Generalized Network Dismantling

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Finding the set of nodes, which removed or (de)activated can stop the spread of (dis)information, contain an epidemic or disrupt the functioning of a corrupt/criminal organization is still one of the key challenges in network science. In this paper, we introduce the generalized network dismantling problem, which aims to find the set of nodes that, when removed from a network, results in a network fragmentation into subcritical network components at minimum cost. For unit costs, our formulation becomes equivalent to the standard network dismantling problem. Our non-unit cost generalization allows for the inclusion of topological cost functions related to node centrality and non-topological features such as the price, protection level or even social value of a node. In order to solve this optimization problem, we propose a method, which is based on the spectral properties of a novel node-weighted Laplacian operator. The proposed method is applicable to large-scale networks with millions of nodes. It outperforms current state-of-the-art methods and opens new directions in understanding the vulnerability and robustness of complex systems.

complex systems | robustness | graph fragmentation

In a hyper-connected world, systemic instability, based on cascading effects, can seriously undermine the functionality of networks (1). The quick global spread of rumors and fake news may be seen as a recent example (2), while the spread of epidemics is a problem that has been around much longer (3–6). It is well-known that the network structure, for example the exponent characterizing scale-free networks, is of particular importance for the controllability of cascading effects (7). Note that, for certain scaling exponents of scale-free network, the variance or mean value of relevant quantities may not be well-defined, which means that unpredictable or uncontrollable behavior may result. For example, it might be impossible to contain epidemic spreading processes. Similar circumstances may also make it impossible to contain the spread of computer viruses or misinformation—a problem that is not only relevant for the quick increase of cyberthreats, but may also undermine the functionality of markets, societal or political institutions.

For over a decade, we are aware that complex systems with a scale-free network structure (8, 9) are more robust to the random attacks than random network structures (10, 11), but at the same time more vulnerable to targeted attacks (12–18). Finding the collective influential nodes or minimal set of nodes such that their removal leaves the network fragmented into components of subcritical size is called the network dismantling problem (19, 20). It belongs to the non-deterministic polynomial-time hard (NP-hard) class. Thus it is the primary reason why this is still one of the mayor challenges in modern network science. Only recently, novel approximations based on spin-glass and optimal percolation theory made new significant contributions (19–26). But still, all these methods (19–26) make the implicit assumption that the cost of removing a node is node-independent. Recently, the effect of costs depending on the node degree was studied (27, 28), but only in random network structures (27) or only with the edge-based strategy (28) that takes the degree cost into the account.

This paper addresses the question, how to select the set of nodes in a network, which when removed or (de)activated can stop the spread of (dis)information, or epidemic spreading or disrupt the functioning of a malicious systems by fragmentation into small components. The approach studied here is the cutting of an over-connected network into smaller pieces, if this is required in a state of emergency. For this purpose, we here discuss a generalized network dismantling problem. In the generalized network dismantling problem, the cost of removing a node can be an arbitrary non-negative real number, which can for example be specified as a function of node centrality properties such as degree, page–rank, betweenness or an external information not captured by network topology. The prime examples of external information associated with the cost of removing a node include: the monetary price for buying or controlling a node in a financial or economic system, or the security protection level in a technological systems.

From a computational perspective, this problem belongs to the NP-hard class and, thus, it is important that we propose a good approximation strategy. For this, we re-formulate the Laplacian spectral partitioning (29, 30) formulation, which dates back to 1970s and was primarily used for edge removal strategies on networks. We then construct a special node–weighted Laplacian operator, which serves to determine the upper bound of the cost of generalized network dismantling problem. Furthermore, we propose an elegant and efficient approximation algorithm for the relaxed formulation of the problem, which is applicable for large-scale networks. Finally, a fine-tunning mechanism for the spectral approximation is introduced, which is based on mapping the spectral solution to the weighted vertex cover problem (31) from graph theory. Overall, the generalized network dismantling formulation opens up new directions in network science, which will enable to increase the level of robustness of real-world systems by first understanding the key relationships between dismantling solutions and their cost(s).

