Home Assignment 1 - SF2955

Sequential Monte-Carlo-based mobility tracking in cellular networks

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Introduction

In this assignment a target moving in $\mathbb{R}^2$ according some dynamics described by the model

$$\mathbf{X}_{n+1} = \mathbf{\Phi}\mathbf{X}_n + \mathbf{\Psi}_z\mathbf{Z}_n + \mathbf{\Psi}_w\mathbf{W}_{n+1}, \quad n \in \mathbb{N} \tag{1}$$

is considered. In the model, $\mathbf{X}_n = (X_n^1, \dot{X}_n^1, \ddot{X}_n^1, X_n^2, \dot{X}_n^2, \ddot{X}_n^2)^\intercal$ is a state vector containing the target's positions (X_n^1, X_n^2) (in m), velocities $(\dot{X}_n^1, \dot{X}_n^2)$ (m s^{-1}) and accelerations $(\ddot{X}_n^1, \ddot{X}_n^2)$ (m s^{-2}) along the x_1 and x_2 directions respectively, for each n. $\{\mathbf{Z}_n\}_{n\in\mathbb{N}^*}$ is the driving command modeled by a bivariate Markov chain taking on the values

$$\{(0,0)^\intercal, (3.5,0)^\intercal, (0,3.5)^\intercal, (0,-3.5)^\intercal, (-3.5,0)^\intercal\}. \tag{2}$$

The chain evolve according to the transition probability matrix

$$\mathbf{P} = \frac{1}{20}\begin{pmatrix} 16 & 1 & 1 & 1 & 1 \\ 1 & 16 & 1 & 1 & 1 \\ 1 & 1 & 16 & 1 & 1 \\ 1 & 1 & 1 & 16 & 1 \\ 1 & 1 & 1 & 1 & 16 \end{pmatrix}.$$

$\{\mathbf{W}_n\}_{n\in\mathbb{N}^*}$ are bivariate, mutually independent normally distributed noise variables. Each $\mathbf{W}_n$ is $N(\mathbf{0}_{2\times 1}, \sigma^2\mathbf{I})$-distributed with $\sigma = 0.5$.

$\mathbf{\Phi}, \mathbf{\Psi}_z$ and $\mathbf{\Psi}_w$ are matrices given by

$$\mathbf{\Phi} = \begin{pmatrix} \tilde{\mathbf{\Phi}} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \tilde{\mathbf{\Phi}} \end{pmatrix} \quad \text{and} \quad \mathbf{\Psi}_\bullet = \begin{pmatrix} \tilde{\mathbf{\Psi}}_\bullet & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 1} & \tilde{\mathbf{\Psi}}_\bullet \end{pmatrix},$$

where

$$\tilde{\mathbf{\Phi}} = \begin{pmatrix} 1 & \Delta_t & \Delta_t^2/2 \\ 0 & 1 & \Delta_t \\ 0 & 0 & \alpha \end{pmatrix}, \quad \tilde{\mathbf{\Psi}}_z = \begin{pmatrix} \Delta_t^2/2 \\ \Delta_t \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{\Psi}}_w = \begin{pmatrix} \Delta_t^2/2 \\ \Delta_t \\ 0 \end{pmatrix}$$

with $\Delta_t = 0.5$ (s) being the sampling discretization period and $\alpha = 0.6$ the correlation between subsequent acceleration values.

The initial state vector $\mathbf{X}_0$ is assumed to be $N(\mathbf{0}_{6\times 1}, \mathrm{diag}(500, 5, 5, 200, 5, 5))$-distributed and the initial driving command $\mathbf{Z}_0$ is supposed to be uniformly distributed over the set (2).

Problem 1

In the first part of the assignment a MATLAB code simulating a trajectory $\{(\mathrm{X}_n^1,\ \mathrm{X}_n^2)\}_{n=0}^m$ of some arbitrary length m is implemented and the trajectory is plotted in figure 1.

