ABC of SUSY

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Abstract

This article is a very basic introduction to supersymmetry. It introduces the two kinds of superfields needed for supersymmetric extensions of the Standard Model, the chiral superfield and the vector superfield, and discusses in detail how to construct supersymmetric, gauge invariant Lagrangians. The main ideas on how to break supersymmetry spontaneously are also covered. The article is meant to provide a platform for further reading.

1 Introduction

This is neither a review article, nor a summary of supersymmetry. There are already many excellent reviews available. The standard reference for a comprehensive introduction and review of supersymmetry has been written by Martin [1]. Recently, an introduction with applications to particle theory has also been written by Peskin [2] and there are earlier articles of Olive [3] and Drees [4], the latter with an extended discussion of quadratic singularities . The Physics Reports of Haber and Kane [5] and Nilles [6] are early review articles about supersymmetry. The former contains a comprehensive discussion of the minimal supersymmetric extension of the Standard Model (MSSM), the latter includes supergravity. An introduction including material for $N > 1$ supersymmetry can be found in the Tasi lecture notes of Lykken [7]. An up-to-date view on breaking supersymmetry is given in the lecture notes of Dine [8] or Intriligator and Seiberg [9]. Needless to say that this list is by no means exhaustive or in any way selective.

As the title suggests, this article is meant to guide the reader through the first few steps of understanding susy. Thus it is for those who have a first go at susy or usually get stuck somewhere between page 2 and page 5 of other introductions and reviews. The hope is that after reading this article the other articles are easier to understand. Accordingly, this article stops where all the others begin in earnest. In particular it does not contain any serious applications to collider physics or cosmology nor does it cover any developments of the past few years or anything beyond $N = 1$ susy. It only covers the very basic concepts of global $N = 1$ susy, but hopefully does so in more detail than the above mentioned articles.

The article assumes a basic understanding of field theory and gauge theory and is meant to provide an as direct as possible path to writing down the MSSM. At the same time it aims to be precise in that nothing essential is left out or swept under the rug. In the main text the basic ideas are given and illustrated. We start in Section 2 with a discussion of symmetries and the extension of the Poincaré symmetry to include susy. In Section 3 the minimal amount of technicalities needed are covered, Weyl spinors (which we use throughout) and Grassmann variables. Section 4 introduces the concepts of superspace and superfields. These will turn out to be indispensable in Section 5 which is the main section and discusses the construction of susy theories. This section concludes with writing down the unbroken MSSM after which we turn to breaking susy in Section 6. The basic possibilities to break susy spontaneously and their problems in realistic applications are discussed and the notion of soft breaking is explained. This is where we stop with our ABC of SUSY and leave the reader to make the steps from D to Z with the help of other articles. It should be possible to follow through the main text without delving into the gory details of conventions and indices. However, for a full understanding these details are required. For the reader willing to get his/her hands dirty, the conventions used in this article are given in Appendix A. Finally, Appendix B presents some sample calculations whose results are used in the main text. These details are often not available in other articles and hopefully provide some help in understanding the technicalities.

2 (Super)Symmetries

A symmetry is a group of transformations that leaves the Lagrangian invariant. Two of the reasons why symmetries are very important are: first according to the Noether theorem, with each continuous symmetry we can associate a conserved quantity and second and even more importantly, nature seems to respect many of them. A continuous symmetry is one that depends continuously on one or several parameters. As an example consider rotations and space translations. To determine a three dimensional rotation completely we need three parameters (angles) which we will denote by $\vec{\vartheta}$. The parameters of the translation are denoted by $\vec{a}$. Under such a transformation

$$\vec{x} \rightarrow \vec{x}' = R(\vec{\vartheta}) \cdot \vec{x} + \vec{a} \tag{2.1}$$

where R is a 3×3 rotation matrix depending on $\vec{\vartheta}$ and $R(\vec{0}) = 1$. In a quantum mechanical system, under such a transformation a state $\psi(\vec{x})$ transforms as

$$\psi(\vec{x}) \rightarrow \psi'(\vec{x}) = e^{-i\,\vec{a}\cdot\vec{P}}\, e^{-i\,\vec{\vartheta}\cdot\vec{J}}\psi(\vec{x}) \tag{2.2}$$

where J_i and P_i, $i \in \{1, 2, 3\}$ are called the generators of the rotations and translations respectively. The explicit form of the generators depends on the precise nature (spin) of the state but in any case they satisfy the familiar commutation relations

$$\begin{align}
\left[P_i, P_j\right] &= 0 \tag{2.3}\\
\left[J_i, J_j\right] &= i\,\epsilon_{ijk}\,J_k \tag{2.4}\\
\left[P_i, J_j\right] &= i\,\epsilon_{ijk}\,P_k \tag{2.5}
\end{align}$$

The remarkable fact is that nature respects rotational and translational symmetry, i.e. the Lagrangian of any fundamental theory has to be invariant under Eq. (2.1). This is a crucial help in constructing theories that have a chance of being realized in nature.

This is all fine and good, but in fact we know we can do better. We can enlarge the symmetry group. The symmetry group that lies at the heart of every Quantum Field Theory (QFT) is the Poincaré group consisting of Lorentz transformations (LT) and translations

$$x^\mu \rightarrow x'^\mu = x^\mu + \omega^{\mu\nu}x_\nu + a^\mu \tag{2.6}$$

where $x^\mu = (t, \vec{x})$ denotes the coordinates in Minkowski space-time. To specify completely an arbitrary Poincare transformation, we need six Lorentz parameters (three boost parameters $\vec{\varphi}$ and three rotation angles $\vec{\vartheta}$), written in terms of an antisymmetric tensor of rank two, $\omega^{\mu\nu} = -\omega^{\nu\mu}$, as well as four translation parameters a^μ. Thus, the LT involves six generators, three for rotations and three for boosts. They are written in terms of an antisymmetric tensor $M^{\rho\sigma} = -M^{\sigma\rho}$, where the Lorentz labels ρ, σ play the role of the label i in J above. The translations require four generators P^ρ, one for each direction. The quantities P^ρ and $M^{\rho\sigma}$ correspond to the 4-momentum and the generalized angular momentum.

The explicit form of the generators depends on the nature of the field they act on. For a spin 1/2 field e.g. we have

$$P^\rho = i\,\partial^\rho\,; \qquad M^{\rho\sigma} = i(x^\rho\partial^\sigma - x^\sigma\partial^\rho) + \frac{i}{4}[\gamma^\rho, \gamma^\sigma]\,; \tag{2.7}$$

whereas for a scalar field, the last term in $M^{\rho\sigma}$, corresponding to the spin, is absent. The transformation of an arbitrary classical field Φ under Eq. (2.6) can now be written as

$$\text{translations} \;:\; \Phi(x) \to \Phi'(x) = e^{i\,a^\rho P_\rho}\Phi(x) \tag{2.8}$$

$$\text{LT} \;:\; \Phi(x) \to \Phi'(x) = e^{\frac{i}{2}\,\omega^{\rho\sigma}M_{\rho\sigma}}\Phi(x) \tag{2.9}$$

The factor 1/2 in Eq. (2.9) is conventional and compensates for the fact that in summing over ρ and σ we count every term twice due to the antisymmetry. The dependence on the nature of the field Φ is only implicit in the representation to be used for the generators. Note that Eqs. (2.8) and (2.9) contain Eq. (2.2) as a special case.

Finally, we can look at the algebra of the Poincaré group, i.e. the commutation relations between the various P^ρ and $M^{\rho\sigma}$. They can be obtained by using Eq. (2.7) and $[x^\rho, P^\sigma] = -i\,g^{\rho\sigma}$ and read

$$[P^\rho, P^\sigma] \;=\; 0 \tag{2.10}$$

$$[P^\rho, M^{\nu\sigma}] \;=\; i(g^{\rho\nu}P^\sigma - g^{\rho\sigma}P^\nu) \tag{2.11}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] \;=\; -i(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}) \tag{2.12}$$

Note that as for Eqs. (2.3)–(2.5), Eqs. (2.10)–(2.12) are independent on the nature/spin of the fields, i.e. on whether or not we include the second term of $M^{\rho\sigma}$ in Eq. (2.7). What is important for us is that all generators mix, in particular, according to Eq. (2.11), the translations and LT are linked together.

Let us pause for a moment to consider what we have done in going from the symmetry under Eq. (2.1) to Eq. (2.6). We have increased the symmetry group from 6 generators to 10 generators. In doing so, we have also increased the number of coordinates that are involved in the transformations from 3 in $\vec{x}$ to 4 in x^μ. Note also, that the "new" generators such as M^{0i} etc. mix in a non-trivial way with the "old" ones such as J_i. The latter are latent in M^{ij}.

Since nature respects Poincaré symmetry, it is natural to ask, whether the symmetry can be extended even further. The answer is obviously yes, since this is precisely what is done in gauge theories. For a certain gauge group, say $SU(N)$ we add generators T^a with $a \in \{1\ldots N^2 - 1\}$. A finite gauge transformation is then specified by $N^2 - 1$ parameters ω^a and is written as $\exp(i\omega^a\,T^a)$. However, such an extension is called *trivial* because the "new" generators all commute with all of the "old" generators

$$[T^a, T^b] \;=\; i\,f^{abc}T^c \tag{2.13}$$

$$[T^a, P^\rho] \;=\; 0 \tag{2.14}$$

$$[T^a, M^{\rho\sigma}] \;=\; 0 \tag{2.15}$$

where f^{abc} are the structure constants of the gauge group. This means that the extended symmetry group is a direct product of the Poincaré group with a gauge (or internal symmetry) group.

Such extensions of the Poincaré group are very successful in describing particle interactions, but not really what we are after. The question is whether we can extend the Poincaré group in a non-trivial way, such that the new generators mix with P^ρ and/or $M^{\rho\sigma}$. The answer to this question is given by the *Coleman-Mandula no-go theorem* [10], which states that any symmetry compatible with an interacting relativistic QFT is of the form of a direct product of the Poincaré algebra with an internal symmetry, such as gauge symmetry.

This would be the end of this article if it was not for the fact that for every no-go theorem there is usually a way around. In the proof of the Coleman-Mandula theorem there was an implicit assumption that only bosonic generators are involved. A bosonic generator is a generator that transforms a bosonic (fermionic) state into another bosonic (fermionic) state. All generators P^ρ, $M^{\rho\sigma}$ and T^a are obviously bosonic since they do not change the spin of the state they act on. What if we allow *fermionic generators*, more precisely generators that change the spin of the state by 1/2? It is clear that such a generator has to have a spinor label α for if it acts e.g. on a scalar (spin 0) state it generates an spin 1/2 state. Thus, denoting the fermionic generator by Q_α we have

$$Q_\alpha|\text{bos}\rangle = |\text{ferm}\rangle_\alpha\,; \qquad Q_\alpha|\text{ferm}\rangle^\alpha = |\text{bos}\rangle\,; \tag{2.16}$$

We will be working with Weyl spinors throughout. To represent a Dirac spinor with four components, we need two Weyl spinors (see Section 3) which are conventionally denoted by Q_α and $\bar{Q}_{\dot\beta}$ with $\alpha, \dot\beta \in \{1, 2\}$. The generators are related by $(Q_\alpha)^\dagger = \bar{Q}_{\dot\alpha}$ and it is simply a matter of notation that Q is written with normal (undotted) indices whereas $\bar{Q}$ is written with dotted indices.

If we allow for one set of such fermionic generators (corresponding to $N = 1$ supersymmetry) according to the *Haag-Lopuszanski-Sohnius theorem* [11] we can in fact extend the Poincaré algebra of Eqs. (2.10)–(2.12) in a non-trivial way to the $N = 1$ super Poincaré algebra:

$$[Q_\alpha, P^\rho] = 0 \tag{2.17}$$

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\sigma^\rho)_{\alpha\dot\beta} P_\rho \tag{2.18}$$

$$[M^{\rho\sigma}, Q_\alpha] = -i(\sigma^{\rho\sigma})_\alpha^{\ \beta} Q_\beta \tag{2.19}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \tag{2.20}$$

We could add another set of fermionic operators, ending up with $N = 2$ supersymmetry, or in fact add even more sets. We will restrict ourselves to $N = 1$ however, because $N > 1$ theories are ruled out as a "low-energy" (i.e. TeV) extension of the Standard Model, as will be explained in Section 4.3.

Note that the relations between two fermionic generators are given by anticommutators, whereas relations involving at least one bosonic operator involve the commutator. We will not delve into the derivation of Eqs. (2.17)–(2.20). We only note that the addition

4

of fermionic generators also implies that we will have to increase the set of coordinates (as we had to when extending Eqs. (2.3)–(2.5)), a point we will come back to in Section 4.

It is important to realize what a strong motivation this provides. We know that symmetries play a crucial role in physics and, in particular, that the Poincaré symmetry is realized in nature. At the same time, the only way to increase the Poincaré symmetry is supersymmetry. It is for this reason that supersymmetry takes a somewhat special status in the many possible scenarios of physics beyond the Standard Model. We also remark that many motivations usually mentioned, in particular the solution to the hierarchy problem, are simply consequences of the increased symmetry in the theory. While other approaches might solve the hierarchy problem as well, susy was not initially introduced to solve this problem (nor to unify gauge couplings).

3   Weyl spinors and Grassmann variables

In this section we present the minimal amount of technicalities required to be able to construct and write down supersymmetric and Lorentz invariant theories in an efficient way. More details on the conventions and notations used are given in Appendix A.

When dealing with fermions, we usually use Dirac spinors $\Psi(x)$ with four components. However, in susy theories it is more convenient to work with Weyl spinors, $\psi(x)$ and $\chi(x)$, each with two components only, writing

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}; \qquad \overline{\Psi} = \begin{pmatrix} \chi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix}; \tag{3.1}$$

Note that the bar over a Dirac spinor and a Weyl spinor mean something different. For the Dirac spinor $\overline{\Psi} \equiv \Psi^\dagger \gamma^0$ denotes the usual Dirac adjoint, whereas for Weyl spinors the bar indicates that if ψ_α transforms with a certain matrix M under LT, $\bar{\psi}_{\dot{\alpha}}$ transforms with the complex conjugate matrix M^*, see Eq. (A.1). Using the explicit form of γ^0, Eq. (A.8), in Eq. (3.1) we find the precise relation between them

$$\bar{\psi}_{\dot{\alpha}} = \left[\psi_\alpha \right]^\dagger; \qquad \chi^\alpha = \left[\bar{\chi}^{\dot{\alpha}} \right]^\dagger; \tag{3.2}$$

The indices α, $\dot{\alpha}$ run from 1 to 2 and, as for the generators Q, it is simply a matter of notation that Weyl spinors corresponding to the first two (last two) components of a Dirac spinor are written with undotted (dotted) indices.

The helicity projection operators acting on a Dirac spinor yield

$$P_L \Psi \equiv \frac{1}{2}(1 - \gamma_5)\Psi = \psi_\alpha; \quad P_R \Psi \equiv \frac{1}{2}(1 + \gamma_5)\Psi = \bar{\chi}^{\dot{\alpha}}; \tag{3.3}$$

Thus, ψ_α and $\bar{\chi}^{\dot{\alpha}}$ are called left-handed and right-handed Weyl spinors respectively. The indices of Weyl spinors can be raised/lowered with the totally antisymmetric ϵ-tensor, Eq. (A.2). The whole machinery is set up such that products of Weyl spinors such as

$$\chi\psi \;\; \equiv \;\; \chi^\alpha \psi_\alpha = \chi^\alpha \epsilon_{\alpha\beta} \psi^\beta \tag{3.4}$$

$$\bar{\chi}\bar{\psi} \;\; \equiv \;\; \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}} \tag{3.5}$$

are Lorentz invariant. Note the different positions of the dotted and undotted indices in the definition of the products.

Having written Dirac 4-spinors in terms of Weyl 2-spinors we have to do the same for Dirac 4×4 matrices. They are written in terms of Pauli 2×2 matrices σ^μ and the related matrices $\bar\sigma^\mu$. The details are given in Eqs. (A.6) and (A.8). What is important for us is that with this setup we are now able to write the bilinear covariants that appear in Lagrangians in terms of Weyl spinors. In particular we have

$$
\begin{aligned}
\overline{\Psi}\Psi &= \chi\psi + \bar\psi\bar\chi \equiv \chi^\alpha\psi_\alpha + \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} \\
\overline{\Psi}\gamma^\mu\Psi &= \chi\sigma^\mu\bar\chi - \psi\sigma^\mu\bar\psi \equiv \chi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} - \psi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\psi^{\dot\alpha}
\end{aligned}
\tag{3.6}
$$

with a more complete list of relations given in Eq. (A.14). Thus the standard Lagrangian for a free Dirac spinor can be written in terms of Weyl spinors as

$$
i\,\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\,\overline{\Psi}\Psi = i\,\chi\sigma^\mu\partial_\mu\bar\chi + i\,\psi\sigma^\mu\partial_\mu\bar\psi - m\,\chi\psi - m\,\bar\psi\bar\chi
\tag{3.7}
$$

where we used integration by parts $-i\,(\partial_\mu\psi)\sigma^\mu\bar\psi = i\,\psi\sigma^\mu\partial_\mu\bar\psi$. Sometimes identities like $\psi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\psi$ are used to write the kinetic part of the Lagrangian such that the r.h.s. of Eq. (3.7) resembles more closely the l.h.s.

A Majorana spinor can be written in terms of a single Weyl spinor as

$$
\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} ; \qquad \overline{\Psi}_M = \begin{pmatrix} \psi^\alpha & \bar\psi_{\dot\alpha} \end{pmatrix} ;
\tag{3.8}
$$

and the standard Lagrangian written in terms of Weyl spinor reads

$$
\frac{i}{2}\overline{\Psi}_M\gamma^\mu\partial_\mu\Psi_M - \frac{m}{2}\overline{\Psi}_M\Psi_M = \frac{i}{2}\left(\psi\sigma^\mu\partial_\mu\bar\psi - (\partial_\mu\psi)\sigma^\mu\bar\psi\right) - \frac{m}{2}\left(\psi\psi + \bar\psi\bar\psi\right)
\tag{3.9}
$$

Of course, we could use integration by parts again, but prefer to write the Lagrangian in symmetric form.

It might seem that we have made a step backwards in introducing Weyl spinors, since the l.h.s. of the above equations clearly are more compact than the r.h.s. However, the theories we are interested in (i.e. supersymmetric extensions of the Standard Model) are intrinsically chiral and it will turn out to be an advantage if this is reflected in our formalism from the beginning. What is important to realize is that expressions that look rather complicated, actually have a very simple behaviour under Lorentz transformations. If all spinor and all Lorentz indices are contracted, the expression is invariant under Lorentz transformations. If there is one free Lorentz index, it transforms as a four vector etc. Thus, simply by looking at the expression we will be able to determine the transformation property. This is an invaluable tool for constructing Lorentz invariant Lagrangians and we want to have a similar formalism for constructing supersymmetric Lagrangians.

In order to achieve this we have to introduce another technical tool, Grassmann variables, or more precisely, Grassmann spinors. A Grassmann variable (or fermionic variable) is like any other variable, except that it anticommutes with other Grassmann

variables (and commutes with ordinary variables). This behaviour is similar to the behaviour of the generators in the Poincare algebra Eqs. (2.17)–(2.20). We can think of Grassmann variables as anticommuting complex numbers.

A Grassmann spinor θ^α or $\bar\theta^{\dot\alpha}$ is made of two Grassmann variables

$$\theta^\alpha = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}; \quad \bar\theta^{\dot\alpha} = \begin{pmatrix} \bar\theta^1 \\ \bar\theta^2 \end{pmatrix}; \tag{3.10}$$

with each entry being a Grassmann variable, i.e. $\{\theta^\alpha, \theta^\beta\} = \{\theta^\alpha, \bar\theta^{\dot\beta}\} = 0$ and, in particular $\theta^\alpha \theta^\alpha = 0$ ($\alpha \in \{1, 2\}$, no summation). Note that in agreement with Eq. (3.4) the product of a Grassmann spinor with itself is given by $\theta\theta = \theta^1\theta_1 + \theta^2\theta_2 = -2\theta^1\theta^2$ and does not vanish. However, adding one more factor of θ^α does give zero. This means that if we Taylor expand an arbitrary function $\phi(\theta)$ in θ and include all terms up to the $\theta\theta$ term, we actually reproduce the full function. Thus we can parameterize any function $\phi(\theta)$ in terms of two constants c and f and a constant Grassmann spinor ζ and write

$$\phi(\theta) = c + \theta\zeta + f\,\theta\theta \tag{3.11}$$

This will be important later on.

