$	0	
$\frac{3\pi}{2} < x < 2\pi$	$-$	Concave Downward

- **Curve:**



Exercises E

1.

a. $f(x) = xe^{-x^2}$

- **Domain:**

Dom $f = \mathbb{R}$

- **Intercept/s:**

x-intercept/s:

$$f(x) = xe^{-x^2} = \frac{x}{e^{x^2}} = 0 \Rightarrow x = 0$$

y-intercept:

$$f(0) = 0$$

- **Asymptote/s:**

VA:

Candidate/s: $e^{x^2} = 0 \Rightarrow$ no solution

$$\Rightarrow \text{no VA}$$

HA:

$$\lim_{x\to+\infty} \frac{x}{e^{x^2}} \quad \left(\frac{+\infty}{+\infty}\right)$$

$$=_{LH} \lim_{x\to+\infty} \frac{1}{e^{x^2}\cdot 2x} \quad \left(\frac{1}{+\infty}\right)$$

$$= 0$$

$$\Rightarrow y = 0 \text{ is a HA}$$

$$\lim_{x\to-\infty} \frac{x}{e^{x^2}} \quad \left(\frac{-\infty}{+\infty}\right)$$

$$=_{LH} \lim_{x\to-\infty} \frac{1}{e^{x^2}\cdot 2x} \quad \left(\frac{1}{-\infty}\right)$$

$$= 0$$

$$\Rightarrow y = 0 \text{ is a HA}$$

- **Critical Number/s:**

$$f'(x) = 1\cdot e^{-x^2} + e^{-x^2}(-2x)x$$

$$= e^{-x^2} - 2e^{-x^2}x^2$$

$$= e^{-x^2}(1-2x^2)$$

$$= \frac{1-2x^2}{e^{x^2}}$$

$$f'(x) = 0 \Rightarrow 1 - 2x^2 = 0$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$$

$$f'(x) \text{ dne} \Rightarrow e^{x^2} = 0$$

$$\Rightarrow \text{no solution}$$

$$f''(x) = \frac{-4x \cdot e^{x^2} - e^{x^2}(2x)(1-2x^2)}{e^{2x^2}}$$

$$= \frac{xe^{x^2}[-4 - 2(1-2x^2)]}{e^{2x^2}}$$

$$= \frac{xe^{x^2}[-6 + 4x^2]}{e^{2x^2}}$$

$$f''(x) = 0 \Rightarrow xe^{x^2}[-6 + 4x^2] = 0$$

$$\Rightarrow x = 0 \text{ or } e^{x^2} = 0 \text{ or } x^2 = \frac{3}{2}$$

$$\Rightarrow x = 0 \text{ or } x = \pm\sqrt{\frac{3}{2}}$$

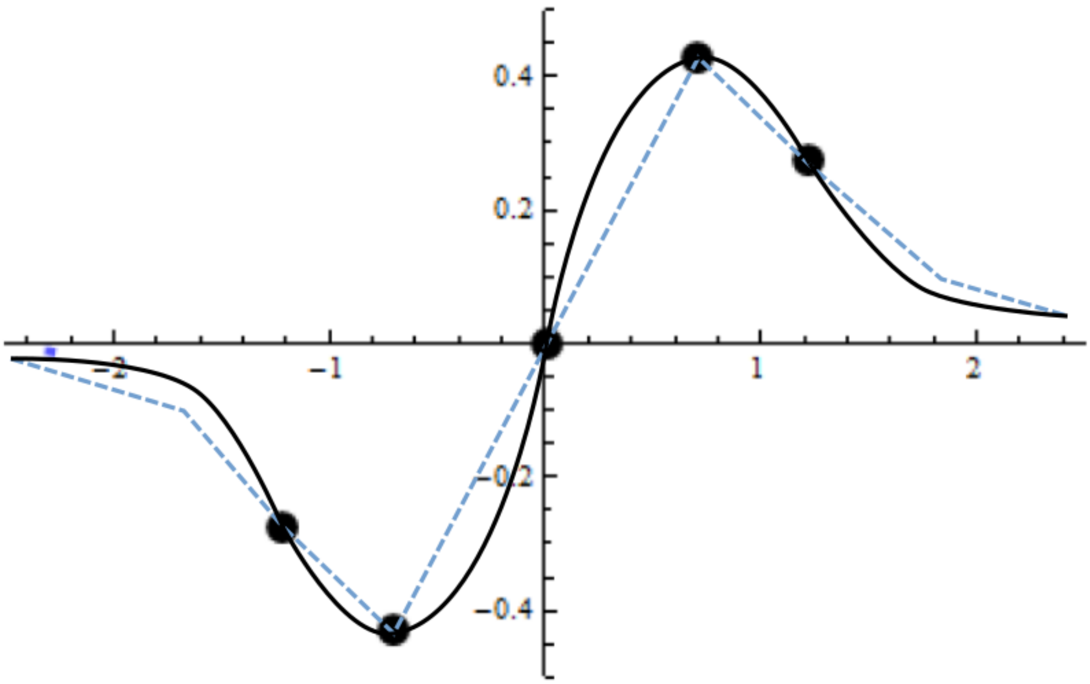
$$f''(x) \text{ dne} \Rightarrow e^{2x^2} = 0 \Rightarrow \text{no solution}$$