Contributions

The main contributions of this paper are listed below:

(i) We introduce a generalized network dismantling problem, which seeks to find the set of nodes that, when removed from a network, results in a network fragmentation into components of subcritical size at minimum cost. Contrary to the previous formulations (19–26) assuming identical costs for the removal of each node, we allow the costs to take arbitrary non-negative real values.

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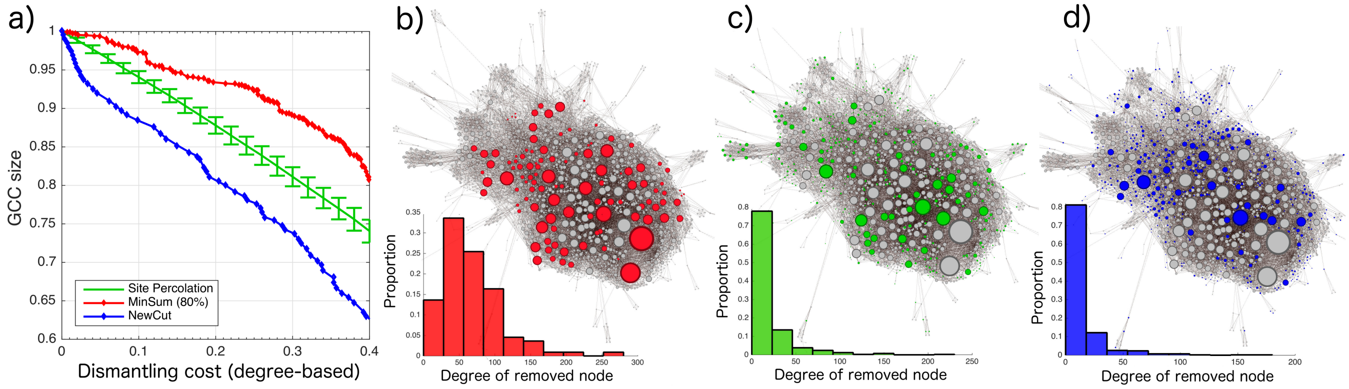


Fig. 1. **(a)** Network dismantling measured by the size of the giant connected component (GCC) with respect to the degree-based definition of the cost for three different strategies: state-of-the-art MinSum (22), random removal (32) (site percolation), our NewCut strategy. Dismantling represents the controlled process of suppressing the spread of disinformation, computer viruses or other harmful contagion effects on the Online social network (Petster-hamster (33)). The MinSum algorithm was set to dismantle the network up to the fixed target size of 80 percent of the network GCC size. The cost of removing a node is proportional to the current degree of a node and the cost of the dismantling is measured with the fraction of removed edges adjacent to the removed nodes. We observe that, for the same dismantling cost 0.4, the MinSum algorithm produces results (red color) which are 5 % worse than the naive random removal (green color) of nodes in a network. However, for the same cost the proposed NewCut strategy (blue color) fragments the network up to 62 % GCC size, which is for 18 % better than the MinSum strategy. **(b,c,d)** Online social network (Petster-hamster (33)) displaying the set of removed nodes according to the MinSum (22) (red color), random removal (green color) and our NewCut strategy (blue color). Although the MinSum strategy removes a small percentage of nodes, the cost is rather high as it targets high degree nodes, which is visible from the histogram of the removed nodes. In contrast, the NewCut strategy avoids the expensive removal of hubs in this scenario and produced much better fragmentation.

(ii) We formulate a novel node–weighted graph cut objective function, which determines the upper bound for the generalized network dismantling cost. We find the analytical solution for the relaxed objective function, which is related to the spectral properties of a novel node-weighted Laplacian matrix.

(iii) For dismantling large-scale networks, we propose an efficient spectral approximation by constructing a Power Laplacian operator, which runs in $O(n \cdot \log^{2+\epsilon}(n))$ complexity. Furthermore, we provide analytical bounds and convergence proofs for the spectral approximation. Finally, we propose a fine-tuning mechanism by mapping it to the weighted vertex cover problem and using the 2-approximation algorithm.