We also remark that with the help of Grassmann spinors we can write the super Poincaré algebra entirely in terms of commutators. In particular we have in place of Eq. (2.18)

$$[\theta Q, \bar\theta\bar Q] \equiv [\theta^\alpha Q_\alpha, \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}] = 2\,\theta\sigma^\mu\bar\theta\,P_\mu \tag{3.12}$$

Finally, we also need to introduce differentiation and integration with respect to Grassmann variables. Derivatives with respect to Grassmann variables are defined in Eq. (A.17) and differentiating e.g. $\phi(\theta)$ as given in Eq. (3.11) with respect to θ^α we get $\partial_\alpha\phi \equiv \partial/\partial\theta^\alpha\,\phi = \zeta_\alpha + 2f\theta_\alpha$. The integration is defined such that it always picks out the highest part in the Taylor expansion of the function. The details are given in Eq. (A.24), but the only important fact is that

$$\int \phi(\theta)\,\mathrm{d}^2\theta \;=\; [\phi(\theta)]_{\theta\theta} = f \tag{3.13}$$

$$\int \Omega(\theta, \bar\theta)\,\mathrm{d}^2\theta\mathrm{d}^2\bar\theta \;=\; [\Omega(\theta, \bar\theta)]_{\theta\theta\,\bar\theta\bar\theta} = d \tag{3.14}$$

with $\phi(\theta)$ as given in Eq. (3.11) and d is the term proportional to $\theta\theta\,\bar\theta\bar\theta$ in the double expansion of the arbitrary function $\Omega(\theta, \bar\theta)$ in θ and $\bar\theta$. We will actually never use the notation with the integral sign and simply think of the operation $[\dots]_{\theta\theta}$ as selecting the $\theta\theta$ component of the argument. It is not a coincidence that the constants in Eqs. (3.13) and (3.14) are denoted by f and d since – as we will see later – this is related to the common terminology of F-terms and D-terms.

4 Superspace and superfields

Our starting point was to consider Poincaré symmetries. More precisely, we write a Lagrangian as a function of fields $\phi(x)$ which have certain transformation properties under

7

Poincaré transformations, Eq. (2.6). We then insist that the Lagrangian is invariant under such transformations.

We also decided to enlarge our symmetry group with fermionic generators. It is clear that in this case we also need some fermionic coordinates that change in a certain way under the enlarged group of transformations. Because we added the generators Q_α and $\bar{Q}_{\dot\alpha}$ we will need a matching set of coordinates which we denote by θ^α and $\bar{\theta}^{\dot\alpha}$. As a consequence, our fields will now not only depend on x^μ but also on θ^α and $\bar{\theta}^{\dot\alpha}$. We will write a generic field as $\Omega(x, \theta, \bar{\theta})$. Such a field is called a *superfield* and the enlarged space is called *superspace* with coordinates $X = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot\alpha})$. This extension of coordinates is similar to the extension from $\vec{x}$ to $x^\mu = (t, \vec{x})$ in Section 2. Note that the mass dimension of the Grassmann coordinates θ and $\bar{\theta}$ is given by $[\theta] = [\bar{\theta}] = -1/2$ whereas obviously $[x] = -1$.

Our ultimate goal is to construct Lagrangians that are invariant under susy transformations. Thus we will need to get a handle on the transformation property of fields. As a first step, we would like to find a representation of the generators in terms of differentiation operators, i.e. equations for Q_α and $\bar{Q}_{\dot\beta}$ that are analogous to $P_\mu = i\partial_\mu$.

Let us consider a susy transformation with $\omega^{\mu\nu}$ of Eq. (2.9) set to zero for simplicity

$$S(a, \zeta, \bar{\zeta}) \equiv e^{i\left(\zeta^\alpha Q_\alpha + \bar{\zeta}_{\dot\alpha} \bar{Q}^{\dot\alpha} + a^\mu P_\mu\right)} \tag{4.1}$$

with parameters a, ζ and $\bar{\zeta}$ and where Q, $\bar{Q}$ and P are operators in Fock space. Note that if we set $\zeta = \bar{\zeta} = 0$ the transformation is simply a translation under which a quantum field transforms as

$$\phi(x) \rightarrow S(a, 0, 0)\phi(x)S^{-1}(a, 0, 0) = e^{ia^\mu P_\mu}\phi(x)e^{-ia^\mu P_\mu} = \phi(x + a) \tag{4.2}$$

If we combine two susy transformations, we obtain

$$S(a, \zeta, \bar{\zeta})S(x, \theta, \bar{\theta}) = S(x^\mu + a^\mu + i\,\zeta\sigma^\mu\bar{\theta} - i\,\theta\sigma^\mu\bar{\zeta},\ \theta + \zeta,\ \bar{\theta} + \bar{\zeta}) \tag{4.3}$$

This can be derived by using the Baker-Campbell-Hausdorff formula which states that if the commutators $[A, [A, B]]$ etc. vanish we have $e^A e^B = e^{A+B+[A,B]/2}$. The only non-vanishing commutators we have in deriving Eq. (4.3) are $[\zeta Q, \bar{\theta}\bar{Q}] = 2\,\zeta\sigma^\mu\bar{\theta}\,P_\mu$ and $[\bar{\zeta}\bar{Q}, \theta Q] = -2\,\theta\sigma^\mu\bar{\zeta}\,P_\mu$. Eq. (4.3) states that even if we set $a^\mu = x^\mu = 0$ we induce a translation. This is a direct consequence of Eq. (2.18). Thus, starting from a point $X = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot\alpha})$ in superspace, under a susy transformation, Eq. (4.1) we have

$$X \rightarrow X' = (x^\mu + a^\mu + i\,\zeta\sigma^\mu\bar{\theta} - i\,\theta\sigma^\mu\bar{\zeta},\ \theta + \zeta,\ \bar{\theta} + \bar{\zeta}) \tag{4.4}$$

This is the generalization of Eq. (2.6).

We now consider a superfield $\Omega(x, \theta, \bar{\theta})$ under a susy transformation Eq. (4.1)

$$\begin{aligned}
\Omega(x, \theta, \bar{\theta}) &\rightarrow e^{i\left(\zeta^\alpha Q_\alpha + \bar{\zeta}_{\dot\alpha}\bar{Q}^{\dot\alpha} + a^\mu P_\mu\right)} \Omega(x, \theta, \bar{\theta})\, e^{-i\left(\zeta^\alpha Q_\alpha + \bar{\zeta}_{\dot\alpha}\bar{Q}^{\dot\alpha} + a^\mu P_\mu\right)} \\
&= \Omega(x^\mu + a^\mu + i\,\zeta\sigma^\mu\bar{\theta} - i\,\theta\sigma^\mu\bar{\zeta},\ \theta + \zeta,\ \bar{\theta} + \bar{\zeta})
\end{aligned} \tag{4.5}$$

8

Since we will need to calculate the transformation of fields several times, we want to find a simple representation for Eq. (4.5). We seek differential operators Q, $\bar{Q}$ and P such that the transformation given in Eq. (4.5) can be written as

$$\Omega(x^\mu + a^\mu + i\,\zeta\sigma^\mu\bar{\theta} - i\,\theta\sigma^\mu\bar{\zeta}, \; \theta + \zeta, \; \bar{\theta} + \bar{\zeta}) = e^{-i\left(\zeta^\alpha Q_\alpha + \bar{\zeta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} + a^\mu P_\mu\right)}\Omega(x, \theta, \bar{\theta}) \qquad (4.6)$$

Note that this is quite some abuse of notation. In Eq. (4.6) Q, $\bar{Q}$ and P are differential operators that act on a function $\Omega(x, \theta, \bar{\theta})$, whereas in Eqs. (4.5) and (4.2) Q, $\bar{Q}$ and P are operators in Fock space (i.e. can be written in terms of creation and annihilation operators) and Ω is a quantum field, i.e. also an operator in Fock space. It is customary but somewhat unfortunate to use the same symbols for these different objects. Note that as far as P is concerned, Eq. (4.6) is in agreement with Eq. (2.8). Indeed, we can combine $\Phi'(x') = \Phi(x)$ with Eq. (2.8) to obtain $\Phi(x + a) = e^{-i\,a^\rho P_\rho}\,\Phi(x)$. But we could change the sign and/or i factors in the coefficients multiplying Q and $\bar{Q}$. This simply would lead to different conventions for Q and $\bar{Q}$ and, unfortunately, many different conventions are used in the literature.

If we assume a, ζ and $\bar{\zeta}$ to be infinitesimally small we can Taylor expand both sides of Eq. (4.6) (see Eq. (A.23))

$$\Omega + \left(a^\mu + i\,\zeta\sigma^\mu\bar{\theta} - i\,\theta\sigma^\mu\bar{\zeta}\right)\partial_\mu\Omega + \zeta^\alpha\partial_\alpha\Omega - \bar{\zeta}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}\Omega$$
$$= \; \Omega - i\left(\zeta^\alpha Q_\alpha + \bar{\zeta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} + a^\mu P_\mu\right)\Omega \qquad (4.7)$$

where $\Omega = \Omega(x, \theta, \bar{\theta})$. By comparing the coefficients of the infinitesimal parameters a^μ, ζ^α and $\bar{\zeta}^{\dot{\alpha}}$ we finally obtain

$$P_\mu \;=\; i\partial_\mu \qquad (4.8)$$
$$Q_\alpha \;=\; i\,\partial_\alpha - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,\partial_\mu \qquad (4.9)$$
$$\bar{Q}_{\dot{\alpha}} \;=\; -i\,\bar{\partial}_{\dot{\alpha}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu \qquad (4.10)$$

It is a useful exercise to check that these representations indeed satisfy Eqs. (2.18) and (2.20). We can now use these expressions to compute the change of a superfield Ω under a susy transformation

$$\Omega \to \Omega' = \Omega + \delta\Omega = \Omega - i\left(\zeta^\alpha Q_\alpha + \bar{\zeta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} + a^\mu P_\mu\right)\Omega \qquad (4.11)$$

For future reference we also introduce *covariant derivatives*

$$D_\alpha \equiv \partial_\alpha - i\,\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,\partial_\mu; \quad \bar{D}_{\dot{\alpha}} \equiv \bar{\partial}_{\dot{\alpha}} - i\,\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu; \qquad (4.12)$$

defined such that they satisfy $\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = 0$, with more relations given in Eq. (A.27). They get their name from the fact that $D_\alpha\Omega$ (and $\bar{D}_{\dot{\alpha}}\Omega$) transform in the same way[1] under susy transformation as Ω, i.e. $D_\alpha\Omega \to (D_\alpha\Omega)' = D_\alpha\Omega + \delta(D_\alpha\Omega)$ with

$$D_\alpha\delta\Omega = \delta(D_\alpha\Omega) = -i\left(\zeta^\alpha Q_\alpha + \bar{\zeta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} + a^\mu P_\mu\right)D_\alpha\Omega \qquad (4.13)$$

---

[1]This is reminiscent of gauge theories, where the (gauge) covariant derivative D_μ is constructed such that a gauge field ψ and $D_\mu\psi$ transform in the same way under gauge transformations.

We should warn the reader again that there are many different conventions used in the literature and the explicit form of the generators Q_α and $\bar{Q}_{\dot{\alpha}}$ and the covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$ is by no means unique.

Let us now expand the most general superfield $\Omega(x,\theta,\bar{\theta})$ in θ and $\bar{\theta}$. According to Eq. (3.11) we expect terms with one or two θ and/or $\bar{\theta}$, but not more. Thus we write

$$
\begin{aligned}
\Omega(x,\theta,\bar{\theta}) &= c(x) + \theta\psi(x) + \bar{\theta}\bar{\psi}'(x) + (\theta\theta)\,F(x) + (\bar{\theta}\bar{\theta})\,F'(x) + \theta\sigma^\mu\bar{\theta}\,v_\mu(x) \\
&+ (\theta\theta)\,\bar{\theta}\bar{\lambda}'(x) + (\bar{\theta}\bar{\theta})\,\theta\lambda(x) + (\theta\theta)\,(\bar{\theta}\bar{\theta})\,D(x)
\end{aligned}
\tag{4.14}
$$

There are several points to be noted. First, the primed fields e.g. $F'(x)$ are not in any way related to the corresponding unprimed fields $F(x)$. They are simply the coefficients in the (terminating) Taylor expansion of Ω in θ and $\bar{\theta}$. Furthermore, it is clear that there are four coefficients of the mixed $\theta^\alpha\,\bar{\theta}^{\dot{\alpha}}$ term. These four coefficients can conveniently be written in terms of a vector field $v^\mu(x)$. Hence, the superfield Ω contains four Weyl spinors ψ, $\bar{\psi}'$, λ and $\bar{\lambda}'$, four scalar fields c, F, F' and D and a vector field v. These fields are called *component fields*. Because a superfield contains a collection of component fields it is often called a *supermultiplet*. There are eight complex fermionic and eight complex bosonic degrees of freedom in Ω. It is of course not a coincidence that the number of bosonic and fermionic degrees of freedom match.

The superfield Ω given in Eq. (4.14) will not be one of the basic blocks that we are going to use to construct supersymmetric theories. We can define simpler building blocks by imposing constraints. This will result in superfields with smaller particle content. In the following two subsections we consider the two important special cases.

4.1   Chiral superfields

A superfield $\phi(x,\theta,\bar{\theta})$ that satisfies the constraint $\bar{D}_{\dot{\alpha}}\,\phi(x,\theta,\bar{\theta}) = 0$, where $\bar{D}$ is the covariant derivative defined in Eq. (4.12), is called a *left-handed chiral superfield* (LHχSF). The reason for the name will become clear in a moment. Note that this constraint is self consistent in the sense that it is invariant under susy transformations. Indeed, after a susy transformation, Eq. (4.11), the superfield still satisfies the constraint. This can be seen using Eq. (4.13).

The constraint imposed reduces the number of degrees of freedom in the superfield. To find the general expression of a LHχSF, analogous to Eq. (4.14), we note that $\bar{D}_{\dot{\alpha}}\theta^\alpha = 0$ and $\bar{D}_{\dot{\alpha}}\,y^\mu = 0$, where we define $y^\mu \equiv x^\mu - i\,\theta\sigma^\mu\bar{\theta}$. Thus, the most general function $\phi(y,\theta) \equiv \phi(y) + \sqrt{2}\,\theta\psi(y) - \theta\theta\,F(y)$ (the $\sqrt{2}$ and the minus sign are simply conventions) satisfies $\bar{D}_{\dot{\alpha}}\,\phi = 0$. Expanding this back in x, θ and $\bar{\theta}$ we obtain

$$
\begin{aligned}
\phi(x,\theta,\bar{\theta}) &= \varphi(x) + \sqrt{2}\theta\psi(x) - i\theta\sigma^\mu\bar{\theta}\,\partial_\mu\varphi(x) + \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi(x)\sigma^\mu\bar{\theta}) \\
&- \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^\mu\partial_\mu\varphi(x) - (\theta\theta)F(x)
\end{aligned}
\tag{4.15}
$$

as the expansion of a LHχSF into component fields. Again, we have the same number of fermionic and bosonic degrees of freedom, with two scalar fields φ and F and a Weyl

10

spinor ψ. It is the left-handed Weyl spinor ψ that lends its name to the whole superfield. The spinors of a LHχSF will be the left-handed quarks and leptons of a susy extension of the Standard Model and the φ fields their supersymmetric partners, the squarks and sleptons. The Higgs bosons and their susy partners will also form chiral superfields. The mass dimension of the various component fields in Eq. (4.15) are $[\varphi] = 1$, $[\psi] = 3/2$ and $[F] = 2$ such that all terms in ϕ have mass dimension 1, i.e. $[\phi] = 1$. Thus, φ and ψ have the expected mass dimension, but F does not have the usual mass dimension of a scalar field. This is a first hint that the F component field is unphysical, an issue we will come back to.

The susy transformation of a superfield, Eq. (4.11) induces transformations of the component fields $\varphi(x) \rightarrow \varphi(x) + \delta\varphi(x)$ etc. Using the explicit representation of Q and $\bar{Q}$, Eqs. (4.9) and (4.10), we find

$$
\begin{aligned}
\delta\varphi &= \sqrt{2}\,\zeta\psi \\
\delta\psi_\alpha &= -\sqrt{2}\,F\,\zeta_\alpha - i\sqrt{2}\,\sigma^\mu_{\alpha\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}\,\partial_\mu\varphi \\
\delta F &= -i\sqrt{2}\,\partial_\mu\psi\sigma^\mu\bar{\zeta} = \partial_\mu\left(-i\sqrt{2}\,\psi\sigma^\mu\bar{\zeta}\right)
\end{aligned}
\tag{4.16}
$$

As expected, the change in the bosonic/fermionic component fields is proportional to the fermionic/bosonic fields. The crucial point is that δF is a total derivative. This will be very important when we construct susy Lagrangians.

We can repeat the whole procedure for right-handed chiral superfields (RHχSF) $\phi^\dagger$, which by definition satisfy the constraint $D_\alpha\,\phi^\dagger = 0$. In terms of component fields they read

$$
\begin{aligned}
\phi^\dagger(x,\theta,\bar{\theta}) &= \varphi^\dagger(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + i\theta\sigma^\mu\bar{\theta}\,\partial_\mu\varphi^\dagger(x) - \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})(\theta\sigma^\mu\partial_\mu\bar{\psi}(x)) \\
&\quad - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial^\mu\partial_\mu\varphi^\dagger(x) - (\bar{\theta}\bar{\theta})F^\dagger(x)
\end{aligned}
\tag{4.17}
$$

The hermitian conjugate of a LHχSF is a RHχSF.

4.2 Vector superfields

The chiral superfields (χSF) introduced above do not have a vector field as component field. Thus, in order to deal with supersymmetric gauge theories, we will also need another superfield, called a *vector superfield* $V(x,\theta,\bar{\theta})$, that contains a spin 1 component field. Such a superfield is defined by the constraint $V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta})$. Again, this constraint is preserved under susy transformations.

The expansion of a vector superfield (VSF) in terms of component fields can be

11

obtained by looking at Eq. (4.14) and enforcing $V = V^\dagger$.

$$
\begin{aligned}
V(x, \theta, \bar{\theta}) =\ & c(x) + i\,\theta\chi(x) - i\,\bar{\theta}\bar{\chi}(x) + \theta\sigma^\mu\bar{\theta}\,v_\mu(x) + i\,(\theta\theta)N(x) - i\,(\bar{\theta}\bar{\theta})N^\dagger(x) \\
& + i\,(\theta\theta)\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\partial_\mu\chi(x)\sigma^\mu\right) - i\,(\bar{\theta}\bar{\theta})\theta\left(\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \\
& + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\left(D(x) - \frac{1}{2}\partial^\mu\partial_\mu c(x)\right)
\end{aligned} \tag{4.18}
$$

Several remarks are in order. First, factors i and some overall signs in the above expansion are simply conventions. Second, the component fields c, D and v are now real, but N is complex. Thus, through the constraint $V^\dagger = V$ the eight complex degrees of freedom in Eq. (4.14) are reduced to eight real bosonic and fermionic degrees of freedom in V. Putting it in other words, in Eq. (4.14), the coefficients of e.g. θ and $\bar{\theta}$, denoted by ψ and $\bar{\psi}'$ were not related. However, in Eq. (4.18) the corresponding coefficients, denoted by χ and $\bar{\chi}$ have to be the same, i.e. there is only one Weyl spinor associated with the θ term. The same is true for the $(\bar{\theta}\bar{\theta})\,\theta$ term. In Eq. (4.14) we denoted the corresponding component field by λ, whereas in Eq. (4.18) we redefine λ such that the coefficient takes a slightly more complicated form. The same remark applies to the $\theta\theta\,\bar{\theta}\bar{\theta}$ term. The reason for this will become clear in Section 5.2 and is related to the fact that V as given in Eq. (4.18) has more degrees of freedom than we bargained for. Apart from the vector field v_μ that we wanted (and that gives the whole superfield its name and will represent gauge bosons in susy extensions of the Standard Model) we might expect some fermions (gauginos). However, we got two fermions, χ and λ and a whole set of scalar fields. A look at the mass dimension of the various component fields, $[c] = 0$, $[\chi] = 1/2$, $[v] = [N] = 1$, $[\lambda] = 3/2$ and $[D] = 2$ reveals that only v and λ have the expected mass dimensions. Indeed, all other component fields will turn out to be unphysical.

As we have done for the LHχSF in Eq. (4.16), we could now determine the transformation properties of the component fields of V. However, as most component fields are unphysical, we refrain from doing this and restrict ourselves to the transformation of the $D(x)$ component field. Under Eq. (4.11), we have $D \to D + \delta D$ with

$$
\delta D = \zeta\sigma^\mu\partial_\mu\bar{\lambda}(x) + \partial_\mu\lambda(x)\sigma^\mu\bar{\zeta} = \partial_\mu\left(\zeta\sigma^\mu\bar{\lambda}(x) + \lambda(x)\sigma^\mu\bar{\zeta}\right) \tag{4.19}
$$

As for the F field of a chiral superfield, the change in the D field of a VSF is a total derivative.

