Digital Modulation 2

– Lecture Notes –

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Contents

References

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[2] J. G. Proakis, *Digital Communications*. McGraw-Hill, 1995.

[3] P. Laurent, "Exact and approximate construction of digital phase modulations by superposition of amplitude modulated pulses (AMP)," *IEEE Trans. Commun.*, vol. 34, no. 2, pp. 150–160, Feb. 1986.

[4] B. Rimoldi, "A decomposition approach to CPM," *IEEE Trans. Inform. Theory*, vol. 34, no. 2, pp. 260–270, Mar. 1988.

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1 Continuous-Phase Modulation

The Classical Representation

Continuous-phase modulation allows for a bandwith-efficient transmission. In this section, it is represented in the classical way. The modern representation is discussed in the following section.

1.1 General Description

The general description of a CPM signal reads

$$X(t;\boldsymbol{\alpha}) = \sqrt{2P}\cos\bigl(2\pi f_0 t + \varphi(t;\boldsymbol{\alpha})\bigr) \tag{1}$$

with the following notation:

The (semi-infinite) *sequence of information-bearing symbols* is

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots)$$

with the symbols

$$\begin{aligned} &\alpha_n \in \mathbb{A} = \{\pm 1, \pm 3, \ldots, \pm M-1\} &&\text{if } M \text{ is even} \\ &\alpha_n \in \mathbb{A} = \{0, \pm 2, \ldots, \pm M-1\} &&\text{if } M \text{ is odd} \end{aligned}$$

The mostly used case is that where M is even. (Or even a power of 2. Why?)

Example: Binary CPM

$M = 2$: $\alpha_n \in \{-1, +1\}$. ◇

The *time-variant information-bearing phase*

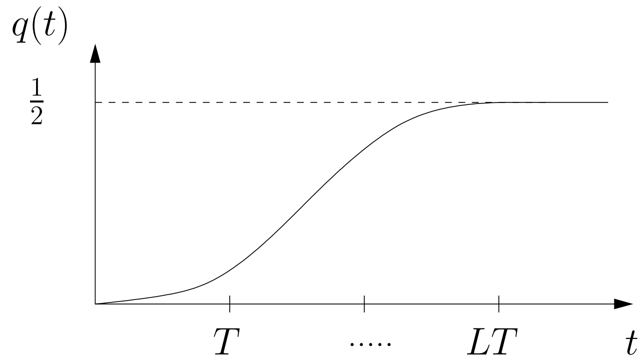
$$\varphi(t;\boldsymbol{\alpha}) = \varphi_0 \;+\; 2\pi h \sum_{n=0}^{\infty} \alpha_n q(t - nT)$$

Remark: In some cases (see later), $L-1$ known symbols are transmitted at the beginning. Then the summation starts with $-(L-1)$.

The *modulation index* is $h = \frac{p_1}{p_2}$, where p_1, p_2 are natural numbers that are relatively prime.

The function $q(t)$ is the *phase reponse.* It satisfies

$$\begin{aligned} q(t) &= 0 && \text{for } t \leq 0, \\ q(t) &= \frac{1}{2} && \text{for } t > LT. \end{aligned}$$



The value L is the *memory of the CPM system*:

$$\begin{aligned} L &= 1 && : \text{ full response CPM} \\ L &> 1 && : \text{ partial response CPM} \end{aligned}$$

The phase φ_0 is arbitrary.
Without loss of generality, we can assume that φ_0 is set such that $\varphi(0;\boldsymbol{\alpha}) = 0$.

The *carrier frequency* is f_0.

The *power of the transmitted signal* is P.

The *symbol rate* is $\frac{1}{T}$.

Instantaneous Frequency of a CPM signal

$$
\begin{aligned}
f(t;\boldsymbol{\alpha}) &= \frac{1}{2\pi}\,\frac{d}{dt}\,\Big[2\pi f_0 t + \varphi(t;\boldsymbol{\alpha})\Big] \\
&= f_0 \;+\; \frac{1}{2\pi}\,\frac{d}{dt}\,\Big[\varphi(t;\boldsymbol{\alpha})\Big] \\
&= f_0 \;+\; h\frac{d}{dt}\,\Big[\sum_{n=0}^{\infty}\alpha_n q(t-nT)\;\Big] \\
f(t;\boldsymbol{\alpha}) &= f_0 \;+\; \underbrace{h\sum_{n=0}^{\infty}\alpha_n q'(t-nT)}_{\Delta f(t;\boldsymbol{\alpha}) \quad \text{(frequency deviation)}}
\end{aligned}
$$

The derivative

$$
q'(t) = \frac{d}{dt}q(t)
$$

of $q(t)$ is called the *frequency response* of the CPM signal.

We can rewrite the CPM signal from (1) as a function of the instantaneous frequency according to

$$
\begin{aligned}
x(t;\boldsymbol{\alpha}) &= \sqrt{2P}\cos\Big(2\pi\int_0^t f(\tilde{t};\boldsymbol{\alpha})\;d\tilde{t}\;\Big) \\
&= \sqrt{2P}\cos\Big(2\pi f_0 t \;+\; 2\pi h\Big[\int_0^t\sum_{n=0}^{\infty}\alpha_n q'(\tilde{t}-nT)\;d\tilde{t}\;\Big]\Big)
\end{aligned}
$$

Properties of the Frequency Response

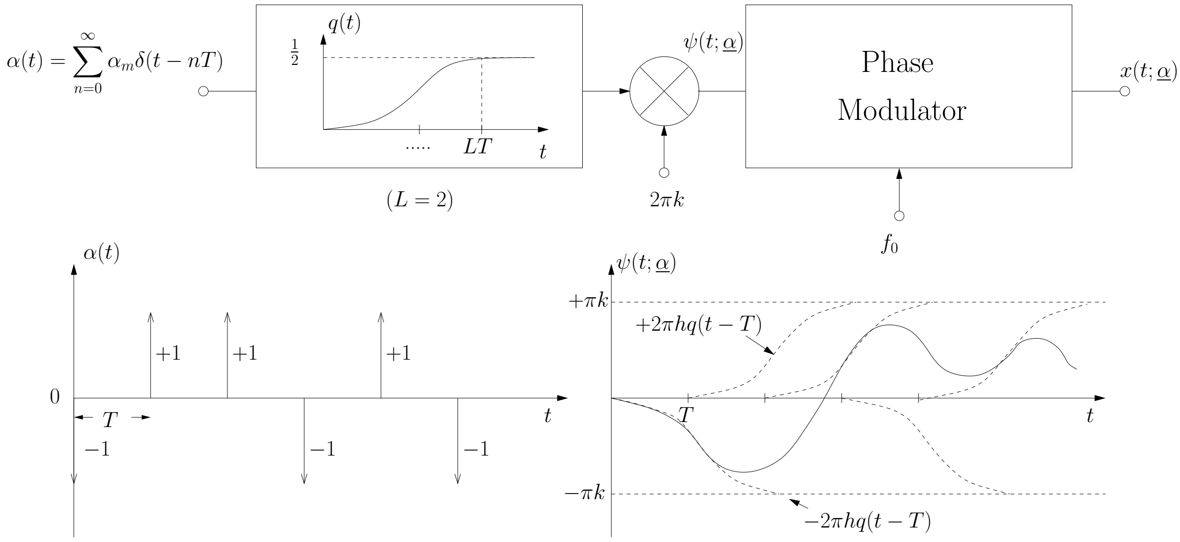
$$
\begin{aligned}
q(t) &= 0,\ t < 0 && \Rightarrow && q'(t) = 0,\ t < 0 \\
q(t) &= \frac{1}{2},\ t > LT && \Rightarrow && q'(t) = 0,\ t > LT \\
q(LT) &= \frac{1}{2} && \Rightarrow && \int_0^{LT} q'(\tilde{t})d\tilde{t} = \frac{1}{2}
\end{aligned}
$$

EXAMPLE:

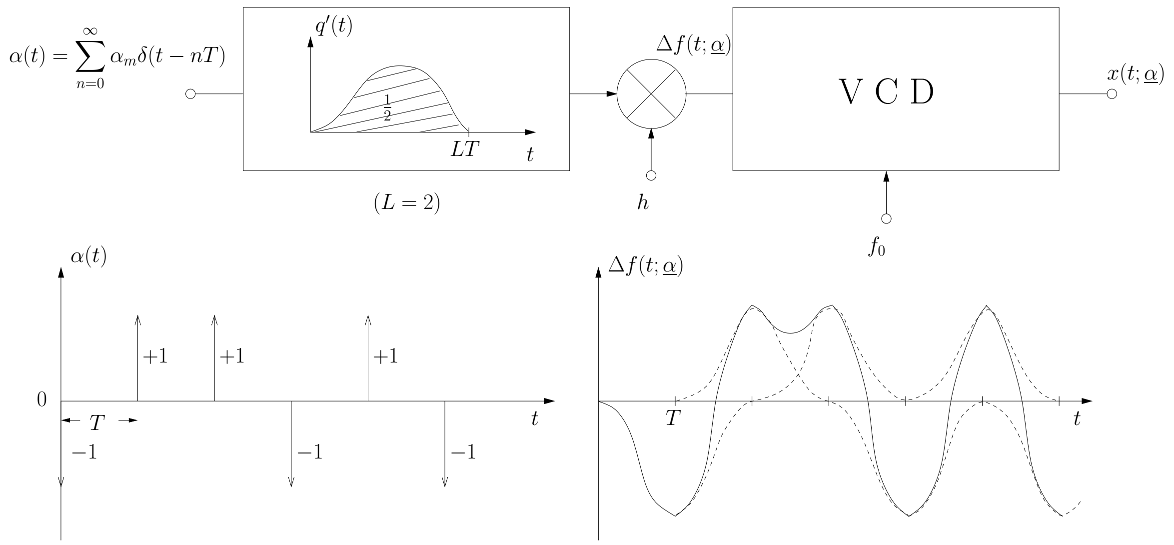
Sketch a valid frequency response for $L = 4$. ◇

Block diagram of a CPM transmitter

(a) Using a phase modulator



(b) Using a frequency modulator

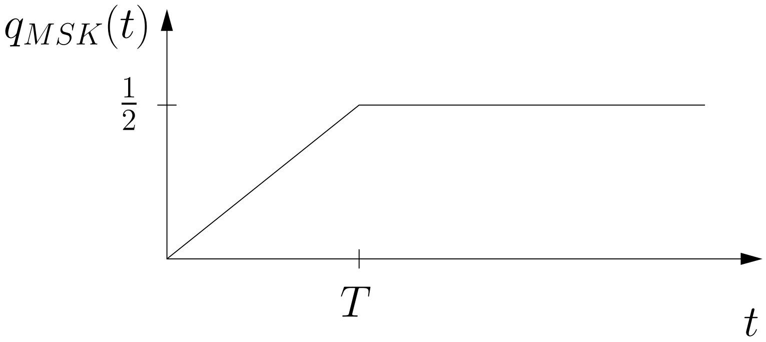


1.2 Minimum-Shift Keying

MSK has the following parameters:

- Binary modulation symbols, i.e., $M = 2$
- Modulation index $h = \frac{1}{2}$
- Phase response

$$q_{MSK}(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} \cdot \frac{t}{T} & 0 \leq t < T \\ \frac{1}{2} & t \geq T \end{cases}$$



Example: MSK

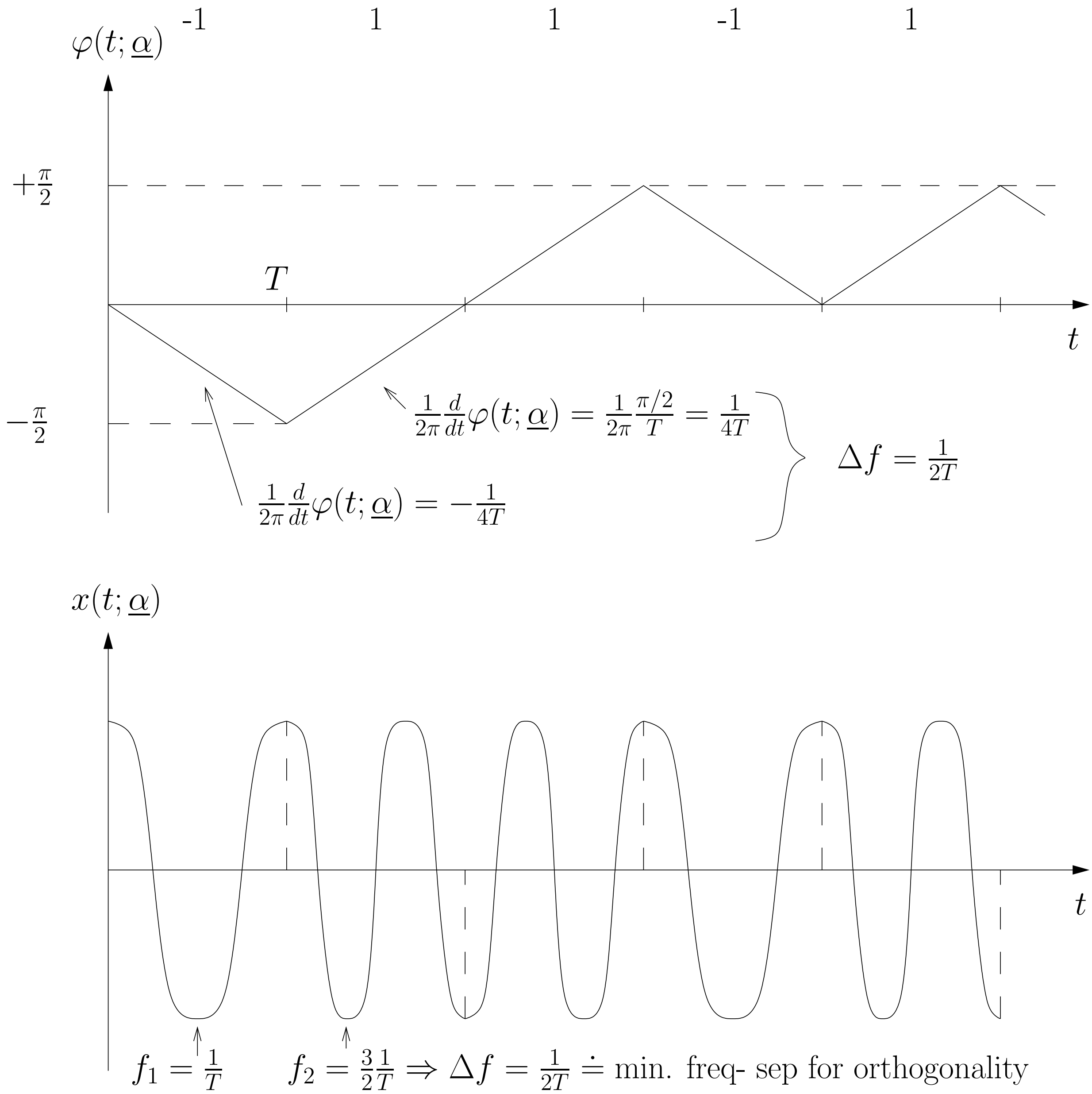
Assume the transmitted symbol sequence

$$\boldsymbol{\alpha} = [-1 + 1 + 1 - 1 + 1].$$

Sketch the phase and the signal. ◇

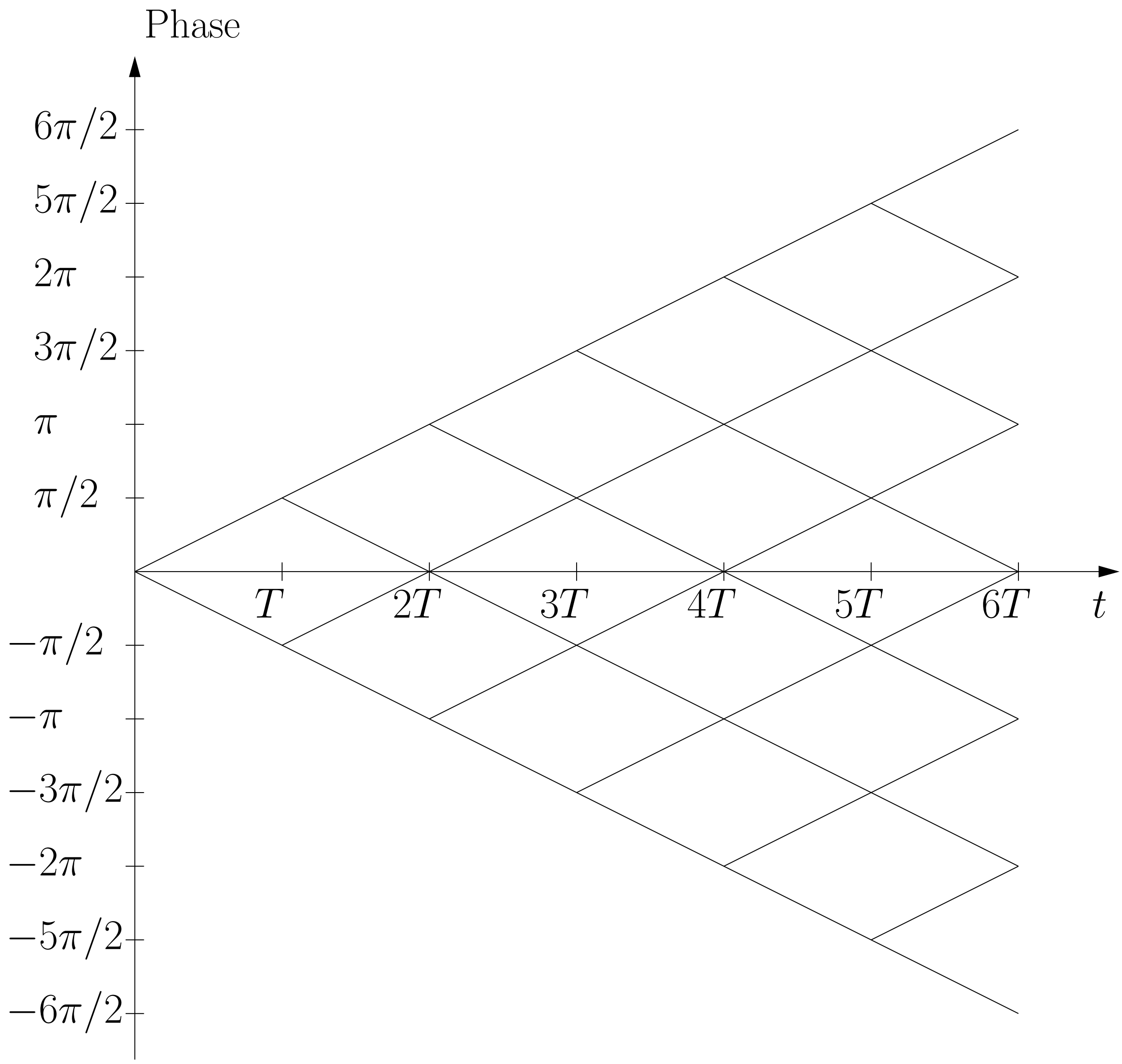
Example: MSK, continued

Transmitted symbols α_n:



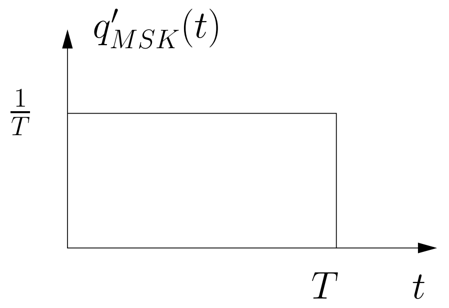
◇

Phase tree of MSK



Instantaneous frequency of MSK

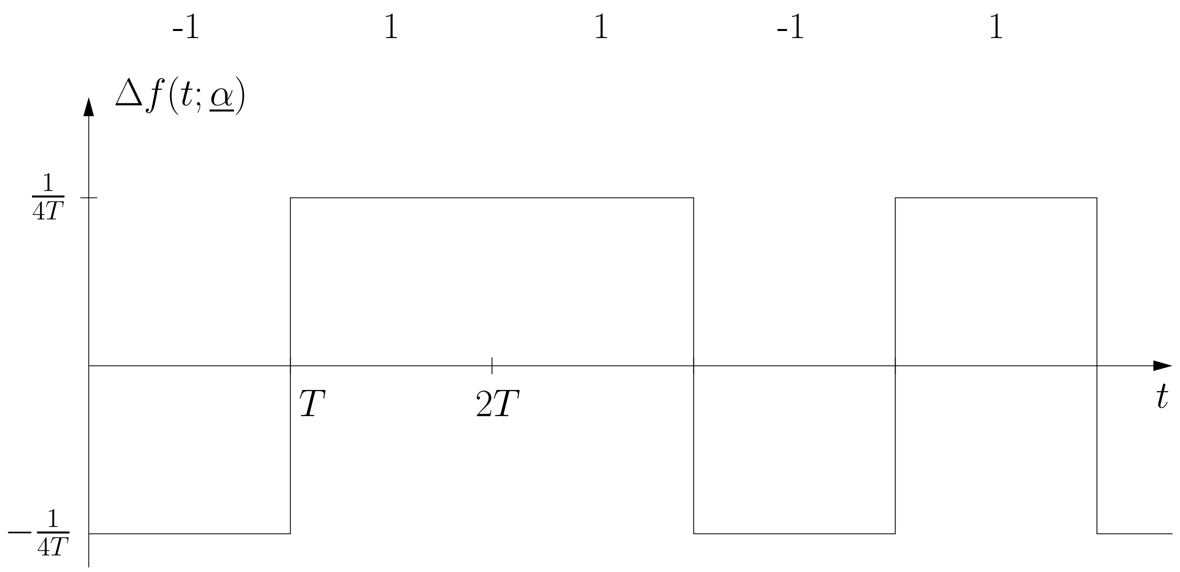
$$q'_{MSK}(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2T} & \text{for } 0 \leq t < T \\ 0 & \text{for } t \geq T \end{cases}$$



Then

$$f(t;\boldsymbol{\alpha}) = f_0 \ + \underbrace{\frac{1}{2}\sum_{n=0}^{\infty} \alpha_n q'_{MSK}(t-nT)}_{\Delta f(t;\boldsymbol{\alpha})}$$

EXAMPLE:



◇

In symbol interval $[nT, (n+1)T)$, we have

$$\begin{aligned} f(t;\boldsymbol{\alpha}) &= f_0 \;+\; \frac{1}{2}\cdot\alpha_n\cdot\frac{1}{2T} \\ &= f_0 \;+\; \frac{\alpha_n}{4T} \end{aligned}$$

with $\alpha_n \in \{-1, +1\}$.

Therefore the frequency offset is

$$\begin{aligned} \Delta f &= \big(f_0 + \frac{1}{4T}\big) \;-\; \big(f_0 - \frac{1}{4T}\big) \\ &= \frac{1}{2T} \end{aligned}$$

1.3 Gaussian Minimum-Shift Keying

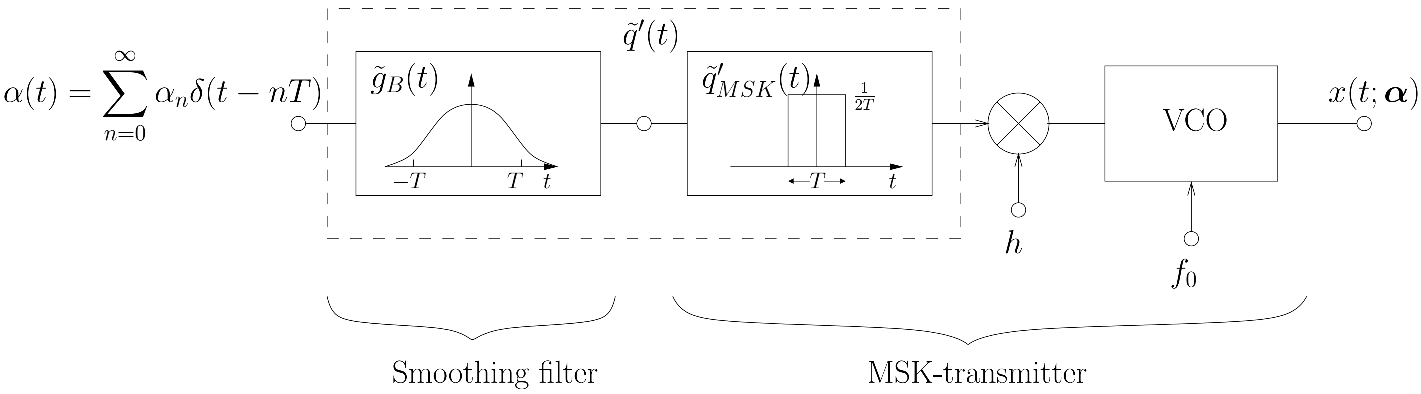
Problem

How to reduce the side lobes in the power spectrum of MSK while keeping the essential features of this modulation scheme that coherent demodulation is possible?

Solution

1. Insert in the MSK VCO-based (non-causal) transmitter a smoothing filter with impulse response $\tilde{g}_B(t)$ in front of the MSK frequency response.

   $\tilde{q}'_{MSK}(t) \longrightarrow$ Reduction of the side lobes!



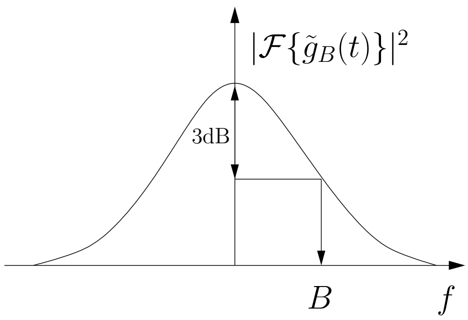
2. Design $\tilde{g}_B(t)$ such that the overall frequency response

$$\tilde{q}'(t) = \tilde{g}_B(t) \ * \ \tilde{q}'_{MSK}(t)$$

   integrates to $\frac{1}{2}$ as does $\tilde{q}'_{MSK}(t)$.

   $\rightarrow$ Coherent demodulation is possible!

The indexing parameter B is the 3 dB bandwidth of the impulse response of the smoothing filter.



Smoothing filter for GMSK

GMSK results by selecting the impulse response of the smoothing filter to be a Gaussian pulse

$$\tilde{g}_B(t) = \frac{1}{\sqrt{2\pi}\sigma T} \exp^{-\frac{t^2}{2\sigma^2 T^2}} \tag{2}$$

with

$$\sigma = \frac{\sqrt{\ln 2}}{2\pi BT}$$

Hence,

$$\tilde{q}'_{GMSK}(t) \;=\; \tilde{g}_B(t) * \tilde{q}'_{MSK}(t) \tag{3}$$

We show now that $\tilde{q}'_{GMSK}(t)$ has the required property:

$$\int_{-\infty}^{+\infty} \tilde{q}'_{GMSK}(t)\ dt \ = \ \frac{1}{2} \tag{4}$$

Proof

We can view $2\tilde{q}'_{MSK}(t)$ and $\tilde{g}_B(t)$ as the probability density functions (pdfs) of two independent random variables, say U and V:

$$\begin{aligned} U \ &\sim \ 2\tilde{q}'_{MSK}(t) \\ V \ &\sim \ \tilde{g}_B(t) \end{aligned}$$

Then from (3), $2\tilde{q}'_{GMSK}(t)$ is the pdf of the sum

$$W \ = \ U \ + \ V$$

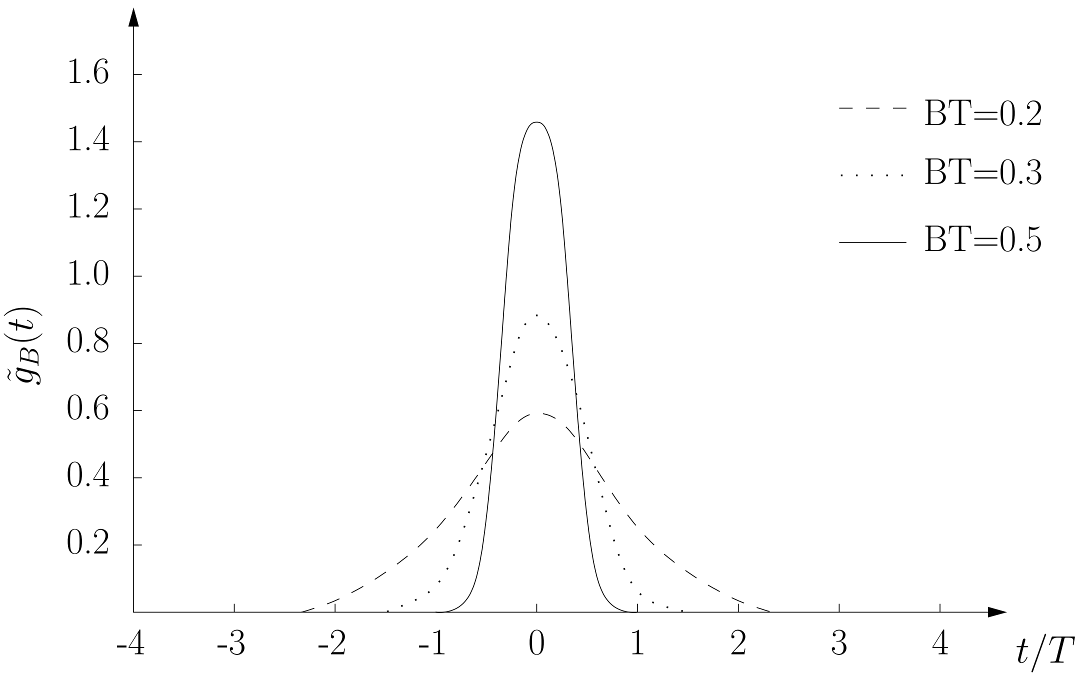
Equ. (4) follows then because the integral of pdfs equals unity.

