Nucleon - nucleon forces

1. and

The deuteron

Nucleon - nucleon Forces

1. The force binding the nucleons together is strong enough to overcome the Coulomb force which repels the protons from one another.

We know this because many nuclei containing many protons are stable.

2. The force has a limited range. e.g. The force is not felt by neighbouring nuclei in a molecule.
[Distance ~ 10^{-10} m]

Typical nucler dimension - the femtometre or Fermi

$1 \text{ fm} = 10^{-15} \text{ m}$
F

3. In particle physics we call the force the strong nuclear force; of particles in the atom, only the neutron and the proton feel the strong nuclear force – not the electron.

[The 4 fundamental forces are

(1) EM force
(2) gravitational force
(3) strong nuclear force
(4) weak nuclear force – which causes β-decay.]

4. The force is charge-independent i.e. It is the same between n & n; p & p; & n & p.

[allowing, of course, for Coulomb repulsion in the case of p & p]

5 The force is strongly spin-dependent i.e. it depends on whether the spins are parallel or opposed.

6. The force has a repulsive core i.e. at a certain distance the force becomes repulsive.
Thus nucleons will have a fixed separation from each other, and this leads to a constant nuclear density.

[In effect, the nucleons behave like attractive billiard balls with a fixed radius.]

7. The force has a component which is non-central and so does not conserve orbital angular momentum.

The Coulomb force

The Coulomb force was investigated and understood classically in the nineteenth century.

Then the simplest system involving individual particles - the hydrogen atom, a two particle system - was investigated spectroscopically giving information about quantum theory - the Bohr atom and forwards onto Quantum theory proper (Heisenberg, Schrödinger).
The hydrogen atom has an infinite number of bound states.

The strong nuclear force

The closest analogy at the nucleon-nucleon level (where we would hope to understand the [strong] nuclear force) is the <u>deuteron</u>, the nucleus of the deuterium atom, which is np

So is there an analogy between H (ep) and the deuteron (np)?

Unfortunately the deuteron has only its ground state – no excited states i.e. it has only <u>one bound state</u>.

Therefore we cannot investigate spectroscopically the transitions between different energy levels.

This is a <u>great hindrance</u> to understanding the nuclear interaction!

The deuteron - what we can find out

1. The energy of the ground state
2. Its angular momentum
3. Its parity
4. Its electric quadrupole moment
5. Its magnetic dipole moment
6. Its size

Scattering

Broadly it may be said that the chief way of investigating nuclear and particle physics is by scattering. A beam of nucleons [or other particles] is incident on a target of nucleons [or other particles] and measurements are made of such quantities as collision probabilities [cross-sections] and angular distributions.

In particular measurements have been made of np & pp scattering.

Here we study what can be learned about nucleon-nucleon forces from studies of the deuteron.

As well as being the simplest two nucleon bound system, the deuteron has the additional advantage that the force can be studied without other complications such as:

(a) Coulomb forces (such as in the p-p system)

(b) Pauli exclusion forces (such as in the p-p and n-n systems)

The Deuteron - Experimental data

(a) Binding Energy

The binding energy of the ~~nucleo~~ deuteron is the energy required to remove each of the nucleons out of the attractive range of the other

Binding energy B given by

$$B = [m(p) + m(n) - m(\text{deuteron})]\,c^2$$

(measured in MeV

[Usually we speak of masses as being "in MeV", implicitly meaning mc^2]

Above we spoke of B in terms of the nuclei of H and D

It is more ~~traditional~~ to use ~~atomic~~ masses

i.e. $B = [m({}^1H) + m(n) - m({}^2H)]\,c^2$

Here we have added and also sub~~tracted~~ an electron mass. In this case there would be no difference in electron binding energy; in other cases there might be a difference, but it would be ignored [measured in eV rather than MeV] appropriate values

	u	MeV
n	1.008665	939.551
p	1.007277	938.258
n + p	2.015942	1877.809
d	2.013554	1875.585
n+p−d	0.002388	2.224 ← B

As we shall see, several methods give much greater accuracy.

Methods

1. Atomic Mass Spectrometer

$B = (2.22463 \pm 0.00004)$ MeV

2. Radiative Capture

Here a proton and a neutron combine to form a deuteron liberating a photon

$$^1H + n \rightarrow {}^2H + \gamma$$

The binding energy is equal to the energy of the γ-ray photon minus a small correction for the recoil of the deuteron.

$B = (2.224589 \pm 0.0000002)$ MeV

3. Photodissociation

This is just the reverse process to radiative capture.

$$\gamma + {}^2H \longrightarrow {}^1H + n$$

After allowance for recoil, the minimum energy of the γ-ray photon necessary to achieve the dissociation is measured, giving a value

$$B = (2.224 \pm 0.002)\ \text{MeV}$$

The deuteron binding energy is thus ~ 2 MeV.

This may be compared with the expression from the binding energy / nucleon curve which is ~ 8 MeV

More fundamentally, though, it may be compared with the leading term in the SEMF for binding energy / nucleon, which is ~ 15.5 MeV

[All later terms - surface term, Coulomb term, symmetry term, pairing term reduce this value]

It must be remembered that, in the latter, each nucleon is in the field of all its nearest neighbours, perhaps 6 – 8. (Z)

Thus ~~it~~ to get a value for the binding energy / pair of interacting nucleons, we must divide the 15.5 Mev by Z, the number of nearest neighbours.
This makes the value comparable with the binding energy of the ~~nucl~~ deuteron.

Thus we may say that the reason the binding energy of the deuteron is low is that each nucleon has only 1 nearest neighbour.

(b) Angular Momentum and Parity

A number of techniques using molecular spectroscopy in the optical, radio frequency and microwave regions of the spectrum may be used to obtain a value of the angular momentum of the deuteron.

In fact $I = 1$.

Parity

Parity is a concept of great importance in atomic, nuclear and particle physics.

If $\psi(x, y, z) = \psi(-x, -y, -z)$, parity is even (positive)

" $\psi(x, y, z) = -\psi(-x, -y, -z)$, " " odd (negative)

Parity is conserved in many types of nuclear process

For many years it was assumed it was always conserved

But see later ....

The proton may be written as

$$\underset{\text{angular momentum}}{\left(\tfrac{1}{2}\right.}{}^{+} \leftarrow \text{parity}$$

[The parity of the proton is actually arbitrary, but, once we have defined it, the parity of other particles may be investigated in terms of that of the proton]

parity of the deuteron

The parity of a nuclear state cannot be directly measured. However studies of nuclear disintegrations and reactions indicate that the parity of the deuteron is even.

Deuteron - components of angular momentum

There are 3 components of the angular momentum of the deuteron:

spin of the neutron S_n
" " " proton S_p
orbital angular momentum l
[as they orbit common centre of mass]

We now add angular momentum components vectorially in the normal way.

First we form a vector for spin angular momentum with quantum no. S

Then $S = (S_n + S_p) \dots |S_n - S_p|$

with $S_n = 1/2$, $S_p = 1/2$

i.e. $S = 1, 0$

Then $I = (l+s) \dots |l-s|$

But $I = 1$

If $l = 0$, S must equal 1 $[S \neq 0]$

If $l = 1$, S can equal 1 or 0

If $l = 2$, S must equal 1 $[S \neq 0]$

It is not possible for l to equal 2, 3, 4...

[To clarify $l = 0, S = 0 \rightarrow I = 0$
$l = 2, S = 0 \rightarrow I = 2$]

But parity associated with orbital angular momentum $= (-1)^l$

But parity = even

$\therefore$ l is even

$\therefore$ $S = 1$

i.e. spins are aligned

(c) Magnetic Dipole Moment

Bohr magneton and nuclear magneton

Classically the angular momentum and magnetic moment of a charged particle are intimately related.

Consider a plane current loop

Magnetic dipole moment $\mu = iS$

where i = current
S = area of loop

For a revolving particle travelling at speed v in circle of radius r and with charge e

$$i = e \frac{v}{2\pi r}$$

Then $\mu = iS = e\frac{v}{2\pi r}\pi r^2 = \frac{evr}{2}$

But angular momentum $l = mvr$

Then $\frac{\mu}{l} = \frac{e}{2m}$

This is usually called the <u>gyromagnetic ratio</u> [better magnetogyric ratio] γ

So $\gamma = \frac{e}{2m}$

<u>Quantum Theory</u>

The same result holds for orbital angular momentum.

But basic unit of angular momentum $= \hbar$ [e.g. Bohr theory]

$\therefore$ Basic unit of magnetic moment $= \gamma\hbar$

$= \frac{e\hbar}{2m}$, electron mass

This is called the <u>Bohr magneton</u> $= \mu_B$

[For spin the factor is doubled

(we often define a $g_l = 1$, $g_s = 2$

Then basic unit of spin angular momentum

$= \hbar/2$

But basic unit of spin magnetic moment

$$= g_s \gamma \frac{\hbar}{2} = 2 \frac{e}{2m} \frac{\hbar}{2} = \mu_B$$

This leads to complications for the total angular momentum of the atom [quantum number J])]

Nuclear magneton

By analogy, the scale for nuclear magnetic ~~moments~~ should be $\dfrac{e\hbar}{2M_p}$

$$= \frac{m}{M_p} \mu_B = \frac{\mu_B}{1836}$$

This is called the <u>nuclear magneton</u> μ_N

This is indeed the scale

BUT we would expect

g as for electron

Magnetic moment of proton $= 1 \mu_N$

" " " neutron $= 0 \mu_N$

Instead

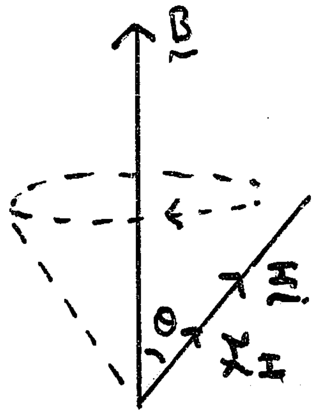
Magnetic moment of proton $= 2.79 \mu_N$

" " " neutron $= -1.91 \mu_N$

The minus sign for the neutron indicates that the magnetic moment is in the opposite direction to the angular momentum.

For a reason for the difference between expectation and experiment, one would have to study the theory of quarks in detail. [See much later.]

<u>Larmor precession</u>



The effect of a magnetic field $\underset{\sim}{B}$ on an object possessing a magnetic dipole moment $\underset{\sim}{\mu}_I$ is to produce a torque

$$\underset{\sim}{\tau} = \underset{\sim}{\mu}_I \times \underset{\sim}{B}$$

By Newton's Second Law, this equals the rate of change of angular momentum

$$= \frac{d\underset{\sim}{I}}{dt}$$

But $\underset{\sim}{\mu}_I = \gamma \underset{\sim}{I}$

$$\therefore \quad \gamma \underset{\sim}{I} \times \underset{\sim}{B} = \frac{d\underset{\sim}{I}}{dt}$$

Because the change of the angular momentum is always at right angles to itself, the result is a <u>precession</u> - the <u>Larmor precession</u>.

Equn. may be written

$$\gamma I B \sin\theta = \frac{dI}{dt}$$

over 1 circuit

$$\int dI = 2\pi I \sin\theta \quad ; \quad \int dt = T$$

$$\Rightarrow \quad \gamma I B \sin\theta = \frac{2\pi I \sin\theta}{T}$$

$$\gamma B = \frac{2\pi}{T}$$

But $\frac{2\pi}{T} = \omega \quad \Rightarrow \quad \omega = \gamma B$

This value is the <u>Larmor precession frequency</u>

$$= \omega_L = \gamma B$$

angular frequency

Quantum-mechanical approach to the Larmor frequency

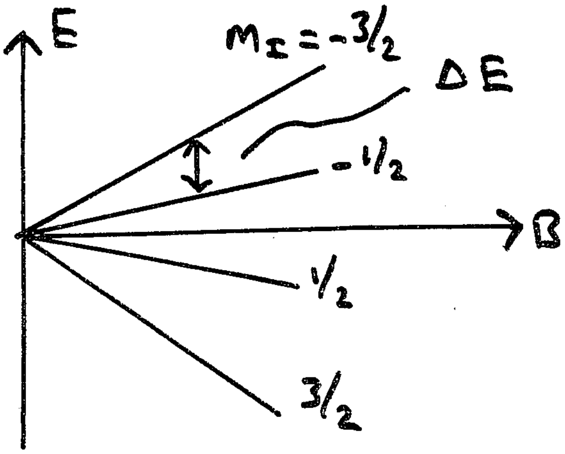
Energy $= -\underset{\sim}{\mu} \cdot \underset{\sim}{B}$ ← along z-axis

← magnetic moment

$= -\hbar \gamma m_I B$

e.g. $I = 3/2$

$m_I = 3/2, 1/2, -1/2, -3/2$



$\Delta E = \hbar \gamma B$

selection-rule

$\Delta m_I = 1$

$$\omega = \frac{\Delta E}{\hbar} = \gamma B = \omega_L$$

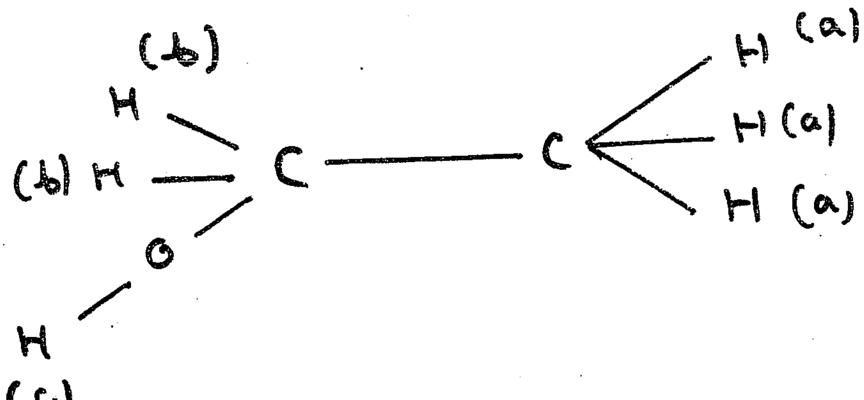
i.e. classically or quantum-mechanically

Energy is absorbed/emitted, and transitions between energy-levels occur when the system is perturbed by an rf field at frequency ω_L - Larmor frequency

This effect is the centre of <u>nuclear magnetic resonance</u>. (NMR)

In this topic, the appropriate frequency is found experimentally for each nuclear species ⇒ γ and hence the magnetic moment.

Also "chemical shifts" may be discovered ⇒ information on solids, molecules, etc.



ethyl alcohol

molecules at (a), (b), (c) all have different frequencies

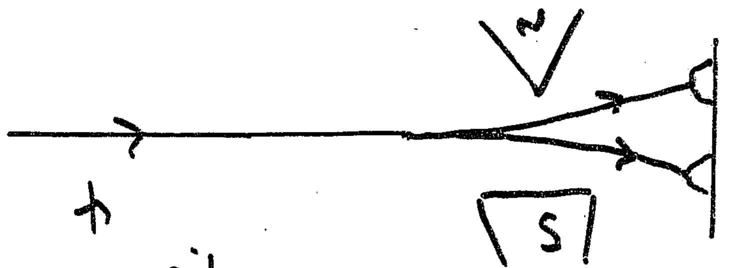
The analogous effects using electron spins are <u>electron spin resonance</u> (ESR)

[microwave region]

NMR & ESR were developed after the second world war with the use of rf and microwave experience developed during the war.

Before the war, a clever technique was used by Rabi (1939 – Nobel Prize 1944)

This was based on the earlier work of Stern and Gerlach (1922 – Nobel Prize for Stern 1943) which demonstrated the existence of the spin of the electron

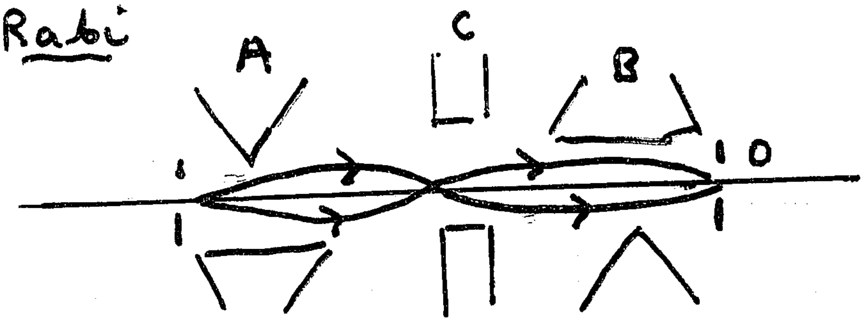


beam of silver atoms with $\ell = 0$; no orbital angular momentum

Two piles of atoms; one with $S_z = 1/2$, the other with $S_z = -1/2$

There is a force on a magnetic dipole in an inhomogeneous field [not in a homogeneous one]

Rabi



molecular-beam resonance

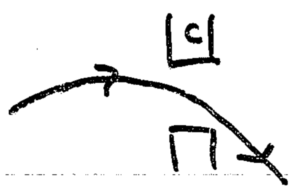
Used for molecules with no resulting atomic angular momentum, so the nuclear spin may be detected

A & B are inhomogeneous magnetic fields, so that, if we ignore C, the molecules follow the path drawn and end up passing through slit D

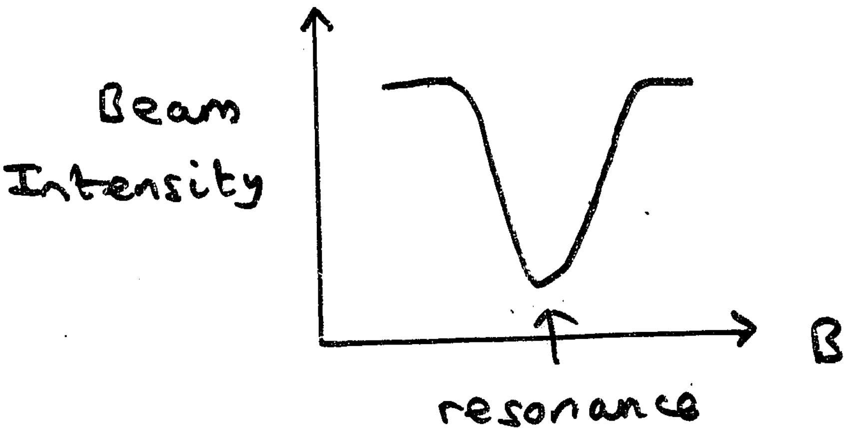
However C is a source of an rf magnetic field, as well as a strong homogeneous field B.

If ω for the rf field is equal to the Larmor frequency, γB, molecules will undergo transitions as they pass through C. They will diverge and not pass through D.

e.g.



In practice ω is kept constant and $\dot{B}$ is varied.



Thus we may get $\gamma = \frac{\mu}{\ell}$

<u>Magnetic moment of deuteron</u>

$\mu_d / \mu_p = 0.307012192 \pm 0.00000015$

$\Rightarrow \mu_d = 0.86 \, \mu_N$

c.f. $\mu_p = 2.79 \, \mu_N$
$\mu_n = -1.91 \, \mu_N$

we see that μ_d is very close to the sum of the proton and neutron magnetic moments.

Since the total magnetic moment is equal to the sum of spin and orbital contributions, this good agreement indicates there is an almost zero orbital contribution. This would be the case if $l = 0$

It turns out that the discrepancy between the sum of the proton and neutron magnetic moments and the measured value for the deuteron is consistent with a mixing of 96% $l=0$ or s-state, and 4% $l=2$ or d-state.

(d) Electric Quadrupole Moment

The electric dipole moment is zero for atoms and nuclei in stationary states.
This is because of the definite parity of the atomic and nuclear states.

$$\text{Whether } \psi(-x,-y,-z) = \psi(x,y,z)$$
$$\text{or } = -\psi(x,y,z)$$

[even or odd parity]

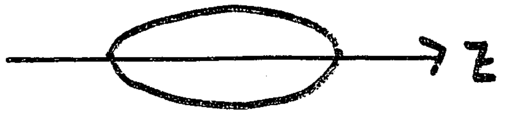
$$\psi^2(-x, -y, -z) = + \psi^2(x, y, z)$$

However the distribution of charge need not be spherical. There is nothing to stop the nucleus from having the shape of an ellipsoid of revolution.

This deviation from spherical symmetry is expressed by a quantity called the <u>Electric Quadrupole Moment</u>, Q

$$Q = \frac{1}{e} \int (3z^2 - r^2)\rho \, d\tau$$

$2z^2 - (x^2 + y^2)$



Q is +ve

A bare proton or neutron has no electric quadrupole moment, so a non-zero value for d, this implies a lack of spherical symmetry.

Molecular - beam resonance on H_2

(Rabi et al) gave a well-understood set of 6 lines

[transitions between $M_S = 0$ & $M_S = 1$, and between $M_S = 0$ & $M_S = -1$, for each of $M_I = -1, 0, +1$]

Molecular - beam resonance on D_2

(Rabi et al)

There were 7 lines, & positions could not be explained by the analysis used for H_2.

The results were explained by the assumption of an electric quadrupole moment for the deuteron

$$q = 0.00282 \times 10^{-28} \, m^2$$
$$= 0.00282 \text{ barns}$$

↑

a good unit for areas in nuclear physics [e.g. cross-sections]

This is small compared to the values for many other nuclei.

e.g. For $_{92}U^{235}$, value is 4.1 barns.

Nevertheless, the non-zero value for Q supports the idea that the deuteron is a combination of s and d states.
[The s-state is spherically symmetric, but states with higher values of ℓ are not.]

(e) Radius of the Deuteron

The most accurate studies of the radii, and the nucleon and charge distribution, of nuclei have been performed by Hofstadter et al in the 1950s

They use scattering or diffraction of high-energy ($\sim$ 183 MeV) electrons

They assumed a charge (or nucleon) density of the form

$$\rho(r) = \frac{\rho_1}{1 + e^{(r-R)/z}}$$

where

$\rho_1 \sim$ density at small values of r

R = radius

z = surface thickness

When $r = 0$,

$$\rho(0) = \frac{\rho_1}{1 + e^{-R/z}}$$

It is assumed that $R \gg z$

so $e^{-R/z} \ll 0$, $\rho(0) = \rho_1$
[as above]

When $r = R$,

$$\rho(r) = \rho_1/2$$

i.e the definition of R is that it is the value of r that gives $\rho = \rho_1/2$

Then when $r = R + Z$

$$\rho = \frac{\rho_1}{1+e}$$

So $R+Z$ is a fair measure of the width of the nucleus, as it gives a value where ρ is considerably reduced from $\rho_1/2$ to $\rho_1/3.7$

Measurements give the following rough values

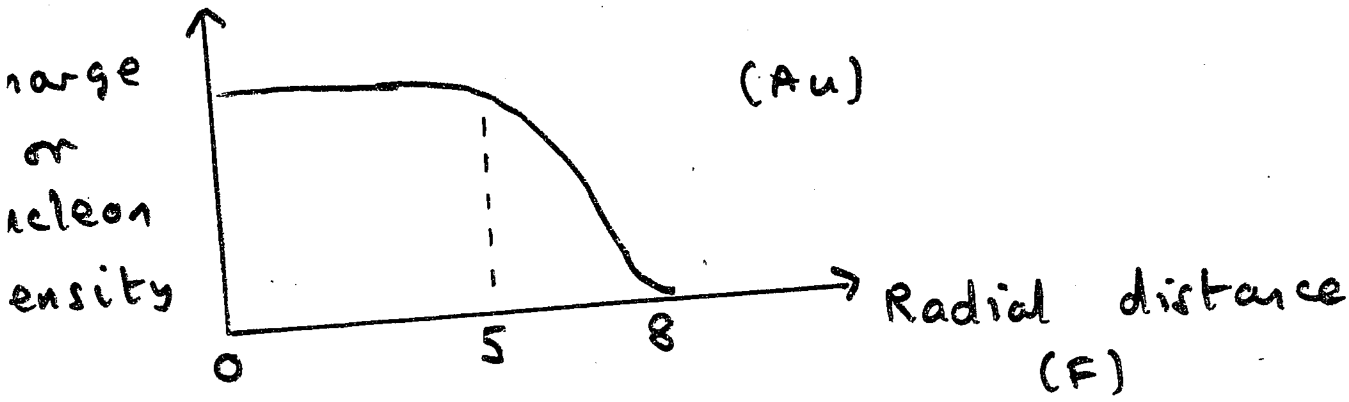
$Z \sim (0.5 \pm 0.1)\,F$ — Fermis / femtometres 10^{-15} m

$R \sim (1.07 \pm 0.02)A^{1/3}\,F$

e.g. If $A \sim 125$, $R \sim 5\,F$

The $A^{1/3}$ factor shows that the volume of the nucleus $\sim A$, so the density of nuclear matter is the same for all nuclei. Each nucleon "has its own volume".

Rough shape :



Values of radius

proton : 0.8 F

deuteron : 2.1 F

Simple theory of the deuteron

We will set up a simple QM model of the deuteron in an attempt to build up an understanding of nuclear forces.

We have 2 particles of very nearly equal masses and so must use the reduced mass

$$m = \frac{M_p M_n}{M_p + M_n} \sim \frac{M_N}{2}$$ mass of nucleon

Then the time-dependent Schrodinger equation is:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

V = the potential giving rise to the force between the bodies

E = the total energy of the system
= − 2.225 MeV (from before)

V is an unknown function of the separation of the particles, and possibly of other variables. It is V we wish to study.

For simplicity we assume V is a function only of the distance between the particles $V = V(r)$
[This assumption will eventually have to be given up.]

Then the S. equation may be separated, and the solutions may be written as follows:

$$\psi = (u_l/r)\, Y_{lm}(\theta, \phi)$$

Here Y_{lm} is a spherical harmonic and u_l is a solution of the equation

$$\frac{d^2u_l}{dr^2} + \frac{2m}{\hbar^2}\left[E - V - \frac{l(l+1)\hbar^2}{2mr^2}\right]u_l = 0$$

The last term in the brackets, $\frac{l(l+1)\hbar^2}{2mr^2}$

acts as an addition to V.

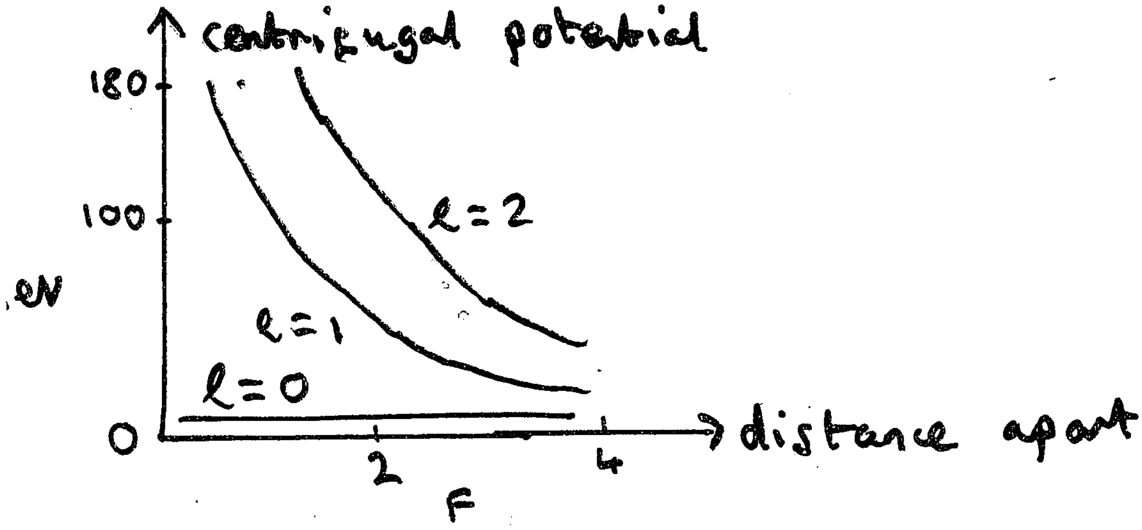
It is called the <u>centrifugal potential</u>

[It gives rise to a force of magnitude $\left|\frac{d}{dr}\frac{l(l+1)\hbar^2}{2mr^2}\right| = \frac{l(l+1)\hbar^2}{mr^3}$

But angular momentum $= [l(l+1)\hbar^2]^{1/2}$
$= mvr$

So $\frac{l(l+1)\hbar^2}{mr^3} = \frac{m^2v^2r^2}{mr^3} = mv^2/r$

= centrifugal force]



The effect of the centrifugal potential is an addition to the potential.

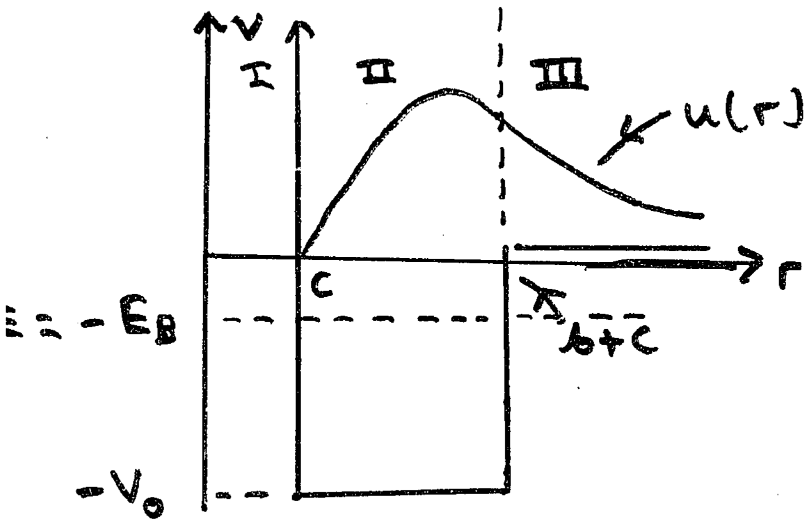
It forces the particles apart.

∴ It hinders binding.

∴ To look for binding functions, we choose $\ell = 0$. (For any spherically symmetric potential this will give the lowest energy.)

The simplest trial potential to investigate is a hard core potential (infinitely high) for $r < c$, and a square well $(V = -V_0)$ for $c < r < c+b$. For $r > c+b$, $V = 0$

c, b and V_0 are adjustable parameters.



