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Quarks and Strings on a Lattice*

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Three lectures describe the lattice version of the color gauge theory of quarks. The string interpretation of the theory is emphasized. The strong coupling expansion is defined by a set of Feynman rules. The dominant diagrams are identified. The result is that for strong quark-gluon coupling, the lattice spacing is about $1/5 \times 10^{-13}$cm, the nucleon has a mass of 1720 MeV/c^2 while the N* mass is 1750 MeV/c. The π and ρ masses are fitted to experiment. The relativistic limit is explained for free field theories on a lattice. For the colored quark theory only a few aspects of the relativistic continuum limit are discussed. It is shown how short wavelength string fluctuations are suppressed. It is shown that the classical limit of the lattice theory is the relativistic continuum color gauge theory.

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I. BASIC IDEAS OF THE LATTICE THEORY

There is at present a unique candidate to be a theory of strong interactions which satisfies Feynman's criteria[1]: it is simple and not obviously wrong. The candidate is the color gauge theory of quarks [2]. In this theory the ordinary fractionally charged quarks (u, d, and s) are each duplicated in three colors (red, yellow, and blue) making 9 quark species total. These quarks interact with an octet of colored gluons through a Yang-Mills interaction. The interaction is invariant to color SU(3) gauge transformations.

The renormalization of the color gauge theory is understood and, in particular, the theory is asymptotically free [3]. This means that at short distances the quarks and gluons interact weakly. The short distance properties of the theory can be calculated explicitly by perturbation theory. In particular, Bjorken scaling is almost correct. Unfortunately there are not many experiments where short distance effects can be disentangled with precision from long distance effects, so there are not many predictions of the short distance behavior that can be tested experimentally. At long distances (of order 10^{-13}cm) the color gauge theory has not been solved due to large infrared logarithms which make very high orders of perturbation theory important. More fundamentally, the problem is that perturbation theory to any order gives only quarks and gluons as the particles of the theory, whereas experiment gives a whole spectrum of multi-quark bound states (mesons and nucleons) and no free quarks or gluons.

In these lectures a version of the color gauge theory will be described which has a meson and nucleon spectrum with no free quarks or gluons. The theory described here is the color gauge theory on a discrete lattice[4]. At the present time, a meson and nucleon spectrum can be obtained from the theory only for <u>large</u>

(bare) quark-gluon coupling g_0. Unfortunately, when g_0 is large, the lattice spacing is also large (of order 1/5 x 10^{-13}cm, see Lecture III) and the spectrum is found to be not very realistic. To achieve a realistic theory g_0 must be small as required by asymptotic freedom. The spectrum of the lattice theory has not been obtained yet for this case due to technical difficulties.

In these lectures the emphasis is on the behavior of the lattice theory for large g_0. The hope for the future is that modern renormalization group techniques will allow one to study a bare Lagrangian with small g_0 and small lattice spacing ($\ll 10^{-14}$cm). Then one generates an effective Lagrangian on a lattice spacing of order $\frac{1}{5}$ x 10^{-13}cm, but such that the particle spectrum and S matrix of the effective Lagrangian is the same as the original Lagrangian. This can be accomplished in principle using "block spin" methods [5]. The effective Lagrangian will depend on an effective coupling which increases when the lattice spacing for the effective Lagrangian is increased (due to asymptotic freedom). In particular, for a lattice spacing of order $\frac{1}{5}$ x 10^{-13}cm one hopes that the effective coupling is large enough so that the strong coupling phenomenology applies. Thus it is not totally absurd to study the strong coupling phenomenology of the lattice theory.

The first lecture is devoted to defining the theory for a fixed lattice spacing. The single particle states on the lattice will be classified; static and dynamic energies will be defined, and it will be shown that excited states can decay and all states can scatter in the lattice theory. The second lecture is concerned with the continuum relativistic limit in a model which is unrealistic but soluble: the model is the π_0 meson as a free elementary particle. Also, a Feynman diagram expansion on a lattice will be defined in the second lecture. In the final lecture Feynman diagrams for quarks and colored strings will be defined. The nature of the continuum limit and the problems involved will be described.

Consider a simple cubic lattice with lattice spacing a (Fig. 1). For this lecture, think of a lattice spacing of order 10^{-13}cm. In later lectures the limit $a \to 0$ will be discussed. The lattice is introduced as a theoretical aid both for defining the quark-string theory and then solving it; but real physics is obtained only in the limit of $a \to 0$. In other words, the lattice is not intended to be present in the final theory. It is used in the same way one uses a discrete mesh when solving partial differential equations on a computer.

On the lattice one can build a simple static picture of quarks and strings which gives a large number of physical bound states. The bound state spectrum is similar but not identical to the spectrum of elementary quark models or the dual resonance model. The static theory is constructed as follows. First one has quarks. A quark must be located at a lattice site. A quark is labelled by a color index (red, blue, or yellow), an SU(3) index (up, down, or strange) and a spin index (up or down). There are also antiquarks. Any number of quarks and antiquarks may exist. Quarks are presumed to satisfy Fermi-Dirac statistics. A quark has a mass: the nonstrange quark mass is denoted m_u or m_d, the strange quark mass is m_s. The masses $m_u = m_d$ and m_s are arbitrary theoretically: they must be fitted to experiment. One can also have charmed quarks or other esoteric quark species, if needed.

Secondly one has colored string bits. A string bit is the shortest length of string the lattice permits, namely a piece of string connecting two nearest neighbor lattice sites (Fig. 2). A string bit is (by decree) rigid; its location is completely determined when its endpoints are fixed. A single string bit is labelled by a color index and an anticolor index. Thus a single string bit state can be written $|\vec{n}, \vec{n} \pm \hat{\mu}, i, j>$ where $\vec{n}$ is the location of the color end of the string bit, $\vec{n} + \hat{\mu}$ is the location of the anticolor end, i is the color index, and j is the anticolor index. A unit vector on the lattice is denoted by $\hat{\mu}$ (μ = 1, 2, or 3).

A string bit is denoted pictorically by a string with an arrow pointing to the anticolor end. The quantum mechanics of the string bit will involve only the string ends, which lie on the lattice sites. The quantum mechanics does not depend on the picture one draws for the string between the sites.

A string bit has a mass μ. The mass μ is arbitrary; it also must be determined by fitting to experiment. As will be explained below, it is related to the typical Regge slope. More formally, μ is related to the coupling g_0 of the continuum theory: see Lecture III.

The lattice theory will be required to satisfy local color gauge symmetry[4], as explained in the opening lecture to this school. This means that only states which are color singlets at every lattice site are able to propagate through the lattice. Hence the only physically observable states are the local color singlet states. It will be shown later how the dynamics is constructed so that only color singlet states propagate.

In the simple static picture, one builds color singlet states by combining quarks, antiquarks, and strings in arbitrary combinations. The masses of these states are obtained by adding the constituent quark and string masses.

In the static picture, SU(6) symmetry is exact in the limit of equal strange and nonstrange quark masses. The reason for this is that in the equal mass limit the quark mass term is invariant to any unitary transformation on the spin and ordinary SU(3) indices of the quark fields. These unitary transformations define the group U(6), which contains as subgroups both SU(6) and baryon number. The string bits are unaffected by these transformations: they are SU(6) singlets. In the static picture there are no other energies. With different strange and nonstrange quark masses one still has SU(6) multiplets but they have mass splittings.

One can now construct a catalog of the physical (local color singlet) single particle states in the static picture. They are as follows:

1) The standard meson SU(6) multiplet containing π, ρ, η, ω, etc. The ρ states in this multiplet consist of a quark and an antiquark at a single lattice site, with no string bits. The mass of these states is the sum of the constituent quark and antiquark masses. There is only one way to form a color singlet state from a colored quark and a colored antiquark state so there are only 36 states possible built from a quark and an antiquark.(2 spin states times 3 ordinary SU(3) states for the quark and similarly for the antiquark). These 36 states consist of an SU(6) singlet (the η' ?) and an SU(6) 35 (the standard meson multiplet).

2) The standard baryon SU(6) multiplet [the spin 1/2 octet (p, n, etc.) and the spin 3/2 decuplet (Δ, Y^*, etc.)]. These states are built from three quarks at the same lattice site in a color singlet state. The SU(3) color singlet state of three triplets is totally antisymmetric in the color indices. Hence to satisfy the Pauli principle, the 3 quark states must be totally symmetric in the ordinary spin and SU(3) indices. The totally symmetric states are the spin 1/2 octet and spin 3/2 decuplet states forming the 56 dimensional representation of SU(6).

3) Excited meson and baryon states. Excited meson states are formed by combining a quark, an antiquark and any number of string bits into local color singlet states. The simplest such states consist of a quark at a lattice site n, an antiquark at a nearest neighbor site $n + \hat{\mu}$, and a single string bit between the two sites. The mass of this state is the sum of the quark mass, the antiquark mass, and the string bit mass. It is an excited meson state because its mass is greater than the corresponding meson ground state. The mass difference between the first excited meson state and the ground state is μ. Phenomenologically this mass difference is determined by the Regge slope parameter; hence μ is related to the Regge slope.