Selected values of B

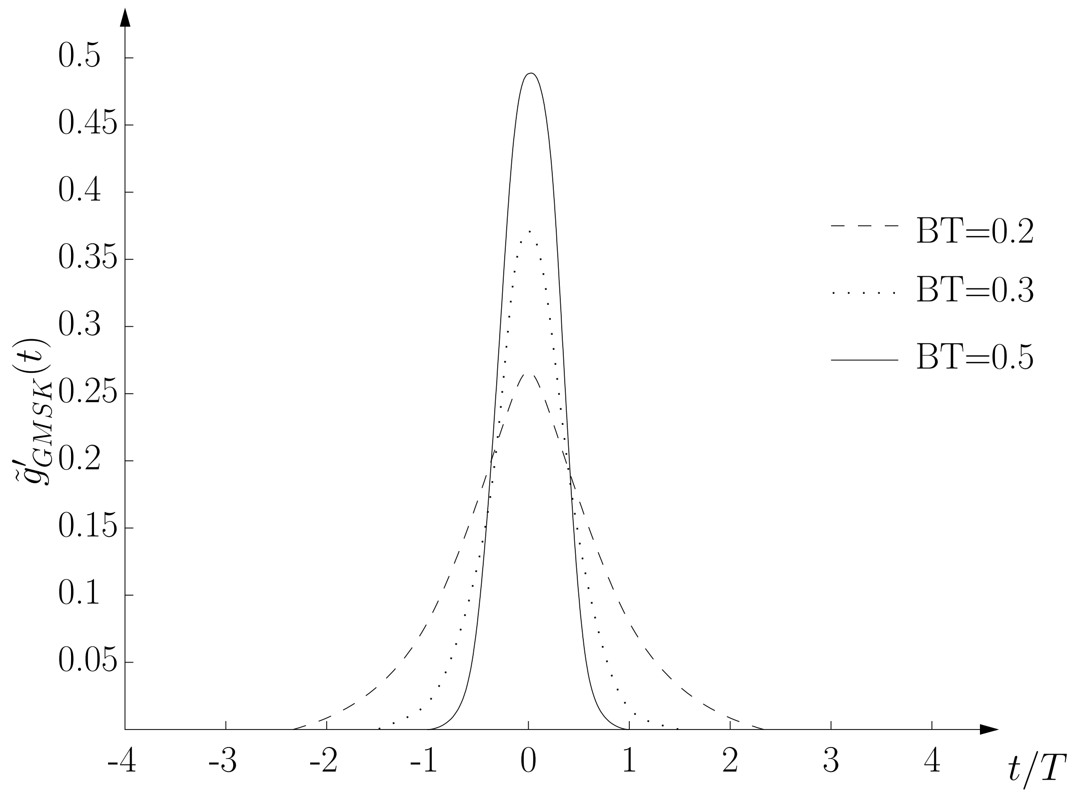
$$\begin{aligned} &\text{GSM}: & BT \ &= \ 0.3 & T \ &= \ \frac{6}{1.625} \ = \ 3.69ms \\ &\text{DECT}: & BT \ &= \ 0.5 \end{aligned}$$

Example

Gaussian Pulse

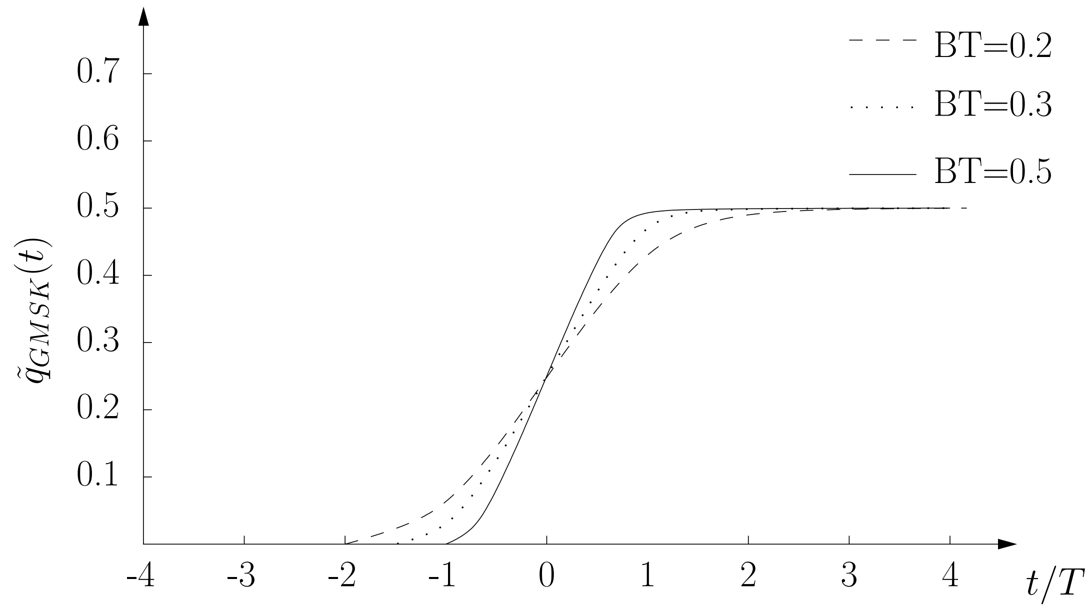


Frequency response



Phase response

$$\tilde{q}_{GMSK}(t) = \int_{-\infty}^{t} \tilde{q}'_{GMSK}(\tilde{t}\,)\, d\tilde{t}$$



Casual implementation of a GMSK transmitter

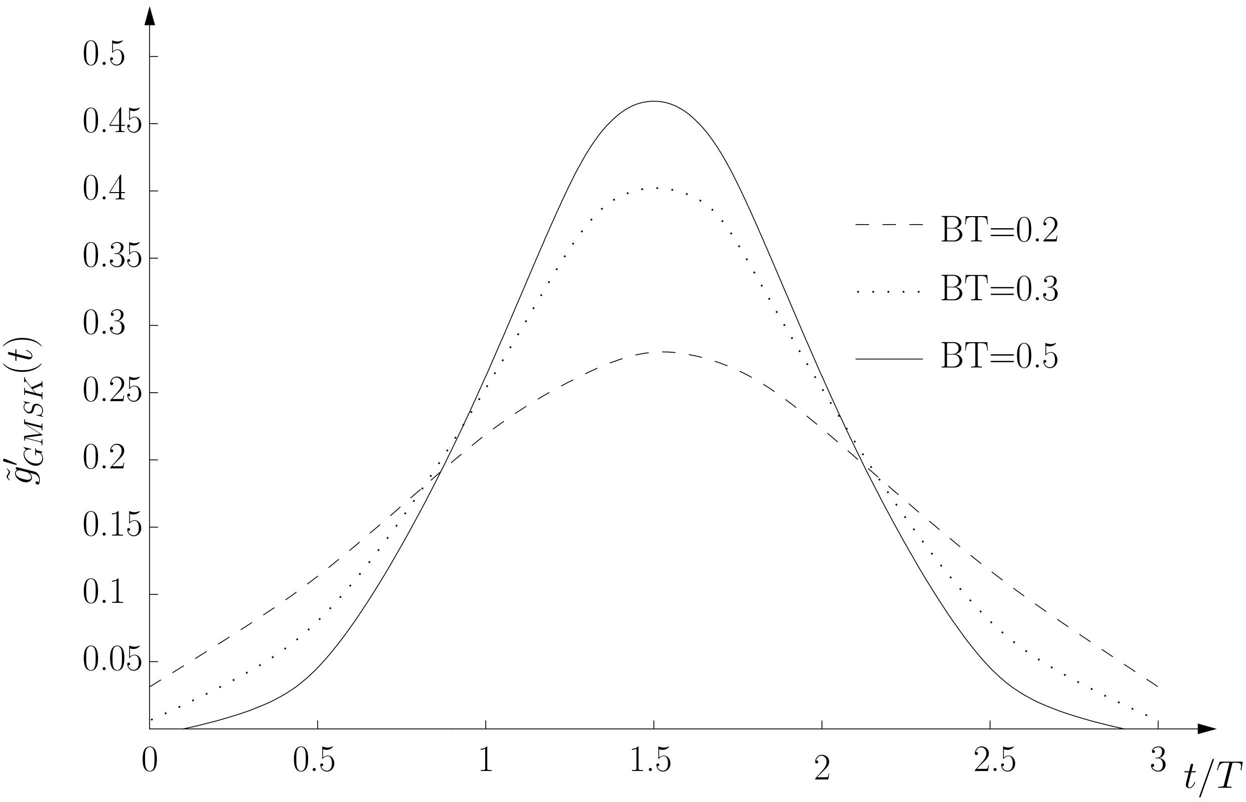
A casual implementation of a GMSK transmitter is obtained by truncating the non-casual frequency response $\tilde{q}_{GMSK}(t)$ outside a centered interval $\left[\ -\frac{LT}{2},\ +\frac{LT}{2}\ \right]$ normalizing to 1/2, and time delaying by $\frac{LT}{2}$:

$$\tilde{q}'_{GMSK}(t) \ = \ \begin{cases} \frac{1}{2}\ \frac{1}{A}\ \tilde{q}'_{GMSK}(t-\frac{LT}{2}) & \text{for } t \in [0, LT) \\ 0 & \text{elsewhere} \end{cases}$$

with

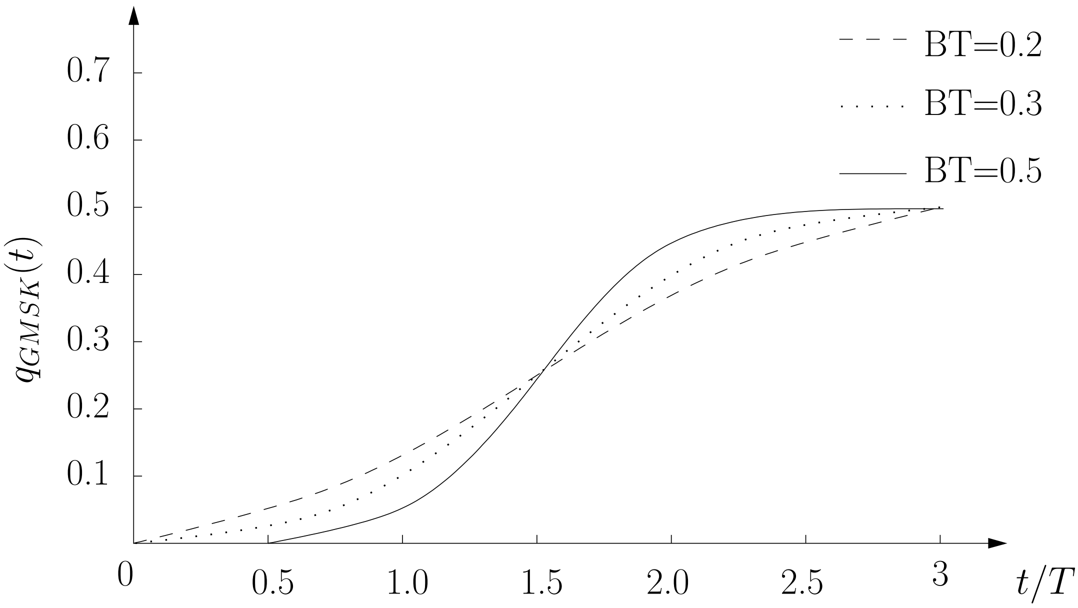
$$A = \int_{-\frac{LT}{2}}^{+\frac{LT}{2}} \tilde{q}'_{GMSK}(\tilde{t})\ d\tilde{t}$$

A practical value is $L = 3$.

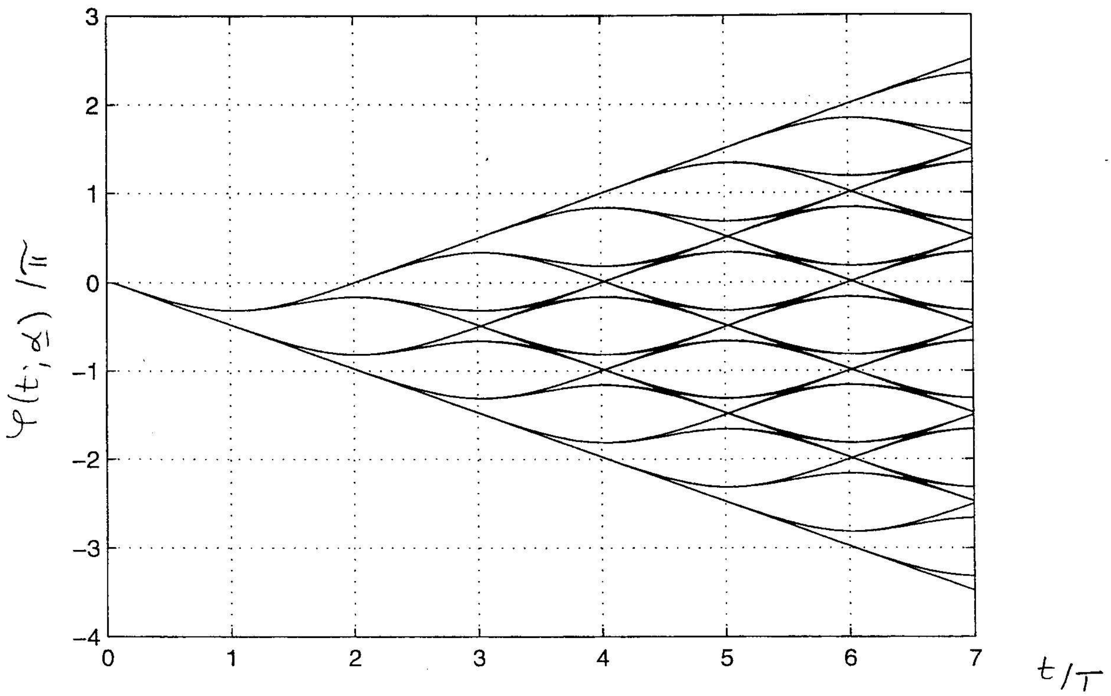


Casual phase response

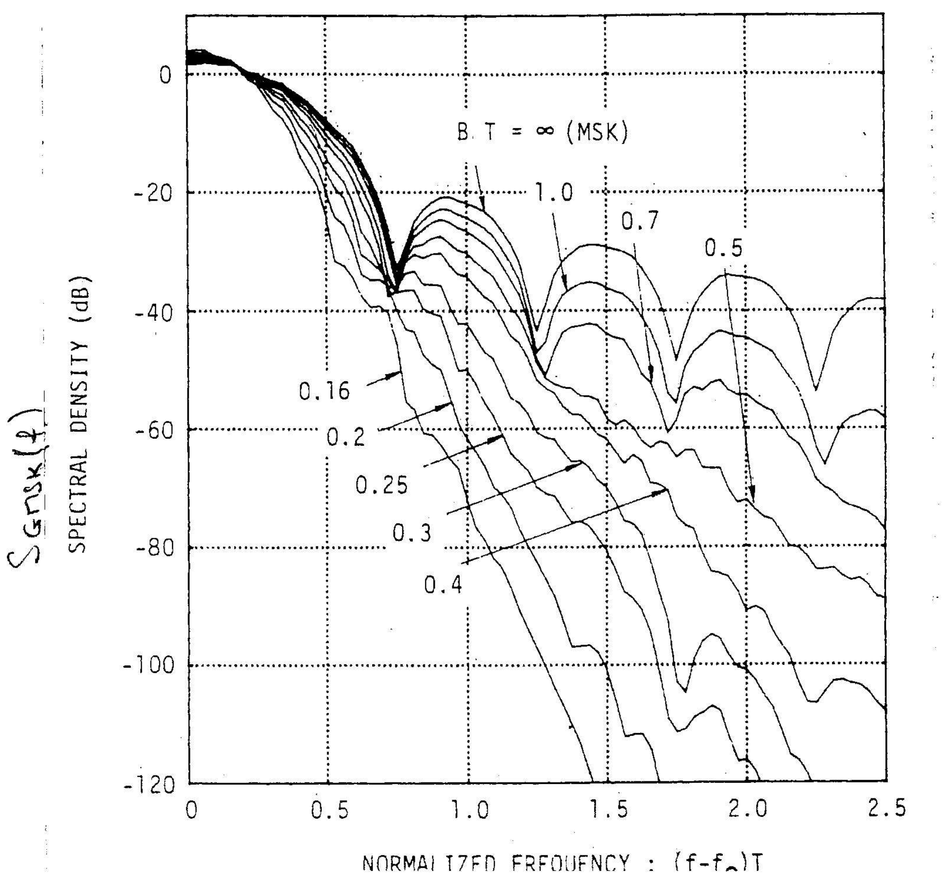
$$q_{GMSK}(t) = \int_{-\infty}^{t} q'_{GMSK}(\tilde{t}\ )\ d\tilde{t}$$



Phase tree of GMSK

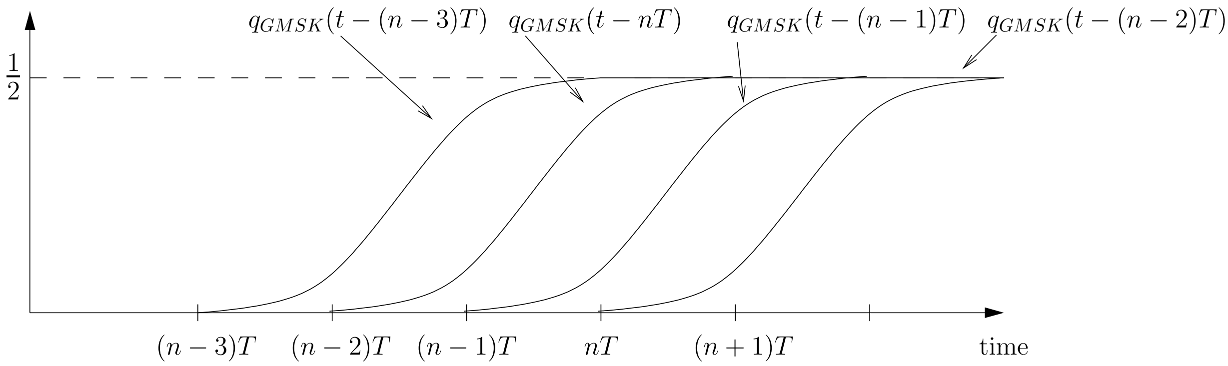


Power spectrum of GMSK



1.4 State and Trellis of CPM

Let us consider the phase of a GMSK signal within some arbitrary signaling interval say $[nT, (n+1)T)$.



We consider first the special case $L = 3$ as depicted above. For $t \in [nT, (n+1)T)$, we can write:

$$\begin{aligned}
\varphi(t;\boldsymbol{\alpha}) &= \varphi_0 \ + \ \pi \sum_{i=0}^{\infty} \alpha_i q_{GMSK}(t - iT) \\
&= \varphi_0 \ + \ \pi \sum_{i=0}^{n-3} \alpha_i \, \frac{1}{2} \ + \ \sum_{i=n-2}^{n} \alpha_i q_{GMSK}(t - iT)
\end{aligned}$$

In the general case where L is arbitrary

$$\begin{aligned}
\varphi(t;\boldsymbol{\alpha}) &= \varphi_0 \ + \ \underbrace{\frac{\pi}{2} \sum_{i=0}^{n-L} \alpha_i}_{\theta_n} \ + \sum_{i=n-(L-1)}^{n-1} \alpha_i q(t - iT) \ + \ \alpha_n q(t - nT) \\
&= \varphi\big(t; \alpha_n, \underbrace{\theta_n, \alpha_{n-1}, \ldots, \alpha_{n-(L-1)}}_{\boldsymbol{S}_n}\big) \\
&= \varphi(t; \alpha_n, \boldsymbol{S}_n)
\end{aligned}$$

for $t \in [nT, (n+1)T)$

Thus, the behaviour of $\varphi(t;\boldsymbol{\alpha})$ in the signaling interval $[nT,(n+1)T)$ is entirely determined by the $(L+1)$-dimensional vector

$$\Big(\alpha_n, \underbrace{\theta_n, \alpha_{n-1}, \ldots, \alpha_{n-(L-1)}}_{\boldsymbol{S}_n \in \mathcal{S}}\Big)$$

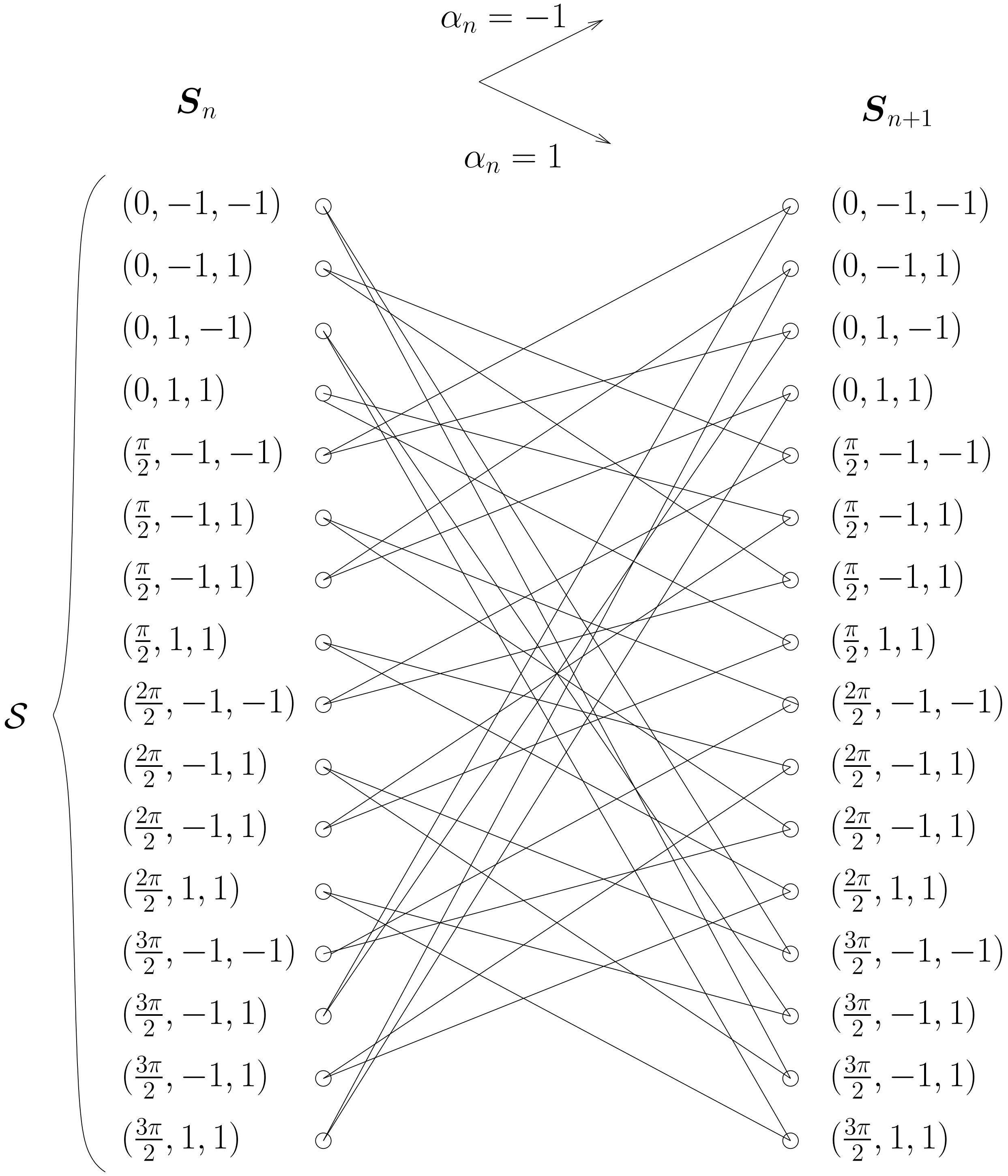
with

$$\begin{aligned} \alpha_n &: \text{current transmitted symbol} \\ \boldsymbol{S}_n &: \text{state of } \varphi(t;\boldsymbol{\alpha}) \text{ in } n\text{-th signaling interval} \\ \mathcal{S} &: \text{state space} \end{aligned}$$

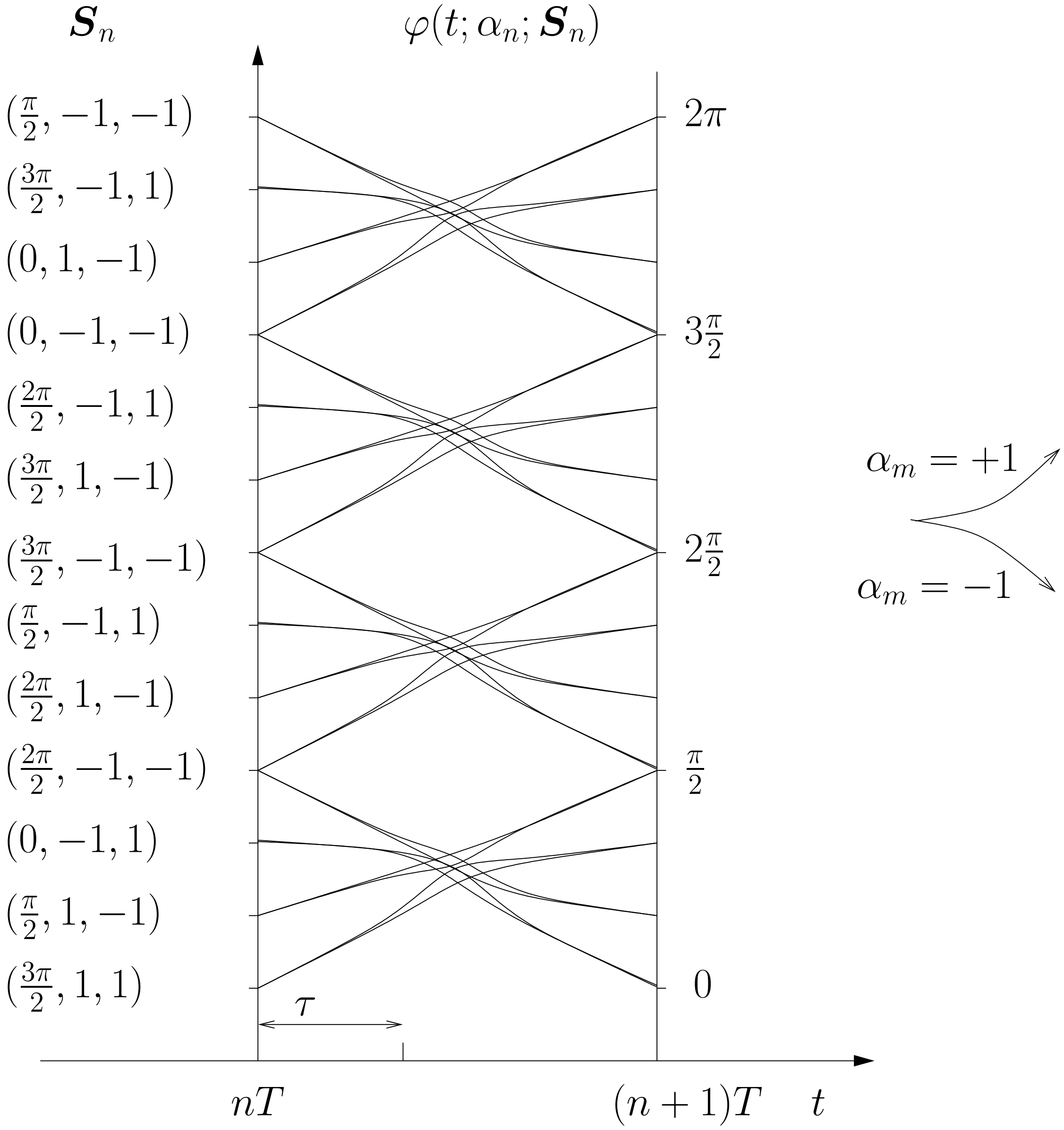
State transitions (trellis) for GMSK with $L = 3$

Notice that

$$\theta_{n+1} \; = \; \theta_n + \frac{\pi}{2}\, \alpha_{n-(L-1)}$$



Phase behavior within one signaling interval as a function of the state transitions for GMSK



Initialization

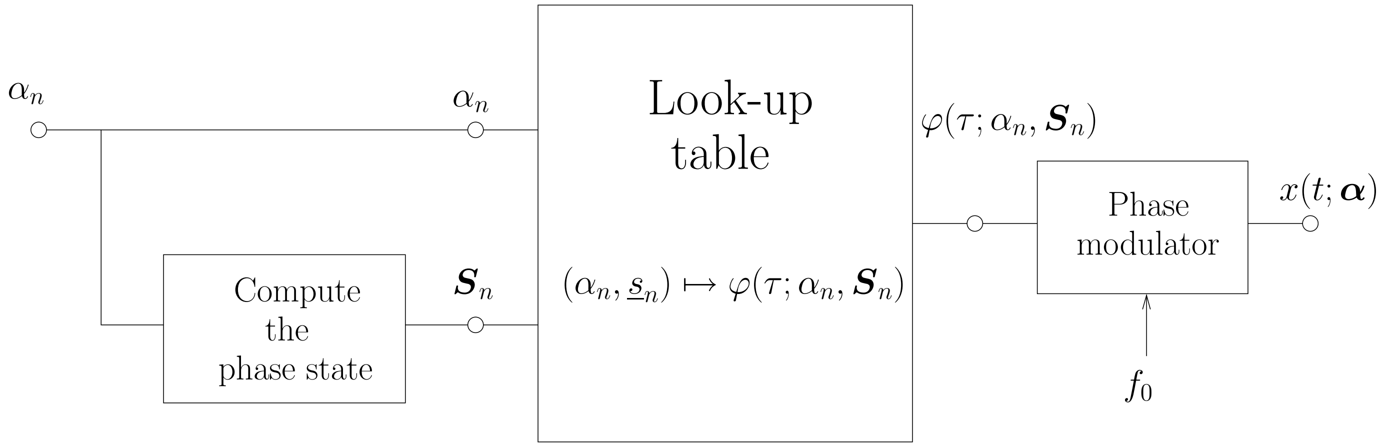
$$\varphi_0 = - \sum_{i=-(L-1)}^{-1} \alpha_i q(t - iT)$$

$$\theta_0 = 0$$

where $\alpha_{-1}, \ldots, \alpha_{-(L-1)}$ are known symbols

Implementation of a CPM transmitter with look-up tables

A. Using a phase modulator



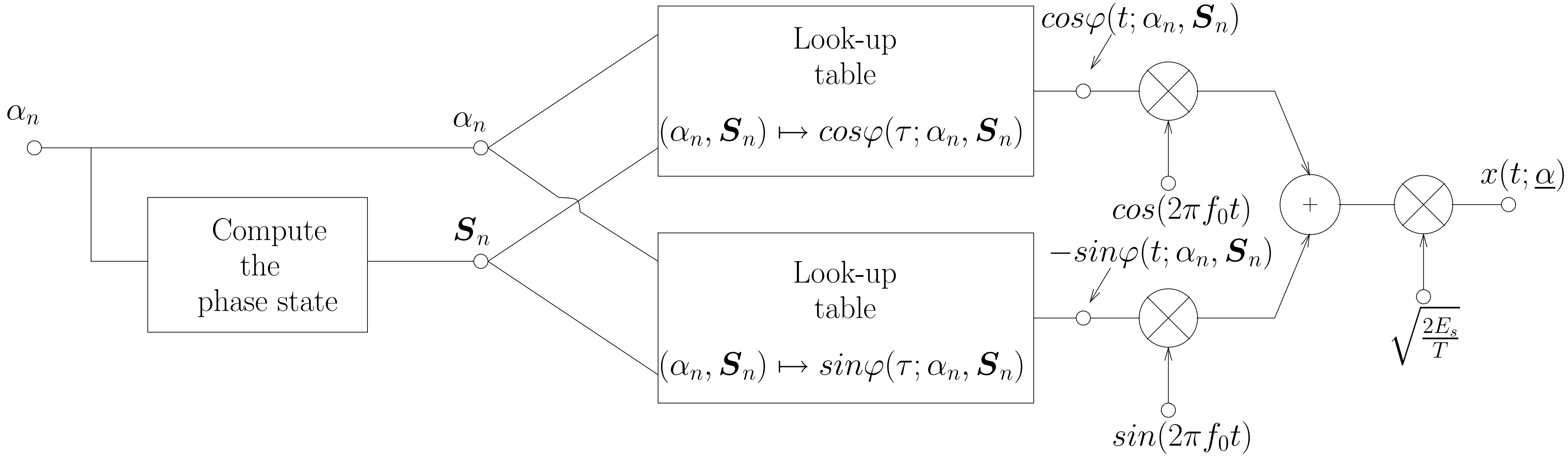
B. Using an inphase/quadrature modulator

Consider the inphase/quadrature representation of $x(t; \boldsymbol{\alpha})$, where we assume $\varphi_0 = 0$:

$$x(t; \boldsymbol{\alpha}) = \sqrt{\frac{2E_s}{T_s}} \cos(2\pi f_0 t + \varphi(t; \boldsymbol{\alpha}))$$

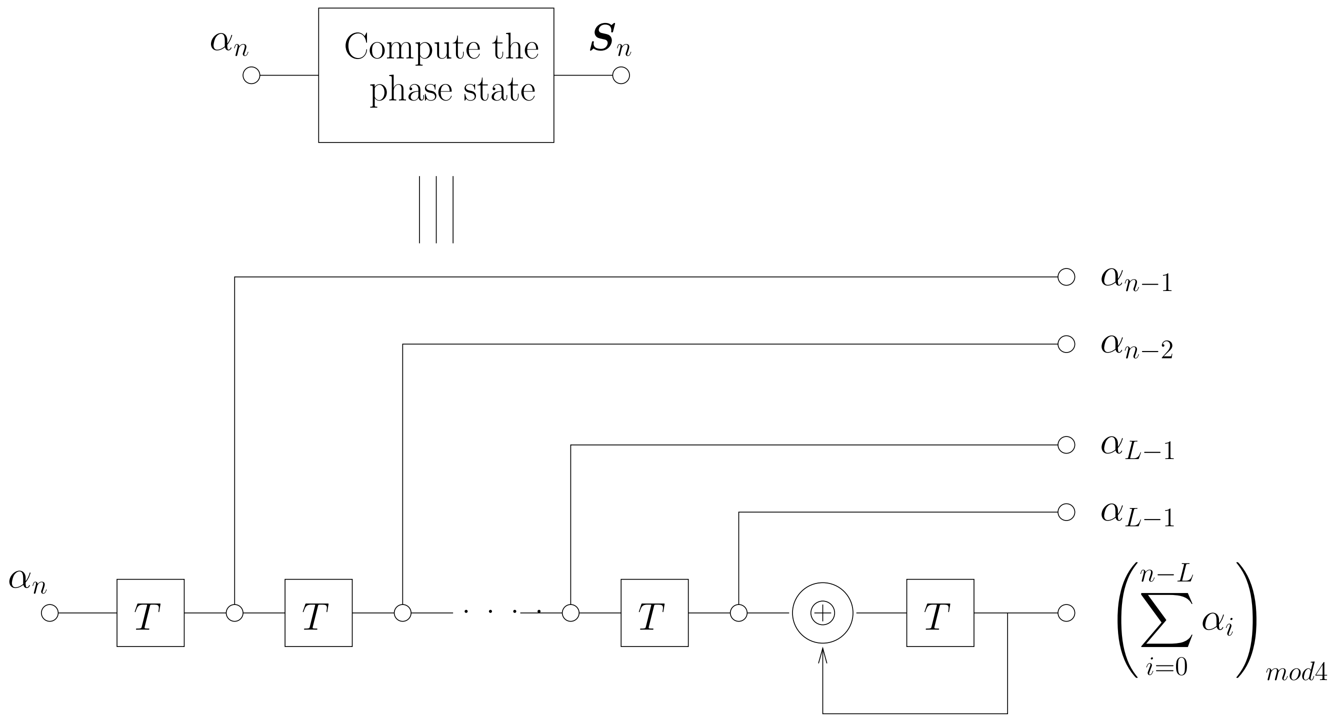
$$= \underbrace{\sqrt{\frac{2E_s}{T_s}} \cos\varphi(t; \boldsymbol{\alpha})}_{\text{In-phase component}} \cos(2\pi f_0 t)$$

$$\underbrace{- \sqrt{\frac{2E_s}{T_s}} \sin\varphi(t; \boldsymbol{\alpha})}_{\text{Quadrature component}} \sin(2\pi f_0 t)$$

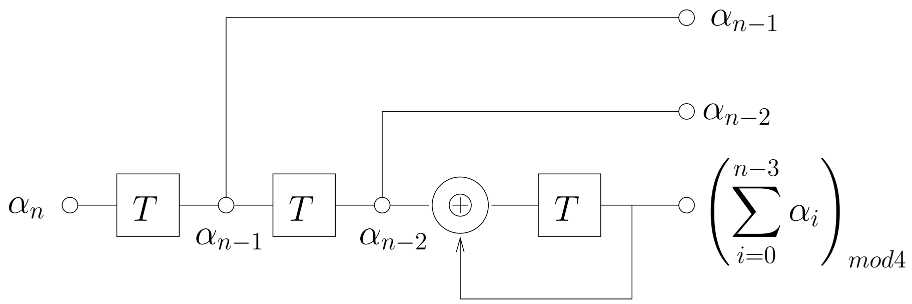


α_n
Compute the phase state
α_n
$\boldsymbol{S}_n$
Look-up table
$(\alpha_n, \boldsymbol{S}_n) \mapsto cos\varphi(\tau; \alpha_n, \boldsymbol{S}_n)$
Look-up table
$(\alpha_n, \boldsymbol{S}_n) \mapsto sin\varphi(\tau; \alpha_n, \boldsymbol{S}_n)$
$cos\varphi(t; \alpha_n, \boldsymbol{S}_n)$
$-sin\varphi(t; \alpha_n, \boldsymbol{S}_n)$
$cos(2\pi f_0 t)$
$sin(2\pi f_0 t)$
$+$
$\sqrt{\frac{2E_s}{T}}$
$x(t; \underline{\alpha})$

Phase state computation



for GMSK ($L = 3$):

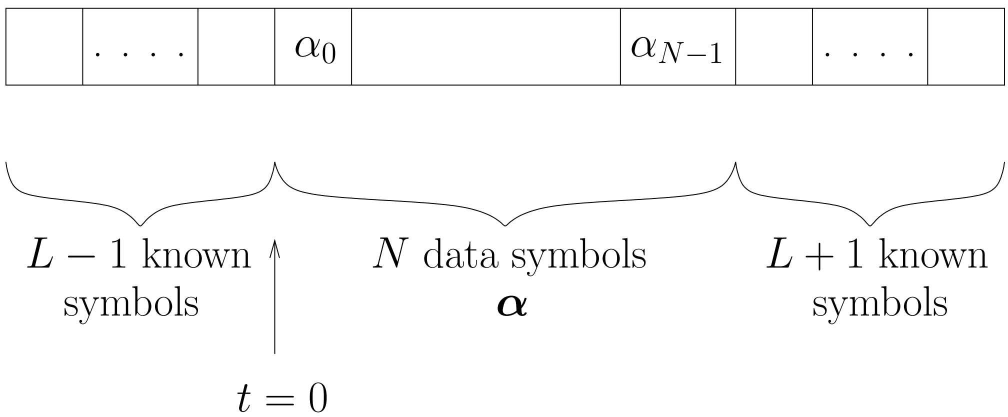


1.5 Coherent Demodulation of CPM

The initial state and the final state of the CPM transmitter are set to known states using the following procedure:

- Before transmitting the data, L known symbols are transmitted to drive the CPM transmitter into a known initial state.
- After transmitting the data, L known symbols are transmitted to to drive the CPM transmitter into a known final state.