4.3   From superfields to particles

Let us pause for a moment an recapitulate what we have done. In increasing the symmetry from the Poincaré group to the super-Poincaré group we also had to increase the coordinate space from Minkowski space with coordinates x^μ to superspace with coordinates $X = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. Thus, our fields now depend on X, i.e. not only on x^μ but also on θ^α and $\bar{\theta}^{\dot{\alpha}}$. In "normal" particle physics, the fields (e.g. the electron or photon field) depend only on x^μ. These "normal" fields are now simply the components of the superfields. Thus, susy forces us to put several "normal" fields together into a superfield.

The most general expression for such a superfield is given in Eq. (4.14). However, such a superfield is not a basic building block for our theory since it contains too many component fields. We have identified the three basic superfields that we will need in the construction of susy extensions of the Standard Model. These are the LHχSF, the RHχSF and the VSF. It will turn out that ultimately the particle (i.e. "normal" field) content of the LHχSF will be a scalar φ and a left-handed fermion ψ only. The other degree of freedom, the F-field will turn out to be unphysical and will be eliminated. Similarly, for the RHχSF the particle content is given by a scalar $\varphi^\dagger$ and a right-handed fermion $\bar{\psi}$. In the case of the VSF, the particle content will consist of a vector boson v^μ and a Weyl spinor λ with is conjugate $\bar{\lambda}$. All other fields will turn out to be unphysical and will be eliminated.

Thus if we want to construct for example a susy version of QED, we have to promote the left-handed (right-handed) electron field into a LHχSF (RHχSF), thereby automatically introducing the scalar partners, the selectrons. The photon field is embedded in a VSF which introduces the fermionic partner of the photon, the photino. In the case of the Standard Model we have

$$\text{left-handed fermions:} \quad \psi_f \;\in\; \phi_f \;=\; (\varphi_f, \psi_f) \tag{4.20}$$

$$\text{right-handed fermions:} \quad \bar{\psi}_f \;\in\; \phi_f^\dagger \;=\; (\varphi_f^\dagger, \bar{\psi}_f) \tag{4.21}$$

$$\text{Higgs boson(s):} \quad \begin{aligned} \varphi_h &\in \phi_h = (\varphi_h, \psi_h) \\ \varphi_h^\dagger &\in \phi_h^\dagger = (\varphi_h^\dagger, \bar{\psi}_h) \end{aligned} \tag{4.22}$$

$$\text{gauge bosons:} \quad v^\mu \;\in\; V \;=\; (v^\mu, \lambda, \bar{\lambda}) \tag{4.23}$$

Thus, the leptons and quarks (ψ_f and $\bar{\psi}_f$) will be part of a χSF (ϕ_f and $\phi_f^\dagger$) and get their scalar partners, the sleptons and squarks (φ_f and $\varphi_f^\dagger$). The gauge bosons (v^μ) will become a part of a VSF (V) and will get their fermionic partners, the gauginos (λ and $\bar{\lambda}$). Finally the Higgs boson(s) (φ_h and $\varphi_h^\dagger$) will be the scalar part of a χSF (ϕ_h and $\phi_h^\dagger$) and get their fermionic partners, the higgsinos (ψ_h and $\bar{\psi}_h$). This will determine to a large extent the particle content of the theory.

What we do not know yet is how to obtain the interactions between the various particles of our theory. We have to make sure that these interactions are compatible with susy. It is here where the superfield formalism is an invaluable help, as we will see in the following section.

Following up from our discussion just after Eq. (2.20), we can now also understand why $N > 1$ susy theories cannot be used as direct low-energy extensions of the Standard Model. The nice feature about $N = 1$ is that it keeps the left-handed and right-handed fermions in separate superfields as given in Eqs. (4.20) and (4.21). This is essential because these fields transform differently under $SU(2)$ gauge transformations. For $N > 1$ the supermultiplets are larger and combine the left-handed and right-handed fermions. This is inconsistent with the weak interactions. Of course it is still possible that at very high energies we have a $N > 1$ theory. But this theory would have to be broken such that at energy scales of a few TeV we have a $N = 1$ susy theory.

5 Supersymmetric Lagrangians

The key observation for the construction of susy theories is that the F-term of a chiral superfield (i.e. the $\theta\theta$ component of a LHχSF or the $\bar{\theta}\bar{\theta}$ component of a RHχSF) and the D-term of a VSF (i.e. the $\theta\theta\,\bar{\theta}\bar{\theta}$ component) transform into themselves plus a total derivative under susy transformations. If the Lagrangian $\mathcal{L}$ changes by a total derivative, the action $\int \mathrm{d}^4x\,\mathcal{L}$ does not change at all. Thus, if we write a Lagrangian as

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_D \tag{5.1}$$

where $\mathcal{L}_F$ is made up of F-terms (of χSF) and $\mathcal{L}_D$ is made up of D-terms (of VSF) we are guaranteed that our theory is invariant under susy transformations. We will use this in the following sections to construct various susy theories.

5.1 The Wess-Zumino Lagrangian

The Wess-Zumino model is the simplest susy Lagrangian and contains only chiral superfields. If we have two LHχSF, ϕ_i and ϕ_j, then the product $\phi_i\phi_j$ is again a LHχSF, because $\bar{D}_{\dot{\alpha}}(\phi_i\phi_j) = (\bar{D}_{\dot{\alpha}}\phi_i)\phi_j + \phi_i(\bar{D}_{\dot{\alpha}}\phi_j) = 0$. Of course, this can be extended to an arbitrary product of LHχSF and an equivalent statement holds for RHχSF. Thus we define the *superpotential*

$$W(\phi_i) \equiv a_i\,\phi_i + \frac{1}{2}m_{ij}\,\phi_i\phi_j + \frac{1}{3!}y_{ijk}\,\phi_i\phi_j\phi_k \tag{5.2}$$

where the sum $\sum_{ijk}$ over all possible combinations of LHχSF is understood and a_i, m_{ij} and y_{ijk} are constants. Then we can write

$$\mathcal{L}_{F,\mathrm{WZ}} = \int \mathrm{d}^2\theta\, W(\phi_i) + \int \mathrm{d}^2\bar{\theta}\, W^\dagger(\phi_i^\dagger) \equiv \Big[W(\phi_i)\Big]_{\theta\theta} + \Big[W^\dagger(\phi_i^\dagger)\Big]_{\bar{\theta}\bar{\theta}} \tag{5.3}$$

The factors $1/2$ and $1/3!$ in Eq. (5.2) could be absorbed into m_{ij} and y_{ijk} but usually are left explicit to take into account the symmetry of the terms. According to Eq. (3.13), the integration $\mathrm{d}^2\theta$ picks out the $\theta\theta$ component, hence $\mathcal{L}_{F,\mathrm{WZ}}$ results in a susy theory. One might think we could add more terms with products of more than three χSF in the superpotential and still end up with a susy theory. However, this would result in a non-renormalizable theory. Indeed, the mass dimension of the various couplings are $[a_i] = 2$, $[m_{ij}] = 1$ and $[y_{ijk}] = 0$ to ensure $[\mathcal{L}_{F,\mathrm{WZ}}] = 4$. Had we added a term $c_{ijkl}\,\phi_i\phi_j\phi_k\phi_l$ in Eq. (5.2) we would have a coupling with negative mass dimension $[c_{ijkl}] = -1$.

We stress that $\mathcal{L}_{F,\mathrm{WZ}}$ contains arbitrary products of LHχSF and arbitrary products of RHχSF but no terms like $\phi_i\phi_j^\dagger$. This is of utmost importance and is due to the fact that the $\theta\theta$ component (or the $\bar{\theta}\bar{\theta}$ component) of a term like $\phi_i\phi_j^\dagger$ does not transform into itself plus a total derivative and hence would break susy. In other words, the superpotential has to be a holomorphic (or analytic) function of the superfields, i.e. it depends only on ϕ_i but not on $\phi_i^\dagger$.

The Lagrangian $\mathcal{L}_{F,\mathrm{WZ}}$ as given in Eq. (5.3) contains mass terms and Yukawa couplings of the component fields, but no kinetic terms, i.e. no terms like $(\partial_\mu\varphi_i)(\partial^\mu\varphi_i)^\dagger$. It is clear that such terms can only come from combinations of $\phi_i\,\phi_i^\dagger$ which we explicitly excluded from the superpotential. On the other hand it is also clear that $\phi_i\,\phi_i^\dagger$ is a vector superfield since $(\phi_i\,\phi_i^\dagger)^\dagger = \phi_i\,\phi_i^\dagger$. Thus we can get a supersymmetric Lagrangian by taking the D-term of $\phi_i\,\phi_i^\dagger$. Such a term has mass dimension 4. Higher products such as $(\phi_i\,\phi_i^\dagger)(\phi_j\,\phi_j^\dagger)$ would lead to non-renormalizable interactions. Thus we write

$$\mathcal{L}_{D,\mathrm{WZ}} = \int \mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\phi_i\,\phi_i^\dagger = \left[\phi_i\,\phi_i^\dagger\right]_{\theta\theta\,\bar\theta\bar\theta} \tag{5.4}$$

and the full Lagrangian $\mathcal{L}_{\mathrm{WZ}} = \mathcal{L}_{F,\mathrm{WZ}} + \mathcal{L}_{D,\mathrm{WZ}}$ has the structure given in Eq. (5.1).

The usefulness of Eqs. (5.3) and (5.4) lies in the fact that a simple glance immediately reveals that the theory is supersymmetric. On the other hand, Eqs. (5.3) and (5.4) are fairly useless if we want information about the particle content and interactions of the theory. To obtain this we will have to express $\mathcal{L}_{\mathrm{WZ}}$ in terms of component fields. Given the explicit expression Eqs. (4.15) and (4.17) this is trivial if slightly tedious (for details see Appendix B). Considering the simplest case with only one chiral superfield (and $a_1 = a$, $m_{11} = m$, $y_{111} = y$) we get

$$\mathcal{L}_{D,\mathrm{WZ}} = F^\dagger F + (\partial_\mu\varphi)\,(\partial^\mu\varphi)^\dagger + \frac{i}{2}\,\psi\sigma^\mu(\partial_\mu\bar\psi) - \frac{i}{2}\,(\partial_\mu\psi)\sigma^\mu\bar\psi \tag{5.5}$$

$$\mathcal{L}_{F,\mathrm{WZ}} = -a\,F - m\,\varphi F - \frac{m}{2}(\psi\psi) - \frac{y}{2}\,\varphi\varphi F - \frac{y}{2}\,\varphi(\psi\psi) + \text{h.c.} \tag{5.6}$$

As expected, the D-term contains the kinetic term of the φ and the ψ component fields (see Eq. (3.9)). Note however, that there is no kinetic term for the F field. This means that the equation of motion for F (and $F^\dagger$) reduces to an algebraic equation

$$0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu F)} - \frac{\partial\mathcal{L}}{\partial F} = -\frac{\partial\mathcal{L}}{\partial F} = -F^\dagger + a + m\,\varphi + \frac{y}{2}\,\varphi\varphi \tag{5.7}$$

We can solve this trivially and eliminate F and $F^\dagger$ from the Lagrangian. The terms containing F and $F^\dagger$ in Eqs. (5.5) and (5.6) then read

$$F^\dagger F - \left(a\,F + m\,\varphi\,F + \frac{y}{2}\,\varphi\varphi\,F + \text{h.c.}\right) = -\left|a + m\,\varphi + \frac{y}{2}\,\varphi\varphi\right|^2 = -\left|\frac{\partial W(\varphi)}{\partial\varphi}\right|^2 \tag{5.8}$$

In the last step, $W(\varphi)$ is the usual superpotential, but it is considered to be a function of the scalar component field φ only, rather than the full superfield ϕ. For writing a Lagrangian in terms of component fields, this is usually more useful.

Performing a shift $\varphi \to \varphi + (M-m)/y$ with $M \equiv \sqrt{m^2 - 2a\,y}$ to eliminate the $a\,\varphi$ term (or simply setting $a = 0$) the Lagrangian reads

$$\begin{aligned}
\mathcal{L}_{\mathrm{WZ}} = {}& (\partial_\mu\varphi)\,(\partial^\mu\varphi)^\dagger + \frac{i}{2}\,\psi\sigma^\mu(\partial_\mu\bar\psi) - \frac{i}{2}\,(\partial_\mu\psi)\sigma^\mu\bar\psi \\
& - |M|^2\varphi\varphi^\dagger - \frac{|y|^2}{4}\varphi\varphi\varphi^\dagger\varphi^\dagger - \left(\frac{M}{2}\psi\psi + \frac{M^*y}{2}\varphi\varphi\varphi^\dagger + \frac{y}{2}\varphi\,\psi\psi + \text{h.c.}\right)
\end{aligned} \tag{5.9}$$

15

This theory contains a spin 0 and a spin 1/2 particle with the same mass. There is a three-point and a four-point interaction between the scalars and a scalar-scalar-fermion interaction. The couplings of these interactions are all related. Of course, this is simply a consequence of susy.

For future reference, let us rewrite the Lagrangian in yet another way. We will do this for the general case with an arbitrary number of chiral superfields.

$$
\begin{aligned}
\mathcal{L}_{\mathrm{WZ}} &= (\partial_\mu \varphi_i)\,(\partial^\mu \varphi_i)^\dagger + \frac{i}{2}\,\psi_i \sigma^\mu (\partial_\mu \bar{\psi}_i) - \frac{i}{2}\,(\partial_\mu \psi_i)\sigma^\mu \bar{\psi}_i \\
&\quad - \sum_i \left| \frac{\partial W(\varphi_i)}{\partial \varphi_i} \right|^2 - \frac{1}{2}\left(\frac{\partial^2 W(\varphi_i)}{\partial \varphi_i\,\partial \varphi_j}\right)\psi_i \psi_j - \frac{1}{2}\left(\frac{\partial^2 W^\dagger(\varphi_i)}{\partial \varphi_i^\dagger\,\partial \varphi_j^\dagger}\right)\bar{\psi}_i \bar{\psi}_j \quad (5.10)
\end{aligned}
$$

The superpotential is as given in Eq. (5.2) but considered to be a function of the scalar component fields φ_i only. Note that the superpotential determines all interactions and the mass terms of the component fields, and thus, the full theory.

5.2 Susy QED

The Wess-Zumino Lagrangian does not contain spin 1 component fields. Thus, to obtain susy gauge theories we will have to extend the field content and include VSF. If we have a VSF $V = V^\dagger$ then V^n is also a VSF and its D-term (i.e. its $\theta\theta\,\bar{\theta}\bar{\theta}$ component) is supersymmetric. However, this will not lead to kinetic terms for the corresponding spin 1 vector field v^μ. As in the case of chiral superfields we will have to add another construct for the kinetic terms. We define[2]

$$
U_\alpha \equiv -\frac{1}{4}\big(\bar{D}\bar{D}\big)D_\alpha V\,; \qquad \bar{U}_{\dot{\alpha}} \equiv -\frac{1}{4}\big(DD\big)\bar{D}_{\dot{\alpha}} V\,; \qquad (5.11)
$$

Because of $\bar{D}_{\dot{\alpha}}\bar{D}\bar{D} = 0$ we know that U_α is a LHχSF, $\bar{D}_{\dot{\alpha}}U_\alpha = 0$. Similarly, $\bar{U}_{\dot{\alpha}}$ is a RHχSF. Forming the products $U^\alpha U_\alpha$ and $\bar{U}_{\dot{\alpha}}\bar{U}^{\dot{\alpha}}$ as in Eqs. (3.4) and (3.5) we obtain a Lorentz invariant expression. Furthermore, the corresponding F-terms are supersymmetric and in fact they do contain the kinetic terms of the component fields v^μ and λ (see Appendix B).

Before we look at this in more detail we have to combine gauge symmetry with susy. After all, our vector bosons are supposed to be gauge bosons. Let us start with a global $U(1)$ gauge symmetry. Under such a symmetry, component fields transform as $\varphi \rightarrowtail \varphi' = e^{-i\Lambda}\varphi$ where Λ is a real constant and has mass dimension $[\Lambda] = 0$. It follows that $\varphi^\dagger \varphi$ is gauge independent. We can easily extend this to superfields by noting that a real constant $\Lambda = \Lambda^\dagger$ is a special case of a chiral superfield. In fact it is actually a LHχSF and a RHχSF at the same time because $\bar{D}_{\dot{\alpha}}\Lambda = D_\alpha \Lambda = 0$. Thus a LH$\chi$SF transforms as $\phi \rightarrowtail \phi' = e^{-i\Lambda}\phi$ with ϕ' still being a LHχSF and $\phi^\dagger$ transforms as $\phi^\dagger \rightarrowtail \phi'^\dagger = e^{i\Lambda^\dagger}\phi^\dagger$ with $\phi'^\dagger$ still being a RHχSF and $\big[\phi^\dagger \phi\big]_{\theta\theta\,\bar{\theta}\bar{\theta}}$ is supersymmetric and invariant under global gauge transformations.

---

[2]In the literature usually the notation W_α and $\bar{W}_{\dot{\alpha}}$ is used in Eq. (5.11). We use U_α and $\bar{U}_{\dot{\alpha}}$ to avoid confusion with the superpotential.

If we want local gauge invariance, then Λ will have to be a function of x. We still want $\Lambda(x)$ ($\Lambda^\dagger(x)$) to be a LHχSF (RHχSF) such that ϕ' ($\phi'^\dagger$) is a LHχSF (RHχSF). However, it is not possible to have a x-dependent superfield that is at the same time a LHχSF and a RHχSF, thus we have $\Lambda(x) \neq \Lambda^\dagger(x)$. As a consequence, under gauge transformations

$$\phi^\dagger\,\phi \longmapsto \phi'^\dagger\,\phi' = \phi^\dagger\,e^{i\Lambda^\dagger(x)}\,e^{-i\Lambda(x)}\,\phi \neq \phi^\dagger\,\phi \tag{5.12}$$

This seems to introduce new particles, the component fields of Λ. However, they have the "wrong" mass dimension. Because Λ appears in the exponent, we must have $[\Lambda] = 0$. This entails mass dimensions 0 and $1/2$ for the scalar and fermion component fields of the χSF Λ, in contrast to the usual dimensions 1 and $3/2$. As we will see, these component fields are unphysical and can be eliminated together with the unphysical component fields of V.

According to Eq. (5.12) $\phi^\dagger\phi$ is invariant under global but not local gauge transformations. This is of course very familiar from standard non-susy theories, where e.g. $(\partial_\mu\varphi)^\dagger(\partial^\mu\varphi)$ is invariant under global but not local gauge transformations. As in these cases, to restore local gauge invariance we have to introduce a gauge VSF, V, transforming under gauge transformations as

$$e^V \longmapsto e^{-i\Lambda^\dagger(x)}\,e^V\,e^{i\Lambda(x)} \tag{5.13}$$

Note that in the abelian case, where all superfields commute, this can be written as

$$V \longmapsto V' = V - i\Lambda^\dagger(x) + i\Lambda(x) \tag{5.14}$$

Then the term

$$\left[\phi^\dagger\,e^V\,\phi\right]_{\theta\theta\,\bar\theta\bar\theta} \longmapsto \left[\phi'^\dagger\,e^{V'}\,\phi'\right]_{\theta\theta\,\bar\theta\bar\theta} = \left[\phi^\dagger\,e^V\,\phi\right]_{\theta\theta\,\bar\theta\bar\theta} \tag{5.15}$$

is supersymmetric and invariant under local gauge transformations.

The general expression of a VSF in terms of component fields is given in Eq. (4.18). We can exploit the gauge transformation Eq. (5.14) to obtain a particularly convenient representation of the gauge VSF. If we choose $\Lambda(x,\theta,\bar\theta)$ as in Eq. (4.15) but with the replacements $\psi \to -\chi/\sqrt{2}$, $F \to N$ and $\text{Im}(\varphi) \to c/2$ we get for $V' \equiv V_{\text{WZ}}$ the simple expression

$$V_{\text{WZ}}(x,\theta,\bar\theta) = \theta\sigma^\mu\bar\theta\,v_\mu(x) + i(\theta\theta)\,\bar\theta\bar\lambda(x) - i(\bar\theta\bar\theta)\,\theta\lambda(x) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\,D(x) \tag{5.16}$$

Note that we can also eliminate one degree of freedom in v_μ through a choice of $\text{Re}(\varphi)$. Thus, we are left with four (three in v_μ one in D) real bosonic and four real fermionic degrees of freedom in $V_{\text{WZ}}(x,\theta,\bar\theta)$. This gauge is called the *Wess-Zumino gauge* and has the nice feature that many unphysical component fields of V (and Λ) are eliminated. In this respect it is reminiscent of the unitary gauge. We should remark however, that this gauge choice is not invariant under susy transformations. Indeed, if we compute the change $\delta V_{\text{WZ}} = -i(\zeta Q + \bar\zeta\bar Q)V_{\text{WZ}}$ under an infinitesimal pure susy transformation, among many others, a term like $-i\,\zeta Q\,(-i\,\bar\theta\bar\theta\,\theta\lambda) = -i\,\theta\theta\,\zeta\lambda$ is generated. Such a term corresponds to a N component field in Eq. (4.18) which is not present in Eq. (5.16).