$u(r) = 0$ in region I

In region II, with $l = 0$ (and the suffix dropped) the equation becomes

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}\left[V_0 - E_B\right]u = 0$$

General Solution is:

$$u_{II} = A \sin k(r-c) + B \cos k(r-c)$$

where

$$k = \sqrt{(2m/\hbar^2)(V_0 - E_B)}$$

At $r = c$, $u_{II} = 0$

$\therefore B = 0$

$$u_{II} = A \sin k(r-c)$$

A is a normalisation constant

In region III, equation becomes

$$\frac{d^2u}{dr^2} - \frac{2m}{\hbar^2} E_B u = 0$$

General solution is:

$$u_{III} = Ce^{-\kappa r} + De^{+\kappa r}$$

where

$$\kappa = (1/\hbar)\sqrt{2mE_B}$$

But u_{III} must tend to infinity as $r \to \infty$

$$\therefore D = 0, \qquad u_{III} = Ce^{-\kappa r}$$

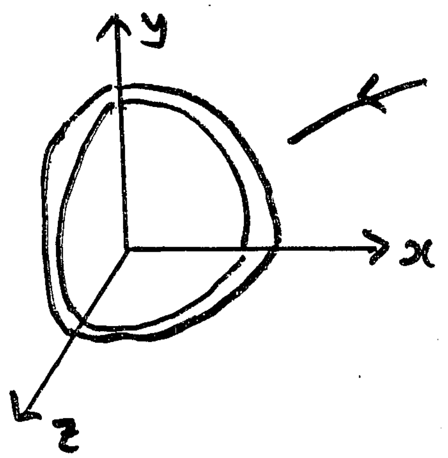
N.B. Quantum-mechanical note

The angular part of the wave-function, Y_{00}, may be ignored, as it is present in all cases, & will cancel in relative probability densities in different regions.

∴ prob. density in region between r and r+dr ~ appropriate element of volume ×

R^*R

(radial component of ψ.



spherical shell between radii r and $r + dr$

$$\text{Vol} = 4\pi r^2 dr$$
$$\sim r^2 dr$$

$\therefore$ probability density $\sim r^2 R^* R$

$$= u^* u$$

This demonstrates a further significance of $u = rR$

[Speaking roughly, in QM, 3D problems with spherical symmetry $(\ell = 0)$ are equivalent to 1D problems with R replaced by u.]

At the boundary between II and III, we must match both u and $\frac{du}{dr}$.

[As a special case, we do not have to match $\frac{du}{dr}$ at the boundary between regions I and II because the potential is infinite in region I.]

At $r = b + c$, we obtain

$$u_{II} = A \sin kb = u_{III} = C e^{-\varkappa(b+c)}$$

and

$$\frac{du_{II}}{dr} = Ak \cos kb = \frac{du_{III}}{dr} = -\varkappa C e^{-\varkappa(b+c)}$$

Dividing both sides, we obtain

$$k \cot kb = -\varkappa$$

Here the binding energy E_B is implicitly related to b and V_0.
[c has been eliminated.]

We have measured E_B, so require one more equation connecting b and V_0 to obtain values for them.

First look for <u>orders of magnitude</u>.

Assume (arbitrarily) $V_0 = 40$ MeV, and use our equation to find a value of b.

For deuteron, reduced mass $= \mu = 0.504u$

General formula for a wave number, k, in F^{-1}:

$$|k| = |p|/\hbar = \frac{1}{\hbar}\sqrt{2m|T|} = 0.2187\sqrt{m|T|}\ F^{-1}$$

with m in u, and T in MeV

Then $k = 0.1555\sqrt{T}\ F^{-1}$

Thus $k = 0.955\ F^{-1}$, $\varkappa = 0.232\ F^{-1}$

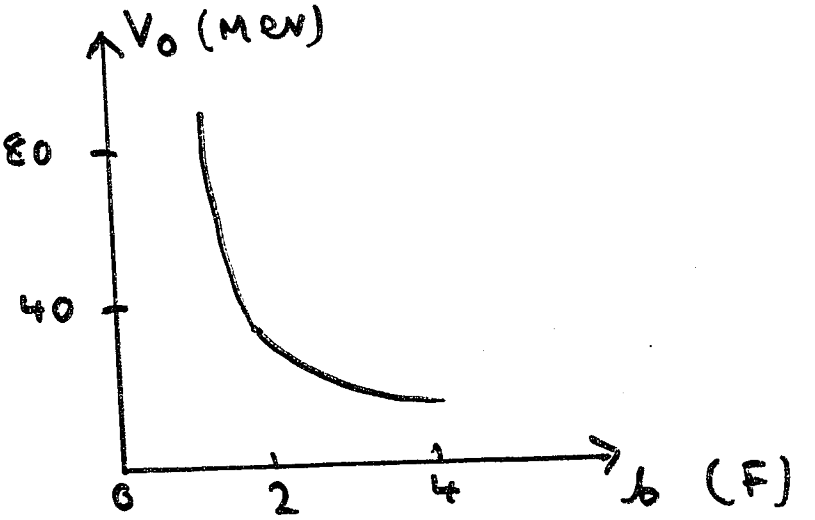
Then $k \cot kb = -\varkappa$

$$\Rightarrow b = \frac{1}{k}\operatorname{arccot}\left(-\frac{\varkappa}{k}\right) = \frac{1}{0.955}\operatorname{arccot}(-0.243)$$

$$= 1.895\ F$$

[Actually, because the cot function is periodic, there are many values of b that satisfy the equation.
For the ground-state, we want the longest possible value of λ, thus the shortest possible value of $|k|$, and hence kb is in the second quadrant.]

Any different value of V_0 will give a different value of b.



Normalisation of the deuteron wave-function - root-mean-square radius

By normalising the wave-function. i.e. by finding the values of A and C we shall obtain the rms value of the deuteron.

$$\int_c^\infty u^* u \, dr = 1$$

$$\Rightarrow \quad A^2 \int_c^{c+b} \sin^2 K(r-c)\, dr + C^2 \int_{c+b}^\infty e^{-2\gamma r} dr = 1$$

Now

$$\int_c^{c+b} \sin^2 k(r-c)\, dr = \frac{1}{2}\int_c^{c+b} \left[1 - \cos 2k(r-c)\right] dr$$

$$= \frac{1}{2}\left\{ \left[r\right]_c^{c+b} - \frac{1}{2k}\left[\sin 2k(r-c)\right]_c^{c+b} \right\}$$

$$= \frac{b}{2} - \frac{1}{4k}\left(\sin 2kb - 0\right) = \frac{1}{2}\left\{b - \frac{1}{2k}\sin 2kb\right\}$$

and

$$\int_{c+b}^{\infty} e^{-2\varkappa r}\, dr = -\frac{1}{2\varkappa}\left[e^{-2\varkappa r}\right]_{c+b}^{\infty}$$

$$= \frac{1}{2\varkappa}\, e^{-2\varkappa(c+b)}$$

$$\therefore\ A^2\left\{b - \frac{1}{2k}\sin 2kb\right\} + \frac{C^2}{2\varkappa}\, e^{-2\varkappa(c+b)} = 1$$

From matching conditions (top of p. 38)

we obtain [tricky algebra]

$$A^2 = \frac{2\varkappa}{1+\varkappa b}\ ; \quad C^2 = \frac{2\varkappa(\sin^2 kb)\, e^{2\varkappa(c+b)}}{1+\varkappa b}$$

Then we may compute the average of the square of the proton to centre-of-mass distance. | This is a measure of the radius r_d of the deuteron

This is half the separation between the proton and nucleus that we have calculated as r above.

$$\therefore\ \langle r_d^2\rangle = \frac{1}{4}\left\{A^2\int_c^{c+b} r^2 \sin^2 k(r-c)\, dr + B^2\int_{c+b}^{\infty} r^2 e^{-2\varkappa r}\, dr\right\}$$

The integrals may be performed by parts (omitted here).

The data may be analysed, and a reasonable set of results emerges:

$$c = 0.4\,\mathrm{F}$$

$$\langle r_d^2 \rangle^{1/2} = 2.22\,\mathrm{F}$$

[compared with a measured value of 2.1F]

Calculated values of V_0 vary greatly - between 35MeV and 75MeV.

We should not be concerned by this:- the square-well model itself is only a rough approximation.

The important point is that, whatever the value, $V_0 \gg B$.

The deuteron is very weakly bound. Since it is the first stage of building up more complicated nuclei, we may be grateful it is bound at all!

Spin - dependence of nuclear forces

Total angular momentum of the deuteron corresponds to $J = 1$

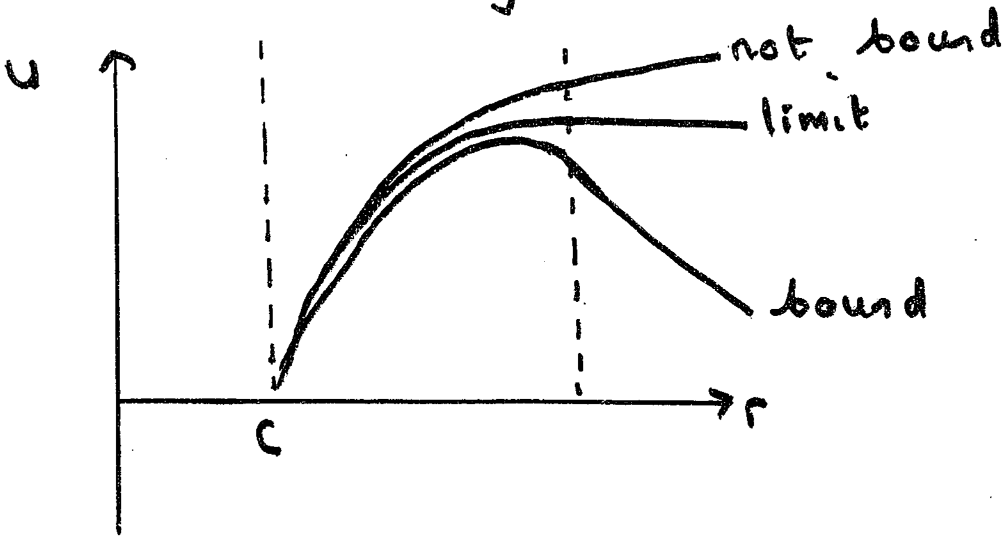
If $L = 0$ is correct, then $S = 1$

This corresponds to an alignment of spins -

$^3S_1$

There is no singlet state $^1S_0$

This indicates that the np force is stronger when the spins are parallel than when they are anti-parallel.



For singlet potential, u is not bound for $V_0 \lesssim 25\,\text{MeV}$
[very rough figure]

It is clear that the effective square-well potential must be much less for the singlet than for the triplet.

In other words we may express the potential as

$$V(r) - \underset{\sim}{S_1} \cdot \underset{\sim}{S_2}\, V'(r)$$

independent of spin

positive if spins aligned
negative " " opposed

<u>Tensor forces</u>

Let us test the hypothesis that the ground state of the deuteron is a pure S-state.

Then only the spins of the p and the n should contribute to the magnetic dipole moment.

Magnetic dipole moments

proton	2.79275 nm
neutron	−1.91350 nm
deuteron expected	0.87925 nm
" observed	0.85735 nm
discrepancy	0.02190 nm

$\sim 2\frac{1}{2}\%$

In principle this discrepancy could be because

(a) the $\genfrac{}{}{0pt}{}{n}{p}$ behaves differently in the presence of the $\genfrac{}{}{0pt}{}{p}{n}$, or

(b) of relativistic effects

However in practise, with the added input of the electric quadrupole moment, it is clear that the state is not a pure S-state.

[The quadrupole moment of an S-state is zero because it is spherically symmetric.]

We saw earlier that we must have a linear combination of S and D. (No P)

$^3S_1$ $^3D_1$

$S=1,\ L=0,\ J=1$ $S=1,\ L=2,\ J=1$

i.e. $\Psi = \alpha_S \Psi_S + \alpha_D \Psi_D$

with $|\alpha_S|^2 + |\alpha_D|^2 = 1$

From the evidence of magnetic dipole moment and electric quadrupole moment, it is estimated that $|\alpha_D|^2$ lies between 0.03 and 0.06.

[α_D itself would be ~ 0.2]

We may express the potential as

$$V(r) \quad - \underset{\sim}{S_1} \cdot \underset{\sim}{S_2}\, V'(r) \quad - \alpha \underset{\sim}{L} \cdot \underset{\sim}{S}$$

$V(r)$ = central potential, with parameters

V_0, c and b

$V'(r)$ = <u>tensor potential</u>, with parameters

V_0', c and b'

← the same as for the central potential

$\alpha \underset{\sim}{L} \cdot \underset{\sim}{S}$ = <u>spin-orbit coupling</u>, which lines up the spin with the orbital angular momentum of the D-component

2. Nuclear decays and reactions

2A. General properties of decay processes

In a collection of unstable nuclei, the no. of decays in unit time is proportional to the no. of nuclei of that species present at that time, $N(t)$

$$dN = -\lambda N(t)\,dt$$

i.e. $$\frac{dN}{dt} = -\lambda N(t)$$

λ = decay constant or decay rate

we obtain $N(t) = N(0)e^{-\lambda t}$

where $N(0)$ = no. of parent nuclei present at $t=0$

Then $$\frac{dN(t)}{dt} = -\lambda N(t) = -\lambda N(0)e^{-\lambda t}$$

halflife, $\tau_{1/2}$

The halflife, $\tau_{1/2}$, is the time at which half the original nuclei have decayed

$$1/2 = e^{-\lambda t} = e^{-\lambda \tau_{1/2}}$$

Taking reciprocals of each side:

$$2 = e^{\lambda \tau_{1/2}}$$

$$\log 2 = \lambda \tau_{1/2}$$

$$\tau_{1/2} = \frac{\log 2}{\lambda}$$

[N.B. log always refers to base e]

<u>mean lifetime, τ</u>

τ is the mean time a nucleus survives after creation.

$$\tau = \int_0^\infty t\,(dN/dt)\,dt \Big/ \int_0^\infty (dN/dt)\,dt$$

$$= \int_0^\infty t\,e^{-\lambda t}\,dt \Big/ \int_0^\infty e^{-\lambda t}\,dt$$

$$\int_0^\infty e^{-\lambda t}\,dt = -\frac{1}{\lambda}\left[e^{-\lambda t}\right]_0^\infty = 1/\lambda$$

$$\int_0^\infty \underset{u}{t}\,\underset{dv/dt}{e^{-\lambda t}}\,dt = \left[-t/\lambda\, e^{-\lambda t}\right]_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\,dt$$

$$v = -\frac{1}{\lambda}e^{-\lambda t} \qquad = 0 + 1/\lambda^2 = 1/\lambda^2$$

$$\therefore \tau = \frac{1/\lambda^2}{1/\lambda} = \frac{1}{\lambda} = \frac{\tau_{1/2}}{\log 2}$$

measurements of lifetimes

For moderate to long lifetimes, the decay rate is measured.

$$\left|\frac{dN(t)}{dt}\right| = \lambda N(0) e^{-\lambda t} \qquad (1)$$

For very long lifetimes [often millions of years, but in practice, any lifetime much longer than the timescale of the experiment], $e^{-\lambda t} \sim 1$

$$\therefore \left|\frac{dN}{dt}\right| = \lambda N(0)$$

$$\tau = 1/\lambda = \frac{N(0)}{|dN/dt|}$$

[e.g. radioactive dating ; see shortly]

moderate lifetimes (e.g. hours)

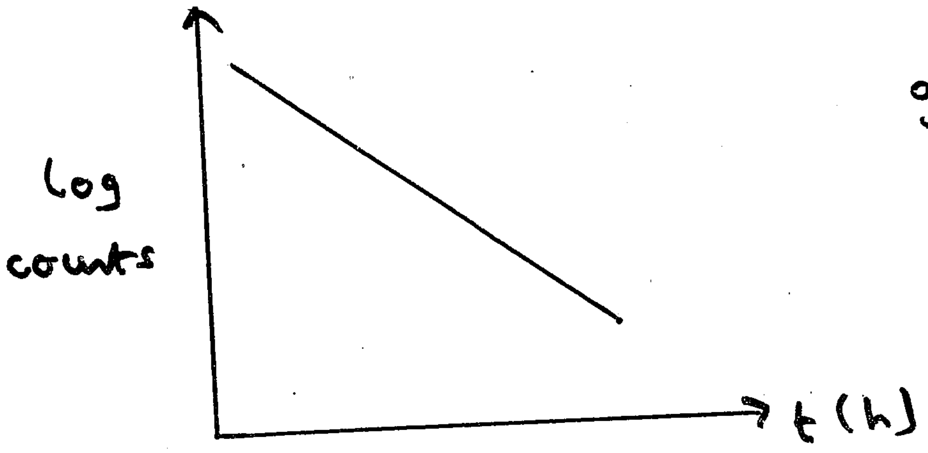
From (1) above write

$$\left|\frac{dN(t)}{dt}\right| = \left|\frac{dN(0)}{dt}\right| e^{-\lambda t}$$

Then

$$\log \left|\frac{dN}{dt}\right|_t = \log \left|\frac{dN}{dt}\right|_0 - \lambda t$$

$$\log\,[\text{counts}]_t = \log\,[\text{counts}]_0 - \lambda t$$



gradient = $-\lambda$

<u>shorter lifetimes</u> (e.g. seconds)

computer - controlled instrumentation

<u>shorter lifetimes still</u> (e.g. $< 10^{-3}$ s)

special techniques

e.g. coincidence techniques

[use of γ-rays emitted during creation and decay of particle]

Multimodal decay

e.g. decay of $^{164}_{67}Ho$

It decays to $^{164}_{68}Er$ by electron capture

$[\tau_1 = 53m \ , \ Q_1 = 1.03 \text{ MeV}]$

and to $^{164}_{66}Dy$ by β^- decay

$[\tau_2 = 60m \ , \ Q_2 = 1.11 \text{ MeV}]$

$\lambda_1 = 1/\tau_1$ and $\lambda_2 = 1/\tau_2$ are partial decay constants

$\frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\frac{\lambda_2}{\lambda_1+\lambda_2}$ are branching ratios.

Fraction decaying to Er $= \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{1/\tau_1}{1/\tau_1 + 1/\tau_2}$

$= \frac{\tau_2}{\tau_1+\tau_2} \sim 53\%$

Similarly fraction decaying to Dy

$= \frac{\tau_1}{\tau_1+\tau_2} \sim 37\%$

Decay equation: $\frac{dN(t)}{dt} = -[\lambda_1 N(t) + \lambda_2 N(t)]$

$\Rightarrow \quad N(t) = N_0 e^{-(\lambda_1+\lambda_2)t}$

we may define

total decay constant $= \lambda = \lambda_1 + \lambda_2$

total lifetime $= \tau = 1/(\lambda_1+\lambda_2)$

branching ratios $= f_1$ and f_2

$= \dfrac{\lambda_1}{\lambda}$ and $\dfrac{\lambda_2}{\lambda}$

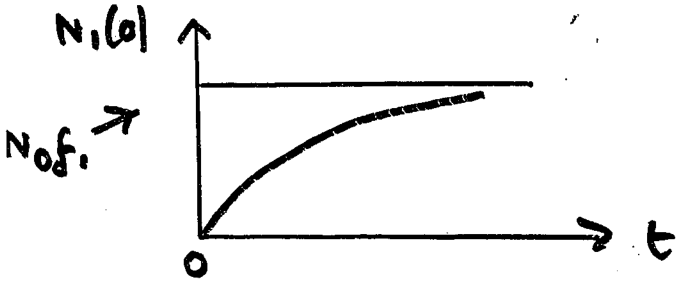
Let us examine the growth of N_1 & N_2

$$\frac{dN_1(t')}{dt} = \lambda_1 N(t') = N_0 \lambda_1 e^{-(\lambda_1+\lambda_2)t'}$$

$$N_1(t) = \int_0^t \frac{dN_1(t')}{dt} = \lambda_1 N_0 \frac{1}{\lambda_1+\lambda_2}\left[-e^{-(\lambda_1+\lambda_2)t'}\right]_0^t$$

$$= N_0 \frac{\lambda_1}{\lambda_1+\lambda_2}\left[1 - e^{-(\lambda_1+\lambda_2)t}\right]$$

$\Rightarrow$ $N_1(0) = 0$; $N_1(\infty) = N_0 \dfrac{\lambda_1}{\lambda_1+\lambda_2} = N_0 f_1$



Similarly, $N_2(t) = N_0 \dfrac{\lambda_2}{\lambda_1+\lambda_2}\left[1 - e^{-(\lambda_1+\lambda_2)t}\right]$

Note that $N_1(t) = f_1\left[N_0 - N(t)\right]$

$N_2(t) = f_2\left[N_0 - N(t)\right]$

All these equations may be generalised to multimodal decays with >2 product nuclei

[N.B. In the above, it is assumed that the product nuclei are themselves stable]

<u>Sequential decays</u>

These occur when a daughter (product) species is itself unstable.

e.g. $\;{}^{218}_{86}\text{Ra}\;(N_1) \xrightarrow[\sim 35\,\text{ms}]{\lambda_1} {}^{214}_{84}\text{Po}\;(N_2) \xrightarrow[\sim 164\,\mu\text{s}]{\lambda_2} {}^{210}_{82}\text{Pb}\;(N_3)$

both by α-decay

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \quad \Rightarrow \quad N_1 = N_0 e^{-\lambda_1 t}$$

[as usual]

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$$

$$= \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N_2$$

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t}$$

1st-order linear differential equation

$$\frac{dN_2}{dt} + PN_2 = Q \qquad \text{[standard form]}$$

integrating factor $= e^{\int P dt} = e^{\int \lambda_2 dt}$
$= e^{\lambda_2 t}$

Equation becomes

$$e^{\lambda_2 t}\frac{dN_2}{dt} + \lambda_2 e^{\lambda_2 t} N_2 = \lambda_1 N_0 e^{(\lambda_2 - \lambda_1)t}$$

$$\frac{d}{dt}\left\{N_2 e^{\lambda_2 t}\right\} = \lambda_1 N_0 e^{(\lambda_2-\lambda_1)t}$$

$$N_2 e^{\lambda_2 t} = \lambda_1 N_0 \frac{e^{(\lambda_2-\lambda_1)t}}{\lambda_2-\lambda_1} + C$$

$$N_2 = \frac{\lambda_1}{\lambda_2-\lambda_1} N_0 e^{-\lambda_1 t} + Ce^{-\lambda_2 t}$$

When $t = 0$, $N_2 = 0 \Rightarrow \frac{\lambda_1}{\lambda_2-\lambda_1} N_0 + C = 0$

$C = -N_0 \frac{\lambda_1}{\lambda_2-\lambda_1}$

$$\therefore N_2(t) = \frac{\lambda_1}{\lambda_2-\lambda_1} N_0 \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$N_2(0) = N_2(\infty) = 0$

<u>check</u> N_2 must always be ≥ 0

e.g. For small t

$$N_2(t) = \frac{\lambda_1}{\lambda_2-\lambda_1} N_0 \left(1 - \lambda_1 t \ldots - 1 + \lambda_2 t \ldots\right)$$

$$= \frac{\lambda_2}{\lambda_2-\lambda_1} N_0\, t\, (\lambda_2-\lambda_1) \ldots$$

$= \lambda_2 N_0 t \ldots$

$$\frac{dN_3}{dt} = \lambda_2 N_2 = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} N_0 \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$$N_3(t) = \int_0^t \frac{dN_3}{dt'} dt'$$

$$= \int_0^t \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} N_0 \left(e^{-\lambda_1 t'} - e^{-\lambda_2 t'}\right)$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} N_0 \left[-\frac{1}{\lambda_1} e^{-\lambda_1 t'} + \frac{1}{\lambda_2} e^{-\lambda_2 t'} \right]_0^t$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} N_0 \left[-\frac{1}{\lambda_1}\left(e^{-\lambda_1 t} - 1\right) + \frac{1}{\lambda_2}\left(e^{-\lambda_2 t} - 1\right) \right]$$

$$= \frac{N_0}{\lambda_2 - \lambda_1} \left\{ \lambda_2 \left(1 - e^{-\lambda_1 t}\right) - \lambda_1 \left\{ 1 - e^{-\lambda_2 t} \right\} \right.$$

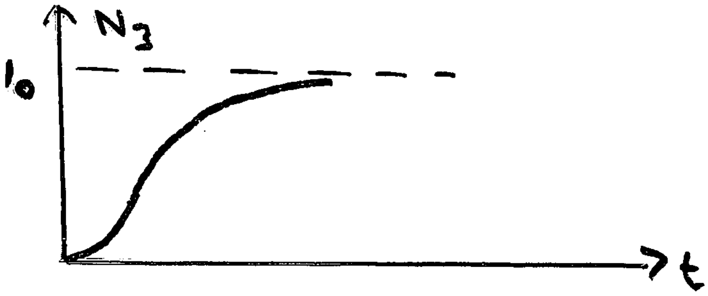
$$N_3(0) = 0 \quad ; \quad N_3(\infty) = \frac{N_0}{\lambda_2 - \lambda_1} (\lambda_2 - \lambda_1) = N_0$$

For small t

$$N_3(t) = \frac{N_0}{\lambda_2 - \lambda_1} \left\{ \lambda_2 \left(1 - 1 + \lambda_1 t - \lambda_1^2 t^2/2 \ldots\right) - \lambda_1 \left(1 - 1 + \lambda_2 t - \lambda_2^2 t^2/2 \ldots\right) \right\}$$

$$= \frac{N_0}{\lambda_2 - \lambda_1} \left\{ -\frac{\lambda_2 \lambda_1^2 t^2}{2} + \frac{\lambda_1 \lambda_2^2 t^2}{2} \ldots \right\}$$

$$= \frac{N_0}{\lambda_2 - \lambda_1} \frac{t^2}{2} \left(\lambda_1 \lambda_2^2 - \lambda_2 \lambda_1^2\right) \ldots = N_0 \lambda_1 \lambda_2 \, t^2/2 \ldots$$



<u>check</u>

$N_1(t) + N_2(t) + N_3(t)$ must equal N_0

$$N_1 + N_2 + N_3 = N_0 \left\{ e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) + \frac{1}{\lambda_2 - \lambda_1}\left[\lambda_2\left(1 - e^{-\lambda_1 t}\right) - \lambda_1\left(1 - e^{-\lambda_2 t}\right)\right]\right\}$$

$$= \frac{N_0}{\lambda_2 - \lambda_1}\left\{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_1 t} + \lambda_1 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t} + \lambda_2 - \lambda_2 e^{-\lambda_1 t} - \lambda_1 + \lambda_1 e^{-\lambda_2 t}\right\} = N_0$$

More complicated situations may be handled usually with considerably more complicated mathematics

Quantum mechanical considerations

Quantum mechanical state represented by

$$\Psi(\underset{\sim}{r}, t) = \psi(r)\, e^{-iE_0 t/\hbar}$$

cap small

$$P(\underset{\sim}{r}, t) = |\Psi(\underset{\sim}{r}, t)|^2$$

To include radioactive decay

$$P(\underset{\sim}{r}, t)^2 = P(\underset{\sim}{r}, 0)^2\, e^{-\lambda t}$$

Then use

$$\Psi(\underset{\sim}{r}, t) = \psi(r)\, e^{-iE_0 t/\hbar}\, e^{-\lambda t/2}$$

This implies (though it is not immediately obvious) that the state corresponds not to a single value of energy E_0, but to a distribution of energies $P(E)$ with a mean value E_0.

To analyse this, we use the <u>Fourier Transform</u>

Time dependence of stable state is $\underset{f(t)}{e^{-iE_0 t/\hbar}}$

To look at energy dependence,

the Fourier transform is

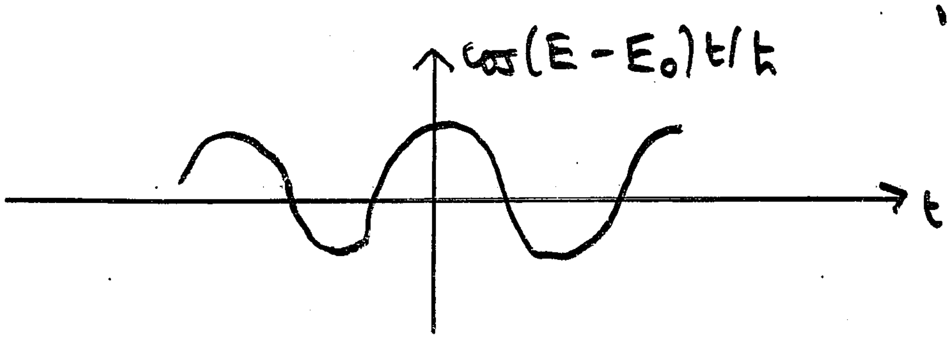
$$\frac{1}{\hbar}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{iEt/\hbar}\, dt = F(E)$$

Here $F(E) = \frac{1}{\hbar}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iEt/\hbar}\, e^{-iE_0 t/\hbar}\, dt$

$$= \frac{1}{\hbar}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 1\, dt = \infty \quad \text{if } E = E_0$$

$$= \frac{1}{\hbar}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\cos(E-E_0)t/\hbar + i\sin(E-E_0)t/\hbar\right\} dt$$

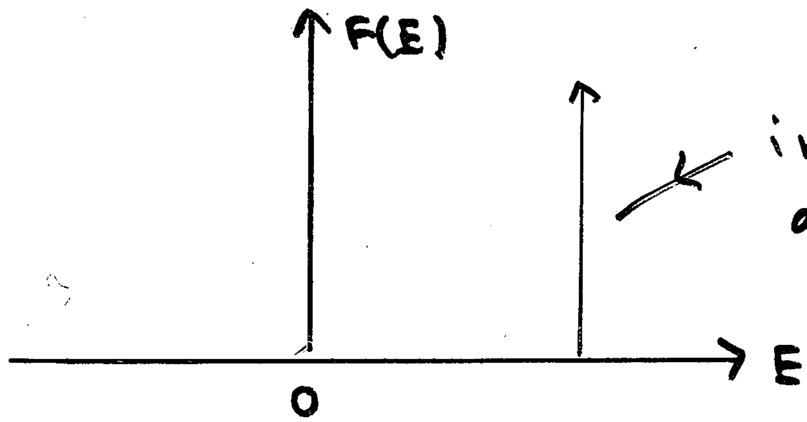
if $E \neq E_0$



Area under curve between $t = -\infty$ and $t = +\infty$ "oscillates about zero"; put equal to 0

Then $F(E) = \infty \quad (E = E_0)$

$\qquad\qquad 0 \quad (E \neq E_0)$



infinite spike at $E = E_0$

This is the <u>Dirac delta function</u>

With the $1/\sqrt{2\pi}$ on the previous page, the area under the spike is <u>unity</u>.