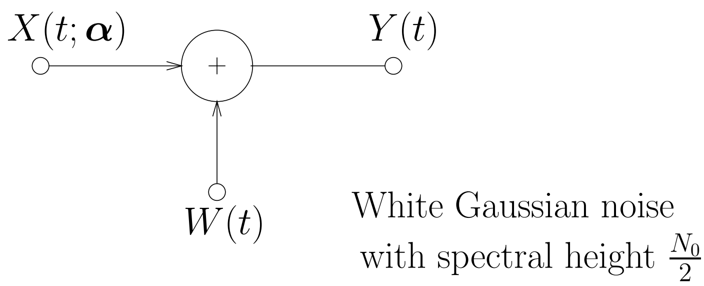
Block of transmitted symbols



ML decoding for the AWGN channel

Received signal:

$$Y(t) = x(t;\boldsymbol{\alpha}) \; + \; W(t)$$



ML estimation

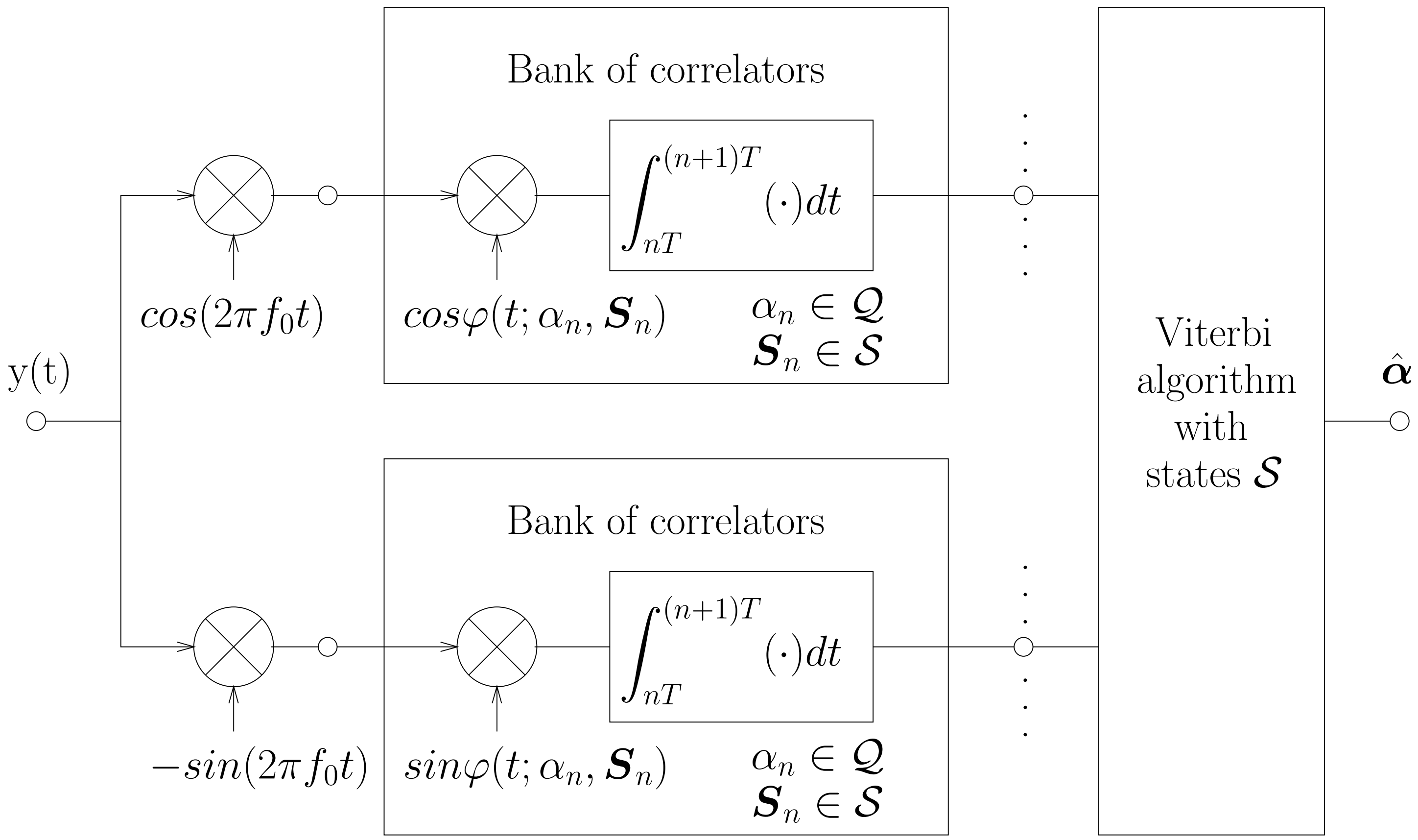
$$\hat{\boldsymbol{\alpha}} = \underset{\boldsymbol{\alpha}\in\mathbb{A}^N}{\operatorname{argmax}} \left\{ \int_0^{(N+L+1)T} y(t)\ x(t;\boldsymbol{\alpha})\ dt \right\}$$

$$= \underset{\boldsymbol{\alpha}\in\mathbb{A}^N}{\operatorname{argmax}} \left\{ \sum_{n=0}^{N+L} \int_{nT}^{(n+1)T} y(t)\ x(t;\boldsymbol{\alpha})\ dt \right\}$$

$$= \underset{\boldsymbol{\alpha}\in\mathbb{A}^N}{\operatorname{argmax}} \left\{ \sum_{n=0}^{N+L} \left[\int_{nT}^{(n+1)T} y(t) \cos\varphi(t;\alpha_n,\boldsymbol{S}_n)\ \cos(2\pi f_0 t)\ dt \right.\right.$$

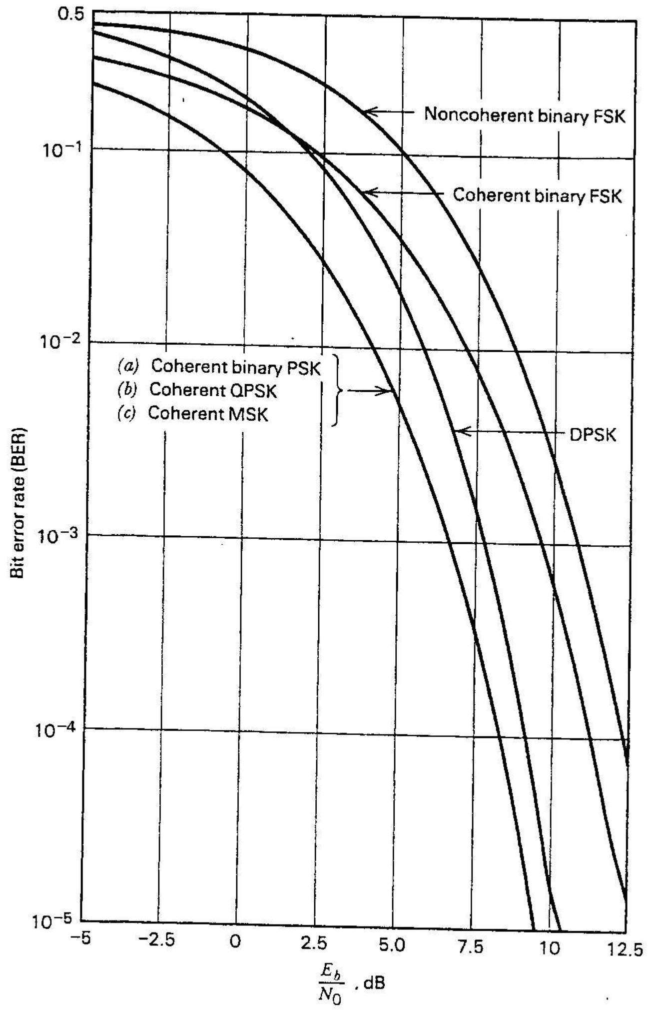
$$\left.\left. - \int_{nT}^{(n+1)T} y(t) \sin\varphi(t;\alpha_n,\boldsymbol{S}_n)\ \sin(2\pi f_0 t)\ dt \right] \right\}$$

ML receiver

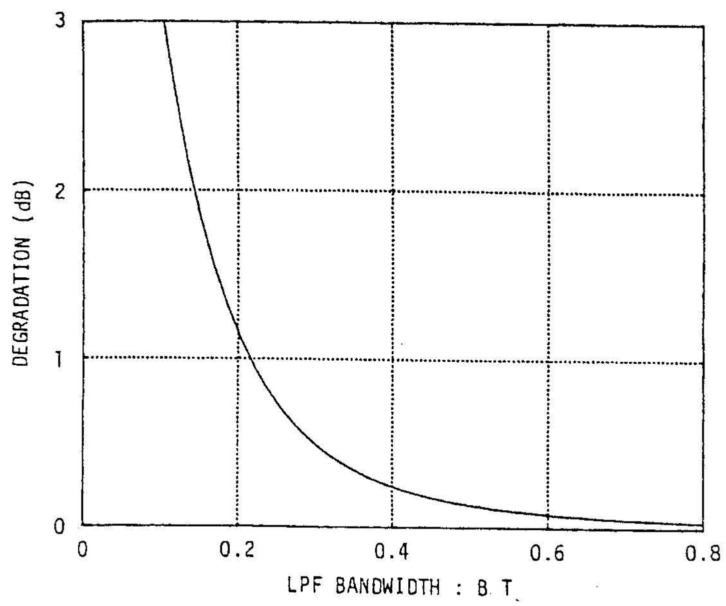


Bank of correlators
$\int_{nT}^{(n+1)T} (\cdot)dt$
$cos(2\pi f_0 t)$
$cos\varphi(t; \alpha_n, \boldsymbol{S}_n)$
$\alpha_n \in \mathcal{Q}$
$\boldsymbol{S}_n \in \mathcal{S}$
y(t)
Viterbi algorithm with states $\mathcal{S}$
$\hat{\boldsymbol{\alpha}}$
Bank of correlators
$\int_{nT}^{(n+1)T} (\cdot)dt$
$-sin(2\pi f_0 t)$
$sin\varphi(t; \alpha_n, \boldsymbol{S}_n)$
$\alpha_n \in \mathcal{Q}$
$\boldsymbol{S}_n \in \mathcal{S}$

Performance of MSK and GMSK

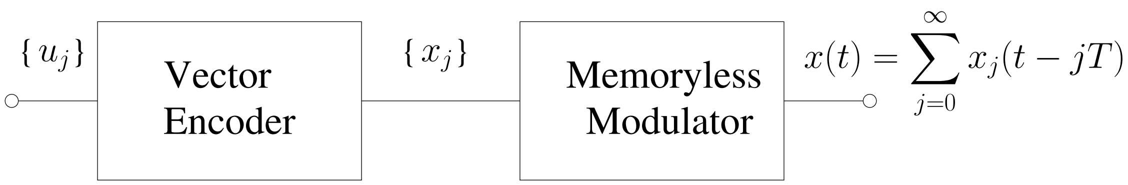


Performace degradation of GMSK with respect to MSK



2 Modern Representation of CPM

In the modern representation of CPM [4], the CPM transmitter is decomposed into a vector encoder with memory and a memoryless modulator.



This modern representation of CPM has already been used in the description of minimum-shift keying (MSK) in the lecture notes for Digital Modulation I. (See therein.)

2.1 Initialization

We assume that the CPM transmitter is initialized with "admissible" symbols. we select

$$\alpha_{-1} = \alpha_{-2} = \ldots = \alpha_{-(L-1)} = -(M-1)$$

Notice that $-(M-1)$ is a symbol common to both alphabets corresponding to M odd and even.

The phase of CPM is then

$$\varphi(t;\boldsymbol{\alpha}) \;=\; \varphi_0 + 2\pi h \sum_{j=-(L-1)}^{\infty} \alpha_j q(t-jT)$$

where $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots)$ is the sequence of information-bearing symbols.

The transmitted CPM signal reads:

$$x(t;\boldsymbol{\alpha}) \;=\; \sqrt{2P}\cos(2\pi f_0 t + \varphi(t;\boldsymbol{\alpha}))$$

Without loss of generality, we assume that φ_0 is selected in such a way that $\varphi(0; \boldsymbol{\alpha}) = 0$. Thus we obtain

$$
\begin{aligned}
\varphi_0 &= -2\pi h \sum_{j=-(L-1)}^{0} \alpha_j q(-jT) \\
&= -2\pi h \sum_{j=0}^{L-1} \alpha_{-j} q(jT) \\
&= -2\pi h \sum_{j=1}^{L-1} \alpha_{-j} q(jT) \\
&= 2\pi h(M-1) \sum_{j=0}^{L-1} q(jT)
\end{aligned}
$$

2.2 The Reference Phase Trajectory

We focus on the phase trajectory corresponding to the sequence

$$\boldsymbol{\alpha}^{-} \;=\; \big(-(M-1), -(M-1), \ldots\big)$$

Inserting yields

$$\varphi(t;\boldsymbol{\alpha}^{-}) \;=\; \varphi_0 - 2\pi h(M-1) \sum_{j=-(L-1)}^{\infty} q(t-jT)$$

We investigate the behaviour of $\varphi(t;\boldsymbol{\alpha}^{-})$ over each signaling interval $[nT,(n+1)T)$. To this end, it is worth introducing the relative time variable $\tau \in [0,T)$. The absolute time variable t within this signaling interval is related to τ according to

$$t = nT + \tau$$

(i) Consider: $n = 0, \;\Rightarrow\; t = \tau$

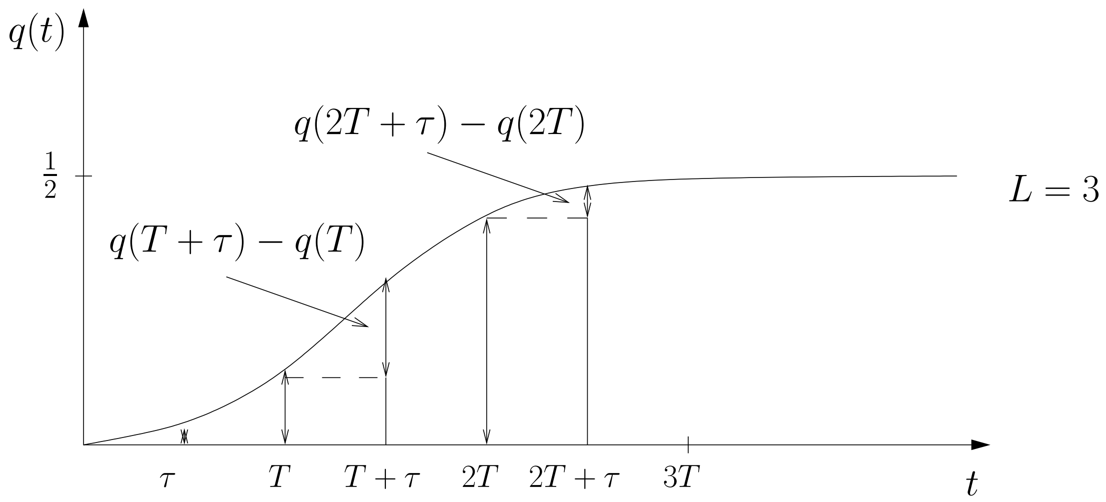
$$\begin{aligned}
\varphi_0(\tau) &= \varphi(\tau,\boldsymbol{\alpha}^{-}) \\
&= \varphi_0 \;-\; 2\pi h(M-1) \sum_{j=-(L-1)}^{0} q(\tau - jT) \\
&= \varphi_0 - 2\pi h(M-1) \sum_{j=0}^{L-1} q(\tau + jT)
\end{aligned}$$

(The index in $\varphi_0(\tau)$ is the value of n.)

Inserting for φ_0, we obtain

$$\varphi_0(\tau) = -2\pi h(M-1) \sum_{j=0}^{L-1} \big(q(\tau+jT) - q(jT)\big)$$

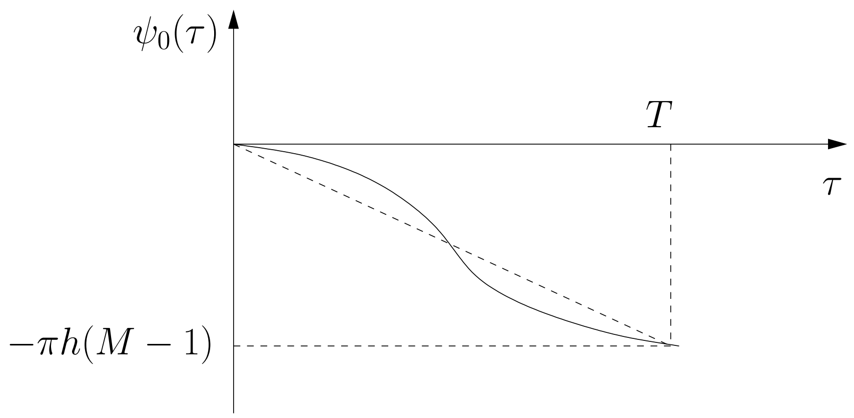
$$\varphi_0(\tau) = -2\pi h(M-1)\sum_{j=0}^{L-1}(q(\tau+jT)-q(jT))$$



Notice that

- $\varphi_0(0) = 0$
- $\varphi_0(T) \; = \; -2\pi h(M-1)q(LT) \; = \; -\pi h(M-1)$

Behavior of $\varphi_0(\tau)$:

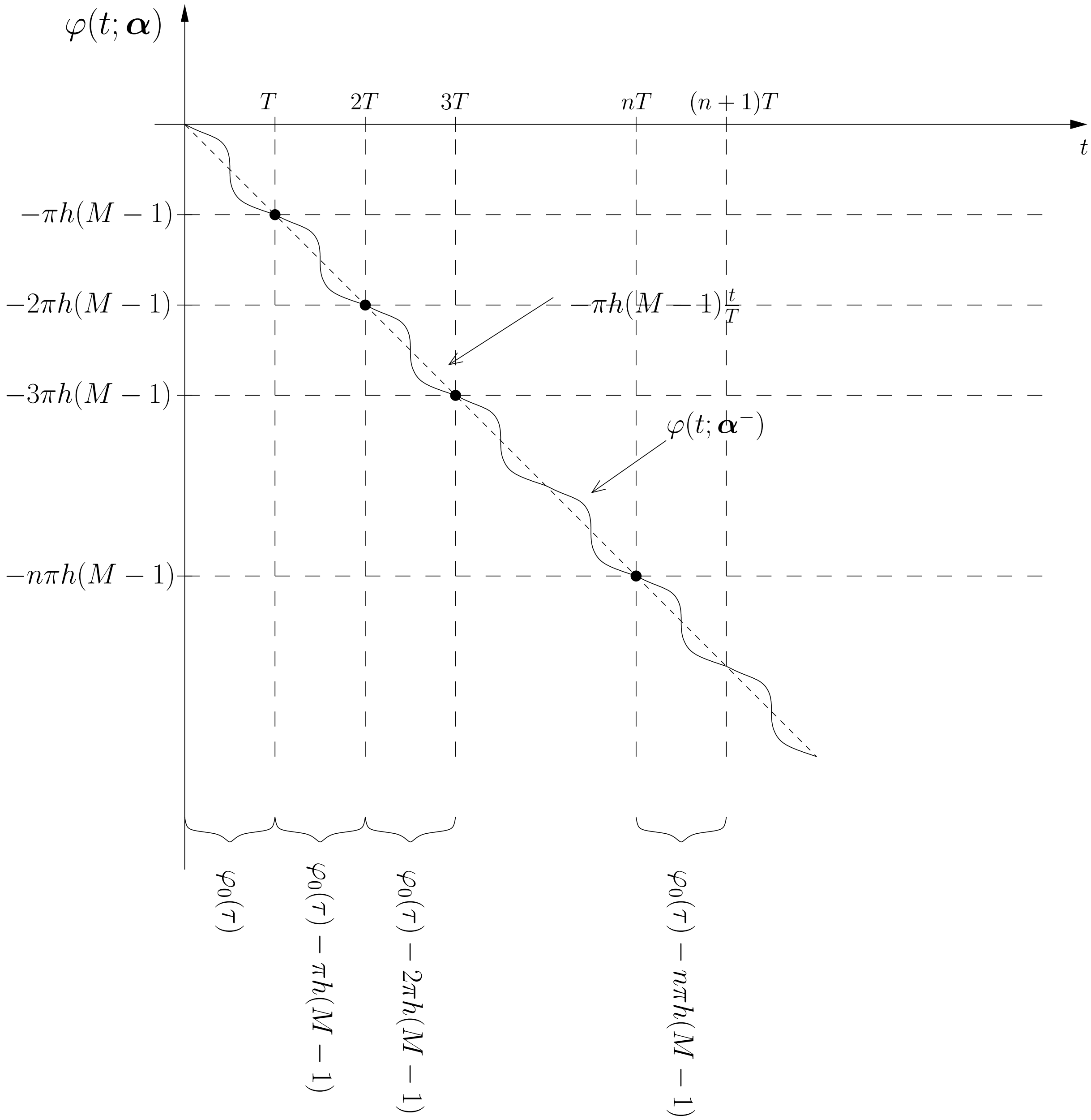


(ii) Consider: $n = 1,\ \Rightarrow\ t = T + \tau$

$$
\begin{aligned}
\varphi_1(\tau) &= \varphi(T+\tau;\boldsymbol{\alpha}^-)\\
&= \varphi_0 - 2\pi h(M-1) \sum_{j=-(L-1)}^{1} q(\tau + T - jT)\\
&= \varphi_0 - 2\pi h(M-1) \sum_{j=-1}^{L-1} q(\tau + (j+1)T)\\
&= \varphi_0 - 2\pi h(M-1) \sum_{j=0}^{L} q(\tau + jT)\\
&= -2\pi h(M-1) \left(\sum_{j=0}^{L-1} q(\tau + jT) - \sum_{j=0}^{L-1} q(jT) \right) - \pi h(M-1)\\
&= -\pi h(M-1) + \varphi_0(\tau)
\end{aligned}
$$

(iii) Consider: $n \geq 1,\ \Rightarrow\ t = nT + \tau$

$$
\begin{aligned}
\varphi_n(\tau) &= \varphi(nT+\tau,\boldsymbol{\alpha}^-)\\
&= \varphi_0 - 2\pi h(M-1) \sum_{j=-(L-1)}^{n} q(\tau + nT - jT)\\
&= \varphi_0 - 2\pi h(M-1) \sum_{j=0}^{(L-1)+n} q(\tau + jT)\\
&= \varphi_0 - 2\pi h(M-1) \sum_{j=0}^{L-1} q(\tau + jT) + n\pi h(M-1)\\
&= n\pi h(M-1) + \varphi_0(\tau)
\end{aligned}
$$


$\varphi(t;\boldsymbol{\alpha})$
t
T
$2T$
$3T$
nT
$(n+1)T$
$-\pi h(M-1)$
$-2\pi h(M-1)$
$-3\pi h(M-1)$
$-n\pi h(M-1)$
$-\pi h(M-1)\frac{t}{T}$
$\varphi(t;\boldsymbol{\alpha}^{-})$
$\varphi_0(\tau)$
$\varphi_0(\tau)-\pi h(M-1)$
$\varphi_0(\tau)-2\pi h(M-1)$
$\varphi_0(\tau)-n\pi h(M-1)$

2.3 The Reference Frequency

For the sake of simplifying the presentation we assume that

$$\varphi(t;\boldsymbol{\alpha}^-) = -\pi h(M-1)\frac{t}{T}$$

or equivalently that

$$\varphi_0(\tau) = -\pi h(M-1)\frac{\tau}{T}$$

In order for $\varphi_0(\tau)$ to exhibit this behavior, $q(t)$ must satisfy the condition

$$-2\pi h(M-1)\sum_{j=0}^{L-1}\big(q(\tau+jT)-q(jT)\big) \;=\; -\pi h(M-1)\frac{\tau}{T}$$

which holds if

$$\sum_{j=0}^{L-1}\big(q(\tau+jT)-q(jT)\big) \;=\; \frac{\tau}{2T} \tag{5}$$

Thus for $\boldsymbol{\alpha}=\boldsymbol{\alpha}^-$, the CPM signal transmitted reads

$$\begin{aligned} x(t;\boldsymbol{\alpha}^-) &= \sqrt{2P}\cos\!\big(2\pi f_0 t - \pi h(M-1)\frac{t}{T}\big) \\ &= \sqrt{2P}\cos(2\pi f_1 t) \end{aligned}$$

with

$$f_1 = f_0 - \frac{h(M-1)}{2T}$$

This frequency f_1 is the retained reference frequency with respect to which the phase of CPM is expressed.

$$x(\boldsymbol{\alpha}) = \sqrt{2P}\cos\Big(2\pi f_1 t + \underbrace{\varphi(t;\boldsymbol{\alpha}) + \pi h(M-1)\frac{t}{T}}_{\phi(t;\boldsymbol{\alpha})}\Big)$$

2.4 The Tilted Phase and the New Symbols

The so called tilted phase $\phi(t;\boldsymbol{\alpha})$ reads

$$\begin{aligned}\phi(t;\boldsymbol{\alpha}) &= \varphi(t;\boldsymbol{\alpha}) + \pi h(M-1)\frac{t}{T}\\ &= \varphi_0 + 2\pi h \sum_{j=-(L-1)}^{\infty} \alpha_j q(t-jT) + \pi h(M-1)\frac{t}{T}\end{aligned}$$

We investigate the behavior of $\phi(t;\boldsymbol{\alpha})$ within each signaling interval.

(i) Consider: $n=0, \;\Rightarrow\; t=\tau\in[0,T)$

$$\begin{aligned}\phi_0(\tau;\boldsymbol{\alpha}) &= \phi(0\cdot T+\tau;\boldsymbol{\alpha})\\ &= \varphi_0 + 2\pi h \sum_{j=0}^{L-1} \alpha_{-j} q(\tau+jT) + \pi h(M-1)\frac{\tau}{T}\end{aligned}$$

As we have (5) due to the assumption regarding $\varphi_0(\tau)$, we conclude that

$$\begin{aligned}\frac{\tau}{T} &= 2\sum_{j=0}^{L-1} q(\tau+jT) - 2\sum_{j=0}^{L-1} q(jT)\\ &= 2\sum_{j=0}^{L-1} q(\tau+jT) - \Big[\pi h(M-1)\Big]^{-1}\varphi_0\end{aligned}$$

Inserting yields

$$\begin{aligned}\phi_0(\tau;\boldsymbol{\alpha}) &= 2\pi h\sum_{j=0}^{L-1}\alpha_{-j}q(\tau+jT) + 2\pi h(M-1)\sum_{j=0}^{L-1} q(\tau+jT)\\ &= 2\pi h\sum_{j=0}^{L-1}\Big(\alpha_{-j} + (M-1)\Big)q(\tau+jT)\end{aligned}$$

We introduce the new information-bearing symbols

$$U_j = \frac{1}{2}\big(\alpha_j + (M-1)\big) \qquad \in \Big\{0, 1, \ldots, M-1\Big\}$$

Notice that the new symbol alphabet is similar regardless of whether M is odd or even.

Making use of these "new" symbols, we can recast $\phi_0(\tau;\boldsymbol{\alpha})$ according to

$$\begin{aligned}\phi_0(\tau;\boldsymbol{\alpha}) &= \phi(\tau;\boldsymbol{\alpha})\\ &= 4\pi h \sum_{j=0}^{L-1} U_{-j} q(\tau + jT)\end{aligned}$$

(ii) Consider: $n \geq 1, \;\Rightarrow\; t = nT + \tau$

$$
\begin{aligned}
\phi_n(\tau;\boldsymbol{\alpha}) &= \phi(nT+\tau;\boldsymbol{\alpha}) \\
&= \varphi_0 + 2\pi h \sum_{j=-(L-1)}^{n} \alpha_j \; q(\tau + nT - jT) \\
&\qquad + \pi h(M-1)\frac{nT+\tau}{T} \\
&= \varphi_0 + 2\pi h \sum_{j=0}^{n+(L-1)} \alpha_{n-j} q(\tau + jT) \\
&\qquad + \pi h(M-1)n + \pi h(M-1)\frac{\tau}{T} \\
&= \varphi_0 + 2\pi h \sum_{j=0}^{L-1} \alpha_{n-j} q(\tau + jT) + \pi h(M-1)\frac{\tau}{T} \\
&\qquad + 2\pi h \sum_{j=L}^{n+(L-1)} \alpha_{n-j}\frac{1}{2} + \pi h(M-1)n \\
&= 4\pi h \sum_{j=0}^{L-1} U_{n-j} q(\tau + jT) + \pi h\Big(\sum_{j=-(L-1)}^{n-L} \big[\alpha_j + (M-1)\big]\Big) \\
&= 4\pi h \sum_{j=0}^{L-1} U_{n-j} q(\tau + jT) + 2\pi h \sum_{j=-(L-1)}^{n-L} U_j \\
&= 4\pi h \sum_{j=0}^{L-1} U_{n-j} q(\tau + jT) + \underbrace{2\pi h \Bigg(\sum_{j=-(L-1)}^{n-L} U_j \Bigg)_{\mathrm{mod}\ p_2}}_{V_n}
\end{aligned}
$$

Remember the definition of the modulation index: $h = p_1/p_2$.

Let us define

$$V_n = \left(\sum_{j=-(L-1)}^{n-L} U_j \right)_{\bmod p_2}$$

Then

$$\begin{aligned}\phi_n(\tau;\boldsymbol{\alpha}) &= 4\pi h \sum_{j=0}^{L-1} U_{n-j}\ q(\tau + jT) + 2\pi h V_n \\ &= \phi\bigl(\tau; U_n, U_{n-1}, \ldots U_{n-(L-1)}, V_n\bigr)\end{aligned}$$

The vector

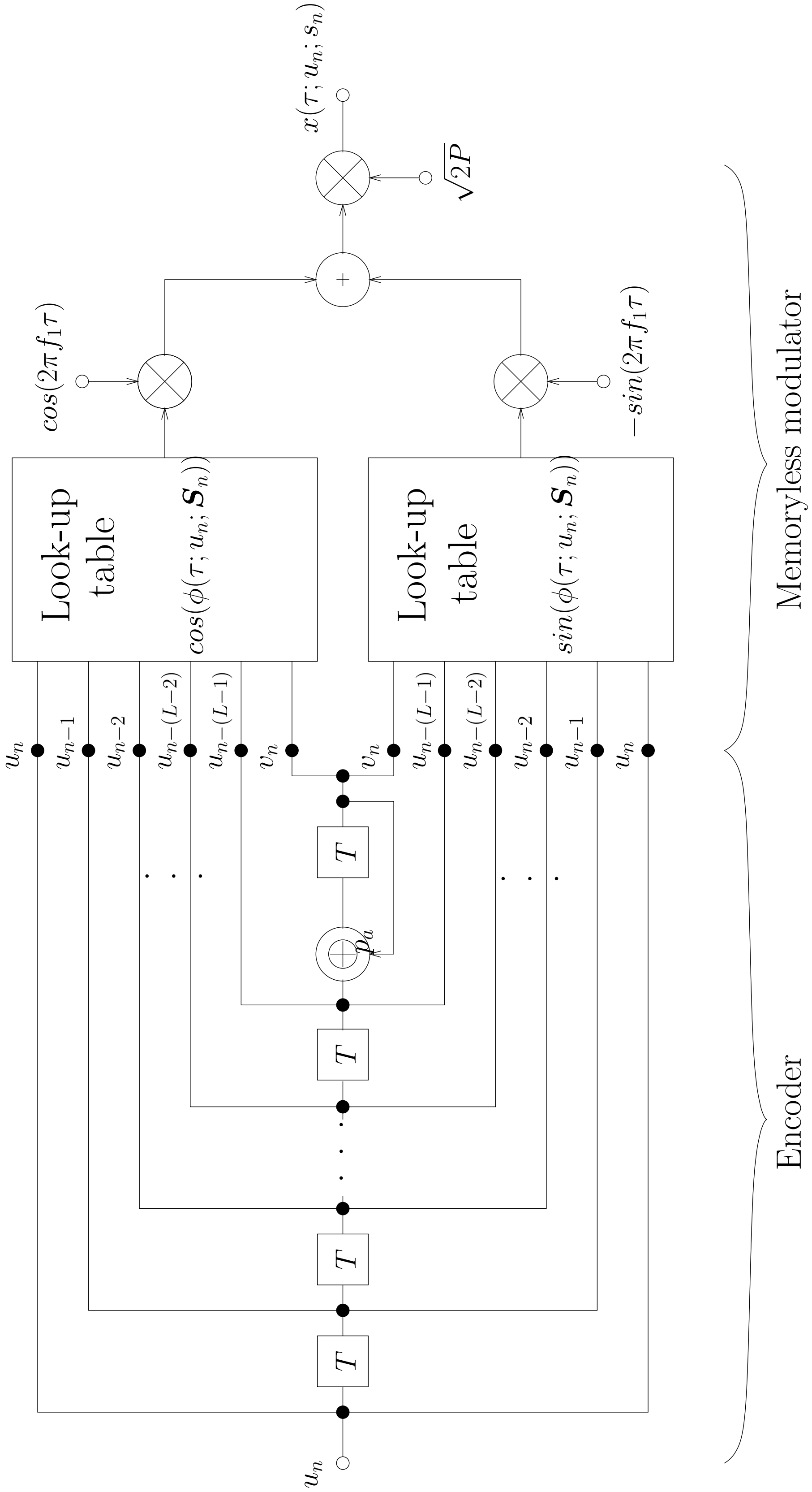
$$S_n \ := \ \bigl(U_{n-1}, \ldots U_{n-(L-1)}, V_n\bigr)$$

defines the state in the new description (modern description).