The complete set of excited meson states form highly degenerate multiplets with a mass spacing μ. Experimentally μ is about 500 MeV/c^2.

Because the lattice does not have full rotational symmetry one cannot determine the orbital angular momentum for most of the excited states. The lattice has a discrete rotational symmetry, namely the symmetry to interchanges of the lattice axes. This symmetry defines the cubic group. The cubic group has only a finite number of distinct representations, including an even parity singlet, an odd parity triplet state, and an even parity doublet. Each of these representations occurs among the first excited meson states. The odd parity triplet can come from any odd L partial wave in a continuum theory, but it seems likely that the first excited odd orbital parity states would choose to be $L = 1$ rather than $L \geq 3$ in the continuum limit. The first excited baryon states are built from two quarks on a site n, a single quark at a nearest neighbor site, and a single string bit with the color end at site n, the antiquark end at the nearest neighbor site. Color singlet states are formed at both sites. The two quarks at the site n satisfy the Pauli principle; they are in an antisymmetric color state so they must be in a symmetric state of their spin and SU(3) indices.

Therefore there are 21 states of the quark pair at site n and 6 states for the quark at the nearest neighbor site, making 126 states total. These states form a 56 and a 70 dimensional representation of SU(6).

In continuum quark models the excited 56 disappears because its wave function depends on the center of mass coordinate -- it is therefore not a static state. This is not true on the lattice. On the lattice, to move the center of mass of a baryon state one must move all three quarks by at least one lattice site. In contrast, in the continuum theory an infinitesimal translation of the center of mass is obtained by multiplying the static wave function by $\vec{x}_1 + \vec{x}_2 + \vec{x}_3$ where $\vec{x}_i$ is the position of the i^{th} quark. This is equivalent to forming a linear combination of three states, such that in the i^{th} state the wave function is multiplied by $\vec{x}_i$. This state involves an infinitesimal translation of the i^{th} quark only. The first excited states on the lattice involve translations of only one quark rather than all three.

Thus if the excited 56 multiplet is to disappear it must do so when we introduce dynamics and take the continuum limit. It is not known yet whether the excited 56 does disappear in the continuum limit.

4) Quarkless states. The lowest mass quarkless state is obtained by combining four string bits around a square (Fig. 3). At each corner of the square there is a color string end meeting an anticolor string end combined into a color singlet state. This state has a mass 4μ, i.e., 4 times the mass difference between the lowest two SU(6) baryon or meson multiplets. This puts the lowest quarkless state at about 2 GeV in the static approximation.

At first sight there would be quarkless states with only two or three string bits connecting a single pair of lattice sites (Fig. 4). However, when the quantum operators of the theory are defined, it will turn out that these states do not exist. See later.

5) Exotic states. These are states built of four or more quarks and/or antiquarks. (There are no other states with less than four quarks and antiquarks, due to the color singlet requirement. For example, there is no way to form color singlet states containing two quarks and no antiquarks, even with string bits). No states are known experimentally which have to contain four or more quarks. For example, a meson with isospin > 1 would contain at least 2 quarks and 2 antiquarks; there are no such states seen. In the static limit exotic states exist, but many of them may disappear when kinetic energy is introduced. See later.

This completes the catalogue of single particle states. There also many particle states, obtained by forming many different single particle states on disconnected sets of lattice sites.

The next stage in the discussion of the lattice theory is to introduce the quantum operators of the theory. The quark operators are standard Dirac fields, restricted to act at the lattice sites. The quark field operators ψ_n and $\bar{\psi}_n$ create or destroy quark (or antiquark) states at the site n.

The string operators are more complicated. The string operators will be defined through the analogy with gauge field theories. There is an important reason for this. The only known way of obtaining Bjorken scaling in a local theory is through asymptotic freedom, which arises only in nonabelian gauge theories. Thus the colored string theory will be formulated so that it becomes the continuum color gauge theory of quarks in the continuum limit. The first step is to make sure that the operators in the lattice theory are related to operators of the gauge theory. In the third lecture the reason for free quark behavior at short distances (the prerequisite for Bjorken scaling) in the lattice gauge theory will be explained. The form of the string operators will be crucial to the existance of free quark behavior at short distances.

To illustrate the construction of string operators, consider first the case that there is a single gauge field $A_\mu(x)$ and an Abelian gauge group. Then a string operator in the continuum gauge theory has the form

$$U(A,B) = \exp \left\{ig_0 \int_A^B A_\mu(x)\, dx^\mu \right\} \tag{1}$$

where the points A and B are the endpoints of the string and the line integral $\int_A^B A_\mu(x)\, dx^\mu$ is a line integral along the path of the string. In particular, note the gauge transformation properties of U(A,B). A gauge transformation can be written

$$A_\mu(x) \to A_\mu(x) - \frac{1}{g_o} \nabla_\mu \phi(x) \tag{2}$$

where $\phi(x)$ is an arbitrary function of x. The effect of this transformation on U(A,B) is

$$U(A,B) \to e^{i\phi(A)} U(A,B) e^{-i\phi(B)} \tag{3}$$

In words, the operator U(A,B) transforms like a field with charge +1 at the endpoint A and charge -1 at the point B. This is what we expect for strings in a theory where ordinary abelian charge replaces color charge.

Now let A and B be nearest neighbor sites on the lattice. The line integral has to be replaced by a lattice sum; for nearest neighbor endpoints the result is

$$U(n, n + \hat{\mu}) = \exp \{ ig_o \, a \, A_{n\mu} \} \tag{4}$$

where $A_{n\mu}$ is the gauge field at the site $\underline{n}$.

The generalization to color is that the string bit operator is a unitary color matrix $U_{n\mu}$ which can be written in terms of the color gauge fields $A_{n\mu}{}^b$ as

$$U_{n\mu} = \exp \{ i \, a \, g_o \sum_{b=1}^{8} A_{n\mu}{}^b T^b \} \tag{5}$$

where the matrices T^b are the 8 Hermiltian generators of SU(3). Thus $U_{n\mu}$ is a matrix with components $U_{n\mu\, ij}$: i is the anticolor index and j is the color index. Each component is a quantum operator which creates or destroys a colored string bit. Thus the unitarity relation

$$\sum_j U_{n\mu\, ij} (U_{n\mu}{}^+)_{jk} = \delta_{ik} \tag{6}$$

is a nonlinear constraint on the quantum operators $U_{n\mu\, ij}$ and $U_{n\mu}{}^+{}_{jk}$. It is this nonlinear constraint which eliminates the color singlet state of two superposed string bits. Furthermore, the $U_{n\mu}$ are SU(3) matrices, which means that the determinant of $U_{n\mu}$ is 1. This is a third order nonlinear constraint; it eliminates the color singlet state of three superposed string bits. The fact that there are 8 gluon fields while there are nine components of $U_{n\mu}$ leads only to the restriction that det $U_{n\mu} = 1$. Thus there is no contradiction in the fact that there are nine string bit species while there are only 8 gluon fields in the continuum theory.

We must specify the color gauge transformation properties of the quark and string bit operators. They are as follows. A gauge transformation is specified by a set of SU(3) transformation matrices Φ_n, one for each lattice site. The transformation of the quark and string bit operators is

$$\psi_n \to \Phi_n \psi_n \tag{7}$$

$$\overline{\psi}_n \to \overline{\psi}_n \Phi_n^+ \tag{8}$$

$$U_{n\mu} \to \Phi_n U_{n\mu} \Phi_{n+\hat{\mu}}^+ \tag{9}$$

Consider now the dynamics of quarks and string bits. Motion of quantum particles on a lattice occurs by transitions. A quark can move from site to site through transitions in which the quark is destroyed at the original site and recreated at a new site. Transitions are caused by terms in the Hamiltonian of the lattice theory. To preserve local color symmetry, the transition operators must be invariant to local color transformations. The simplest operators which generate transitions and are locally color invariant are the following:

1) $\overline{\psi}_n U_{n\mu} \psi_{n+\hat{\mu}}$ or $\overline{\psi}_{n+\hat{\mu}} U_{n\mu}^+ \psi_n$

These operators cause transitions which move quarks or antiquarks from one lattice site to a nearest neighbor site. However, when, for example, a quark is moved from the site n to the site $n + \hat{\mu}$ a string bit is created (Fig. 5) such that the final state is still a color triplet at the site n (the open end of the string bit) while the site $n + \hat{\mu}$ is in a color singlet state. This means that a meson moves from one lattice site to the next by making two transitions. The first transition is a transition to an excited state: the quark, for example, may move to a nearest neighbor site, creating a string bit. The second transition is back to the ground state: the antiquark moves to the new site and the string bit is destroyed. There are also operators of the form $\bar{\psi}_n \, U_{n\mu} \, \gamma_\mu \, \psi_{n+\hat{\mu}}$; these operators create or destroy quark-antiquark pairs on nearest neighbor sites with a string bit in between.