In this sense the Dirac delta function is normalised

[All this is <u>extremely</u> awkward mathematically. Do not try to understand the mathematical details too closely!]

What it tells us is that the function $e^{-iE_0t/\hbar}$ is all 'at' energy E_0 [or all 'at' frequency $E_0/\hbar$]

Fairly obvious!

Now look at $e^{-iE_0t/\hbar}\,e^{-\lambda t/2} = f(t)$

Then $F(E) = \frac{1}{\hbar}\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)\,e^{iEt/\hbar}\,dt$

N.B.

$$= \frac{1}{\hbar}\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i(E-E_0)t/\hbar} \; e^{-\lambda t/2} \, dt$$

$$= \frac{1}{\hbar\sqrt{2\pi}} \frac{\left[e^{i(E-E_0)t/\hbar} \; e^{-\lambda t/2} \right]_0^\infty}{i(E-E_0)t/\hbar \; - \lambda/2}$$

$$= \frac{1}{\hbar\sqrt{2\pi}} \frac{1}{\lambda/2 - i(E-E_0)t/\hbar}$$

$$= \frac{i}{\sqrt{2\pi}} \frac{1}{(E_0 - E) + i\hbar\lambda/2}$$

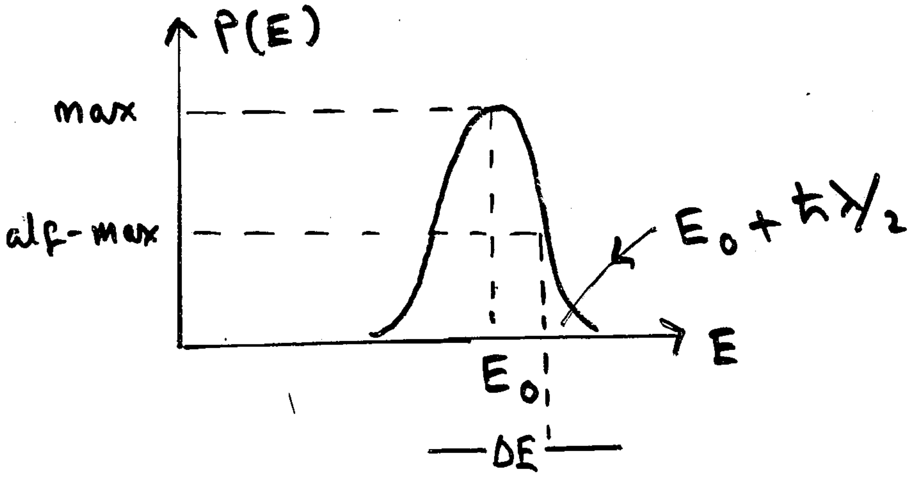
The the probability of finding the particle with a particular value of energy, E, is proportional to $P(E) = |F(E)|^2$

$$P(E) = \frac{1}{2\pi} \frac{1}{(E-E_0)^2 + \hbar^2\lambda^2/4}$$

This function is the <u>Lorentzian function</u> (bell-shaped).

Its peak is at $E = E_0$, and its height is half its maximum when

$$\frac{\hbar^2\lambda^2}{4} = (E-E_0)^2 ; \quad E = E_0 \pm \underbrace{\frac{\hbar\lambda}{2}}_{\Delta E}$$



max value of $P(E) = \frac{1}{2\pi} \frac{4}{\hbar^2\lambda^2} = \frac{2}{\pi\hbar^2\lambda^2}$

Half-max " " " $= \frac{1}{\pi\hbar^2\lambda^2}$

So $\Delta E = \frac{\hbar\lambda}{2}$

But $\lambda = 1/\tau \Rightarrow (\Delta E)\tau \sim \hbar$

We may regard τ as an uncertainty in the time of decay i.e. $\tau = (\Delta t)$

$\Rightarrow (\Delta E)(\Delta t) \sim \hbar$

a form of the <u>Heisenberg principle</u>

What we have demonstrated is that the fact that the system is unstable to radioactive decay means that the state has a spread in energies, or the energy-level has a width.

Of course, if $\tau \to \infty$, so the state becomes stable, the <u>Lorentzian</u> loses its width and tends to the <u>Dirac delta function</u>

<u>Normalisation of the Lorentzian</u>

As yet the Lorentzian itself is not normalised.

To normalise it we require

$$\int_{-\infty}^{\infty} P(E)\, dE = 1$$

with
$$P(E) = 2\pi\alpha \frac{1}{(E-E_0)^2 + \hbar^2\lambda^2/4}$$
α normalisation constant

$$\int_{-\infty}^{\infty} \frac{dE}{(E-E_0)^2 + \hbar^2\lambda^2/4} = \frac{1}{(\hbar\lambda/2)} \tan^{-1}\left[\frac{E-E_0}{(\hbar\lambda/2)}\right]_{-\infty}^{\infty}$$

$$= \frac{2}{\hbar\lambda}\left[\pi/2 - (-\pi/2)\right] = \frac{2\pi}{\hbar\lambda}$$

$$\left[\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \right]$$

$$\int_{-\infty}^{\infty} P(E)\,dE = 2\pi\,\alpha\,\frac{2\pi}{\hbar\lambda} = 1$$

$$\Rightarrow \quad \frac{4\pi\alpha}{\hbar\lambda} = 1, \quad \alpha = \frac{\hbar\lambda}{4\pi}$$

So $\quad P(E)$ (normalised) $= \dfrac{\hbar\lambda/2}{(E-E_0)^2 + \hbar^2\lambda^2/4}$

N.B.

If $\tau > 10^{-15}$ s, $\quad \Delta E < 10^{-6}$ MeV

Typically energy of decay $\sim 10^{-2}$ MeV or 10^{-1} MeV

So widths of energy-levels $\ll$ differences between energy-levels

Radioactive Dating

This may be used to determine (a) the age of organic systems or (b) the age of mineralogical samples.

(a) use of ^{14}C (lifetime 8270 years) for ages up to 10^4 to 10^5 years

(b) various nuclides for ages up to $\sim 10^9$ years

technique

sample produced at $t=0$ with a number of parent nuclei that decay to daughter nuclei with a known lifetime $\tau = 1/\lambda$

Assume

1. The sample contains no daughter nuclei at $t=0$
2. The daughter nuclei are produced only by this process.
3. The daughter nuclei are stable.

If $N_1(t)$ = no. of parent nuclei
$N_2(t)$ = " " daughter nuclei

Then $N_1(t) + N_2(t) = N_1(0)$ (x)

(t = the present)

But $N_1(t) = N_1(0)\, e^{-\lambda t}$

$$\Rightarrow \{N_1(t) + N_2(t)\} = N_1(t)\, e^{\lambda t}$$

$$1 + \frac{N_2(t)}{N_1(t)} = e^{+\lambda t}$$

$$t = \frac{1}{\lambda} \log \left\{ 1 + \frac{N_2(t)}{N_1(t)} \right\}$$

All the information on the RHS may be found by measurement.

Then t may be calculated.

Unfortunately in most cases, some daughter nuclei are present at the formation of the sample.

So (x) is replaced by

$$N_1(t) + N_2(t) = N_1(0) + N_2(0)$$

and we cannot obtain t

However very often an additional stable isotope of the daughter nuclide is also present at the time of formation.

$$N_S(t) = N_S(0)$$

It is also assumed that samples with a common origin should have the same value of $\left.\frac{N_2(0)}{N_s(0)}\right\}$ ~ known, call it β

Then

$$N_2(t) = N_2(0) + \underbrace{N_1(0) - N_1(t)}$$

$$= N_2(0) + N_1(t)\left\{e^{\lambda t} - 1\right\}$$

$$\frac{N_2(t)}{N_s(t)} = \frac{N_1(t)}{N_s(t)}\left\{e^{\lambda t} - 1\right\} + \underbrace{\left(\frac{N_2(0)}{N_s(0)}\right)}_{\beta}$$

$$\frac{N_1(t)}{N_s(t)}\left\{e^{\lambda t} - 1\right\} = \frac{N_2(t)}{N_s(t)} - \beta \qquad (y)$$

$$e^{\lambda t} = 1 + \left[\left\{\frac{N_2(t)}{N_s(t)} - \beta\right\} \Big/ \frac{N_1(t)}{N_s(t)}\right]$$

$$t = \frac{1}{\lambda}\log\left[1 + \left[\left\{\frac{N_2(t)}{N_s(t)} - \beta\right\} \Big/ \frac{N_1(t)}{N_s(t)}\right]\right]$$

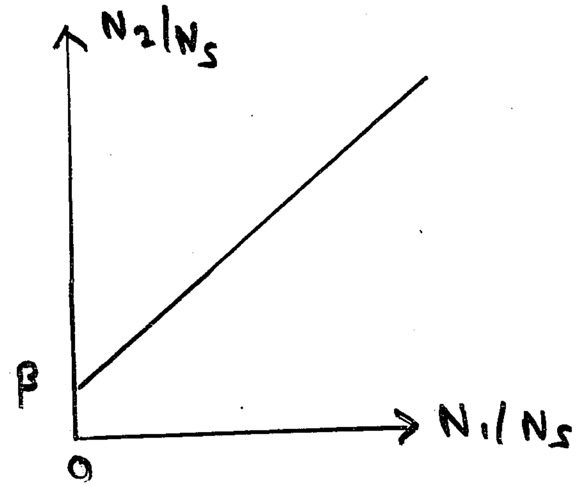
$\frac{N_2(t)}{N_s(t)}$ and $\frac{N_1(t)}{N_s(t)}$ may both be measured

e.g The β^- decay of ${}^{87}_{37}Rb$ to ${}^{87}_{38}Sr$

Another isotope of Sr exists - ${}^{86}_{38}Sr$

$$\left.\begin{aligned} N_1(t) &= \text{no. of } {}^{87}Rb \\ N_2(t) &= \text{no. of } {}^{87}Sr \\ N_s(t) &= \text{no. of } {}^{86}Sr \end{aligned}\right\}$$

From (y) above, plot ${}^{87}Sr / {}^{86}Sr$ (N_2/N_s) against ${}^{87}Rb/{}^{86}Sr$ (N_1/N_s) for samples with a common origin



gradient $= e^{\lambda t} - 1$

$$N_2/N_s = N_1/N_s\,(e^{\lambda t} - 1) + \beta$$

For 5 meteorites, $\Rightarrow$ age $\sim 4.4 \times 10^9$ years

2B Alpha Decay

General equation:

He nucleus
2n2P

$$ {}^{A}_{Z}X^{N} \rightarrow {}^{A-4}_{Z-2}Y^{N-2} + \alpha $$

occurs for a very few light nuclides and many heavy nuclides

Energy released $\mathcal{E}$ given by

$$ \mathcal{E} = \left\{ M_N\left({}^{A}_{Z}X^{N} \right) - M_N\left({}^{A-4}_{Z-4}Y^{N-2} \right) - M_\alpha \right\} c^2 $$

nuclear masses

It is convenient to work with atomic masses, so we may add Z, $Z-4$ and 4 electron masses to the terms in the above equation. Then

$$ \mathcal{E} = \left\{ m\left({}^{A}_{Z}X^{N} \right) - m\left({}^{A-4}_{Z-4}Y^{N-2} \right) - m\left({}^{4}He \right) \right\} c^2 $$

[all atomic masses]

Here we have ignored electron binding energy (~ eV ; we are concerned with MeV)

If Q is negative, process is endothermic and cannot occur spontaneously.

If Q is positive, process is exothermic and can occur spontaneously (from energy point of view; it may be prohibited for other reasons).

The XS energy is given to KE of the α and the daughter nucleus.

Binding energies

Because the number and types of nucleons do not change, we may write the equation for Q in terms of binding energy (with minus signs)

binding energy

$$\left[\text{e.g. } m({}^4He) = 2m_p + 2m_n - \frac{B({}^4He)}{c^2}\right]$$

$$\therefore Q = B({}^4He) + B\left({}^{A-4}_{Z-2}Y^{N-2}\right) - B\left({}^{A}_{Z}X^{N}\right)$$

28.3 MeV

very high because the α is a double magic nucleus

Because $B(^4He)$ is very high,

α-emission is very often energetically possible for a large nucleus, while emission of other light nuclei is extremely unlikely.

e.g. ^{235}U

For α-emission, $Q = +4.68$ MeV

For emission of $n, p, {}^2H, {}^3H$ etc.

Q negative [table, Dunlap, p.95]

Values of Q

These may be calculated on the basis of measured atomic masses

[N.B. The binding energy nucleon / curve has its max at $A \sim 55$

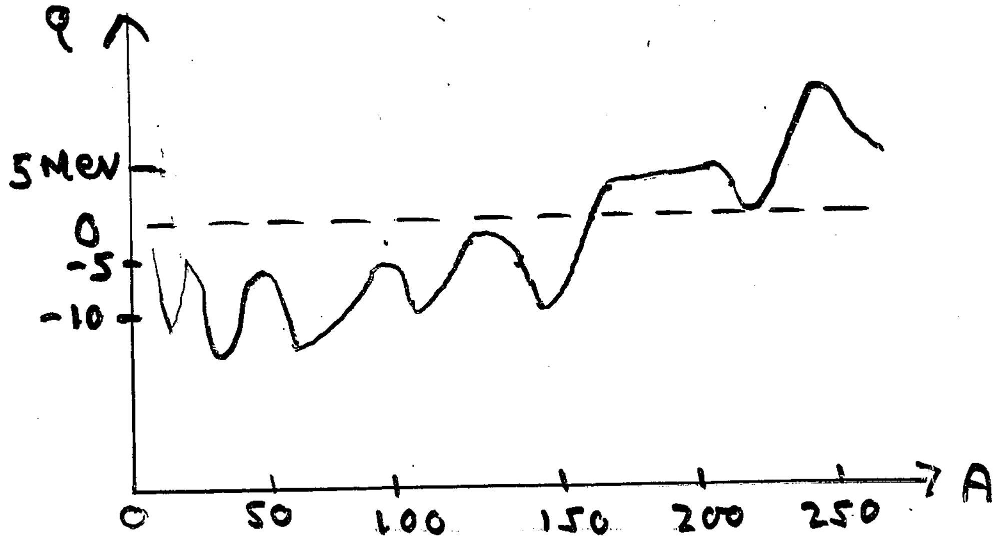
$$\therefore B\left({}^{A-4}_{Z-2}Y^{N-2}\right) > B\left({}^{A}_{Z}X^{N}\right)$$

for $A \gtrsim 55$, but the requirement is more stringent than this because of the binding energy of the α.
[much lower than larger nuclei]]

Results

Q crosses from -ve to +ve around $A = 150$

There are local minima & maxima due to shell effects



[general ~~form~~ of results omitting all detail]

The kinetic energy of the α-particle may be measured.

The KE is shared between the α-particle and the daughter nucleus D

~~The~~ momenta of α and D are equal in magnitude and opposite

i.e. $M_\alpha V_\alpha = M_D V_D$

$$V_D = \frac{M_\alpha V_\alpha}{M_d}$$

Then $\left.\begin{aligned} T_\alpha &= \tfrac{1}{2} m_\alpha V_\alpha^2 \\ T_D &= \tfrac{1}{2} m_D V_D^2 \end{aligned}\right\}$ $\quad T_D + T_\alpha = Q$

$$T_D = \tfrac{1}{2} m_D V_D^2 = \frac{1}{2} m_D V_\alpha^2 \frac{m_\alpha^2}{m_D^2} = T_\alpha \frac{m_\alpha}{m_D}$$

Then $T_\alpha \left\{ 1 + \frac{m_\alpha}{m_D} \right\} = Q$

$$T_\alpha = \frac{Q}{1 + m_\alpha / m_D}$$

Typically $\frac{m_\alpha}{m_D} \sim 0.02$

So the α-particle takes $\sim 98\%$ of Q

the recoil of the daughter takes $\sim 2\%$

<u>Relation between energy and lifetime</u>

<u>Geiger - Nuttall rule</u>

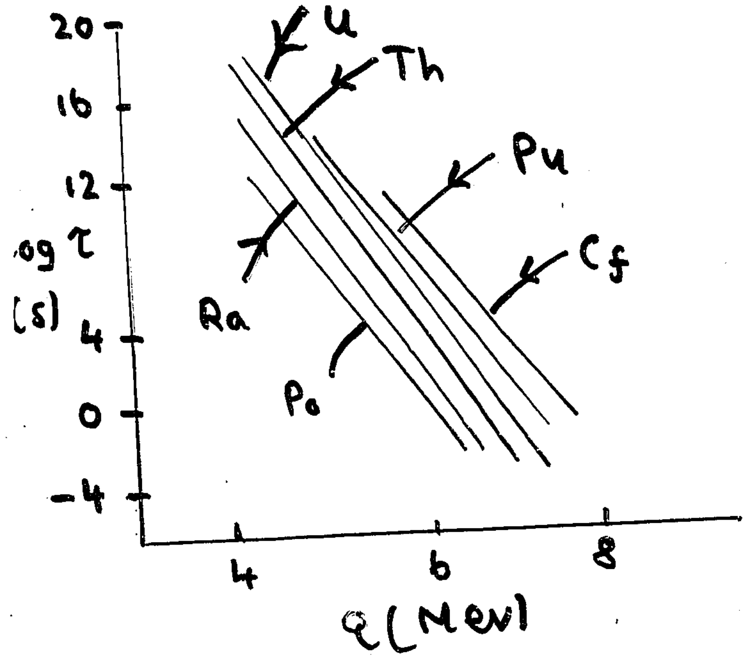
Lifetime decreases <u>drastically</u> with energy

[<u>Many</u> orders of magnitude for comparatively small changes of energy.]

e.g. ^{232}Th, $Q = 4.08$ MeV, $\tau = 6 \times 10^{17}$ s

^{218}Th, $Q = 9.85$ MeV, $\tau = 1.4 \times 10^{-7}$ s

[i.e (broadly) if $Q \lesssim 4$ MeV, decay not observed]



(sketch)

even-even nuclei

[each line joins data for different isotopes of the same atom]

[even-odd, odd-even & odd-odd nuclei show same general behaviour, though with extra features - see below]

Theory of α-decay

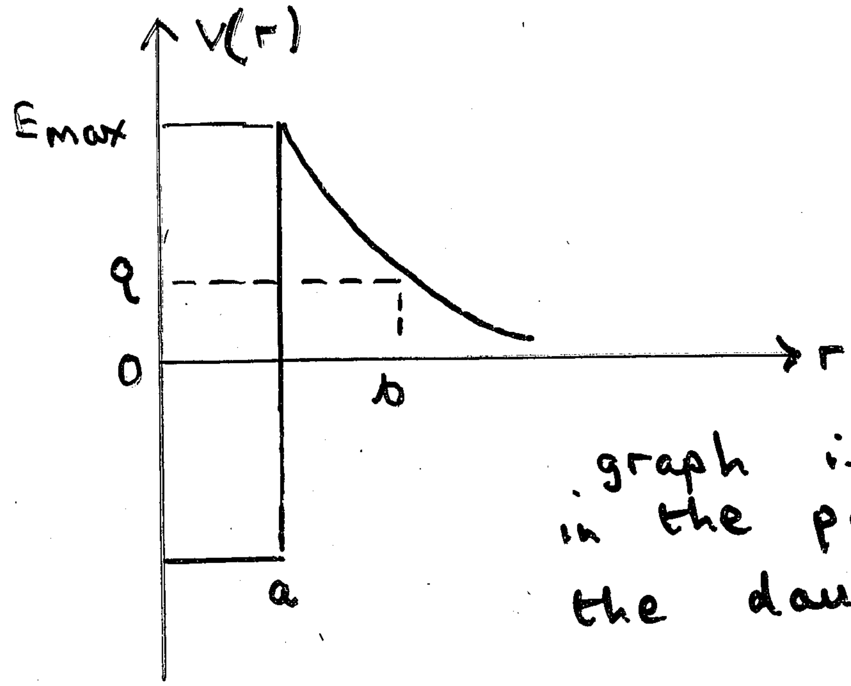
[Must explain Geiger-Nuttall]

Basic theory considers the probability that 2n and 2p become bound, creating an α-particle, and this particle then escapes.

If timescale for α-formation $= \tau_0$ and probability that the α-particle having been formed will escape is P then lifetime $\tau = \frac{\tau_0}{P}$

(We will consider τ_0 later, briefly)

Escape probability P



potential (more or less) spherical

graph is of the α-particle in the potential well of the daughter nucleus

We may say

$a = R_D + R_\alpha$
  └ daughter nucleus

For $r > a$, $V(r) = \dfrac{2 Z_D e^2}{4\pi\varepsilon_0 r}$

For typical heavy nucleus

$\dfrac{e^2}{4\pi\varepsilon_0} = 1.44$ MeV F (Dunlap Appendix A)

$a \sim 6$F , $Z_D \sim 75$

$E_{max} = \dfrac{2 Z_D e^2}{4\pi\varepsilon_0 a} \sim 35$ MeV

Since $Q \ll E_{max}$, classically the α-particle could never escape

Alpha-decay may be explained by QM barrier penetration

Inside the nucleus, $r<a$, $V=$ constant

Solutions are sinusoidal [as for infinite or finite potential well]

Outside the nucleus

The radial part of the Schrödinger equation obeys:

$$-\frac{\hbar^2}{2m}\underbrace{\left[\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right]}_{\text{radial part of } \nabla^2}+\left[\frac{2Z_De^2}{4\pi\varepsilon_0 r}+\frac{\ell(\ell+1)\hbar^2}{2mr^2}\right]R=ER$$

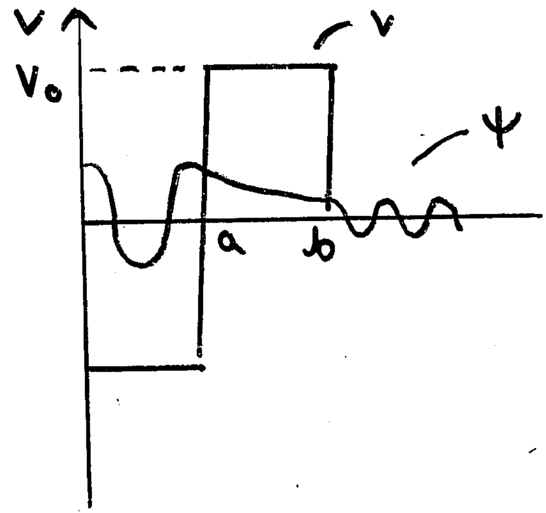
$$m = \text{reduced mass} = \frac{M_\alpha M_D}{M_\alpha + M_D}$$

We define b as the value of r at which the α-particle may exist classically. (see previous diagram.)

Then $$b = \frac{2Z_De^2}{4\pi\varepsilon_0 E}$$

The problem is difficult!

Simplified version



consider $\underline{\ell = 0}$

In this case (from WKB - Wentzel-Kramers-Brillouin theory)

$$R = \frac{u}{r} = \frac{A}{r} \exp\left(-\int F(r')\,dr'\right) \qquad a < r < b$$

where

$$F(r') = \left\{ \frac{2m}{\hbar^2} \left(\underbrace{\frac{2 Z_0 e^2}{4\pi \varepsilon_0 r'}}_{V} - \varepsilon \right)\right\}$$

[Essentially we are putting V equivalent to V_0 in the diagram above integrating from a to b]

Then $$p = \frac{4\pi b^2 |R(b)|^2}{4\pi a^2 |R(a)|^2}$$

$$= \frac{\frac{A^2}{b^2} 4\pi b^2 \exp\left(-2 \int_0^b F(r')\,dr'\right)}{\frac{A^2}{a^2} 4\pi a^2 \exp\left(-2 \int_0^a F(r')\,dr'\right)}$$

$$P = \exp\left(-2 \left\{ \int_a^b F(r')\,dr' \right\}\right).$$

Clearly the value of the integral in the above equation depends markedly on the value of ϵ from the expression for $F(r')$ above.

The exponential factor then means that P will decrease extremely fast as ϵ decreases. $\Rightarrow$ Geiger - Nuttall

We do not yet have an expression for τ since $\tau = \tau_0 P$, and we do not know τ_0.

However, let us consider similar nuclei with similar masses and all even-even. We will assume that τ_0 is a constant for such nuclei

We may start from ^{230}Th and deduce a value of τ_0 from the measured value of τ and the calculated value of P.

[In fact we obtain $\tau_0 = 6.3 \times 10^{-23}$ s]

We then use the <u>same</u> value of τ_0 and calculated values of P to obtain values of τ for other nuclei. These may be compared with experiment with

<u>enormous success</u>.

<u>Parent</u>	<u>Daughter</u>	<u>$\mathcal{E}$(MeV)</u>	<u>τ_{meas}(s)</u>	<u>τ_{calc}(s)</u>
^{238}U	^{234}Th	4.27	2.0×10^{17}	3.0×10^{17}
^{234}U	^{230}Th	4.86	1.1×10^{13}	1.0×10^{13}
^{230}Th	^{226}Ra	4.77	3.5×10^{12}	3.5×10^{12}
			by construction	
^{226}Ra	^{222}Rn	4.87	7.4×10^{10}	6.6×10^{10}
^{222}Rn	^{218}Po	5.59	4.8×10^{5}	3.8×10^{5}
^{218}Po	^{214}Pb	6.11	2.6×10^{2}	1.4×10^{2}
^{214}Po	^{210}Pb	7.84	2.3×10^{-4}	1.0×10^{-4}
^{210}Po	^{206}Pb	5.41	1.7×10^{7}	5.2×10^{5}

Historically this was extremely important as the analysis by Gamow and by Gurney and Condon independently was the first demonstration (1928) that quantum theory could be applied to nuclei.

N.B. For even-even nuclei, decay is almost always to the ground state of the daughter.

This is natural as, for decay to an excited state, Q would be reduced, and because τ is such a sensitive function of Q, it would be very much increased.

α-decay in even-odd, odd-even & odd-odd nuclei

Here τ may be as much as 3 orders of magnitude longer than for even-even nuclei with similar mass and Q.

This is to do with the formation of the α-particle

In even-even nuclei, paired neutrons and protons readily form an α-particle

However in the other types of nucleus, the least-bound n or p (or both) is unpaired, and does not readily take part in the formation of an α-particle.

So in these nuclei, lower lying paired protons and neutrons take part preferentially in formation of the α-particle.

This leaves the daughter nucleus in an excited state, and hence increases the lifetime.

Angular Momentum Considerations

So far we have assumed that $\ell = 0$.

Conservation laws impose restrictions:

α-decay processes are only allowed if total angular momentum and parity are conserved.

With J_D, J_P and ℓ the appropriate quantum numbers for daughter, parent and alpha, we must have

$$J_D + J_P \geq \ell \geq |J_D - J_P|$$

e.g. $\quad {}^{237}Np\ (5/2+) \longrightarrow {}^{233}Pa\ (3/2-) + \alpha$

angular momentum $\quad \ell = 4, 3, 2, 1$

parity $\quad \alpha$ must have odd parity

parity $= (-1)^{\ell}$

$\therefore \ \ell = 3$ or 1

centrifugal potential (or angular momentum barrier)

For $\ell > 0$, the potential barrier that the α-particle must travel through is increased.

Thus the decay constant will be decreased substantially.

With $\lambda(\ell = 0) = 1.0$

ℓ	λ
0	1.0
1	0.7
2	0.37
3	0.137
4	0.037
5	0.0071
6	0.0011

e.g. Rotational levels of heavy nuclei

Branching ratios for the α-decay of $^{244}Cm\,(0^+)$ (even - even) to the ground state and lower excited states of ^{240}Pu

state of ^{240}Pu	Energy (MeV)	Branching Ratio
0^+	0	77 %
2^+	0.043	23 %
4^+	0.142	0.23 %
6^+	0.296	0.036 %

The decrease is the combined result of

(1) Increase of angular momentum barrier

(2) Decrease in α-particle energy

For even-even transitions, the 0^+ to 0^+ is always prefered, as it maximises the transition energy, and minimises the angular momentum term.

For odd-odd transitions, things are not as straightforward.

e.g. $^{243}_{95}Am\left(\frac{5}{2}^-\right)$ to $^{239}_{93}Np$

state of ^{239}Np	Energy (MeV)	Branching Ratio
$5/2^+$	0	0.16%
$7/2^+$	0.031	0.12%
$5/2^-$	0.075	88%
$7/2^-$	0.118	10.6%
$9/2^-$	0.172	1.1%

The $(5/2^-)$ state is preferred because there is no angular momentum barrier $[(5/2^-)$ to $(5/2^-)]$

The lower energy states are ruled out because the change in parity rules out the 0^+ state of the α-particle.

2C Beta Decay

β^--decay

$${}^A_Z X^N \rightarrow {}^A_{Z+1} Y^{N+1} + e^- + \tilde{\nu}_e$$

β^+-decay

$${}^A_Z X^N \rightarrow {}^A_{Z-1} Y^{N+1} + e^+ + \nu_e$$

RHS the same apart from the e^+ in β^+-decay

electron capture

$${}^A_Z X^N + e^- \rightarrow {}^A_{Z-1} Y^{N+1} + \nu_e$$

→ capture of a $\underbrace{\text{K-shell}}_{K=1}$ electron ['K capture']

[the basis of all is $n \rightarrow p$ or $p \rightarrow n$]

energetics

β^--decay

$$Q = \left[m_N({}^A_Z X^N) - m_N({}^A_{Z+1} Y^{N-1}) - m_e \right] c^2$$

Here the mass of the $\tilde{\nu}_e$ is assumed to be irrelevant, and it is also assumed that the transition is between ground states.

We may translate to atomic masses, using

$$m({}^A_Z X^N) = m_N({}^A_Z X^N) + Z m_e$$

When we do this, we ignore electron binding energy because it is actually the difference between the binding energy of the initial and final nucleus that is relevant, and this is negligible.