(Cf. definition pf state in the traditional description.)

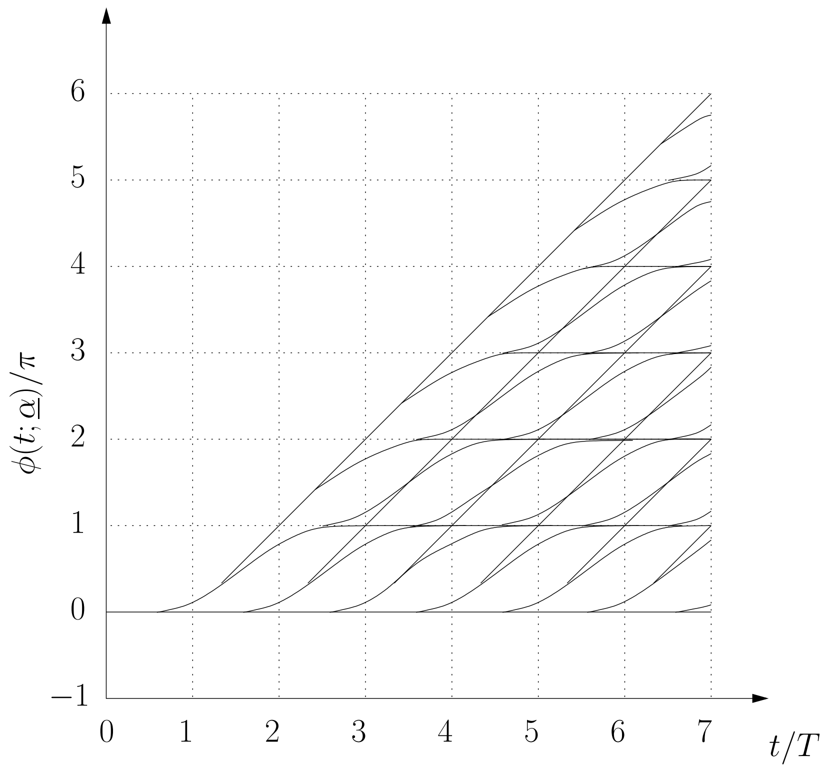
2.5 Block Diagram of the CPM Transmitter

The previous discussions lead to the following block diagram.

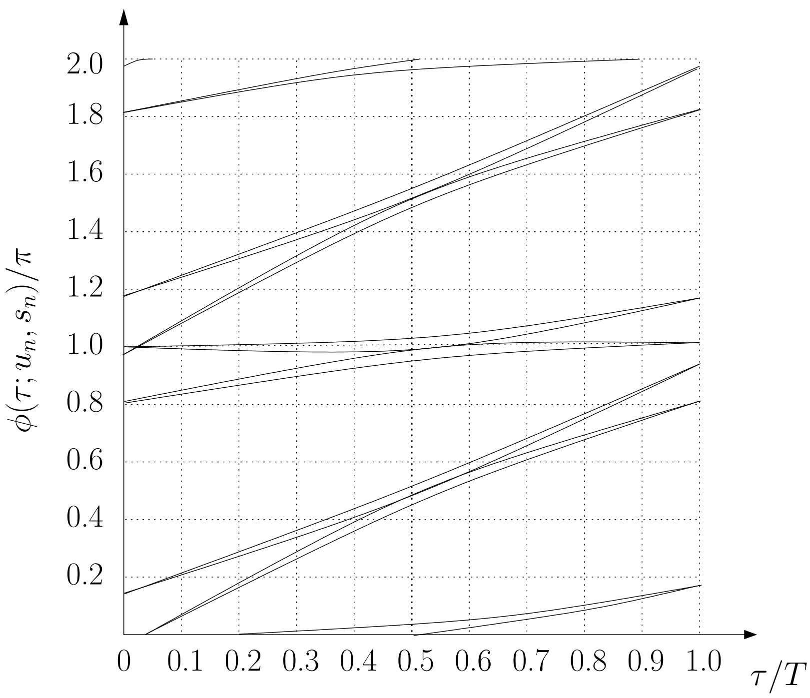


EXAMPLE: GMSK with $L = 3$

Tilted phase tree:



Trajectories of $\phi\big(\tau; U_n, S_n\big)$:



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3 Laurent Representation of CPM

CPM signals can be equivalently represented by a superposition of PAM signals [3]. In this section we use only the first term of this representation as a (fairly) good approximation of the CPM signal.

3.1 Minimum Shift Keying

The MSK Signal

The MSK signal is given by

$$x(t; \boldsymbol{\alpha}) = \sqrt{\frac{2E_b}{T}} \cos\Big(2\pi f_0 t + \varphi(t; \boldsymbol{\alpha})\Big)$$

with the information-bearing phase

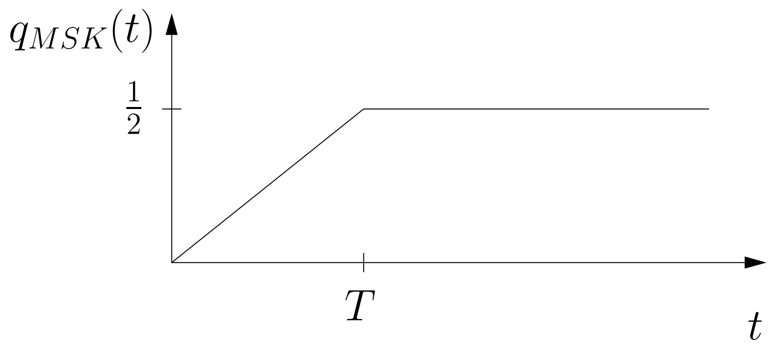
$$\varphi(t; \boldsymbol{\alpha}) = \pi \sum_{n=0}^{N-1} \alpha_n \, q_{MSK}(t - nT),$$

$t \in [0, NT)$.

The information symbols are

$$\alpha_n \in \Big\{-1, +1\Big\}, \qquad n = 0, \ldots, N-1.$$

The phase pulse $q_{MSK}(t)$ has the following shape:



In the nth signaling interval, $t \in [nT, nT + T)$, $t = nT + \tau$, $\tau \in [0, T)$, we have

$$\varphi_n(\tau; \boldsymbol{\alpha}) = \underbrace{\frac{\pi}{2} \sum_{i=0}^{n-1} \alpha_i}_{\theta_n} + \frac{\pi}{2} \frac{\tau}{T} \alpha_n.$$

Remember that θ_n is the phase at the beginning of the nth signaling interval.

Notice that

$$\theta_n \in \begin{cases} \{0, \pi\} & \text{for } n \text{ even} \\ \{\frac{\pi}{2}, \frac{3\pi}{2}\} & \text{for } n \text{ odd} \end{cases}$$

The initialization is $\theta_0 = 0$.

In-phase and quadrature components of MSK

Consider $t \in [0, NT)$. The MSK signal can be decomposed into its inphase and its quadrature components:

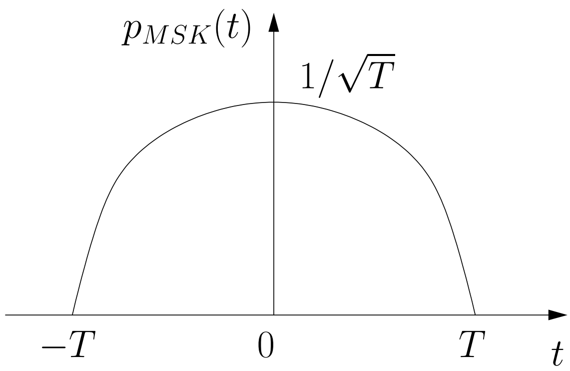
$$x(t;\boldsymbol{\alpha}) = \underbrace{\sqrt{\frac{E_b}{T}}\cos\Big(\varphi(t;\boldsymbol{\alpha})\Big)}_{x_I(t;\boldsymbol{\alpha})}\cdot\sqrt{2}\cos\big(2\pi f_0 t\big)$$
$$- \underbrace{\sqrt{\frac{E_b}{T}}\sin\Big(\varphi(t;\boldsymbol{\alpha})\Big)}_{x_Q(t;\boldsymbol{\alpha})}\cdot\sqrt{2}\sin\big(2\pi f_0 t\big)$$

Consider now the pulse

$$p_{MSK}(t) = \begin{cases} \frac{1}{\sqrt{T}}\ \cos(\frac{\pi t}{2T}) & \text{for } t \in [-T, +T], \\ 0 & \text{elsewhere.} \end{cases}$$

It is normalized such that it has energy one, i.e.,

$$\int_{-T}^{+T} \Big(p_{MSK}(t)\Big)^2 dt = 1.$$



Using this pulse, the inphase component $x_I(t;\boldsymbol{\alpha})$ and the quadrature component $x_Q(t;\boldsymbol{\alpha})$ can be written as follows:

$$
\begin{aligned}
x_I(t;\boldsymbol{\alpha}) &= \sqrt{\frac{E_b}{T}} \cos\Big(\varphi(t;\boldsymbol{\alpha})\Big) \\
&= \sum_{\substack{n=0 \\ n \text{ even}}}^{N} x_{1,n} \cdot p_{MSK}(t-nT) \\
x_{1,n} &= \sqrt{E_b}\cos(\theta_n) \qquad \in \big\{-\sqrt{E_b}, +\sqrt{E_b}\big\}
\end{aligned}
$$

$$
\begin{aligned}
x_Q(t;\boldsymbol{\alpha}) &= \sqrt{\frac{E_b}{T}} \sin\Big(\varphi(t;\boldsymbol{\alpha})\Big) \\
&= \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} x_{2,n} \cdot p_{MSK}(t-nT) \\
x_{2,n} &= \sqrt{E_b}\sin(\theta_n) \qquad \in \big\{-\sqrt{E_b}, +\sqrt{E_b}\big\}
\end{aligned}
$$

The symbol α_N is a known suffix symbol.

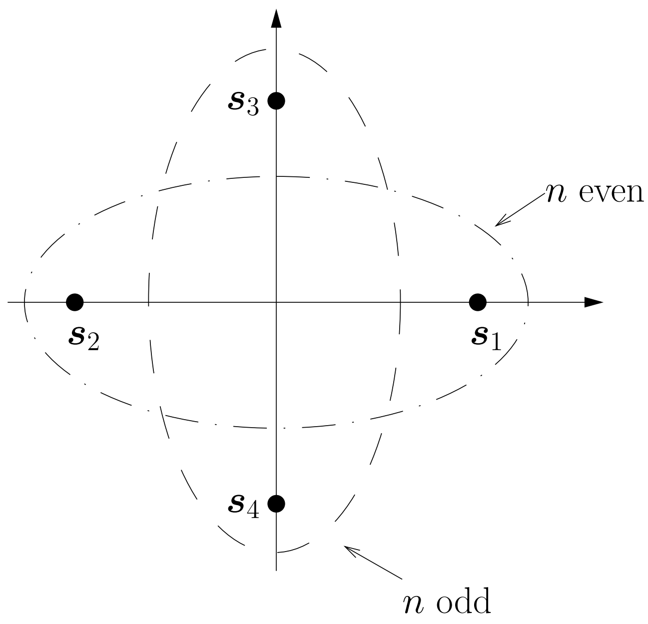
The modulation symbols can thus be considered as two-dimensional symbols

$$
\boldsymbol{X}_n = \begin{pmatrix} X_{1,n} \\ X_{2,n} \end{pmatrix} \in \mathcal{S}
$$

out of the signal constellation

$$
\mathcal{S} = \Bigg\{ \underbrace{\underbrace{\begin{pmatrix} +\sqrt{E_b} \\ 0 \end{pmatrix}}_{\boldsymbol{s}_1}, \underbrace{\begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}}_{\boldsymbol{s}_2}}_{n \text{ even}}, \underbrace{\underbrace{\begin{pmatrix} 0 \\ +\sqrt{E_b} \end{pmatrix}}_{\boldsymbol{s}_3}, \underbrace{\begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}}_{\boldsymbol{s}_4}}_{n \text{ odd}} \Bigg\}
$$

Illustration of the signal constellation:



Example: Phase Trajectories

Draw the phase and the trajectories of the inphase and quadrature components for the following example.

n	α_n	θ_n	$\cos\theta_n$	$\sin\theta_n$
0	$+1$	$\theta_0 = 0$(init)	1	0
1	-1	$\frac{\pi}{2}$	0	1
2	$+1$	0	1	0
3	$+1$	$\frac{\pi}{2}$	0	1
$N-1=4$	$+1$	π	-1	0
Suffix $N=5$	$(+1)$	$\frac{3\pi}{2}$	0	-1
			(1)	

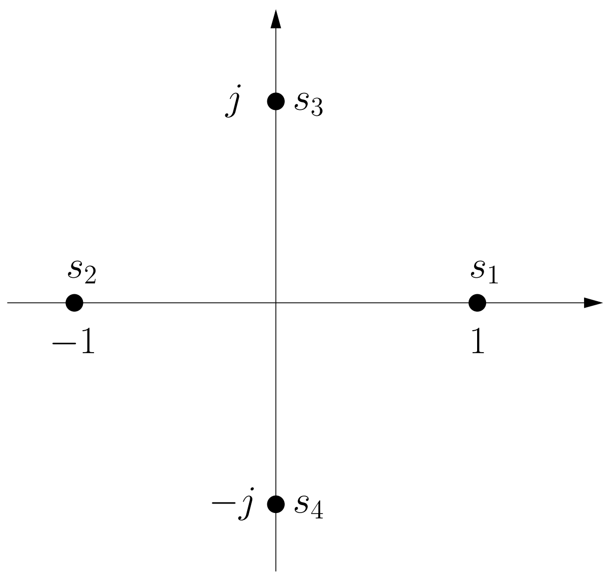
◇

Complex Representation

The Vector $\boldsymbol{X}_n = (X_{1,n}\ ,\ X_{2,n})^T$ can be conceived as the complex number

$$X_n = X_{1,n} + jX_{2,n}$$
$$= \sqrt{E_b}\ \exp(j\theta_n) \qquad\qquad \theta_n = \frac{\pi}{2}\sum_{i=0}^{n-1}\alpha_i$$

The (complex-valued) signal constellation is thus



The values of the complex-valued modulation symbol can be recursively computed:

$$X_n = \sqrt{E_b}\cdot\exp\Big(j\frac{\pi}{2}\sum_{i=0}^{n-1}\alpha_i\Big)$$
$$= \exp\Big(j\frac{\pi}{2}\alpha_{n-1}\Big)\cdot\sqrt{E_b}\ \exp\Big(j\frac{\pi}{2}\sum_{i=0}^{n-2}\alpha_i\Big)$$
$$= j\alpha_{n-1}\cdot X_{n-1}$$

Notice that

$$\exp\left(j\frac{\pi}{2}\alpha_{n-1}\right) = (j)^{\alpha_{n-1}} = j\alpha_{n-1}$$

as $\alpha_{n-1} \in \{-1, +1\}$.

Thus the MSK signal can be written as

$$X(t;\boldsymbol{\alpha}) = \Re\left\{ \left[\sum_{n=0}^{N} X_n \; p_{MSK}(t - nT) \right] \sqrt{2}\exp(j2\pi f_0 t) \right\}$$

for $t \in [0, NT)$.

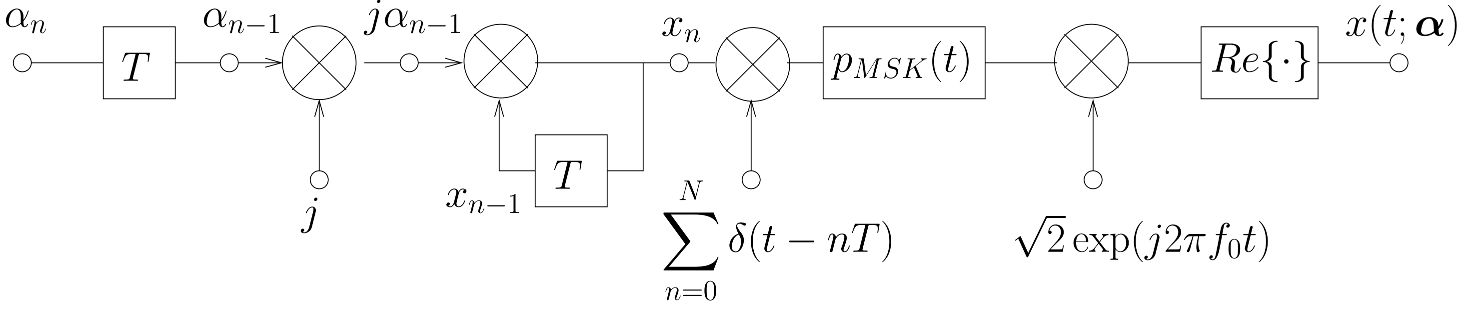
The initialization is

$$X_0 = j \; \alpha_{-1} \; X_{-1} \; = \sqrt{E_b}.$$

This can be achieved by

$$\begin{aligned} X_{-1} &= j \; \sqrt{E_b}, \\ \alpha_{-1} &= -1. \end{aligned}$$

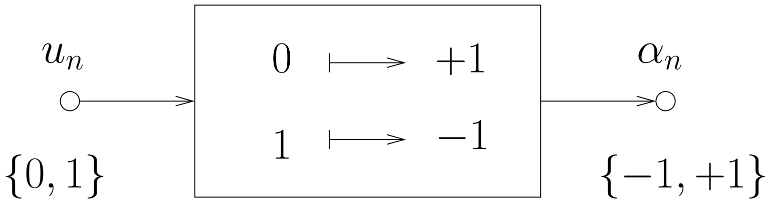
Block Diagram



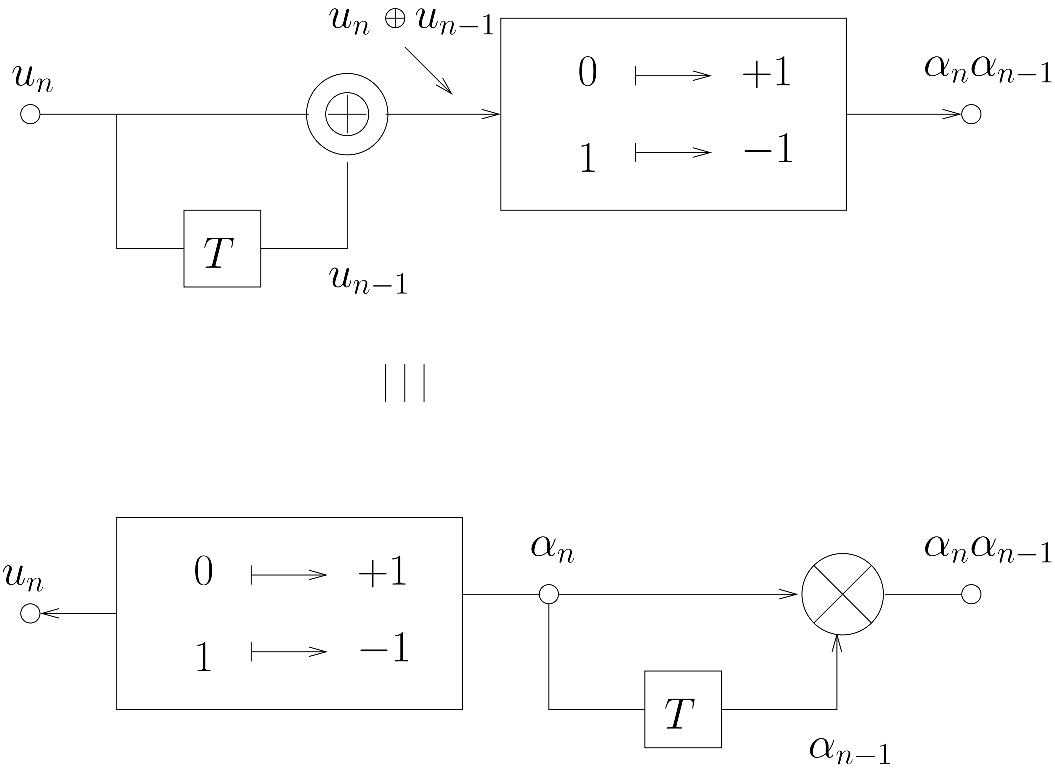
3.2 Precoded MSK

Precoding

The information symbols are mapped from the $\{0, 1\}$ representation to the $\{+1, -1\}$ representation: $u_n \mapsto \alpha_n$



The precoding can be done in both representations:



Notice that $u_{n-1} \oplus u_n$ (addition in $\mathbb{F}_2 \equiv GF(2)$) corresponds to $\alpha_{n-1} \cdot \alpha_n$ (multiplication in $\mathbb{R}$). This justifies the particular mapping.

Complex Representation

Transmitted symbols:

$$X_n = j(\alpha_{n-1}\ \alpha_{n-2})X_{n-1}, \qquad\qquad n = 0, \ldots, N-1$$

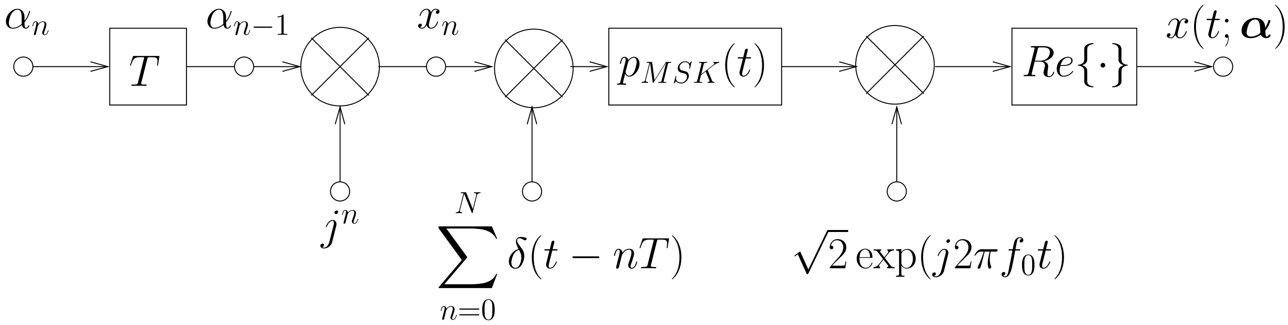
Initialization:

$$\alpha_{-2} = \alpha_{-1} = -1$$
$$X_{-1} = j\sqrt{E_b}$$

The symbols are generated as follows:

n	X_n	
0	$j\ j\sqrt{E_b} = \sqrt{E_b}$	$= j^0\ \alpha_{-1}\ \sqrt{E_b}$
1	$j\ \alpha_0\ \alpha_{-1}\ j^0\ \alpha_{-1}\ \sqrt{E_b}$	$= j^1\ \alpha_0\ \sqrt{E_b}$
2	$j\ \alpha_1\ \alpha_0\ j^1\ \alpha_0\ \sqrt{E_b}$	$= j^2\ \alpha_1\ \sqrt{E_b}$
3	$j\ \alpha_2\ \alpha_1\ j^2\alpha_1\ \sqrt{E_b}$	$= j^3\ \alpha_2\ \sqrt{E_b}$
$\vdots$		$\vdots$
n		$j^n\ \alpha_{n-1}\ \sqrt{E_b}$

This gives the block diagram of precoded MSK using the complex representation of the symbols:



Vector Representation

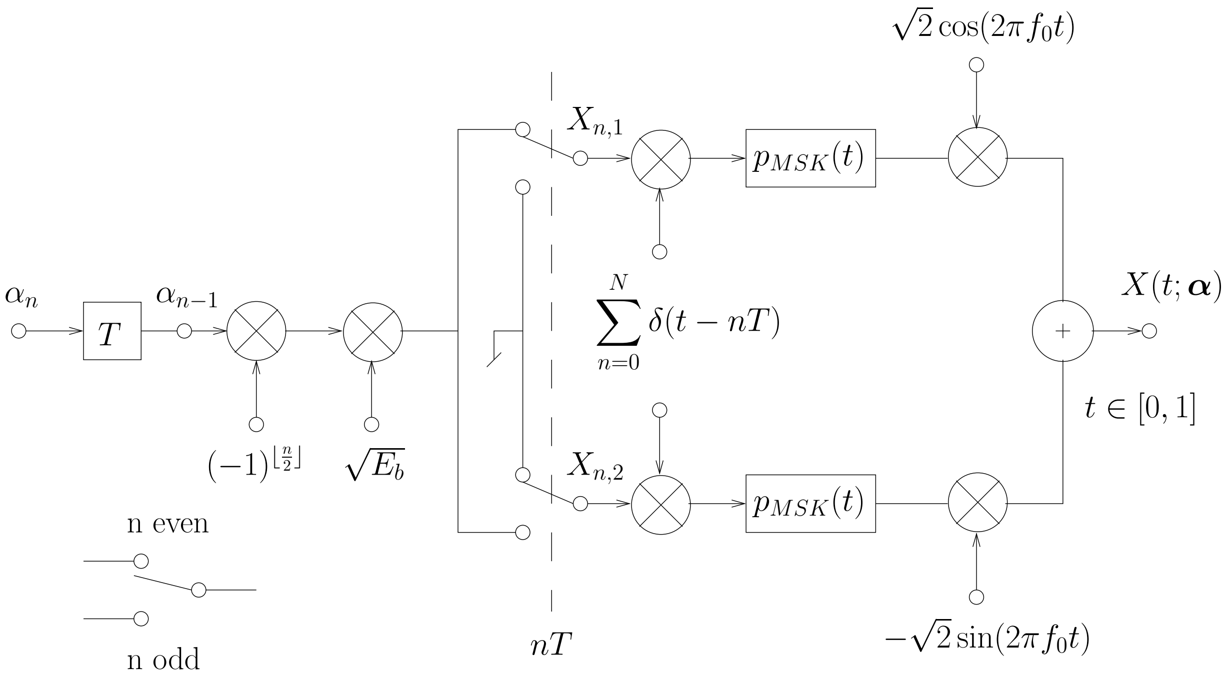
$$
\begin{aligned}
n \text{ even:} \quad X_n &= (j)^n \sqrt{E_b}\,\alpha_{n-1} \\
&= (j^2)^{\frac{n}{2}} \sqrt{E_b}\,\alpha_{n-1} \\
&= (-1)^{\frac{n}{2}} \sqrt{E_b}\,\alpha_{n-1} \qquad\qquad \text{(real-valued)}
\end{aligned}
$$

$$
\boldsymbol{X}_n = \Big(\underbrace{(-1)^{\frac{n}{2}} \sqrt{E_b}\,\alpha_{n-1}}_{X_{n,1}}, \underbrace{0}_{X_{n,2}}\Big)^T
$$

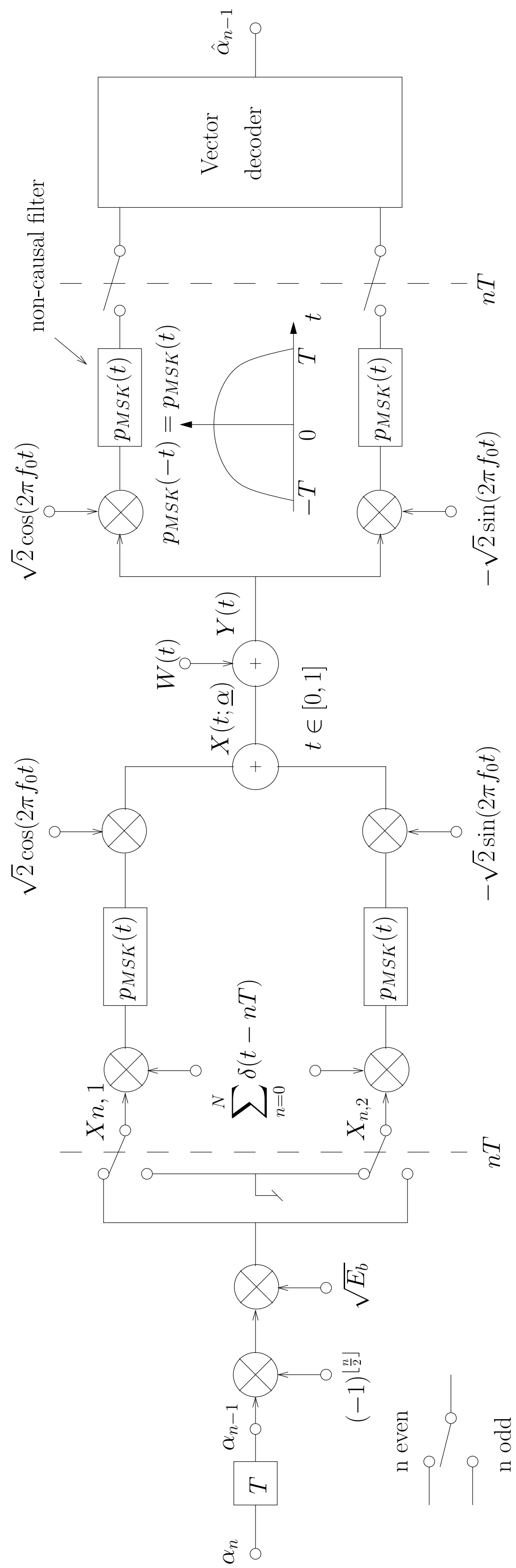
$$
\begin{aligned}
n \text{ odd:} \quad X_n &= (j)^n \sqrt{E_b}\,\alpha_{n-1} \\
&= j(j)^{n-1} \sqrt{E_b}\,\alpha_{n-1} \\
&= j(-1)^{\frac{n-1}{2}} \sqrt{E_b}\,\alpha_{n-1} \qquad\qquad \text{(imaginary-valued)}
\end{aligned}
$$

$$
\boldsymbol{X}_n = \Big(\underbrace{0}_{X_{n,1}}, \underbrace{(-1)^{\frac{n-1}{2}} \sqrt{E_b}\,\alpha_{n-1}}_{X_{n,2}}\Big)^T
$$

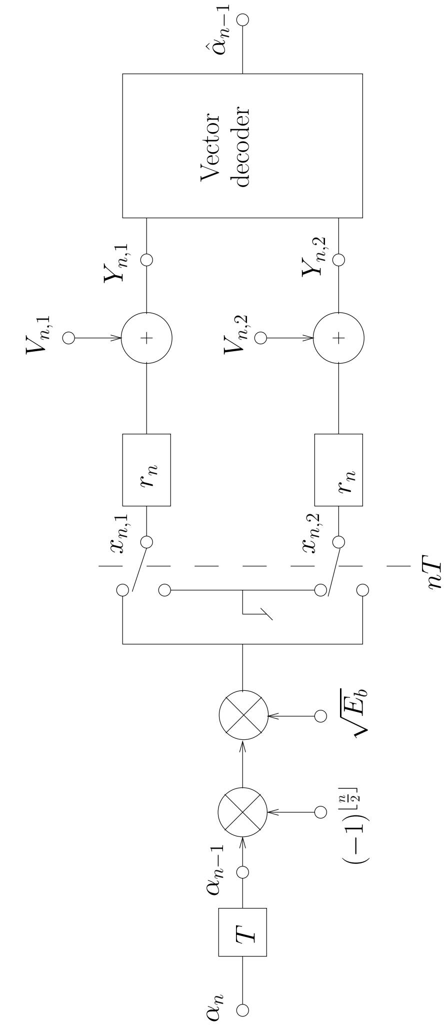
This results in the following block diagram:



Demodulation in the AWGN Channel



$$\equiv$$



$$r_n = R_{pp}(nT)$$

$$R_{pp}(\tau) = \int_{-\infty}^{+\infty} p_{MSK}(t) p_{MSK}(t+\tau) dt$$

$$E[V_{n,i} V_{m,j}] = \delta_{ij} \frac{N_0}{2} r_{m-n}$$

The effective discrete-time impulse response is

$$\begin{aligned} r_n &= R_{pp}(nT), \\ R_{pp}(\tau) &= \int p_{MSK}(t)\, p_{MSK}(t+\tau)\, dt. \end{aligned}$$

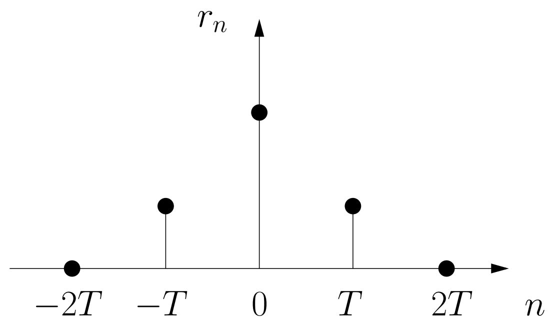
The noise after sampling has the covariance (mean is zero)

$$E\Big[V_{n,i} V_{M,j}\Big] = \delta_{i,j} \frac{N_0}{2} r_{m-n}.$$

The optimum vector decoder for MSK can be simplified. To see this, consider the impulse response:

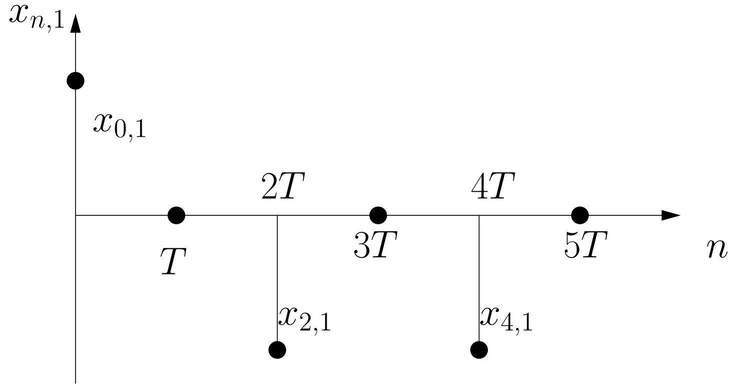
$$r_n = R_{pp}$$

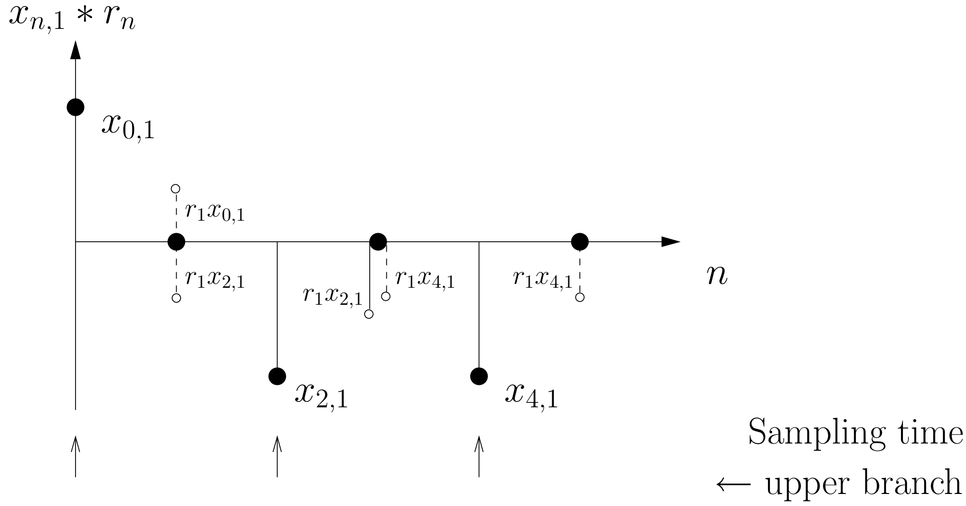
$$r_n = \begin{cases} 1 & \text{for } n = 0, \\ r_1 & \text{for } |n| = 1, \\ 0 & \text{elsewhere.} \end{cases}$$



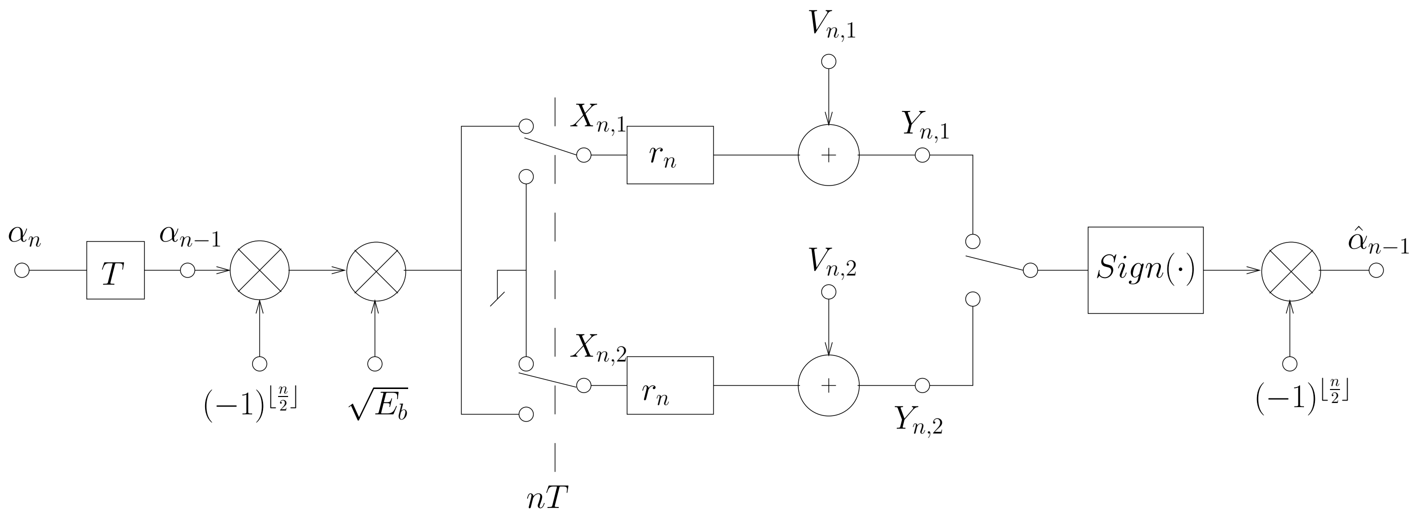
Thus, in the upper (lower) branch, symbols are effectively transmitted at even (odd) time indices.