2) Transition operators for string bits. The simplest non-trivial color singlet operators involves 4 string bit operators around a square. These operators can create or destroy the simplest color singlet quarkless states, or they can cause transitions between different quarkless states which result in motion in higher orders. The operators have the form

$$\mathrm{Trace} \left\{ U_{n\mu} \, U_{n+\hat{\mu},\nu} \, U_{n+\hat{\nu},\mu}^{\ +} \, U_{n\nu}^{\ +} \right\}$$

One can write more complicated transition operators but they are not needed.

When the transition operators are included in the Hamiltonian for quarks and string bits, a number of effects occur. Firstly, the particle states acquire a spectrum (a spectrum means a dependence of energy on momentum). The states with a particle at a lattice site (or set of sites) are no longer eigenstates of the Hamiltonian; the eigenstates are plane wave states and the energy of the eigenstate depends on its momentum. However, these spectra are not relativistic except in the continuum limit (see later lectures).

Secondly, it is possible for particle states to become unstable. Consider the ρ meson, for example. This can make a transition to two π mesons, in two steps. The first step is to make a transition to an excited meson state with a string bit separating the quark and antiquark. The second step is to destroy the string bit and create a new quark-antiquark pair on top of the old pair. Then one has two quark-antiquark pairs. This transition only causes decay when it is possible for two π mesons to have a total energy equal to the ρ mass. This is not possible in the static limit where the ρ mass is equal to a single π mass (since they are built of the same type of quarks). Thus the transition matrix elements must be of order the static energies before this decay becomes possible. In contrast the decay of exotic states becomes possible even for infinitesimal transition matrix elements since, for example, the mass of a two quark, two antiquark state is twice the mass of a meson state in the static limit. It is not known yet whether this difference between ordinary and exotic states means that exotic states are absent in the continuum limit.

Finally, scattering processes occur, for example, by quark exchange. Two mesons can arrive at the same lattice site, exchange their quarks (but not their antiquarks) and separate again. This exchange will cause scattering. There are also mechanisms involving creation and destruction of string.

Quarks are confined in the static picture. They cannot separate infinitely far because to do so requires an infinite length of string. Since each string bit has a mass μ, an infinite length of string has infinite mass.

Lecture II

In this lecture the continuum limit and the Feynman expansion on the lattice will be discussed for a soluble model: a free scalar particle. There will be no strings or string bits in this lecture, only the scalar particle.

The reason for studying a free particle is that it moves on the lattice by making transitions. These transitions are similar to the transitions which occur in the quark-string model, but in a simpler form. Thus it is useful to study the free particle transitions. In particular one can learn how relativistic invariance is recovered in the continuum limit. Likewise the methods for solving theories with transitions can be illustrated in the free particle example.

In this lecture we will abandon the Hamiltonian framework which was assumed in the first lecture (for a detailed development of the Hamiltonian framework see Kogut and Susskind [4]). Instead, a Euclidean space-time formalism will be used. In particular a time lattice will be introduced. It is useful to discuss very briefly the meaning of a Euclidean time lattice in quantum mechanics (for a more thorough discussion see Section X of Wilson and Kogut [6]). The operator which propagates a quantum state through a real time interval t is e^{-iHt}. It follows that the operator which propagates a state through an imaginary time interval $t = -ia$ is e^{-Ha}. This is not physical propagation; however, the operator e^{-Ha} is well defined, and by "propagating" states with this operator we will be able to compute eigenvalues e^{-Ea} of e^{-Ha}. Once an eigenvalue e^{-Ea} of e^{-Ha} is known one easily computes the eigenvalue $E = -\frac{1}{a}\ln(e^{-Ea})$ of H. In general, finding the eigenvalues and eigenvectors of e^{-Ha} is equivalent to finding the eigenvalues and eigenvectors of H itself. Thus we lose <u>no information</u> by using the Euclidean time lattice: it is simply an alternative theoretical procedure for solving the quantum mechanics. In particular the operator e^{iHt} for any real t can be computed once the eigenvalues and eigenvectors of e^{-Ha} are known.

There are two benefits gained by using the Euclidean space-time lattice as compared to a pure spatial lattice. The first benefit is that one can impose a discrete space-time symmetry, namely the symmetry to interchanges of space and time axes. This symmetry reduces the number of arbitrary parameters in the theory: in particular the static energy parameters and the strengths of the kinematic terms will be related. This symmetry also makes it easier to see the full four dimensional Euclidean symmetry appear in the continuum limit. The second benefit is that there is a Feynman diagram expansion that one can define on the lattice which is easily carried to high orders.

To begin with the Feynman rules for a free scalar particle on the space-time lattice will be stated. These rules define diagrams which give an expansion of the scalar particle propagator. The expansion parameter is the strength of the transition amplitude. The rules will be justified later. The rules are as follows:

1) Diagrams are built from nearest neighbor links.

2) There is a factor K for each link.

3) To compute the position space propagator D_n sum all distinct graphs which consist of a single line (built of nearest neighbor links) connecting the origin to the site n.

4) To compute the propagator D(p) in momentum space one calculates the sum of all distinct lines from the origin to any point n, multiplied by $e^{-ip\cdot na}$.

5) There may be multiple links superposed on the same physical location. There are no extra factors for this case provided one counts distinct configurations of lines, not distinct configurations of links. This means that the links along a line are regarded as distinct when counting configurations of lines.

The result of these rules is that the propagator satisfies a simple recursion formula. The formula is obtained by thinking of a single link as analogous to a self energy diagram in ordinary quantum field theory. The recursion formulae are the following

$$D_n = \delta_{no} + K \sum_{\mu=o}^{3} \{ D_{n+\hat{\mu}} + D_{n-\hat{\mu}} \} \tag{10}$$

$$D(p) = 1 + K \sum_{\mu=o}^{3} \{ e^{ip_\mu a} + e^{-ip_\mu a} \} D(p) \tag{11}$$

The first formula says that any line contributing to D_n must either

(a) be the line of zero length, which contributes only for n= 0, giving

δ_{no}

or

(b) contain a final link ending at the site n. The final link must begin at a nearest neighbor site, namely either $n + \hat{\mu}$ or $n - \hat{\mu}$ for some choice of μ. The sum of all lines with a final link from $n - \hat{\mu}$ to n is a product of the final link times the sum of all lines from 0 to $n - \hat{\mu}$. This latter sum is $D_{n-\hat{\mu}}$. Thus the first equation contains the sum over all nearest neighbor sites of products of K (for the last link) times $D_{n\pm\hat{\mu}}$. The second equation is the Fourier transform of the first equation where

$$D(p) = \Sigma_n \, e^{-ip\cdot na} \; D_n \tag{12}$$

Eq. (10) can be used as a <u>definition</u> of D_n, that is, the Feynman rules can be derived from this equation. Now it will be shown that Eq. (10) is the discrete version of the Klein-Gordon equation.

Start with a continuum propagator D(x) satisfying the Euclidean Klein-Gordon equation

$$\left\{ m^2 - \sum_\mu \frac{\partial^2}{\partial x_\mu^2} \right\} D(x) = \delta^4(x) \tag{13}$$

This equation can be approximated by a finite difference equation. Let

$$D_n = D(na) \tag{14}$$

Where a is the spacing for finite differences. The finite difference approximation to a second derivative is

$$\frac{\partial^2 D(x)}{\partial x_\mu^{\ 2}} = \frac{D(na - \hat{\mu}a) + D(na + \hat{\mu}a) - 2D(na)}{a^2} + O(a^2) \tag{15}$$

where x= na. This form can be verified using the Taylor's series expansion of $D(na \pm \hat{\mu}a)$ about D(na).

The discrete approximation to the delta function is δ_{no}, apart from normalization constant which will be ignored. Thus the discrete approximation to the Klein-Gordon equation is

$$m^2 D_n - \Sigma_\mu \frac{1}{a^2} \left\{ D_{n-\hat{\mu}} + D_{n+\hat{\mu}} - 2D_n \right\} = \delta_{no} \tag{16}$$

This equation can be rewritten

$$(8 + m^2a^2)\, D_n - \sum_\mu \left\{ D_{n+\hat{\mu}} + D_{n-\hat{\mu}} \right\} = a^2\, \delta_{no} \tag{17}$$

or

$$D_n - \frac{1}{8+m^2a^2} \sum_\mu \left\{ D_{n+\hat{\mu}} + D_{n-\hat{\mu}} \right\} = \frac{a^2}{8+m^2a^2}\ \delta_{no} \tag{18}$$

This equation is the same, apart from the normalization of the δ_{no} term, as the equation derived for the diagram expansion. One must set

$$K = \frac{1}{8+m^2a^2} \tag{19}$$

The relativistic continuum limit is obtained by letting $a \to 0$. In this limit $K \to 1/8$. This is the remarkable feature of the lattice expansion: K, the expansion parameter, does not go to ∞ , or even to 1, but only to 1/8 in the continuum limit. For a> 0, K is less than 1/8.