17

In order to complete the construction of an abelian supersymmetric gauge theory, we note that U_α and $\bar{U}_{\dot\alpha}$ are gauge independent. This can be verified by using Eq. (5.14) in Eq. (5.11) and using $\bar{D}_{\dot\alpha}\Lambda = D_\alpha\Lambda^\dagger = 0$ (see Appendix B). Thus we have an abelian gauge invariant and susy Lagrangian

$$\mathcal{L} = \frac{1}{4}\left[U^\alpha U_\alpha\right]_{\theta\theta} + \frac{1}{4}\left[\bar{U}_{\dot\alpha}\bar{U}^{\dot\alpha}\right]_{\bar\theta\bar\theta} + \left[\phi_i^\dagger\, e^{2g\,V}\, \phi_i\right]_{\theta\theta\,\bar\theta\bar\theta} + \left[W(\phi_i)\right]_{\theta\theta} + \left[W^\dagger(\phi_i^\dagger)\right]_{\bar\theta\bar\theta} \qquad (5.17)$$

as long as we make sure the superpotential is gauge independent. In particular, the fields present in the term $a_i\,\phi_i$ in Eq. (5.2) have to be gauge singlets. In Eq. (5.17) g denotes the gauge coupling and the normalization of the various terms has been chosen such that we will recover the standard normalization if we rewrite Eq. (5.17) in terms of the component fields.

If we consider QED, the χSF would correspond to a superfield for each charged lepton. Thus we have a LHχSF, ϕ_1, containing the left-handed electron (as ψ) and its susy partner, the "left-handed" selectron (as φ). Note that the term left-handed for the selectron is widely used but misleading, because the spin of the selectron is 0. There is also the corresponding RHχSF, $\phi_1^\dagger$, containing the right-handed electron (as $\bar\psi$) and its susy partner, the "right-handed" selectron (as $\varphi^\dagger$). If we want to include the second and third family, we have to introduce ϕ_2 and ϕ_3 as well as $\phi_2^\dagger$ and $\phi_3^\dagger$ containing the muons and taus respectively. In this theory there cannot be a term $a_i\,\phi_i$ because none of the fields is a gauge singlet. We could introduce one (or three) LHχSF for the neutrino(s). Since they are singlets under $U_{QED}(1)$, a linear term in the superpotential with these LHχSF would be allowed. However, it is clear that introducing a neutrino field in QED is not particularly interesting.

Let us consider the structure of the Lagrangian Eq. (5.17) and its form in terms of the component fields. The first two terms of Eq. (5.17) contain only the gauge boson v^μ (the photon), its susy partner λ (the photino) and the scalar D field. As we will see below, these terms are nothing but the kinetic terms of the photon and photino. The third term of Eq. (5.17) can be split into two parts. If we take the leading part of $e^{2g\,V} = 1 + \ldots$, we see that this terms coincides with Eq. (5.4) which in component form is given in Eq. (5.5). Thus it contains the kinetic terms of the leptons and sleptons. The higher order terms in $e^{2g\,V} = 2g\,V + \ldots$ contain the interactions between the leptons (and sleptons) with the photon (and photino). Finally, the last two terms of Eq. (5.17) are again equivalent to the corresponding terms discussed in Section 5.1 and contain the interactions involving only component fields of the χSF. In the case of QED, the total charge of each term has to vanish to preserve gauge invariance.

Let us consider the kinetic terms of the photon and photino in more detail. The most tedious part of the calculation is to obtain an expression for U_α in terms of the component fields. For this (details are given in Appendix B) it is convenient to write x^μ in terms of y^μ, as used in the derivation of Eq. (4.15) or $\bar{y}^\mu \equiv x^\mu + i\,\theta\sigma^\mu\bar\theta$ which satisfies $D_\alpha\,\bar{y}^\mu = 0$ and we obtain

$$U_\alpha = -i\,\lambda_\alpha(y) - \theta\theta\,\sigma^\nu_{\alpha\dot\beta}\,\partial_\nu\bar\lambda^{\dot\beta}(y) - \frac{i}{2}\,\theta_\beta\,(\sigma^\mu\bar\sigma^\nu)_\alpha^{\ \beta}F_{\mu\nu}(y) + \theta_\alpha\,D(y) \qquad (5.18)$$

where $F_{\mu\nu} \equiv \partial_\mu v_\nu - \partial_\nu v_\mu$ is the usual field strength tensor and the component fields are functions of $y^\mu = x^\mu - i\theta\sigma^\mu\bar\theta$. Thus the first two terms of Eq. (5.17) in terms of the component fields are given by

$$\frac{1}{4}\big[U^\alpha U_\alpha\big]_{\theta\theta} + \frac{1}{4}\big[\bar{U}_{\dot\alpha}\bar{U}^{\dot\alpha}\big]_{\bar\theta\bar\theta} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i}{2}(\partial_\mu\lambda)\sigma^\mu\bar\lambda + \frac{i}{2}\lambda\sigma^\mu(\partial_\mu\bar\lambda) + \frac{1}{2}D^2 \qquad (5.19)$$

and, indeed, contain kinetic terms for v_μ and λ. However, there is no kinetic term for the D component field. This field is an auxiliary field, similar to the F component field of χSF, and will be eliminated using the equation of motion. Before we can do this, we have to find all other terms containing D. They are in the third term of Eq. (5.17). Note that in the Wess-Zumino gauge $e^{2gV} = 1 + 2gV + 2g^2V^2$, i.e. we need at most two factors of V, because V_{WZ}^3 and higher powers vanish. We postpone the derivation of the full interaction term to Section 5.3 and write here only the term containing the D component field

$$\big[\phi_i^\dagger\, e^{2gV}\,\phi_i\big]_{\theta\theta\,\bar\theta\bar\theta} = g\,\varphi_i^\dagger\,\varphi_i\, D + \text{terms without } D \qquad (5.20)$$

In this context we mention that we can add another susy and gauge invariant term to Eq. (5.17). We know already from Eq. (4.19) that the $\theta\theta\,\bar\theta\bar\theta$ component of a VSF is susy. In the case of an abelian gauge field, this term is also gauge invariant. Indeed, Eq. (5.14) reveals that under a gauge transformation the $\theta\theta\,\bar\theta\bar\theta$ component of a VSF transforms into itself plus a total derivative, because the $\theta\theta\,\bar\theta\bar\theta$ component of a χSF (Λ and $\Lambda^\dagger$ of Eq. (5.14)) are total derivatives. Thus we could add a term

$$\mathcal{L}_{\text{FI}} = 2\big[kV\big]_{\theta\theta\,\bar\theta\bar\theta} = kD \qquad (5.21)$$

to the Lagrangian Eq. (5.17), where k is a constant (often denoted by ξ in the literature) with mass dimension $[k] = 2$ and the factor 2 is added for convenience. Such a term is called a *Fayet-Iliopoulos term* [14] and will be important later on when we discuss spontaneous breaking of susy. For the moment we simply note that this term also depends on the component field D as indicated in Eq. (5.21).

The full Lagrangian $\mathcal{L} + \mathcal{L}_{\text{FI}}$ does not contain terms involving $\partial_\mu D$. Thus the equation of motion for D is algebraic and can be solved trivially, resulting in

$$0 = \frac{\partial\mathcal{L}}{\partial D} = \frac{\partial}{\partial D}\left(\frac{D^2}{2} + g\,\varphi_i^\dagger\varphi_i\, D + kD\right) = D + g\,\varphi_i^\dagger\varphi_i + k \qquad (5.22)$$

As for the F component field, we can solve this and eliminate the D component field from the Lagrangian. We obtain

$$\frac{D^2}{2} + D\left(g\,\varphi_i^\dagger\varphi_i + k\right) = -\frac{1}{2}\left(g\,\varphi_i^\dagger\varphi_i + k\right)^2 \qquad (5.23)$$

for the terms containing the D field in Eqs. (5.19), (5.20) and (5.21). This is analogous to Eq. (5.8).

We refrain from writing down the full Lagrangian in terms of the component fields. This will be done in the next section for a non-abelian gauge theory from which the abelian limit can easily be taken.

5.3 Susy QCD

The construction of supersymmetric non-abelian gauge theories is slightly more complicated, as expected. Without loss of generality we will start by looking at $SU(3)$ with the eight generators T^a and the corresponding gauge superfields (containing the gluon) V^a. We also introduce $V \equiv V^a T^a$ (where the sum $\sum_a$ with $a \in \{1 \ldots 8\}$ is understood) with the generators in the adjoint representation and the gauge coupling g. The gauge transformation is as given in Eq. (5.13) with $\Lambda \equiv \Lambda^a T^a$. Note, however, that Eq. (5.14) is not applicable any longer, due to non-commuting terms in the Baker-Campbell-Hausdorff formula (see remark after Eq. (4.3)).

We have to modify the kinetic terms, because U_α as defined in Eq. (5.11) is not gauge invariant in the non-abelian case. Instead we define

$$U_\alpha \equiv -\frac{1}{8\,g}\bar{D}\bar{D}\,e^{-2g\,V}D_\alpha\,e^{2g\,V}\;; \qquad \bar{U}_{\dot\alpha} \equiv \frac{1}{8\,g}DD\,e^{2g\,V}\bar{D}_{\dot\alpha}\,e^{-2g\,V}\;; \tag{5.24}$$

where again $U_\alpha \equiv U^a_\alpha T^a$ and $\bar{U}_{\dot\alpha} \equiv \bar{U}^a_{\dot\alpha} T^a$. Using the expansion of the exponentials with $V^3_{\mathrm{WZ}} = V^2_{\mathrm{WZ}}(D_\alpha V_{\mathrm{WZ}}) = 0$ and $D_\alpha V^2_{\mathrm{WZ}} = (D_\alpha V_{\mathrm{WZ}})V_{\mathrm{WZ}} + V_{\mathrm{WZ}}(D_\alpha V_{\mathrm{WZ}})$ we can write Eq. (5.24) as

$$U_\alpha \equiv -\frac{1}{4}\bar{D}\bar{D}\,(D_\alpha V + g[D_\alpha V, V])\;; \qquad \bar{U}_{\dot\alpha} \equiv -\frac{1}{4}DD\,(\bar{D}_{\dot\alpha}V - g[\bar{D}_{\dot\alpha}V, V])\;; \tag{5.25}$$

Thus, in the abelian case Eq. (5.24) reduces to Eq. (5.11), but in the non-abelian case there is a difference due to $[T^a, T^b] \neq 0$, resulting in $[D_\alpha V, V] \neq 0$. Note that U_α and $\bar{U}_{\dot\alpha}$ as given in Eq. (5.24) are not invariant under non-abelian gauge transformations, but they transform like (see Appendix B)

$$U_\alpha \rightarrowtail e^{-2ig\,\Lambda}\,U_\alpha\,e^{2ig\,\Lambda}\;; \qquad \bar{U}_{\dot\alpha} \rightarrowtail e^{-2ig\,\Lambda^\dagger}\,\bar{U}_{\dot\alpha}\,e^{2ig\,\Lambda^\dagger}\;; \tag{5.26}$$

such that the trace (over the gauge group indices), $\mathrm{Tr}\,U^\alpha U_\alpha = 1/2\,(U^a)^\alpha\,U^a_\alpha$ is gauge invariant[3]. This is completely analogous to the non-susy case, where the field-strength tensor $F_{\mu\nu}$ itself is invariant in the abelian case, but in the non-abelian case only the trace $\mathrm{Tr}\,F^{\mu\nu}F_{\mu\nu} = 1/2\,(F^a)^{\mu\nu}\,F^a_{\mu\nu}$ is invariant, with $F_{\mu\nu} \equiv F^a_{\mu\nu}T^a$.

In the derivations above we have tacitly assumed that we can use the Wess-Zumino gauge again. However, this is not clear a priori. After all, Eq. (5.14) is not applicable in the non-abelian case. If we use the Baker-Campbell-Hausdorff formula in Eq. (5.13) we see that the non-abelian generalization of Eq. (5.14) reads

$$V \rightarrowtail V' = V + i(\Lambda - \Lambda^\dagger) - \frac{i}{2}[\Lambda + \Lambda^\dagger, V] + \ldots \tag{5.27}$$

where we have left out an infinite tower of higher commutators $[V, [V \ldots [V, (\Lambda - \Lambda^\dagger)]]]$. Thus the relation between V and Λ and $\Lambda^\dagger$ in the Wess-Zumino gauge fixing is more complicated, but we can still arrange Λ and $\Lambda^\dagger$ such that V' takes the form given in Eq. (5.16).

---

[3]We use the normalization $\mathrm{Tr}\,T^a T^b = \delta^{ab}/2$

It might not be obvious that U_α as defined in Eq. (5.24) has the structure $U_\alpha^a T^a$. But the situation is again very similar to $F^{\mu\nu}$. Performing an explicit computation (see Appendix B), we get terms involving commutators $[T^a, T^b]$ which are written in terms of the structure constants, using Eq. (2.13), and we get

$$U_\alpha^a = -\frac{i}{2}\,\theta_\beta(\sigma^\mu\bar{\sigma}^\nu)_\alpha^{\;\beta}\,F_{\mu\nu}^a - \theta\theta\,\sigma_{\alpha\dot{\alpha}}^\mu(D_\mu\bar{\lambda}^a)^{\dot{\alpha}} - i\,\lambda_\alpha^a + \theta_\alpha\,D^a \tag{5.28}$$

where the explicit form of the field-strength tensor and the (gauge) covariant derivatives are given by

$$F_{\mu\nu}^a \;\equiv\; \partial_\mu v_\nu^a - \partial_\nu v_\mu^a - g f^{abc}\,v_\mu^b\,v_\nu^c \tag{5.29}$$

$$(D^\mu\bar{\lambda}^a)^{\dot{\alpha}} \;\equiv\; (\partial^\mu\bar{\lambda}^a)^{\dot{\alpha}} - g f^{abc}\,(v^b)^\mu(\bar{\lambda}^c)^{\dot{\alpha}} \tag{5.30}$$

and the component fields are functions of $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$. Note that the normalization and the details of the definition in Eq. (5.24) have been chosen such that Eq. (5.28) agrees with Eq. (5.18) in the abelian limit $f^{abc} \to 0$.

We can now proceed as in non-susy gauge theories and introduce an arbitrary number of matter fields, in our case χSF, that transform under a certain representation

$$\phi_i \longrightarrow \phi_i' = \left(e^{i\Lambda^a T^a}\right)_{ij}\phi_j \tag{5.31}$$

where T^a are the generators in the chosen representation and i and j are the corresponding indices. In the case of susy QCD these would be the χSF containing the quarks, transforming in the fundamental representation of $SU(3)$, i.e. $i, j \in \{1, 2, 3\}$. The Lagrangian then reads

$$\mathcal{L} = \frac{1}{4}\big[U^a U^a\big]_{\theta\theta} + \frac{1}{4}\big[\bar{U}^a\bar{U}^a\big]_{\bar{\theta}\bar{\theta}} + \big[\phi_i^\dagger\,\big(e^{2gV}\big)_{ij}\,\phi_j\big]_{\theta\theta\,\bar{\theta}\bar{\theta}} + \big[W(\phi_i)\big]_{\theta\theta} + \big[W^\dagger(\phi_i^\dagger)\big]_{\bar{\theta}\bar{\theta}} \tag{5.32}$$

where the products of the χSF U^a and $\bar{U}^a$ are defined as in Eqs. (3.4) and (3.5).

The next task is to rewrite Eq. (5.32) in terms of the component fields (details are given in Appendix B). Starting with the first two terms, we note that they take the same form as Eq. (5.19) with the exception that the normal derivatives ∂_μ have to be replaced by the (gauge) covariant derivatives D_μ, Eq. (5.30), and the explicit form of $F_{\mu\nu}^a$ takes the "non-abelian" form given in Eq. (5.29). This can be seen by comparing Eq. (5.28) with Eq. (5.18). Thus the first two terms contain the kinetic terms of the gluons and gluinos as well as their self interactions due to the non-abelian nature of the gauge group. Thus susy forces a non-abelian gluino-gluino-gluon interaction on us through the term $\sim \lambda\sigma^\mu D_\mu\bar{\lambda}$.

The superpotential terms in Eq. (5.32) are familiar from the Wess-Zumino models. This leaves us with the term $\phi^\dagger\left(e^{2gV}\right)\phi$. Expanding the exponential, the leading term $\phi^\dagger\phi$ is again familiar from the Wess-Zumino models and contains the kinetic terms of the squarks and quarks. The remaining terms, $2g\,\phi^\dagger V\phi$ and $2g^2\,\phi^\dagger V V\phi$ contain the gauge interactions of the squarks and quarks with the gluons and gluinos.

Putting everything together, the supersymmetric Lagrangian in the Wess-Zumino gauge for chiral superfields ϕ_i (with component fields φ_i, ψ_i) and vector superfields V^a (with component fields v_μ^a, λ^a) for a general gauge group is given by

$$
\begin{aligned}
\mathcal{L} &= (D_\mu\varphi)_i^\dagger(D^\mu\varphi)_i + \frac{i}{2}\psi_i\sigma^\mu(D_\mu\bar\psi)_i - \frac{i}{2}(D_\mu\psi)_i\sigma^\mu\bar\psi_i \\
&- \frac{1}{4}F_{\mu\nu}^a(F^a)^{\mu\nu} + \frac{i}{2}\lambda^a\sigma^\mu(D_\mu\bar\lambda)^a - \frac{i}{2}(D_\mu\lambda)^a\sigma^\mu\bar\lambda^a \\
&- \sqrt{2}ig\,\bar\psi_i\bar\lambda^a T_{ij}^a\varphi_j + \sqrt{2}ig\,\varphi_i^\dagger T_{ij}^a\psi_j\lambda^a \\
&- \frac{1}{2}\frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2 W^\dagger}{\partial\varphi_i^\dagger\partial\varphi_j^\dagger}\bar\psi_i\bar\psi_j - V(\varphi_i,\varphi_j^\dagger)
\end{aligned} \tag{5.33}
$$

The potential is the sum of the F-terms, Eq. (5.8), and D-terms, Eq. (5.23), and reads

$$
V(\varphi_i,\varphi_j^\dagger) = F_i^\dagger F_i + \frac{1}{2}(D^a)^2 = \sum_i\left|\frac{\partial W}{\partial\varphi_i}\right|^2 + \frac{1}{2}\sum_a(g\,\varphi_i^\dagger\,T_{ij}^a\,\varphi_j + k^a)^2 \tag{5.34}
$$

where the Fayet-Iliopoulos term $\mathcal{L}_{\text{FI}} = 2\sum_a k^a[V^a]_{\theta\theta\bar\theta\bar\theta}$ can be present only for $U(1)$ gauge fields. The most general superpotential W is given by (see Eq. (5.2))

$$
W(\varphi_i) = a_i\,\varphi_i + \frac{1}{2}m_{ij}\,\varphi_i\varphi_j + \frac{1}{3!}y_{ijk}\,\varphi_i\varphi_j\varphi_k \tag{5.35}
$$

The requirement of gauge invariance imposes constraints on the coefficients a_i, m_{ij} and y_{ijk}. Finally, the (gauge) covariant derivatives act as follows:

$$
\begin{aligned}
(D_\mu\varphi)_i &= \partial_\mu\varphi_i + ig\,v_\mu^a T_{ij}^a\varphi_j \\
(D_\mu\psi)_i &= \partial_\mu\psi_i + ig\,v_\mu^a T_{ij}^a\psi_j \\
(D_\mu\lambda)^a &= \partial_\mu\lambda^a - gf^{abc}v_\mu^b\lambda^c
\end{aligned} \tag{5.36}
$$

Eq. (5.33) is our master equation for the Lagrangian of a susy gauge theory. Note that at this point we can forget about superfields and superspace if we want. These concepts have been extremely useful in deriving Eq. (5.33), but are not required any longer once we have the Lagrangian.