- **Table of Signs:**

x **Values**	**Sign of** $f'(x)$	**Conclusion for** $f(x)$
$x < -\sqrt{\frac{1}{2}}$	$-$	Decreasing
$x = -\sqrt{\frac{1}{2}}$	0	Relative Minimum
$-\sqrt{\frac{1}{2}} < x < \sqrt{\frac{1}{2}}$	$+$	Increasing
$x = \sqrt{\frac{1}{2}}$	0	Relative Maximum
$x > \sqrt{\frac{1}{2}}$	$-$	Decreasing

x Values	Sign of $f''(x)$	Conclusion for $f(x)$
$x < -\sqrt{\frac{3}{2}}$	$-$	Concave Downward
$x = -\sqrt{\frac{3}{2}}$	0	Inflection Point
$-\sqrt{\frac{3}{2}} < x < 0$	$+$	Concave Upward
$x = 0$	0	Inflection Point
$0 < x < \sqrt{\frac{3}{2}}$	$-$	Concave Downward
$x = \sqrt{\frac{3}{2}}$	0	Inflection Point
$x > \sqrt{\frac{3}{2}}$	$+$	Concave Upward

- **Curve:**



3. $f(x) = x^3 + ax^2 + b$, relative extremum at (2,3)

Solution.

- $3 = f(2) = 2^3 + a \cdot 2^2 + b = 8 + 4a + b$

  $\Rightarrow 4a + b + 8 = 3$

  $\Rightarrow \quad 4a + b = -5$

- $f'(x) = 3x^2 + 2ax$

  $f'(2) = 3 \cdot 2^2 + 2a \cdot 2$

$0 \ = 12 + 4a$

$a = -3$

- $4a + b = -5$ and $a = -3 \Rightarrow b = -5 - 4a = 7$

$\boxed{a = -3, b = 7}$

4. $f(x) = ax^3 + bx^2$, inflection point at $(1, 2)$

<u>Solution</u>.

- $2 = f(1) = a + b$

$\Rightarrow a + b = 2$

- $f'(x) = a \cdot 3x^2 + b \cdot 2x$

$= 3ax^2 + 2bx$

$f''(x) = 3a \ \cdot 2x + 2b \cdot 1$

$= 6ax + 2b$

$f''(1) = 6a + 2b$

$0 = 6a + 2b$

- $a + b = 2$ and $6a + 2b = 0$:

$a + b = 2$

$a = 2 - b$

$6a + 2b = 0$

$6(2 - b) + 2b = 0$

$12 - 6b + 2b = 0$

$b = 3 \Rightarrow a = 2 - 3 = -1$

$\boxed{a = -1, b = 3}$

Other Features of a Graph

1. $f(x) = x^2 + \cos x$ is an even function since $x^2 + \cos x = (-x)^2 + \cos(-x)$.

2. $f(x) = \sin x + \tan x$ is an odd function since $-[\sin x + \tan x] = \sin(-x) + \tan(-x)$.

3. $f(x) = \tan x$ is a periodic function with period π since $\tan x = \tan(x + \pi)$.