Example for the upper branch:





The optimum vector decoder for the AWGN channel can thus be described as follows.



Remark: More information about the discrete-time representation of QAM-like signals is provided in the appendix.

3.3 Precoded Gaussian MSK

The lenght of the phase pulse is set to $L = 3$, as before.

Precoded GMSK Signal

The signal reads

$$x(t;\boldsymbol{\alpha}) = \sqrt{\frac{2E_b}{T}}\ \cos\Big(2\pi f_0 t + \varphi(t;\boldsymbol{\alpha})\Big)$$

with

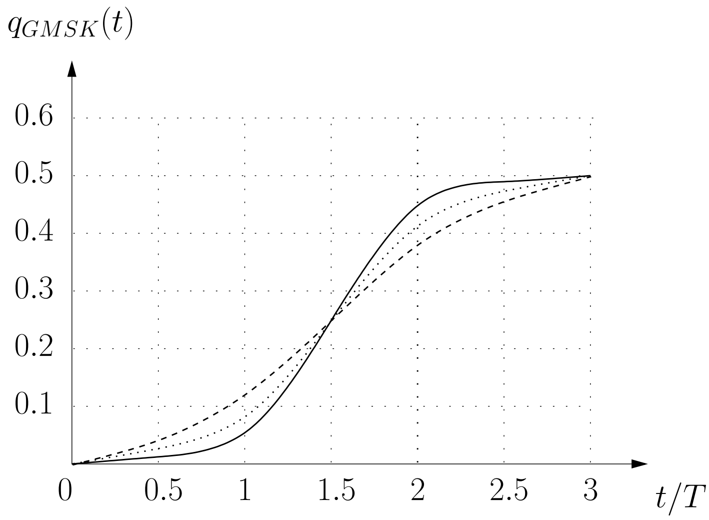
$$\varphi(t;\boldsymbol{\alpha}) = \pi \sum_{n=-2}^{N-1} (\alpha_n \alpha_{n-1})\ q_{GMSK}(t - nT) + \varphi_0$$

where

- $\alpha_n \in \{-1, +1\}$;

  prefix symbols (GSM): $\alpha_n = -1$ for $n = -3, -2, -1$,
  sufffix symbols (GSM): $\alpha_n = -1$ for $n = N, N+1, N+2$

- GMSK phase pulse $q_{GMSK}(t)$:



- φ_0 is set so that $\varphi(0; \boldsymbol{\alpha}) = 0$. Hence,

$$\varphi_0 = -\pi \sum_{n=1}^{2} q_{GMSK}(nT)$$

Within the n-th signaling interval, i.e., for
$t \in [nT, nT + T)$, $n = 0, \ldots, N - 1$.

$$\begin{aligned}\varphi_n(\tau; \boldsymbol{\alpha}) = \varphi_0 & \\ & + \frac{\pi}{2} \sum_{i=-2}^{n-3} (\alpha_i \, \alpha_{i-1}) \\ & + \sum_{i=n-2}^{n} (\alpha_i \, \alpha_{i-1}) \; q_{GMSK}(\tau)\end{aligned}$$

In-phase and Quadrature Components

The signal can be decomposed as

$$\begin{aligned}X(t; \boldsymbol{\alpha}) = & \underbrace{\sqrt{\frac{E_b}{T}} \cos \varphi(t; \boldsymbol{\alpha})}_{X_I(t; \boldsymbol{\alpha})} \cdot \sqrt{2} \cos(2\pi f_0 t) \\ & - \underbrace{\sqrt{\frac{E_b}{T}} \sin \varphi(t; \boldsymbol{\alpha})}_{X_Q(t; \boldsymbol{\alpha})} \cdot \sqrt{2} \sin(2\pi f_0 t).\end{aligned}$$

The inphase component and the quadrature component can be approximated as follows:

$$X_I(t;\boldsymbol{\alpha}) \approx \sum_{\substack{n=-2 \\ n \text{ even}}}^{N+2} X_{1,n} \cdot p_{GMSK}(t-nT)$$

$$X_{1,n} = (-1)^{\lfloor \frac{n}{2} \rfloor} \sqrt{E_b}\, \alpha_{n-1}$$

$$X_Q(t;\boldsymbol{\alpha}) \approx \sum_{\substack{n=-1 \\ n \text{ odd}}}^{N+1} X_{2,n}\, p_{GMSK}(t-nT)$$

$$X_{2,n} = (-1)^{\lfloor \frac{n}{2} \rfloor} \sqrt{E_b}\, \alpha_{n-1}$$

The pulse $p_{GMSK}(t)$ is often denoted as $c_0(t)$ pulse, referring to [3], and it is defined as follows.

Let LT, $L = 3$, be the length of the truncated GMSK pulse $q_{GMSK}(t)$. Then

$$c_0(t) = s_0(t) \cdot s_1(t) \cdot s_2(t)$$

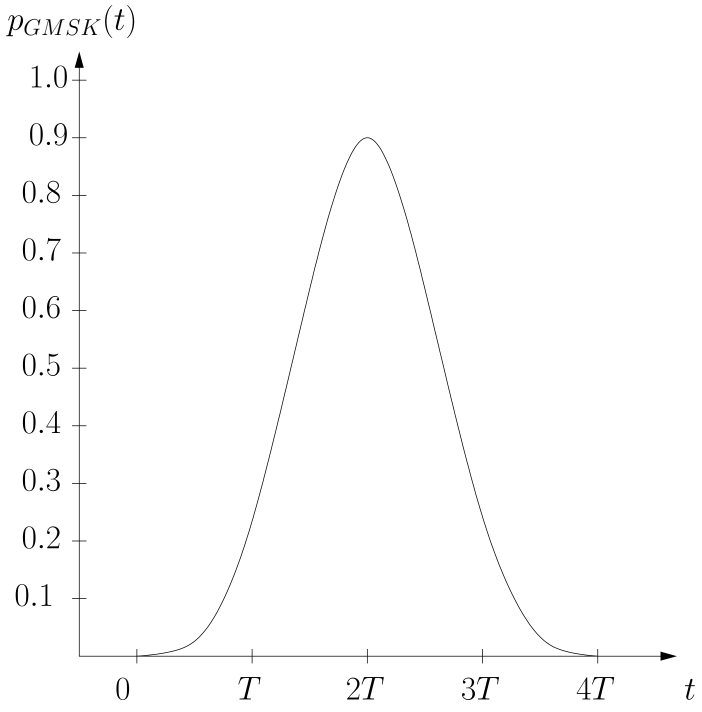
(for larger L, more pulses have to be multiplied) with

$$s_l(t) = \sin\Big(\phi(t+lT)\Big)$$

$$\phi(t) = \begin{cases} \pi \cdot q_{GMSK}(t) & \text{for } t < LT \\ \frac{\pi}{2} - \pi \cdot q_{GMSK}(t) & \text{for } t \geq LT \end{cases}$$

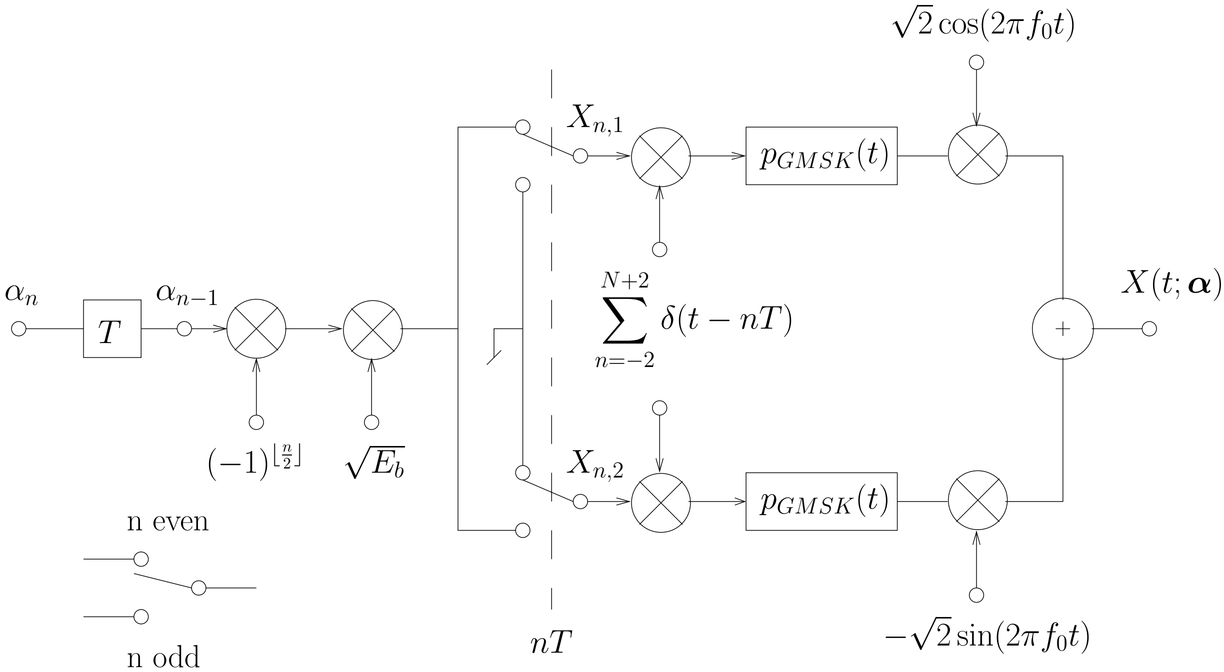
The resulting pulse $c_0(t)$ has length $(L+1)T = 4T$.

The resulting pulse is depicted for $L = 3$:

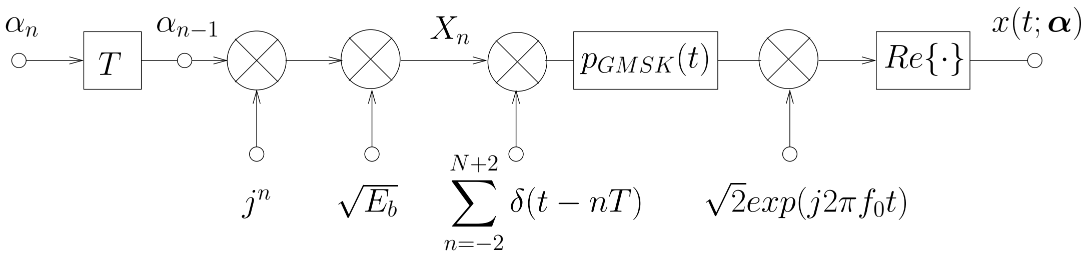


Block diagram of precoded GMSK

Using the vector representation:



Using the complex representation:



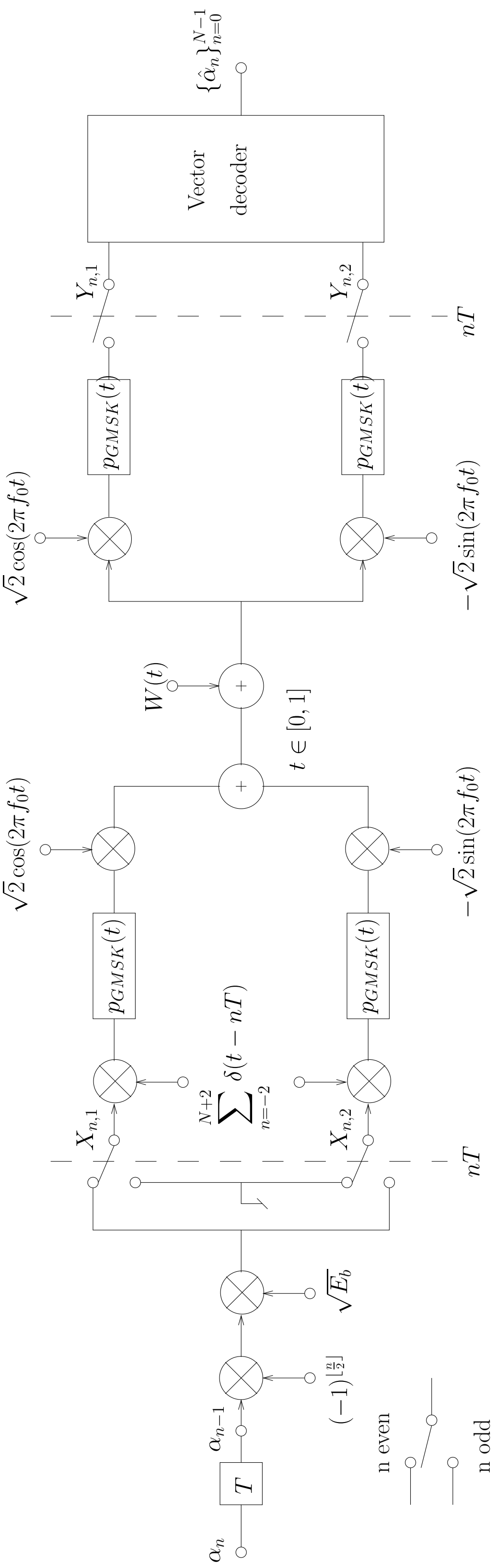
Initialization (prefix symbols):

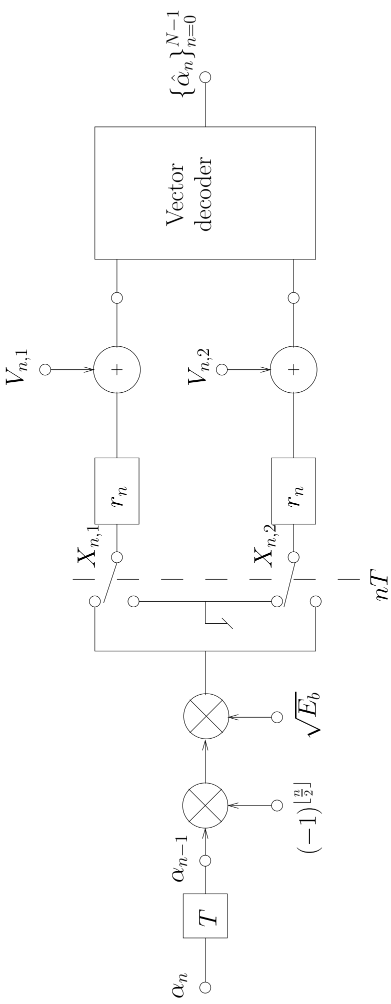
$$\alpha_{-3} = \alpha_{-2} = \alpha_{-1} = -1,$$
$$X_{-2} = j^{-2}\alpha_{-3}\sqrt{E_b} = \sqrt{E_b}.$$

Termination (suffix symbols):

$$\alpha_N = \alpha_{N+1} = \alpha_{N+2} = +1.$$

Demodulation of precoded GMSK in the AWGN channel



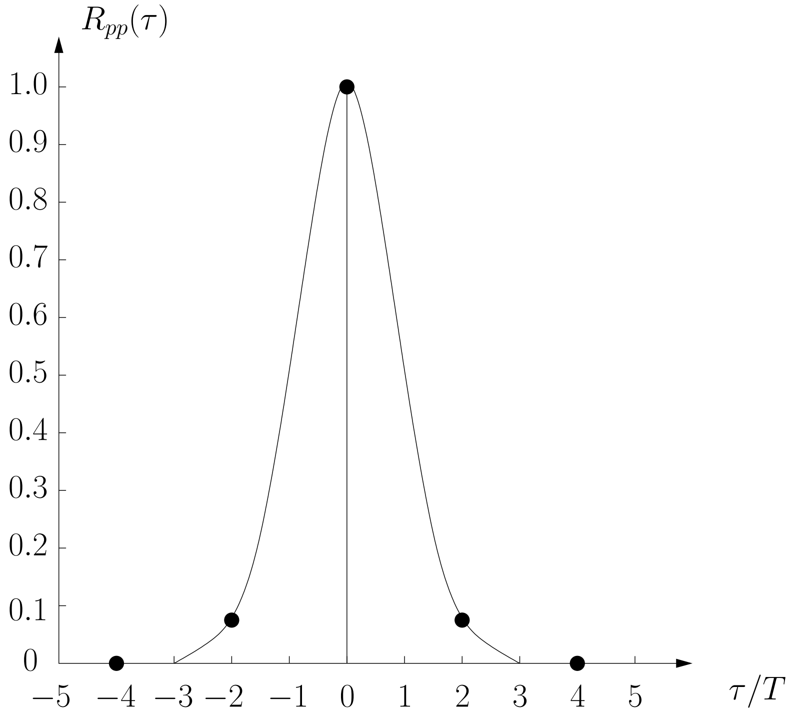


$$r_n = R_{pp}(nT)$$

$$R_{pp}(\tau) = \int_{-\frac{LT}{2}}^{\frac{LT}{2}} p_{GMSK}(t) p_{GMSK}(t+\tau) dt$$

The effective impulse response has the (deterministic) autocorrelation function:

$$R_{pp}(\tau) = \int p_{GMSK}(t)\, p_{GMSK}(t+\tau)\, dt$$

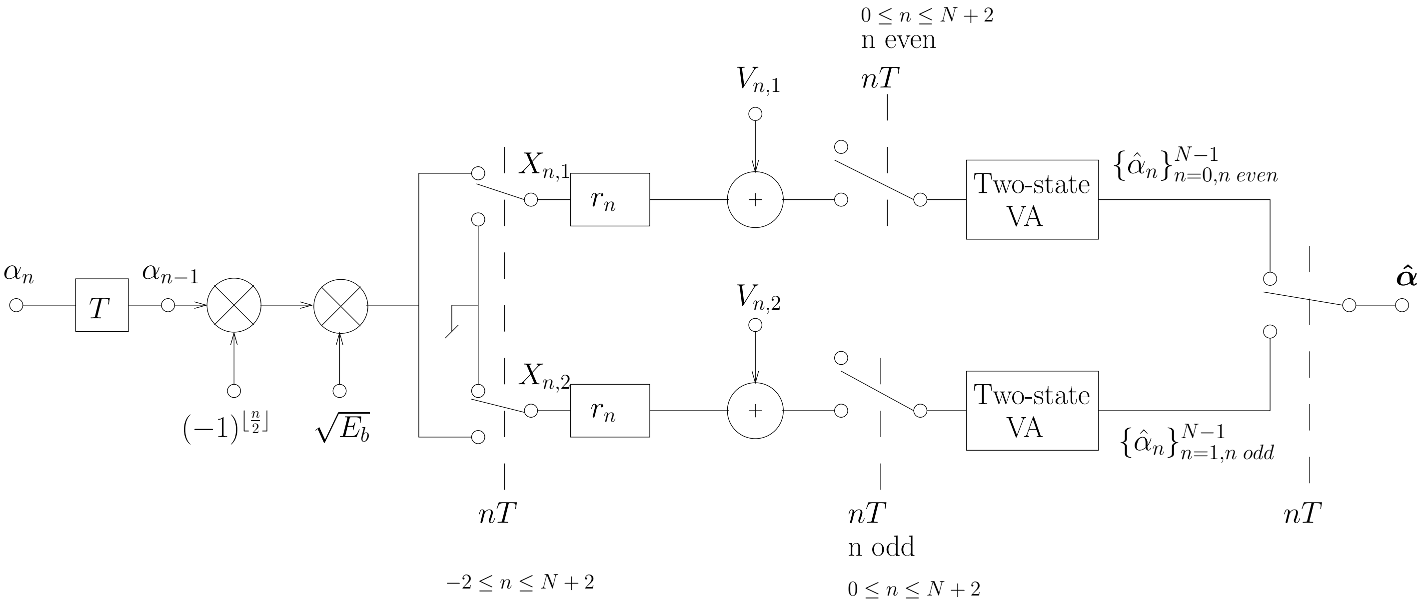


We have approximately

$$r_n = \begin{cases} 1 & \text{for } n = 0 \\ 0.05 & \text{for } |n| = 2 \\ 0 & \text{for } |n| > 2 \end{cases}$$

Hence, ISI occurs in each branch. For optimum decoding, the Viterbi algorithm (VA) may be applied.

(Close-to-optimum) Vector Decoder



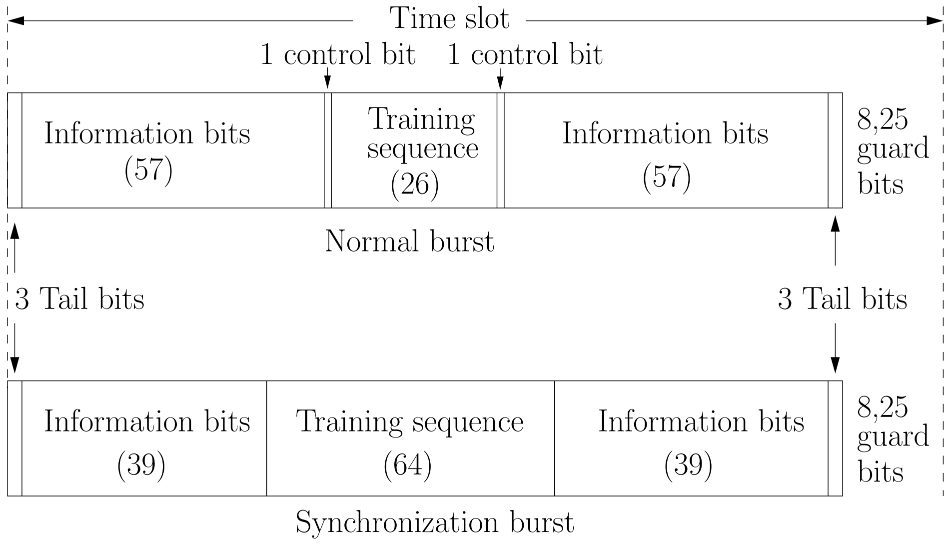
Metrics to be used in the upper-branch VA (even n):

$$\lambda(\boldsymbol{\alpha}_e) = \sum_{\substack{n=0 \\ n \text{ even}}}^{N-1} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \sqrt{E_b}\, \alpha_{n-1} \cdot \cdot \left[y_n - r_1 (-1)^{\left\lfloor \frac{n}{2} - 1 \right\rfloor} \sqrt{E_b}\, \alpha_{n-3} \right]$$

Metrics to be used in the lower-branch VA (odd n):

$$\lambda(\boldsymbol{\alpha}_o) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{N-1} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \sqrt{E_b}\, \alpha_{n-1} \cdot \cdot \left[y_n - r_1 (-1)^{\left\lfloor \frac{n}{2} - 1 \right\rfloor} \sqrt{E_b}\, \alpha_{n-3} \right]$$

GSM Burst Structure



3.4 Modulation in EDGE

Objective

Modify the GSM format in order to increase the bit rate while keeping the spectrum requirement of this modulation scheme.

Selected Solution

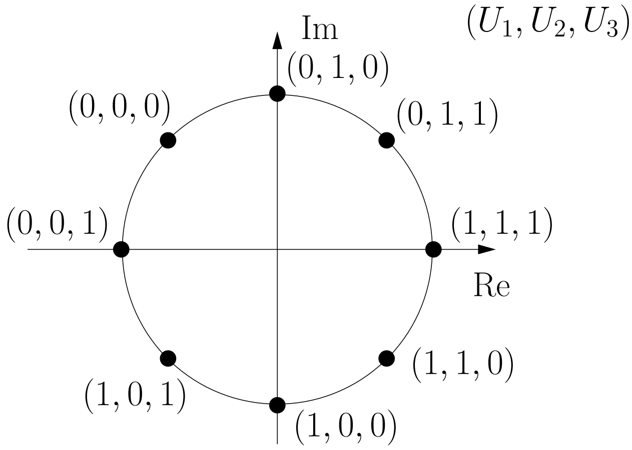
- Extend the signal space in the Laurent representation of GMSK

  $\longrightarrow$ 8PSK, i.e., 3 bits / symbol

- Include an additional rotation of the symbols

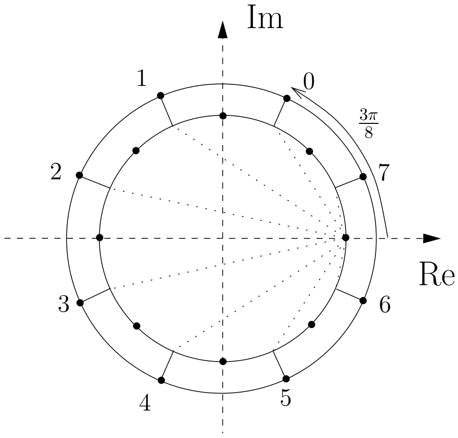
  $\longrightarrow$ rotation of $3\pi/8$ in each signaling interval

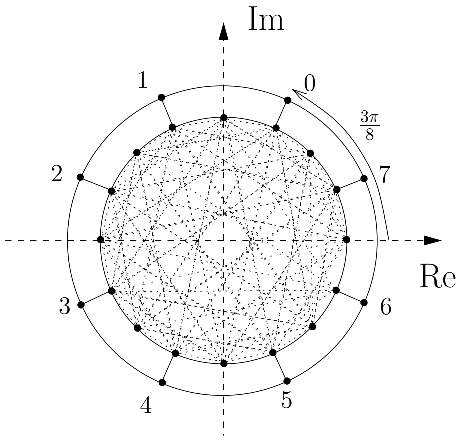
Signal Constellation and Mapping



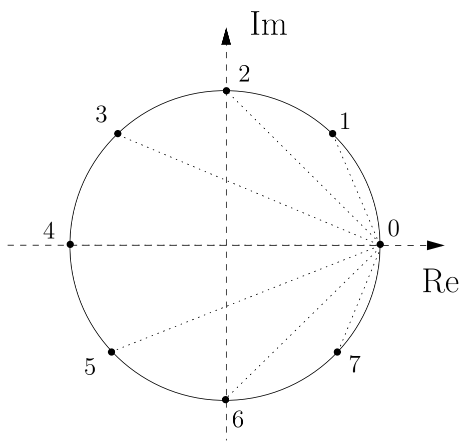
Phase Rotation

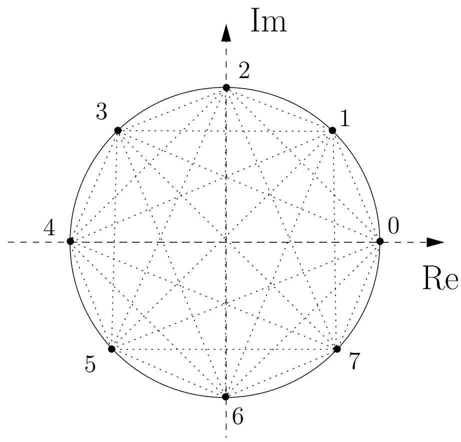
Rotation by $\frac{3\pi}{8}$ in each signaling interval





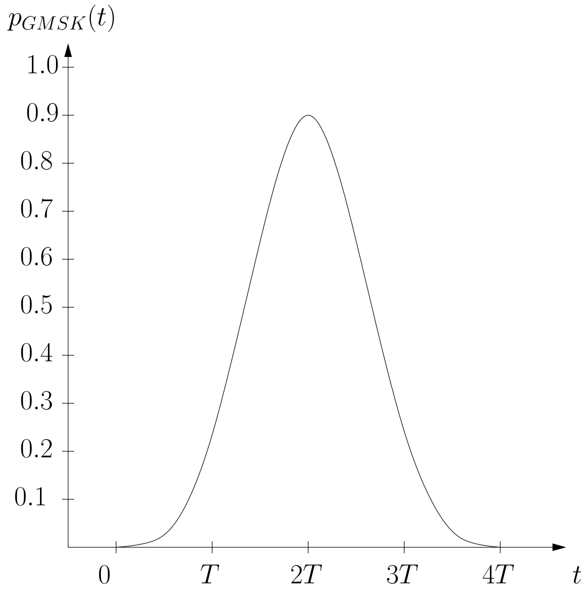
By comparison, for 8 PSK without such phase rotation:



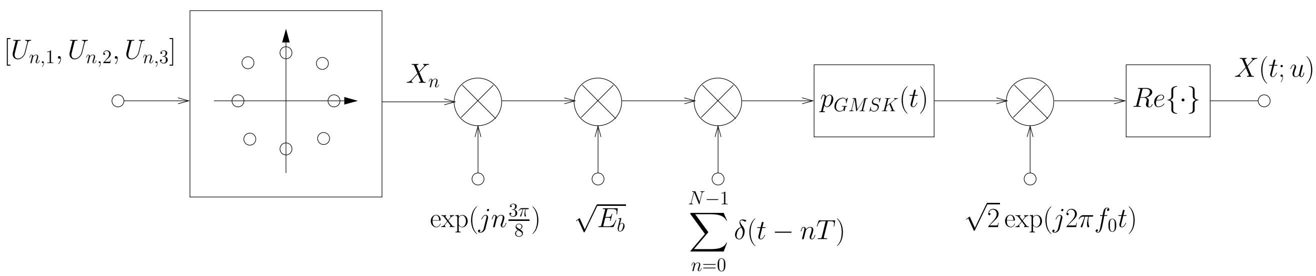


Pulse Shaping

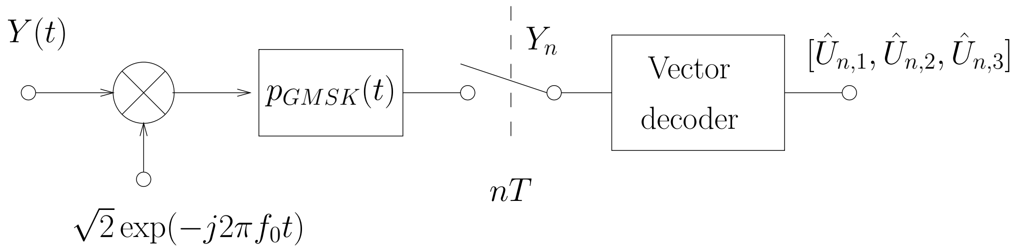
$$p(t) = p_{GMSK}(t)$$



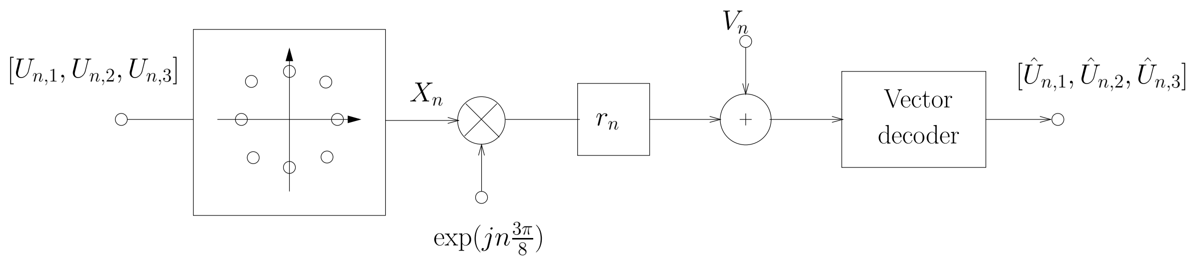
EDGE Transmitter



EDGE Receiver



Discrete-time Representation of the EDGE System



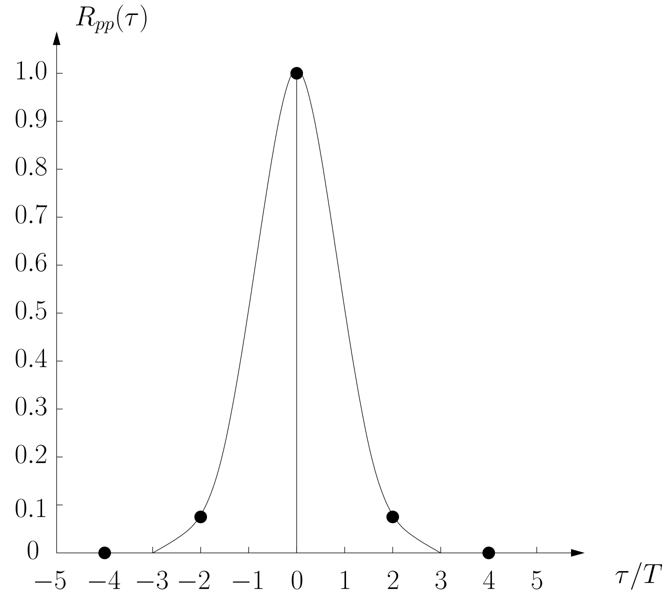
The effective impulse response is

$$r_n = R_{pp}(nT)$$

$$R_{pp}(\tau) = \int_0^{LT} p_{GMSK}(t)\, p_{GMSK}(t+\tau)\, dt$$

The noise has the covariance

$$E\Big[V_n V_{n+l}\Big] = N_0 r_l.$$



Optimum Vector Decoder for EDGE

For optimum vector decoding of EDGE in the AWGN channel, the Viterbi algorithm may be applied. The corresponding trellis has

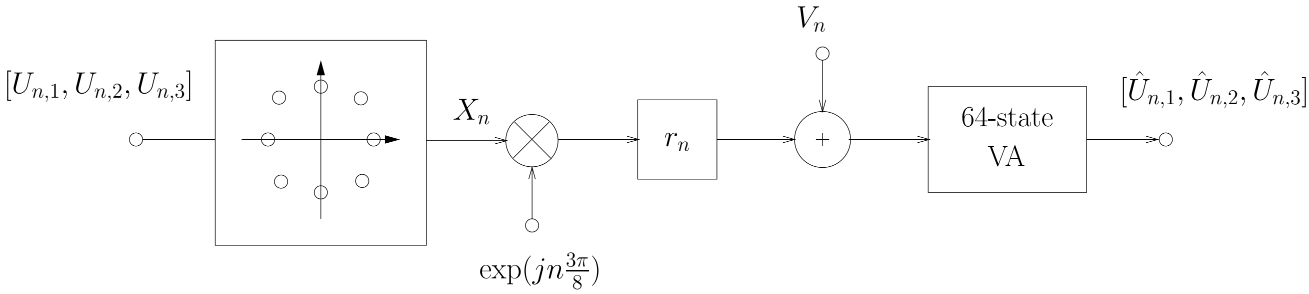
- $J = 8^2 = 64$ states and
- 8 transitions per state.