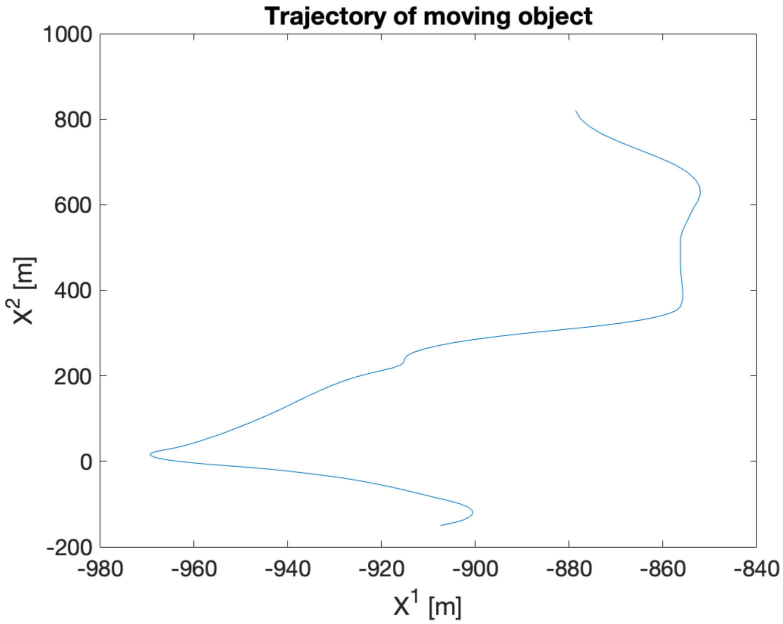

Figure 1: Trajectory of a moving object with length $m = 100$.

For all $n \in \mathbb{N}$ we have that $\tilde{\mathbf{X}}_n = (\mathbf{X}_n^\intercal, \mathbf{Z}_n^\intercal)$. We want to investigate if $\{\mathbf{X}_n\}_{n\in\mathbb{N}}$ and $\{\tilde{\mathbf{X}}_n\}_{n\in\mathbb{N}}$ are Markov chains. An important property of Markov chains is that the distribution for subsequent states only depend on the current state and not any previous states. Therefore $\{\mathbf{X}_n\}_{n\in\mathbb{N}}$ is not a Markov chain since it does not fulfill this property. If we assume that the current state is $\mathbf{X}_n$ and use equation 1 to describe it. By rearranging the equation we can derive the most probable previous driving command $\mathbf{Z}_{n+1}$.

$$\mathbf{X}_n - \mathbf{\Phi}\mathbf{X}_{n-1} - \mathbf{\Psi}_w\mathbf{W}_n = \mathbf{\Psi}_z\mathbf{Z}_{n-1}$$

The transition probability for $\mathbf{Z}_{n-1}$ is known and therefore we can obtain the most likely current driving command $\mathbf{Z}_n$ which will have an direct impact on the future state $\mathbf{X}_{n+1}$. Thus the future state $\mathbf{X}_{n+1}$ is dependent both on the current state $\mathbf{X}_n$ and the previous state $\mathbf{X}_{n-1}$. Therefore we gain more information about future states by taking into account previous states. However for $\{\tilde{\mathbf{X}}_n\}_{n\in\mathbb{N}}$ all factors that directly impact the future state are being regarded in the current state. We cannot derive more information about the future state by taking into account the previous state since $\mathbf{W}_n$ is memoryless. Therefore the Markov property is fulfilled. We can conclude that $\{\mathbf{X}_n\}_{n\in\mathbb{N}}$ is not a Markov chain while $\{\tilde{\mathbf{X}}_n\}_{n\in\mathbb{N}}$ is.