4. $f(x) = \dfrac{x^5 + 1}{x^4 - 1} = x + \dfrac{x + 1}{x^4 - 1} \Rightarrow y = x$ is an oblique asymptote:

$$\lim_{x \to \pm\infty} \left[\frac{x^5 + 1}{x^4 - 1} - x\right] = \lim_{x \to \pm\infty} \left[\left(x + \frac{x + 1}{x^4 - 1}\right) - x\right]$$

$$= \lim_{x \to \pm\infty} \left[\frac{x + 1}{x^4 - 1}\right]$$

$$= 0$$

Illustration 9

- **Domain:**

  Dom $f = \mathbb{R}$

- **Intercepts/s:**

  x-intercepts: $x = -4.6, 0$

  y-intercept: $y = 0$

- **Asymptotes:**

  VA: none

  HA: none

- **Critical Numbers:**

  $f'(x) = 0 \Rightarrow x = -3, 2, 4$

  $f'(x)$ dne $\Rightarrow$ no solution

  $f''(x) = 0 \Rightarrow x = -1.25, 2.9$

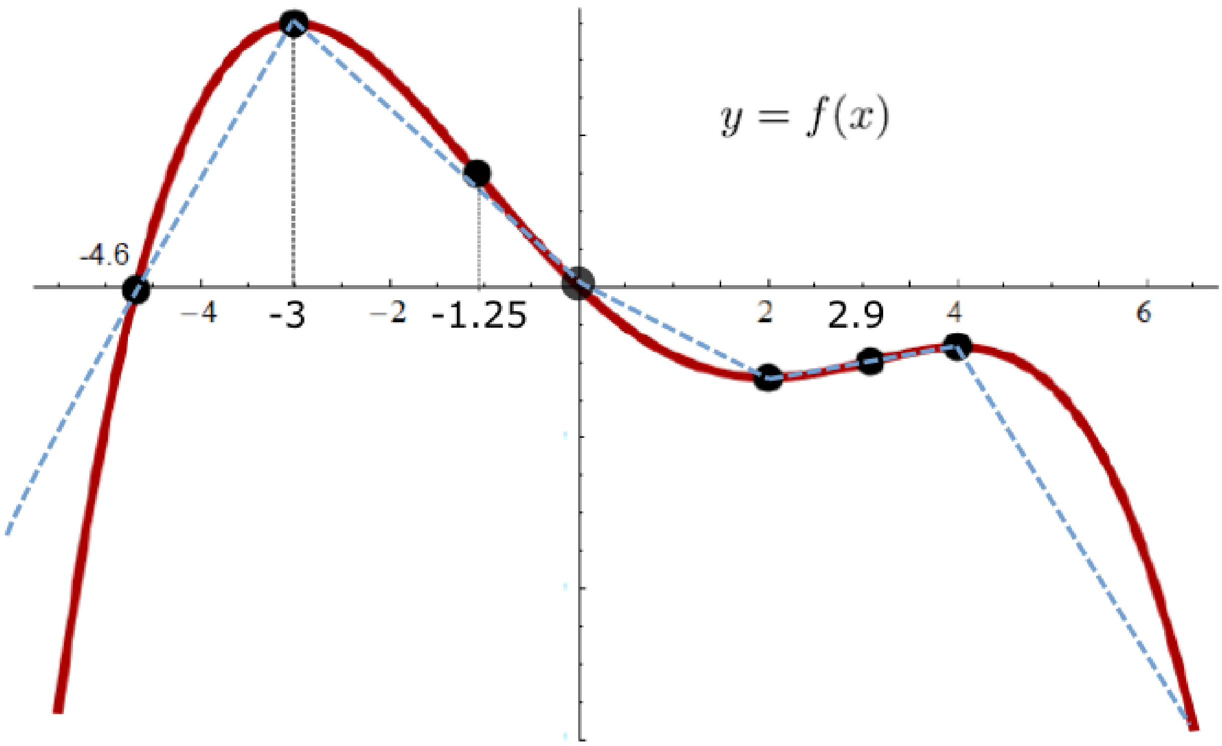
  $f''(x)$ dne $\Rightarrow$ no solution

- **Table of Signs:**

x **Values**	**Sign of** $f'(x)$	**Conclusion for** $f(x)$
$x < -3$	$+$	Increasing
$x = -3$	0	Relative Maximum
$-3 < x < 2$	$-$	Decreasing
$x = 2$	0	Relative Minimum
$2 < x < 4$	$+$	Increasing
$x = 4$	0	Relative Maximum
$x > 4$	$-$	Decreasing

x **Values**	**Sign of** $f''(x)$	**Conclusion for** $f(x)$
$x < -1.25$	$-$	Concave Downward
$x = -1.25$	0	Inflection Point
$-1.25 < x < 2.9$	$+$	Concave Upward
$x = 2.9$	0	Inflection Point
$x > 2.9$	$-$	Concave Downward

- **Curve:**



Exercises F

1.