To understand the situation better, consider the momentum space propagator. The solution of Eq. (11) is

$$D(p) = \frac{1}{1 - K \sum_{\mu=0}^{3} \left(e^{ip_\mu a} + e^{-ip_\mu a}\right)} \tag{20}$$

We would like to use the propagator to find the scalar particle spectrum. The particle energy is determined by looking for a pole in the propagator, for given particle momentum. In order to return to the Lorentz metric from the Euclidean metric, one replaces p_o by $+iE$. Thus one is looking for poles of

$$D(E, p) = \frac{1}{1 - K(e^{Ea} + e^{-Ea}) - K \sum_{\mu=1}^{3} \left(e^{ip_\mu a} + e^{-ip_\mu a}\right)} \tag{21}$$

Two cases will be discussed. The first is the near-static limit: $K \ll 1/8$. The only way one can obtain a pole for very small K is for e^{Ea} to be large . In this case the pole occurs for

$$e^{Ea} \simeq \frac{1}{K} - 2 \sum_{\mu=1}^{3} \cos p_\mu a \tag{22}$$

The term depending on the spatial momentum $\vec{p}$ is small compared to $\frac{1}{K}$ but it is kept in this equation to give the momentum dependence of E. From this equation one obtains

$$E \simeq + \frac{1}{a} \ln \frac{1}{K} - \frac{2K}{a} \sum_{\mu=1}^{3} \cos p_\mu a \tag{23}$$

It is clear from this equation that the particle has a large mass ($\gg 1/a$) when K is very small and that the kinetic energy is small (of order K/a) for any momentum $\vec{p}$. Thus for K small one has a near-static situation. Note that a single parameter K determines both the particle mass and the strength of the particle's kinetic energy, apart from the factor 1/a which is present to make the dimensions correct.

Consider now the near-relativistic case: $K \simeq 1/8$. In this case there is a pole for Ea and $\vec{p}a$ small as can be seen as follows. For small Ea and $\vec{p}a$ one has

$$D(p) \simeq \frac{1}{1-8K - K(E^2a^2 - \vec{p}^2a^2)} \tag{24}$$

neglecting terms of order $(Ea)^4$ or $(\vec{p}\ a)^4$. Thus there is a pole at

$$E^2 = \vec{p}^2 + \frac{1-8K}{Ka^2} \tag{25}$$

This is the relativistic formula, provided the mass is

$$m^2 = \frac{1-8K}{Ka^2} \tag{26}$$

Since Ea and $\vec{p}a$ are small, ma must also be small, which requires 8K to be near 1. If one lets $a \to 0$ holding m^2 fixed then the a^4 terms disappear and the relativistic formula becomes exact. Eq. (26) is equivalent to the earlier formula (19).

For $K \simeq 1/8$, the kinetic energy can be much larger than the mass. In particular, when $\vec{p}a$ is of order 1, then Ea is also of order 1, i.e., E is of order 1/a. However, m is much less than 1/a. Nevertheless, the expansion parameter K is only 1/8. How does this happen? The reason K is not large is that the K expansion is <u>not</u> an expansion in the complete kinetic energy. Rather, the expansion parameter K is related to the ratio of the fluctuation in kinetic energy relative to the average total energy (total energy being kinetic plus potential). More precisely, K measures the ratio of the contribution of one axis to the fluctuation in kinetic energy to the mean total energy. Thus, even when the kinetic energy can be very large, the mean energy is also large and K does not become large.

Now look at what happens to the expansion in K as K ranges from 0 to 1/8. For K near 0, the expansion is rapidly convergent in the Euclidean domain.

However, one is interested in a pole in the propagator; clearly the expansion cannot converge at a pole. The pole occurs for $e^{Ea} \sim 1/K$ and the dominant graphs are lines along the time axis (Fig. 6). For these graphs there is a factor e^{Ea} compensating each factor of K and the sum of all these graphs is the geometric series 1 + 1 + 1 + . . . which diverges. In contrast, for $K \simeq \frac{1}{8}$, the pole occurs for $e^{Ea} \simeq 1$. In this case what happens is that there are eight possible directions for each link in a line, so for each factor K there is a factor 8 from counting the different directions. Since 8K= 1 the series is again 1 + 1 + 1 + . . ., which is divergent. Note that in both cases the pole occurs at the radius of convergence of the K expansion for fixed E and $\vec{p}$.

For the free particle example studied here the expansion in K is a simple geometric series. Thus even the pole in the propagator can be determined knowing only the first few terms in the K expansion. The location of the pole is determined by using the term of order K (by rewriting the series 1 + K A + . . . as 1/(1-KA)); the terms of order K^2, K^3, etc. can be used to verify that the series is geometric.

To conclude this lecture the theory of free quarks on a lattice will be discussed, with emphasis on the complications caused by spin. The essential complication of spin is that the Dirac quark field has four spin components while the quark has only two spin components.

The Feynman rules for free quarks on a lattice are as follows. As in the scalar case the quark propagator is built of lines going from the origin to a site n. The lines are built of nearest neighbor links which are now labelled with an arrow (the direction of the line containing the link). For a link from site n to site n $+\hat{\mu}$ one substitutes $K(1 + \gamma_\mu)$, while for a line from site n + $\hat{\mu}$ to site n one substitutes $K(1-\gamma_\mu)$.

The γ matrices are Euclidean, that is,

$$\left\{ \gamma_\mu, \gamma_\nu \right\} = \delta_{\mu\nu} \tag{27}$$

instead of $g_{\mu\nu}$. The reason for the factor $1 \pm \gamma_\mu$ is simple. For propagation along the positive time axis the factor is $1 + \gamma_0$. This is a projection operator. It is present so that only two quark spin states propagate forwards in time instead of all four Dirac states. For a link in the reverse time direction the factor is $1-\gamma_0$ which permits only antiquark states to propagate backwards in time. An alternative explanation of the need for the factors $1\pm\gamma_0$ will be given later. Then to preserve space-time symmetry one uses factors $1\pm\gamma_\mu$ for the direction μ. [The factors $1\pm\gamma_\mu$ were not reported in the author's original paper [4]; the original paper incorrectly uses $\pm\gamma_\mu$ for $1\pm\gamma_\mu$].

The quark propagator derived from these rules is

$$S(p) = \frac{1}{1 - K \sum_\mu \left\{ (1+\gamma_\mu)\, e^{-ip_\mu a} + (1-\gamma_\mu)\, e^{ip_\mu a} \right\}} \tag{28}$$

This becomes the relativistic Dirac propagator in the following limit:

$p_i a \ll 1$ (p_i fixed, a small)

$p_0 a = iEa \ll 1$

$K \simeq 1/8$

Then

$$S(p) \simeq \frac{1}{1 - 8K - 2K\gamma_0 Ea + 2Ki\,\vec{p}\cdot\vec{\gamma} + O(a^2)} \tag{29}$$

This is proportional to the Dirac propagator with a quark mass

$$m = \frac{1-8K}{2Ka} \tag{30}$$

One must replace $\vec{\gamma}$ by $-i\vec{\gamma}$ to return completely to the Lorentz metric. With this replacement one has

$$S(p) = \frac{-1}{2Ka} \quad \frac{1}{\gamma_0 E - \vec{\gamma} \cdot \vec{p} - m} \tag{31}$$

which is the usual propagator apart from an unimportant normalization factor. For the quark propagator on a lattice, therefore, the continuum limit is $a \to 0$, $K \to 1/8$ with $m = (1-8K)/2Ka$ fixed.

Return to the general lattice propagator, and to illustrate the importance of using the projection operators $1 \pm \gamma_\mu$, suppose these were replaced by $c \pm \gamma_\mu$ where c is arbitrary. The propagator (with $p_o = -iE$) then can be written

$$S(p) = \frac{1}{A + \sum_\mu B_\mu \gamma_\mu} \tag{32}$$

$$\text{with } A = 1 - 2cK \cosh Ea - 2Kc \sum_{\mu=1}^{3} \cos p_\mu a \tag{33}$$

$$B_o = -2K \sinh Ea \tag{34}$$

$$B_\mu = -2iK \sin(p_\mu a) \quad (1 \leq \mu \leq 3) \tag{35}$$

The propagator can be rationalized, giving

$$S(p) = \frac{A - \sum_\mu B_\mu \gamma_\mu}{A^2 - B^2} \tag{36}$$

Hence there are poles whenever $A^2 = B^2$. This condition becomes

$$c^2K^2 (e^{Ea} + e^{-Ea})^2 - 4cK \cosh Ea(1 - 2cK \sum_{\mu=1}^{3} \cos p_\mu a)$$
$$+ (1 - 2cK \sum_{\mu=1}^{3} \cos p_\mu a)^2$$
$$= K^2 (e^{Ea} - e^{-Ea})^2 - 4K^2 \sum_\mu \sin^2 p_\mu a \tag{37}$$

This equation is to be solved to find the eigenvalues E for given $\vec{p}$. If c= 1 this is easy: one obtains

$$\cosh Ea = \frac{4K^2 + (1 - 2K \sum_{\mu=1}^{3} \cos p_\mu a)^2 + 4K^2 \sum_{\mu=1}^{3} \sin^2 p_\mu a}{4K (1 - 2K \sum_{\mu=1}^{3} \cos p_\mu a)} \tag{38}$$

which has a unique positive solution for E (the equal and opposite solution for negative E is the antiquark pole). However, if c is not equal to 1, the equation involves $\cosh^2 Ea$, and hence there are <u>two roots</u> for cosh Ea, i.e., two quark energies E instead of one. For each solution there are two spin states (when A = B the numerator $1- \frac{B_\mu \gamma_\mu}{A}$ is a projection operator with the same properties as $1- \gamma_0$: it projects out two states out of the four possible). Thus for $c \neq 1$ there are 4 quark states corresponding to the four spin components of the Dirac matrices. For c = 1 there are only two states which is what one wants.