Problem 2

As the target is moving, it measures online the pilot signal strengths from the basis stations of a cellular network. The network comprises $s = 6$ basis stations (BS), whose positions $\{\boldsymbol{\pi}_l\}_{l=1}^s$ in the plane are known. The RSSI (measured in dB) that the mobile unit receives from the lth BS at time $n \in \mathbf{N}$ can be modeled as

$$Y_n^l = v - 10\eta \log_{10} \left\| (X_n^1, X_n^2)^\intercal - \boldsymbol{\pi}_l \right\| + V_n^l,$$

where $\|\bullet\|$ denotes the Euclidean distance, $v = 90$ (dB) is the base station transmission power, $\eta = 3$ is the so-colled slope index and $\{V_n^l\}_{l=1}^s$ are independent Gaussion noise variables with mean zero and standard deviation $\varsigma = 1.5$ (dB). We denote by $\mathbf{Y}_n = (Y_n^1, ..., Y_n^s)^\intercal$ the RSSIs received at time n from all the BSs in the network.

$\{\tilde{\mathbf{X}}_n, \mathbf{Y}_n\}_{n\in\mathbb{N}}$ forms a hidden Markov model since $\{\tilde{\mathbf{X}}_n\}_{n\in\mathbb{N}}$ is a Markov model and $\mathbf{Y}_n$ is the observation from the Markov model. Since we have noise, $\mathbf{Y}_n$ is not observed directly and therefore $\{\tilde{\mathbf{X}}_n, \mathbf{Y}_n\}_{n\in\mathbb{N}}$ forms a hidden Markov model.

For each l we have that

$$Y_n^l \sim N(v - 10\eta\log_{10}\left\|(X_n^1, X_n^2) - \boldsymbol{\pi}_l\right\|, \varsigma^2)$$

thus it follows that

$$\mathbf{y}_n|\tilde{\mathbf{x}}_n \sim N([v - 10\eta\log_{10}\left\|(X_n^1, X_n^2) - \boldsymbol{\pi}_l\right\|]_{1\times 6}, \varsigma^2\mathbf{I})$$

Problem 3

The vector $\mathbf{y}_{0:m} = (\mathbf{y}_0, ..., \mathbf{y}_m)$, $m = 500$ contains RSSI measurements on a moving target with an unkonwn trajectory. We want to estimate sequentially the expected positions

$$\tau_n^1 = \mathbb{E}[X_n^1|\mathbf{Y}_{0:n} = \mathbf{y}_{0:n}] \text{ and } \tau_n^2 = \mathbb{E}[X_n^2|\mathbf{Y}_{0:n} = \mathbf{y}_{0:n}] \tag{3}$$

for $n = 0, 1, 2, ...$ Sequential Monte Carlo methods (SMC) will be used since the model is nonlinear. We will evolve a particle sample $\{(\tilde{\mathbf{X}}_{0:n}^i, \omega_n^i)\}_{i=1}^N$ targeting sequentially, as new measurements appear for $n = 0, 1, 2, ...$ the densities

$$f(\tilde{\mathbf{x}}_{0:n}|\mathbf{y}_{0:n}) = \frac{f(\tilde{\mathbf{x}}_{0:n}, \mathbf{y}_{0:n})}{f(\mathbf{y}_{0:n})} = \frac{q(\tilde{\mathbf{x}}_0)p(\mathbf{y}_0|\tilde{\mathbf{x}}_0)\prod_{k=1}^n p(\mathbf{y}_k|\tilde{\mathbf{x}}_k)q(\tilde{\mathbf{x}}_k|\tilde{\mathbf{x}}_{k-1})}{f(\mathbf{y}_{0:n})}$$

of the smoothing distributions $\tilde{\mathbf{X}}_{0:n}|\mathbf{Y}_{0:n}$, where $q(\tilde{\mathbf{x}}_k|\tilde{\mathbf{x}}_{k-1})$ and $q(\tilde{\mathbf{x}}_0)$ denote the transition density and initial distribution of $\{\tilde{\mathbf{X}}_n\}_{n\in\mathbb{N}}$, respectively. Since

$$\tau_n^1 = \int x_n^1 f(x_n^1|\mathbf{y}_{0:n})dx_n^1$$

we may use the components $\{X_n^{1,i}\}_{i=1}^N$ of the last particle generation $\{\tilde{\mathbf{X}}_n^i\}_{i=1}^N$ for approximating τ_n^1 and similarly for τ_n^2.