- **Domain:**

  Dom $f = \mathbb{R}$

- **Intercepts/s:**

  x-intercept/s: to be assigned later
  y-intercept: to be assigned later

- **Asymptotes:**

  VA: none

  HA: none

- **Critical Numbers:**

  $f'(x) = 0 \Rightarrow x = 1$
  $f'(x)$ dne $\Rightarrow x = 0$

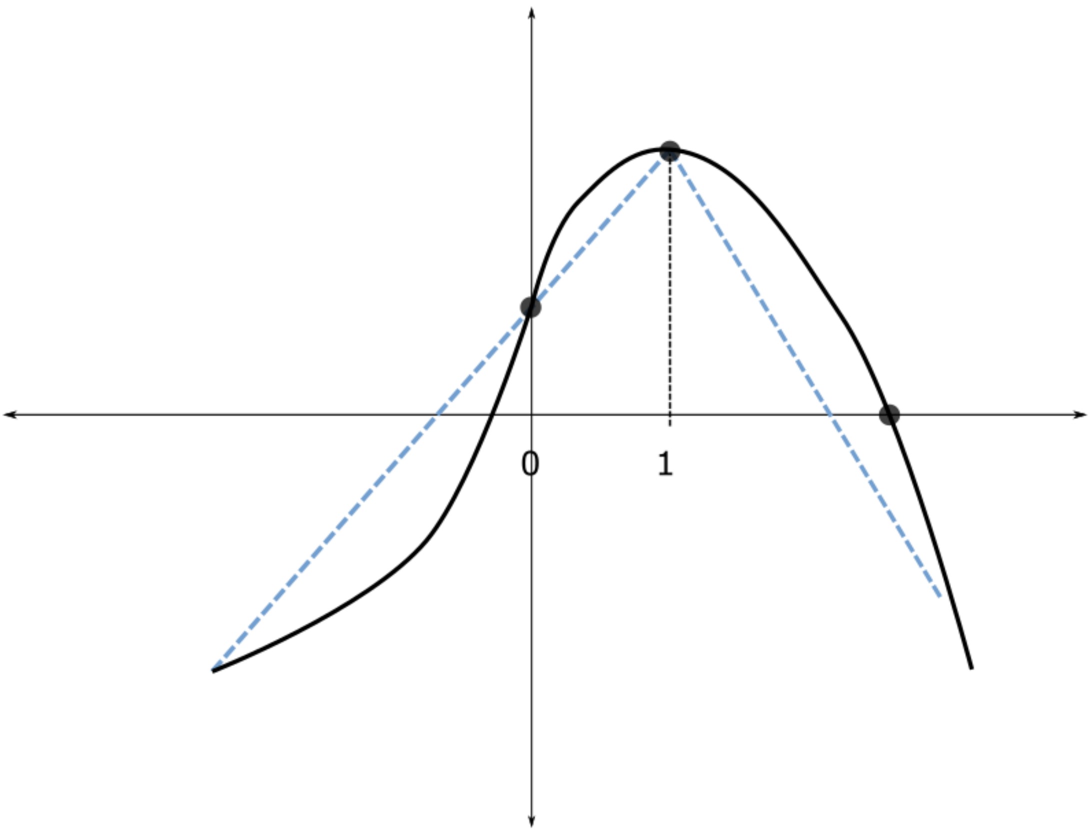
  $f''(x) = 0 \Rightarrow$ no solution
  $f''(x)$ dne $\Rightarrow x = 0$

- **Table of Signs:**

x **Values**	**Sign of** $f'(x)$	**Conclusion for** $f(x)$
$x < 0$	$+$	Increasing
$x = 0$	dne	
$0 < x < 1$	$+$	Increasing
$x = 1$	0	Relative Maximum
$x > 1$	$-$	Decreasing

x **Values**	**Sign of** $f''(x)$	**Conclusion for** $f(x)$
$x < 0$	$+$	Concave Upward
$x = 0$	dne	Inflection Point
$x > 0$	$-$	Concave Downward

- **Curve:**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