Then

$$Q = \left[m\left({}^{A}_{Z}X^{N}\right) - Zm_e - m\left({}^{A}_{Z+1}Y^{N-1} + (Z+1)m_e - m_e\right]c^2$$

$$= \left[m\left({}^{A}_{Z}X^{N}\right) - m\left({}^{A}_{Z+1}Y^{N-1}\right)\right]c^2$$

β^+ - decay

$$Q = \left[m_N\left({}^{A}_{Z}X^{N}\right) - m_N\left({}^{A}_{Z-1}Y^{N-1}\right) - m_e\right]c^2$$

$$= \left[m\left({}^{A}_{Z}X^{N}\right) - Zm_e - m\left({}^{A}_{Z-1}Y^{N-1}\right) + (Z-1)m_e - m_e\right]c^2$$

$$= \left[m\left({}^{A}_{Z}X^{N}\right) - m\left({}^{A}_{Z-1}Y^{N+1}\right) - 2m_e\right]c^2$$

electron capture

In this case, as well as the binding energy terms that approximately cancel, we have the binding energy, b_c, of the captured electron. As this is usually an inner shell electron ($n=1$), it is of order 10^{-2} MeV, and so is appreciable enough to be included.

Then

$$Q = \left[m_N \left({}^{A}_{Z}X^{N} \right) + m_e - \frac{b_c}{c^2} - m_N \left({}^{A}_{Z-1}Y^{N+1} \right) \right] c^2$$

$$= \left[m\left({}^{A}_{Z}X^{N} \right) - Zm_e + m_e - \frac{b_c}{c^2} - m\left({}^{A}_{Z-1}Y^{N+1} \right) + (Z-1)m_e \right] c^2$$

$$= \left[m\left({}^{A}_{Z}X^{N} \right) - m\left({}^{A}_{Z-1}Y^{N+1} \right) \right] c^2 - b_c$$

When Q is positive, the process is exothermic and can proceed spontaneously.

When Q is negative, the process is endothermic and is not energetically favourable.

β^+-decay/electron capture

We see that Q is larger for electron capture than for β^+-decay.
There are nuclei for which Q is positive for the first, but negative for the second.
$\therefore$ The first process occurs but the second does not.

[N.B. $m_e c^2 = 0.51$ MeV]

Energy distribution

The difference in the energy of the initial and final nucleus for β^- or β^+-decay is fixed, and is equal to the sum of

(a) $M_e c^2$ – known and obviously > 0.

(b) $M_\nu c^2$ – for a long time suspected to be exactly zero, but now known to be > 0 but very small ($\sim$ eV)

(c) E_e

(d) E_ν [or $E_{\bar{\nu}}$]

$E_e + E_\nu$ must equal $(\underbrace{\Delta m}_{\text{nuclei}} - m_e - m_\nu)c^2$,

but each of E_e will E_ν vary.

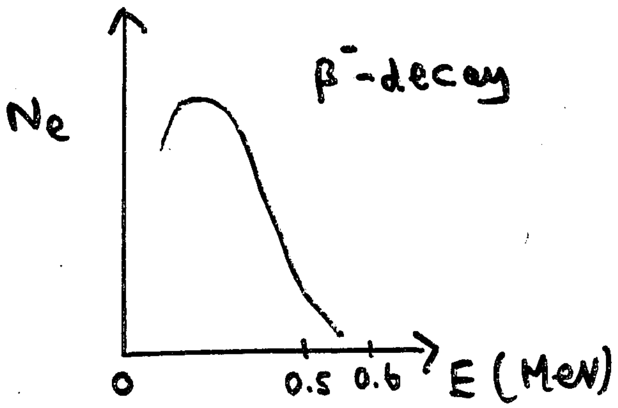
Observationally, of course, we detect only the e and measure only E_e.

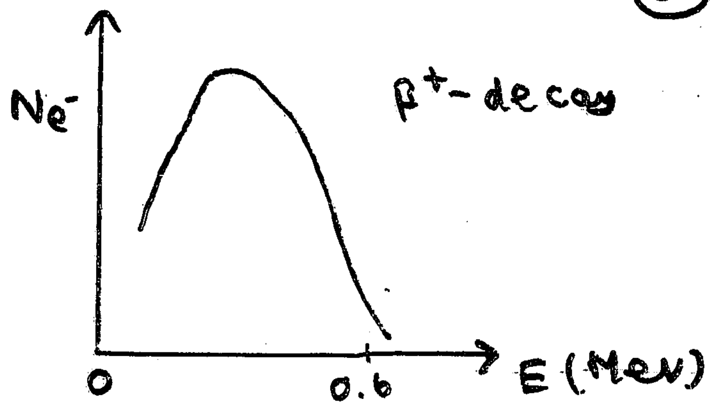
Historically it was the fact that E_e varied from 0 to $E_e(max)$ that led Pauli to hypothesise the existence of the neutrino.

It was the fact that $E_{e(max)}$ was, within experimental error, equal to $\Delta E - m_e c^2$ that showed that m_ν was (at most) extremely small, and convinced nearly everybody that it must be zero.

[N.B. In the above we have ignored the recoil energy of the daughter nucleus. This is because, as we saw in the previous section, it is $\sim \frac{m_e}{M_N} \times E_e$ (ignoring, for the moment the neutrino, which, if its mass was 0, would need to be treated relativistically in mechanics).
This amount of energy might be $\sim 10^{-5} \times E_e$ and is well within any experimental uncertainty.]

We look at the case of $^{64}_{29}Cu$ which interestingly can decay to $^{64}_{30}Zn$ by β^--decay, and to $^{64}_{28}Ni$ by β^+-decay.





E_e (max) = endpoint energy
= 0.579 MeV
(from atomic masses)

E_{e^-} (max)
= 0.653 MeV
(from atomic masses)

[The precise shape of the curve around $E_e = 0$ and $E_e = E_e(\max)$ also indicates that the neutrino mass is extremely small.]

Fermi Theory of Beta Decay

[To attempt to explain the above]

The decay rate for a decay process is the transition rate from the initial to the final state.

Time dependent perturbation theory gives the result for transitions between states (Fermi's golden rule) as

$$\lambda = \frac{2\pi}{\hbar} |H_{if}|^2 \frac{dn}{dE}$$

where

$\frac{dn}{dE}$ = density (in Energy) of final states

H_{if} = a matrix element involving the

interaction operator, H

$$H_{if} = G \int \psi_f^* \, H \, \psi_i \, d^3 \underline{r}$$

where

ψ_i and ψ_f are the wave-functions of the initial and final states

G is a constant representing the strength of the interaction.

(β^--decay)

wave-function of the initial state is that of the parent nuclear state P

wave-function of the final state is that of the daughter nuclear state D, the electron and the anti-neutrino.

so

$$H_{if} = G \int \psi_D^* \psi_e^* \psi_{\bar{\nu}}^* \, H \, \psi_P \, d^3 \underline{r}$$

(similarly for β^+-decay)

ψ_e and $\psi_{\bar{\nu}}$ may be taken to be plane waves (as they are free particles)

For convenience they may be normalised over the nuclear volume, V

$$\left.\begin{aligned} \psi_e(\underset{\sim}{r}) &= \frac{1}{\sqrt{V}}\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{r}/\hbar} \\ \psi_{\bar{\nu}}(\underset{\sim}{r}) &= \frac{1}{\sqrt{V}}\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}/\hbar} \end{aligned}\right\}$$

where p are q the momenta of e and $\bar{\nu}$

The exponentials may be expanded:

$$e^{i\underset{\sim}{p}\cdot\underset{\sim}{r}/\hbar} = 1 + \frac{i}{\hbar}\,\underset{\sim}{p}\cdot\underset{\sim}{r} \ldots$$

(and similarly for $\bar{\nu}$)

For typical β-decay energies, $p \sim 1\,\mathrm{MeV}/c$

" r about the nuclear radius

$$\frac{pr}{\hbar} \sim \frac{1\,\mathrm{MeV}/c \times 7\,\mathrm{fm}}{2\times10^{2}\,\mathrm{MeV\,fm}} \sim 0.03$$

∴ We may take the leading term in the exponentials i.e they both equal 1

so

$$H_{if} = G M_{if}/V$$

where

$$M_{if} = \int \psi_D^{*}\, H\, \psi_P\, d^3\underset{\sim}{r}$$

Density of final states

free-particle Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

For a 3D infinite well in Cartesians, momentum states are categorised as

$$p_x = \frac{n_x \pi\hbar}{L}\ ;\ p_y = \frac{n_y\pi\hbar}{L}\ ,\ p_z = \frac{n_z\pi\hbar}{L}$$

(n_x, n_y, n_z all positive

i.e. density of states along each Cartesian direction $= L/\pi\hbar$ (x, y, z positive)

Density in 3D $= (L/\pi\hbar)^3 = \dfrac{V}{\pi^3\hbar^3}$

$\therefore$ Total no. of allowed states with momentum less than p

= Vol. in p-space $\times \frac{1}{8}$ [because we are restricted to x, y, z being positive

$\times$ density of states

$$= \left(\frac{4}{3}\pi p^3\right) \times \left(\frac{1}{8}\right) \times \frac{V}{\pi^3\hbar^3}$$

$$= \frac{4/3\,\pi p^3 V}{(2\pi\hbar)^3} = n_e$$

$$\frac{dn_e}{dp} = \frac{4\pi V p^2}{(2\pi\hbar)^3} \; ; \qquad dn_e = \frac{4\pi V p^2}{(2\pi\hbar)^3}\, dp$$

similarly

$$\frac{dn_{\tilde{\nu}}}{dq} = \frac{4\pi V q^2}{(2\pi\hbar)^3} \; ; \qquad dn_{\tilde{\nu}} = \frac{4\pi V q^2}{(2\pi\hbar)^3}\, dq$$

From equation for λ, we may write
(p.89)

$$d\lambda = \frac{2\pi}{\hbar} \frac{G^2 |M_{if}|^2}{V^2} \frac{dn_{\tilde{\nu}_e}\, dn_e}{dE_f}$$

$$= \frac{2\pi}{\hbar} \frac{G^2 |M_{if}|^2}{V^2} \frac{16\pi^2 p^2 q^2}{64\pi^6 \hbar^6} \frac{dp\, dq}{dE_f} V^2$$

$$= \frac{G^2 |M_{if}|^2 p^2 q^2}{2\pi^3 \hbar^7} \frac{dp\, dq}{dE_f}$$

[The V factor, which is somewhat arbitrary has conveniently cancelled.]

What we require is not so much λ as $d\lambda/dp$ - this will give us the spectrum of electron energies which is directly observable.

$$\frac{d\lambda}{dp} = \frac{G^2 |M_{if}|^2 p^2 q^2}{2\pi^3 \hbar^7} \frac{dq}{dE_f}$$

Since q and dq/dE_f are not measurable, we must express them in terms of measurable quantities.

N.B. We will ignore the mass of the neutrino since it is extremely small. Thus we treat it as a purely relativistic particle,

$E_\nu = qc$.

momentum of the neutrino here

Then $E_f = Q + m_e c^2$ (cap)

where $Q = T_e + qc$ (cap)

So $E_f = T_e + m_e c^2 + qc$

With T_e, $m_e c^2$ constant,

$$\frac{dq}{dE_f} = \frac{1}{c}$$

and $$q = \frac{Q - T_e}{c}$$

Then $$\frac{d\lambda}{dp} = \frac{G^2}{2\pi^3 \hbar^7 c^3} |M_{if}|^2 p^2 (Q - T_e)^2$$

We require $\frac{d\lambda}{dT_e} = \frac{d\lambda}{dp} \frac{dp}{dT_e}$

Now

$$T_e = E_{rel} - m_e c^2$$
$$= \sqrt{p^2c^2 + m_e^2 c^4} - m_e c^2$$
$$\therefore \quad p^2c^2 + m_e^2 c^4 = (T_e + m_e c^2)^2$$
$$= T_e^2 + 2T_e m_e c^2 + m_e^2 c^4$$

$$p^2c^2 = T_e^2 + 2T_e m_e c^2$$
$$\text{ll } p^2 = \frac{T_e^2 + 2T_e m_e c^2}{c^2}$$

differentiating

$$2p\,dp = \frac{1}{c^2}(2T_e + 2m_e c^2)$$
$$dp = \frac{1}{c^2 p}(T_e + 2m_e c^2) \leftarrow \text{N.B.}$$

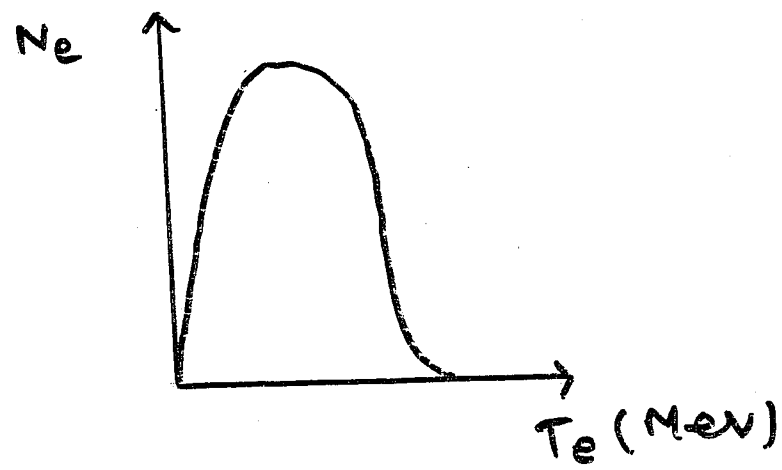
$$\therefore \frac{d\lambda}{dp} = \frac{G^2}{2\pi^3 \hbar^7 c^3} |M_{if}|^2 (\mathcal{E} - T_e)^2 (T_e + 2m_e c^2) \frac{p}{c^2}$$

$$= \frac{G^2}{2\pi^3 \hbar^7 c^6} |M_{if}|^2 (\mathcal{E} - T_e)^2 (T_e + 2m_e c^2)(T_e^2 + 2T_e m_e c^2)^{1/2}$$

Comparison with experiment

For simplicity we may assume M_{if} is independent of T_e. (This is reasonable in many cases.)

Then we can calculate an energy spectrum:



So far this should be correct for electron and positron emission.

In practice

(1) It does not agree with either.

(2) They themselves differ.

The reason is that the departing e^-/e^+ is attracted/repelled by the positive charge on the nucleus.

Thus the electron spectrum is pulled to lower energies, while the positron spectrum is pushed to higher energies.

[The low energy section is augmented for electrons, diminished for positrons.]

Mathematically we may handle this by multiplying our expression for $\frac{d\lambda}{dp}$ by a factor $F(Z_D, T_e)$ known as a Coulomb factor or Fermi function

good at low energies where the effect is most important

A non-relativistic calculation gives

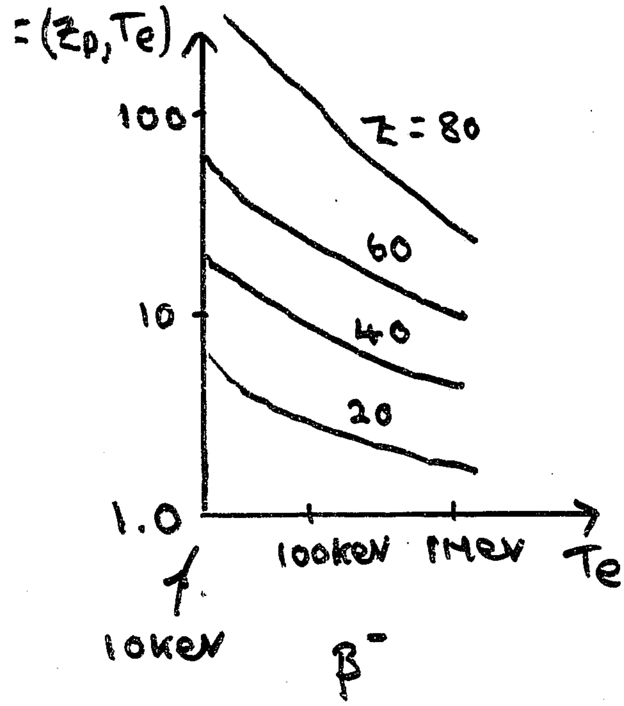
$$F(Z_D, T_e) = \frac{2\pi\eta}{1 - \exp(-2\pi\eta)}$$

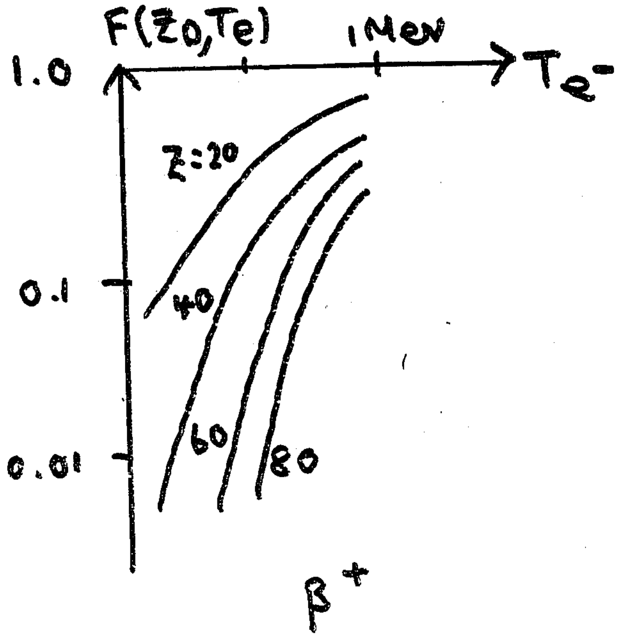
where

$$\eta = \pm \frac{Z_D e^2}{4\pi\varepsilon_0 \hbar v} \quad \begin{cases} \beta^- \\ \beta^+ \end{cases}$$

Here

v = velocity of β^-/β^+





Fermi-Kurie Plots

The theory may be checked as follows

We may write

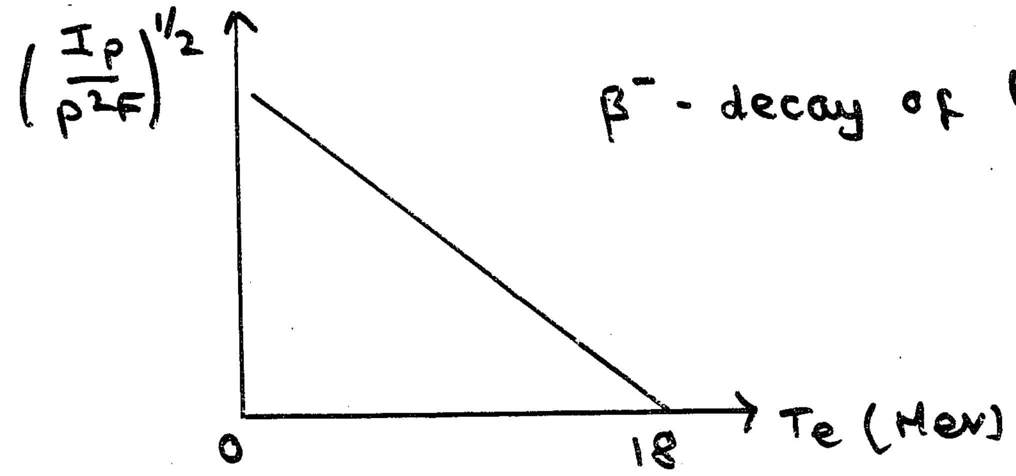
$$I(p) = \frac{d\lambda}{dp} = \frac{G^2}{2\pi^3 \hbar^7 c^3} |M_{if}|^2 p^2 (\mathcal{E} - T_e)^2 F(Z_D, T_e)$$

N.B.

If we plot $\left(\frac{I_p}{p^2 F}\right)^{1/2}$ against T_e, we should obtain a straight line. The intercept on the T_e axis should be $\mathcal{E}$.

If the decay is to an excited state of the daughter nucleus, this must be taken into account.

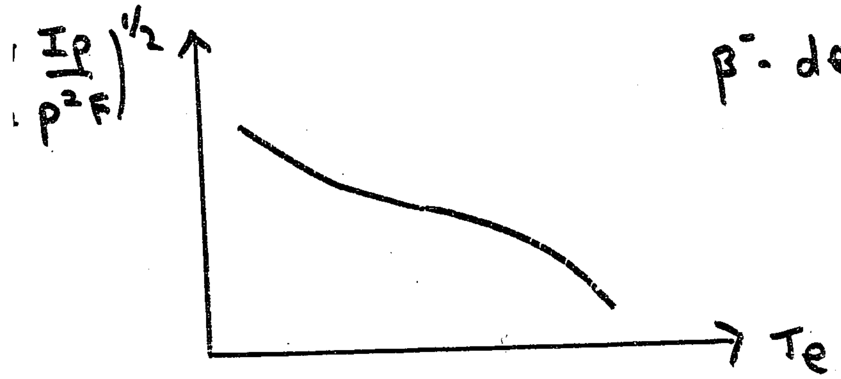
β^- - decay of tritium (^3H)



Transitions which give a straight line on the Fermi-kurie plot are called allowed transitions [very confusing terminology!]

The ~~transition~~ plotted below is called a first-forbidden ~~transition~~ as it does not give a straight line.

β^- - decay of $^{91}_{39}Y$



We may attempt to explain this effect by including extra terms in the wave-functions of β^- and ν

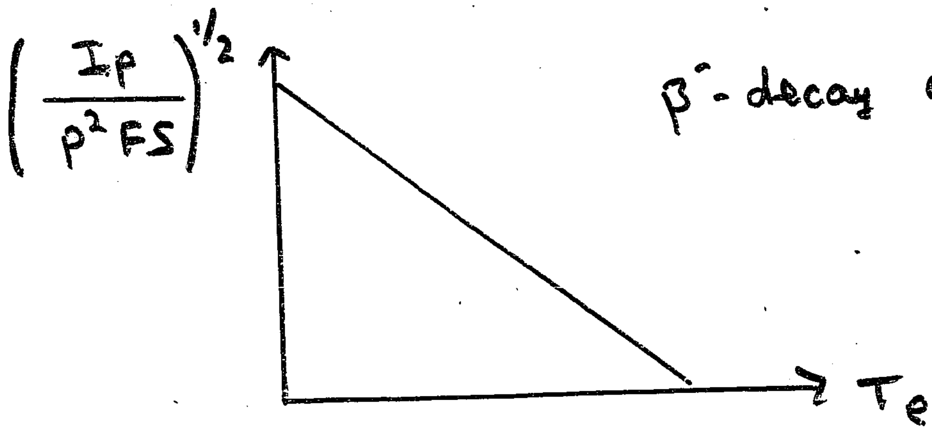
$$e^{i\underline{k}\cdot\underline{r}/\hbar} = 1 + \underbrace{i\underline{k}\cdot\underline{r}/\hbar}_{\lambda} \ldots$$

and analogously for ν

To cater for this we may multiply I_p by a further factor, a <u>shape factor</u>

$$S(p,q) \sim (p^2 + q^2)$$

Then we get



β^--decay of ${}^{91}_{39}Y$

<u>Mass of neutrino</u>

Return to $d\lambda/dp$, evaluated for $m_\nu = 0$. This agrees well with experiment, especially for behaviour at $E=0$ & $E=E_{max}$

We may include a non-zero m_ν in the calculations, and obtain different behaviour. This demonstrates that m_ν is either exactly zero or of order eV.

Allowed and Forbidden Transitions

We must consider conservation of angular momentum.

The elimination of the higher terms in the expansion of $e^{i\underline{p}\cdot\underline{r}/\hbar}$ and $e^{i\underline{q}\cdot\underline{r}/\hbar}$ is justified in the limit as $r \to 0$.

In this limit, the β^- and $\tilde{\nu}$ may be viewed as being created at $r = 0$, & so carry no orbital angular momentum.

$\therefore$ Total angular momentum of the $\beta^-, \tilde{\nu}$ pair is therefore the resultant of their individual spins.

$$S_e = 1/2 \,,\; S_\nu = 1/2 \;\therefore\; S = (S_e + S_\nu) \ldots |S_e - S_\nu| \ldots 0$$
$$= 1, 0$$

$S = 0$ is called a <u>Fermi decay</u>

$S = 1$ " " " <u>Gamow - Teller decay</u>

By conservation of angular momentum

$J_P = J_D + S \ldots |J_D - S|$ <u>Gamow - Teller</u>

<u>If $S=1$</u> } $J_P = J_D + 1, J_D, J_D - 1$

$$\Delta J = J_D - J_P = 1, 0, -1$$

[exception If $J_D = 0$, $J_P = S = 1$
$\Delta J = -1$]

OR If $J_P \geq 1$, $\Delta J = 1, 0, -1$
" $J_P = 0$, $\Delta J = 1$ Gamow - Teller

If
$S=0$ } $J_P = J_D$
If $J_P = 0, \geq 1$, $\Delta J = 0$
Fermi

Parity

Parity of $(e^-, \bar{\nu})$ pair $= (-1)^{L \to 0} =$ even
$\therefore$ No parity change for nuclei

All the above is for allowed transitions

Forbidden decays

'Forbidden decays' are not really forbidden, but are less probable than 'allowed decays' and so correspond to larger values of τ.

If the matrix-element as calculated above is equal to zero, higher order terms must be included in the expansion of $e^{i\underline{p}\cdot\underline{r}/\hbar}$ and $e^{i\underline{q}\cdot\underline{r}/\hbar}$. This is a <u>first forbidden decay</u>

If the result is still zero, the next terms again must be included. This will be a <u>second forbidden decay</u>. And so on.

For each type of forbidden decay, an appropriate shape factor must be included in a Fermi-Kurie plot

<u>decay</u>	<u>$S(p,q)$</u>
1st forbidden	$(m_0c)^{-2}\,(p^2+q^2)$
2nd "	$(m_0c)^{-4}\,[p^4+q^4+(10/3)\,p^2q^2]$
3rd "	$(m_0c)^{-4}\,[p^6+q^6+7p^2q^2(p^2+q^2)]$

<u>parity</u>

The most obvious case where a first-forbidden transition can occur is where there is a change of parity between parent and daughter nuclei.

It this case, the electron-antineutrino pair must have odd parity.

Since parity $= \pi = (-1)^L$, L cannot be zero.

Thus r cannot be approximated to zero, and higher order ~~terms~~ must be included.

first-forbidden decays

$L(e^-, \bar{\nu}$ pair$) = 1$, $\Delta J = 0, \pm 1, \pm 2$

change in parity of nucleus.

continuing ...

Decay	L	ΔJ	nuclear parity change
2nd forbidden	2	$\pm 1, \pm 2, \pm 3$	no
3rd "	3	$\pm 2, \pm 3, \pm 4$	yes
4th "	4	$\pm 3, \pm 4, \pm 5$	no

etc.

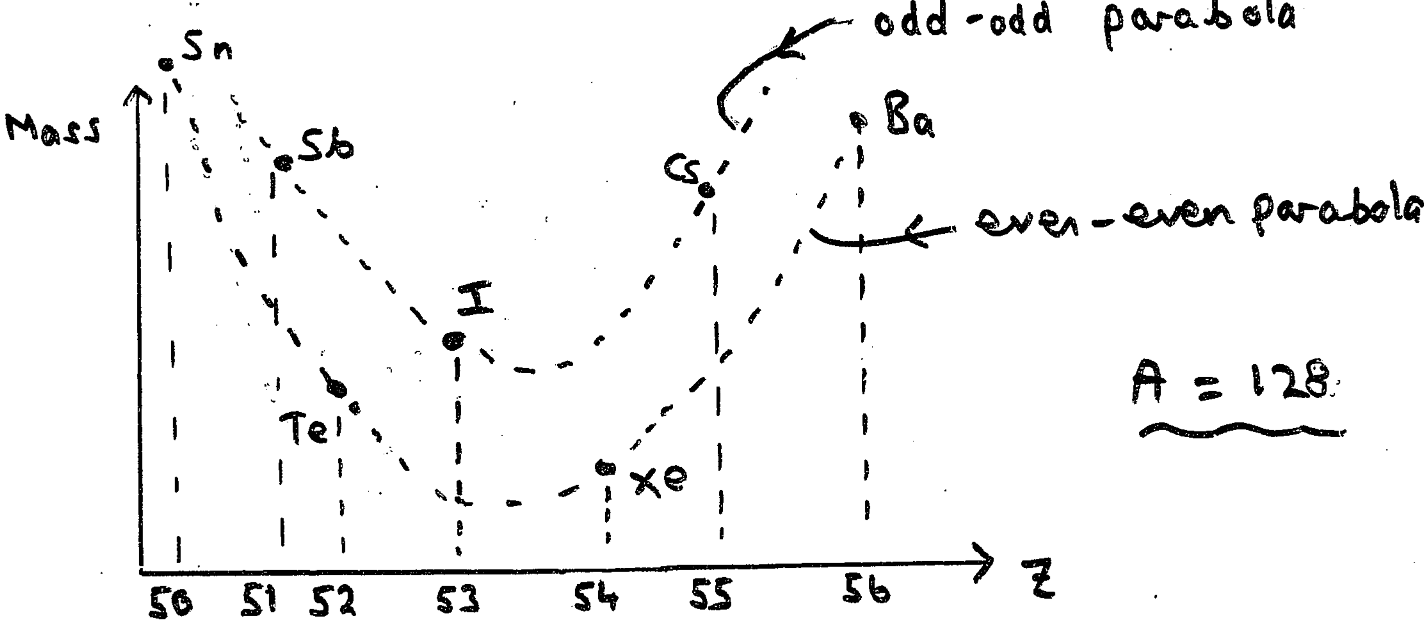
Parity Violation in Beta Decay

A very surprising discovery in beta decay (c. 1958) was that parity was not conserved in some cases.

We shall consider this in the work on Particle Physics.

Double Beta Decay

There are situations where a nucleus may be stable to β^- or β^+-decay, but potentially unstable to <u>double beta-decay</u>



$^{128}_{52}Te$ will not decay to $^{128}_{53}I$ or $^{128}_{51}Sb$.

But it is energetically favourable for it to decay to $^{128}_{54}Xe$ by double β-decay

$$^{128}_{52}Te \longrightarrow {}^{128}_{54}Xe + 2e^- + 2\tilde{\nu}_e$$

(see below)

Here $Q = 0.876\,\text{MeV}$ and the barrier height $= 1.33\,\text{MeV}$

Lifetimes will be long enough that observation must be extremely difficult.