In the third part of the assignment the sequential importance sampling algorithm (SIS) is implemented for sampling from $f(\tilde{\mathbf{x}}_n|\mathbf{y}_{0:n})$, $n = 0, 1, ..., m$, for the observation stream. The expected positions $\{(\tau_n^1, \tau_n^2)\}_{n=0}^m$ are plotted in figure 2 together with the locations of the basis stations. The particle sample size is $N = 10000$.

The weights are calculated for each iteration and a histogram of the weights at iteration $n = 1, 10, 40$ are plotted in figures 3 a, 3 b and 3c.

The efficient sample size was calculated for a number of iterations. The efficient sample size is given by

$$ESS = \frac{N}{1 + CV_N^2}, \qquad CV_N^2 = \frac{1}{N}\sum_{i=1}^N \big(N\frac{\omega_n^i}{\sum_{j=1}^N \omega_n^j} - 1\big)^2$$

The ESS is displayed in table 1.

Number of iterations	n=1	n=10	n=20	n=20	n=80
ESS	9988.7	9462.3	3156	132.33	NaN

Table 1: The efficient sample sizes for iteration $n = 1, 10, 20, 60, 80$.

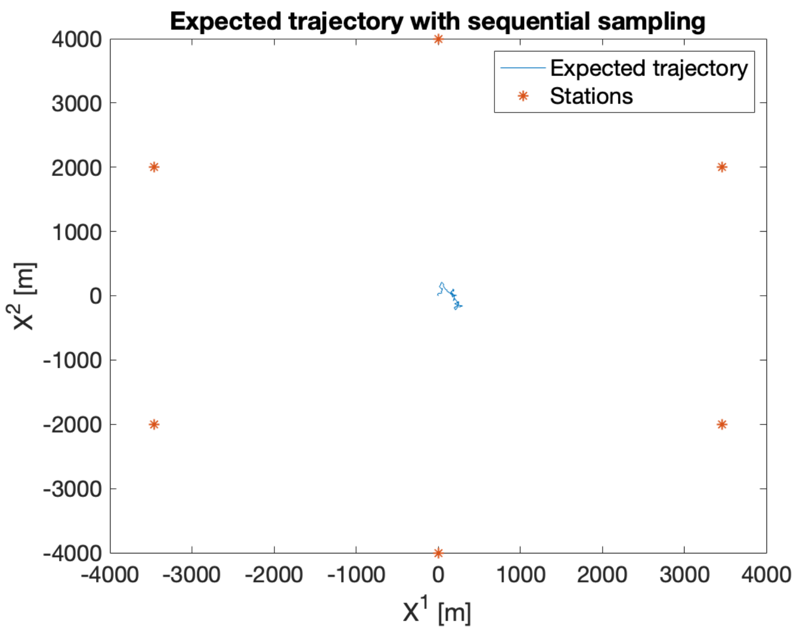

Figure 2: Expected trajectory of the moving object simulated by sequential importance sampling algorithm.

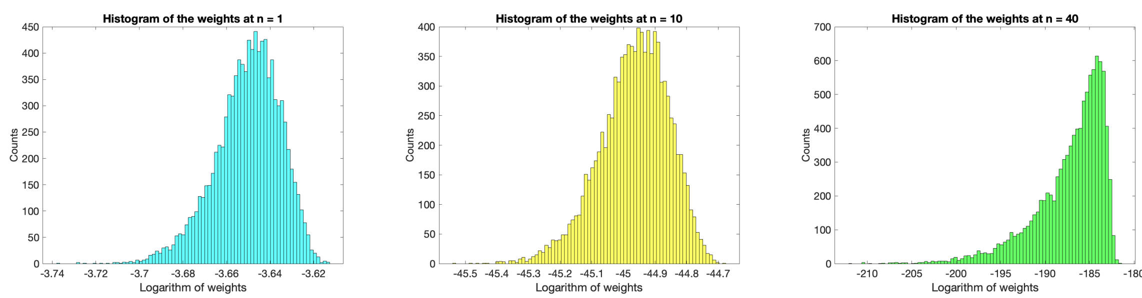

Figure 3: Histograms of the logarithm of the weights after iteration $n = 1, 10, 40$.