If c = 0, that is $c \pm \gamma_\mu$ becomes simply $\pm\gamma_\mu$ (as proposed incorrectly in Wilson [4]) there is an additional difficulty, namely there are quark states with low energy when the momentum is of order of the cutoff. In this case the equation for E is

$$4K^2 \sinh^2 Ea = 1 + 4K^2 \sum_\mu \sin^2 p_\mu a \tag{39}$$

In this equation there is no distinction between the case $p_\mu = \frac{\pi}{a}$ and $p_\mu = 0$ since $\sin p_\mu a$ is 0 for both cases. Hence a quark with momentum π/a has the same energy as a quark at rest. The state with momentum π/a is a distinct state: the quark wave function has alternate signs at alternate lattice sites. In interacting theories quarks with momentum $\pm\pi$ can be produced in pairs; in the limit of small a these quarks will appear as a separate quark species in low energy experiments, since quarks with intermediate momenta (of order $\frac{\pi}{2a}$) have very high energy and cannot be produced. There will be eight quark species when all the possibilities are considered (the possibilities are $p_1 = 0$, or π/a, $p_2 = 0$ or π/a, and $p_3 = 0$ or π/a).

Lecture III

In this lecture the full quark-string theory will be discussed. First the Feynman rules for the quark-string theory will be stated. These rules define a strong coupling expansion, that is, an expansion in $1/g_0$ where g_0 is the bare quark-gluon coupling. There are a simple set of diagrams which dominate when g_0 is reasonably large or infinite. These diagrams will be computed explicitly for the meson propagator; results will also be given for baryons. Finally the continuum limit of the lattice theory will be discussed. In particular the way that free quark behavior arises for short distances will be explained, including the role of string fluctuations. The classical continuum limit of the lattice theory will be explained in Appendix A.

The Feynman rules for quarks and colored strings will be stated for a Euclidean space-time lattice as in Lecture II. The simplest calculation one can perform using the rules to compute the meson and baryon propagators and use the poles of these propagators to locate the meson and baryon masses. One can also compute excited state propagators, vertex functions and scattering amplitudes. In this lecture the Feynman rules will be stated explicitly for the meson propagator.

Diagrams contain nearest neighbor quark links, gauge loop squares, and gauge field contractions. The substitution for a quark link from n to $n + \hat{\mu}$ is $K(1 + \gamma_\mu)\, U^+_{n\mu}$ where $U_{n\mu}$ is a unitary SU(3) matrix. The substitution for a quark link from $n + \hat{\mu}$ to n is $K(1 - \gamma_\mu)\, U_{n\mu}$. The factor K is the same quantity that appears in the free quark theory and is related to the quark mass. Hence, there are separate constants $K_u = K_d$ and K_s for the three ordinary SU(3) species of quarks.

Quark links are connected together to form quark lines. In the case of the meson propagator there are two kinds of quark lines possible, illustrated in Fig. 7. The graph of Fig. 7a includes a quark line from the origin to the site n and a second line returning from n to o. These two lines form a single quark loop. Associated with the loop there is a color SU(3) trace and a γ matrix trace. (The matrices in these traces run from right to left on the page as one goes around the loop in the direction of the arrows.) There is an extra γ matrix in the trace for each external point: for the pseudoscalar meson propagator the extra γ matrix is γ_5; for the vector meson propagator the external γ matrix is γ_μ with $\mu = 1, 2,$ or 3. Each quark line also carries a definite ordinary SU(3) index. The graph of Fig. 7b contains two separate quark loops: one for each external point. There is a separate color and γ matrix trace for each loop. Both kinds of diagrams can contain any number of additional internal quark loops. For each internal loop there is a color trace, a γ matrix trace, and a sum over the ordinary SU(3) index. For an internal loop containing ℓ links there is a factor $\frac{1}{\ell}$ (a similar factor appears in continuum Feynman rules). Also there is a factor $\frac{1}{m!}$ whenever there are m identical superposed loops. A gauge loop square is an oriented elementary square (cf Fig. 3). For each elementary square one substitutes $\frac{1}{2g_0^2}$ times a color trace. The color trace involves four U or U^+ matrices; the color trace is formed exactly as if the gauge loop square were a quark loop. There is a factor $\frac{1}{m!}$ whenever there are m superposed squares with the same orientation.

Finally, there are contractions. Contractions define numerical substitutions for products of color SU(3) matrices between a given pair of sites. The simplest contraction is written graphically in Fig. 8. What it stands for is the following.

For each nearest neighbor pair of sites n and $n + \hat{\mu}$ there is an invariant SU(3) group integration over all unitary matrices $U_{n\mu}$. The contraction shown in Fig. 8a represents the group integral

$$\int_{U_{n\mu}} (U_{n\mu})_{ij}\ (U^{+}_{n\mu})_{k\ell} = \frac{1}{3}\,\delta_{i\ell}\,\delta_{jk} \tag{40}$$

The integral can be performed explicitly by setting up a generalized Euler angle representation for SU(3) unitary matrices and determining explicitly the form of the group integral in terms of the eight generalized Euler angles. The details will not be reported here. The simple integrations can also be determined from invariance arguments alone. For normalization, one sets the group integral of 1 equal to one. Invariance arguments show that the integral of $(U_{n\mu})_{ij}\ (U^{+}_{n\mu})_{k\ell}$ must be proportional to $\delta_{i\ell}\ \delta_{jk}$ and the unitarity condition (6) then requires the coefficient to be 1/3. A graphical notation for the Kronecker delta's is shown in Fig. 8b.

The integrals of $(U_{n\mu})_{ij}$ or the product $(U_{n\mu})_{ij}\ (U_{n\mu})_{k\ell}$ vanish. This is a consequence of color invariance. The next nonzero integral is

$$\int (U_{n\mu})_{ij}\ (U_{n\mu})_{kl}\ (U_{n\mu})_{mp} = \frac{1}{6}\, e(ikm)\ e(j\ell p) \tag{41}$$

where e(123) = 1, e(132) = -1, etc. The factor 1/6 follows from the requirement that det $U_{n\mu} = 1$. Higher order contractions are reported in Appendix B.

An example of a diagram calculation will now be given. The diagram is shown in Fig. 9. Assume this is a π^{+} meson graph. Then the amplitude for the diagram is (before performing the color SU(3) integrations)

$$\begin{aligned} &K_u^4 \left(\frac{1}{2g_o^{\,2}} \right) \mathrm{Tr}\ \gamma_5 (1-\gamma_\mu)\ (1-\gamma_\nu)\ (1+\gamma_\mu)\ \gamma_5 (1+\gamma_\nu) \\ &\qquad \mathrm{Tr}\ (U_{o\mu}\ U_{\hat{\mu},\nu}\ U^{+}_{\hat{\nu},\mu}\ U^{+}_{o\nu}) \\ &\qquad \mathrm{Tr}\ (U^{+}_{o\mu}\ U_{o\nu}\ U_{\hat{\nu},\mu}\ U^{+}_{\hat{\mu},\nu}) \end{aligned}$$

There are four color integrations (over $U_{o\mu}$, $U_{\hat{\mu},\nu}$, $U_{\hat{\nu},\mu}$, and $U_{o\nu}$). Each integration gives a factor 1/3. The Kronecker δ's result in four separate color sums, one for each lattice site in the diagram, each giving a factor 3. The γ matrix trace is easily determined to be 8. Thus the total amplitude is

$$8\, K_u^4 \left(\frac{1}{2g_o^2}\right)$$

If one is computing the meson propagator in momentum space, this amplitude is multiplied by $e^{ip_\nu a}$ (since $n = \hat{\nu}$). Then one wants to sum over all possible ways of orienting the graph on a lattice, i.e., sum over all μ and ν with $\nu \neq \mu$, plus sum over the reflections $\mu \rightarrow -\mu$ and $\nu \rightarrow -\nu$. In the zero momentum or zero lattice spacing limit, the result of this sum is an amplitude

$$48 \times 8\, K_u^4 \left(\frac{1}{2g_o^2}\right)$$

This is to be compared with the zeroth order term in the meson propagator which contains a color trace and a γ matrix trace for the case of no internal links: the γ matrix trace is $\mathrm{Tr}\,\gamma_5^2 = 4$ and the color trace is 3. So the zeroth order amplitude is 12. In the free relativistic limit ($K_u = 1/8$) the ratio of the diagram just calculated to the zeroth order result is $(1/128) \times (1/2g_o^2)$. This is a rather small result unless $1/2g_o^2$ is enormous. See later.