There are many candidates, all from an <u>even-even</u> nucleus to another <u>even-even</u> nucleus. ($0^+ \to 0^+$ if ground states are involved)

The <u>barrier height</u> is the energy barrier formed by the intervening <u>odd-odd</u> nucleus.

($^{128}_{53}I$ in this case)

experimental evidence for double beta decay

From geological observations of rock samples. It will be observed that an xs of the daughter nuclide will be found compared to other isotopes of that element.
[Analysis along the lines of radioactive dating]

decay	Lifetime (years)	decays/year/g
$^{130}_{52}Te \to ^{130}_{54}Xe$	3.2×10^{21}	0.503
$^{128}_{52}Te \to ^{128}_{54}Xe$	5.0×10^{24}	0.00029
$^{82}_{34}Se \to ^{82}_{36}Kr$	3.7×10^{20}	1.8

laboratory experiments

Many are taking place - very difficult because of very long lifetimes.

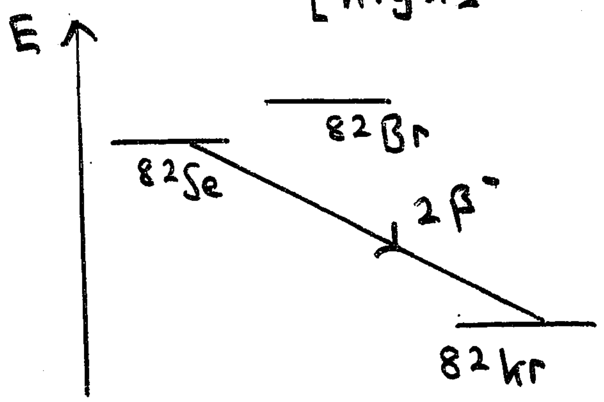
To avoid background radiation from, for example, cosmic rays, many are performed in mines. [see experiments to study the possible decay of the proton under Grand Unified Theories (GUTs) at the end of the course.]

an interesting example

$$^{82}_{34}Se \longrightarrow \, ^{82}_{36}Kr$$

This is interesting because

$Q = 3.00$ MeV ; barrier energy $= 0.09$ MeV
[high] [low]



Double beta-decay observed

Energy spectrum agrees (quite well) with calculated spectrum for double beta-decay

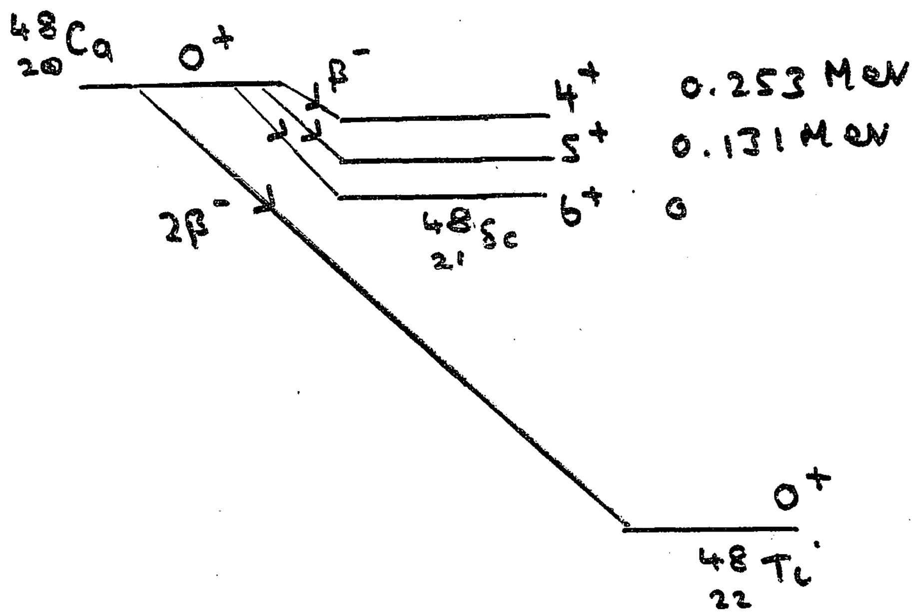
Decay rate agrees (broadly) with geological observations

another interesting possibility

Another interesting situation is where single β-decay is energetically possible but the spins and parities of parent and daughter nuclei make the transition <u>highly forbidden</u>.

If a double β-decay is allowed, this might occur in preference.

e.g. $^{48}_{20}Ca$ to $^{48}_{22}Ti$



Energy for transition to ^{48}Sc ground state = 0.28 MeV
6th forbidden

to 1st excited state = 0.15 MeV
5th forbidden

to 2nd excited state = 0.03 MeV
4th forbidden

decay-rates expected to be low

A possible alternative is the higher energy allowed double β-decay to ^{48}Ti.

This is allowed $(0^+ \rightarrow 0^+)$ and $\varepsilon = 4.27$ MeV

Experiments have been performed and values obtained for the energy spectrum agree broadly with calculations.

2D Gamma Decay

Energetics

A nucleus in an excited state can decay to a lower energy state by gamma emission or internal conversion.

Such excited states can readily be formed in the daughter nucleus of an α- or β- decay process, so we may expect to see chains of decays including α-, β- and γ- processes.

Excited states may also be formed by nuclear reactions. [See next section.]

Internal conversion is a process in which γ-decay energy liberates an atomic electron rather than producing a photon (γ). It is analogous to an Auger process in atomic transitions.

Transition processes involving protons may naturally produce radiation.
So may processes involving neutrons because, although the neutron does not have an electric charge, it does have a magnetic moment.

In fact many nuclear transitions are not between single particle states, but between multiple excitation states, rotational states or vibrational states.

We have

$$M_i c^2 = M_f c^2 + E_\gamma + E_R$$

where

E_γ = energy of the γ-ray
E_R = recoil energy of the nucleus

Conservation of momentum gives

$$P_R = E_\gamma / c$$

So $$E_R = \frac{P_R^2}{2M_f} = \frac{E_\gamma^2}{2M_f c^2}$$

[non-relativistic because E_R is small]

With $E_\gamma \sim 1$ MeV, $A = 100$

$$E_R \sim 5 \text{ eV}$$

This is much smaller than γ-ray energies, & within the accuracy of γ-ray experiments.
So we may ignore, and we use

$$E_\gamma = (M_i - M_f) c^2$$

However it should be noted that the γ-ray is reduced by this amount, and it is much larger than the typical Heisenberg width of an excited state.

$\therefore$ Consideration of the nuclear recoil energy can be very important for γ-resonance experiments such as Mössbauer effect spectroscopy.

[In resonance experiments, 2 frequencies must be exactly the same.]

Classical Theory of Radiative Processes

A simple model of a γ-ray process can be derived on the basis of classical electrodynamics.

A static electric field is produced by a distribution of charges, and the charge distribution can be described in terms of a multipole expansion - monopole [simple charge], dipole, quadrupole, octupole, etc

A time varying distribution of charges produces a time varying electric field and this gives rise to the emission of radiation.

When the time variation is periodic (e.g. sinusoidal) a radiation field at the same frequency is produced.

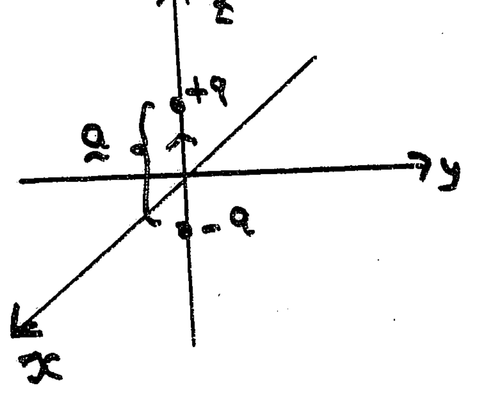
Like the static case, this radiation field may be described in terms of a multipole expansion.

e.g. Consider an electric dipole

~~A simple model~~ consists of equal +ve and -ve charges separated by a distance $\underset{\sim}{a}$.

This constitutes an electric dipole moment

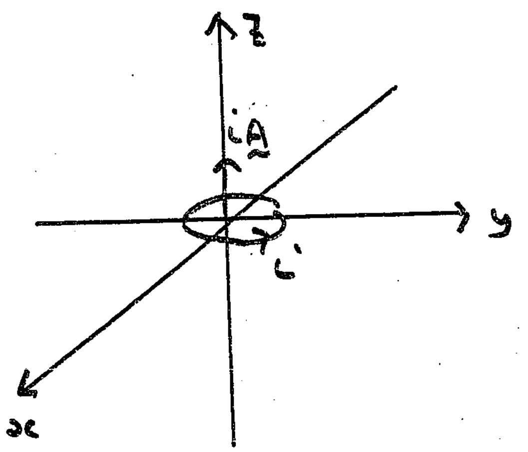
$\underset{\sim}{d} = q\,\underset{\sim}{a}$ along the z-axis



$\uparrow \underset{\sim}{d} = q\,\underset{\sim}{a}$

Similarly we may set up a multipole expansion of the magnetic moments of the system.

These moments can be modelled on the basis of currents, and the simplest case of the magnetic dipole moment may be thought of as arising from a single loop of current.



current loop in xy plane
current = i
normal area = $\underset{\sim}{A}$ in z-direction

magnetic dipole moment
$= \underset{\sim}{\mu} = i\underset{\sim}{A}$

Then if current is varied sinusoidally

$$\underset{\sim}{\mu}(t) = i\underset{\sim}{A}\sin\omega t$$

and there is a radiation field at frequency ω

The basic properties of the radiation fields may be obtained from classical electrodynamics;

total radiated power

electric dipole $\quad P_e = \dfrac{1}{12\pi\varepsilon_0} \dfrac{\omega^4 d^2}{c^3}$

magnetic dipole $\quad P_m = \dfrac{\mu_0}{12\pi} \dfrac{\omega^4 \mu^2}{c^3}$

higher order multipoles

For more general charge/current distributions, higher order multipole moments exist

e.g. 4 charges may be used to construct an electric quadrupole moment

$+q$ ● ● $-q$

$-q$ ● ● $+q$

The order of the moment is defined as

$L = 1$ dipole

$L = 2$ quadrupole

$L = 3$ octopole etc

The power radiated by an electric multipole moment of order L oscillating at frequency ω is:

$$P_e(L) = \frac{2(L+1)c}{\varepsilon_0 L\left[(2L+1)!!\right]^2} \left(\frac{\omega}{c}\right)^{2L+2} Q_L^{\,2}$$

where
see below
Q_L is the generalised electric multipole moment of order L

e.g. $6!! = 6 \times 4 \times 2$, $7!! = 7 \times 5 \times 3 \times 1$ } etc

Inseting $L=1$ into formula for $P_e(L)$

$$\Rightarrow P_e(1) = \frac{4c}{\varepsilon_0 \times 9} \left(\frac{\omega}{c}\right)^4 d^2$$

$$= \frac{4}{9\varepsilon_0} \left(\frac{\omega^4 d^2}{c^3}\right)$$

N.B. The constant term is different from above because the generalised moment is defined slightly differently from the simple definition given above.

Similarly

$$P_m(L) = \frac{2(L+1)\mu_0 c}{L\,[(2L+1)!!]^2} \left(\frac{\omega}{c}\right)^{2L+2} M_L^2$$

where M_L is the generalised magnetic multipole moment of order L

parity

The parity of the radiation field may be understood by investigating the effect of the transformation $\underset{\sim}{r} \rightarrow -\underset{\sim}{r}$ [or $x \rightarrow -x$, $y \rightarrow -y$, $z \rightarrow z$] for the corresponding multipole moment.

electric dipole moment

$\underset{\sim}{d} \rightarrow -\underset{\sim}{d}$

radiation field has odd parity (-1)

magnetic dipole moment

reflection in x-axis and y-axis yields no change in $\underset{\sim}{\mu}$

$\therefore$ radiation field has even parity $(+1)$

electric quadrupole moment

no change $\therefore$ even parity

In general, the parity of an electric multipole radiation field of order L is

$$\pi_L = (-1)^L$$

and for a magnetic multipole field

$$\pi_L = (-1)^{L+1}$$

The radiation produced by an electric dipole of order L is written as EL, and that produced by a magnetic dipole of order L is written as ML

e.g. electric dipole E1
" quadrupole E2
" octupole E3
" hexadecapole E4
(similarly M1, M2, M3 ...)

we now need to relate these ideas to a quantum mechanical context.

Quantum Mechanical Description of Gamma Decay

Quantum mechanically, the radiation given off by a varying electric or magnetic field is in the form of photons with energy $E_\gamma = \hbar\omega$.

So decay constants are related to the power radiated by

$$\lambda_e(L) = \frac{P_e(L)}{\hbar\omega} \; ; \quad \lambda_m(L) = \frac{P_m(L)}{\hbar\omega}$$

Then we may use our previous expressions, replacing <u>multipole moments</u> by appropriate <u>multipole operators</u>

So

$$\lambda_e(L) = \frac{2(L+1)c}{\varepsilon_0 L[(2L+1)!!]^2}\left(\frac{E_\gamma}{\hbar c}\right)^{2L+2}\frac{1}{\hbar\omega}\,|\mathcal{E}_{if}(L)|^2$$

$$= \frac{2(L+1)}{\varepsilon_0 \hbar L[(2L+1)!!]^2}\left(\frac{E_\gamma}{\hbar c}\right)^{2L+1}|\mathcal{E}_{if}(L)|^2$$

where

$$\mathcal{E}_{if}(L) = \int \psi_f^* \, \mathcal{E}(L)\, \psi_i \, d^3\underset{\sim}{r}$$

electric multipole operator

The electric multipole operator is $e r^L Y^*_{L,M}$

where $Y^*_{L,M}$ is a spherical harmonic determined by ψ_i and ψ_f

In fact we assume

$$\psi_i = R_i(r)\, Y_{LM}(\theta, \phi)$$

$$\psi_f = R_f(r)$$ ← i.e. no angular dependence an s state

and we assume the radial parts of the wave-functions, $R_i(r)$ and $R_f(r)$, are constant over the nuclear volume and zero outside the nucleus.

These (rough) estimates are called the <u>Weisskopf estimates</u>

Then we may write

$$\varrho_{if}(L) = e \int_0^{R_0} r^L \, d^3\underset{\sim}{r} \Big/ \int_0^{R_0} d^3\underset{\sim}{r}$$

But $d^3\underset{\sim}{r} = 4\pi r^2 dr$, so

$$\varrho_{if}(L) = e \int_0^{R_0} r^{L+2} dr \Big/ \int_0^{R_0} r^2 dr$$

$$= e \left[\frac{r^{L+3}}{L+3} \right]_0^{R_0} \Big/ \left[\frac{r^3}{3} \right]_0^{R_0}$$

$$= e \, \frac{R_0^{\,L+3}}{L+3} \Big/ \frac{R_0^{\,3}}{3} = e \, \frac{3}{L+3} R_0^{\,L}$$

Then, using

$$R_0 = R_1 A^{1/3}$$

where

$$R_1 = 1.2 \text{ fm}$$

[from general ideas of nuclear structure]

$$\lambda_e(L) = \frac{2e^2(L+1)}{\varepsilon_0 \hbar L \left[(2L+1)!!\right]^2} \left[\frac{3}{L+3} \right]^2 \left(\frac{E_\gamma}{\hbar c} \right)^{2L+1} R_1^{\,2\ell} A^{2\ell/3} .$$

Similarly we may define a <u>magnetic multipole operator</u>

$$M_{if}(L) = \int \psi_f^* \, M(L) \, \psi_i \, d^3\underset{\sim}{r}$$

$M(L)$ is proportional to $r^L Y_{LM}^*(\theta,\phi) \nabla.(\underset{\sim}{r} \times \underset{\sim}{j})$ where $\underset{\sim}{j}$ is the current density inside the nucleus.

It is also necessary to add a term to account for the intrinsic spin of any unpaired nucleon.

Then, making use of the <u>weisskopf estimates</u>, we obtain

$$\lambda_m(L) = \frac{20 e^2 \hbar (L+1)}{\varepsilon_0 c^2 m_p^2 L [(2L+1)!!]^2} \left[\frac{3}{L+3}\right]^2 \left(\frac{E_\gamma}{\hbar c}\right)^{2L+1} R_1^{2L-2} A^{\frac{2L-2}{3}}$$

Numerically, for any L

$$\lambda_m(L) = 0.308 \, A^{-2/3} \lambda_e(L)$$

i.e. (roughly)

$\lambda_m(L)$ 2 orders of magnitude $<< \lambda_e(L)$

Also for a given transition energy there is a substantial decrease in λ with increasing L.

Selection Rules

These determine which transitions actually take place.

Conservation of Angular Momentum

$$J_i = J_f + L, \ \ldots\ldots \ |J_f - L|$$

J_i	J_f	L
0	0	—
1/2	1/2	1
1	0	1
2	0	2
3/2	1/2	1, 2
5/2	1/2	2, 3
2	1	1, 2, 3
3/2	5/2	1, 2, 3, 4

← examples

N.B. $L = 0$ is not allowed because the photon has an intrinsic spin of 1

parity

A $\pi = -1$ photon (L odd) can only be emitted if there is a change in the parity of the nucleus.

A $\pi = 1$ photon (L even) can only be emitted if there is no change in the parity of the nucleus.

So we may complete the chart above.

J_i^*	J_f^*	Nuclear Parity Change	L	Allowed Transitions
0^+	0^+	no	–	none
$1/2^+$	$1/2^-$	yes	1	E1
1^+	0^+	no	1	M1
2^+	0^+	no	2	E2
$3/2^-$	$1/2^+$	yes	1, 2	E1, M2
$5/2^+$	$1/2^-$	yes	2, 3	M2, E3
2^+	1^+	no	1, 2, 3	M1, E2, M3
$3/2^-$	$5/2^+$	yes	1, 2, 3, 4	E1, M2, E3, M4
$5/2^+$	$3/2^-$	no	1, 2, 3, 4	M1, E2, M3, E4

N.B. $0^+ \to 0^+$ is forbidden, because it requires $L=0$ which is not allowed (above)

There are a small number of cases where the ground state and first excited state are both 0^+. e.g. ^{40}Ca

In these cases, the transition from first excited state to ground state must occur by internal conversion.

Often more than one type of decay is allowed.
Usually the lowest order multipolar radiation dominates.

e.g. E1 + M2 (low energy)
E1 dominates

But if lowest order is M
e.g. M2 + E3 (high energy)
contribution from E3 will be significant

Internal Conversion

For a decay from an excited state to a lower energy state, internal conversion competes with γ-emission.
For $0^+ \to 0^+$ transition, that is an E0 transition, γ-decay is not allowed and internal conversion is the only possibility.

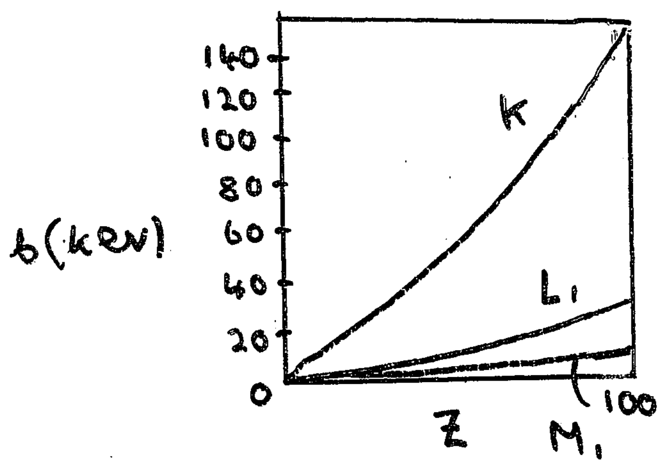
In internal conversion, the energy involved in the nucleus dropping from an excited level to a lower level is given to an atomic electron.

If the energy involved is greater than the binding energy, b_n, then the electron is liberated with a KE

$$T_e = E_\gamma - b_n$$

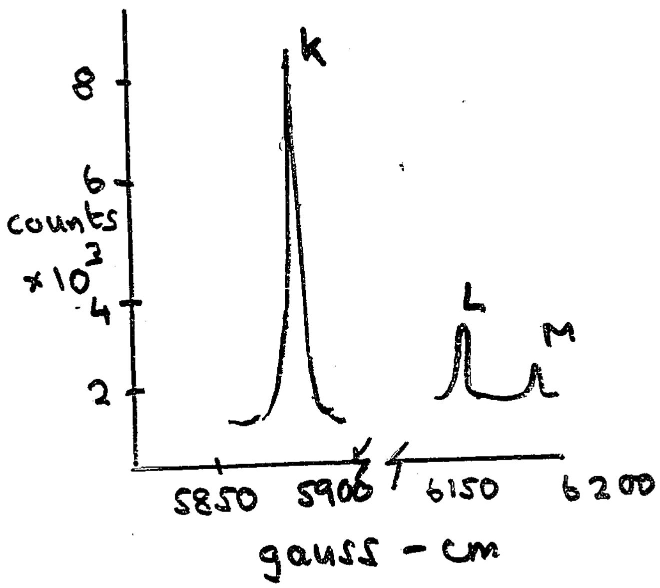
We also require to conserve angular momentum.

It is nearly always electrons from the K, L and M shells that are involved ($n = 1, 2, 3$), and the electrons are the s-electrons, because these are the electrons that have substantial probabilities of being at the nucleus.



N.B. $L_1 \equiv n=2, \ell=0$

We will expect the internal conversion electron spectrum to consist of peaks corresponding to different electron shells.



internal conversion electron spectrum for the 1.416 MeV E0 transition in $^{214}_{84}Po$

[increasing magnetic field corresponds to increasing electron energy]

Interpretation not always as straightforward as here.

There may be more than one excited state so transitions with several values of E_γ so multiple sets of peaks.

Also the excited state may have been populated by a β^--decay, so the internal conversion electron spectrum will be superimposed on the β-decay spectrum.

The total decay constant for γ-decay and internal conversion is given by

$$\lambda = \lambda_\gamma + \lambda_e$$

internal conversion

Here

$$\lambda_e = \lambda_K + \lambda_L + \lambda_M \ldots$$

The internal conversion coefficient α is

$$\alpha = \frac{\lambda_e}{\lambda_\gamma}$$

The K-shell internal conversion coefficient is

$$\alpha_K = \frac{\lambda_K}{\lambda_\gamma} \quad \text{etc}$$

so

$$\alpha_c = \alpha_K + \alpha_L + \alpha_M + \ldots$$

<u>calculation of coefficients</u>

(non-relativistic so not correct for high energies, but gives the importance of the various factors)

$$\alpha(EL) = \frac{Z^3}{n^3}\left(\frac{L}{L+1}\right)\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^4\left(\frac{2m_ec^2}{E_\gamma}\right)^{L+5/2}$$

$$\alpha(ML) = \frac{Z^3}{n^3}\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^4\left(\frac{2m_ec^2}{E_\gamma}\right)^{L+3/2}$$

principal quantum no.
$n=1$ for K etc.

α is greatest for large Z, small E_γ and small n.

2E

Nuclear Reactions

A nuclear reaction is a process that results from the interaction between a nucleus and a particle incident on it.

In general a nuclear reaction can be represented by a particle, a, incident on a nucleus, A, producing another particle, b, and a resulting nucleus, B.

$$a + A \longrightarrow b + B$$

or

$$A\,(a, b)\,B$$

Nuclear processes can be roughly divided into 2 categories:
scattering in which the incident and the emitted particle are the same, and
reactions in which they are different.

Scattering processes can be elastic or inelastic.

Elastic scattering resulting from coulombic interactions is referred to as coulombic or Rutherford scattering. Such processes conserve KE

Inelastic scattering refers to situations where KE is not conserved. The scattered particle loses KE, and the nucleus may be left in an excited state.

Nuclear reactions may be

(1) Direct reactions in which the incident particle interacts only with a limited no. of valence electrons in the target nucleus.
This is most likely when the de Broglie wavelength of the incident particle is of the order of the size of an individual nucleon (c. 1 fm) rather than that of the whole nucleus (c. 10 fm).
This is when the energy of the incident particle is relatively high.

(2) Compound nucleus reactions in which the incident particle becomes bound to the nucleus forming a compound nucleus before the reaction continues.

Thus the interactions between all the nucleons is important.
It is only the properties of the compound nucleus and not how it was formed that determine the way it decays.

(3) Resonance reactions are intermediate between (1) and (2). The incident particle becomes 'quasi-bound' to the nucleus.

[N.B. The distinction between (1), (2) and (3) is not always clear.]

conservation laws

Mass / energy & momentum must be conserved.

KE is often not conserved

At high energies, new types of particle may be produced (from $E = mc^2$). This is the central process in particle physics [or high energy nuclear physics or just high energy physics]

At lower energies [low energy nuclear physics] the number and identity of particles is normally conserved.
The number of p and n will not change unless the weak interaction (that causes β-decay) is important.

Charge must be conserved, but will, of course, be if numbers of different types of particle do not change.

Here we consider relatively low energy reactions involving the strong interaction (that, apart from being involved in reactions, binds nucleons together to form nuclei, or, at a higher level, binds quarks together to form nucleons and mesons; see later in particle physics.)
or the electromagnetic interaction.

[N.B. The strong and weak interactions are more formally called the strong and weak nuclear interactions.]

Low energy processes usually involve incident particles that are n or p or bound systems containing a small no. of nucleons :

symbol	name	nucleus of	identity
n	neutron	—	neutron
p	proton	^1H	proton
d	deuteron	^2H	n + p
t	triton	^3H	2n + p
^3He	—	^3He	n + 2p
^4He	α-particle	^4He	2n + 2p

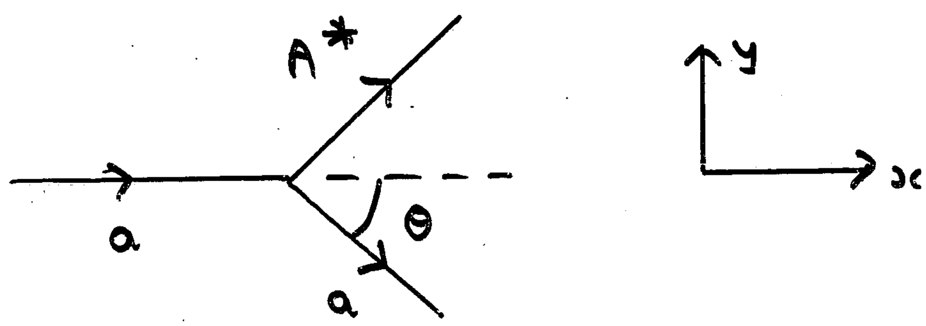
tritium (unstable)

Inelastic scattering

Consider the event $A(a,a)A^*$

the nucleus is left in an excited state

A particle a scatters from a nucleus inelastically, losing energy and leaving the nucleus in an excited state.



We shall treat non-relativistically (since it is low energy)

Before collision:

$$E = E_i + m_a c^2 + M_A c^2$$

KE of a

After collision:

$$E = E_f + E_{A^*} + m_a c^2 + M_A c^2$$

final KE of a

$\therefore \Delta E$ [of nucleus] $= (M_{A^*} - M_A)c^2 = E_i - E_f - E_{A^*}$

In terms of momenta:

$$\Delta E_i = \frac{p_i^2}{2M_a} - \frac{p_f^2}{2M_a} - \frac{p_{A*}^2}{2M_{A*}}$$

Conservation of angular momentum:

$$\left. \begin{aligned} p_{A*x} &= p_i - p_f \cos\theta \\ p_{A*y} &= p_f \sin\theta \end{aligned} \right\}$$

Substituting these into the expression for ΔE_i gives:

$$\Delta E_i = \frac{p_i^2}{2m_a} - \frac{p_f^2}{2m_a} - \frac{1}{2m_{A*}}\left[p_i^2 + p_f^2 - 2p_i p_f \cos\theta\right]$$

In terms of the initial and final energies of the scattered particle:

$$\Delta E = E_i\left(1 - \frac{m_a}{m_A^*}\right) - E_f\left(1 + \frac{m_a}{m_{A*}}\right) + 2\frac{m_a}{m_A^*}\sqrt{E_i E_f}\cos\theta$$

Elastic scattering is a special case of this expression with $\Delta E = 0$; for a given scattering angle, θ, the difference between E_i and E_f accounts for the KE of recoil of the nucleus.

Since ΔE is equal to $(m_{A*} - m_A)c^2$, it is clear that m_A^* appears on both sides of the previous and calculation will be awkward.

However, to the accuracy of experimental measurements, it will be appropriate to replace m_{A^*} on the RHS by m_A, the known ground state mass.

While the equation has been set up in terms of nuclear masses, since there is no change in the identity of the nucleus, it will be appropriate to add the mass of the associated electrons and use *atomic masses*.

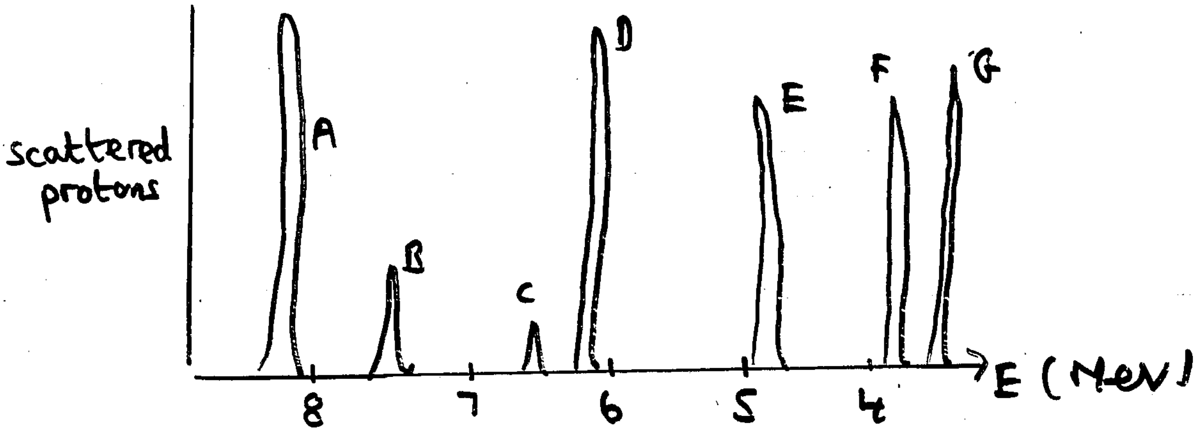
Experimentally, the excited state energies of a nucleus may be investigated by allowing a monoenergetic beam of particles to be incident on the nucleus, and measuring the energy spectrum of particles scattered at a particular angle.

e.g. Consider the inelastic scattering of 10.02 MeV protons from a sample containing ^{10}B nuclei, $^{10}B(p,p)^{10}B^*$

For a fixed scattering angle, say 90°, protons that have a final energy E_f will give up an energy ΔE to the ^{10}B nucleus.