Problem 4

In the fourth part of the assignment, the expected trajectory of the moving object is calculated with Sequential importance sampling with re-sampling. The expected trajectory of the moving object $\{(\tau_n^1, \tau_n^2)\}_{n=0}^m$ is plotted in figure 4. Also here the particle sample size is $N = 10000$.

Furthermore, the most probable driving command at all time points are plotted in figure 5. It is found that the most probable driving command is north.

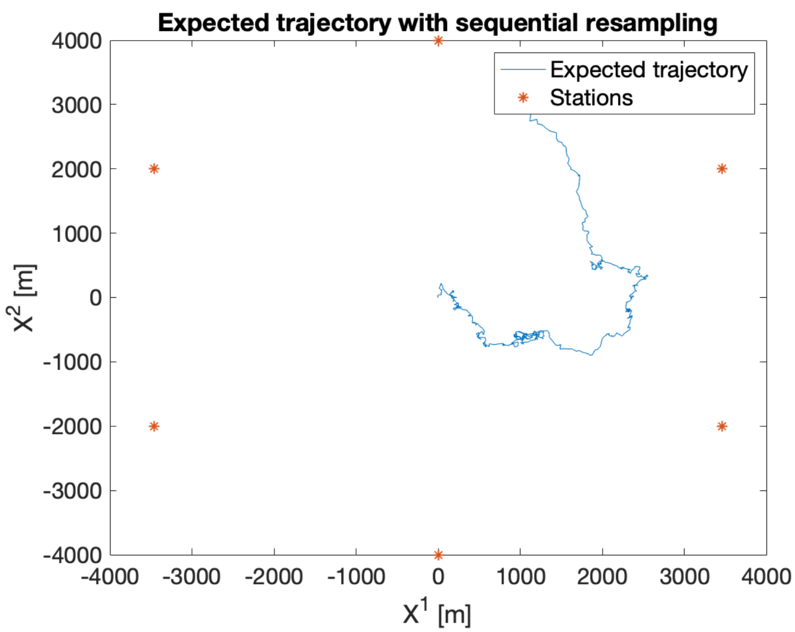

Figure 4: Expected trajectory of the moving object simulated by sequential importance re-sampling algorithm.

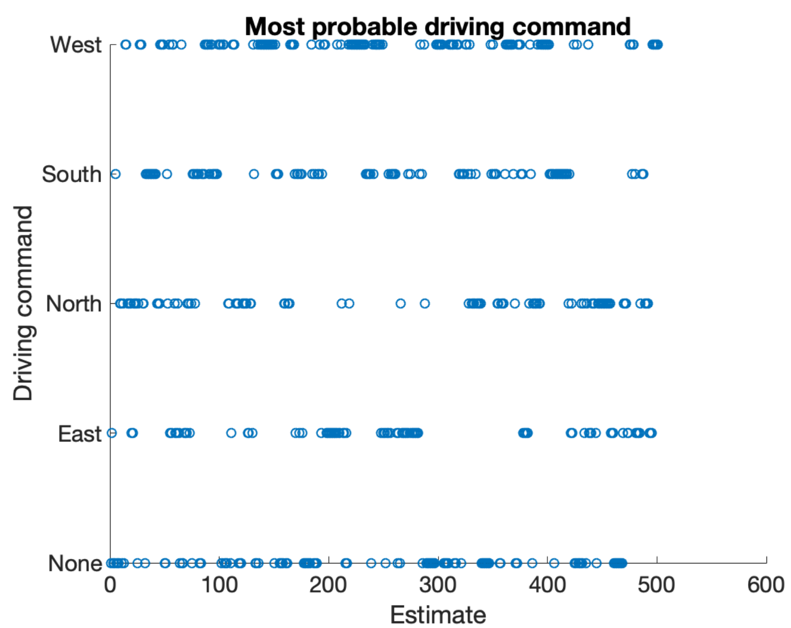

Figure 5: The most probable driving command calculated at all time points.