The branch metric is

$$\lambda(u) = \sum_{i=0}^{N-1} \Re\Big\{ x_n^* \exp\Big(-jn\frac{3\pi}{8}\Big) \cdot \\ \cdot \Big[y_n - r_1 x_{n-1} \exp\Big(j(n-1)\frac{3\pi}{8}\Big) \\ - r_2 X_{n-2} \exp\Big(j(n-2)\frac{3\pi}{8}\Big)\Big]\Big\}$$

Discrete-time Model

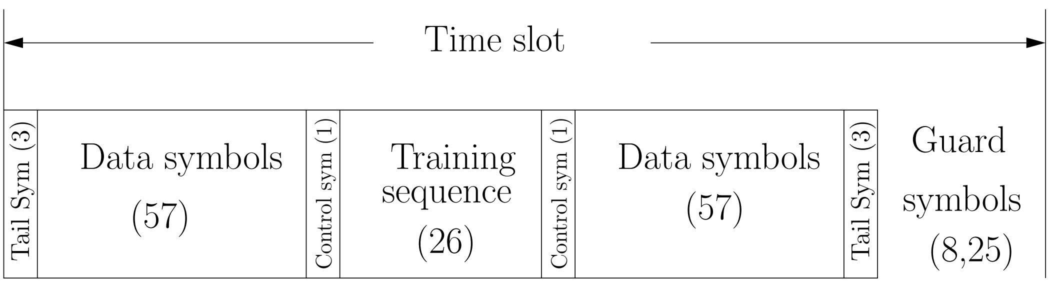
Using the complex-valued representation, the discrete-time model of EDGE is give by the following block diagram:



The vector with information bits looks like

$$U = \Big[\big[U_{0,1} \; , \; U_{0,2} \; , \; U_{0,3} \big], \ldots, \big[U_{N-1,1} \; , \; U_{N-1,2} \; , \; U_{N-1,3} \big] \Big]$$

Burst format of EDGE

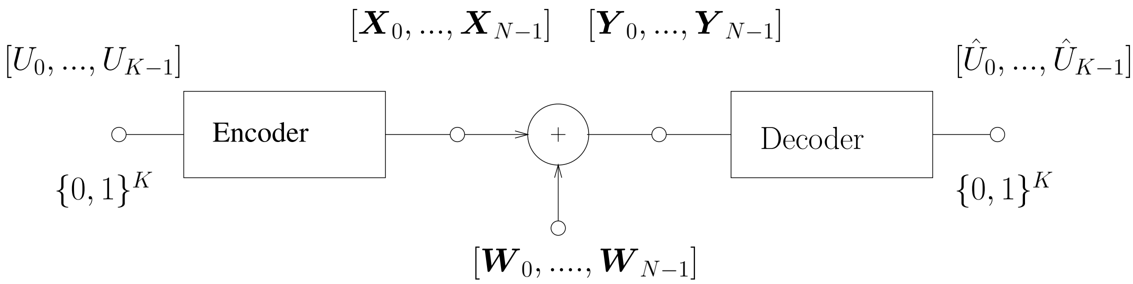


Normal burst

4 Trellis-Coded Modulation

4.1 Motivation

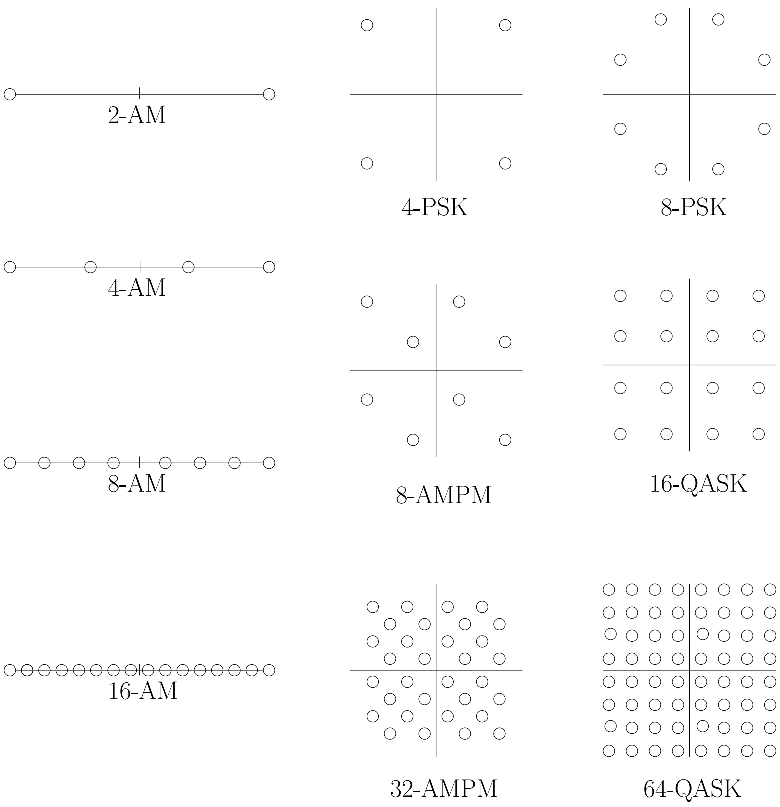
We consider the vector representation of a digital communication system transmitting across the AWGN channel.



We restrict our attention to the case where the dimension of the signal constellation $\mathbb{X} = \{\boldsymbol{S}_0 \ldots, \boldsymbol{S}_{M-1}\}$ is one or two.

EXAMPLE:

MPAM: dim $\mathbb{X} = 1$ MPSK $(M > 2)$ and QPAM : dim $\mathbb{X} = 2$.



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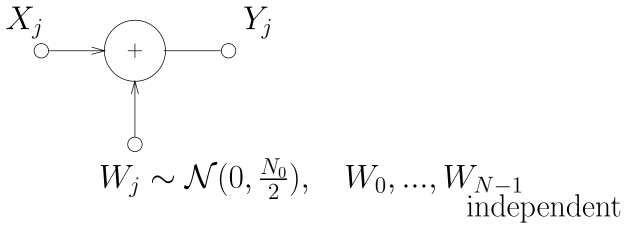
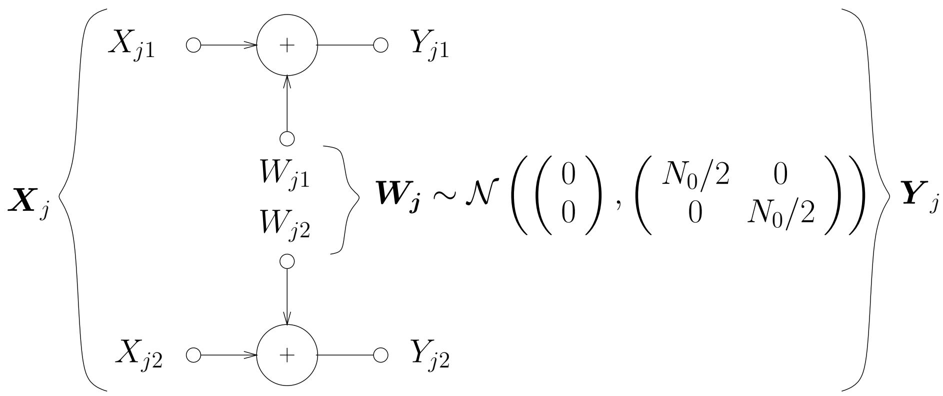
For these two cases, the vector channel is either of the form.

- $\dim \mathbb{X} = 1$



- $\dim \mathbb{X} = 2$



In both cases, the noise samples are independent.

Rate of the encoder

The Encoder maps bit-blocks onto modulation symbols

$$\boldsymbol{x} \in \mathbb{X} = \{\boldsymbol{s}_0, \ldots, \boldsymbol{s}_{M-1}\}.$$

Its rate is

$$R = \frac{K}{N} \quad \text{bits/symbol}$$

Two equivalent necessary conditions for the encoder to be one-to-one are

$$2^K \leq M^N \qquad \Leftrightarrow \qquad R = \frac{K}{N} \leq \log_2(M)$$

In the case of uncoded modulation, we have
$N = 1$, $K = \log_2(M) \Rightarrow R = \log_2(M)$

Bit Error Probability

The BER is defined as

$$P_b = \frac{1}{K} \sum_{j=0}^{K-1} P\big[U_j \neq \widehat{U}_j\big]$$

Capacity of the AWGN Channel

With the constraint that the transmitted vectors belong to a finite signal constellation $\mathbb{X} = \{\boldsymbol{s}_1 \ldots, \boldsymbol{s}_M\}$:

$$C_{\mathbb{X}} = \max_{p(\boldsymbol{s}_1),\ldots,p(\boldsymbol{s}_M)} \sum_{m=1}^{M} p(\boldsymbol{s}_m) \cdot$$
$$\cdot \left(\int p\big(\boldsymbol{y}|\boldsymbol{s}_m\big) \; \log_2\Big(\frac{p\big(\boldsymbol{y}|\boldsymbol{s}_m\big)\; p(\boldsymbol{s}_m)}{\sum_{m'=1}^{M} p\big(\boldsymbol{y}|\boldsymbol{s}_{m'}\big)\; p(\boldsymbol{s}_{m'})} \Big) d\boldsymbol{y} \right)$$

Remark

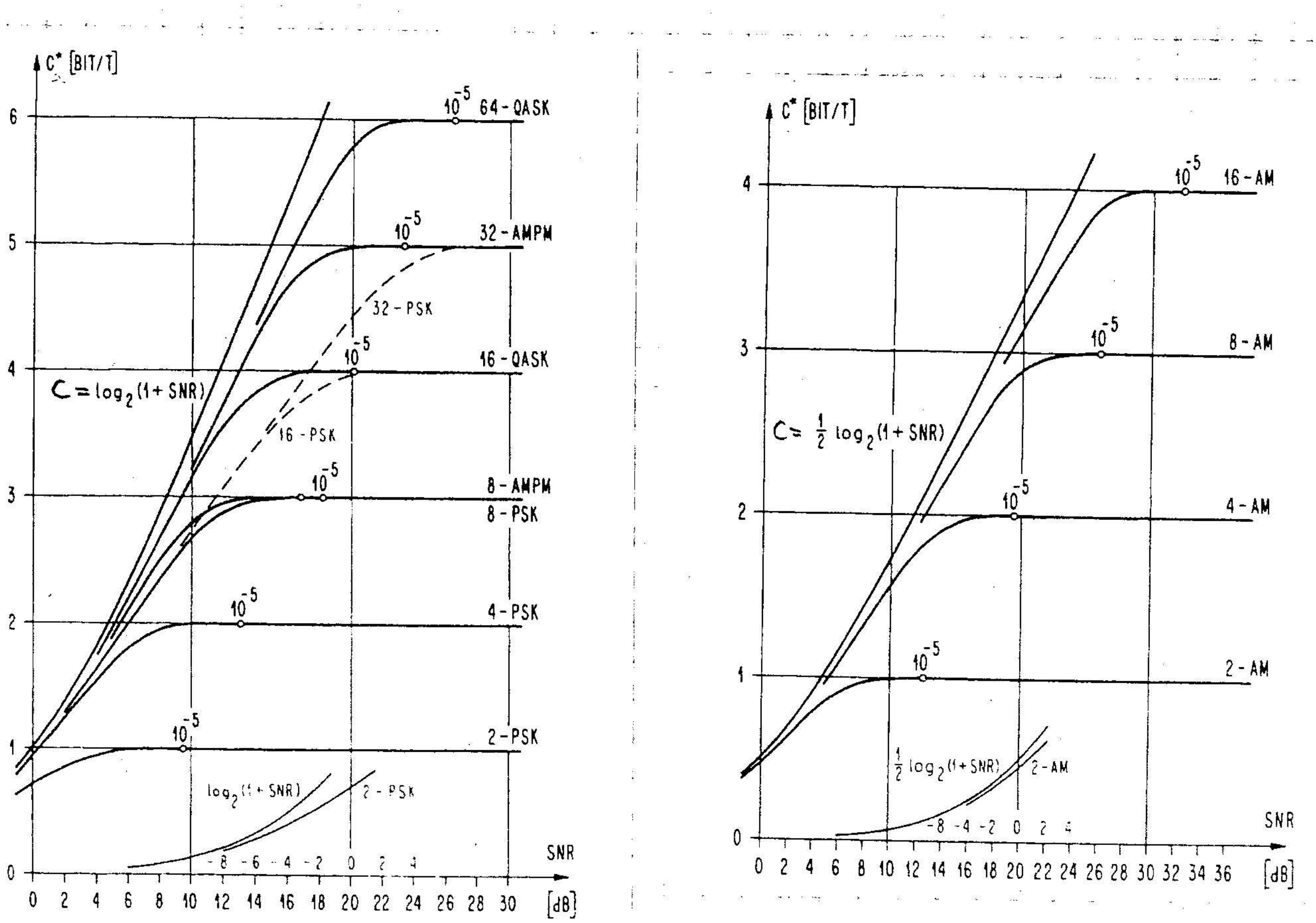
The capacity $C_{\mathbb{X}}$ depends on the signal constellation $\mathbb{X}$.

Dropping the restriction that $\mathbb{X}$ is finite while assuming a given average power $\overline{E}_s = E\left[|X_j|^2\right]$,

$$C := \log_2\big(1 + \mathrm{SNR}\big) \quad , \qquad\qquad \mathrm{SNR} = \frac{\overline{E}_s}{N_0}$$

The dimension of C is bit / transmitted symbol.

Tight lower bounds for the capacities of the signal constellations given on page 72:



Channel Coding Theorem

Provided $R < C$, P_b can be made arbitrarily small by selecting appropriate encoders.

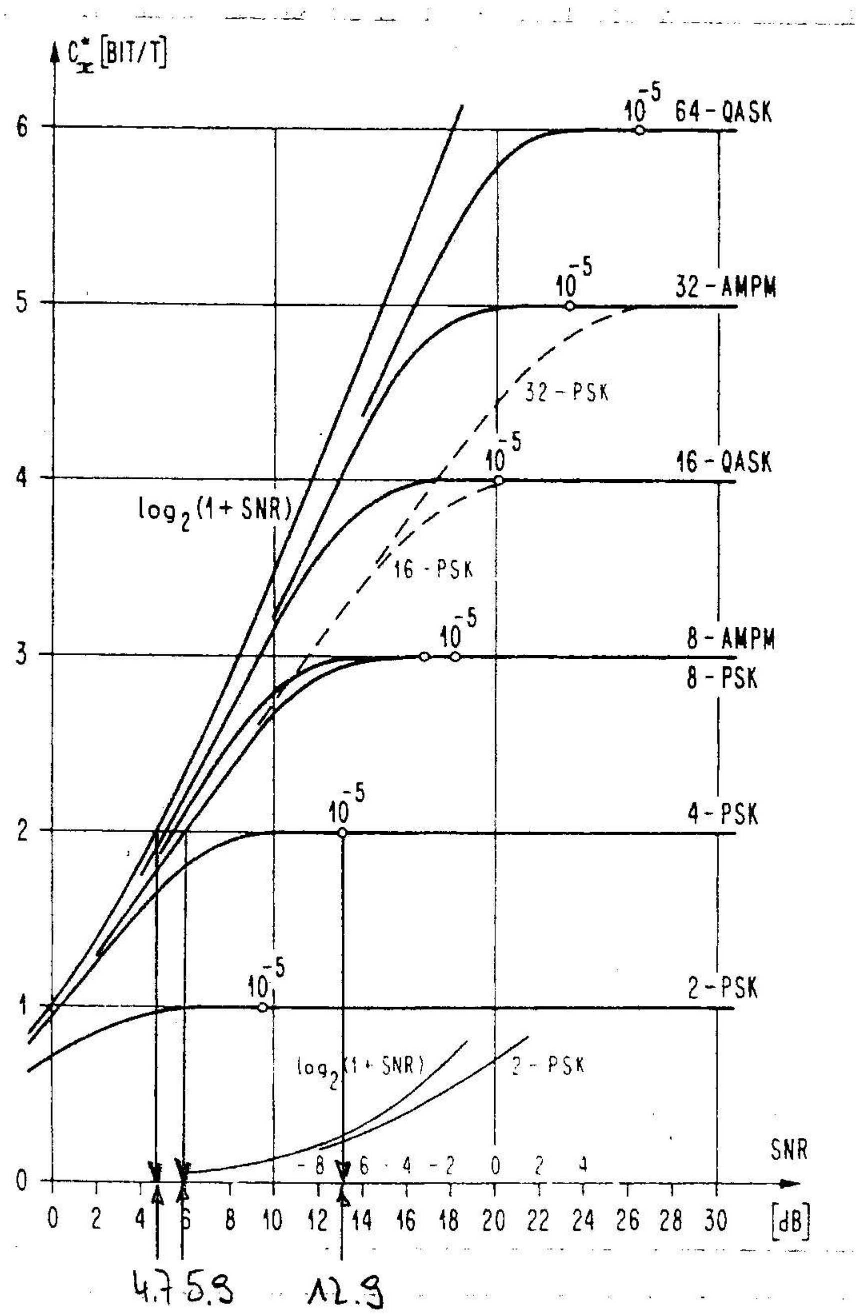
In other words, reliable communication is possible at rates $R < C$.

Converse of the Channel Coding Theorem

If $R > C$, reliable communication is impossible.

Example: Some comparisons

- Uncoded 4−PSK:
  - $R \; = \; 2$ bits/T
  - $P_b = 10^{-5}$ at $SNR = 12.9 \; dB$
- Coded 8−PSK:



Remark: $C^* = 2$ bits/symbols at 5.9 dB

Thus, with coded 8PSK it is theoretically possible to transmit reliably 2 bits/symbol at $SNR = 5.9\ dB$.

- Coding gain of coded 8PSK transmitting 2 bits/symbol versus uncoded 4PSK at $BER = 10^{-5}$.

$$12.9 - 5.9 = 7\ dB$$

- Coding gain of coded digital transmission across the AWGN channel
  - with no restriction on the signal constellation except the average transmitted power,
  - transmitting (on average) 2 bits/symbol

  versus coded 4PSK at $BER = 10^{-5}$:

$$12.9 - 4.7 = 8.2\ dB$$

- Similar figures for the above coding gains hold when other signal constellations are considered.

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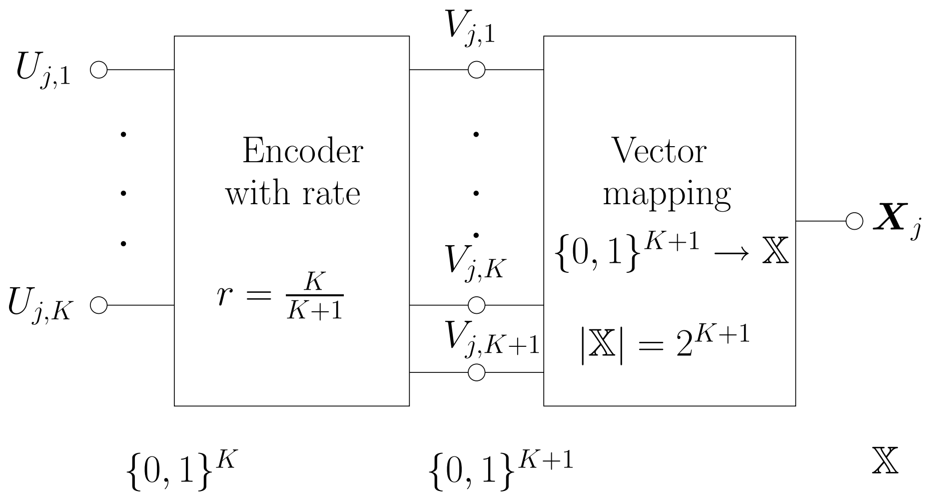
Remark

By doubling the number of vectors in a signal constellation $\mathbb{X}$, containing 2^K vectors, almost all is achieved in terms of coding gain for reliable transmission of K bits/T versus uncoded transmission using the signal constellation $\mathbb{X}$. (The transmission rates are the same!)

A way to realize the above coding gains by doubling the number of vectors in the signal constellation is as follows:



Open issues:

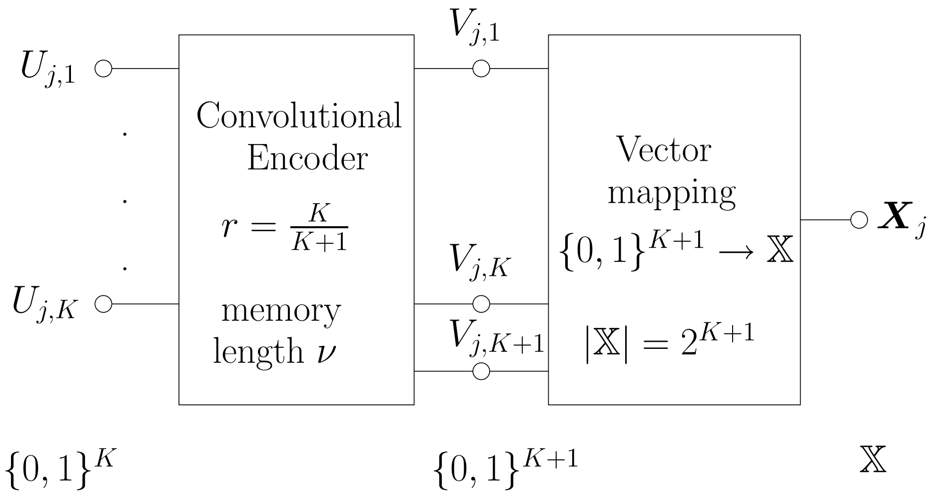
1. how to select the encoder, and
2. how to design the vector mapping,

to obtain a scheme for which P_b is small.

For the encoder, block coding or convolutional coding may be used. In many schemes, convolutional codes are applied due to their low decoding complexity. (The Viterbi algorithm is applicable.)

A schemes that consists of the combination of a convolutional encoder with a vector mapping device is called Trellis-Coded Modulation (TCM).

Block Diagram of a TCM Scheme

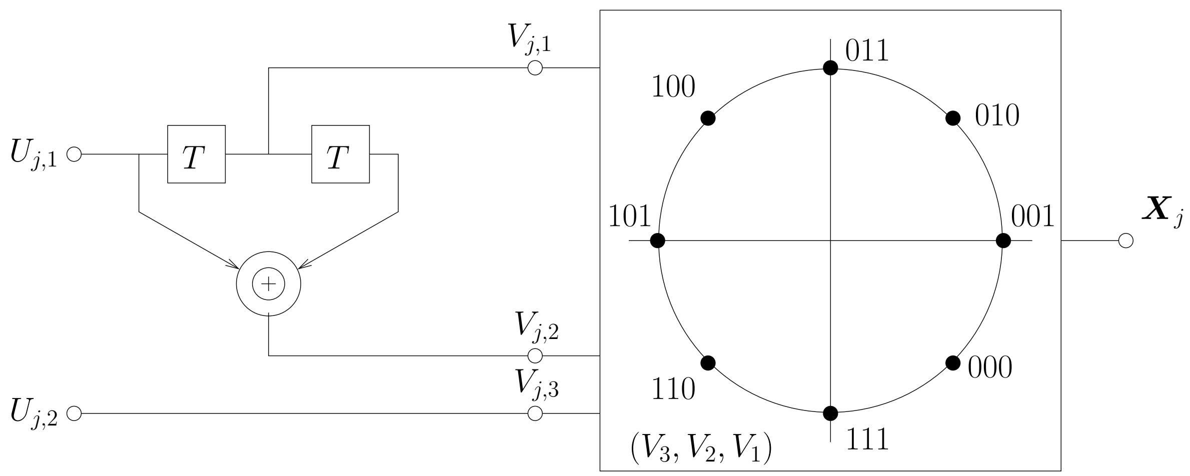


The overall rate is given by

$$R = r \cdot \log_2(K+1) = K$$

Objective
Find TCM schemes that achieve a good bit-error-rate performance.

EXAMPLE: Coded 8PSK

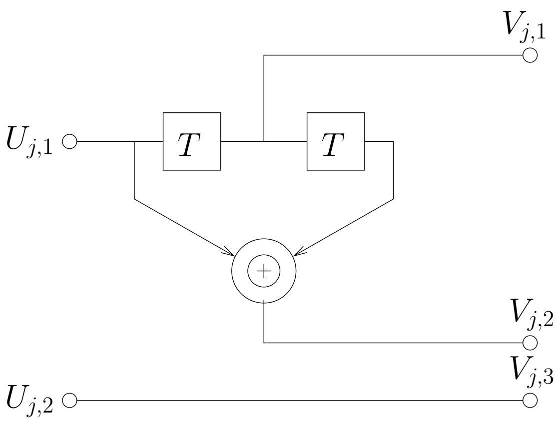


The parameters are $r = 2/3$, $\nu = 2$, $R = 2/3 \cdot \log_2 8 = 2$. $\diamond$

4.2 Convolutional Codes

Two examples of convolutional encoders. (See e.g. Proakis or Morelos-Zaragoza for a general description.)

EXAMPLE: 1



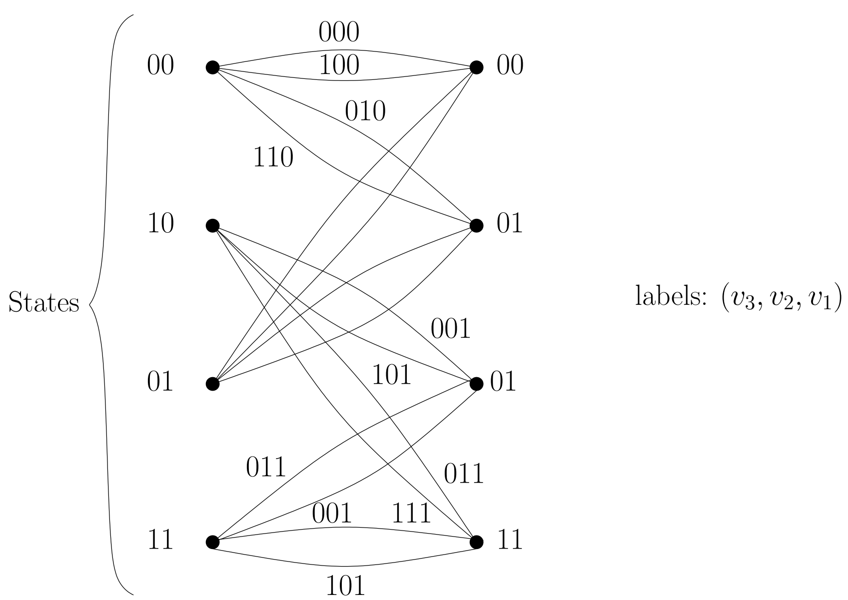
Code rate:

$$r = \frac{\text{number of input bits}}{\text{number of output bits}} = \frac{2}{3}$$

Memory length:

$$\nu = \text{number of shift registers} = 2$$

Trellis:



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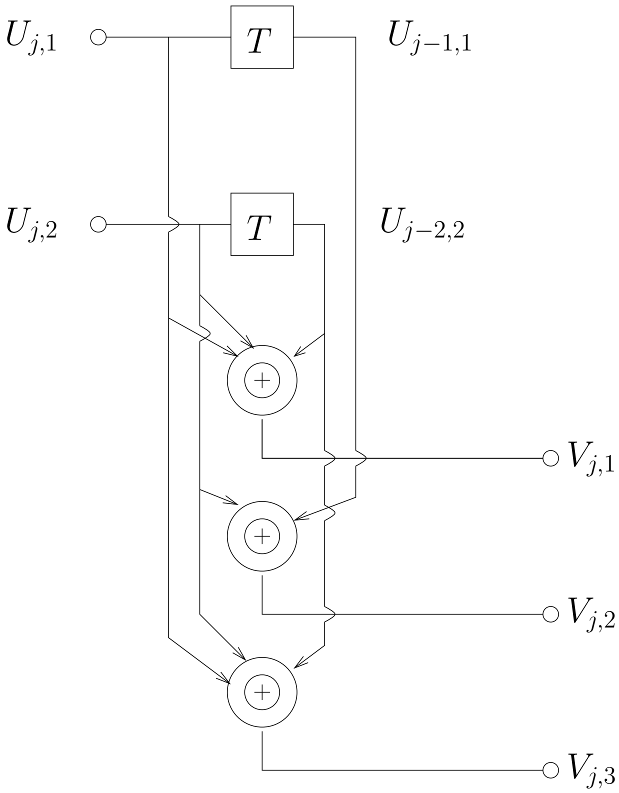
Example: 2



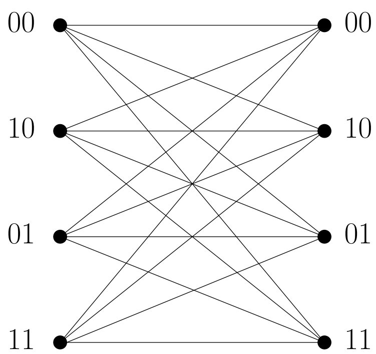
Code rate:

$$r = \frac{2}{3}$$

Memory length:

$$\nu = 2$$

Trellis:



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4.3 Goodness of a TCM Scheme

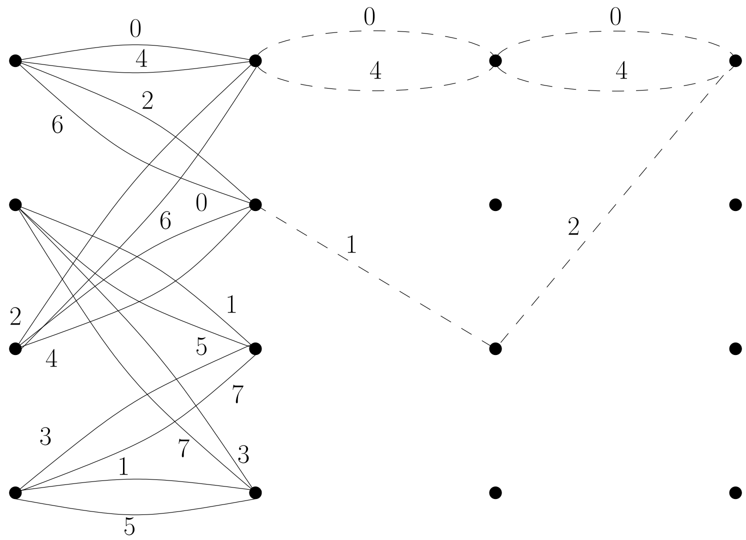
Admissible sequences

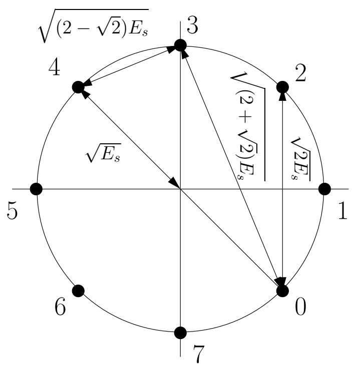
Let $\mathbb{A}_\infty$ denote the set of sequences $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ generated by a given TCM scheme. These sequences are called **admissible** for the considered TCM scheme.