If ΔE is equal to the energy difference between the ^{10}B ground state and one of the ^{10}B excited states then the energy given up by the proton can cause an excitation of the ^{10}B nucleus.

Since the nucleus has a number of well-defined excited states, we expect the protons scattered at a specific angle will have a series of well-defined energies or resonances, related to the excited state energies by the previous equation.



A $\Delta E = 0$ (elastic scattering)

B $= 0.72$ MeV

C $= 1.74$ MeV

D $= 2.15$ MeV

E $= 3.59$ MeV

F $= 4.77$ MeV

G $= 5.11$ MeV

} excited state energy levels for ^{10}B

Nuclear reactions

Different situations

1. An incident nucleon is absorbed by the nucleus leaving the nucleus in an excited state; the emitted particle is a γ-ray resulting from the de-excitation of the nucleus. e.g. the (n, γ) reaction

2. An incident nucleon is absorbed by the nucleus and a different nucleon is emitted e.g. the (n, p) reaction

3. The incident particle loses one or more of its nucleons to the nucleus e.g. the (d, p) reaction. This is known as a <u>stripping reaction</u> - see shortly

4. The incident particle gains one or more nucleons from the nucleus e.g. the (d, α) reaction. This is known as a <u>pick-up reaction</u>

5. The reaction causes the component nucleons of the incident particle to become unbound. e.g. the (d, np) reaction, where np means a neutron proton pair that is <u>not</u> bound.

e.g. a deuteron incident on an ^{16}O nucleus

possible reactions:

$$d + {}^{16}O \rightarrow {}^{16}O + d$$
$$d + {}^{16}O \rightarrow {}^{18}F$$
$$d + {}^{16}O \rightarrow {}^{17}O + p$$
$$d + {}^{16}O \rightarrow {}^{17}F + n$$
$$d + {}^{16}O \rightarrow {}^{14}N + \alpha$$
$$d + {}^{16}O \rightarrow {}^{15}O + t$$
$$d + {}^{16}O \rightarrow {}^{15}N + {}^{3}He$$
$$d + {}^{16}O \rightarrow {}^{16}O + np$$

The energetics may be established on the basis of measured atomic masses, taking care of any change in the number of electrons in the reaction.

For ^{18}F, $^{17}O + p$, $^{14}N + \alpha$
($Q \sim 7$ MeV) (~ 2 MeV) (~ 3 MeV)

Q is positive [Final mass < initial mass]

reaction is exothermic (take place automatically)

For $^{17}F + n$, $^{15}O + t$, $^{15}N + {}^{3}He$, $^{16}O + np$
($Q \sim -2$ MeV) (~ -9 MeV) (~ -7 MeV) (~ -2 MeV)

Q is negative [Final mass > initial mass]

reaction is endothermic

(does not take place unless extra mass/energy is provided in the ~~form~~ of KE of the incident particle.

It is KE in the centre-of-mass frame that is relevant

$$E_{CM} = \frac{E_{lab}}{1 + \frac{m_a}{M_A}}$$

(N.B. $E_{CM} < E_{lab}$

Energy above the minimum to cause a reaction to proceed may (a) yield KE of the products, or (b) leave the resulting nucleus in an excited state.

So any excited state at or below an energy

$$E = E_{CM} + Q$$

is accessible.

This requires the existence of an excited state in this range in the resulting nucleus

The density of excited states increases with increasing nuclear mass.

The density in a given nucleus increases with increasing energy.

(All ~~understood~~ on the basis of the shell model)

The lowest lying states are often single nucleon states.

Higher energy states usually involve multiple nuclear excitations.

As the number of nucleons participating increases, so does the number of combinations of state occupancies that will give similar but slightly different state energies.

Thus for heavy nuclei, and for cases where $E = E_{CM} + Q$ is large (or both) the spacing between resonances is small. [See later]

The angular dependence of the energy spectrum of emitted particles in a nuclear reaction may be calculated in a similar way to the inelastic scattering case. The relativistic result is:

$$\Delta E = E_i\left(1 - \frac{m_a}{m_B}\right) - E_f\left(1 + \frac{m_b}{m_B}\right) + 2\,\frac{\sqrt{m_a m_b E_i E_f}}{m_B}\cos\theta + Q$$

where nucleus B may be left in an excited state. This is very important in <u>deuteron stripping reactions</u> ((d, n), (d, p))

Deuteron Stripping Reactions

The reaction

$$d + {}^{16}O \rightarrow {}^{17}O + p$$

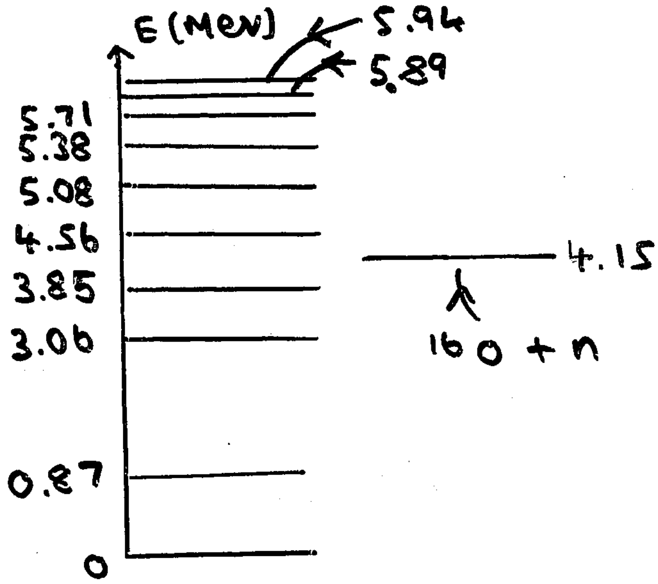
may be used to study the excited states of ^{17}O.

Since the reaction is exothermic, even low energy deuterons may excite any ^{17}O levels below 1.9 MeV.
(Coulombic effects must be taken into account.)

Additional deuteron KE (in CoM frame) can excite higher ^{17}O levels.

e.g. Peaks in the energy spectrum of protons emitted by the $^{16}O(d,p)^{17}O$ reaction at 25° with respect to a beam of 10 MeV lab-frame deuterons.

	E_f (MeV)	ΔE (MeV)	E_f (MeV)	ΔE (MeV)
(of proton	11.69	0.00	6.50	5.08
	10.81	0.871	6.19	5.38
	8.58	3.06	5.85	5.71
	7.77	3.85	5.66	5.89
	7.05	4.56	5.60	5.94



(resulting) <u>energy level diagram for ^{17}O</u>

<u>neutron separation energy</u> = 4.15 MeV

Excited states above 4.15 MeV may be thought of as n-^{16}O resonances as the n is only quasibound to the ^{16}O nucleus.

Since ^{17}O is β-stable, excited states below 4.15 MeV may decay only by γ-decay or internal conversion.

However states above 4.15 MeV may decay by γ-decay or by neutron emission.

The interaction between the n and the ^{16}O nucleus may be understood in the context reactions involving incident neutrons. (See next section.)

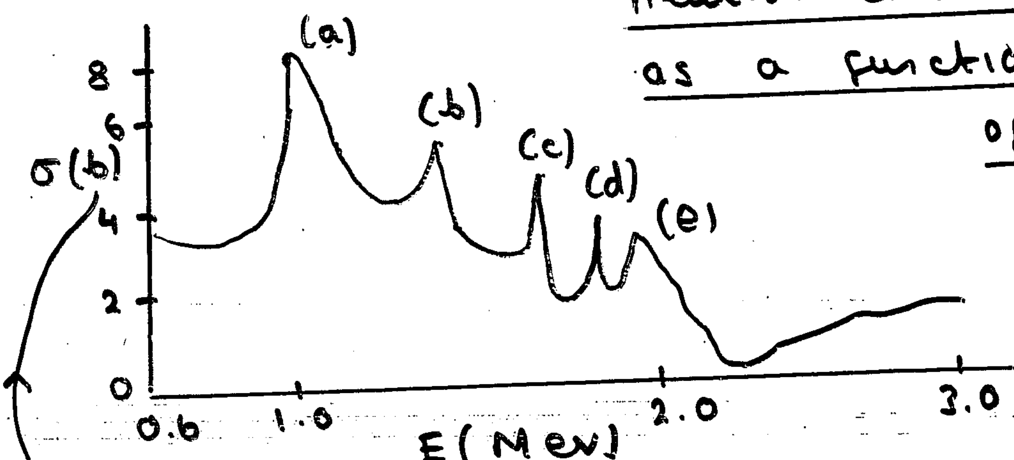
Neutron Reactions

A neutron incident on a ^{16}O nucleus (for example) may be absorbed forming ^{17}O.

The available energy is $E_{cm} + Q$, which is here $E_{cm} + 4.15$ MeV.

If this energy is equal to the energy of a ^{17}O excited state, a resonance condition occurs, and the neutron absorption cross section is greatly enhanced.

neutron cross section for ^{16}O as a function of centre-of-mass energy



barn (see later)

Only states with energies > 4.15 MeV are accessible

For peaks above, add 4.15 MeV to give energy of excited state

(a) $0.93 + 4.15 = 5.08$ MeV
(b) $1.23 + 4.15 = 5.38$ MeV
(c) $1.56 + 4.15 = 5.71$ MeV
(d) $1.74 + 4.15 = 5.89$ MeV
(e) $1.79 + 4.15 = 5.94$ MeV

Once ^{17}O has been formed in an excited state above 4.15 MeV, it may decay by neutron emission or by γ-decay.

If decay is by γ-decay to an energy level below 4.15 MeV, the ^{17}O nucleus can no longer decay by neutron emission, and the neutron is said to have been captured.

This is the (n,γ) reaction and it has important applications for the study of excited state structure, for the formation of radioactive nuclides and for chemical analysis by neutron activation analysis

cross-sections

The barn is defined as $100\ fm^2$

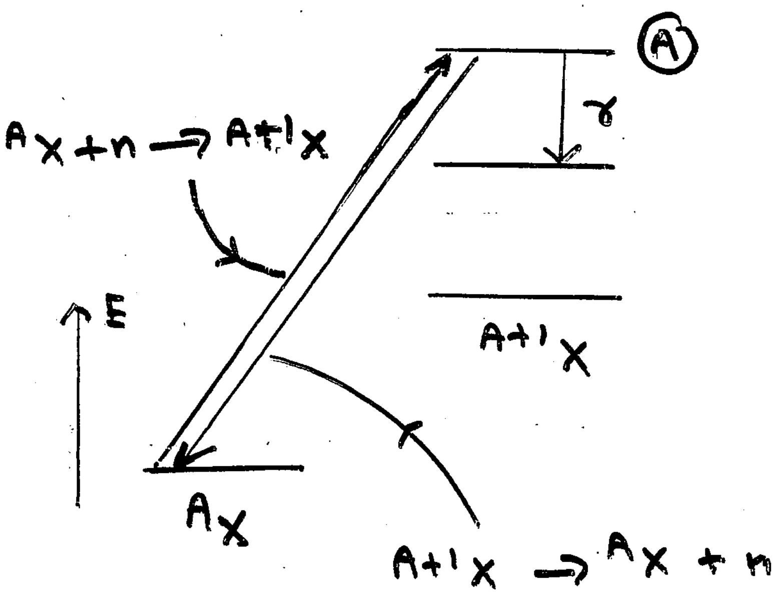
The radius of a ^{16}O nucleus is ~ 3 fm, so cross-sectional area $\sim 9\ fm^2$ or 0.1 barn

Cross-sections here up to 100 times as large.

(n,γ) cross-sections for some nuclei may be as large as 10^5 barns or higher.

So cross-section should be thought of as a physical cross sectional area, but as a quantum mechanical result of the interaction between the neutron and the nucleus; it is a measure of <u>reaction probability</u>

As well as giving information about the energies of excited states, the energy dependence can also give information about the <u>stability</u> of the states.



State (A) may decay by 2 modes or <u>channels</u>

γ- decay or <u>neutron emission</u>

└ the <u>incident channel</u>

e total width, Γ, of the ${}^{A+1}X$

uted state, (A), will be

$$\Gamma = \Gamma_\gamma + \Gamma_i \leftarrow \text{incident}$$

then expect an energy dependent cross

tion:

$$\sigma_i(E) = \frac{C}{(E-E_0)^2 + \Gamma^2/4}$$

ere C is related to the density of

tes in the system.

find C:

e reaction is assumed to take place

the volume of the nucleus, V.

en (p.92) the density of states per

t momentum for fermions within this

lume is

$$\frac{dn}{dp} = \frac{4\pi p^2}{(2\pi\hbar)^3} V$$

the neutron, moving with speed v

Then the decay rate per unit momentum is the fraction of V swept out per unit time multiplied by the density of states.

$$d\lambda = \frac{\sigma_i v}{V} dn = \sigma_i v \frac{4\pi p^2}{(2\pi\hbar)^3} dp$$

Then

$$\lambda = \frac{4\pi}{(2\pi\hbar)^3} \int_{-\infty}^{\infty} v \sigma_i p^2 dp$$

In fact we obtain

$$\lambda = \frac{8\pi m E C \hbar}{\Gamma}$$

where m is the reduced mass of the system

$$m = \frac{M_n M_x}{M_n + M_x}$$

So $\quad C = \dfrac{\Gamma\lambda}{8\pi m E \hbar}$

In a system with a large number of neutrons and nuclei, the rate of formation of the excited state would be balanced by the rate of decay back through the incident channel

This is

$$\lambda = \frac{\Gamma_i}{\hbar}$$

So $$C = \frac{\Gamma \Gamma_i}{8\pi m E \hbar^2}$$

and
$$\sigma_i(E) = \frac{1}{8\pi m E \hbar^2} \frac{\Gamma_i \Gamma}{(E-E_0)^2 + \Gamma^2/4}$$

A more detailed analysis shows that Γ must be replaced by $g\Gamma$ where the statistical factor g is

$$g = \frac{2J+1}{(2S_n+1)(2S_x+1)}$$

where S_n and S_x are the spins of the incident neutron and the target nucleus, and J is the total angular momentum of the excited state.

The formula for σ_i is the Breit-Wigner formula, and allows the analysis of resonance line shapes.

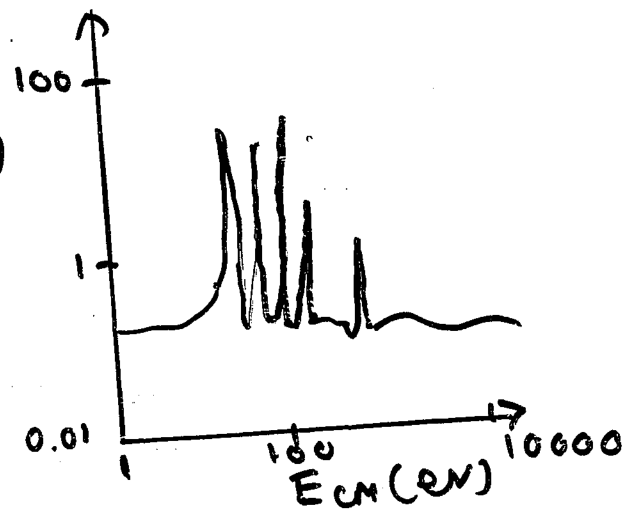
[It has a basically Lorentzian structure,

$$\sim \frac{1}{x^2 + a^2}$$]

The (n,γ) process is important for low energy neutrons as it is then typically the energy levels near or below the neutron separation energy that are populated directly leaving γ-decay the likely decay mode.

For cases where A is large, and the value of Q places the excited nucleus well above the ground state, we will expect the density of excited states to be high

e.g.



Total neutron cross section for ${}^{238}_{92}U$ as a function of centre-of-mass energy

spacing between peaks ~ tens of eV rather than the hundreds of keV for ${}^{16}O$

Neutrons for experiments can be produced from neutron-emitting sources.

Unlike charged particles, neutrons cannot be accelerated by electric fields.

However higher energy neutrons produced by a radioactive source may be slowed by scattering mechanisms in an appropriate material called a moderator

The ability to reduce the KE of a neutron from high energy (a few MeV) to a few eV or less is crucial in the operation of a fission reactor.

(see later.)

neutrons	Class	typical E
	thermal	0.02 eV
	epithermal	1 eV
	slow	1 keV
	fast	> 100 keV

[N.B. $k = 8.6 \times 10^{-5}$ eV K^{-1}

∴ At room temp. $kT \sim 8.6 \times 10^{-5}$ eV $K^{-1} \times 3 \times 10^{2}$ K

$\sim 2.5 \times 10^{-2}$ eV

Thermal neutrons

$$E \sim 0.02\,\text{eV} \sim 0.03 \times 10^{-19}\,\text{J} \sim 3 \times 10^{-21}\,\text{J}$$

$$p = \sqrt{2Em} \sim \left(2 \times 3 \times 10^{-21} \times 10^{-30}\right)^{1/2} \text{kg m s}^{-1}$$

$$\sim \left(6 \times 10^{-51}\right)^{1/2} \text{kg m s}^{-1}$$

$$\sim 10^{-25}\,\text{kg m s}^{-1}$$

Then $\lambda \sim \dfrac{10^{-33}}{10^{-25}}\,\text{m} \sim 10^{-8}\,\text{m}$

i.e. crystal diffraction is possible

Neutron cross sections at high energies can be complex.

At higher energies, resonance peaks become close and closer, and as the energy difference between peaks becomes comparable to their half-width, the concept of distinct energy levels becomes ambiguous.

(This is important in fission.)

Thermal effects

For low energy neutrons, the motion of the atoms in the target influences the centre-of-mass energy of the incident neutrons.

If the neutron has velocity $\underset{\sim}{V_n}$ and the particular atom has velocity $\underset{\sim}{V_A}$, then the centre-of-mass energy is

$$E = \tfrac{1}{2} m \left(\underset{\sim}{V_n} - \underset{\sim}{V_A} \right)^2$$

where m is the reduced mass.

This may be written in terms of the neutron centre-of-mass energy of the neutron, E_{CM}, and the thermal energy of the atom, $E_A = K_B T$, as

$$E = \tfrac{1}{2} m V_n^2 + \tfrac{1}{2} m V_A^2 - m \underset{\sim}{V_n} \cdot \underset{\sim}{V_A}$$

$$= E_{CM} + \frac{m}{M_A} E_A - 2 \left[\frac{m}{M_A} E_{CM} E_A \right]^{1/2} \cos\theta$$

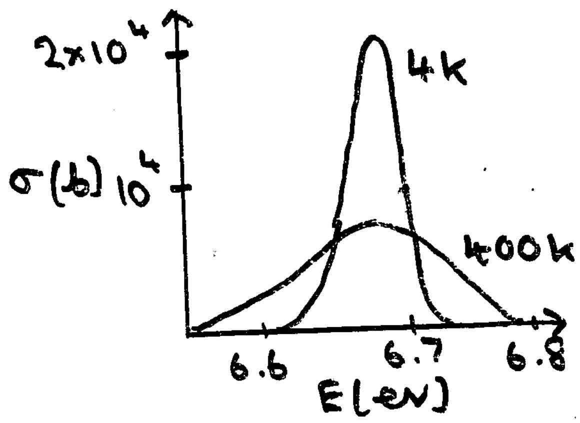
In general, m/M_A is small, and, except for very low energy electrons, $E_A \ll E_{CM}$.
$\therefore$ The second term can be ignored.

$\cos\theta$ takes values from -1 to $+1$, so the actual energy may be reduced or increased relative to the value of E_{CM}.

This results in a broadening of the resonance peak, the <u>Doppler broadening</u>.

The increased width due to thermal effects

$$\Delta E \sim 2 \sqrt{\frac{m}{M_A} E_{CM} k_B T}$$



<u>Effect of Doppler broadening on the (n,γ) resonance at 6.67eV in ^{238}U</u>

<u>Coulombic Effects</u>

In contrast to neutron reactions, reactions involving charged particles are affected by Coulombic interactions.

This is true if the incident particle or the emitted particle is charged (or both).

The interaction potential between the charged nucleus and the charged particle forms a Coulomb barrier as in α-decay.

unless the energy of the particle > barrier height, the calculation of cross section is a quantum mechanical tunnelling problem.

At low energies, charged particles incident on a nucleus predominantly undergo elastic scattering because of their inability to tunnel through the Coulomb barrier.

With increasing energy, the tunnelling probability, and hence the reaction cross section increases.

The calculation of the tunnelling probability for charged particles incident upon a nucleus is similar to that for α-decay.

We write the cross section due to Coulombic interaction at low energies as

$$\sigma(E) = \frac{S(E)}{E} e^{-G}$$

where

$\frac{S(E)}{E}$ is the Breit-Wigner function, and it is modified by e^{-G} to account for the tunnelling probability.

Here G is proportional to $E^{-1/2}$ as in the expression for G in α-decay, with Q replaced by the incident particle energy E.

At low energies, S(E) varies very little if at all, as long as there are no resonances in the energy range.

Then

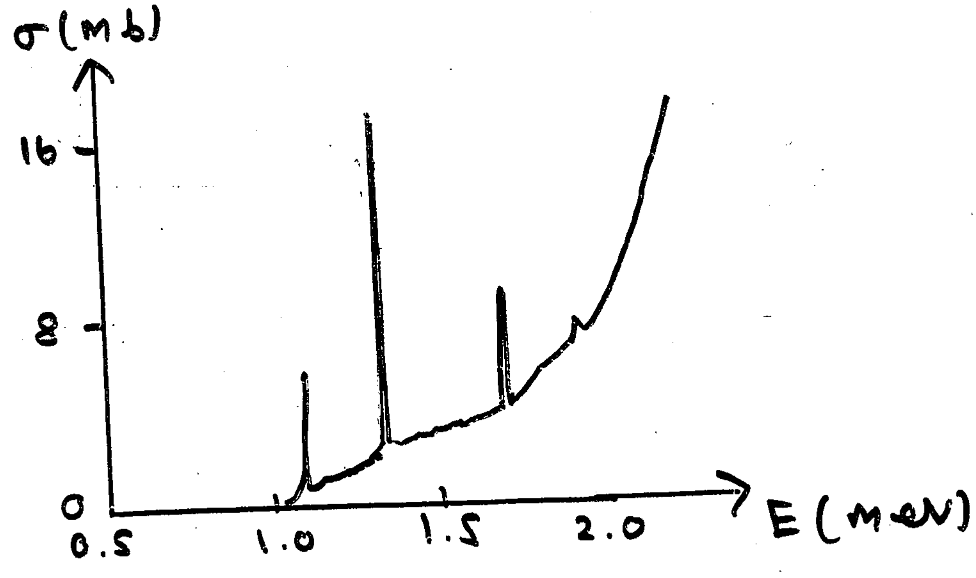
$$\sigma(E) = \frac{S(0)}{E} e^{-\alpha/\sqrt{E}}$$

where α is analogous to the related quantity in α-decay, and includes nuclear and particle masses, charges and dimensions.

The result is that, for a given process, there is a threshold energy below which the reaction cross section is virtually zero because the tunnelling probability is virtually zero.

The resonant peaks corresponding to the population of certain states appear at higher energies superimposed on a background given by the preceding equation.

e.g.



cross section for the reaction $^{13}C(\alpha, n)^{16}O$

2F Fission Reactions

Fission is the splitting of a relatively heavy nucleus into 2 lighter nuclei.

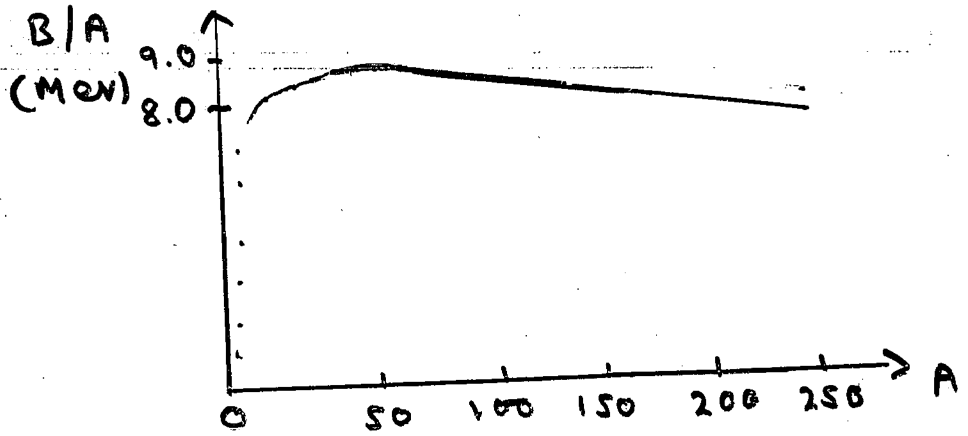
It can be a spontaneous process or it can be induced by the reaction of an incident particle, usually a neutron.

Alpha decay may be regarded as an extreme case of spontaneous fission, though the term is usually used when the 2 fragments are of similar mass.

A simple fission process is written as:

$$^{A}X \longrightarrow {}^{B}Y + {}^{A-B}Z$$

The binding energy/nucleon curve:



This tells us that if a heavy nucleus

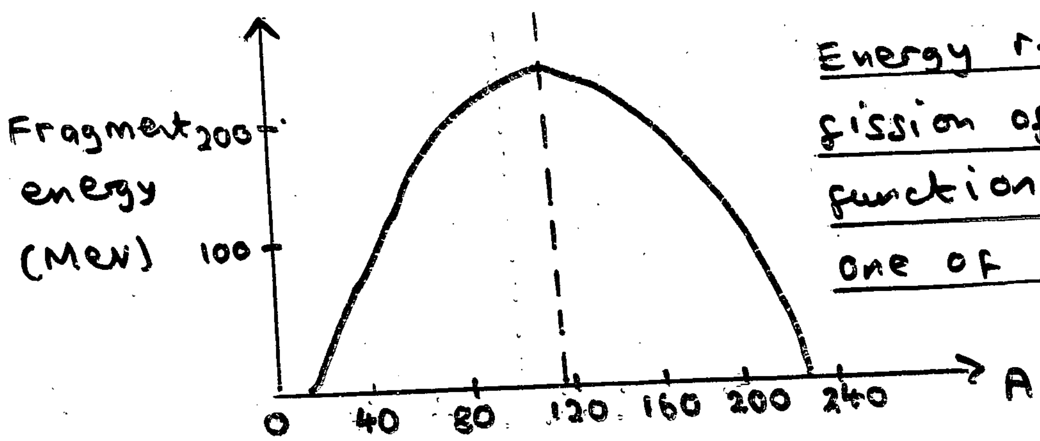
breaks up into 2 lighter nuclei, then xs binding energy will be released i.e. the process will be exothermic

e.g. A nucleus with $A = 236$ has a binding energy (BE) of 7.6 MeV/nucleon, or a total BE of 1.78 GeV.

If it breaks into 2 equal fragments then the 2 nuclei, with $A = 117$, have BEs of 8.5 MeV/nucleon, or a total of 1.99 GeV.

This gives a net energy release of 210 MeV per fission.

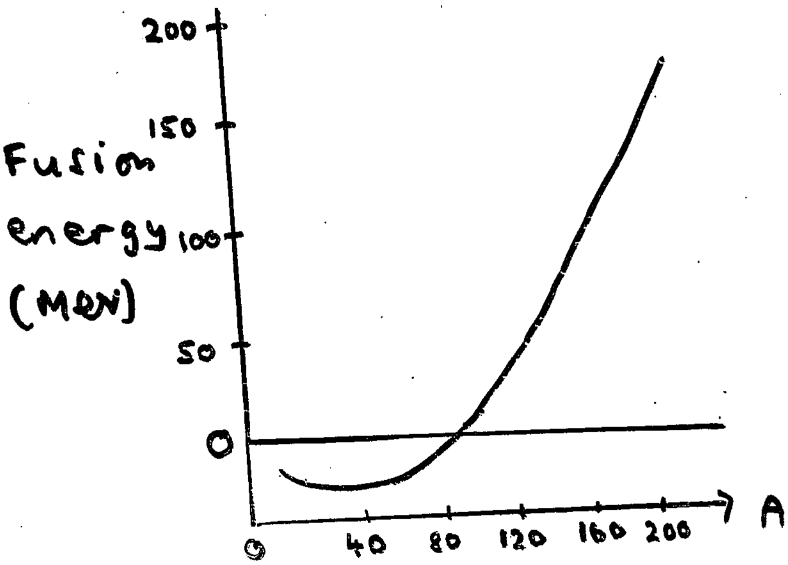
The semi-empirical mass formula (SEMF) can be used to estimate the net energy release as a function of the size of the fragments



Energy release from the fission of ^{236}U as a function of A of one of the fragments

This picture indicates that the energy release is greatest if the fragments are of equal size.

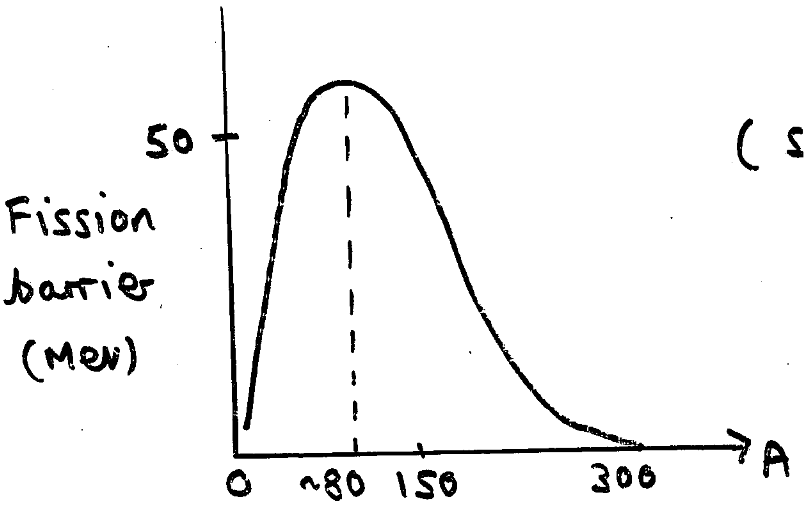
We can also calculate the energy for symmetric fission as a function of A:



This confirms that if the binding energy/nucleon ~~for~~ a nucleus with A/2 nucleons becomes less than that for A nucleons, the process will not produce XS energy, so is not energetically favourable.