We close this section by looking at the interactions induced by the various terms of Eq. (5.33). Starting with the terms containing kinetic terms (propagators) we have

$$(D_\mu\varphi)_i^\dagger(D^\mu\varphi)_i \quad\rightarrow \tag{5.37}$$

$$\frac{i}{2}\psi_i\sigma^\mu(D_\mu\bar\psi)_i + \text{h.c.} \quad\rightarrow \tag{5.38}$$

$$-\frac{1}{4}F_{\mu\nu}^a(F^a)^{\mu\nu} \quad\rightarrow \tag{5.39}$$

$$\frac{i}{2}\lambda^a\sigma^\mu(D_\mu\bar\lambda)^a + \text{h.c.} \quad\rightarrow \tag{5.40}$$

Dashed lines represent scalars, solid lines superimposed with wavy lines represent gauginos. The hermitian conjugate of the various diagrams are not shown. Grey vertices are present only in non-abelian gauge theories. Turning to the remaining interactions with no kinetic terms we have

$$- \sqrt{2} i g \, \bar{\psi}_i \bar{\lambda}^a T^a_{ij} \varphi_j + \text{h.c.} \quad \rightarrow \tag{5.41}$$

$$\frac{1}{2} (g \, \varphi^\dagger_i T^a_{ij} \, \varphi_j)^2 \quad \rightarrow \tag{5.42}$$

$$-\frac{1}{2} \frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} \psi_i \psi_j + \text{h.c.} \quad \rightarrow \tag{5.43}$$

$$\left| \frac{\partial W}{\partial \varphi_i} \right|^2 \quad \rightarrow \tag{5.44}$$

The terms introduced through the superpotential are familiar from the Wess-Zumino model. Indeed, Eq. (5.43) corresponds to the terms $-M/2 \, \psi\psi$ and $y/2 \, \varphi \, \psi\psi$ of Eq. (5.9) respectively, whereas Eq. (5.44) is responsible for the terms $-|M|^2 \varphi\varphi^\dagger$, $-M^* y/2 \, \varphi\varphi\varphi^\dagger$ and $-|y|^2/4 \, \varphi\varphi\varphi^\dagger\varphi^\dagger$. The first terms in Eqs. (5.43) and (5.44) represent mass terms for the component fields of the χSF and the masses have to be equal in a susy theory. There are no mass terms for the gauge bosons and the gauginos. This is to be expected since in an unbroken gauge theory the gauge bosons are massless. Due to susy, the gauginos have to be massless as well. To give mass to gauge bosons we have to break gauge invariance. A simple example is discussed in Section 6.2. To give mass to gauginos, we can either break gauge invariance (and keep susy) such that the gauginos get the same non-vanishing mass as the gauge bosons, or we can keep gauge invariance (i.e. still have massless gauge bosons) but break susy. In the MSSM, this is done with soft breaking terms as will be discussed in Section 6.3.

5.4   The unbroken MSSM

With the results of the previous sections we can now go ahead and write down the susy extension of the Standard Model. We do this by introducing a χSF for every fermion of the Standard Model, a VSF for every gauge boson of the Standard Model and finally two chiral superfields for the Higgs bosons (the reason for having to introduce two Higgs superfields will be explained below). By doing this we introduce the scalar partners of the quarks and leptons, the squarks and sleptons, and the fermionic partners of the gauge bosons, the gauginos. We also get a richer Higgs sector, with fermionic partners. The latter will mix with (some of the) gauginos to produce the neutralinos and charginos. The χSF and the VSF are listed in Tables 1 and 2 respectively. The superscripts $\pm$ and 0 indicate the electric charge Q_{em} with the convention $Q_{\text{em}} = T_3 + Y$, where T_3 is the third component of isospin.

	LHχSF	spin 0	spin $\frac{1}{2}$	$(SU(3), SU(2), U_Y(1))$
squarks and quarks	Q	$(\tilde{u}_L, \tilde{d}_L)$	(u_L, d_L)	$(3, 2, \frac{1}{6})$
	U	$\tilde{u}_R^\dagger$	$u_R^\dagger$	$(\bar{3}, 1, -\frac{2}{3})$
	D	$\tilde{d}_R^\dagger$	$d_R^\dagger$	$(\bar{3}, 1, \frac{1}{3})$
sleptons and leptons	L	$(\tilde{\nu}, \tilde{e}_L)$	(ν, e_L)	$(1, 2, -\frac{1}{2})$
	E	$\tilde{e}_R^\dagger$	$e_R^\dagger$	$(1, 1, 1)$
higgs and higgsinos	H_u	(h_u^+, h_u^0)	$(\tilde{h}_u^+, \tilde{h}_u^0)$	$(1, 2, \frac{1}{2})$
	H_d	(h_d^0, h_d^-)	$(\tilde{h}_d^0, \tilde{h}_d^-)$	$(1, 2, -\frac{1}{2})$

Table 1: Chiral superfields of the MSSM with their particle content. The transformation property under $SU(3) \times SU(2)$ and the value of $U_Y(1)$ is given in the last column. There are three copies of the quark and lepton superfields, one for each family.

It is clear that constructing such a theory by using Eq. (5.33) will result e.g. in squarks and sleptons with the same mass as the corresponding quarks and leptons. Since this is in clear contradiction to observation, we will have to find a way to break susy to make the model phenomenologically acceptable. This issue will be addressed in Section 6. Here we focus on the simpler task of writing down the strictly susy extension of the Standard Model.

Following Eq. (5.33) we see that after having fixed the list of χSF and VSF, i.e. the matter fields and the gauge group, the only freedom we have is in choosing the superpotential $W(\phi_i)$. This completely fixes the Lagrangian. As stated repeatedly, we have to make sure that W is gauge invariant and that it is an analytic function of the LHχSF. It is for this reason that in Table 1 we have listed all χSF as LHχSF, i.e. we take the hermitian conjugate of the right-handed fields to obtain a LHχSF.

Let us start with a term in the superpotential, W_1, that gives rise to down-type quark masses. As in the Standard Model this is done by coupling the quark fields to a Higgs field with a non-vanishing vacuum expectation value (vev). The term is given by

$$W_1(\phi_i) = - y_d \, D \, Q \, H_d \equiv - \, (y_d)_{f_i f_j} \, D^{f_i} \, Q_a^{f_j} \epsilon^{ab} \, (H_d)_b \qquad (5.45)$$

and is usually written as in the l.h.s. of Eq. (5.45). On the r.h.s. we have introduced (nearly) all labels. First, f_i, $f_j \in \{1, 2, 3\}$ label the family/flavour. Second, a, $b \in \{1, 2\}$ are $SU(2)$ labels. The ϵ^{ab} is needed to make the term $Q \, H_d$ a singlet under $SU(2)$. Since D is also a singlet under $SU(2)$ the whole term is gauge invariant with respect to $SU(2)$. The gauge invariance with respect to $SU(3)$ is trivial (which is why we omitted colour labels on the r.h.s. of Eq. (5.45)), since $3 \times \bar{3} = 1 + 8$ contains a singlet and H_d is a

	VSF	spin $\frac{1}{2}$	spin 1	$(SU(3), SU(2), U_Y(1))$
gluinos and gluons	G	$\tilde{g}$	g	$(8, 1, 0)$
winos and W-bosons	W	$\widetilde{W}^\pm, \widetilde{W}^0$	$W^\pm, W^0$	$(1, 3, 0)$
bino and B-boson	B	$\widetilde{B}$	B	$(1, 1, 0)$

Table 2: Vector superfields of the MSSM with their particle content. The transformation property under $SU(3) \times SU(2)$ and the value of $U_Y(1)$ is given in the last column.

singlet. The hypercharges of the three χSF add to zero, thus the term is indeed gauge invariant under $SU(3) \times SU(2) \times U_Y(1)$.

Giving the Higgs a non vanishing vev then results in a mass term for the down-type quarks. More precisely, writing the term Eq. (5.45) in terms of its scalar component fields, as required for Eq. (5.33), we get

$$W_1(\varphi_i) = - (y_d)_{f_i f_j} \, (\tilde{d}_R^\dagger)^{f_i} \left((\tilde{u}_L)^{f_j} \, h_d^- - (\tilde{d}_L)^{f_j} \, h_d^0 \right) \tag{5.46}$$

If the neutral component of the Higgs gets a vev, $\langle h_d^0 \rangle = v_d$, we obtain a mass term for the fermions through the term

$$-\frac{1}{2} \frac{\partial^2 W_1}{\partial \varphi_i \partial \varphi_j} \, \psi_i \psi_j + \text{h.c.} \Rightarrow -\frac{1}{2} v_d \, (y_d)_{f_i f_j} \, (d_R^\dagger)^{f_i} \, d_L^{f_j} + \text{h.c.} \tag{5.47}$$

where on the l.h.s. we have given the general expression as in Eq. (5.33) and on the r.h.s. the explicit expression we obtain from W_1 as given in Eq. (5.45) with $\varphi_i = (\tilde{d}_R^\dagger)^{f_i}$, $\varphi_j = (\tilde{d}_L)^{f_j}$, $\psi_i = (d_R^\dagger)^{f_i}$ and $\psi_j = d_L^{f_j}$. Thus we have a mass matrix in family space, $m_{f_i f_j} = v_d (y_d)_{f_i f_j}$ which we have to diagonalize to obtain the masses of the three down-type quarks. The squarks obtain their mass from the term

$$-\sum_i \left| \frac{\partial W_1}{\partial \varphi_i} \right|^2 \Rightarrow - v_d^2 \, |(y_d)_{f_i f_j}|^2 \left(\tilde{d}_L^{f_i} (\tilde{d}_L^{f_j})^\dagger + \tilde{d}_R^{f_i} (\tilde{d}_R^{f_j})^\dagger \right) \tag{5.48}$$

which results in the same masses for the squarks and quarks. Note that both, the squarks and quarks get their masses from a non-zero vev of the scalar component field of the neutral Higgs boson. Charged fields or fermionic fields cannot get a vev without violating charge conservation or Lorentz invariance.

Of course, there are more terms associated with the superpotential term W_1. If we insert $W_1 = - y_d \, D \, Q \, H_d$ into Eq. (5.33) we get interactions of Higgs bosons with fermions e.g $h^0 \to d \, \bar{d}$ and $h^- \to \bar{u} \, d$ or interactions of squarks with higgsinos and quarks, e.g. $\tilde{d} \to \tilde{h}^- u$. These interactions correspond to those exemplified in Eq. (5.43). There are also four-point scalar interactions such as $\tilde{d}\tilde{u} \to \tilde{d}\tilde{u}$ as shown in Eq. (5.44). The

25

higgsinos actually mix with the fermionic partners of gauge bosons to form charginos and neutralinos. For a more complete discussion and a list of interactions with Feynman rules we refer to Refs. [5, 12].

The charged leptons obtain their mass in exactly the same way, i.e. by introducing the term $W_2 = -y_e \, E \, L \, H_d$. The $SU(2)$ doublets are combined as in Eq. (5.45) to obtain a gauge invariant term. The gauge invariance with respect to $SU(3)$ and $U_Y(1)$ is obvious. Giving mass to the up-type quarks is not so easy. In the Standard Model, this is done with the same Higgs boson, by introducing a term $\sim UQH^\dagger$. However, this term violates susy, because it contains $H^\dagger$ and therefore the superpotential is not an analytic function of the LHχSF any longer. Thus we have no other choice than to introduce a second Higgs doublet H_u with the neutral component field that gets a vev in the $T_3 = -1/2$ position of the doublet, $\langle h_u^0 \rangle = v_u$. Then we can write the gauge invariant term $W_3 = y_u \, U \, Q \, H_u$ which gives a mass to the up-type quarks. The presence of the second Higgs doublet also ensures the cancellation of anomalies.

Having a second Higgs doublet allows us to construct another gauge invariant term, $W_4 = \mu \, H_u \, H_d$, such that the MSSM superpotential reads

$$W_{\mathrm{MSSM}} = y_u \, U \, Q \, H_u - y_d \, D \, Q \, H_d - y_e \, E \, L \, H_d + \mu \, H_u \, H_d \qquad (5.49)$$

These are all the terms we want but, most unfortunately, not all the terms we get. There are many more gauge invariant terms that can be included in the superpotential and, unless there is a good reason to leave them out, from a theoretical point of view we have to include them.

Looking at Table 1 we see that the following terms are also all gauge invariant

$$W_{\not R} = \frac{1}{2} \lambda \, E \, L \, L + \lambda' \, D \, L \, Q + \mu' \, L \, H_u + \frac{1}{2} \lambda'' \, U \, D \, D \qquad (5.50)$$

The factors $1/2$ are introduced to account for the symmetry. The gauge invariance under $SU(3)$ of the term $U \, D \, D$ implies that we have to take the completely antisymmetric colour combination, i.e. $\epsilon_{ijk} U^i \, D^j \, D^k$, where i, j and k are colour indices. Thus this is the same colour combination as e.g. in a antiproton. Note that the gauge invariant term $E \, H_d \, H_d$ vanishes due to the ϵ^{ab} in the combination of the two weak doublets, as detailed in Eq. (5.45). We also remind the reader that terms with more than three χSF lead to a non-renormalizable theory and therefore are left out. The problem with the terms in Eq. (5.50) is that they violate lepton number (the first three terms) and baryon number (the last term). This leads to serious problems with proton decay (see e.g. Ref [1]).

These problems can be avoided by pulling another symmetry out of a hat. Usually this is R-parity, a multiplicative quantum number defined in terms of baryon number B, lepton number L and spin s as $R \equiv (-1)^{3B+L+2s}$ such that Standard Model particles (including the Higgs bosons) have $R = 1$, whereas all superpartners have $R = -1$. Note that the various component fields of a superfield have different R-parity due to the spin contribution. Thus we cannot associate R-parity to a superfield and it is not immediately obvious that the terms in Eq. (5.50) violate R-parity. From this point of view a more convenient symmetry is matter parity, defined as $(-1)^{3B+L}$. Due to angular momentum

conservation matter parity conservation and R-parity conservation are equivalent. The former has the advantage that it is defined for a superfield. The lepton and quark superfields have matter parity -1, whereas the Higgs and vector superfields have matter parity $+1$. Keeping in mind that this is a multiplicative quantum number, it is now immediately obvious that all terms in Eq. (5.49) have matter parity $+1$, whereas all terms in Eq. (5.50) have matter parity -1.

Another option to avoid problems with proton decay is to impose baryon or lepton number conservation, leading to R-parity violating scenarios. In either case, it is disturbing that in the MSSM an additional symmetry has to be introduced to avoid these problems. In the Standard Model, such problematic terms are absent accidentally, i.e. without any further requirements.

6 Breaking supersymmetry

The MSSM Lagrangian of Section 5.4 leads to superpartners with the same mass as the corresponding Standard Model particles. Obviously this is not in accord with Nature and therefore not acceptable. Thus, we have to break susy in such a way as to give the superpartners a larger mass. It is also clear that we must not break susy by brute force. The situation is similar to the case of gauge theories, where gauge symmetry implies massless gauge bosons in contrast with experiment. As is well known, this problem can be solved by breaking gauge symmetry spontaneously, i.e. the Lagrangian is still gauge invariant but the ground state of the theory does not share this symmetry. This gives mass to the W and Z bosons while maintaining the wanted features of the symmetry. We want to do the same for susy.

Before we look at the various possibilities explicitly, let us make a few general considerations. Let us start with Eq. (2.18) and multiply it by $(\bar{\sigma}^0)^{\dot{\beta}\alpha}$. On the l.h.s. we use the fact that $(\bar{\sigma}^0)^{\dot{\beta}\alpha}$, as defined in Eq. (A.6), is simply the unit 2×2 matrix. On the r.h.s. we use Eq. (A.9) and thus obtain

$$Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2 = 4\, g^{0\mu} P_\mu = 4\, P^0 = 4\, H \tag{6.1}$$

where H is the Hamiltonian and we used $(Q_\alpha)^\dagger = \bar{Q}_{\dot{\alpha}}$. From Eq. (6.1) we see that susy theories have the remarkable property that H is bounded from below, i.e. for any state $|b\rangle$ we have $\langle b|H|b\rangle \geq 0$.

Let us specialize to the ground state $|0\rangle$ of our theory. If susy is not spontaneously broken, the ground state shares the symmetry of the Lagrangian, i.e. $|0\rangle$ is invariant under susy. This means $S(0, \zeta, \bar{\zeta})|0\rangle = |0\rangle$, with $S(0, \zeta, \bar{\zeta})$ as given in Eq. (4.1), which entails $Q_\alpha|0\rangle = 0$ and $\bar{Q}_{\dot{\alpha}}|0\rangle = (Q_\alpha)^\dagger|0\rangle = 0$. From Eq. (6.1) we then immediately conclude $\langle 0|H|0\rangle = 0$. Again this is a remarkable property of unbroken susy. Compare this for example to the normal harmonic oscillator, where the ground state energy is $1/2 \neq 0$. In a susy harmonic oscillator, the fermionic part cancels this contribution and the ground state energy is zero.

If susy is spontaneously broken, the ground state does not share the symmetry of the

Lagrangian, i.e. $|0\rangle$ is not invariant under susy. This implies that $Q_\alpha |0\rangle \neq 0$ and thus $\langle 0|H|0\rangle > 0$. This is the crucial criteria for the construction of spontaneously broken susy.

In order to obtain a strictly positive ground state energy the potential V has to satisfy $V|_{\min} > 0$. According to Eq. (5.34) the potential has two terms, a F-term given by $V_F = |\partial W/\partial \varphi_i|^2$ and a D-term, $V_D = 1/2 \left(g\, \varphi_i^\dagger\, T_{ij}^a\, \varphi_j + k^a \right)^2$ and obviously satisfies $V \geq 0$ in agreement with $\langle 0|H|0\rangle \geq 0$. If we want spontaneous symmetry breaking we either need $V_F|_{\min} > 0$ (*F-term breaking*) or $V_D|_{\min} > 0$ (*D-term breaking*) or a combination of both. We will look at explicit examples of the two cases in turn.

6.1   F-term breaking

The canonical example of F-term breaking is the O'Raifeartaigh (OR) model [13]. Consider the case where we have three χSF and the superpotential

$$W_{\mathrm{OR}}(\phi_i) = -a\,\phi_1 + m\,\phi_2\phi_3 + \frac{y}{2}\phi_1\phi_3^2 \tag{6.2}$$

The potential is then given by

$$V_{\mathrm{OR}} = \sum_i \left| \frac{\partial W_{\mathrm{OR}}(\varphi_i)}{\partial \varphi_i} \right|^2 = \left| a - \frac{y}{2}\varphi_3^2 \right|^2 + |m\,\varphi_3|^2 + |m\,\varphi_2 + y\,\varphi_1\varphi_3|^2 \tag{6.3}$$

Looking at the first two terms of V_{OR} we conclude $V_{\mathrm{OR}} > 0$, which is precisely what we want. The potential has three extrema. If we assume $a < m^2/y$, the absolute minimum of the potential is at $\varphi_2 = \varphi_3 = 0$ and arbitrary φ_1. In this case $V_{\mathrm{OR}}|_{\min} = a^2 \neq 0$.

To verify that susy has been broken, let us compute the masses of the fermions and scalars in this theory. For each of the three χSF we have two real scalars φ_i^{Re} and φ_i^{Im} and a Weyl spinor ψ_i, i.e. two real bosonic and two real fermionic degrees of freedom. To compute the fermion mass, we first obtain the mass matrix

$$-\frac{1}{2}\frac{\partial^2 W_{\mathrm{OR}}}{\partial \varphi_i \partial \varphi_j}\,\psi_i\psi_j + \mathrm{h.c.} = -\frac{1}{2}\left(2m\,\psi_2\psi_3 + y\langle\varphi_1\rangle\,\psi_3\psi_3\right) + \mathrm{h.c.} \tag{6.4}$$

These are all bilinear in ψ_i terms we get for $\langle\varphi_2\rangle = \langle\varphi_3\rangle = 0$ and $\langle\varphi_1\rangle \neq 0$. There is no mass term at all for ψ_1, resulting in a massless fermion. This is not surprising. We know from gauge theories that spontaneous breaking of a global (bosonic) symmetry results in a massless Goldstone boson. Here we have the spontaneous breaking of global susy, a fermionic symmetry, thus we get a massless Goldstone fermion, usually called *goldstino*. Linear combinations of the other two fermions, ψ_2 and ψ_3 have mass m.

Let us now compute the mass of the scalars. To do this we expand the potential around $\varphi_i = 0$ and consider the bilinear terms in φ_i^{Re} and φ_i^{Im}. We get two massless scalars, φ_1^{Re} and φ_1^{Im} and two scalars of mass m, φ_2^{Re} and φ_2^{Im}. This is still completely susy, as these masses agree with the corresponding fermion masses. However, the breaking of susy manifests itself in the remaining scalar masses, φ_3^{Re} and φ_3^{Im} which are found

to be $\sqrt{m^2 - a\,y}$ and $\sqrt{m^2 + a\,y}$. Thus the scalar masses differ from the masses of the corresponding fermions, a clear sign that susy is broken.

The problem with this mechanism is that it does not provide what we want from a phenomenological point of view. We would like to break susy such that all of the (yet undiscovered) scalars get a larger mass than the fermions. In the example above, one of the scalars has a higher mass than the corresponding fermion, the other has a lower mass. Unfortunately, this is a general feature [15] and can be written as

$$\text{STr}\, M^2 \equiv \sum (-1)^s (2s + 1)\, m_s^2 = 0 \tag{6.5}$$

In the above relation the *supertrace* sums over all component fields, s denotes the spin and m_s is the mass associated with the real component field of spin s. This implies that with this mechanism we will always get a symmetric shift in the masses, i.e. some superpartners are heavier and others have smaller mass than the Standard Model particles such that the average mass remains the same.