(iv) Finally, we show that on real-world network systems, our approach outperforms current state-of-the-art methods (19, 20, 22, 24, 25) in non-unit and unit cost scenarios. These results are practically relevant for the robustness of real-world systems under dismantling with different cost functions.

Generalized network dismantling problem

Let us define a network $G(V, E)$ as the set of nodes V which are connected via a set of edges E. The network dismantling problem aims to find the minimal subset of nodes $S \subseteq V$ that, when removed from the network, would result in a fragmentation into components of subcritical size. More formally, a set S is called a C-dismantling set, if the largest connected component of a network contains at most C nodes (19, 34). Finding the minimum number of removed nodes i.e. the smallest set S, is a non-deterministic polynomial-time hard (NP-hard) computational problem. Up to this point for this task, no polynomial-time algorithm has been found, and only recently different state-of-the-art appoximations were proposed (19, 20, 22, 24, 25).

Current state-of-the-art methods make the implicit assumption that the cost of node removal is same for all nodes in a network, regardless of their importance. Here thus, we generalize the network dismantling problem in such a way that the cost of removing a node i can be an arbitrary non-negative number $w_i \in \mathcal{R}$ instead of a unit value. More formally, for a given network $G(V, E)$, we want to find the subset of nodes $S \subseteq V$ with the minimal cost of removal, which will result in fragmentation into components of size C. Depending on the system of interest, the cost w_i could indicate the amount of energy needed to remove a node, monetary cost of buying or controlling a node or some other network measure such as the node importance or influence. The presented methodology works for arbitrary non-negative weights, but in the absence of other information, the node degree is used as a proxy for node importance and the associated removal cost. Note, that in case of unit costs, the problem becomes equivalent to the standard network dismantling problem (19, 20, 22, 24, 25).

Node–weighted spectral cut. Let us assume that we want to partition the network $G = (V, E)$ in such a way that the nodes from a set $M \subseteq V$ are not connected to the nodes from the complement set $\overline{M} = V \setminus M$. Whether a node i belongs to the set $M \subseteq V$ is represented with the following vector $v \in R^n$:

$$v_i := \begin{cases} +1 & if\ i \in\ M, \\ -1 & otherwise. \end{cases} \qquad [1]$$

The classical spectral bisection of a graph aims to minimize the number of edges that has to be removed between the clusters M and $\overline{M}$. However, in this paper we propose a novel node–weighted spectral cut objective function, where the cost of cutting the edge (i, j) has the upper bound of the cost of removing both, nodes i and j. We define the associated cost of removing the edge (i, j) as $-\frac{1}{2}(v_i v_j - 1) A_{i,j} (w_i + w_j)$. Motivation for this definition comes from the fact that we do not want to remove edges that do not effect the fragmentation. Therefore, if the edge (i, j) connects nodes from different clusters, the associated cost is $w_i + w_j$, as $v_i v_j = -1$ and $A_{i,j} = 1$. Contrary, if the edge (i, j) connects nodes from the same cluster, the associated cost is zero, as $v_i v_j = 1$. With this definition, we can find the partition to clusters M and $\overline{M}$ that minimizes the associated cost of removing both nodes along the edges that connect nodes from different clusters.

Without the loss of generality, we assume that the proxy for the weight is proportional to the degree centrality $w_i \propto d_i$. Then the upper bound for the cost of removing a subset of nodes that are adjacent to the edges separating cluster M from cluster $\overline{M}$ is:

$$\frac{1}{2}\sum_{i,j} -\frac{1}{2}\left(v_i v_j - 1\right) A_{i,j}\left(d_i + d_j - 1\right). \qquad [2]$$

The term $(d_i + d_j - 1)$, contains the element -1, in order to correct for double counting of links that connects i and j. This leads to more elegant matrix notation. Now, if we define the matrix B by the elements $B_{i,j} = A_{i,j}\left(d_i + d_j - 1\right)$, in matrix notation the optimization problem can be written as:

$$\min \frac{1}{4} v^T L_W v, \qquad [3]$$

subject to

$$\mathbf{1}^T v = 0, \qquad [4]$$

$$v_i \in \{+1, -1\}, i \in \{1, 2, ..., n\}, \qquad [5]$$

where $L_w = D_B - B$ is the Laplacian of the matrix $B = AW + WA - A$. Matrices W and D_B are diagonal matrices with the elements $W_{ii} = d_i$ and $(D_B)_{ii} = \sum_{j=1}^{n} B_{ij}$. For more details about the objective function, see section 1 of the SI. We name the matrix L_w a node–weighted Laplacian.