Problem 5

In the fifth part of this assignment, the variance of the observed signals, ς, is calibrated. It is known that $\varsigma \in (0, 3)$. So we divide the interval in steps of 0.1 and for each ς_j a Monto Carlo estimate of the log-likelihood is performed. Then the best ς is found by maximizing $l_m^N(\varsigma_j) = m^{-1}\ln(L_m^N(\varsigma, \mathbf{y}_{0:m}))$, where L is the log-likelihood function. The log-likelihood can be estimated as

$$L_n^N(\varsigma, \mathbf{y}_{0:m}) = \frac{1}{N^{m+1}} \prod_{k=0}^{m} \Omega_k,$$

where

$$\Omega_k = \sum_{i=1}^{N} \omega_k^i.$$

Therefore, the aim is to maximize

$$\frac{1}{m}\ln\Big(\frac{1}{N^{m+1}} \prod_{k=0}^{m} \Omega_k\Big) = \frac{1}{m} \sum_{k=0}^{m} \ln\Big(\frac{\Omega_k}{N}\Big) = \frac{1}{m} \sum_{k=0}^{m} (\ln(\Omega_k) - \ln(N)).$$

Thus, maximizing $\sum_{k=0}^{m} \ln(\Omega_k)$ is equivalent to maximizing $l_m^N(\varsigma_j)$. The log-likelihood function is plotted in Figure 6, together with the expected trajectory for the best ς.

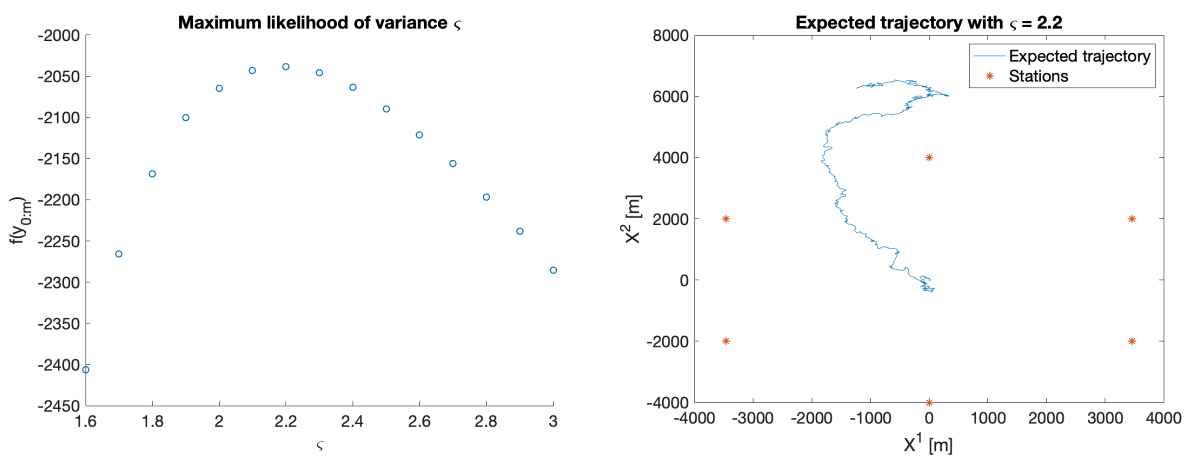

Figure 6: The maximum of the likelihood function is plotted in the left figure. In the right figure, the expected driving trajectory is plotted with $\varsigma = 2.2$

The best ς is found to be $\varsigma = 2.2$.