It is important to note that this relation holds only at tree level and is in general violated by loop corrections. This does not help directly, as loop corrections will never be able to shift e.g. the selectron mass from below the electron mass to something like 100 GeV. But it does leave a window for F-term spontaneous susy breaking. If we have F-term breaking not directly in the MSSM, but in a hidden sector, then it is possible to mediate the susy breaking by loop effects into the MSSM and avoid the constraint of Eq. (6.5).

6.2   D-term breaking

Let us come back to the abelian susy gauge theory discussed in Section 5.2. For simplicity we assume there is only one χSF and the superpotential vanishes, $W(\phi) = 0$. However, we have a Fayet-Iliopoulos term. After eliminating the D component field the Lagrangian reads

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{4}\left[U^\alpha U_\alpha\right]_{\theta\theta} + \frac{1}{4}\left[\bar{U}_{\dot\alpha}\bar{U}^{\dot\alpha}\right]_{\bar\theta\bar\theta} + \left[\phi^\dagger e^{2gV}\phi\right]_{\theta\theta\,\bar\theta\bar\theta} + \left[2k\,V\right]_{\theta\theta\,\bar\theta\bar\theta} \\
&= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - \frac{1}{2}\left(g\,\varphi^\dagger\varphi + k\right)^2 \\
&\quad - \left(\frac{i}{2}(\partial_\mu\lambda)\sigma^\mu\bar\lambda + \frac{i}{2}(D_\mu\psi)\sigma^\mu\bar\psi + \sqrt{2}ig\,\bar\psi\bar\lambda\,\varphi + \text{h.c.}\right)
\end{aligned}
\tag{6.6}
$$

with $D^\mu = \partial^\mu + ig\,v^\mu$.

Let us focus on the potential $V = D^2/2 = (g\,\varphi^\dagger\varphi + k)^2/2$ which holds the key to spontaneous breaking of susy. We would like V to be strictly positive $V|_{\min} > 0$. In order to see whether we can achieve this we have to distinguish two cases. Either the scalar field φ gets a vev or it does not.

Starting with the first scenario we see that the presence of k does not prevent $D = 0$. What can happen is that φ gets a vev such that $V|_{\min} = 0$, i.e. $\langle\varphi^\dagger\varphi\rangle = -k/g > 0$. Thus

what we actually achieve is not spontaneous breaking of susy but rather spontaneous breaking of gauge invariance. Indeed, the term $(D_\mu \varphi)^\dagger (D^\mu \varphi)$ will result in a gauge boson mass term $g^2 v^\mu v_\mu \langle \varphi^\dagger \varphi \rangle = -k\, g\, v^\mu v_\mu = (m_v^2/2)\, v^\mu v_\mu$, i.e the gauge boson mass is $m_v = \sqrt{-2k\,g}$. The additional degree of freedom associated with the mass of the gauge boson comes from one of the scalars. This can be seen by writing $\varphi = (\varphi^{\mathrm{Re}} + i\,\varphi^{\mathrm{Im}})/\sqrt{2}$ and expanding the potential around $\langle \varphi^{\mathrm{Im}} \rangle = \sqrt{-2\,k/g}$. The scalar field φ^{Im} gets a mass term $k\,g\,(\varphi^{\mathrm{Im}})^2 = -(m_\varphi^2/2)\,(\varphi^{\mathrm{Im}})^2$, i.e. the same mass as the gauge boson. However, the other scalar field, φ^{Re} does not get a mass term. This is the Goldstone boson that gets absorbed by the initially massless gauge boson. The fermion ψ of the χSF and the gaugino λ form a Dirac spinor $\Psi = (\psi, \bar{\lambda})$, as in Eq. (3.1), and also get a mass term from $-\sqrt{2}i\,g\,\bar{\psi}\bar{\lambda}\,\langle \varphi \rangle + \mathrm{h.c} = \sqrt{-2g\,k}\,\overline{\Psi}\Psi$. In fact the mass of the gauge boson is the same as the gaugino mass as it has to be since susy is not broken. For the same reason, the (massive) scalar and the other fermion also have the same mass. This scenario is simply the susy generalization of the Higgs mechanism, where a massless χSF and a massless VSF combine to a massive VSF.

To achieve what we set out for we have to prevent $D = 0$. If $\langle \varphi \rangle = 0$ we have $V|_{\min} = k^2/2 \neq 0$. The gauge boson v_μ, its partner the gaugino λ as well as the fermion ψ of the χSF all remain massless. The only particle that gets a mass is the scalar, through the term $-g\,k\,\varphi^\dagger \varphi$ from the potential. This corresponds to a mass $m_\varphi = \sqrt{g\,k}$ for the two (real) scalar fields φ^{Re} and φ^{Im}.

That is precisely what we wanted to achieve! Thus the key for D-term susy breaking is to prevent the scalar fields to develop a vev. We can achieve this by giving the scalar fields large masses through superpotential terms. Therefore we now consider a non-vanishing superpotential. To get a gauge invariant superpotential we need a pair of χSF, ϕ_i, $i \in \{1, 2\}$ with opposite charges q_i with respect to the $U(1)$ under consideration. More precisely, the fields have to have gauge transformations like $\phi_i \rightarrowtail e^{-iq_i\Lambda}\phi_i$ with $q_1 = -q_2$. This enables us to write a gauge invariant term $W = m\,\phi_1\phi_2$ in the superpotential. The scalar potential then also gets a F-term contribution and reads

$$V = |m|^2 \sum_{i=1}^{2} \varphi_i^\dagger \varphi_i + \frac{1}{2}\Big(k + g \sum_{i=1}^{2} q_i\,\varphi_i^\dagger \varphi_i\Big)^2 \tag{6.7}$$

with $\varphi_i^\dagger \varphi_i = 1/2\,((\varphi_i^{\mathrm{Re}})^2 + (\varphi_i^{\mathrm{Im}})^2)$. If we choose $|m|^2$ large enough, $|m|^2 > g|q_i|k$, the minimum of the potential is at $\langle \varphi_i^{\mathrm{Re}} \rangle = \langle \varphi_i^{\mathrm{Im}} \rangle = 0$ and we have $V|_{\min} = k^2/2 \neq 0$.

Let us try to apply this mechanism to the MSSM with $U_Y(1)$ as the abelian group. Now we immediately face a problem. We can give large masses to the Higgs scalars through the superpotential term $\mu\,H_u\,H_d$ but not to the other scalars. There are no gauge invariant terms corresponding to $W = m\,\phi_1\phi_2$ in the MSSM superpotential given in Eqs. (5.49) and (5.50). Thus, however nice the D-term susy breaking mechanism is, it cannot be applied to the MSSM. What would happen is that e.g. the squark fields develop a vev, rather than susy being broken. This is not acceptable as it would break electric charge and colour conservation, the last thing we want.

As for F-term breaking, in order for D-term breaking to be phenomenologically acceptable, it would have to happen in a hidden sector, with a new $U(1)$ group. The

breaking then would have to be mediated to the visible sector, the MSSM.

Let us close this section with a remark concerning the supertrace formula Eq. (6.5). In our initial D-term breaking example with only one χSF the two real scalars of the χSF obtain a mass shift, whereas all other particles remain massless. This clearly violates the supertrace formula. In fact, Eq. (6.5) can be generalized by writing the r.h.s. as $\sum q_i^2 g \langle D \rangle$. However, as we have seen, for a realistic (gauge invariant) example we need the χSF to come in pairs with opposite charges. Thus for every mass shift $\delta m_{\varphi_1} = q_1 \sqrt{g\,k}$ of a scalar component field we get an opposite mass shift $\delta m_{\varphi_2} = q_2 \sqrt{g\,k} = -\delta m_{\varphi_1}$ and Eq. (6.5) holds again.

6.3 Soft breaking and the hierarchy problem

In the previous two sections we have seen that while it is possible to break susy spontaneously either through F-term or D-term breaking, neither option works directly for the MSSM. The standard procedure then is to introduce a hidden sector, break susy in the hidden sector and mediate the breaking to the visible sector, the MSSM, either through gravity, gauge interactions or by other means. If we did know the details of the hidden sector and the mediation we could compute the induced breaking in the visible sector. Sadly, we don't. Thus we have to parameterize our ignorance. If we choose the latter option we introduce susy breaking terms by hand. The idea is to measure these parameters and hopefully, once a consistent picture arises, to infer from these measurements the theory behind susy breaking.

Inserting susy breaking terms by hand we have to be careful not to destroy all the nice features of susy. One of these features is the much celebrated cancellation of quadratic divergences and its relation to the hierarchy problem.

To understand this let us start by considering a fermion, say the electron, and recapitulate some basic properties about renormalization. In the Lagrangian we have a term $m_0 \overline{\Psi}\Psi$, where m_0 is the bare mass. The parameter m_0 is related in a particular way (depending on the precise definition of the mass) to the (renormalized) theoretical mass m_{th}. At tree level, we have $m_0 = m_{\mathrm{th}}$, at one loop we have $m_{\mathrm{th}} = m_0 + \delta m$, where the one-loop corrections $\delta m \sim \alpha\, m_0\, K$. Here α is the (electromagnetic) coupling and K a calculable coefficient, depending on the regularization and precise definition of the mass. Due to the presence of ultraviolet singularities in loop integrals, δm is actually divergent. If we use dimensional regularization in $D = 4 - 2\epsilon$ dimensions, δm contains a pole $1/\epsilon$. The physical reason for this divergence is the breakdown of our field-theory picture at large energies because for instance it does not include gravity. In order to be able to proceed in our field theory approach we absorb our ignorance into a counterterm and relate it to an experimentally measured value, in our case, the electron mass m_{exp}. Thus we set $m_{\mathrm{th}} = m_0 + \delta m = m_{\mathrm{exp}}$ and thereby determine m_0. Once we have done this for the electron mass (and a few more quantities) we can then go and predict any other quantity within our field theory approach.

Since m_{th} is finite but δm is divergent, we know m_0 has to be divergent as well. Thus we have an infinite fine tuning in that two infinite quantities, m_0 and δm, conspire

to give a value $m_{\text{th}} = m_0 + \delta m = 0.5$ MeV (in the case of the electron). Due to the above mentioned reason i.e. our accepted ignorance of what is happening at very large energy scales, nobody is worried about this. However, we certainly want our theory to be valid up to a certain scale Λ. Thus, if we replace the usual dimensional regularization by a more physical regularization, which consists of introducing a cutoff Λ in our loop integrals, we would hope not to have this fine tuning problem.

In the case of the electron, or any fermion, this is the case. Considering the power counting of the one-loop diagram that contributes to δm_F, the correction to the fermion mass m_F, we get from the fermion and photon propagator four powers of the integration momentum k in the denominator and one in the numerator.

$$\Rightarrow \quad \delta m_F \simeq \alpha \int^\Lambda d^4k \, \frac{\{k\}}{k^2 \, (k^2 - m_F^2)} \simeq \alpha \, m_F \log \frac{\Lambda}{m_F} \qquad (6.8)$$

This seems to lead to a linear divergence in Λ, i.e. $\delta m_F \sim \alpha \Lambda$. However, the linear term in the numerator always vanishes upon integration, as indicated by the curly brackets, if our regulator does not break Poincaré invariance. Thus we are left with only a logarithmic divergence $\delta m_F \sim \alpha \, m_F \log(\Lambda/m_F)$. As a result even for very large values of $\Lambda \sim 10^{15}$ GeV, we have $\delta m_F \lesssim m_F$, i.e. the correction is of the order of m_F and there is no fine tuning required.

Let us now repeat this exercise for any gauge boson. In general, we have two kinds of one-loop diagrams contributing to δm_G^2, fermion loops and gauge boson loops and we generically denote the masses of particles in the loop by m_L. In both cases, we get four powers of k in the denominator and two in the numerator. Those in the numerator are either from the fermion propagator or the gauge-boson interaction vertices.
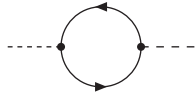
$$+ \quad \Rightarrow \quad \delta m_G^2 \simeq \alpha \int^\Lambda d^4k \, \frac{\{k^2\}}{k^2 \, k^2} \simeq \alpha \, m_L^2 \log \frac{\Lambda}{m_L} \qquad (6.9)$$

From power counting we would expect a quadratic divergence in δm_G^2. However, as indicated by the curly brackets, there is a cancellation of these quadratic singularities and the final answer is only logarithmically divergent. As for δm_F, even for very large values of Λ we have δm_G^2 and m_G^2 of the same order and no fine tuning is required. Thus our theory potentially could be valid up to very large energy scales.

The cancellation of quadratic singularities is not a coincidence. It is a symmetry that ensures this cancellation. In an unbroken gauge theory the gauge boson remains massless to all orders, so the cancellation is actually even stronger. Not only do the quadratic singularities cancel, but $\delta m_G^2 = 0$. In a spontaneously broken gauge theory this is no longer the case, but gauge symmetry still ensures the cancellation of the quadratic singularities. Also in the case of the fermion there is a symmetry that protects the fermion mass from large corrections. In fact, if the fermion is initially massless there is an additional symmetry, chiral symmetry, which prevents the generation of a non-vanishing mass.

So far so good, but what about scalars, i.e. the Higgs boson in the Standard Model. Let us consider the correction δm_S^2 due to a fermion loop. Again, we expect two powers

of k in the numerator and four powers of k in the denominator.

$$\cdots\!-\!\bigcirc\!-\!\cdots \quad \Rightarrow \quad \delta m_S^2 \simeq \alpha \int^{\Lambda} d^4k \, \frac{k^2}{(k^2 - m_L^2)\,(k^2 - m_L^2)} \simeq \alpha\Lambda^2 \qquad (6.10)$$

In this case, there is no cancellation of quadratic singularities. Thus if we expect our theory to be valid up to say $\Lambda \sim 10^{15\pm5}$ GeV we would need an incredible fine tuning between $m_S^2 \simeq 10^4$ GeV^2 (which is the typical Higgs mass) and $\delta m_S^2 \simeq \alpha\,10^{30\pm10}$ GeV^2. While this is not inconsistent as such it is not what is expected, even more so as this fine tuned cancellation would have to be repeated order by order in perturbation theory. Thus we are nudged towards thinking that the Standard Model may be valid only up to values of $\Lambda \sim 10^3$ GeV such that m_S^2 and δm_S^2 are of the same order.

What happens above $\Lambda \sim 10^3$ GeV? This is where susy comes into play. We have seen that in the fermion and gauge boson case it was a symmetry that protected the masses from quadratic singularities. In the case of the scalars this role can be played by susy. For each fermion loop there are diagrams with a susy scalar partner in the loop and adding them all up, the quadratic singularities cancel. It is not surprising that this works. After all, we have seen that fermionic masses are protected from quadratic singularities in any theory. Since susy relates scalar masses to fermion masses, in a susy theory scalar masses have to be protected as well.

If there are susy partner particles, they should show up at about $\sim 10^3$ GeV in order to be of any use in the solution of the hierarchy problem. If there are susy partners at $\sim 10^3$ GeV an unnaturally small tree-level value for the Higgs mass would be protected from large radiative corrections. However, there is still no explanation, why the mass is small in the first place. This is connected to another problem, the μ-problem. Looking at Eq. (5.49), the natural value for the couplings y_u, y_d and y_e are $\mathcal{O}(1)$ and the natural value of μ would be of the order of the largest scale, i.e. the cutoff scale where our theory breaks down. We just convinced ourselves that thanks to the cancellation of quadratic singularities this could well be $\sim 10^{15\pm5}$ GeV. However, the term $\mu\,H_u\,H_d$ governs the electroweak symmetry breaking, thus is of the order of the weak scale. Finding an explanation (and there are many proposed) why μ is so small would be a solution to the μ-problem.

When breaking susy by hand we want to make sure that we do not disturb the cancellation of quadratic singularities. We call this *soft breaking* of susy. To get a rough understanding which terms are allowed in soft breaking let us perform a simple dimensional analysis. The correction to the scalar mass squared all have to have mass dimension $[\delta m_S^2] = 2$. They can take the form

$$\delta m_S^2 = c^{(2)}\Lambda^2 + c^{(0)}\log(\Lambda/m_L) \qquad (6.11)$$

where $c^{(i)}$ is the product of the two couplings in the self-energy diagram of the Higgs. Note that $[c^{(2)}] = 0$ and $[c^{(0)}] = 2$ and that there is no term $c^{(1)}\Lambda$ as discussed in the fermion case. We want to prevent terms of the form $c^{(2)}\Lambda^2$. This can be done by allowing only susy breaking terms with mass dimension strictly larger than 0, such that $[c] > 0$. This eliminates the possibility of a term $c^{(2)}\Lambda^2$ in Eq. (6.11). Thus we can have scalar

mass terms $-m_{ij}^2\,\varphi_i^\dagger\varphi_j$, gaugino mass terms $-(m_{ij}/2)\,\lambda_i\lambda_j$+h.c. and bilinear and trilinear scalar couplings $b_{ij}\,\varphi_i\varphi_j$ and $a_{ijk}\,\varphi_i\varphi_j\varphi_k$ as long as they respect gauge invariance.

Writing down the general form of all terms with mass dimension equal to or larger than one, taking into account the gauge symmetry of the MSSM we get

$$
\begin{aligned}
\mathcal{L}_{\text{soft}} \;=\; & -\frac{1}{2}\Big(M_1\,\widetilde{B}\widetilde{B} + M_2\,\widetilde{W}\widetilde{W} + M_3\,\tilde{g}\tilde{g}\Big) + \text{h.c.} \\
& -\; m_{H_u}^2\,h_u^\dagger h_u - m_{H_d}^2\,h_d^\dagger h_d - (b\,h_u\,h_d + \text{h.c.}) \\
& -\; \Big(a_u\tilde{u}_R\,\tilde{q}\,h_u - a_d\,\tilde{d}_R\,\tilde{q}\,h_d - a_e\,\tilde{e}_R\,\tilde{l}\,h_u\Big) + \text{h.c.} \\
& -\; m_Q^2\,\tilde{q}^\dagger\,\tilde{q} - m_L^2\,\tilde{l}^\dagger\,\tilde{l} - m_u^2\,\tilde{u}_R^\dagger\,\tilde{u}_R - m_d^2\,\tilde{d}_R^\dagger\,\tilde{d}_R - m_e^2\,\tilde{e}_R^\dagger\,\tilde{e}_R \qquad (6.12)
\end{aligned}
$$

where a_i and m_i^2 are 3×3 matrices in family space. Thus, the term $m_u^2\,\tilde{u}_R^\dagger\,\tilde{u}_R$ for example stands for $(m_u^2)_{f_i f_j}\,(\tilde{u}_R^\dagger)^{f_i}\,(\tilde{u}_R)^{f_j}$. The doublets with only the scalar component fields are denoted by $h_u = (h_u^+, h_u^0)$ and $h_d = (h_d^0, h_d^-)$ for the Higgs bosons and $\tilde{q} = (\tilde{u}_L, \tilde{d}_L)$ and $\tilde{l} = (\tilde{\nu}, \tilde{e}_L)$ for the matter multiplets. They are combined into $SU(2)$ gauge invariant expressions as in Eq. (5.45).

Note that there are no terms $m_{ij}\,\psi_i\psi_j + \text{h.c.}$ and/or $c_{ijk}\,\varphi_i^\dagger\,\varphi_j\varphi_k + \text{h.c.}$ These two terms are related since e.g. the first can be written as a standard superpotential term plus $\varphi^\dagger\varphi$ and $\varphi^\dagger\varphi\varphi + \text{h.c.}$ terms. According to our simple minded analysis they (or at least one of them) should be present, since $[m_{ij}] = [c_{ijk}] = 1 > 0$. However, a more careful analysis [16] shows that these terms could lead to quadratic divergences. It is remarkable that the soft terms can also be explained as arising through a low energy effective theory of spontaneously broken supergravity.

The terms in $\mathcal{L}_{\text{soft}}$ give rise to additional interactions, not listed in Eqs. (5.37)–(5.44). The scalar mass terms are not new as they have the same form as the first interaction in Eq. (5.44) and the others are given by

$$
-\frac{1}{2}\Big(M_1\,\widetilde{B}\widetilde{B} + M_2\,\widetilde{W}\widetilde{W} + M_3\,\tilde{g}\tilde{g}\Big) + \text{h.c.} \quad \rightarrow \qquad \text{<image: wavy line with X vertex>} \qquad (6.13)
$$

$$
-(b\,h_u\,h_d + \text{h.c.}) \quad \rightarrow \qquad \text{<image: dashed line with X vertex>} \qquad (6.14)
$$

$$
-\Big(a_u\tilde{u}_R\,\tilde{q}\,h_u - a_d\,\tilde{d}_R\,\tilde{q}\,h_d - a_e\,\tilde{e}_R\,\tilde{l}\,h_u\Big) + \text{h.c.} \quad \rightarrow \qquad \text{<image: dashed vertex diagram>} \qquad (6.15)
$$

We have not depicted the hermitian conjugate of the various terms.