When the weight matrix equals the identity matrix ($W = I$), we get the unweighted Laplacian, which corresponds to the classical bisection problem (29, 30, 35). The additional constraint $\mathbf{1}^T v = 0$ enforces that the clusters are of the same size. Unfortunately, the optimization problem is NP-hard. Therefore, we relax the integer constraint from $v_i \in \{+1, -1\}$ to $v_i \in \mathcal{R}$. Additionally, we introduce the constraint $v^T v = n$, to prevent the vector v from becoming a zero vector. The relaxed constrained minimization problem is analytically solved by the second smallest eigenvector of the node–weighted Laplacian $\lambda_2 v^{(2)} = L_W v^{(2)}$. A more detailed derivation of this solution is presented in section 1 of the SI. We remove all nodes i for which the corresponding element in the second smallest eigenvector is positive ($v_i^{(2)} > 0$) and break the network into two components M and $\overline{M}$. Recursively, the node–weighted spectral cut is applied to M and $\overline{M}$ until the network is sufficiently fragmented into small subnetworks of maximum size C.

Spectral approximation. In order to find the second smallest eigenvectors for large–scale networks, we propose the following simple and elegant approximation algorithm. Note, that the L_w is a real, symmetric and positive semidefinite matrix. Therefore, it has real non-negative eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ with the eigenvectors $v^{(1)}, ..., v^{(n)}$, which form an orthonormal basis of $\mathbb{R}^n$. In section 2 of the SI, we show that $0 = \lambda_1$ and $\lambda_n \leq 6 \cdot d_{max}^2$, where d_{max} is the maximum degree of any node of the network. Furthermore, in section 2 of the SI, we also give spectral bound for general non-negative weights. So, in order to compute $v^{(2)}$, we consider the matrix $\tilde{L} = 6 \cdot d_{max}^2 \cdot I - L_w$, which has the same eigenvectors