EXAMPLE: 1 continued

$$\mathbf{A}_\infty \in \begin{cases} \{\boldsymbol{x}_j\} &= \big(\boldsymbol{s}_0\ ,\ \boldsymbol{s}_0\ ,\ \boldsymbol{s}_0\ ,\ \boldsymbol{s}_0\ ,\ \boldsymbol{s}_0, \ldots\big) \\ \{\boldsymbol{x}'_j\} &= \big(\boldsymbol{s}_0\ ,\ \boldsymbol{s}_4\ ,\ \boldsymbol{s}_0\ ,\ \boldsymbol{s}_0\ ,\ \boldsymbol{s}_0, \ldots\big) \\ \{\boldsymbol{x}''_j\} &= \big(\boldsymbol{s}_2\ ,\ \boldsymbol{s}_1\ ,\ \boldsymbol{s}_2\ ,\ \boldsymbol{s}_0\ ,\ \boldsymbol{s}_0, \ldots\big) \end{cases}$$

$$\mathbf{A}_\infty \notin \Big\{ \{\boldsymbol{x}'''_j\} = \big(\boldsymbol{s}_0\ ,\ \boldsymbol{s}_5\ ,\ \boldsymbol{s}_3\ ,\ \boldsymbol{s}_2\ ,\ \boldsymbol{s}_0, \ldots\big)$$





◇

Euclidean Distance

Let $\{\boldsymbol{x}_j\}$ and $\{\boldsymbol{x}'_j\}$ denote two sequences in $\mathbf{A}_\infty$.
The Euclidean distance between $\{\boldsymbol{x}_j\}$ and $\{\boldsymbol{x}'_j\}$ is

$$d\Big(\{\boldsymbol{x}_j\}, \{\boldsymbol{x}'_j\}\Big) = \left(\sum_{j=0}^{\infty} \left|\boldsymbol{x}_j - \boldsymbol{x}'_j\right|^2\right)^{1/2}$$

Example: 1 continued

$$d\Big(\{\boldsymbol{x}_j\}, \{\boldsymbol{x}'_j\}\Big) = \sqrt{4E_s} \;=\; 2\sqrt{E_s}$$
$$d\Big(\{\boldsymbol{x}_j\}, \{\boldsymbol{x}''_j\}\Big) = \sqrt{\big(2 + (2 - \sqrt{2}) + 2\big)\; E_s} \;=\; 2.1414\sqrt{E_s}$$

◇

Free Euclidean Distance

The minimum Euclidean distance between pairs of admissible sequences of a TCM scheme is called the free Euclidean distance of the TCM scheme:

$$d_f = \min_{\substack{\{\boldsymbol{x}_j\},\{\boldsymbol{x}'_j\} \,\in\, \mathbf{A}_\infty \\ \boldsymbol{x}_j \neq \boldsymbol{x}'_j}} d\Big(\{\boldsymbol{x}_j\}, \{\boldsymbol{x}'_j\}\Big)$$

Example: 1 continued

It can be shown that

$$d_f \;=\; d\Big(\{\boldsymbol{x}_j\}, \{\boldsymbol{x}'_j\}\Big) \;=\; 2\sqrt{E_s}$$

◇

Probability of a Sequence Error

Probability that the Viterbi decoder makes a wrong decision on the path in the trellis:

$$P_e \;\geq\; N(d_f) \;\cdot\; Q\Big(\frac{d_f}{\sqrt{2N_0}}\Big)$$

where

(i) $Q(z)$ is the Gaussian error function

$$Q(z) = \int_z^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\,u^2\right\} du$$

(ii) $N(d_f)$ is the average number of sequence pairs in $\mathbf{A}_\infty$ that have the Euclidean distance d_f.

Remark
The above lower bound for P_b is tight for high SNR. The reason is that in this case an erroneously detected path has a sequence likely to have an Euclidean distance to the true transmitted sequence equal to d_f, when ML decoding is applied.

Remember that given a finite observed sequence, $\{\boldsymbol{y}_j\} = \big(\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{L-1}\big)$, the Viterbi algorithm (which realizes the ML decoder) searches for the finite sequence generated by the trellis (with initial state 0) which minimizes

$$\sum_{j-0}^{L-1} \left|\boldsymbol{y}_j - \boldsymbol{x}_j\right|^2$$

Optimality Criterion

Optimality criterion for finding a vector mapping for a given convolutional code:

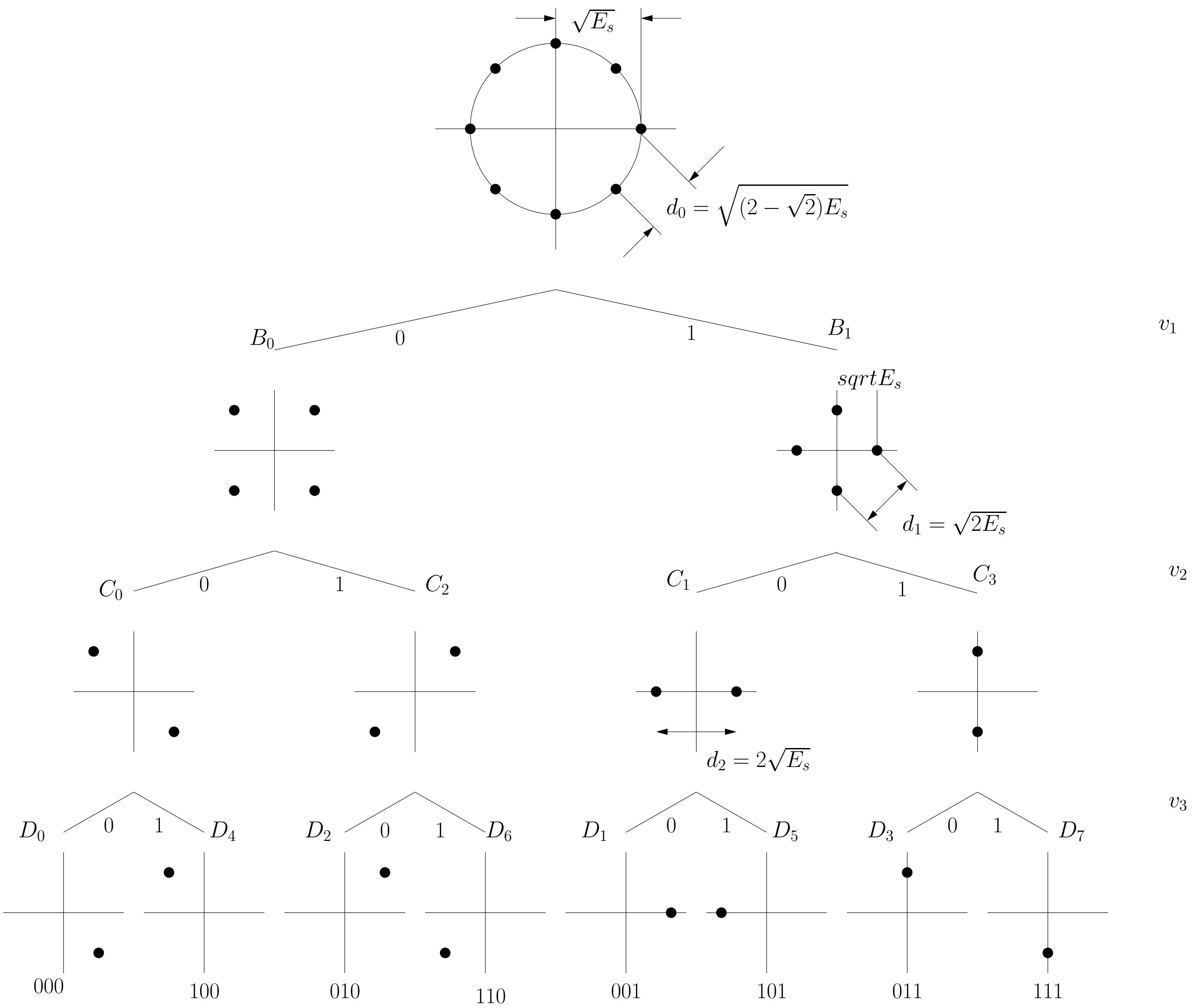
“Maximize the free Euclidean distance !”

TCM schemes maximizing the free Euclidean Distance among all TCM with a specified memory length are called **optimal**.

The method proposed by Ungerboeck to identify ”good” vector mapping given a certain convolutional encoder (or equivalently, given a certain trellis) relies on the technique of set partitioning.

4.4 Set Partitioning

EXAMPLE: 8PSK



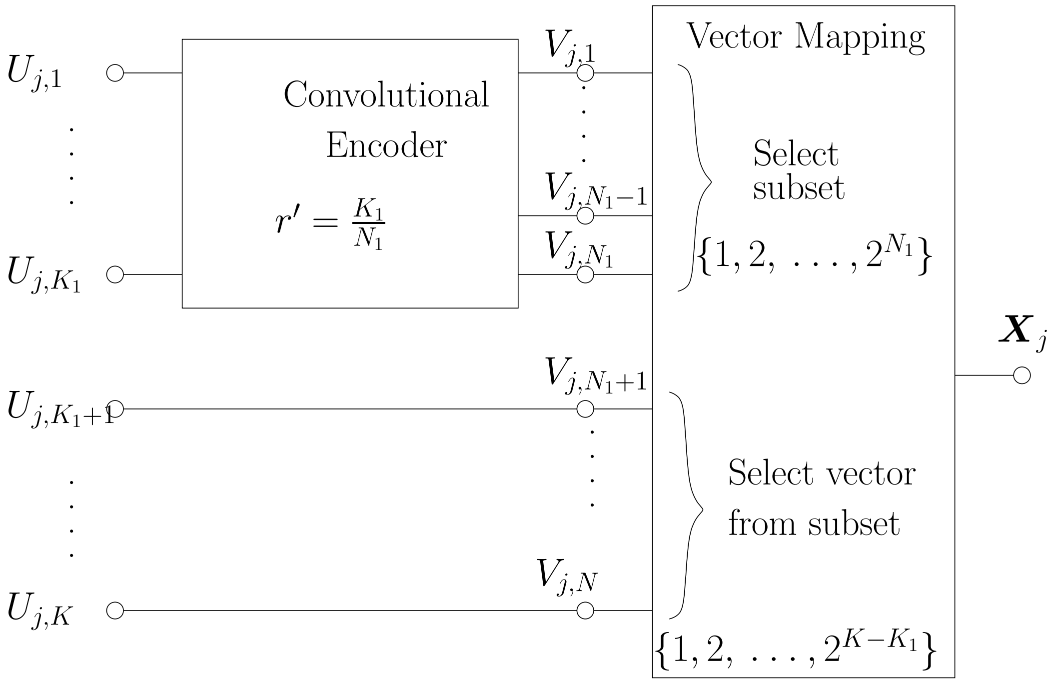
◇

Goals of set partitioning:

1. The subsets should be similar .
2. The points inside each subset should be maximally separated.

4.5 Trellis Coded Modulation

General form of the encoding process:

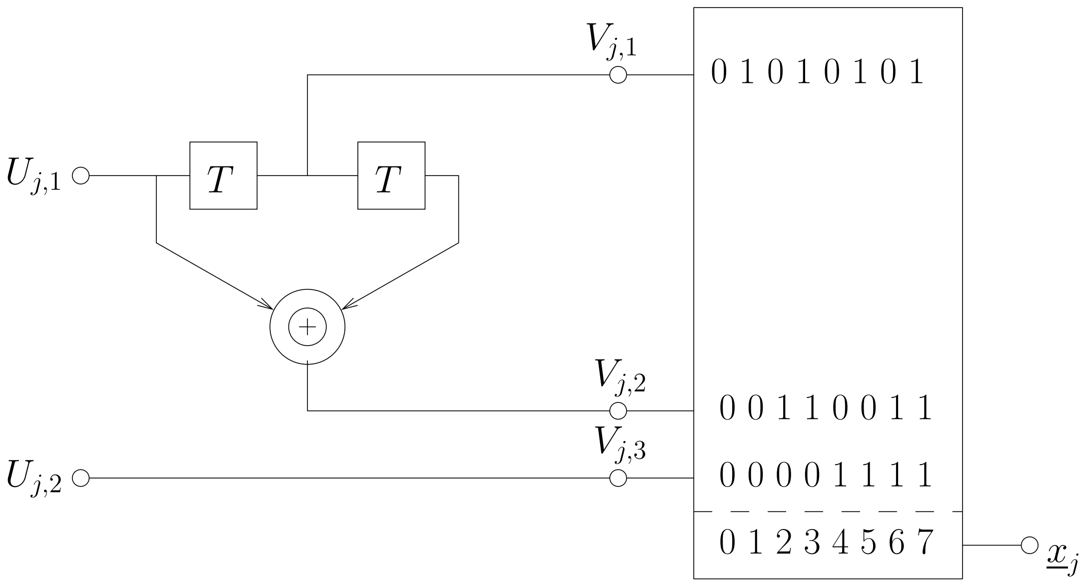


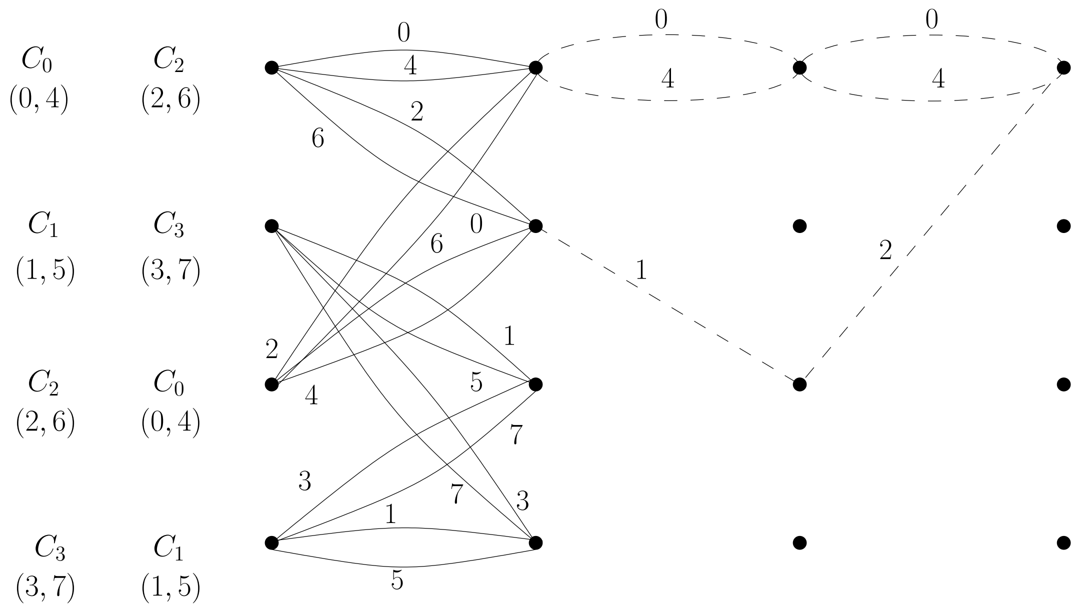
Remarks

1. Number of states in the trellis : 2^{ν}
2. Number of branches leaving each state : 2^{K_1}
3. Number of parallel branches in each transition : 2^{K-K_1}

Example: 1 continued

Parameters: $K = 2,\ K_1 = 1,\ N_1 = 2,\ N = 3$





◇

Rules for the Assignments

Experimentally inferred rules for assigning the signal vectors to the transitions:

1. All vectors in $\mathbb{X}$ are used with the same frequency.
2. The vectors assigned to branches originating from or merging to the same state belong to the same subset at level 1.
3. Vectors with maximum Euclidean distances are assigned to parallel branches.

Remarks

- Rule 1 guarantees that the trellis code has a regular structure.
- Rule 2 and Rule 3 guarantee that the minimum distance between paths in the trellis which diverge from any state and re-emerge later exceeds the free Euclidean distance of the uncoded modulation scheme.

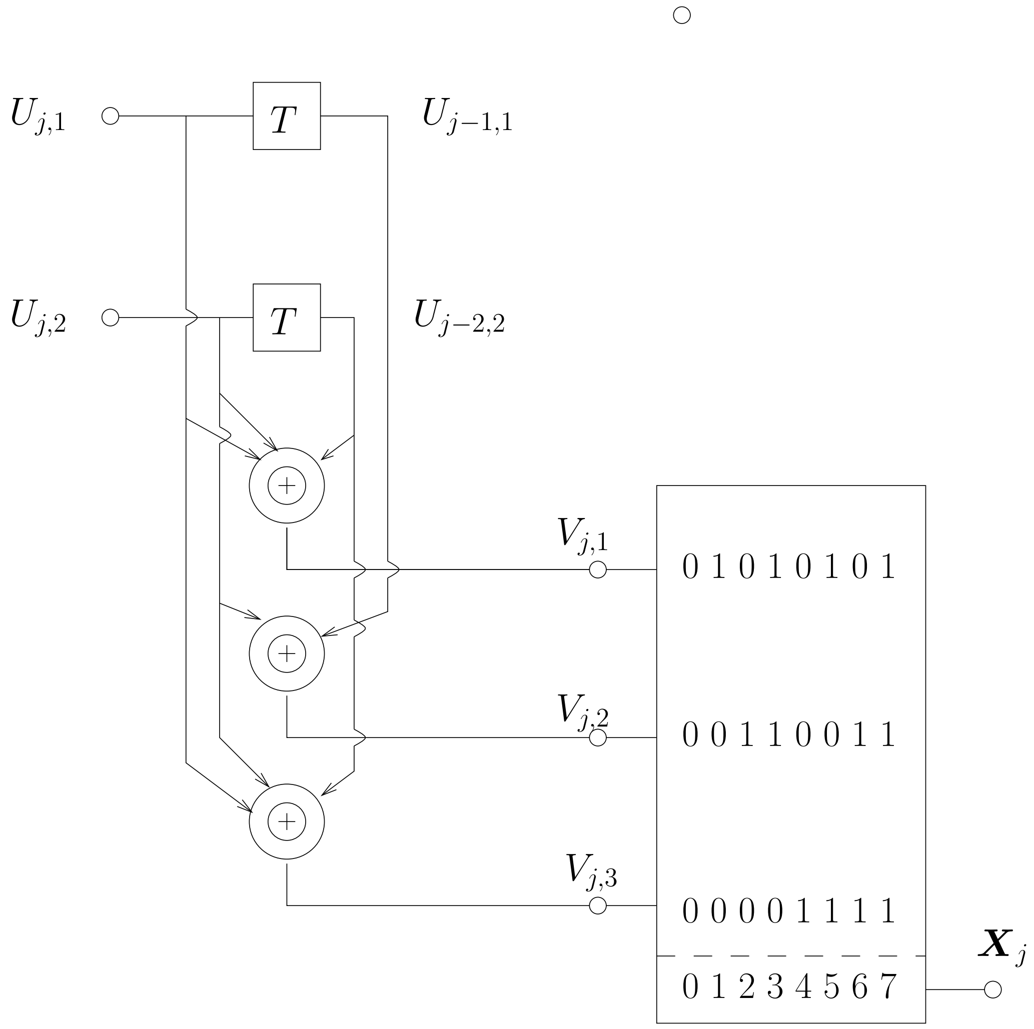
Example: 1 continued

$$(d_f)^2_{\substack{\text{coded}\\ \text{8PSK}}} = 4E_s \quad \geq \quad 2E_s = (d_f)^2_{\substack{\text{uncoded}\\ \text{4PSK}}}$$

◇

EXAMPLE: 2 continued

Consider the following trellis code:



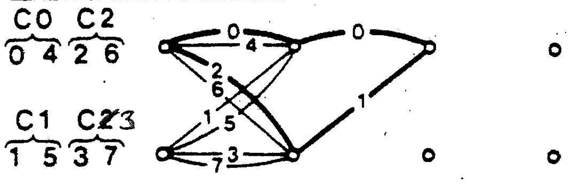
Despite the fact that this trellis code has no parallel branches, its free Euclidean distance is smaller than that of the trellis code of Example 1. (See Exercise 4.1). ◇

4.6 Examples for TCM

In the following, some examples of specific TCM schemes are given and their error probabililites are discussed.

TCM with 8PSK and 2 bits/T

2 TRELLIS STATES

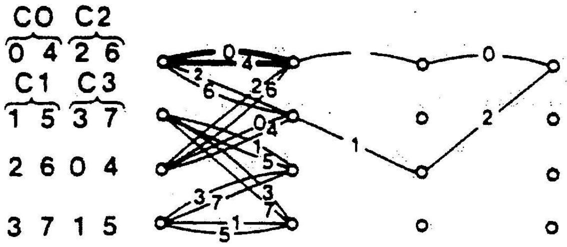


$d_{free} = \sqrt{d_1^2 + d_0^2} = 1.608$

(1.1 dB GAIN OVER 4-PSK).

$Pr(e) \geq 2.Q(d_{free}/2\sigma)$.

4 TRELLIS STATES

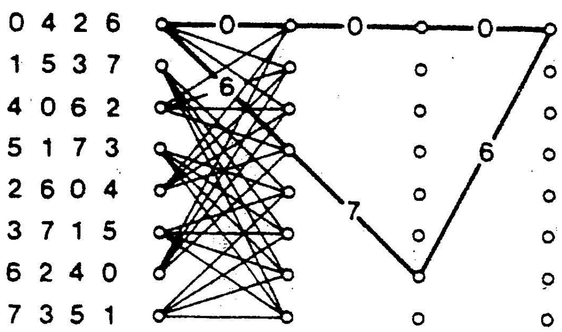


$d_{free} = d_2 = 2.000$

(3.0 dB GAIN OVER 4-PSK).

$Pr(e) \geq 1.Q(d_{free}/2\sigma)$.

8 TRELLIS STATES

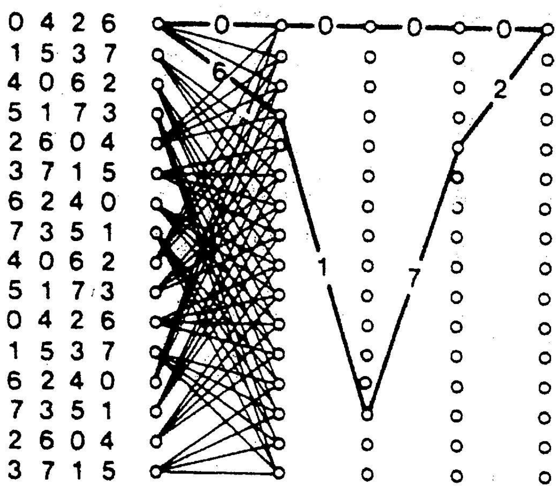


$d_{free} = \sqrt{d_1^2 + d_0^2 + d_1^2} = 2.141$

(3.6 dB GAIN OVER 4-PSK).

$Pr(e) \geq 2.Q(d_{free}/2\sigma)$.

16 TRELLIS STATES

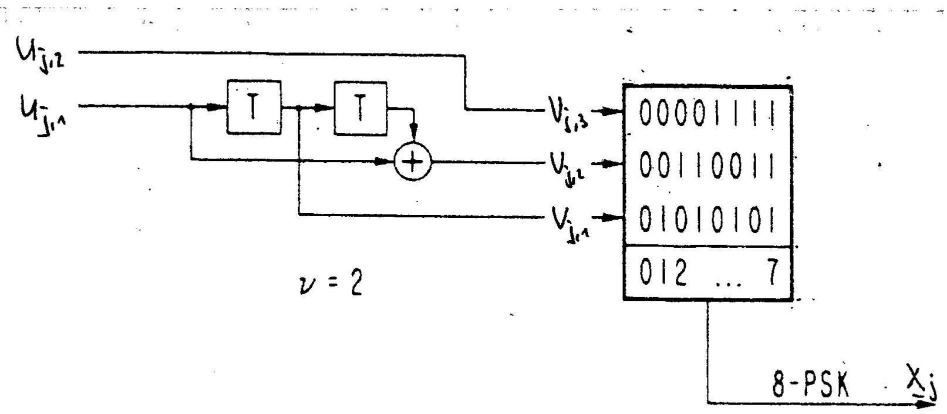


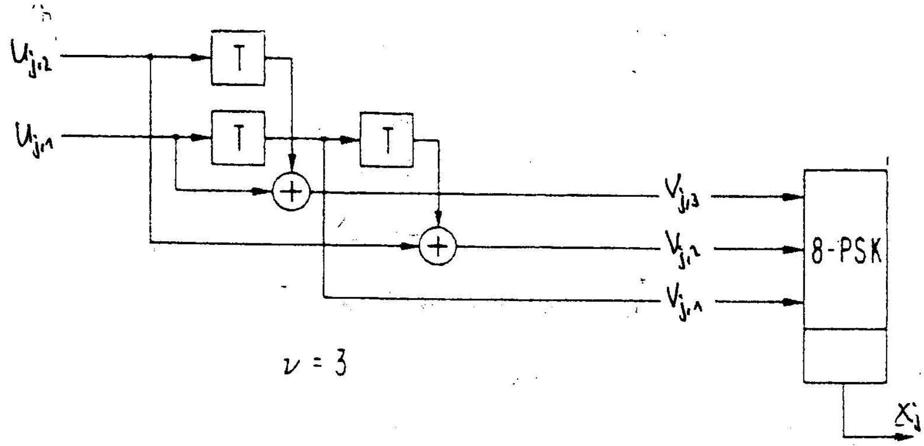
$d_{free} = \sqrt{d_1^2 + d_0^2 + d_0^2 + d_1^2} = 2.274$

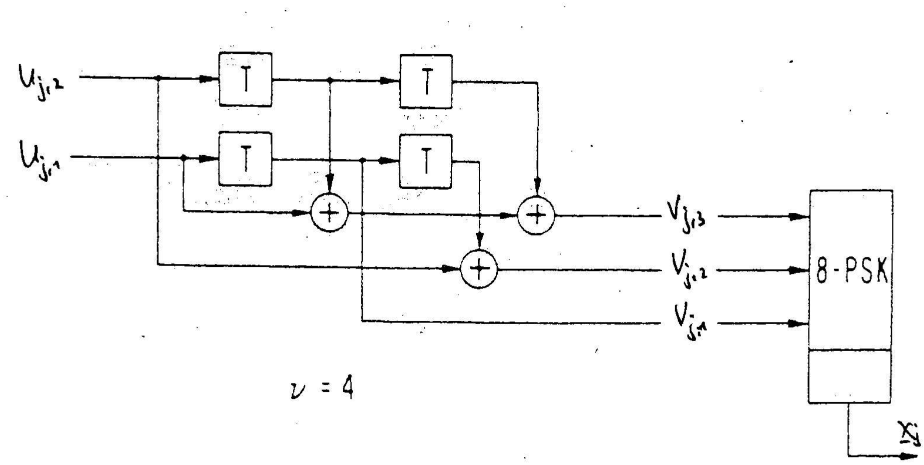
(4.1 dB GAIN OVER 4-PSK).

$Pr(e) \geq (3?)Q(d_{free}/2\sigma)$.

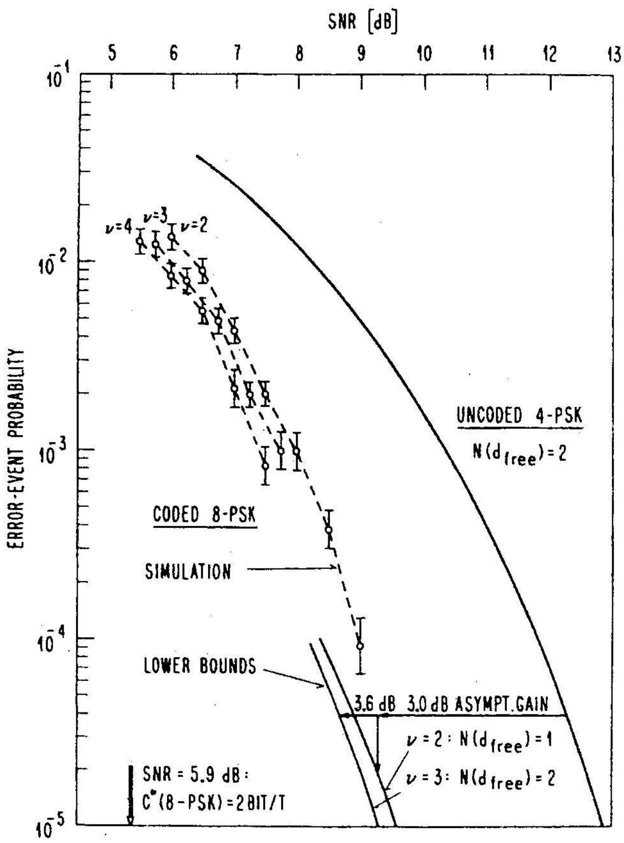
Implementation



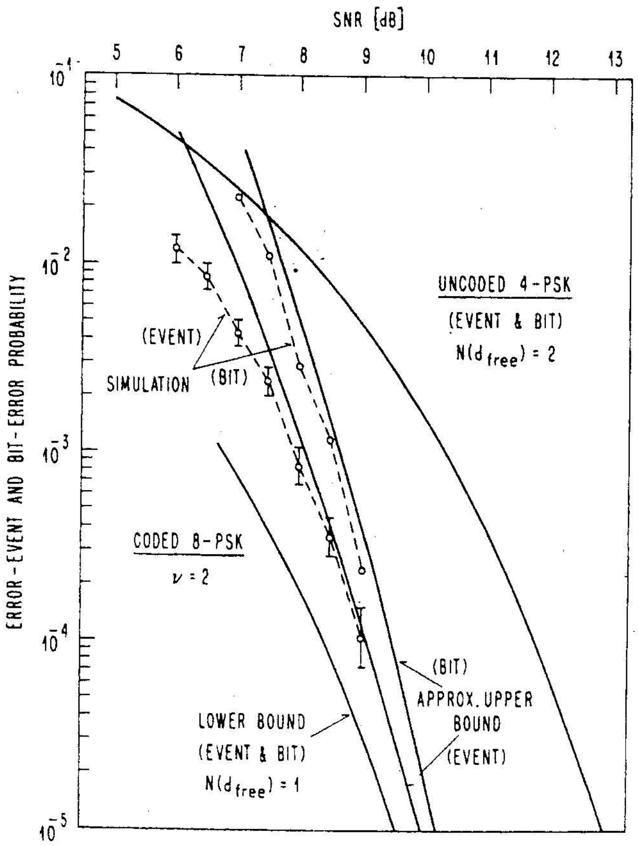




Probability of error-event and of bit-error



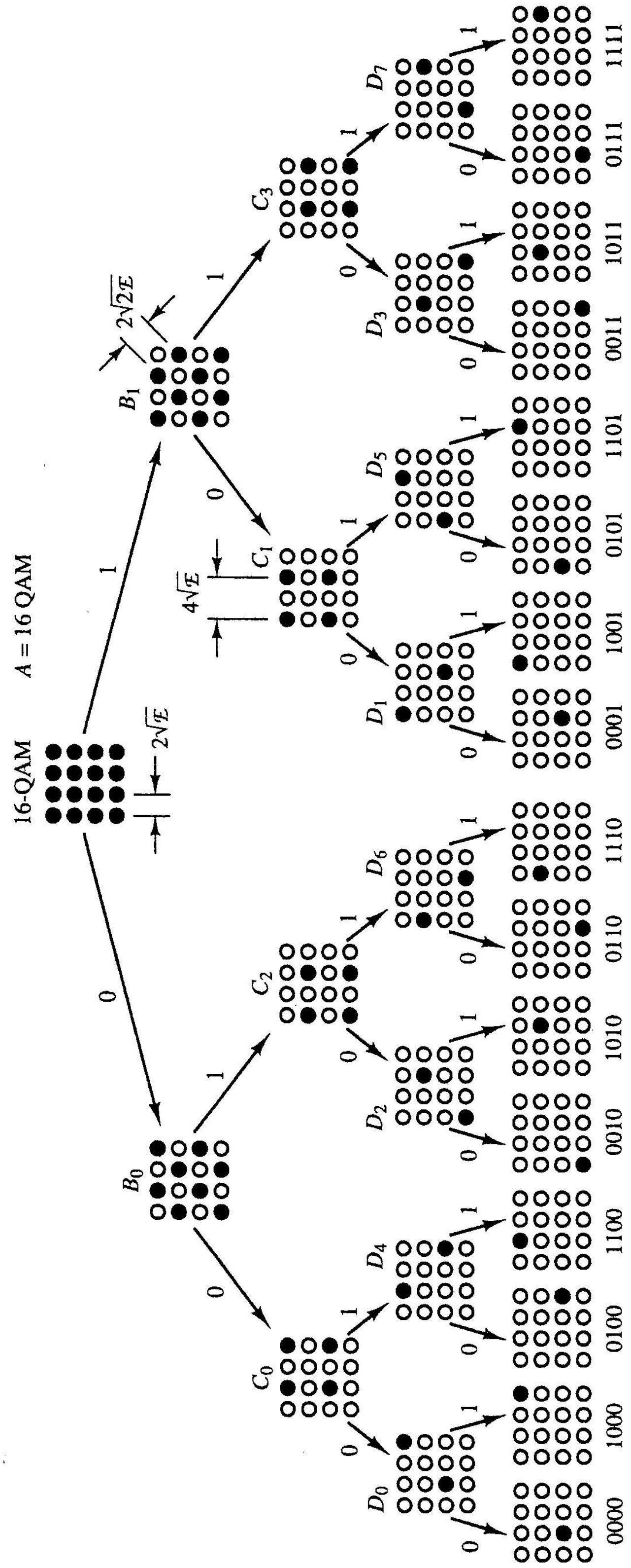
14. Error-event performance of coded 8-PSK versus uncoded 4-PSK 2 bit/T.