The cancellation of quadratic divergences discussed above is simply a special case of so-called *non-renormalization theorems*. In fact, in a perfectly supersymmetric theory there are even stronger cancellations. For example, it can be shown that the parameters of the superpotential do not receive any quantum corrections at all. Thus if a particle is massless at tree level, there will be no mass generated at any order in perturbation theory, reminiscent of the situation of gauge boson masses in gauge theories. Or if

we choose a (small) value of μ in the superpotential term $\mu\,H_u\,H_d$, susy protects this value to all orders. All there is is wave-function renormalization of χSF and VSF, or alternatively gauge coupling renormalization. In these renormalization factors there are only logarithmic singularities. There is a similar theorem related to the Fayet-Iliopoulos term, Eq. (5.21). If we set $k = 0$ such a term will not be generated at higher orders if we make the additional requirement that the trace over the charges (associated with the $U(1)$ under consideration) vanishes. As we have seen in Section 6.2 this can have important implications for the breaking of susy.

6.4 Towards the bigger picture

The Lagrangian of the MSSM is given by Eq. (5.33), adapted to the gauge group $SU(3)\times SU(2)\times U(1)$ with the superpotential given in Eq. (5.49) and supplemented by the soft breaking terms Eq. (6.12). As the name suggests, in the unconstrained MSSM, there are no constraints put on the soft breaking terms. This introduces a large number of parameters. At the same time, for arbitrary values of the parameters in Eq. (6.12) we run very quickly in conflict with experimental constraints. For example we get unacceptably large flavour-changing neutral currents or large CP-violating effects (see e.g. Ref [1]). If the terms in Eq. (6.12) follow a rather striking pattern in that they basically are proportional to the Standard Model values (i.e. the matrices m_i^2 are proportional to the identity matrix and a_i are proportional to the Yukawa coupling matrices) these dangerous terms are absent. This also reduces the number of parameters drastically. A full 3×3 matrix is replaced by a single parameter. We can go even further and assume that some of these parameters are actually the same at some high energy scale, leading to more and more constrained versions of the MSSM.

The ultimate goal would be to understand the theory behind susy breaking. In the top-down approach we start with a theory that is valid up to very large scales. Such a theory then predicts all soft-breaking parameters by considering the corresponding low-energy effective theory. In this context, low-energy means TeV energy scales. This would also have to include the gravitino, the susy partner of the spin 2 graviton. We have not mentioned this at all in this article because it leads to non-renormalizable theories and is beyond the scope of this article, but the fact that local susy is directly related to gravity is a strong hint that for a full understanding of susy breaking, gravity might play an important role.

Alternatively, in the bottom-up approach we try to determine as many parameters of the MSSM as precisely as possible, in the hope that this will provide sufficient information to hint towards the theory that is behind susy breaking. With the LHC about to start in earnest this approach gains momentum. But first of all, we have to find at least some of the susy partner particles. If you still hear the sentence "susy is just around the corner" in a few years from now, you most probably wasted your time reading this article.

Acknowledgement

It is a pleasure to thank Alan Martin for suggesting to write this article and for his efforts and comments to make sure it was (hopefully) kept on an accessible level.

A   Notation and Conventions

Minkowski indices are denoted by $\mu, \nu, \kappa, \rho \ldots$; $(\mu \in \{0, 1, 2, 3\})$, spinor indices by $\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta} \ldots$; $(\alpha \in \{1, 2\})$, two-component Grassmann spinors by $\theta, \zeta, \xi, \bar{\theta}, \bar{\zeta}, \bar{\xi}$ and Weyl spinors by ψ, χ, λ. The summation convention is always used.

metric: $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}$

Pauli matrices: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Dirac spinor: $\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$;   adjoint Dirac spinor: $\overline{\Psi} \equiv \Psi^+ \gamma_0 = \begin{pmatrix} \chi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix}$

Grassmann spinor: $\theta^\alpha = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}$; $\bar{\theta}^{\dot{\alpha}} = \begin{pmatrix} \bar{\theta}^1 \\ \bar{\theta}^2 \end{pmatrix}$;

Antisymmetric ϵ-tensor: $\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} \equiv 1$; $\epsilon^{11} = \epsilon^{22} = \epsilon_{11} = \epsilon_{22} = 0$;

where θ^1 etc. are Grassmann variables, i.e. anticommuting c-numbers. The bar on $\bar{\psi}_{\dot{\alpha}}$ as well as the dotted index denote hermitian conjugation, i.e. $\bar{\psi}_{\dot{\alpha}} = [\psi_\alpha]^\dagger$ and $\chi^\alpha = [\bar{\chi}^{\dot{\alpha}}]^\dagger$.

The two component Weyl spinors ψ_α (left-handed) and $\bar{\psi}^{\dot{\alpha}}$ (right-handed) transform under Lorentz transformations as follows:

$$
\begin{aligned}
\psi'_\alpha &= M_\alpha{}^\beta \psi_\beta; & \bar{\psi}'_{\dot{\alpha}} &= (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} \\
\psi'^\alpha &= \psi^\beta (M^{-1})_\beta{}^\alpha; & \bar{\psi}'^{\dot{\alpha}} &= \bar{\psi}^{\dot{\beta}} (M^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}
\end{aligned} \tag{A.1}
$$

where $M = \exp(i\frac{\vec{\sigma}}{2}(\vec{\vartheta} - i\vec{\varphi}))$ and $\vec{\vartheta}$ and $\vec{\varphi}$ are the three rotation angles and boost parameters respectively. The indices can be raised/lowered through the totally antisymmetric ϵ-tensor. This holds for the two-component spinors as well as for any Grassmann variables:

$$
\begin{aligned}
\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta; &\quad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta; \quad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}; \quad \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}; \\
\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta; &\quad \theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta; \quad \bar{\theta}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}; \quad \bar{\theta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\beta}};
\end{aligned} \tag{A.2}
$$

we also have

$$
\begin{aligned}
\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} &= \delta^\alpha{}_\beta; & \epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} &= \delta_\alpha{}^\beta; \\
\epsilon^{\alpha\gamma}\epsilon_{\beta\delta} &= \delta^\alpha_\delta \delta^\gamma_\beta - \delta^\alpha_\beta \delta^\gamma_\delta
\end{aligned} \tag{A.3}
$$

The product of two Grassmann spinors is defined through

$$
\begin{aligned}
\theta\zeta &\equiv \theta^\alpha\zeta_\alpha = \theta^\alpha\epsilon_{\alpha\beta}\zeta^\beta = -\zeta^\beta\epsilon_{\alpha\beta}\theta^\alpha = \zeta^\beta\epsilon_{\beta\alpha}\theta^\alpha = \zeta^\beta\theta_\beta = \zeta\theta \\
\bar{\theta}\bar{\zeta} &\equiv \bar{\theta}_{\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}} = -\bar{\zeta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}} = \bar{\zeta}\bar{\theta}
\end{aligned} \tag{A.4}
$$

In particular, using $\theta_1 = -\theta^2, \theta_2 = \theta^1$ we have

$$\theta\zeta = -\theta^1\zeta^2 + \theta^2\zeta^1 = \theta_2\zeta_1 - \theta_1\zeta_2$$
$$\bar{\theta}\bar{\zeta} = +\bar{\theta}_1\bar{\zeta}_2 - \bar{\theta}_2\bar{\zeta}_1 = -\bar{\theta}^2\bar{\zeta}^1 + \bar{\theta}^1\bar{\zeta}^2 \tag{A.5}$$

Using the Pauli matrices, we define

$$(\sigma^\mu)_{\alpha\dot{\alpha}} \equiv \{1, \sigma^1, \sigma^2, \sigma^3\}_{\alpha\dot{\alpha}}; \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \equiv \{1, -\sigma^1, -\sigma^2, -\sigma^3\}^{\dot{\alpha}\beta}; \tag{A.6}$$

Note that σ^μ has lower undotted-dotted indices, whereas $\bar{\sigma}^\mu$ has upper dotted-undotted indices. These two set of matrices are also related by

$$(\sigma^\mu)_{\alpha\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\alpha}}\epsilon_{\alpha\beta}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}; \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = \epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\alpha\dot{\beta}}; \tag{A.7}$$

The bar on σ is a well established but maybe somewhat misleading notation. We have $(\theta\sigma^\mu\bar{\zeta})^\dagger = \zeta\sigma^\mu\bar{\theta} \neq \zeta\bar{\sigma}^\mu\theta$ and $\theta\sigma^\mu\bar{\zeta} = -\zeta\bar{\sigma}^\mu\bar{\theta}$.

The γ-matrices are defined as

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}; \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{A.8}$$

and have the usual commutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ which is a simple consequence of the identity

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\alpha} = \text{Tr}(\sigma^\mu\bar{\sigma}^\nu) = 2g^{\mu\nu} \tag{A.9}$$

We also need

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta; \qquad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}; \tag{A.10}$$

$$(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha^\beta; \qquad (\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}}; \tag{A.11}$$

$$\text{Tr}\,(\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\kappa) = 2\,(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} + i\,\epsilon^{\mu\nu\rho\kappa}) \tag{A.12}$$

$$\text{Tr}\,(\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho\sigma^\kappa) = 2\,(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} - i\,\epsilon^{\mu\nu\rho\kappa}) \tag{A.13}$$

where we have $\epsilon^{0123} = +1$. Using

$$\Phi = \begin{pmatrix} \lambda_\alpha \\ \bar{\phi}^{\dot{\alpha}} \end{pmatrix}; \quad \overline{\Psi} = \begin{pmatrix} \chi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix}$$

the Lorentz covariant expressions can be written in two-component notation as follows:

$$\begin{aligned}
\overline{\Psi}\Phi &= \chi\lambda + \bar{\psi}\bar{\phi} = \chi^\alpha\lambda_\alpha + \bar{\psi}_{\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} \\
\overline{\Psi}\gamma^5\Phi &= -\chi\lambda + \bar{\psi}\bar{\phi} = -\chi^\alpha\lambda_\alpha + \bar{\psi}_{\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} \\
\overline{\Psi}\gamma^\mu\Phi &= \chi\sigma^\mu\bar{\phi} - \lambda\sigma^\mu\bar{\psi} = \chi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} - \lambda^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} \\
\overline{\Psi}\gamma^\mu\gamma^5\Phi &= \chi\sigma^\mu\bar{\phi} + \lambda\sigma^\mu\bar{\psi} = \chi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} + \lambda^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} \\
\overline{\Psi}\gamma^\mu\gamma^\nu\Phi &= \chi\sigma^\mu\bar{\sigma}^\nu\lambda + \bar{\psi}\bar{\sigma}^\mu\sigma^\nu\bar{\phi} = \chi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma})^{\dot{\alpha}\beta}\lambda_\beta + \bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(\sigma^\nu)_{\beta\dot{\beta}}\bar{\phi}^{\dot{\beta}}
\end{aligned} \tag{A.14}$$

With the help of

$$\theta^\alpha \theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}(\theta\theta)\,; \qquad \bar{\theta}^{\dot\alpha}\bar{\theta}^{\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}(\bar\theta\bar\theta)\,; \tag{A.15}$$

we can derive the following frequently used identities:

$$\begin{aligned}
(\theta\sigma^\mu\bar\theta)\,\theta^\alpha\sigma^\nu_{\alpha\dot\alpha} &= \frac{1}{2}(\theta\theta)\,\bar\theta_{\dot\beta}(\bar\sigma^\mu\sigma^\nu)^{\dot\beta}{}_{\dot\alpha} \\[4pt]
(\theta\sigma^\mu\bar\theta)\,(\sigma^\nu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} &= \frac{1}{2}(\bar\theta\bar\theta)\,(\sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta\theta_\beta \\[4pt]
(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) &= \frac{1}{2}g^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta) \\[4pt]
(\theta\zeta)(\theta\xi) &= -\frac{1}{2}(\theta\theta)(\zeta\xi) \\[4pt]
(\bar\theta\bar\zeta)(\bar\theta\bar\xi) &= -\frac{1}{2}(\bar\theta\bar\theta)(\bar\zeta\bar\xi)
\end{aligned} \tag{A.16}$$

The last two identities are known as Fierz rearrangement formulæ.

The derivatives with respect to a Grassmann variable are defined as follows:

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}\,; \quad \partial^\alpha \equiv \epsilon^{\alpha\beta}\partial_\beta\,; \quad \bar\partial_{\dot\alpha} \equiv \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\,; \quad \bar\partial^{\dot\alpha} \equiv \epsilon^{\dot\alpha\dot\beta}\bar\partial_{\dot\beta}\,; \tag{A.17}$$

Note that the rules for raising/lowering indices imply that

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta\,; \quad \partial^\alpha\theta_\beta = -\delta_\beta^\alpha\,; \quad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}\,; \quad \bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = -\delta_{\dot\beta}^{\dot\alpha}\,; \tag{A.18}$$

$$\partial^\alpha\theta^\beta = \epsilon^{\alpha\beta}\,; \quad \partial_\alpha\theta_\beta = -\epsilon_{\alpha\beta}\,; \quad \bar\partial^{\dot\alpha}\bar\theta^{\dot\beta} = \epsilon^{\dot\alpha\dot\beta}\,; \quad \bar\partial_{\dot\alpha}\bar\theta_{\dot\beta} = -\epsilon_{\dot\alpha\dot\beta}\,; \tag{A.19}$$

Furthermore, the derivatives also anticommute with other Grassmann variables. For example

$$\partial_\alpha(\theta\theta) = (\partial_\alpha\theta^\beta)\theta_\beta - \theta^\beta(\partial_\alpha\theta_\beta) = \delta_\alpha{}^\beta\theta_\beta - \theta^\beta(-\epsilon_{\alpha\beta}) = \theta_\alpha + \epsilon_{\alpha\beta}\theta^\beta = 2\theta_\alpha \tag{A.20}$$

and similarly

$$\partial^\alpha(\theta\theta) = 2\theta^\alpha\,; \quad \bar\partial_{\dot\alpha}(\bar\theta\bar\theta) = -2\bar\theta_{\dot\alpha}\,; \quad \bar\partial^{\dot\alpha}(\bar\theta\bar\theta) = -2\bar\theta^{\dot\alpha}\,; \tag{A.21}$$

The minus signs in Eqs. (A.18), (A.19) and (A.21) seem strange at first, however they are required if we insist on raising and lowering spinorial indices with the ϵ-tensor. As a consequence, the Taylor expansion in Grassmann variables also have some unexpected minus signs. For infinitesimal ζ and $\bar\zeta$ we have

$$\begin{aligned}
\phi(\theta+\zeta) &= \phi(\theta) + \zeta\partial\,\phi(\theta) + \mathcal{O}(\zeta\zeta) = \phi(\theta) + \partial\zeta\,\phi(\theta) + \mathcal{O}(\zeta\zeta) \tag{A.22} \\
\phi(\bar\theta+\bar\zeta) &= \phi(\bar\theta) - \bar\zeta\bar\partial\,\phi(\bar\theta) + \mathcal{O}(\bar\zeta\bar\zeta) = \phi(\bar\theta) - \bar\partial\bar\zeta\,\phi(\bar\theta) + \mathcal{O}(\bar\zeta\bar\zeta) \tag{A.23}
\end{aligned}$$

where ϕ is an arbitrary function. Note that Taylor expansions in Grassmann spinors terminate after the second term, since expressions like $\zeta^\alpha\zeta\zeta = 0$.

Integration with respect to Grassmann variables is defined through

$$\int \mathrm{d}\theta^1\theta^1 = -\int \theta^1\mathrm{d}\theta^1 = 1\,; \qquad \int \mathrm{d}\theta^1 = 0\,; \tag{A.24}$$

38

thus, it is really the same as differentiation. We define

$$\mathrm{d}^2\theta \;\;\equiv\;\; -\frac{1}{4}\epsilon_{\alpha\beta}\,\mathrm{d}\theta^\alpha\mathrm{d}\theta^\beta \tag{A.25}$$

$$\mathrm{d}^2\bar\theta \;\;\equiv\;\; -\frac{1}{4}\epsilon_{\dot\alpha\dot\beta}\,\mathrm{d}\bar\theta^{\dot\alpha}\mathrm{d}\bar\theta^{\dot\beta} \tag{A.26}$$

This has been arranged such that Eqs. (3.13) and (3.14) hold.

Finally, our convention for the generators and covariant derivatives are given in Eq. (4.9), (4.10) and (4.12). Thanks to our conventions for the derivatives with respect to Grassmann variables, Eq. (A.17), they satisfy $Q^\alpha = \epsilon^{\alpha\beta}Q_\beta$ etc. and, in particular, $\theta Q = Q\theta$. They have the following anticommutation relations:

$$
\begin{aligned}
&\{Q_\alpha, D_\beta\} = \{\bar Q_{\dot\alpha}, \bar D_{\dot\beta}\} = \{Q_\alpha, \bar D_{\dot\beta}\} = 0\\
&\{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0\\
&\{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0\\
&\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu = 2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu\\
&\{D_\alpha, \bar D_{\dot\alpha}\} = -2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu = -2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu
\end{aligned}
\tag{A.27}
$$

B Sample Calculations

In this Appendix we present the details of some of the calculations referred to in the main text.

B.1 Anitcommutation relation

As an example for an anticommutation relation we consider $\{D_\alpha, \bar D_{\dot\alpha}\}$. Using the definitions Eq. (4.12) we would expect to get four terms. However the expression reduces immediately to two terms since $\{\partial_\alpha, \bar\partial_{\dot\alpha}\} = \{\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}, \theta^\beta\sigma^\mu_{\beta\dot\alpha}\} = 0$, due to the Grassmann nature of ∂_α and θ^β etc. Thus

$$
\begin{aligned}
\{D_\alpha, \bar D_{\dot\alpha}\} &= \{\partial_\alpha, -i\,\theta^\beta\sigma^\mu_{\beta\dot\alpha}\,\partial_\mu\} + \{-i\,\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\,\partial_\mu, \bar\partial_{\dot\alpha}\}\\
&= -i\,\sigma^\mu_{\beta\dot\alpha}\,\partial_\mu\{\partial_\alpha, \theta^\beta\} - i\,\sigma^\mu_{\alpha\dot\beta}\,\partial_\mu\{\bar\partial_{\dot\alpha}, \bar\theta^{\dot\beta}\}\\
&= -i\,\sigma^\mu_{\beta\dot\alpha}\,\partial_\mu\,\delta^\beta_\alpha - i\,\sigma^\mu_{\alpha\dot\beta}\,\partial_\mu\,\delta^{\dot\beta}_{\dot\alpha}\\
&= -2\,i\,\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu \tag{B.1}
\end{aligned}
$$

where we used $\{\partial_\alpha, \theta^\beta\} = \partial_\alpha\theta^\beta + \theta^\beta\partial_\alpha = [\partial_\alpha\theta^\beta] - \theta^\beta\partial_\alpha + \theta^\beta\partial_\alpha = \delta^\beta_\alpha$. Here it is understood that in the term $[\partial_\alpha\theta^\beta]$ the derivative acts only within the brackets, but in all other terms it acts to everything on the right. The minus sign in the product rule for differentiation is as e.g. in Eq. (A.20) due to the Grassmann nature of θ^β and ∂_α.

In a similar way we immediately get relations like $\{\partial_\alpha, \bar\theta^{\dot\beta}\} = 0$ which have to be used to verify the remaining relations listed in Eq. (A.27).

B.2   Gauge transformation of U_α

Let us verify Eq. (5.26). Using the definition of U_α, Eq. (5.24), and the gauge transformation of the VSF, Eq. (5.13), we get

$$
\begin{aligned}
U_\alpha \;\longmapsto\; & -\frac{1}{8\,g}\bar{D}\bar{D}\,e^{-2i\,g\,\Lambda}e^{-2g\,V}e^{2i\,g\,\Lambda^\dagger}D_\alpha\,e^{-2i\,g\,\Lambda^\dagger}e^{2g\,V}e^{2i\,g\,\Lambda} \\
= & -\frac{1}{8\,g}\bar{D}\bar{D}\,e^{-2i\,g\,\Lambda}e^{-2g\,V}D_\alpha\,e^{2g\,V}e^{2i\,g\,\Lambda} \\
= & -\frac{1}{8\,g}e^{-2i\,g\,\Lambda}\bar{D}\bar{D}\,e^{-2g\,V}\left(\left[D_\alpha\,e^{2g\,V}\right]+e^{2g\,V}D_\alpha\right)e^{2i\,g\,\Lambda} \\
= & \;\;e^{-2i\,g\,\Lambda}\,U_\alpha\,e^{2i\,g\,\Lambda}-\frac{1}{8\,g}e^{-2i\,g\,\Lambda}\bar{D}\bar{D}\,D_\alpha\,e^{2i\,g\,\Lambda} \quad\quad\text{(B.2)}
\end{aligned}
$$

In the first step we used $D_\alpha\Lambda^\dagger = 0$ since $\Lambda^\dagger$ is a RHχSF. In the second step we applied the product rule for D_α and it is understood that in $\left[D_\alpha\,e^{2g\,V}\right]$ the derivative acts only within the brackets. We also used $\bar{D}_{\dot\alpha}\Lambda = 0$. In the last step we used the definition of U_α. What remains to be done is to show that the second term in the last line of Eq. (B.2) vanishes. To do this we use Eq. (A.27) to anticommute $\bar{D}$ through to act on $e^{2i\,g\,\Lambda}$.

$$
\begin{aligned}
\bar{D}\bar{D}\,D_\alpha\,e^{2i\,g\,\Lambda}=-\bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha}\,D_\alpha\,e^{2i\,g\,\Lambda} \;=\; & -\bar{D}^{\dot\alpha}\{\bar{D}_{\dot\alpha},D_\alpha\}\,e^{2i\,g\,\Lambda}+\bar{D}^{\dot\alpha}\,D_\alpha\bar{D}_{\dot\alpha}\,e^{2i\,g\,\Lambda} \\
= & \;\;2(\sigma^\mu)_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}P_\mu\,e^{2i\,g\,\Lambda}+0 \\
= & \;\;2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu\bar{D}^{\dot\alpha}\,e^{2i\,g\,\Lambda} \\
= & \;\;0 \quad\quad\text{(B.3)}
\end{aligned}
$$

In the second last step we used $[\bar{D}^{\dot\alpha},P_\mu]=0$. This completes the proof of Eq. (5.26). Note that in the abelian case U_α commutes with Λ (they do not contain non-commuting generators of a non-abelian gauge theory) and thus we obtain the result that U_α is gauge invariant.