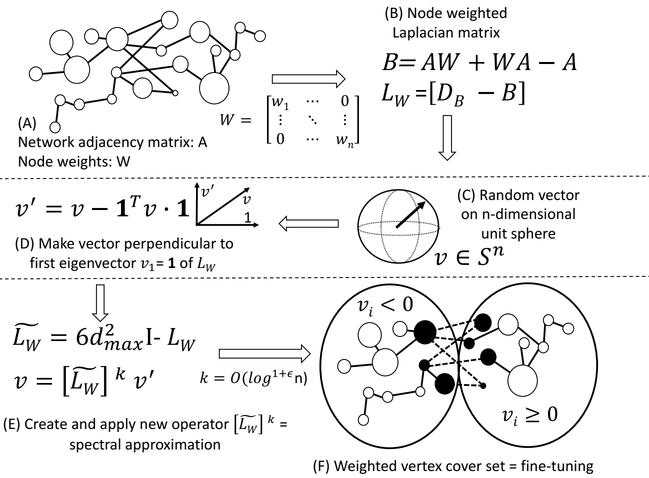


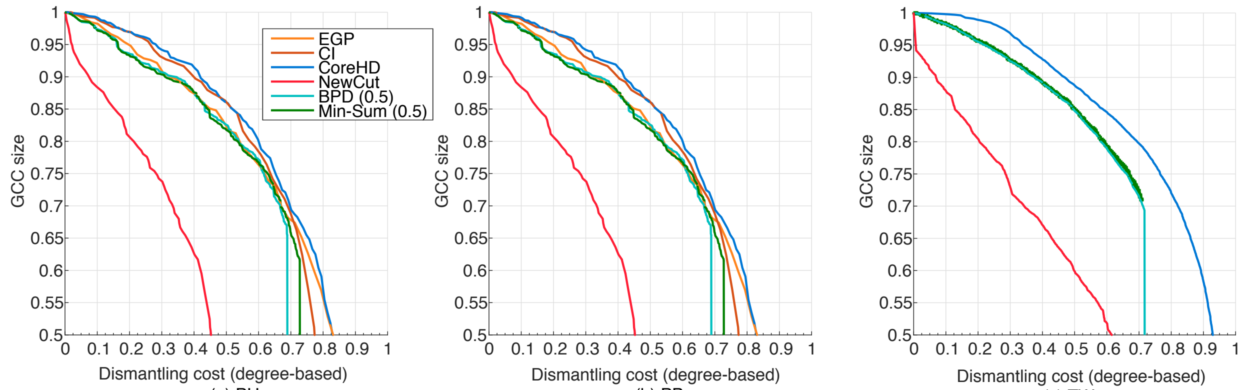
Fig. 2. Schematic diagram of the proposed methodology for generalized network dismantling. **(A)** The input network is defined by the adjacency matrix A. The costs for nodes removals are represented by the diagonal matrix W and visualized by different node sizes in the network. **(B)** Construction of the cost weighted network defined with the adjacency matrix B and it's corresponding node weighted Laplacian L_W. **(C)** Initialization of the random solution v to the generalized dismantling problem as a unit vector on the n-dimensional sphere. **(D)** As the weighted Laplacian operator L_W always has the first eigenvector $v_1 = (1, 1, ..., 1)$ of all-ones, it is uninformative for dismantling and thus removed. **(E)** Construction of the Power Laplacian operator $\tilde{L}^k$, which is applied to the random vector from the previous step, gives an approximate solution to the generalized network dismantling into two components $\{i : v_i < 0\}$ and $\{i : v_i \geq 0\}$. The operator $\tilde{L}^k$ is constructed from the node–weighted Laplacian (L_W) and the scaled identity matrix **I** with the maximal degree of a node in the network d_{max}. **(F)** Fine-tuning of the spectral solution is done with the weighted vertex cover on the subgraph of nodes (represented in black) that contains edges between components.

$v^{(1)}, ..., v^{(n)}$ as L_w. Now the corresponding eigenvalues are shifted such that $\tilde{\lambda_1} = 6 \cdot d_{max}^2 \geq ... \geq \tilde{\lambda_n} = 6 \cdot d_{max}^2 - \lambda_n \geq 0$. Let $v^{(1)}$ correspond to eigenvector with the largest eigenvalue and $v^{(2)}$ to eigenvector with the second largest eigenvalue. Then, we find the eigenvector of L_w associated with the eigenvalue λ_2 via the following steps: (i) start with a random vector v uniformly drawn from the unit sphere S^n, (ii) force it to be perpendicular to the first eigenvector $v_1 = (1, ..., 1)^T$ of the weighted Laplacian L_w and (iii) apply the linear operator $\tilde{L}^k$ with unit normalization to our vector v. We call this operator $\tilde{L}^k$, the Power Laplacian operator.

1. Draw v randomly with uniform distribution on the unit sphere.
2. Set $v = v - v_1^T v \cdot v_1$.
3. For $i = 1$ to $k = \eta(n)$
   $v = \frac{\tilde{L}v}{\|\tilde{L}v\|}$

The intuition that the random vector v converges exponentially to some eigenvector of L_w with eigenvalue λ_2 is closely related to the spectral properties of operator $\tilde{L}^k$. Note, that we can represent our random vector v in the orthonormal eigenvector basis as $v = \sum_{i=1}^{n} \psi_i v^{(i)}$. The second step of orthogonalization ensures $\psi_1 = 0$ and $\psi_2 \neq 0$ (almost surely). Finally, by applying the linear operator $\tilde{L}^k$ on vector v we get:

$$\tilde{L}^k v = \sum_{i=2}^{n} \psi_i \tilde{\lambda_i}^k v^{(i)} \propto \psi_2 v^{(2)} + \sum_{i=3}^{n} \psi_i \left(\frac{\tilde{\lambda_i}}{\tilde{\lambda_2}}\right)^k v^{(i)}. \qquad [6]$$