The diagrammatic rules stated above are derived from a Feynman path integral formalism on the lattice. This formalism is explained elsewhere ([4], [6]) and will not be explained here. The action which appears in the path integral for the quark-string theory is

$$\begin{aligned} A = & -\sum_n \bar{\psi}_n \psi_n + K \sum_n \sum_\mu \bar{\psi}_n (1-\gamma_\mu)\, U_{n\mu}\, \psi_{n+\hat{\mu}} \\ & + K \sum_n \sum_\mu \bar{\psi}_{n+\hat{\mu}} (1+\gamma_\mu)\, U^+_{n\mu}\, \psi_n \\ & + \frac{1}{2g_o^2} \sum_n \sum_\mu \sum_{\nu \neq \mu} \mathrm{Tr}\,(U_{n\mu}\, U_{n+\hat{\mu},\nu}\, U^+_{n+\hat{\nu},\mu}\, U^+_{n\nu}) \end{aligned} \qquad (42)$$

The first two lines in the action are the same as the action for free quarks on the lattice except for the factors $U_{n\mu}$ and $U^{+}_{n\mu}$. Hence the only modification of the free quark rules for links is to include the factors $U_{n\mu}$ and $U^{+}_{n\mu}$. The final term in the action is the term which generates gauge loop squares. The Feynman path integral involves integration over all classical values of the variables in the action, hence one performs a group integration for each $U_{n\mu}$ matrix. One uses invariant group integration in order to maintain local color invariance. It is these group integrations which ensure that only color singlet states propagate. One also defines integrations over the Fermi fields ψ_n and $\bar{\psi}_n$ but the only effect of these integrations is to force the quark links to combine into external lines or internal closed loops as already discussed.

The complete set of graphs for the meson propagator is the set of all connected, topologically distinct diagrams containing arbitrarily many quark links and gauge loop squares. All quark links and all sides of gauge loop squares must be contained in contractions.

When K_u, K_s, and $1/2g_0^2$ are small (g_0 is large) the space time diagrams give the same static picture described in the first lecture. This will be illustrated briefly. The diagrams which give the π meson pole for small K^2 and $1/2g_0^2$ are shown in Fig. 10. They sum up to give

$$D_\pi(p) = 6 + \frac{6}{1-4K^2e^{Ea}} \tag{43}$$

Hence the π meson mass in this approximation is

$$m_\pi = \frac{1}{a}\ln(1/4K_u^2) = \frac{2}{a}\ln(1/2K_u) \tag{44}$$

This means $\frac{1}{a}\ln(1/2K_u)$ is the non-strange quark mass for small K_u. Similarly, the diagrams which give propagation of an excited meson state with one string bit are shown in Fig. 11. For these diagrams there is a factor $4K_u^2\ (1/2g_0^2)\ e^{Ea}$ for each time step. Thus for the sum of the diagrams to diverge one must have

$$E = \frac{1}{a}\ln\ (1/4K_u^2) + \frac{1}{a}\ln\ (2g_0^2) \qquad (45)$$

Hence the string bit mass is $\frac{1}{a}\ln\ (2g_0^2)$. With these formulae for quark and string bit masses, the masses of states in the static limit are the sum of constituent quark and string bit masses.

As K_u, K_s and $1/2g_0^2$ increase away from the static limit, other diagrams become important. Fortunately, except for quite small g_0 a simple class of diagrams dominates at least for qualitative calculations. The dominant graphs for the meson propagator are the graphs shown in Fig. 10 but generalized so that different segments (a segment is a contracted pair of links) can lie along different axes (spacelike or timelike). In other words, the graphs are the same as for the free scalar meson discussed in Lecture II except that the single free meson link is replaced by a pair of contracted quark links. The dominant graphs for the baryon propagator are shown in Fig. 12. They are again similar to free field diagrams. For the meson propagator, the dominance of these graphs was established by a computer calculation of all graphs containing up to 5 squares and up to 8 quark links, although the calculation of the graph with one square already shows the unimportance of graphs with squares. The effect making the graphs of Fig. 10 or 12 dominant is the large dimensionality of space and the corresponding small values of K_u and K_s. Because K_u and K_s are small (no larger than 1/4: see later) one must have many ways of locating each quark link on the lattice for a graph to be important. Simple graphs with closed loops do not have the necessary flexibility -- they have a large power of K_u or K_s without a sufficiently large numerical factor in front. This was shown explicitly for the example given earlier.

More complex diagrams, say with 16 quark links and of order 10 squares, become important when g_0 is small enough. In this case a straight summation of diagrams is inadequate. Better methods of calculating are under study.

The results of computing the dominant graphs (for an example, see later) is the following:

1) The π and the ρ masses can be fitted to experiment. This determines K_u to be about 1/4 and the lattice spacing a to be about $(1\ \text{GeV})^{-1}$. (Strange quarks and strange particles were not studied.)

2) The nucleon and the N^* have a mass close to 1700 MeV/c^2 with a mass difference about 30 MeV/c^2.

3) The ω is close to the ρ and the η is close to the π.

The η is above the π ; however, the splitting is of order K^{16}. The reason for this is that the diagrams which distinguish η from π are the kind shown in Fig. 7b where there is a separate quark loop for each external point of the η-meson propagator. Each quark loop contains an external γ_5. For the loop trace not to vanish the internal links in the loop must contribute another γ_5. This is possible only if all four directions on the lattice are present in the loop. In order to close the loop every link must be accompanied by a reverse link, so there must be 8 links minimum in each quark loop. This means the graph is of order K^{16}.

The calculation for the π and ρ mesons is as follows. There is a spin splitting between the π and ρ because spacelike propagation of quarks involves spacelike γ matrices. The dominant diagrams can be summed by writing an equation very similar to the equation for the free scalar propagator of the previous lecture. However, the equation is now a matrix equation. For simplicity we put the spatial momentum equal to zero. The last segment of a diagram now contains two quark links, one entering and one leaving the external point n (Fig. 13).

Assume the last segment runs between $n-\hat{\mu}$ and n. Then associated with this segment there is a piece of a quark loop trace: this piece is (for the π-η case)

$$K_u^2 (1 - \gamma_\mu)\gamma_5 (1 + \gamma_\mu)$$

This γ matrix product is inside a full trace containing other γ matrices from other links. Also associated with the last segment is a color contraction giving in particular a factor 1/3. There is also a color trace for the site n giving a factor 3. The factor 3 and 1/3 cancel each other. We want to relate the diagram with the last segment present to the diagrams without this last segment. We will do this in momentum space. Then the diagram to the site $n-\hat{\mu}$ is multiplied by $e^{-ip\cdot(n-\hat{\mu})a}$, the diagram to n by $e^{-ip\cdot na}$. So there is a factor $e^{-ip_\mu a}$ for the last segment. For the case $\vec{p} = 0$, $p_o = + iE$ this means a factor 1 for $\mu = 1, 2$, or 3 and e^{Ea} for $\mu = 0$. If the segment runs from $n + \hat{\mu}$ to n rather than $n - \hat{\mu}$ to n, the factor is 1 or e^{-Ea} respectively.

One can now sum over all possible directions of the last segment. The result is that there is an insertion into the γ matrix trace of the form

$$K_u^2 (1 - \gamma_o)\, \gamma_5\, (1 + \gamma_o)\, e^{Ea} + K_u^2 (1 + \gamma_o)\, \gamma_5\, (1 - \gamma_o)\, e^{-Ea}$$
$$+ K_u^2 \sum_{\mu=1}^{3} [(1 - \gamma_\mu)\gamma_5 (1 + \gamma_\mu) + (1 + \gamma_\mu)\, \gamma_5\, (1 - \gamma_\mu)]$$

This reduces to

$$\{ 2K_u^2 (e^{Ea} + e^{-Ea}) + 12K_u^2 \}\, \gamma_5 + 2K_u^2 (e^{Ea} - e^{-Ea})\, \gamma_5\, \gamma_o$$

Hence we must define a propagator with $\gamma_5\, \gamma_o$ at the site n in place of γ_5. Let D be propagator with γ_5, and let D_{50} be the propagator with $\gamma_5\, \gamma_o$. The full recursion equations come out to be

$$D(p) = 12 + \{ 2K_u^2 (e^{Ea} + e^{-Ea}) + 12K_u^2 \} D(p)$$

$$+ 2K_u^2 (e^{Ea} - e^{-Ea}) D_{50}(p) \tag{46}$$

$$D_{50}(p) = 2K_u^2 (e^{Ea} - e^{-Ea}) D(p) + 2K_u^2 (e^{Ea} + e^{-Ea}) D_{50}(p) \tag{47}$$