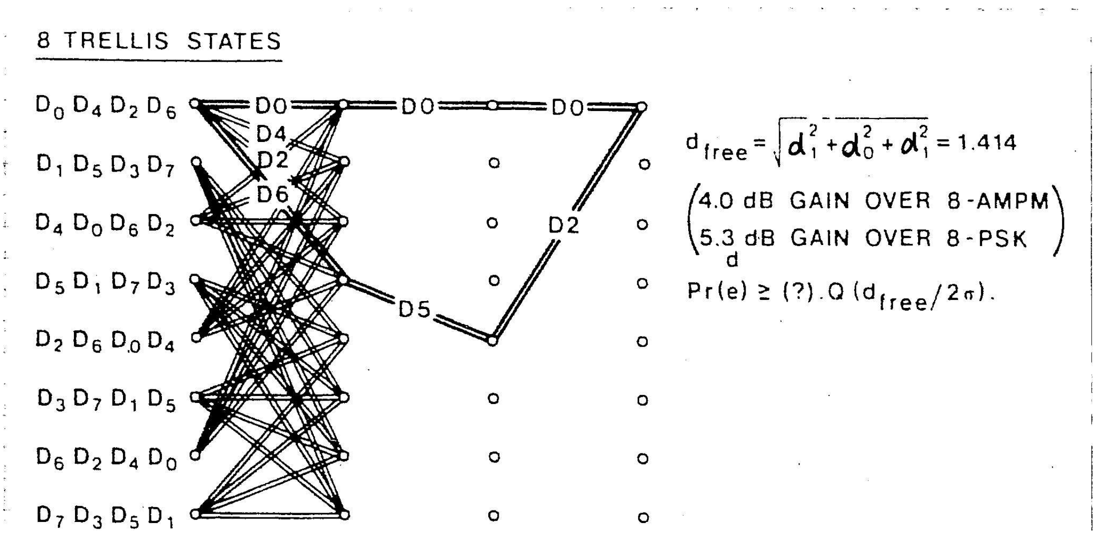


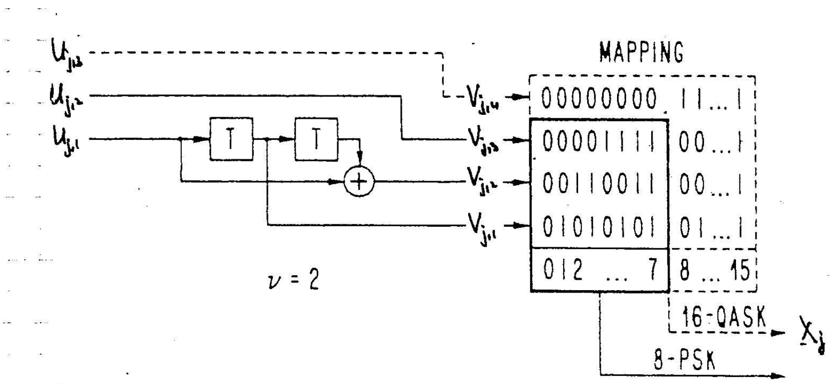
16. Error-event and bit-error performance of coded 8-PSK ($\nu = 2$, minimal encoder) and uncoded 4-PSK, 2 bit/T.

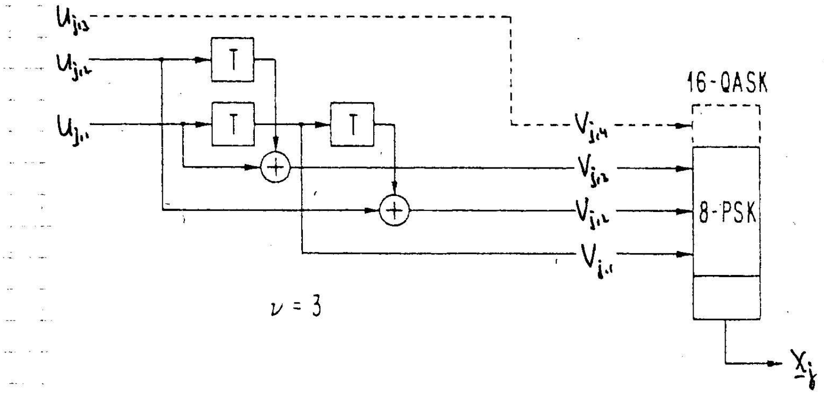
TCM with 16QAM and 3 bits/T

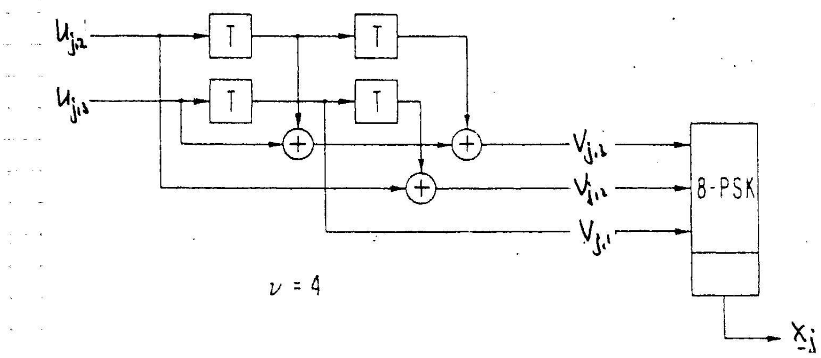
Set Partitioning



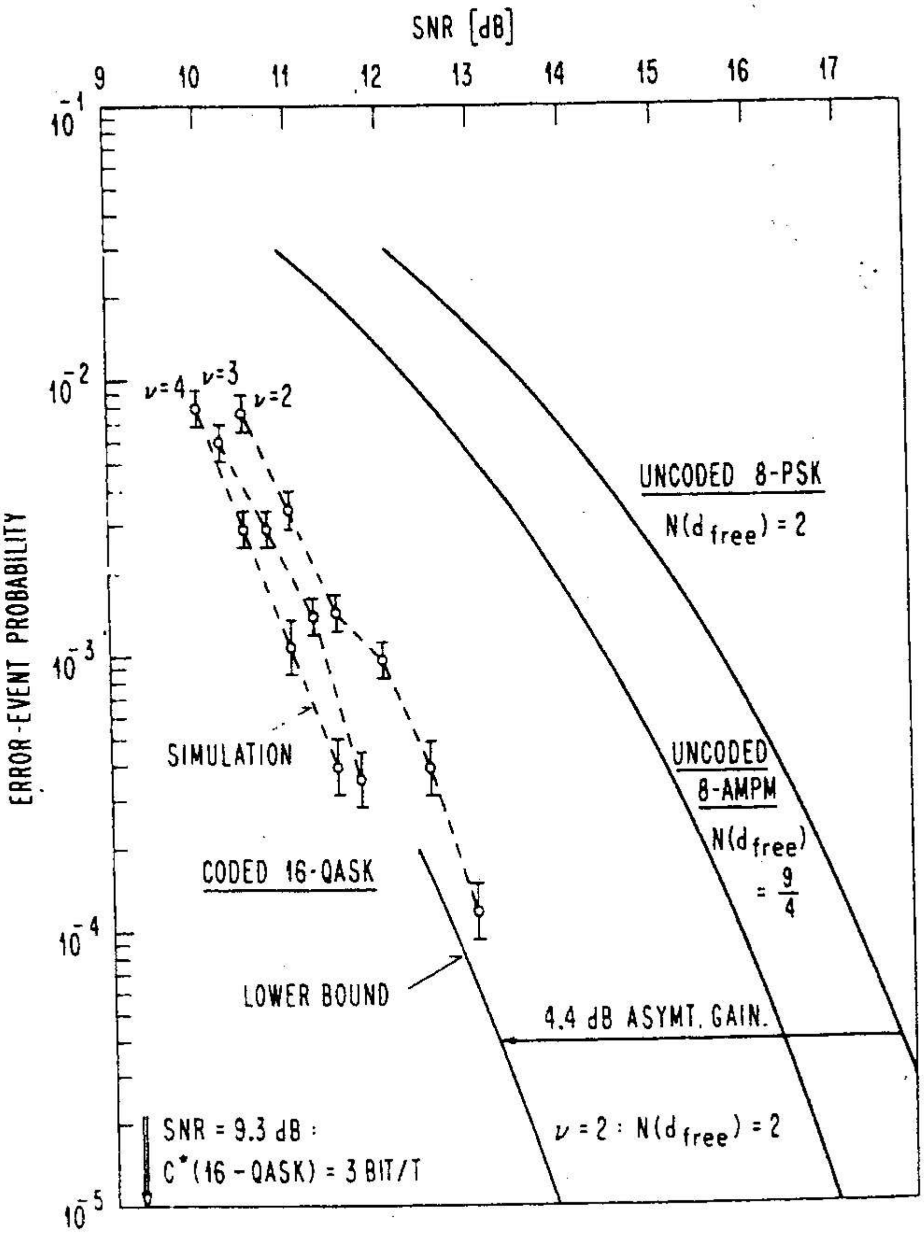

8 TRELLIS STATES
D0 D4 D2 D6
D1 D5 D3 D7
D4 D0 D6 D2
D5 D1 D7 D3
D2 D6 D0 D4
D3 D7 D1 D5
D6 D2 D4 D0
D7 D3 D5 D1
D0
D4
D2
D6
D5
D2
d free = √(d1² + d0² + d1²) = 1.414
(4.0 dB GAIN OVER 8-AMPM
5.3 dB GAIN OVER 8-PSK)
Pr(e) ≥ (?).Q (d free/2σ).


MAPPING
00000000 11...1
00001111 00...1
00110011 00...1
01010101 01...1
012 ... 7 8 ... 15
16-QASK
8-PSK
ν = 2


16-QASK
8-PSK
ν = 3


8-PSK
ν = 4

Error event probability

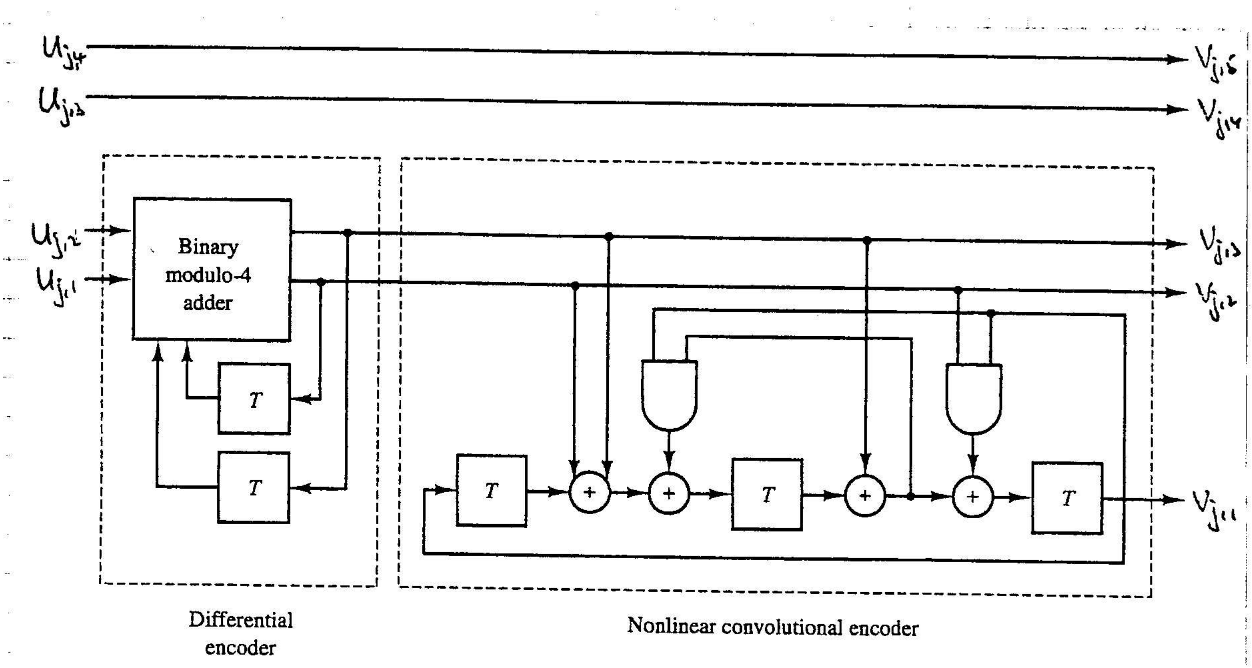


15. Error-event performance of coded 16-QAM versus uncoded 8-PSK and 8-AMPM, 3 bit/T.

CCITT V.32 standard

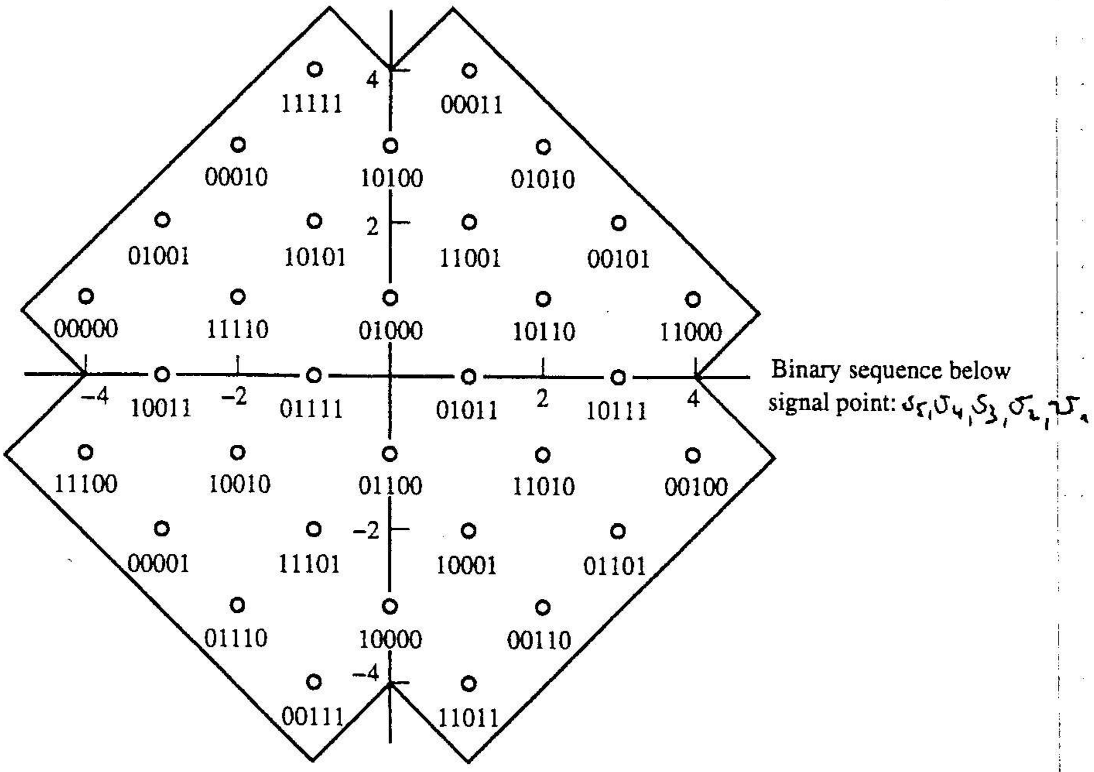
CCITT V.32 standard for 9.6 Kbit/s and 14.4 bit/s modems.

Convolutional Coder



The code is designed to guarantee invariance to phase rotations of $\pm\pi/2$ and π. These values are the phase ambiguities which result when a phase-locked loop (PLL) is employed for carrier-phase estimation.

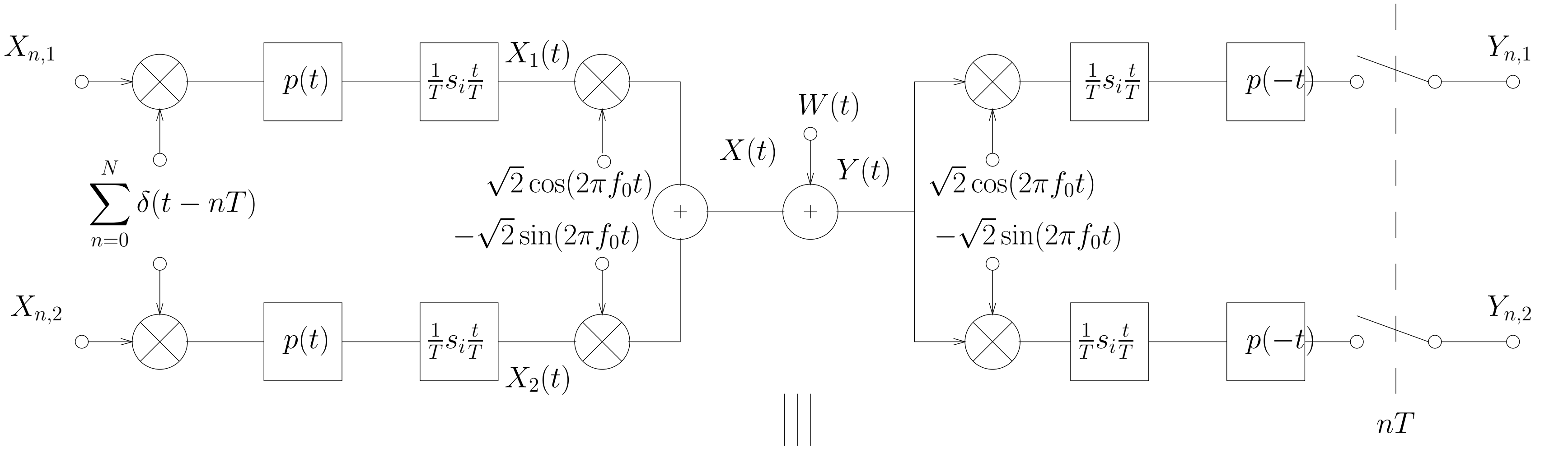
Mapping


11111
4
00011
00010
10100
01010
01001
10101
2
11001
00101
00000
11110
01000
10110
11000
−4
10011
−2
01111
01011
2
10111
4
Binary sequence below
signal point:
11100
10010
01100
11010
00100
00001
11101
−2
10001
01101
01110
10000
00110
00111
−4
11011

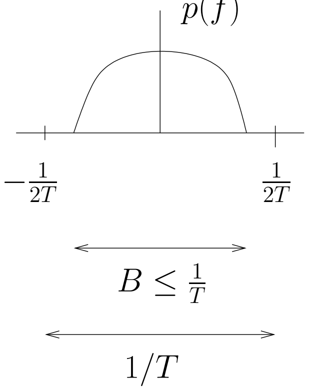
A ECB Representation of a Band-Pass Signal

Band-pass signals can equivalently be represented as complex-valued base-band signals. This is often called the equivalent complex base-band (ECB) representation.

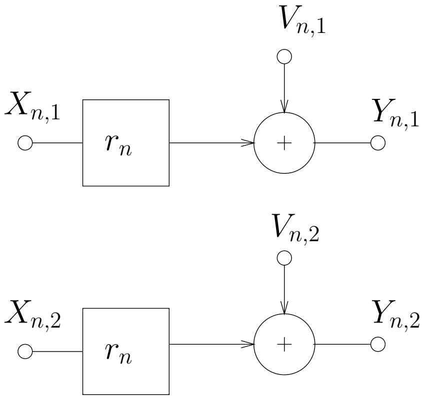
This concept is discussed in more detail in the following.



Assumption:



$f_0 \geq \frac{1}{2T}$



$Y_{n,1} = r_n * X_{n,1} + V_{n,1}$

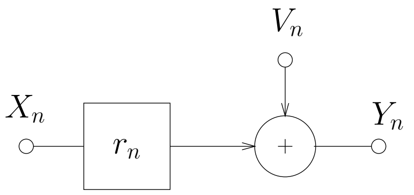
$Y_{n,2} = r_n * X_{n,2} + V_{n,2}$

Complex discrete-time representation:

$X_n = X_{n,1} + jX_{n,2}$

$Y_n = Y_{n,1} + jY_{n,2}$

$V_n = V_{n,1} + jV_{n,2}$



$$Y_n = r_n * X_n + V_n$$

A.2 The Signal

Consider a real-valued BP signal:

$$\begin{aligned}
x(t) &= x_1(t)\sqrt{2}\cos(2\pi f_0 t) - x_2(t)\sqrt{2}\sin(2\pi f_0 t) \\
&= \Re\Big\{ \big[\underbrace{x_1(t) + jx_2(t)}_{=\widetilde{x}(t)}\big] \big[\underbrace{\sqrt{2}\cos(2\pi f_0 t) + j\sqrt{2}\sin(2\pi f_0 t)}_{\sqrt{2}\exp(j2\pi f_0 t)}\big] \Big\} \\
&= \Re\Big\{ \widetilde{x}(t)\sqrt{2}\exp(j2\pi f_0 t) \Big\}
\end{aligned}$$

The signal $\widetilde{x}(t)$ is a complex-valued LP signal, and it is called the ECB representation of $x(t)$.

The spectra

$$X(f) = \mathcal{F}\Big\{x(t)\Big\} \qquad\qquad \widetilde{X}(f) = \mathcal{F}\Big\{\widetilde{x}(t)\Big\}$$

are related by

$$X(f) = \frac{\sqrt{2}}{2}\Big[\widetilde{S}(f - f_0) + \widetilde{S}(-f - f_0)\Big]$$

The energies of the signals are

$$E_s = E_{\widetilde{s}}$$

A.3 Effect of QA Mod/Demod on Signal

$$
\begin{aligned}
x(t) &= x_1(t)\sqrt{2}\cos(2\pi f_0 t) - x_2(t)\sqrt{2}\sin(2\pi f_0 t)\\
x_1(t) &= \sum_i x_{i,1}\,\delta(t-iT) * p(t)\\
x_2(t) &= \sum_i x_{i,2}\,\delta(t-iT) * p(t)
\end{aligned}
$$

Assumption: $p(t)$ is a LP signal, i.e., its Fourier transform $P(f) = \mathcal{F}\{p(t)\}$ has the property

$$P(f) = 0 \quad \text{for} \quad |f| > f_0$$

Tricks:

$$
\begin{aligned}
2\cos^2 x &= 1 + \cos 2x\\
2\sin^2 x &= 1 - \cos 2x\\
2\sin x\ \cos x &= \sin 2x
\end{aligned}
$$

In-phase component:

$$
\begin{aligned}
z_1(t) &= x(t)\sqrt{2}\cos(2\pi f_0 t) * p(-t)\\
&= \Big[x_1(t)\ \underbrace{2\cos^2(2\pi f_0 t)}_{1+\cos(4\pi f_0 t)} - x_2(t)\,\underbrace{2\sin(2\pi f_0 t)\cos(2\pi f_0 t)}_{\sin(4\pi f_0 t)}\Big] * p(-t)\\
&= x_1(t) * p(-t) + \underbrace{\underbrace{\Big[x_1(t)\cos(4\pi f_0 t) - x_2(t)\sin(4\pi f_0 t)\Big]}_{\mathrm{BP}} * \underbrace{p(-t)}_{\mathrm{LP}}}_{=0}
\end{aligned}
$$

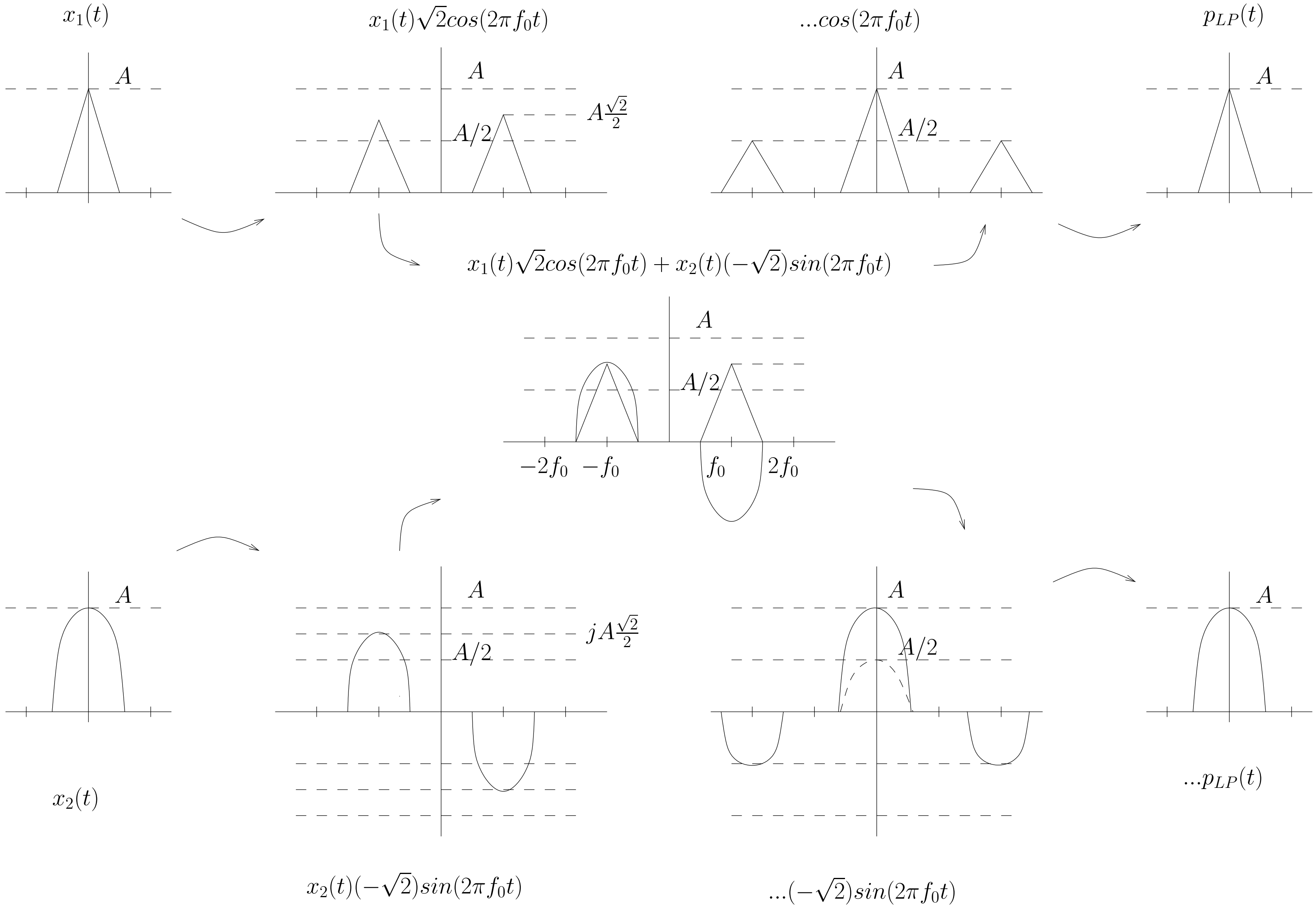
$$
\begin{aligned}
z_1(t) &= x_1(t) * p(-t) \\
&= \sum_i x_{i,1}\ \delta(t - iT) * \underbrace{p(t) * p(-t)}_{= R_p(t)} \\
&= \sum_i x_{i,1}\ R_p(t - iT)
\end{aligned}
$$

After sampling:

$$
\begin{aligned}
z_{1,n} &= z_1(nT) \\
&= \sum_i x_{i,1} \qquad \underbrace{R_p(nT - iT)}_{R_p\big((n-i)T\big) = r_{n-i}} \\
&= \sum_i x_{i,1} \ \cdot r_{n-i} = x_{1,n} * r_n \\
r_n &= R_p(nT)
\end{aligned}
$$

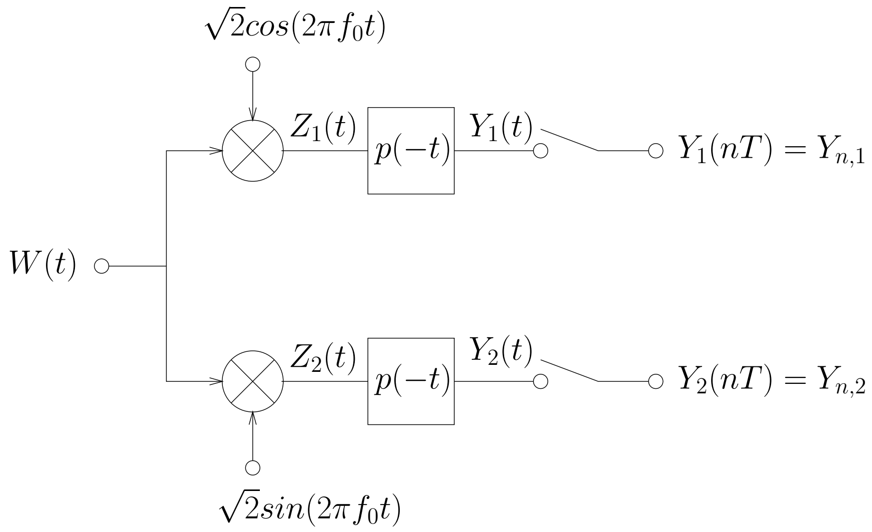
Quadrature component:
(...similar calculations...)

$$
z_{n,2} = z_2(nT) = X_{n,2} * r_n
$$


$x_1(t)$
$x_1(t)\sqrt{2}cos(2\pi f_0 t)$
$...cos(2\pi f_0 t)$
$p_{LP}(t)$
A
$A/2$
$A\frac{\sqrt{2}}{2}$
$x_1(t)\sqrt{2}cos(2\pi f_0 t) + x_2(t)(-\sqrt{2})sin(2\pi f_0 t)$
$-2f_0$ $-f_0$ f_0 $2f_0$
$jA\frac{\sqrt{2}}{2}$
$x_2(t)$
$...p_{LP}(t)$
$x_2(t)(-\sqrt{2})sin(2\pi f_0 t)$
$...(-\sqrt{2})sin(2\pi f_0 t)$

A.4 Effect of QA Demodulation on Noise

Block Diagram



$w(t)$: WGN with $S_w(f) = \frac{N_0}{2}$

Noise before LP filtering

$$Z_1(t) = W(t)\sqrt{2}\cos(2\pi f_0 t)$$

$$\begin{aligned}
E\big[Z_1(t)\ Z_1(t+\tau)\big] &= \\
&= \underbrace{E\big[W(t)W(t+\tau)\big]}_{=\frac{N_0}{2}\delta(\tau)}(\sqrt{2})^2\ \cos\big(2\pi f_0 t\big)\ \cos\big(2\pi f_0(t+\tau)\big) \\
&= \frac{N_0}{2}\ \delta(\tau)\quad \underbrace{2\cos^2(2\pi f_0 t)}_{1+\cos(4\pi f_0 t)} \\
&= \frac{N_0}{2}\ \delta(\tau) + \underbrace{\frac{N_0}{2}\ \delta(\tau)\ \cos(4\pi f_0 t)}_{\text{time-variant past}} = R_{Z_1}(\tau; t)
\end{aligned}$$

$\rightarrow$ Time-variant (periodic) auto-correlation function

Noise after LP filtering

Assumption: $p(t)$ is a LP signal, i.e., its Fourier transform $P(f) = \mathcal{F}\{p(t)\}$ has the property

$$P(f) = 0 \quad \text{for} \quad |f| > f_0$$

Autocorrelation function of $Y_1(t)$:

$$Y_1(t) = W(t)\sqrt{2}\cos(2\pi f_0 t) * p(t)$$

$$\begin{aligned}
E\Big[Y_1(t)Y_1(t+\tau)\Big] &= \\
&= E\Big[\int W(t-s)\sqrt{2}\cos\big(2\pi f_0(t-s)\big)p(s)ds \cdot \\
&\quad \cdot \int W(t+\tau-s')\sqrt{2}\cos\big(2\pi f_0(t+\tau-s')\big)p(s')ds'\Big] \\
&= \int\int \underbrace{E\Big[w(t-s)\; w(t+\tau-s')\Big]}_{\frac{N_0}{2}\delta(\tau-s'+s)} \cdot \\
&\qquad \cdot 2 \cdot \cos\big(2\pi f_0(t-s)\big) \cdot \cos\big(2\pi f_0(t+\tau-s')\big) \cdot \\
&\qquad \cdot p(s)\; p(s')\; ds\; ds'
\end{aligned}$$

Integrate w.r.t. $s' \Rightarrow \neq 0$ for $\tau - s' = -s \Leftrightarrow s' = \tau + s$

$$= \frac{N_0}{2}\int 2 \underbrace{\cos^2\big(2\pi f_0(t-s)\big)}_{1+\cos\left(4\pi f_0(t-s)\right)} \; p(s)\; p(\tau+s)\; ds$$

$$= \frac{N_0}{2}R_p(\tau) + \frac{N_0}{2}\underbrace{\int \cos\big(4\pi f_0(t-s)\big)\; p(s)\; p(\tau+s)\; ds}_{=\cos\left(4\pi f_0(t-\tau)\right)\; p(\tau)*p(-\tau)}$$

$$E\Big[Y_1(t)\ Y_1(t+\tau)\Big] = \frac{N_0}{2} R_p(\tau) + \frac{N_0}{2} \underbrace{\int \cos\big(4\pi f_0(t-s)\big)\ p(s)\ p(\tau+s)\ ds}_{(1)}$$

$$R_p(\tau) = p(\tau) * p(-\tau)$$

$$(1) = \Big[\cos\big(4\pi f_0(t-\tau)\big)\ p(\tau)\Big] * p(-\tau) \tag{6}$$

The Fourier transform w.r.t. τ of

$$\cos\big(4\pi f_0(t-\tau)\big) = \cos\big(4\pi f_0(\tau-t)\big) = \cos\big(4\pi f_0\tau\big) * \delta(\tau-t)$$

is

$$\frac{1}{2}\Big[\delta(f-2f_0) + \delta(f+2f_0)\Big]\ \exp(-j2\pi f t) =$$
$$= \frac{1}{2}\Big[\delta(f-2f_0)\ \exp(-j4\pi f_0 t) + \delta(f+2f_0)\ \exp(j4\pi f_0 t)\Big] = (3)$$

Consider the Fourier transform w.r.t. τ of (6):

$$\begin{aligned}(2) &= \Big[(3) * P(f)\Big]\ \cdot\ P(-f)\\ &= \frac{1}{2}\Big[P(f-2f_0)\ \exp(-j4\pi f_0 t)\\ &\qquad + P(f+2f_0)\ \exp(j4\pi f_0 t)\Big]\ \cdot\ P(-f)\\ &= 0\end{aligned}$$

because $P(f) = 0$ for $|f| > f_0$ according to the assumption.

$$\Rightarrow \underbrace{E\Big[Y_1(t)\; Y_1(t+\tau)\Big]}_{R_{Y_1}(\tau)} = \frac{N_0}{2}\, R_p(\tau)$$

$$\Rightarrow R_{Y_1}(\tau) = \frac{N_0}{2}\, R_p(\tau)$$

A.5 Noise After LP Filtering and Sampling

$$\begin{aligned} R_{Y_1}(kT) &= \frac{N_0}{2}\, \underbrace{R_p(kT)}_{=\,r_k} \\ &= \frac{N_0}{2}\, r_k \end{aligned}$$