(a) PH (b) PB (c) TW

Fig. 3. Size of the GCC versus degree-based dismantling cost for three different networks: (a) Online social network (Petster-hamster (33)), (b) information network (Political blogs (36)) and (c) social media network (Twitter (37)). The dismantling represents creating firewalls for stopping the spread of disinformation, malicious cyber data, quarantines for epidemic and computer virus spreading, engineered breaking points for criminal and corruption networks, etc. The cost of removing a node is proportional to the current degree of a node and the dismantling cost is measured with the fraction of removed edges adjacent to the removed nodes. The network fragmentation was measured for a fixed realistic size from 0.5 to 1.0 (partial dismantling). We observe the for the fixed GCC target size that the NewCut strategy significantly outperforms current state-of-the-art strategies. On the largest Twitter network ($8.1 * 10^4$ nodes and $1.7 * 10^6$ edges), only NewCut, MinSum, BPD and CoreHD methods could be computed due to the scalability issues.

Because for every i with $\lambda_i > \lambda_2$ we have $|\frac{\tilde{\lambda_i}}{\lambda_2}| < 1$, $\left(\frac{\tilde{\lambda_i}}{\lambda_2}\right)^k \psi_i v_i \to 0$ with exponential speed.

Furthermore, we have the following upper bound for this spectral approximation:

$$|\lambda_2 - \frac{v^T L_w v}{v^T v}| \leq \frac{12 \cdot d_{max}^2 \cdot n}{\psi_2^2} \cdot |\frac{6 \cdot d_{max}^2 - \lambda_3}{6 \cdot d_{max}^2 - \lambda_2}|^{2\eta(n)}. \quad [7]$$

Note, when $v = (v_1, ..., v_n)$ is uniformly distributed on the unit sphere S^n, then its elements are distributed like a Beta distribution $v_i^2 \sim \beta_{\frac{1}{2}, \frac{n-1}{2}}$. This allows us to state that in expectation the vector v converges to some eigenvector of L_w with eigenvalue λ_2:

$$\mathbb{E}\left[|\lambda_2 - \frac{v^T L_w v}{v^T v}|\right] \to 0, \quad [8]$$

when the power k of operator $\tilde{L}^k$ is in the following range $\mathcal{O}(\log(n)^{1+\epsilon})$ for every real number $\epsilon > 0$. Formal proofs of the convergence and bounds are given in section 3 of the SI.

Recursively applying this procedure to smaller and smaller partitions, we get to the complexity of $O(n \cdot \eta(n) \cdot \log(n))$ for sparse networks. Due to the fast convergence, one can expect asymptotically good partitions when $\eta(n) = k = \log(n)^{1+\epsilon}$ and $\epsilon > 0$, which finally ends in the complexity of $O(n \cdot \log^{2+\epsilon}(n))$ for sparse networks. Details about the asymptotic complexity are given in the SI section 4. A general overview of our proposed method is given in Fig. 2.

Fine-tuning of the spectral solution. After we obtain the spectral approximation, we proceed with the fine-tuning of the solution by mapping it to the weighted vertex cover set problem of graph theory (31). All nodes i for which $v_i \geq 0$ belong to group M, otherwise they belong to group $\overline{M}$. Let us denote the set of edges that connect nodes from group M to $\overline{M}$ as E^*. The set of nodes that are incident to set E^* are denoted with V^*. Note that if we remove nodes from set $V^+ \subseteq V^*$ for which $v_i \geq 0$, a partition of the network G into two groups M and $\overline{M}$ will result. However, we can further optimize the solution by reducing it to the **weighted vertex cover set** problem on graph $G^* = (V^*, E^*)$ with weights $w_i = \sum_j A_{i,j}$ according to the degrees in the original network $G = (V, E)$. Simply, we want to find a subset of nodes Y^* such that all edges E^* that divide components $\overline{M}$ and M are connected with Y^*. This becomes the following optimization problem:

