Active Contours

Internal Energies

Internal Energies

- E_{int} is the internal energy of the snake
- It tries to force the snake to be small and smooth

- $E_{int} = \frac{1}{2}(\alpha(s)|v_s(s)|^2 + \beta(s)|v_{ss}(s)|^2)$

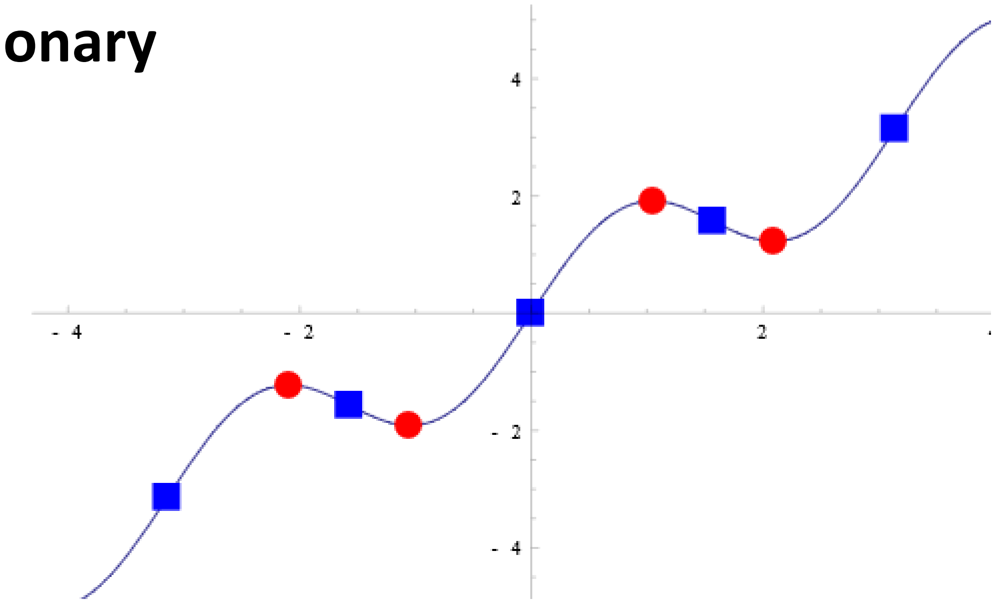
- First-order term $\alpha(s)|v_s(s)|^2$
  - magnitude of first derivative larger for longer snakes
- Second-order term $\beta(s)|v_{ss}(s)|^2$
  - magnitude of second derivative larger for sharper bends

Euler-Lagrange equation

- Minimize using **Euler-Lagrange** equation

- E-L in short
  - Second-order partial differential equation
  - Finds solutions that make the function **stationary**

- Stationary
  - Functions derivative is equal to 0
  - Red dots →



Source: https://en.wikipedia.org/wiki/Stationary_point

Conversion

- $E_{int} = \frac{1}{2}(\alpha(s)|v_s(s)|^2 + \beta(s)|v_{ss}(s)|^2)$

- Magic… (a lengthy conversion)

- $\alpha s(x,y)'' + \beta s(x,y)'''' + \frac{\partial E_{ext}}{\partial x,y} = 0$

Conversion

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- $\alpha s(x,y)'' + \beta s(x,y)'''' + \boxed{\frac{\partial E_{ext}}{\partial x,y}} = 0$ $\longrightarrow$ External Forces: F_{ext}

Solve with Gradient Descent

1. Set 0 to be derivative of s over time

- $\alpha s'' + \beta s'''' + F_{ext} = \frac{\partial s}{\partial t}$

- Logic
  - When the snake has converged to minimum, there will be no further change
  - Derivative of s over time will be 0

  → Equation is satisfied

Solve with Gradient Descent

2. Finite differences to get derivatives on N spline points

- E.g. for x direction:
  - $x''(i) = x(i-1) - 2x(i) + x(i+1)$
  - $x''''(i) = x(i-2) - 4x(i-1) + 6x(i) - 4x(i+1) + x(i+2)$