The problem of <u>spontaneous fission</u> is much like that of α-decay, because the charged fragments must overcome a Coulombic barrier before they can separate.

The calculation of the barrier height can be achieved with the SEMF.

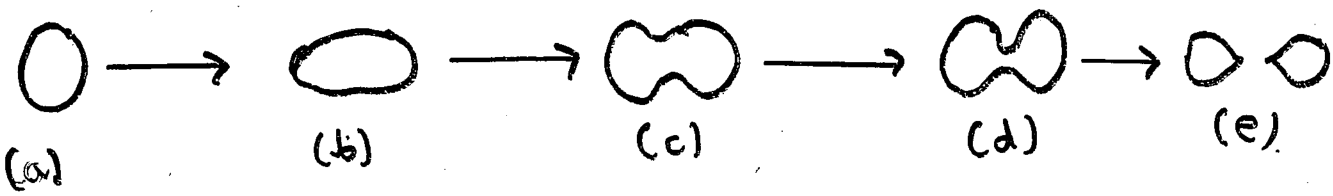


(symmetric fission)

[A more detailed calculation involving shell effects shows some added structure on the curve near magic numbers.]

i.e. the barrier height drops to zero at $A \sim 300$, so nuclei with masses greater than this are unstable and will undergo spontaneous fission with a very short lifetime.

schematic illustration



This may be analysed in the context of the liquid drop model

(b)

Any oscillation in the shape of the drop will grow and cause the drop to fragment if the deformation from spherical symmetry makes the drop more stable.

We calculate the energy associated with nuclear deformation (or stretching in the diagram above) using nuclear binding energies

$$\Delta E = B(\varepsilon) - B(\varepsilon = 0)$$

where ε is a deformation parameter

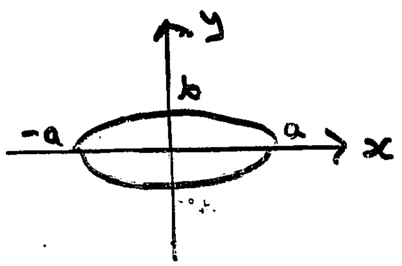
If it is assumed that the deformations are elliptical, then ε is the eccentricity

Since nuclear matter is incompressible, the volume of the nucleus does not change during deformation.

Therefore we can equate the volume of the spherical nucleus of radius R_0 before stretching to the volume of the stretched ellipsoidal nucleus:

$$\frac{4\pi}{3} R_0^{\,3} = \frac{4\pi}{3} ab^2$$

where a and b are the semimajor and semiminor axes of the ellipsoid respectively.



We have
$$\left.\begin{aligned} a &= R_0(1+\varepsilon) \\ b &= R_0(1+\varepsilon)^{-1/2} \end{aligned}\right\}$$

Using the SEMF, we can now calculate the energy associated with the deformation.

The changes in binding energy will be associated with the surface and Coulomb terms of the SEMF.

The surface area of the ellipsoid is

$$S = 4\pi R_0^2 \left(1 + \frac{2}{5}\varepsilon^2 + \ldots\right)$$

The Coulomb energy of the ellipsoid relative to that of a sphere of the same volume is:

$$\frac{E_{\text{ellipse}}}{E_{\text{sphere}}} = 1 - \frac{1}{5}\varepsilon^2 \ldots$$

[So in a distortion, the surface energy goes up but the Coulomb energy goes down.]

The relevant terms in the SEMF are

N.B.

$$\underline{\text{binding energy}} = -a_S A^{2/3} - \frac{a_c Z(Z-1)}{A^{1/3}} \quad \text{[+ other terms]}$$

$A^{2/3} \sim$ surface area

$\therefore$ ~~term~~ surface term becomes $-a_S A^{2/3}\left(1 + \frac{2}{5}\varepsilon^2 \ldots\right)$

Coulomb term becomes $-\frac{a_c Z(Z-1)}{A^{1/3}}\left(1 - \frac{1}{5}\varepsilon^2 \ldots\right)$

$$\therefore \Delta BE = -a_S A^{2/3}\left(\frac{2}{5}\varepsilon^2 \ldots\right) + \frac{a_c Z(Z-1)}{A^{1/3}}\left(\frac{1}{5}\varepsilon^2 \ldots\right)$$

To order ε^2

$$\Delta BE = \left[-\frac{2}{5} a_s A^{2/3} + \frac{1}{5} a_c Z(Z-1) A^{-1/3}\right] \varepsilon^2$$

ΔBE will be positive and the nucleus will be unstable to stretching if

$$\frac{Z(Z-1)}{A} > \frac{2a_s}{a_c}$$

$$\frac{Z(Z-1)}{A} \gtrsim \frac{33.6}{0.72} \quad \text{i.e.} \gtrsim 50$$

Heavy nuclei have $Z/A \sim 0.4$

So nucleus is unstable if

$$0.4\,Z \gtrsim 50$$

$$Z \gtrsim 125$$

$$A \gtrsim 300$$

[In agreement with graph of barrier heights]

Induced Fission

In some nuclei, fission can be induced by bombardment with neutrons.

Uranium is an important example as it is commonly used for fuel in commercial fission reactors.

Natural U consists of ~ 0.72% ^{235}U & ~ 99.27% ^{238}U.

A low energy neutron can be captured by either of these nuclei:

$$\left.\begin{aligned} n + {}^{235}U &\rightarrow {}^{236}U + 6.46\,\text{MeV} \\ n + {}^{238}U &\rightarrow {}^{239}U + 4.78\,\text{MeV} \end{aligned}\right\}$$

In heavy nuclei, the density of states, particularly at energies a few MeV above the ground state, is usually quite high, so there are many excited states that may be occupied.

For $A = 236$, the barrier energy is about 6.2 MeV

∴ Following neutron capture, the ^{236}U nucleus has enough energy to induce fission without any need to tunnel through the Coulomb barrier.

However, even after neutron capture, the ^{239}U has not got enough energy to induce fission ; an extra 1.4MeV is required, which may be provided as extra KE of the neutron.

Nuclides such as ^{235}U in which fission can be induced by thermal neutrons are called fissile

Nuclides such as ^{238}U in which thermal neutrons cannot induce fission are called nonfissile

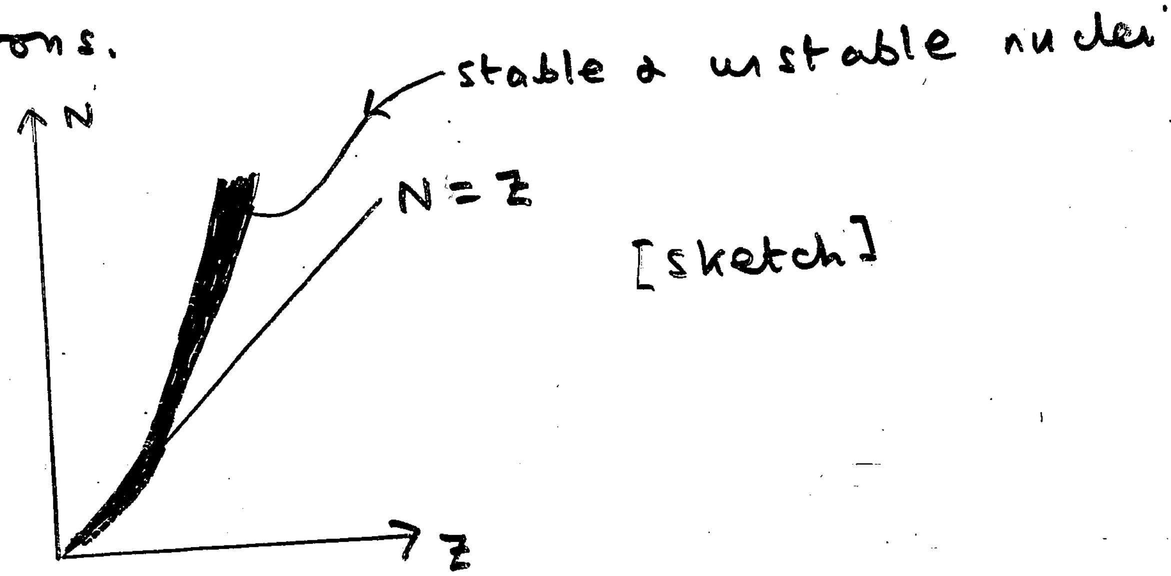
For heavy nuclei, odd-A nuclides are generally fissile, while even-even nuclides are nonfissile [as in the example above]

This is a result of the pairing term in the SEMF.

This is a negative contribution to the energy for even-even nuclei, positive for odd-A nuclei.

In other words, even-even nuclei are more strongly bound than odd-A nuclei.

In a sample of fissile material, a spontaneous fission produces xs neutrons.



[sketch]

This is because, since heavier nuclei have a higher N/Z ratio than the stable nuclei to be produced after fission.

Thus the fission equation should be written

$$^{A}X \rightarrow {}^{B}Y + {}^{A-B-\nu}Z + \nu n$$

Even fragments Y and Z are typically too neutron rich (even after these xs neutrons have been given off) so there is further decay towards the β-stability line.

As these nuclides approach the β-stability line, the lifetimes become longer, and this is the source of the long-lived radioactive waste produced by fission reactors

The neutrons produced in the above equation may induce further fissions, and under appropriate conditions, a chain reaction may occur

Fission processes in uranium

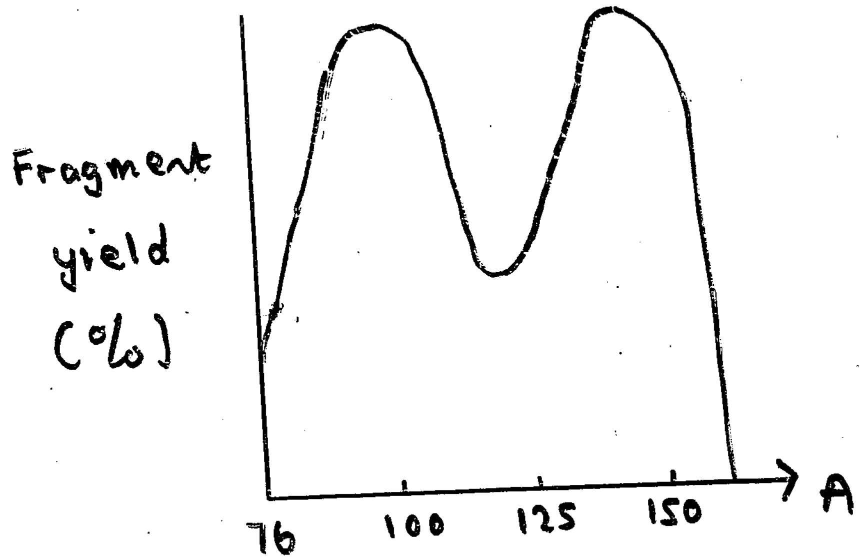
In most reactors it is fission of uranium nuclei that supplies the energy, and this is primarily a result of the behaviour of ^{235}U.

The induced fission process is

$$n + {}^{235}U \rightarrow {}^{B}Y + {}^{236-B-\nu}Z + \nu n$$

On average, $\nu \sim 2.5$
These xs neutrons are referred to as <u>prompt neutrons</u> and are given off on the time scale of the fission process, $\sim 10^{-14}$ s

The distribution of masses of the fission fragments is :



i.e. The fission process does not usually end up in equal-sized fragments.

<u>N.B.</u> The graph is (nearly) symmetric because each fission producing a fragment of $A/2 + x$ must also produce one of (nearly) $A/2 - x$.
The 'nearly' is because of the xs neutrons.

The fission fragments are normally left in excited states that γ- decay to their ground states.

These γ- rays are called prompt γ-rays.

The energy that is immediately released by the fission process is distributed between the KE of the fission fragments, the KE of the prompt neutrons, and the energy of the prompt γ- rays.

It is primarily this energy that becomes available as heat and can be extracted from the reactor.

The neutron-rich fragments decay by a series of β- and γ-decays ~~toward~~ β-stability.

This results in a delayed release of energy due to electron and antineutrino KE (the latter lost into space) and γ-ray energy.

In some cases, a nuclide in the β-decay sequence is left in an excited state above the neutron separation threshold and can therefore emit a neutron. These neutrons are called delayed neutrons. On average, there are 0.02 neutrons per fission given off in a time scale of seconds after the fission event.

Although these represent a negligible amount of energy, they are very important in controlling the chain reaction

Distribution of Energy from Fission of ^{238}U

Source of Energy	E (MeV)
fission fragment KE	167
prompt neutron KE	5
prompt γ-ray energy	6
delayed β-decay energy	8
delayed γ-ray energy	7
delayed antineutron energy	12
total	205

Neutron Cross Sections for Uranium

To understand the details of induced fission in U, it is essential to investigate the n cross section as a function of energy.

There are a number of possible processes by which ns can interact with nuclei:

Elastic Scattering

Elastic scattering conserves KE, but, as the target nucleus gains ~~some~~ energy when it recoils, the n loses a small amount of energy.

Inelastic Scattering

In this process, the n gives up some KE to the target nucleus, leaving it in an excited state.

This process has a threshold energy equal to the energy of the first excited state above the ground state of the nucleus.

For ^{235}U and ^{238}U, the inelastic scattering threshold is 14 keV and 44 keV respectively

Radiative Capture

In the (n,γ) reaction, an n is absorbed and the resulting nucleus decays by γ-emission to a state below the neutron separation energy, thereby capturing the n.

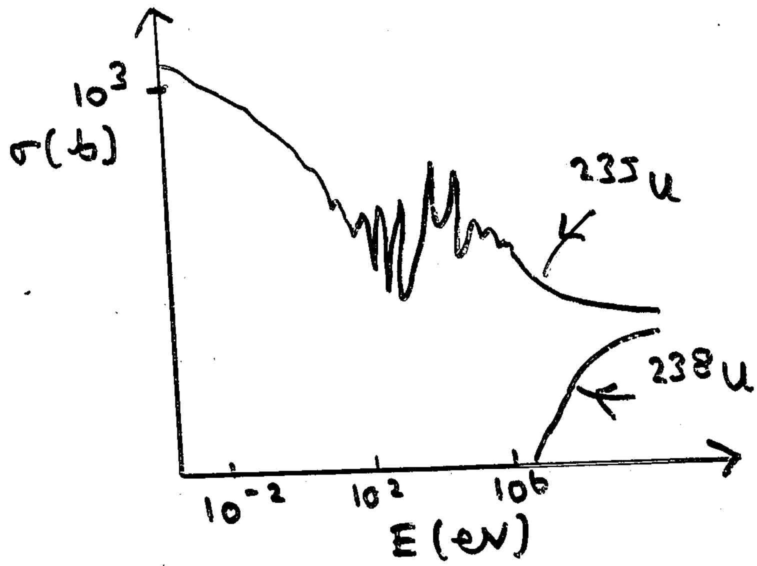
The cross section for (n,γ) process is characterised by a series of resonances. The lowest energy resonance occurs at an energy equal to the difference between the neutron separation energy and the energy of the next available state.

Fission

In this process the n leaves the nucleus in an energy state above the fission barrier and fission proceeds spontaneously on a very short time scale.

The energy available is the sum of the n KE and the Q for the process.

The energy dependent n induced fission cross sections for ^{235}U and ^{238}U are on the next sheet.



Fission cross sections for neutrons incident on ^{235}U and ^{238}U as a function of energy

The total n cross section is the sum of the 4 contributions.

Cross Sections for ^{235}U

E<1eV 85% of the cross section is due to fission. The remainder is mostly due to (n,γ) processes as a result of a ^{236}U resonance lying just below E=0.

1eV<E<100eV (n,γ) resonances dominate, with the remainder being due to fission, and, to a lesser extent, elastic scattering.
(Scattering reduces the energy of the ns while (n,γ) processes absorb them.)

<u>100 eV < E < 1 MeV</u> (n, γ) processes are still important, but the energy levels of the excited states overlap considerably so the resonances no longer show discrete peaks. Above the 14 keV threshold, inelastic scattering processes are important. Fission becomes less important as the energy decreases.

<u>E > 1 MeV</u> Similar to previous range but with a decreasing probability of (n, γ) processes.

<u>Cross Sections for ^{238}U</u>

<u>E < 1 eV</u> Only elastic scattering is possible.

<u>1 eV < E < 100 eV</u> Resonances due to (n, γ) processes and some elastic scattering. (see graph on p. 150).

<u>100 eV < E < 1 MeV</u> Unresolved (n, γ) resonances, some elastic scattering and inelastic scattering above the threshold energy of 44 keV

<u>E > 1 MeV</u> Predominantly inelastic scattering and a reduced probability for (n, γ) reactions. Fission above a threshold energy of ~ 1.4 MeV.

Critical Mass for Chain Reactions

Neutrons that are given off during the fission process in ^{235}U or ^{238}U have a $KE \sim 2MeV$. These of course may now interact with U nuclei.

In a sample of U, a chain reaction is produced if the $(\nu-1) \times S$ neutrons produced by a fission event induce more than one additional fission.

In a controlled chain reaction the neutrons produced by one fission will induce exactly one more fission. The remaining neutrons will be lost either by (n, γ) processes or by exiting the sample.

Consider a piece of pure ^{235}U. If each fission neutron produces, on average, νq neutrons, then there will be a net gain of $(\nu q - 1)$ neutrons on the time scale of the fission, τ.

The value of q is <1 and q accounts for the neutrons that are lost.

So, if $n(t)$ neutrons are present at time t then at time $t+dt$ the number will be

$$n(t+dt) = n(t)\left[1 + (\nu q - 1)\frac{dt}{\tau}\right]$$

$$\Rightarrow \quad \frac{dn}{dt} = \frac{(\nu q-1)\,n(t)}{\tau}$$

$$\Rightarrow \quad n(t) = n(0)\exp\left[\frac{(\nu q-1)\,t}{\tau}\right]$$

If $\nu q - 1$ is negative, the exponential is decaying, and $n(\infty) = 0$.

If $\nu q - 1$ is positive, then the exponential is increasing, $n(t)$ becomes very large very fast, and the chain reaction is uncontrolled.

When $\nu q = 1$ (exackly) the number of neutrons remains constant and the chain reaction is controlled.

Since $\nu \sim 2.5$, we require $q \sim 0.4$ for a controlled chain reaction

To study νq, we must determine how large τ is and how far the neutrons travel during that time.

From graph on p. 174, we see that the fission cross section for a 2 MeV neutron in $^{235}U \sim 1$ barn.

The total neutron cross section at this energy ~ 7 barns, the remainder being mainly inelastic scattering.

Thus on average, a neutron undergoes 1 fission per 7 interactions.

The distance the neutron travels between interactions (the mean free path) is

$$l = \frac{1}{\rho \sigma}$$

where

ρ = the number density of nuclei

σ = total cross section

For U, $\rho \sim 4.8 \times 10^{28}\,m^{-3}$, $\sigma \sim 7$ barns $= 7 \times 10^{-28}\,m^2$

so $l \sim 0.03\,m$

A 2MeV neutron traverses this distance in $\sim 1.5 \times 10^{-9}$s.

$\therefore$ Time between fissions $\sim 7 \times 1.5 \times 10^{-9}$s

$\sim 10^{-8}$s

[we may ignore the time taken by the actual fission because it is much smaller than this.]

The total distance from the initial point will not be 0.03m × 7, because at each inelastic scattering event, the direction of travel changes in a random way.

This is a <u>random walk</u> or <u>drunkard's walk</u>

Then average distance travelled $\sim 0.03\text{m} \times \sqrt{7}$

~ 0.8m

Since the (n,γ) reaction does not contribute significantly to neutron loss in this energy range, then 0.8m is a <u>critical radius</u>

A sphere of ^{235}U with radius $\lesssim 0.8$m will have $q \ll 1$, and a chain reaction will not develop.

However if $r \gtrsim 0.8$m, q will be close to 1, and the reaction will be uncontrolled.

This is a rough calculation, but a more accurate calculation gives a value of 0.08m for the critical radius, and the corresponding critical mass is around 52kg.

Moderators and Reactor Control

In a natural mixture of ^{235}U and ^{238}U, there is a small probability of fission being induced by 2MeV neutrons in ^{238}U.

Since the concentration of ^{235}U is small (0.72%) the probability of fission in ^{235}U is also small.

The great majority of neutrons will scatter inelastically with ^{238}U nuclei until their energy is below the fission threshold of 1.4 MeV.

The greatest possibility of inducing fission is at low energy in ^{235}U. Even though the fraction of ^{235}U is very small, the fission cross section is 3 orders of magnitude higher than for either isotope at high energy.

In a reactor using natural U, or U only slightly enriched in ^{235}U, there are 2 major design problems:

(1) How to reduce the energy of the high energy neutrons to ~ 1 eV without losing neutrons to the (n, γ) process, primarily in the ^{238}U

(2) How to sustain the chain reaction in a controlled way.

The most common design of a nuclear reactor is the thermal reactor that uses natural or only slightly enriched ^{235}U as a fuel.

This reactor design utilises a core consisting of a number of uranium fuel rods, each less than the critical mass (typically about 1 cm in diameter).

The fuel rods are bundled to form fuel assemblies in arrays of e.g. 16 × 16 or 8 × 8 rods.

The fuel assemblies are surrounded by a moderator contained in the reactor vessel and are separated by control rods

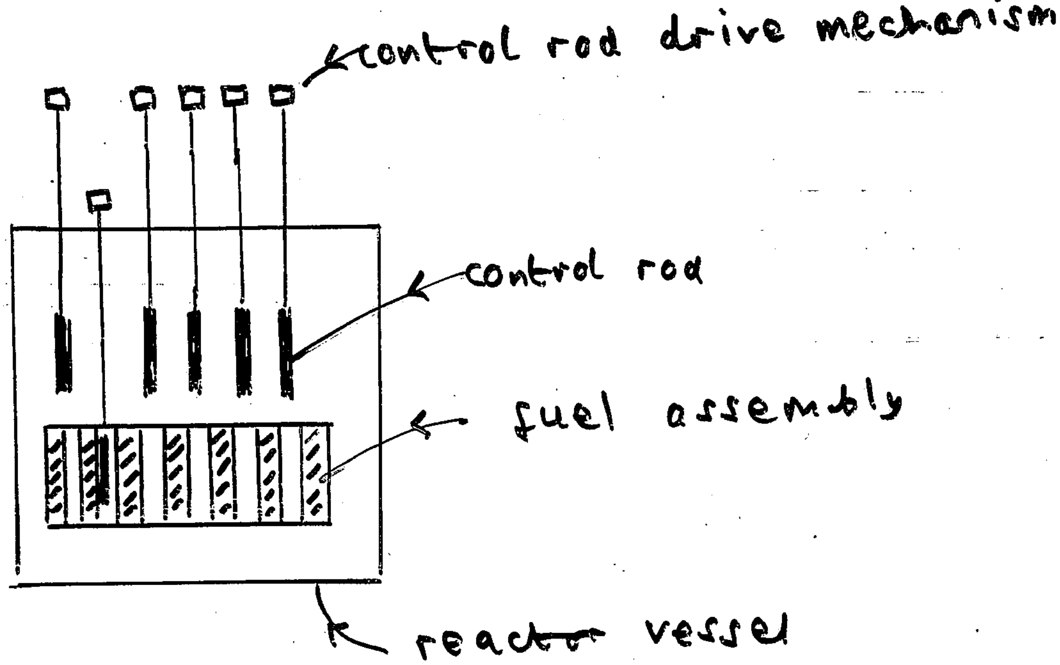
---

Fast neutrons within a fuel rod have a small probability of inducing fission within the rod, but are much more likely to ~~be~~ emitted to the moderator while their energy is in the MeV range.

The <u>moderator</u> is a material which decreases the energy to a fraction of an eV.

These <u>thermalised neutrons</u> pass into another fuel rod where they will have a very high probability of inducing a fission process in ^{235}U because of the large cross section at these energies.

The <u>control rods</u> are made of a material that will effectively stop the neutrons and are used to limit the number of neutrons travelling between fuel rods. [controlling the value of q]



Moderator

The purpose of the moderator is to allow the neutron energy to be reduced in the region outside the fuel rod, thus avoiding neutron loss from (n,γ) reactions in the uranium.

The moderator is a material that interacts with the neutrons across a wide range of energies by elastic and/or inelastic scattering, thus reducing the neutron energy without reducing the number of neutrons.

Desirable criteria are:

1. Inexpensive and easily obtained
2. Small (n,γ) cross section across a wide range of energies.
3. Small nuclear mass, in order to maximise the energy transfer per scattering event.
4. Reasonably high density
5. Chemically stable with a low level of toxicity.

Various materials satisfy some of the criteria; no material is ideal.

Graphite satisfies all but 3, meaning that a large quantity of moderator material is required.

Water satisfies all but 2, and requires the use of ^{235}U enriched fuel.

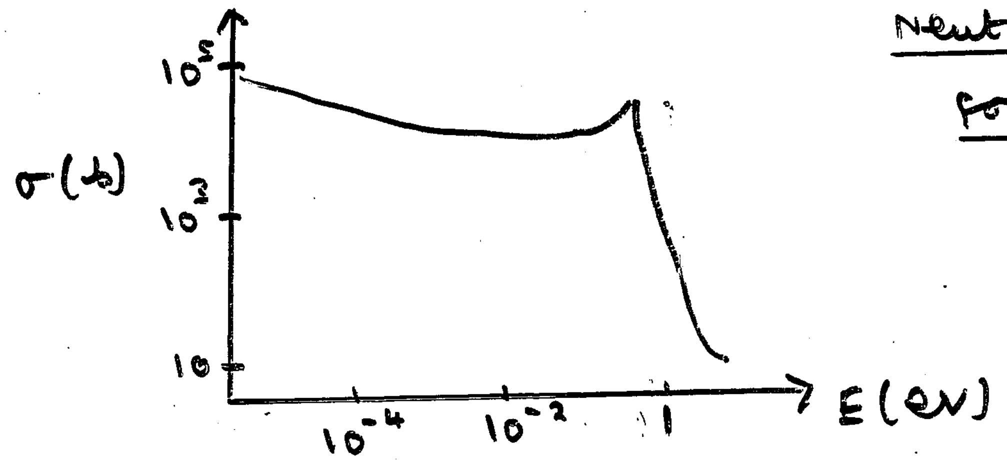
Heavy water (D_2O) has a lower (n,γ) cross section than H_2O, but the processes that do occur ($n + d \rightarrow t + \gamma$) produce radioactive tritium as an undesirable byproduct.

The properties of the moderator determine how much must be used, and how much the U must be enriched in ^{235}U for the reactor to operate efficiently.

Control Rods

The control rods should have a large neutron absorption (i.e. (n, γ) cross section especially at low energies.

Cadmium metal is most commonly used.



Neutron cross section for natural Cd

In figure, the large peak at a fraction of an eV is due to an (n, γ) process in ^{113}Cd that constitutes 12.3% of natural Cd.

Reactor Stability

At first sight it would seem straightforward to control the number of neutrons by use of the control rods.

However there is a problem with the timescale.

The average lifetime of a fission neutron in ^{235}U is $\sim 10^{-8}$ s.

The neutrons spend much of their time in the moderator and their lifetime may be several orders of magnitude longer than this, perhaps 10^{-4} s.

Nevertheless this is still too short to be controlled mechanically by the use of the control rods.

The key to reactor control is with the delayed neutrons

Each fission process produces ν' delayed neutrons (where $\nu' \sim 0.02$) as well as the ν prompt neutrons.

The critical condition is thus

$$(\nu + \nu')\, q = 1$$

If the reactor is designed so that $\nu q = 0.99$ then the control rods can be utilised in a timescale of seconds to regulate the delayed neutrons.

An important factor in reactor stability is the influence of temperature on q.

As fission proceeds, the temperature naturally rises.
For safety reasons, it is important that

$$\frac{dq}{dT} < 0.$$

otherwise, temperature increases could lead to an increase in q and could result in an uncontrolled chain reaction.

One of the main factors influencing $\frac{dq}{dT}$ is the Doppler broadening of the (n, γ) resonances in the fuel rods.

Most of the neutrons that are lost are absorbed in the control rods, but a few pass through a fuel rod when their energy is in the range of the $^{238}U(n, \gamma)$ resonances and are lost by radiative capture.

As T increases, these resonances broaden [Doppler broadening], and the probability that a neutron will have the correct energy to be absorbed increases. This additional probability of neutron absorption corresponds to a decrease in q as T increases.

Reactor Designs

The simplest design is the boiling water reactor.