$$\min \sum_{i \in V^*} w_i y_i \quad [9]$$

subject to

$$y_i + y_j \geq 1, \forall (i,j) \in E^*. \quad [10]$$

$$y_i \in \{0, 1\}, \forall i \in V^*, \quad [11]$$

where y_i indicates whether node i is taken into the fine-tuning solution. However, this subproblem is also an NP-hard problem, but fortunately there exists a 2-approximation efficient solution for it. This implies that the cost of the approximated solution for the subproblem $\sum_{i \in V^*} w_i y_i$ is at most two times larger than the optimal fine-tuning. Upon obtaining the approximation of the weighted vertex cover, we remove all nodes for which $y_i = 1$. Further details about the fine-tuning approximation are provided in section 5 of the SI.

Results

In order to demonstrate the applicability of the proposed generalized network dismantling framework to a realistic scenario, we start with a real-world network system and demonstrate that the current state-of-the-art dismantling strategy (22) delivers different results from what is expected under the non-unit cost definition. We make the following two realistic assumptions: (i) the cost of removing a node is not a constant, but proportional to the importance of the node, measured here by its current degree and (ii) we concentrate on the partial dismantling of the system's giant connected component (GCC). The first assumption that the cost of removing a node is non-unit was already motivated in this paper before. The second assumption reflects the fact that, in practical applications, a partial dismantling of the system size to say, 80 % or 50 %

of the original GCC size is more realistic than the complete dismantling, as the budget is usually limited such that only a partial dismantling is possible. The degree-based cost of the dismantling is measured by the number of removed edges adjacent to the removed nodes, which is normalized with the total number of edges in a network. In a case of unit-costs, the cost of the dismantling is the fraction of the removed nodes and total number of nodes in the network.

In Fig 1, we show the effects of the dismantling that represents the controlled process of suppressing the spread of disinformation, computer viruses or other harmful contagion processes on the Online social network (Petster-hamster (33)). The cost for the partial dismantling (reduction to the 80 % of the original GCC size) with the the state-of-the-art MinSum strategy (22) is 0.4. However, although the MinSum algorithm removes only 5 % of nodes to dismantle the network to 80 % of the original size, it's cost is rather large. The reason of this high costs becomes more clear if we observe the degree distribution of the removed nodes Fig .1b, where we notice that all large hubs are removed. In contrast, the random removal of nodes, also known as site percolation process with the same cost of 0.4 achieves fragmentation to approximately 75 % of the original GCC size. Finally, for the same cost of 0.4 our new method proposed here fragments the network to 62 % of the original GCC size and for the target size of 80 % of GCC size the corresponding cost is only 0.2.

Next, we study the partial dismantling up to 50 % of GCC size on three different real-world systems for 5 different state-of-the-art methods: Equal Graph Partitioning (EGP) (38), Collective Influence (CI) (20), Min-Sum (22), CoreHD (19) and Belief propagation-guided decimation (BPD) (23). The dismantling of these networks enables the efficient immunization strategies against harmful contagion effects such as creating firewalls for stopping the spread of disinformation, malicious cyber data, quarantines for epidemic and computer virus spreading, engineered breaking points for criminal and corruption networks, etc. The real world systems include: (i) information networks of political blogs (36), an undirected social network with 1222 nodes and 16714 edges; (ii) online social network – Petster-hamster (33), an undirected social network, which contains 2466 nodes and 16631 edges; (iii) social media Twitter network (37), consists of 'circles' from Twitter, that was crawled from public sources and it contains $8.1 * 10^4$ nodes and around $1.7 * 10^6$ edges. In Fig. 3, we observe that for the partial dismantling to 50 % of the original GCC size the proposed methodology (NewCut) achieves the same fragmentation level with much smaller cost e.g. 0.45 (NewCut) vs 0.7 (BPD) for Online Social Petster network, 0.55 (NewCut) vs 0.65 (EGP) for information network of Blogs and 0.61 (NewCut) vs 0.71 for Twitter network (BPD). When we changed the definition of the cost as a unit one, our approach is still better or comparable to other approaches (see SI section 6, for more details).