Of course, the corresponding equations for $\bar{U}_{\dot\alpha}$ can be obtained in a completely analogous way.

B.3 U_α in terms of component fields

This calculation is somewhat tedious and can be tackled in several ways. One option is to first make the computation in the abelian case and verify Eq. (5.18). Then Eq. (5.25) can be used to identify the additional terms required in the non-abelian case. Here, we perform the calculation directly for the non-abelian case and verify Eq. (5.28). We will repeatedly use the fact that $\theta_\alpha\,\theta\theta=\bar{\theta}_{\dot\alpha}\,\bar{\theta}\bar{\theta}=0$ and the identities listed in Eqs. (A.15) and (A.16).

We first write $e^V=1+V+V^2/2$ with $V=V^aT^a$ and have to keep in mind that we have to rescale all component fields at the end by a factor $2\,g$ to obtain $e^{2g\,V}$. Using

Eq. (5.16) we find

$$
\begin{aligned}
e^V &= 1 + \theta\sigma^\mu\bar\theta\, v_\mu(x) + i\,\theta\theta\,\bar\theta\bar\lambda - i\,\bar\theta\bar\theta\,\theta\lambda + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\, D + \frac{1}{2}\theta\sigma^\mu\bar\theta\, v_\mu\,\theta\sigma^\nu\bar\theta\, v_\nu \\
&= 1 + \theta\sigma^\mu\bar\theta\, v_\mu(\bar y) + i\,\theta\theta\,\bar\theta\bar\lambda - i\,\bar\theta\bar\theta\,\theta\lambda + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\left(D + \frac{1}{2}v^\mu v_\mu - i\,\partial^\mu v_\mu\right) \quad \text{(B.4)}
\end{aligned}
$$

where we used $x^\mu = \bar y^\mu - i\,\theta\sigma^\mu\bar\theta$ and in the last line of Eq. (B.4) all component fields are functions of $\bar y^\mu$. The advantage of this is that $D_\alpha\,\bar y^\mu = 0$, thus the covariant derivative acts only as ∂_α on the explicit θ appearing in Eq. (B.4). The change from x^μ to $\bar y^\mu$ matters only in the one term where the argument is explicitly indicated. The colour matrices are understood, i.e. $\lambda = \lambda^a T^a$ etc. and $v^\mu v_\mu = T^a T^b\,(v^a)^\mu v_\mu^b$. Acting with D_α we get

$$
\begin{aligned}
D_\alpha\, e^V &= (\sigma^\mu\bar\theta)_\alpha\, v_\mu(\bar y) + 2\,i\theta_\alpha\,\bar\theta\bar\lambda(\bar y) - i\bar\theta\bar\theta\,\lambda_\alpha + \theta_\alpha\,\bar\theta\bar\theta\left(D + \frac{1}{2}v^\mu v_\mu - i\,\partial^\mu v_\mu\right) \\
&= (\sigma^\mu\bar\theta)_\alpha\, v_\mu(y) + i\,\bar\theta\bar\theta(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\,\partial_\nu v_\mu + 2\,i\theta_\alpha\,\bar\theta\bar\lambda(y) - (\sigma^\nu\partial_\nu\bar\lambda)_\alpha\,\theta\theta\,\bar\theta\bar\theta \\
&\quad - i\,\bar\theta\bar\theta\,\lambda_\alpha + \theta_\alpha\,\bar\theta\bar\theta\left(D + \frac{1}{2}v^\mu v_\mu - i\,\partial^\mu v_\mu\right) \quad \text{(B.5)}
\end{aligned}
$$

where in the last step we have changed coordinates once more, this time from $\bar y^\mu$ to $y^\mu = \bar y^\mu - 2i\,\theta\sigma^\mu\bar\theta$ in order to be able to exploit $\bar D_{\dot\alpha}\, y^\mu = 0$ in what follows. This affects only the terms where the dependence on $\bar y^\mu$ or y^μ is explicitly given. Before we can act with $\bar D\bar D$ we have to multiply Eq. (B.5) by $e^{-V} = 1 - V + \ldots$. The omitted terms have no effect. Also, we can directly use y^μ since the change from x^μ to y^μ again results only in vanishing terms. Thus, restoring the colour labels and colour matrices, we obtain

$$
\begin{aligned}
e^{-V^a T^a} D_\alpha\, e^{V^b T^b} &= T^a\bigg((\sigma^\mu\bar\theta)_\alpha\, v_\mu^a + i\,\bar\theta\bar\theta(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\,\partial_\nu v_\mu^a + 2\,i\theta_\alpha\,\bar\theta\bar\lambda^a \quad\text{(B.6)} \\
&\qquad - (\sigma^\nu\partial_\nu\bar\lambda^a)_\alpha\,\theta\theta\,\bar\theta\bar\theta - i\,\bar\theta\bar\theta\,\lambda_\alpha^a + \theta_\alpha\,\bar\theta\bar\theta\,(D^a - i\,\partial^\mu v_\mu^a)\bigg) \\
&\quad + \frac{1}{2}T^a T^b\bigg(\theta_\alpha\,\bar\theta\bar\theta\,(v^a)^\mu v_\mu^b - \bar\theta\bar\theta\,(\sigma^\nu\bar\sigma^\mu\theta)_\alpha\, v_\mu^a v_\nu^b - i\,\theta\theta\,\bar\theta\bar\theta(\sigma^\mu\bar\lambda^b)_\alpha\, v_\mu^a + i\,\theta\theta\,\bar\theta\bar\theta(\sigma^\mu\bar\lambda^a)_\alpha\, v_\mu^b\bigg)
\end{aligned}
$$

The last two terms actually form a commutator $[T^a, T^b] = if^{abc}T^c$ if we reshuffle colour indices $a \leftrightarrow b$ in one of the terms. This will give rise to the non-abelian covariant derivative Eq. (5.30). If we use Eq. (A.11) in the form

$$
\begin{aligned}
\theta_\alpha\,(v^a)^\mu v_\mu^b - (\sigma^\nu\bar\sigma^\mu\theta)_\alpha\, v_\mu^a v_\nu^b &= \theta_\beta\, v_\mu^a v_\nu^b\left(\delta_\alpha^\beta\, g^{\mu\nu} - (\sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta\right) \\
&= \theta_\beta\, v_\mu^a v_\nu^b\,\frac{1}{2}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)_\alpha{}^\beta \quad \text{(B.7)}
\end{aligned}
$$

we can also write the other two terms proportional to $T^a T^b$ as a commutator and we start to recover the non-abelian field-strength tensor Eq. (5.29). The same game has to be played to combine the terms $i\,\bar\theta\bar\theta(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\,\partial_\nu v_\mu^a$ and $-i\,\theta_\alpha\,\bar\theta\bar\theta\,\partial^\mu v_\mu^a$, proportional to T^a. After this, we simply act with $\bar D\bar D$. Since all component fields are functions of y^μ, this

implies to only act with $\bar\partial\bar\partial$ on the explicit terms with $\bar\theta$. Thus terms not containing $\theta\bar\theta$ will vanish and using $\bar\partial\bar\partial\,\bar\theta\bar\theta = -4$ we get

$$\bar D\bar D\, e^{-V^a T^a} D_\alpha\, e^{V^b T^b} \;=\; 2i\,(\sigma^\mu\bar\sigma^\nu\theta)_\alpha \left(T^a\,\partial_\mu v^a_\nu - T^a\,\partial_\nu v^a_\mu + \frac{i}{2}\,[T^a, T^b]v^a_\mu v^b_\nu \right) \tag{B.8}$$

$$+\; 4\,\theta\theta \left(T^a(\sigma^\mu\partial_\mu\bar\lambda^a)_\alpha + \frac{i}{2}\,[T^a, T^b](\sigma^\mu\bar\lambda^b)_\alpha\, v^a_\mu \right) + 4i\,T^a\lambda^a_\alpha - 4\,\theta_\alpha\, T^a D^a$$

Restoring all factors (including the rescaling factor $2g$ for each component field) we recover Eq. (5.28). The corresponding expression for $\bar U_{\dot\alpha}$ reads

$$\bar U^a_{\dot\alpha} = \frac{i}{2}\,\bar\theta_{\dot\beta}(\bar\sigma^\mu\sigma^\nu)^{\dot\beta}{}_{\dot\alpha}\,F^a_{\mu\nu} + \bar\theta\bar\theta\,(D_\mu\lambda^a)^\alpha\sigma^\mu_{\alpha\dot\alpha} - i\,\bar\lambda^a_{\dot\alpha} - \bar\theta_{\dot\alpha}\,D^a \tag{B.9}$$

To obtain the abelian expression we simply set $f^{abc} = 0$.

What remains to be done is to compute the Lorentz invariant product $U^a U^a \equiv U^a_\alpha (U^a)^\alpha = \epsilon^{\alpha\beta}\, U^a_\alpha U^a_\beta$ and take its supersymmetric $\theta\theta$ component. The only slightly tricky part is the $F^{\mu\nu} F_{\mu\nu}$ term. Using Eq. (A.3), results in two traces which are evaluated with the help of Eqs. (A.9), (A.12) and (A.13). We find

$$\big[U^a U^a\big]_{\theta\theta}$$
$$= \; -\frac{1}{8}\Big(\mathrm{Tr}(\sigma^\kappa\bar\sigma^\rho)\,\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) - \mathrm{Tr}(\sigma^\kappa\bar\sigma^\rho\sigma^\mu\bar\sigma^\nu)\Big)F^a_{\kappa\rho}\,F^a_{\mu\nu} + 2i\,(\lambda^a\sigma^\mu D_\mu\bar\lambda^a) + D^a D^a$$
$$= \; -\frac{1}{2}(F^a)^{\mu\nu}F^a_{\mu\nu} + \frac{i}{4}\,\epsilon^{\mu\nu\kappa\rho}F^a_{\kappa\rho}\,F^a_{\mu\nu} + 2i\,(\lambda^a\sigma^\mu D_\mu\bar\lambda^a) + D^a D^a \tag{B.10}$$

In a similar manner we get

$$\big[\bar U^a\bar U^a\big]_{\bar\theta\bar\theta} = -\frac{1}{2}(F^a)^{\mu\nu}F^a_{\mu\nu} - \frac{i}{4}\,\epsilon^{\mu\nu\kappa\rho}F^a_{\kappa\rho}\,F^a_{\mu\nu} - 2i\,(D_\mu\lambda^a\sigma^\mu\bar\lambda^a) + D^a D^a \tag{B.11}$$

Thus the terms containing $\epsilon^{\mu\nu\kappa\rho}F^a_{\kappa\rho}\,F^a_{\mu\nu}$ cancel in the sum of the two terms Eqs. (B.10) and (B.11) and adding all prefactors we obtain the non-abelian version of Eq. (5.19).

B.4 $\phi^\dagger\phi$

Let us verify Eq. (5.5) and compute $\phi^\dagger\phi\big|_{\theta\theta\,\bar\theta\bar\theta}$, taking Eqs. (4.15) and (4.17) as input. Neglecting terms that do not contain two θ and two $\bar\theta$ spinors we get

$$\phi^\dagger\phi \;=\; \big[\varphi^\dagger\big]\big[-\frac{1}{4}\,\theta\theta\,\bar\theta\bar\theta\,\partial^\mu\partial_\mu\varphi\big] + \big[\sqrt{2}\,\bar\theta\bar\psi\big]\big[\frac{i}{\sqrt{2}}\,\theta\theta\,(\partial_\mu\psi\sigma^\mu\bar\theta)\big]$$

$$+\; \big[i\,\theta\sigma^\mu\bar\theta\,(\partial_\mu\varphi)^\dagger\big]\big[-i\,\theta\sigma^\nu\bar\theta\,\partial_\nu\varphi\big] + \big[-\frac{i}{\sqrt{2}}\,\bar\theta\bar\theta\,(\theta\sigma^\mu\partial_\mu\bar\psi)\big]\big[\sqrt{2}\,\theta\psi\big]$$

$$+\; \big[-\frac{1}{4}\,\theta\theta\,\bar\theta\bar\theta\,(\partial^\mu\partial_\mu\varphi)^\dagger\big]\big[\varphi\big] + \big[-\bar\theta\bar\theta\,F^\dagger\big]\big[-\theta\theta\,F\big] \tag{B.12}$$

42

We now use the reshuffling identities listed in Eq. (A.16). In particular we make use of $\bar{\theta}\bar{\psi}(\partial_\mu\psi\sigma^\mu\bar{\theta}) = -\frac{1}{2}\bar{\theta}\bar{\theta}(\partial_\mu\psi\sigma^\mu\bar{\psi})$ and $\theta\psi(\theta\sigma^\mu\partial_\mu\bar{\psi}) = -\frac{1}{2}\theta\theta(\psi\sigma^\mu\partial_\mu\bar{\psi})$ and obtain

$$
\begin{aligned}
\phi^\dagger\,\phi \;=\; & \theta\theta\,\bar{\theta}\bar{\theta}\Big(-\frac{1}{4}\varphi^\dagger\,\partial^\mu\partial_\mu\varphi - \frac{i}{2}(\partial_\mu\psi\sigma^\mu\bar{\psi}) + \frac{1}{2}(\partial^\mu\varphi)^\dagger\partial_\mu\varphi \\
& + \frac{i}{2}(\psi\sigma^\mu\partial_\mu\bar{\psi}) - \frac{1}{4}(\partial^\mu\partial_\mu\varphi)^\dagger\,\varphi + F^\dagger F\Big)
\end{aligned}
\tag{B.13}
$$

Finally, using integration by parts $(\partial^\mu\partial_\mu\varphi)^\dagger\,\varphi = -(\partial^\mu\varphi)^\dagger\partial_\mu\varphi$ we obtain the result given in Eq. (5.5).

B.5 $\phi^\dagger\,e^{2gV}\,\phi$

In this section we compute the gauge interaction terms between matter fields and gauge fields in a non-abelian gauge theory, i.e. the third term in Eq. (5.32). We assume the χSF transform under a certain representation of the gauge group and T^a_{ij} are the generators in this representation. Expanding e^{2gV} in the Wess-Zumino gauge we get

$$
\left(e^{2g\,V^a T^a}\right)_{ij} = \delta_{ij} + 2g\,V^a T^a_{ij} + 2g^2\,V^a V^b\,T^a_{ik}T^b_{kj}
\tag{B.14}
$$

and higher terms vanish due to the presence of terms $\theta^\alpha\,\theta\theta = 0$ or $\bar{\theta}_{\dot{\alpha}}\,\bar{\theta}\bar{\theta} = 0$.

The insertion of the first term on the r.h.s. of Eq. (B.14) has already been computed in Section B.4. For the second term on the r.h.s. of Eq. (B.14) we use Eq. (5.16) as well as Eqs. (4.15) and (4.17). Ignoring terms that do not contain two θ and two $\bar{\theta}$ spinors we get

$$
\phi^\dagger_i\,2g\,V^a T^a_{ij}\,\phi_j =
\tag{B.15}
$$
$$
\begin{aligned}
& \left[\varphi^\dagger_i\right]\left[2g\,\theta\sigma^\mu\bar{\theta}\,v^a_\mu T^a_{ij}\right]\left[-i\,\theta\sigma^\nu\bar{\theta}\,\partial_\nu\varphi_j\right] + \left[\sqrt{2}\,\bar{\theta}\bar{\psi}_i\right]\left[2g\,\theta\sigma^\mu\bar{\theta}\,v^a_\mu T^a_{ij}\right]\left[\sqrt{2}\,\theta\psi_j\right] \\
+ & \left[i\,\theta\sigma^\nu\bar{\theta}\,(\partial_\nu\varphi_i)^\dagger\right]\left[2g\,\theta\sigma^\mu\bar{\theta}\,v^a_\mu T^a_{ij}\right]\left[\varphi_j\right] + \left[\varphi^\dagger_i\right]\left[-i\,2g\,\bar{\theta}\bar{\theta}\,\theta\lambda^a T^a_{ij}\right]\left[\sqrt{2}\,\theta\psi_j\right] \\
+ & \left[\sqrt{2}\,\bar{\theta}\bar{\psi}_i\right]\left[i\,2g\,\theta\theta\,\bar{\theta}\bar{\lambda}^a T^a_{ij}\right]\left[\varphi_j\right] + \left[\varphi^\dagger_i\right]\left[\frac{1}{2}\theta\theta\,\bar{\theta}\bar{\theta}\,D^a T^a_{ij}\right]\left[\varphi_j\right]
\end{aligned}
$$

Proceeding as in the derivation of Eq. (B.13) we get

$$
\begin{aligned}
\phi^\dagger_i\,2g\,V^a T^a_{ij}\,\phi_j\big|_{\theta\theta\,\bar{\theta}\bar{\theta}} = & \tag{B.16} \\
- & ig\,\varphi^\dagger_i\,v^a_\mu\,T^a_{ij}\,\partial^\mu\varphi_j + ig\,(\partial^\mu\varphi_i)^\dagger\,v^a_\mu\,T^a_{ij}\,\varphi_j + g\,\psi_j\sigma^\mu\bar{\psi}_i\,v^a_\mu\,T^a_{ij} \\
+ & ig\,\sqrt{2}\,\varphi^\dagger_i\,(\lambda^a\psi_j)\,T^a_{ij} - ig\,\sqrt{2}\,\varphi_j\,(\bar{\lambda}^a\bar{\psi}_i)\,T^a_{ij} + g\,\varphi^\dagger_i\,D^a\varphi_j\,T^a_{ij}
\end{aligned}
$$

Finally we compute the insertion of the third term on the r.h.s. of Eq. (B.14). We get only one term proportional to $\theta\theta\,\bar{\theta}\bar{\theta}$

$$
\begin{aligned}
\phi^\dagger_i\,2g^2\,V^a T^a_{ik}\,V^b T^b_{kj}\,\phi_j\big|_{\theta\theta\,\bar{\theta}\bar{\theta}} \;=\; & \left[\varphi^\dagger_i\right]\left[2g^2\,(\theta\sigma^\mu\bar{\theta})\,v^a_\mu T^a_{ik}\,(\theta\sigma^\nu\bar{\theta})\,v^b_\nu T^b_{kj}\right]\left[\varphi_j\right]\big|_{\theta\theta\,\bar{\theta}\bar{\theta}} \\
=\; & g^2\,\varphi^\dagger_i\,v^a_\mu(v^b)^\mu\,T^a_{ik}T^b_{kj}\,\varphi_j
\end{aligned}
\tag{B.17}
$$

which represents a four-point interaction between the scalars and the gauge bosons.

Combining Eqs. (B.13), (B.16) and (B.17) we see that the various terms combine into gauge invariant parts. The terms with $\partial_\mu \varphi$ and $\partial_\mu \psi_i$ combine to $(D_\mu \varphi_i)^\dagger D^\mu \varphi_i$ and $\frac{i}{2}(\psi_i \sigma^\mu (D_\mu \bar{\psi})_i - (D_\mu \psi)_i \sigma^\mu \bar{\psi}_i)$ respectively, where the (gauge) covariant derivatives are given in Eq. (5.36). This leaves us with the $F^\dagger F$ term of Eq. (B.13) (which is eliminated through its equation of motion as discussed in Section 5.1) and the interaction terms $ig\sqrt{2}\, T^a_{ij}(\varphi^\dagger_i \lambda^a \psi_j - \varphi_j \bar{\lambda}^a \bar{\psi}_i)$ of Eq. (B.16) which are explicitly written in Eq. (5.33).

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