The equations for the vector propagator involve a propagator with $\gamma_3 \cdot \gamma_0$ at the site n as well as the propagator with γ_3 at site N. Call these propagators D_V and D_{V0} (for γ_3 and $\gamma_3\, \gamma_o$ respectively). Then

$$D_V(p) = 12 + \{2K_u^2 (e^{Ea} + e^{-Ea}) + 8K_u^2 \} D_V(p)$$

$$+ 2K_u^2 (e^{Ea} - e^{-Ea}) D_{V0}(p) \tag{48}$$

$$D_{V0}(p) = 2K_u^2 (e^{Ea} + e^{-Ea}) D_V(p) + \{2K_u^2 (e^{Ea} + e^{-Ea}) + 4K_u^2 \} D_{V0}(p) \tag{49}$$

The pseudoscalar propagator has a pole whenever

$$\{1 - 2K_u^2 (e^{Ea} + e^{-Ea}) - 12K_u^2 \} \quad \{1 - 2K_u^2 (e^{Ea} + e^{-Ea})\} = 4K_u^4 (e^{Ea} - e^{-Ea})^2 \tag{50}$$

The vector propagator has a pole whenever

$$\{1 - 2K_u^2(e^{Ea} + e^{-Ea}) - 8K_u^2 \} \quad \{1 - 2K_u^2 (e^{Ea} + e^{-Ea}) - 4K_u^2 \}$$

$$= 4K_u^4 (e^{Ea} - e^{-Ea})^2 \tag{51}$$

These equations reduce to

$$e^{Ea} + e^{-Ea} = \frac{1 - 12K_u^2 + 16K_u^4}{4K_u^2 - 24K_u^4} \quad \text{(pseudoscalar case)} \tag{52}$$

$$e^{Ea} + e^{-Ea} = \frac{1 - 12K_u^2 + 48K_u^4}{4K_u^2 - 24K_u^4} \quad \text{(vector case)} \tag{53}$$

When $K_u = 1/4$, for example, the pseudoscalar pole occurs at $E = 0$ while the vector propagator has a pole at $2 \cosh(Ea) = 2.8$ giving $Ea = .87$ about. The difference between the vector and pseudoscalar equations originates in the difference between the products $(1 \pm \gamma_3)\, \gamma_5\, (1 \mp \gamma_3)$ and $(1 \pm \gamma_3)\, \gamma_3\, (1 \mp \gamma_3)$, the latter being 0.

The final topic of these lectures is the continuum limit of the lattice theory. The continuum limit is a limit in which $g_0 \to 0$, K_u and $K_s \to 1/8$, and $a \to 0$. One must specify in detail how g_0 approaches 0 and how K_u and K_s approach 1/8; this will not be discussed here. In this limit the theory can be partially solved by an expansion in g_0. Unfortunately, it is technically complicated to set up the weak coupling expansion, and so this will not be done in these lectures (see Appendix A). The result of the weak coupling analysis is that for small g_0 the lattice gauge theory becomes the continuum gauge theory apart from a cutoff at high momenta caused by the lattice. In particular the string operators $U_{n\mu}$ are close to 1 for small g_0 and can be approximated by

$$U_{n\mu} \simeq 1 + ig_0\, a \sum_b A_{n\mu}^b\, T^b \tag{54}$$

(see Eq. 5). When this approximation is used consistently (see Appendix A) one obtains a lattice approximation to the continuum Yang-Mills-type action, starting from the lattice action of Eq. (42). One must then apply convention renormalization theory to this action. Assuming that the usual results of renormalization theory for the continuum color gauge theory [3] apply to the lattice type of cutoff, one concludes that the theory is asymptotically free and therefore that renormalization requires that g_0 goes to 0 proportional to $1/\ln a$ as $a \to 0$. Furthermore, conventional renormalization theory suggests that the renormalized theory is fully Lorentz invariant for $a \to 0$ even though the lattice cutoff is noncovariant. The reason for this is that the effects of a sufficiently large cutoff momentum (covariant or noncovariant) can be removed by counter terms. For the cutoff to cause noncovariant effects (at momenta well below the cutoff momentum) there must be noncovariant counter terms. However, one finds that there are no noncovariant counter terms.

Because a full analysis including renormalization for small g_0 is complicated, it is not at all certain yet that the conclusions given above are correct. Hence the conclusions relating the lattice theory to the continuum theory are only tentative.

In these lecture notes only the simple aspects of the continuum limit will be explained. The simple aspects are the following.

1) The existence of free quark behavior at small distances for small g_0. This will be explained with special reference to string fluctuations for small g_0. This analysis is important because in the continuum string models (dual resonance models) short distance string fluctuations are disastrous.

2) The problem of local gauge symmetry in quantum electrodynamics.

3) Properties of the expansion in K of the meson propagator for small g_0. It is easy to show that one must go to very high orders in K (somewhere around K^{16} to K^{30}) to obtain the dominant terms giving the meson pole (or whatever else may occur) for small g_0. This result will help explain the unimportance of simple graphs containing gauge loop squares, since simple graphs are not 16^{th} order or higher in K.

To give a preliminary understanding of how the continuum limit might work, a simplified version of the diagrammatic rules will be discussed. In the simplified rules

1) one omits internal quark loops

2) one omits the graphs of Fig. 7b

3) for each configuration of the quark lines one constructs only one surface of squares to complete the graph, namely the surface (or one of the surfaces) with minimal area.

4) the coefficient for each square will be called J instead of $1/2g_0^2$ (for technical reasons :see Appendix C).

5) color is neglected. The contraction of two oppositely directed links is 1 and all other contractions are 0.

Free quark behavior is easily obtained in the simplified rules: one puts $J = 1$. In this case the number of squares in a diagram does not affect the amplitude for a diagram. Thus the meson propagator is the sum of all graphs containing two quark lines with no restriction on where the lines are located, except that they begin and end at the external points 0 and n. With $J = 1$, the rules for diagrams are the same as for free quarks. Hence the sum of these graphs is the composite $\overline{\psi}\,\gamma_5\psi$ propagator for the free quark theory. The quarks are relativistic if $K \simeq \frac{1}{8}$. In other words, the continuum relativistic free quark limit is at $J = 1$ and $K_u = K_s = \frac{1}{8}$. For these values the quarks are free at all distances. We want the quarks to be free only for distances small compared to 10^{-13}cm. Consider a small but finite lattice spacing a, say 10^{-16}cm. Then we want the quarks to be free only for distances less than 1000a, that is, distances less than 1000 lattice spacings. This is accomplished by letting J be slightly less than 1. Suppose $J = 1 - 10^{-6}$, for example. Then for small size quark loops, namely loops containing much less than 10^6 squares, the J factors can be replaced by 1 and the quarks are free. In contrast, a loop containing 10^6 squares has a factor J^{10^6} which is 1/e; this factor is no longer 1. A loop containing many more than 10^6 squares is very much suppressed. A simple loop containing 10^6 squares has a size about 10^3 lattice spacings, i.e., a size of about 10^{-13}cm when $a = 10^{-16}$cm. Thus $J = 1 - 10^{-6}$ is about the right value to give free quark behavior only for distances less than 10^{-13}cm.

Thus the continuum limit is a limit in which $J \to 1$ and $K \to 1/8$ as $a \to 0$, in such a way that J can be replaced by 1 for short distance calculations but the difference 1-J is important for distances of order 10^{-13}cm regardless of how small a is.

The above discussion is drastically oversimplified but should help to understand a more realistic analysis.

Now consider the problem of obtaining free quark behavior in the full theory. The basic problem we want to discuss is the following. For a given quark loop there are very many distinct surfaces of gauge loop squares that can be constructed. These surfaces represent strings propagating in time; the many different surfaces define propagation of many different locations for the string. In other words, the sum over surfaces is a sum over string fluctuations in space and time. In continuum string models the string fluctuations have disastrous effects on form factors and other properties of currents; in particular one expects short wave length string fluctuations to completely spoil Bjorken scaling because quarks can exchange energy instantaneously with these fluctuations [7]. In the lattice theory the short wavelength fluctuations do <u>not</u> spoil Bjorken scaling provided the continuum limit of g_0 is 0. Since $1/2g_0^2$ is the expansion parameter of the lattice theory, letting $g_0 = 0$ would seem to maximize the role of string fluctuations, but this is not true. Instead, the fluctuations are killed in the limit $g_0 \to 0$. Before showing this for the full quark-string theory, a simple example will be discussed which shows the basic idea. Consider a single nearest neighbor link. Consider uncolored string bits at this link. "Uncolored" means that the string bit operator has the form $e^{i\phi}$ where ϕ is a simple phase. This means one way of representing string bit states is as functions $u(\phi)$. In this representation the state with no string bits is $u(\phi) = 1$. The state with one string bit is $u(\phi) = e^{i\phi}$. The n string bit wave function is $e^{in\phi}$ (if n is negative then there are $|n|$ string bits with the reverse direction). There are also linear combinations of these states. For example, $u(\phi) = e^{-\cos\phi}$ is an acceptable wave function which is a linear combination of states with any number of string bits.