If, however, one were interested in the complete dismantling of a malicious system e.g. a criminal or corruption network, the same methodology is able to produce good dismantling solutions too. As the proposed solution is offering a recursive solution, some of the nodes from early stages of fragmentation do not contribute to the final stage of complete fragmentation. Therefore, in order to produce better dismantling solutions (NewCut-reduced) for the complete fragmentation, we reduce some of the nodes from the final dismantling set with two simple rules: (1) nodes that connect components whose total size is still under a target size or (ii) nodes from an early stage that don't affect the final GCC size. In Fig. 4, we demonstrate the fragmentation curve for the complete dismantling (approximated with the 1 % of the original GCC size) for different weighting: (a) degree-based cost and (b) unit costs.

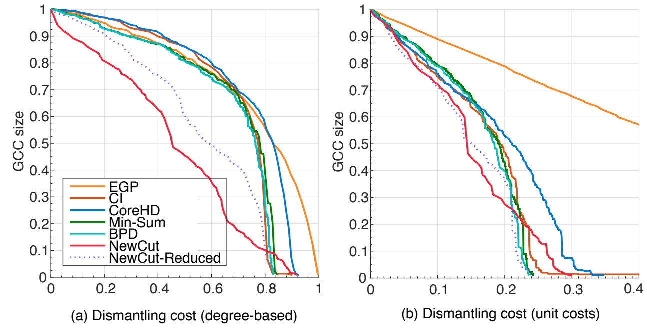


Fig. 4. Size of the GCC versus the dismantling cost for complete dismantling (target size 0.01). Dismantling represents the controlled process of suppressing the spread of disinformation, computer viruses or other harmful contagion processes on the Online social network (Petster-hamster (33)). The cost of removing a node is: (a) proportional to the current degree of a node or (b) equal for all of the nodes corresponding to unit costs. We observe that even for unit costs and complete dismantling the presented methodology (NewCut-Reduced) provides good solutions.

Conclusion

In this paper, we introduce the generalized network dismantling problem, which seeks to find the set of nodes allowing to dismantle a network into components of subcritical size in the most cost-effective way. We do not make the assumption (19–26) that the cost of removing nodes is the same. Compared to recent studies (27, 28) which use topological properties to determine costs, we allow the cost to include non-topological properties related to the price, protection level or even a social value. Our proposed methodology is based on the blend of the spectral properties of a novel node-weighted Laplacian operator, randomized approximations and weighted vertex cover approximations. We demonstrate that, for the partial dismantling of the networked systems, under more realistic non-unit costs e.g. degree-based, current state-of-the-art methods do not produce near-optimal results and sometimes behave even worse than random baseline strategy. This raises serious questions regarding how to reorder current socio-technical systems under different realistic costs definitions. The origin of this vulnerabilites comes from the fact that the problem is NP-hard and that non-unit definitions of costs change the landscape of possible solutions of the dismantling problem.

We believe that this paper raises the awareness for some current open problems regarding the system robustness. Understanding the theory behind efficient partial or complete network dismantling opens new research directions and allows to design more robust and resilient systems in the future, and to make more efficient immunization strategies for epidemic processes.

Ethics. It is clear that the approach, when not applied in appropriate contexts, may also be misused to undermine the proper functionality of networks. We therefore point out that related ethical issues must always be addressed (39, 40), before the method is applied. The method we investigate, may be justified to stop harmful cascading failures such as epidemic spreading, malicious cyber data or disinformation spreading or to dismantle criminal organizations or corruption networks. It may also be used to identify more resilient networks and system designs.

Note that the use of dismantling strategies to contain disinformation can be potentially problematic, as it may result in censorship if a government agency or company or news agency is authorized to decide what is disinformation or not. Sufficient openness for independent verification or falsification is necessary. Transparency is key, and it must be possible to challenge decisions what constitutes disinformation and what not. Independent experts should be involved, which take into the account: the distinction of information into facts, advertisements, opinions, rating, reputation and qualification mechanisms.

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