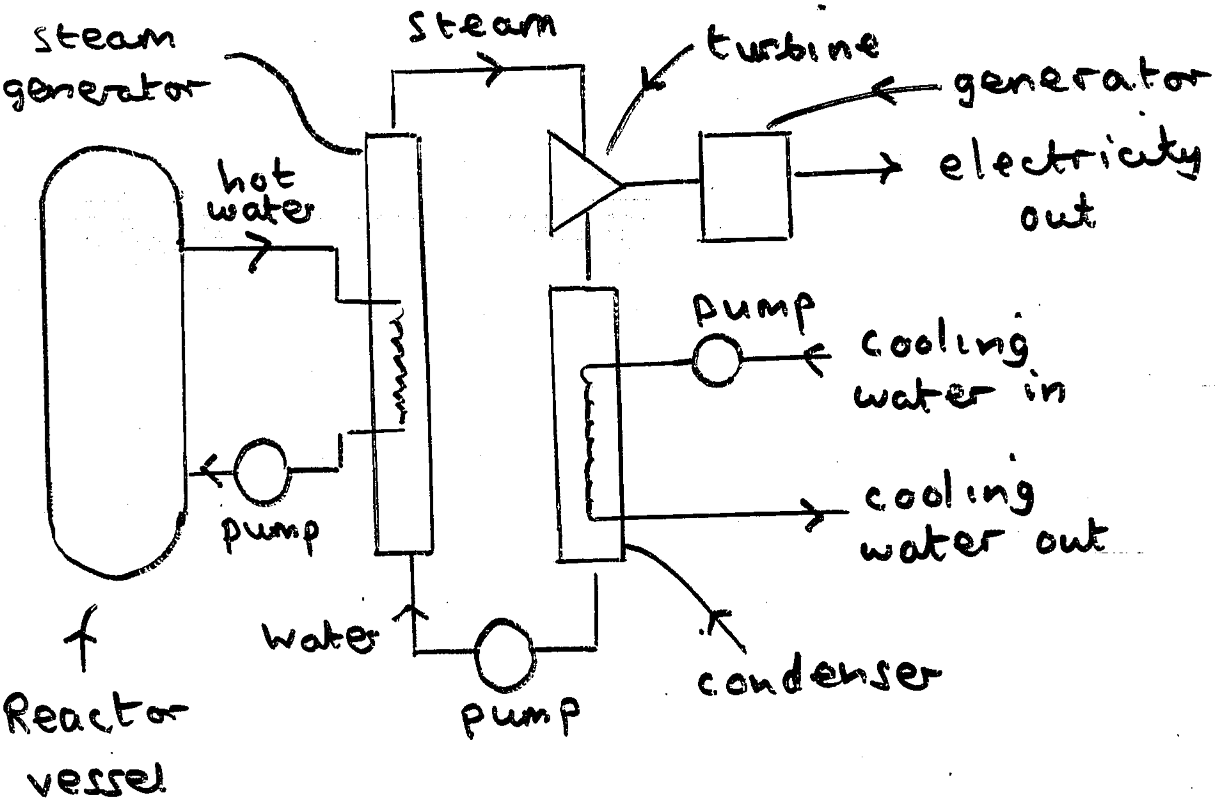
Here the moderator is H_2O and is allowed to boil, the steam being used to run a turbine that drives an electric generator.

Disadvantage: circulating the moderator outside the containment vessel could lead to the spread of radioactive contamination.

A preferable design uses pressurised H_2O in a closed system as moderator, and transfers heat through a heat exchanger to a steam generator.

The use of H_2O means that the U must be enriched to 2-3% ^{235}U.

This design is the most common in the US.



A similar design used in Canada (CANDU) uses D_2O in a closed system as moderator. This design can use natural U as a fuel.

In these designs, the H_2O or D_2O serves both as moderator and to transfer heat to a steam generator. This latter function is necessary not only to extract energy from the reactor but also to cool the reactor core and prevent it from over-heating.

In GB, graphite moderator reactors are mostly used. These use a closed system of circulating He gas to cool the core and transfer heat to a heat exchanger.

Breeder Reactors

An obvious drawback in the reactor above is that it makes use of the fission energy only from ^{235}U not from ^{238}U.
Thus only $\sim 1\%$ of the potential energy from natural U is extracted.

In a <u>breeder reactor</u>, xs fission neutrons are used to generate fissile fuel from non fissile ^{238}U.

Neutron capture produces ^{239}U that decays by β-decay:

$$^{239}U \longrightarrow {}^{239}_{93}Np + e^- + \bar{\nu}_e$$

($\tau \sim 40$ minutes) neptunium

This is followed by plutonium

$$^{239}Np \longrightarrow {}^{239}_{94}Pu + e^- + \bar{\nu}_e$$

($\tau \sim 4$ days)

^{239}Pu is β-stable and has fission properties similar to those of ^{235}U, so is suitable for use in a reactor.

Historical Perspective

1960s, 1970s Many reactors built.
Many operating safely today.

However safety concerns:

- Windscale 1957
- ThreeMileIsland 1979
- Chernobyl 1986
- Operations
- Nuclear waste

put an effective stop to new reactors in the late 1970s. [Also cost of decommisioning]

More recently

New interest and a number of reactors planned or being constructed though not in UK or USA.
[Not breeder reactors, because there is no proven safety record.]

Safety

Improved safety is a major consideration
Many planned reactors utitise very small fuel pellets less than 1mm in diameter covered in ceramic.

The latter prevents spread of radioactive material in the event of overheating.
The pellets are embedded in a graphite matrix (acting as moderator).
Cooling and heat transfer is by circulating He gas.

More recently still

Carbon emission has become a hot issue.
Nuclear fission has the potential to make a major contribution.
(UK) It has split the environmental movement between those who have embraced it and those who still detest it.

The UK government has decided in principle to go ahead with a new generation of fission reactors.

26 Fusion Reactors

Potential advantages over fission:

1. Inexpensive and plentiful supply of heat
2. Reactions that are inherently easier to control and are therefore much safer.
3. Substantially reduced environmental hazards from reactor byproducts.

Unfortunately at present not technologically feasible (despite 50 years plus of sustained work).

This is because of fundamental differences between the fission and the fusion process.

For a fissile material, fission is induced by a thermal neutron because the resulting nuclear state is above the Coulomb barrier.

For fusion, the Coulomb barrier is between a few MeV and a few tens of MeV and is always important.

It is certainly straightforward to accelerate nuclei to these energies, even in small accelerators, and to collide them with other nuclei to produce fusion reactions.

BUT the energy expenditure is much greater than the energy gain!

Thus in a practical sense, fusion always involves energies below the Coulomb barrier so it involves a tunnelling process.

The treatment will be analogous to that of α-decay:

$$\text{Barrier height} = V_c = \frac{e^2}{4\pi\varepsilon_0}\frac{Z_1 Z_2}{R_1+R_2}$$

where Z_i and R_i are the charges and radii of the 2 nuclei.

To maximise the tunnelling probability, the barrier height must be minimised.

Therefore nuclei with small values of Z must be used. They are involved in fusion processes in the sun, and also in those most considered on earth.

The simplest possibility would seem to be the fusion of 2 protons - the p-p process.

However 2 protons cannot ~~form~~ a bound state ; one of the protons is converted to a neutron to give

$$p + p \longrightarrow {}^2H + e^+ + \nu_e$$

The energy release is 0.42 MeV per fusion. Normally an additional 1.02 MeV becomes available due to the annihitation of the e^+.

The process of the ν_e indicates that the weak iteration is responsible. As a result the cross section is very small.

Fusion processes involving <u>deuterons</u> are extremely important.

The simplest is:

$$d + p \longrightarrow {}^3He + \gamma$$

$$(Q = 5.49\,MeV)$$

This is important in stars.

The most obvious process involving 2 <u>deuterons</u> is the formation of ^4He

$$d + d \longrightarrow {}^4He + \gamma$$

$$(Q = 23.8\,MeV)$$

However this process is unlikely as the energy release is well above the neutron and proton separation energies of ^4He. There are 2 possible modes of d-d fusion:

$$d + d \rightarrow {}^3H + p \qquad (Q = 4.0\,MeV)$$

and

$$d + d \longrightarrow {}^3He + n \qquad (Q = 3.3\,MeV)$$

A final process of importance is the fusion of a deuteron with a triton (d-t fusion)

$$d + t \longrightarrow {}^4He + n$$

$$(Q = 17.6 \text{ MeV})$$

Q is high, and this process is very important in controlled fusion reactors.

Distribution of fusion energy

In the process above, there are 2 relatively massive particles produced

Consideration of conservation of momentum and KE (as before) yields

$$\left.\begin{aligned} E_n &= Q\frac{m_{^4He}}{m_{^4He} + m_n} \\ E_{^4He} &= Q\frac{m_n}{m_{^4He} + m_n} \end{aligned}\right\}$$

i.e. the n carries away 80% of the energy.

Since the n has a low reaction cross section, the method of extracting the energy must be considered carefully.

Fusion Cross Sections and Reaction Rates

The cross section for fusion follows from the cross section for charged particle reactions (p. 155):

$$\sigma \sim 1/v^2 \, e^{-G}$$

G is determined by the probability of tunnelling through the Coulomb barrier between the 2 nuclei.

From the discussion of α-decay, this is:

$$G = \frac{2 Z_1 Z_2 e^2}{4\pi\varepsilon_0 \hbar} \sqrt{\frac{2m}{E}} \left[\cos^{-1}\sqrt{\frac{a}{b}} - \sqrt{\frac{a}{b}\left(1-\frac{a}{b}\right)} \right]$$

m is the reduced mass

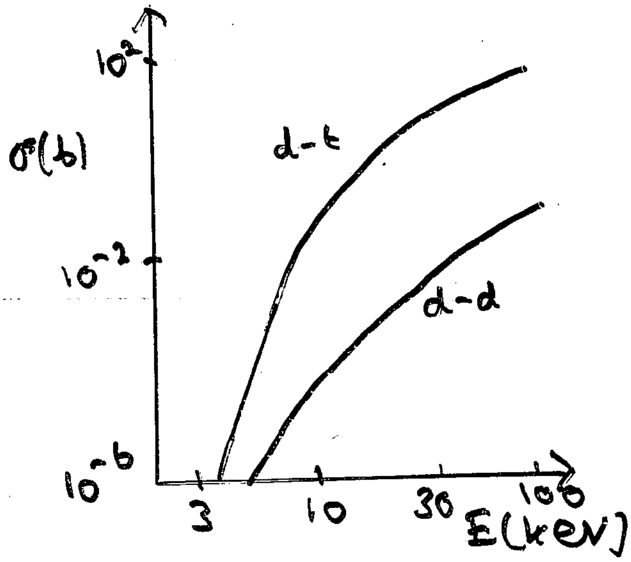
$$a = R_1 + R_2 \quad ; \quad b = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 E}$$

Since $a \ll b$ [high barrier height]

$$\cos^{-1}\sqrt{\frac{a}{b}} \sim \cos^{-1} 0 = \pi/2$$

Then $G = \dfrac{2 Z_1 Z_2 e^2 \pi/2}{4\pi\varepsilon_0 \hbar} \sqrt{\dfrac{2m}{1/2\, m v^2}}$

$$= \frac{2 Z_1 Z_2 e^2 \pi}{4\pi\varepsilon_0 \hbar v}$$



reaction cross sections for d-d and d-t fusion

The reaction rate for fusion can be determined by considering a beam of particles of species "1" with a flux ϕ_1 incident on a collection of particles of species "2" with a number density n_2 and a relative cross section σ.

The reaction rate for fusion per unit volume will be

$$R = \sigma \phi_1 n_2$$

For a plasma of 2 types of particles, as for the interior of a star, or the fuel of a fusion reactor, the flux may be written as

$$\phi_1 = n_1 v$$

where n_1 is the number density of species "1", and v is the relative particle velocity.

Then $$R = \frac{n_1 n_2 \langle \sigma v \rangle}{1 + \delta_{12}}$$

where
$\delta_{12} = 0$ if the 2 species are different
$= 1$ " " " " " the same

[This avoids double counting in the latter case.]

Since both σ and v are functions of the particle energy, it is appropriate to evaluate $\langle \sigma v \rangle$ in terms of the energy distribution of the particles in the plasma.

The <u>Maxwell-Boltzmann</u> distribution gives the probability of finding a particle with speed between v and $v + dv$ at T:

$$P(v)\,dv = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{M}{kT}\right)^{3/2} v^2 e^{-E/kT} \, dv$$

So $\langle \sigma v \rangle = \int \sigma v \, P(v) \, dv$ (*)

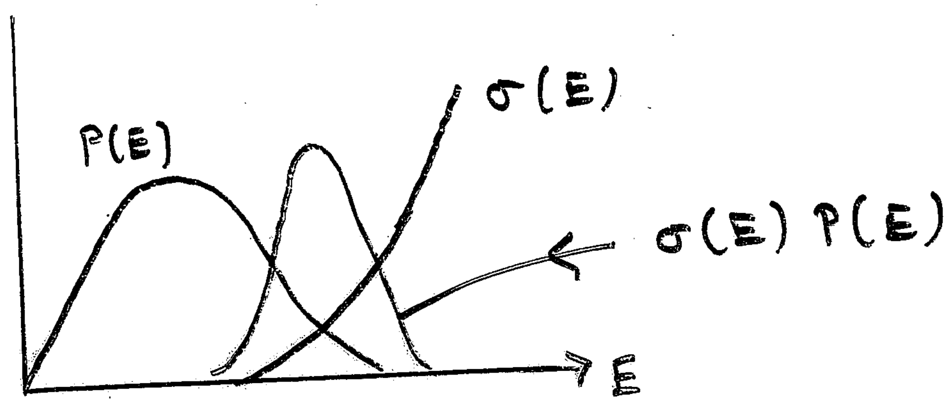
or $\langle \sigma v \rangle \sim \int \frac{1}{v^2} e^{-G} v^2 e^{-E/kT} v \, dv$

$\sim \int e^{-G} e^{-E/kT} v \, dv$

But $dE \sim v \, dv$

$\therefore \langle \sigma v \rangle \sim \int e^{-G} e^{-E/kT} dE$

(*) $\Rightarrow$ $\langle \sigma v \rangle \approx \int \sigma \, P(E) \, dE$



There is a max in $\sigma(E) P(E)$

In fact in most cases for stellar fusion or fusion reactors, energies are comparatively low and it is the portion of the curve with positive gradient that is relevant

energies may be between 1 keV and 10^3 keV

σv " " " 10^{-28} and 10^{-21} $m^3 s^{-1}$

Stellar Fusion Processes

Like most stars, the sun produces energy by fusing 4 hydrogen nuclei into 1 α-particle.

Fusion processes involving heavier nuclei are important in some stars, but relatively unimportant in the sun because:

1. The concentration of heavier nuclei is relatively small - the sun is roughly 92% H, 8% He in terms of number of atoms.

2. Reaction rates for heavier nuclei are small because of the larger Coulomb barriers involved.

The fusion of 4 p's is:

$$4p \longrightarrow \alpha + 2e^{+} + 2\nu_e$$

2 of the fusing protons being converted into neutrons by β^{+}-decay.

To convert to atomic masses, we add 4 electrons to each side:

$$4\,{}^{1}H \longrightarrow {}^{4}He + 2e^{-} + 2e^{+} + 2\nu_e$$

The 4 hydrogen nuclei do not fuse simultaneously to form He; rather there are a series of steps.

First, 2 protons fuse:

$$p + p \longrightarrow {}^{2}H + e^{+} + \nu_e$$

In principle, 2 deuterons could then fuse to form a ${}^{4}He$ nucleus

However the low density of deuterons in the sun and the large value of ε make this process extremely unlikely.

Instead, p-d fusion is more likely

$$d + p \rightarrow {}^{3}He$$

At this stage, because of the high concentration of protons, the most logical process involving ${}^{3}He$ would seem to be the formation of ${}^{4}Li$.

However, ^4Li (3p, 1n) does not form a bound state and leads to the process

$$^3He + p \rightarrow {}^4Li \rightarrow {}^3He + p$$

(No use!)

The reaction of ^3He with a deuteron is unlikely as the deuterons that are formed relatively quickly fuse with protons to form more ^3He.

So a ^3He nucleus is most likely to react with another ^3He to form ^4He:

$$^3He + {}^3He \rightarrow {}^4He + 2\,{}^1H + \gamma$$

$(Q = 12.86\,MeV)$

The overall process described above is the most common method of energy production in the sun and is called the <u>proton-proton cycle</u>.

It has the net result of fusing 4 ps, and with 2 β-decays, produces a ^4He nucleus.

The total energy associated with the process is $Q = 26.7$ MeV.

Most of the energy is eventually converted into solar radiation; a small amount is carried away as KE of the neutrinos and lost.

The rate at which the process proceeds is limited by the weak interaction cross section.

other reactions ultimately resulting in the fusing of 4 H into 1 He are possible:

$$\left.\begin{array}{l} p+p \rightarrow d+e^{+}+\nu_e \\ p+e^{-}+p \rightarrow d+\nu_e \end{array}\right\} \rightarrow d+p \rightarrow {}^3He+\gamma$$

$$\left\{\begin{array}{l} {}^3He+{}^3He \rightarrow {}^4He+2p \quad [\text{as before}] \\ \quad [85\ \%] \\ {}^3He+{}^4He \rightarrow {}^7Be+\gamma \\ \quad [15\ \%] \end{array}\right.$$

$\downarrow$

$^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_e$ OR $^{7}Be + p \rightarrow {}^{8}B + \gamma$

[99.9 %] [0.1 %]

$\downarrow$ $\downarrow$

$^{7}Li + p \rightarrow 2(^{4}He)$

$^{8}B \rightarrow {}^{8}Be + e^{+} + \nu_e$

$\downarrow$

$^{8}Be \longrightarrow 2(^{4}He)$

In stars with higher temperatures, particle energies are higher and there is a probability of a fusion process involving heavier nuclei where Coulomb barriers are higher.

e.g. carbon-nitrogen-oxygen (CNO) cycle

$$^{12}C + p \rightarrow {}^{13}N + \gamma$$

$$^{13}N \rightarrow {}^{13}C + e^{+} + \nu_e$$

$$^{13}C + p \rightarrow {}^{14}N + \gamma$$

$$^{14}N + p \rightarrow {}^{15}O + \gamma$$

$$^{15}O \rightarrow {}^{15}N + e^{+} + \nu_e$$

$$^{15}N + p \rightarrow {}^{12}C + {}^{4}He$$

This cycle is equivalent to the <u>proton-proton cycle</u> since it ultimately fuses 4 H into 1 He.

As in the proton-proton cycle, 2 β-decay processes are included.

Since C is neither created nor destroyed but merely acts as a <u>catalyst</u>, Q for the process is the same as for the proton-proton cycle

This process requires more energy to overcome the Coulomb barrier, but it is not limited by the rate of deuteron production.

Thus the CNO cycle is <u>less</u> likely than the proton-proton cycle at low temperatures, but becomes <u>more</u> likely at high temperatures.

The cross-over point is around 1.5×10^7 K

Internal temperature of the sun $\sim 10^7 K$
$\therefore$ Energy production in the sun is dominated by the proton - proton cycle.

Stars that have depleted their supply of H can no longer produce energy by the proton - ~~proton~~ or the CNO cycle.

This frequently happens in the central region of a star where temperatures and hence hydrogen fusion rates are greater resulting in a core made up mostly of ^4He.

If the temperature is sufficiently high ($\gtrsim 10^8 K$) further energy can be produced by He fusion:

$$^4He + {}^4He \longrightarrow {}^8Be + \gamma$$

The ^8Be is unstable, and decays by α - decay

$$^8Be \longrightarrow {}^4He + {}^4He$$

However the lifetime for this process is 7×10^{-17} s.

Before decaying, some ^{8}Be will fuse with another ^{4}He:

$$^{8}Be + {}^{4}He \rightarrow {}^{12}C + \gamma$$

This process is known as a triple α-process. It has a Q of 7.27 MeV

Heavy nuclei can be synthesised in the interiors of stars by processes involving neutron capture and α-particle capture.

Fusion is no longer energetically favourable for $A \gtrsim 55$, so the relative abundance of elements heavier than Fe in the universe is much less than that of elements lighter than Fe.

Fusion Reactors

Fuel	Reaction	Mass (g) of reactive component per kg of fuel	Energy (J) per kg of fuel
chemical	chemical		6×10^{6}
natural uranium	^{235}U fission	7.2	6×10^{11}
natural water	p-p fission	110	7×10^{12}
natural water	d-d fission	0.016	3×10^{9}

N.B. A chemical reaction is the burning of coal or oil.
For fission it is assumed that all available fission energy can be extracted from ^{235}U, but none from ^{238}U.

In all cases fission/fusion give much higher yields of energy/kg than chemical reactions.
This is because electron binding energies are in eV, nuclear ones in MeV

p-p fusion gives a very high yield because of the large no. of protons in 1 kg of water.

d-d fusion gives a much lower yield because of the much lower natural abundance of ^2H.

On this basis, it would be desirable to use p-p fusion in reactors.

However, because the reaction is dominated by the weak interaction, its cross section is small.

∴ Though it is important in the sun, it is unsuitable for a fusion reactor.

The d-d reactions make this an attractive energy source.

However (Figure on p. 201), the reaction rate for d-t fusion is 2 orders of magnitude higher, so current fusion reactor research uses this scheme

To gain energy from a sustained fusion reaction, it is essential that the fusion energy output be greater than the sum of the energy input required to heat the fuel and the energy losses in the system.

The energy gained per unit time per unit volume (power per unit volume) can be expressed as

$$P_f = RQ$$

where R is the reaction rate (from before) and Q is the energy per fusion.

N.B. Because of the required high temperatures, the working substance will inevitably be in the plasma state - fully ionised.

The fusion power per unit volume can be determined from the values of $\langle \sigma v \rangle$ shown before.

energy losses

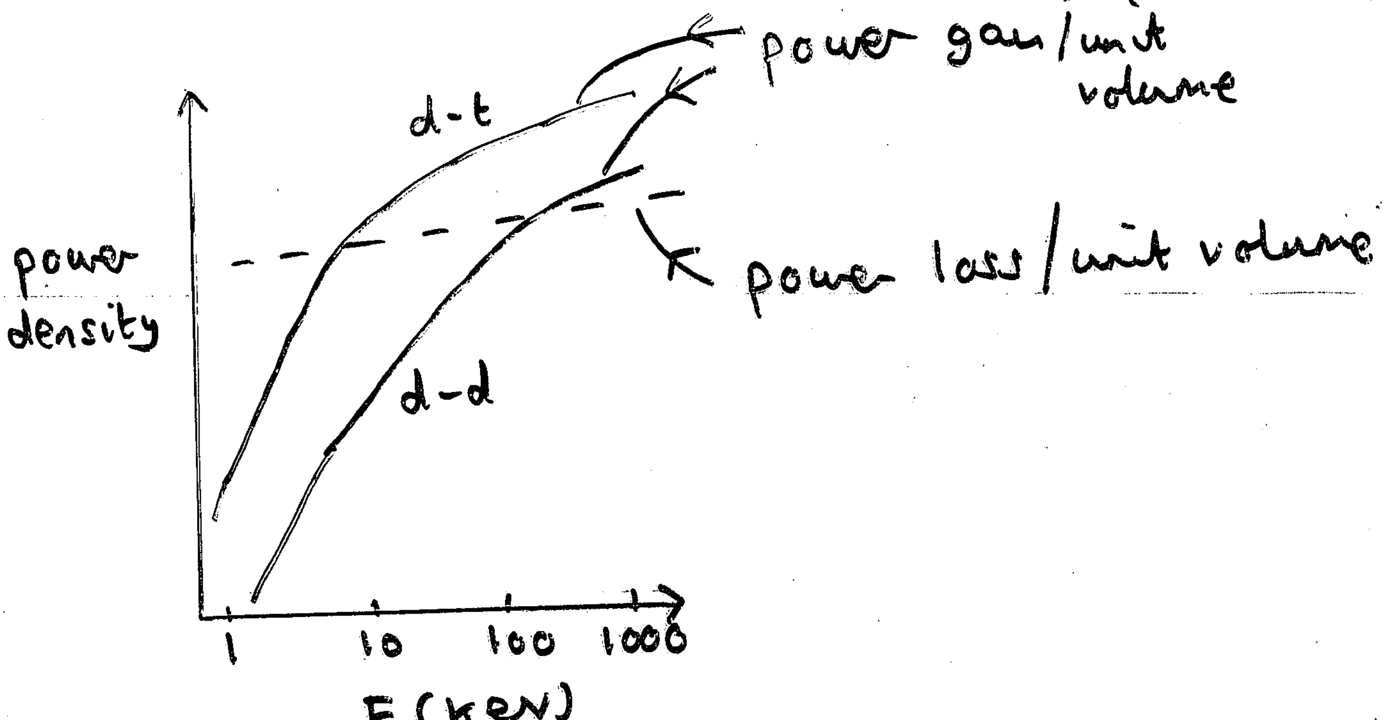
A plasma loses energy by bremsstrahlung (braking radiation). This loss results primarily from the behaviour of the electrons, which, being much less massive than the ions, experience greater acceleration during electronic interactions.

The power radiated per unit volume is:

$$P_r = \frac{4\pi n n_e Z^2 e^6}{3(4\pi\varepsilon_0)^3 c^3 \hbar} \left(\frac{3kT}{m_e^3} \right)^{1/2}$$

where n and n_e are the ion and electron densities.

This is the same for d-d and d-t plasmas since the value of Z is the same.



This tells us the ~~temperature~~ requirement much more challenging for d-d than for d-t.

Now compare fusion energy produced with thermal energy needed to heat the plasma

Fission energy per unit volume:

$$E_f = \frac{n^2 \langle \sigma v \rangle}{4} Q\tau \quad \text{[from equation on p. 214]}$$

where τ is the time for which the plasma is held at the required temperature and pressure.

[we have considered d-d fusion where $n_1 = n_2 = n/2$.]

The thermal energy needed to heat the plasma is the sum of the energy needed to heat the electrons and the ions.

If $n_e = n$, then the total thermal energy per unit volume is

$$E_{th} = \frac{3}{2} nkT + \frac{3}{2} nkT = 3nkT$$

For a plasma above the temperature corresponding to the bremsstrahlung crossing point, radiative losses may be ignored.

Then we require

$$\frac{n^2 \langle \sigma v \rangle \mathcal{E} \tau}{4} > 3nkT$$

$$n\tau > \frac{12\,kT}{\langle \sigma v \rangle \mathcal{E}}$$

This is the Lawson criterion

The LHS " " Lawson parameter

The value of the Lawson parameter for which the reactor produces net energy depends on the operating temperature and specific fusion reaction.

Nevertheless, the equation is a general guideline for reactor design criteria.

It indicates that the achievement of a useful controlled fusion reaction requires that a sufficient plasma density must be obtained for a sufficient period of time.

Plasma Confinement

The plasma must be confined
If it touches the wall of the vessel, it will:

1. melt the vessel
2. lose energy

Methods of Confinement

1. Magnetic confinement
2. Inertial confinement

Magnetic Confinement

Control motion of ions and electrons by suitable magnetic fields

1. Linear Geometry

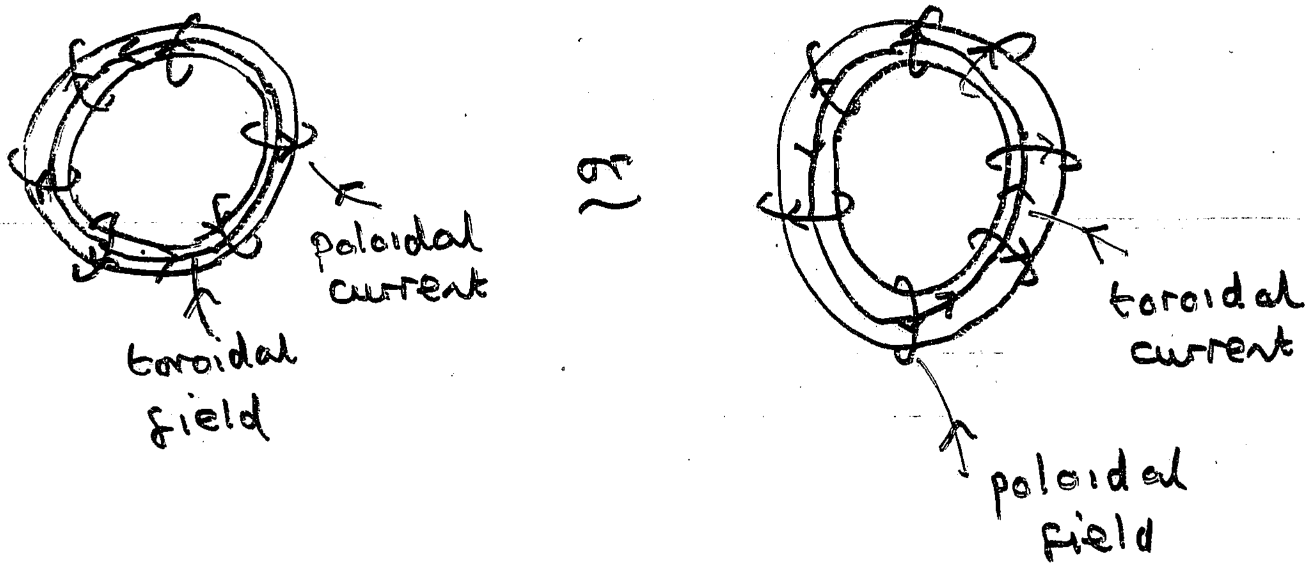
This uses a plasma column pinched at ends. There is an axial magnetic field. The particles travel in a region of low field along the column and are reflected by a region of high field at the ends.

"Magnetic mirror confinement"

"θ - pinch", "z - pinch" etc

In general not as successful as other methods in approaching the Lawson criterion

2. Toroidal Geometry



"Stellerator"

"Tokamak"

Inertial Confinement

Here the fusion fuel is confined by inertial forces in the plasma itself.

Most experiments in this category are laser fusion experiments.

A pellet of fuel (usually a mixture of deuterium and tritium about a millimetre in diameter) is bombarded from several directions by high energy laser beams.

The fuel pellet heats rapidly to a temperature suitable for fusion to ~~take~~ place.

In more detail:
When the laser radiation is absorbed by the fuel pellet, it heats it from the outside.
The heat propagates through the pellet, ~~transforming~~ the outer section to a plasma.
This outer plasma atmosphere is driven off as it heats and expands. This is called ablation.

The remaining core of the pellet is compressed and heated by the inertial forces resulting from the expanding planetary atmosphere.

The lasers are pulsed and the duration of the process is $\sim 10^{-9}$ s.

During this time T is very high and the density of the pellet core can be compressed to densities several thousand times the density of water. So although τ is short, n may be large enough to aim at achieving the Lawson criterion.

N.B. Even when a functioning fusion reactor has been constructed, it is still a major task to design and construct a viable reactor that can produce energy on a commercial scale.

[fission
First experimental reactor 1942
" commercial " ~1962]

Estimated World Energy Reserves

fossil fuels		200 years
fission (^{235}U non breeder reactor)		10^4 "
"	(" breeder ")	10^6 "
d-t	fusion	10^7 "
d-d	fusion (from sea water)	10^{10} "