→ $\frac{\partial x_t(i)}{\partial t} = \alpha\big(x(i-1) - 2x(i) + x(i+1)\big) + \beta(x(i-2) - 4x(i-1) + 6x(i) - 4x(i+1) + x(i+2)) + f_x$

Solve with Gradient Descent

3. Formulate A

$$\frac{\partial x_t(i)}{\partial t} = \alpha\big(x(i-1) - 2x(i) + x(i+1)\big) + \beta(x(i-2) - 4x(i-1) + 6x(i) - 4x(i+1) + x(i+2)) + f_x$$

- For N = 2: $\frac{\partial x_t(2)}{\partial t} = \alpha\big(x(1) - 2x(2) + x(3)\big) + \beta(x(0) - 4x(1) + 6x(2) - 4x(3) + x(4)) + f_x$
- For N = 3: $\frac{\partial x_t(3)}{\partial t} = \alpha\big(x(2) - 2x(3) + x(4)\big) + \beta(x(1) - 4x(2) + 6x(3) - 4x(4) + x(5)) + f_x$
- …

- This can be rewritten in matrix form!

Solve with Gradient Descent

$$\frac{\partial x_t(i)}{\partial t} = \alpha\big(x(i-1) - 2x(i) + x(i+1)\big) + \beta(x(i-2) - 4x(i-1) + 6x(i) - 4x(i+1) + x(i+2)) + f_x$$

- We observe the following dependencies:
  - $x(i-2)$: β
  - $x(i-1)$: $\alpha - 4\beta$
  - $x(i)$: $-2\alpha + 6\beta$
  - $x(i+1)$: $\alpha - 4\beta$
  - $x(i+2)$: β

Solve with Gradient Descent

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  - $x(i+2)$: β

$$\frac{\partial X_t}{\partial t} = AX_t + f_x(X_t, Y_t)$$

A = NxN, N is the number of discrete points for the snake

$$A = \begin{bmatrix} -2\alpha+6\beta & \alpha-4\beta & \beta & 0 & 0 & 0 & \cdots & \beta & \alpha-4\beta \\ \alpha-4\beta & -2\alpha+6\beta & \alpha-4\beta & \beta & 0 & 0 & \cdots & 0 & \beta \\ \beta & \alpha-4\beta & -2\alpha+6\beta & \alpha-4\beta & \beta & 0 & \cdots & 0 & 0 \\ 0 & \beta & \alpha-4\beta & -2\alpha+6\beta & \alpha-4\beta & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Solve with Gradient Descent

4. Iterate

- We found: $\frac{\partial X_t}{\partial t} = AX_t + f_x(X_t, Y_t)$
- Now we assume that the movement of the snake in each iteration is very small
- $\frac{(X_t - X_{t-1})}{\gamma} = AX_t + f_x(X_t, Y_t)$

Solve with Gradient Descent

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- $X_t - X_{t-1} = \gamma AX_t + \gamma f_x(X_t, Y_t)$
- $X_t = \gamma AX_t + X_{t-1} + \gamma f_x(X_t, Y_t)$

Solve with Gradient Descent

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- $X_t = \gamma AX_t + X_{t-1} + \gamma f_x(X_t, Y_t)$
- $X_t - \gamma AX_t = X_{t-1} + \gamma f_x(X_t, Y_t)$
- $(I - \gamma A)X_t = X_{t-1} + \gamma f_x(X_t, Y_t)$

Solve with Gradient Descent

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- $X_t - \gamma AX_t = X_{t-1} + \gamma f_x(X_t, Y_t)$
- $(I - \gamma A)X_t = X_{t-1} + \gamma f_x(X_t, Y_t)$
- $X_t = (I - \gamma A)^{-1}(X_{t-1} + \gamma f_x(X_t, Y_t))$

Solve with Gradient Descent

4. Iterate

- $X_t = (I - \gamma A)^{-1}(X_{t-1} + \gamma f_x(X_t, Y_t))$
- $Y_t = (I - \gamma A)^{-1}(Y_{t-1} + \gamma f_y(X_t, Y_t))$