Consider now the wave function $e^{-(\cos\phi)/g_0^2} = u(\phi)$. If g_0 is large then the wave function has a simple expansion in $1/2g_0^2$:

$$u(\phi) = 1 - \frac{1}{2g_0^2} \left(e^{i\phi} + e^{-i\phi}\right) + 0\left[\left(\frac{1}{2g_0^2}\right)^2\right] \tag{55}$$

When g_0^2 is large, therefore, mostly there is no string bit at the site, occasionally there is one. This is analogous to the case of large g_0 in the full theory where the dominant states contain only a few string bits at most. From the string viewpoint, this is the small fluctuation situation.

Consider next the case of small g_0. According to the expansion in $1/2g_0^2$, many string bits are present and the number of string bits has a large range. In other words, there are large fluctuations in the number of strings present. However, when the wave function is viewed as a function of ϕ, the fluctuations in ϕ are small when g_0 is small. To be precise, the wave function is negligible for $\phi^2 >> g_0^2$. We shall see that it is the analogue of ϕ fluctuations (rather than string fluctuations) that destroys free quark behavior; as long as the ϕ fluctuations are small the fluctuations in the number of string bits does not matter. In fact, small fluctuations in ϕ necessarily imply large fluctuations in the number of string bits, as can easily be seen.

The full analysis for small g_0 will be done using a partial summation of the quark-string diagrams. Namely, we shall use an expansion in K but not in $1/2g_0^2$. Consider in particular a contribution shown in Fig. 14. This consists of a quark-antiquark loop of 6 links. It contributes to the meson propagator. The amplitude corresponding to this graph is

$$A = \frac{K^6}{Z_U} \operatorname{Tr} \gamma_5 (1 - \gamma_\mu)(1 - \gamma_\mu)(1 - \gamma_\nu)\gamma_5 (1 + \gamma_\mu)(1 + \gamma_\mu)(1 + \gamma_\nu) \prod_{n\pi} \int U_{n\pi}$$

$$\times \operatorname{Tr} (U_{0,\mu} U_{\hat{\mu},\mu} U_{2\hat{\mu},\nu} U^{+}_{\mu+\hat{\nu},\mu} U^{+}_{\hat{\nu},\mu} U^{+}_{0,\nu})$$

$$\times \exp\left\{\frac{1}{2g_0^2} \sum_n \sum_\pi \sum_\sigma \operatorname{Tr} (U_{n\pi} U_{n+\hat{\pi},\sigma} U^{+}_{n+\hat{\sigma},\pi} U^{+}_{n\sigma})\right\} \tag{56}$$

where Z_U is the gauge field integral over the final exponential alone. This amplitude comes from expanding the complete Feynman path integral for the meson propagator using the action of Eq. (42) in powers of K but not $1/2g_0^2$ and picking out a simple term of order K^6. The denominator Z_U removes disconnected diagrams.

The limit $g_0 \to 0$ is a classical limit. The exponent in Eq. (56) has a large coefficient when g_0 is small. Therefore, the gauge field integrations are dominated by values of $U_{n\pi}$ which maximize the exponent. It is easily seen that the maximum of the exponent is obtained when all the matrices $U_{n\pi}$ are 1. The reason for this is that the product $U = U_{n\pi} U_{n+\hat{\pi},\sigma} U^{+}_{n+\hat{\sigma},\pi} U^{+}_{n,\sigma}$ is itself a unitary 3 x 3 matrix. The diagonal elements of a unitary matrix cannot be greater than 1. Hence the trace of the unitary matrix U has its maximum value (namely, 3) when U is the unit matrix. When all the matrices $U_{n\pi}$ are 1, then all the products like U appearing in the exponent are also 1.

The maximum of the exponent is not unique. Any gauge transformation on the matrices $U_{n\pi}$ leaves the exponent unchanged; hence the exponent stays at its maximum for any gauge transformation applied to the initial choice $U_{n\pi} = 1$.

After such a transformation, $U_{n\pi}$ has the form $\Phi_n \Phi_{n+\hat{\pi}}^{+}$ where the matrices Φ_n are (for each n) arbitrary unitary unimodular matrices.

Apart from the freedom to make gauge transformations, the choice $U_{n\pi} = 1$ is the unique choice which maximizes the exponent in Eq. (56). In other words, there is no way of choosing the matrices $U_{n\pi}$ so that the trace for every elementary square is 3 and yet one cannot convert all the matrices $U_{n\pi}$ into 1 by a gauge transformation. (This is not true on a finite periodic lattice. The most general configuration (apart from configurations generated by gauge transformations) maximizing the exponent on a periodic lattice is shown in Fig. 15. The unitary matrices W_μ in Fig. 15 must commute. These configurations do not change any of the results obtained below).

The trace of gauge field matrices outside the exponent is also the trace of the unit matrix when all the $U_{n\pi}$'s are 1. This trace is gauge invariant. Hence the trace outside the exponent is 3 whenever the exponent itself has its maximum value.

Once the trace of gauge field matrices outside the exponential has been replaced by the constant 3, the gauge field integrations in the numerator of A are identical to the integrations defining Z_u, and therefore

$$A = 3K^6 \,\mathrm{Tr}\, \gamma_5(1-\gamma_\mu)(1-\gamma_\mu)(1-\gamma_\nu)\gamma_5(1+\gamma_\mu)(1+\gamma_\mu)(1+\gamma_\nu) \tag{57}$$

This amplitude makes no reference to the gauge field. It is a term in the expansion of the composite propagator $\langle\Omega|\, T\, \overline{\psi}_n \gamma_5 \psi_n \overline{\psi}_0 \gamma_5 \psi_0 |\Omega\rangle$ in a free quark theory with action

$$A = -\sum_n \psi_n \psi_n + K \sum_n \sum_\mu \left\{ \psi_n (1-\gamma_\mu)\psi_{n+\hat{\mu}} + \psi_{n+\hat{\mu}}(1+\gamma_\mu)\psi_n \right\} \tag{58}$$

There was nothing special about the particular term discussed above. The general result is that when $g_0 \to 0$, the π meson propagator becomes three times the composite propagator of a free quark model on a lattice with the action A being given by Eq. (58).

The quarks behave like free quarks when $g_0 \to 0$ because the fluctuations in the matrices $U_{n\pi}$ are suppressed by the nearly classical action. In particular the trace of matrices $U_{n\pi}$ associated with a quark loop has the value 3 and does not vary. In the string representation there appear to be very large fluctuations because the string representation is obtained from an expansion in $1/g_0^2$ and this expansion has many terms when g_0 is small. However, it is the fluctuations in $U_{n\pi}$ rather than the number of string configurations that determines whether the quarks are free.

If g_0 is small, but not zero, one still expects small quark loops to behave as if the quark loops were free. What about large loops? For large loops one has a trace of very many gauge field matrices (the trace around the loop). If g_0 is small but nonzero then each matrix $U_{n\pi}$ must be close to 1 but not 1 precisely. The product of a sufficiently large number of U's can now be quite far away from 1. Thus one again expects sufficiently large quark loops to behave differently from free quark loops.

There is a serious problem with the analysis of this lecture. For quantum electrodynamics, the analysis fails. The situation will be summarized without details. Quantum electrodynamics has a local gauge symmetry. It can be formulated on the lattice in the same form as discussed here for the quark theory: one simply replaces quark fields by electron fields, and the string variables are simple phase factors instead of 3 x 3 matrices. The quark-gauge field coupling constant g_0 is replaced by the bare electron charge e_0. However, electrons are not confined. This does not contradict the string picture; it means the string mass must be zero. If the string mass is zero then an electron and a positron connected by a string can separate infinitely far without using an infinite mass of string.

However, if the strings in electrodynamics have zero mass then one might fear that the strings of the quark color gauge theory also have zero mass.

There is no chance of the string mass being zero if g_0 or e_0 is large enough (in the lattice theory). When g_0 and e_0 are large the string mass can be calculated explicitly from the expansion in $1/g_0$ or $1/e_0$; the string mass is not zero and electrons and quarks are confined. Unfortunately, the interesting case is the case of small g_0. In the case of electrodynamics, small e_0 is precisely the range of e_0 that leads to a string mass equal to 0.

The hope for confinement for the quark theory for small g_0 lies in the different consequences of renormalization for the quark theory as opposed to quantum electrodynamics. The conventional weak coupling renormalization theory for quantum electrodynamics shows that electrons at large distances behave <u>more</u> like free electrons than at small distances. This contradicts the intuitive analysis of this paper. That is, for electrons, large loops are <u>less</u> affected by the gauge field fluctuations than small loops. This fact prevents any kind of confinement occurring. For the quark color gauge theory, however, the recent studies of renormalization show that the quarks do behave less like free quarks at large distances than at small distances, in agreement with the analysis of this lecture. Unfortunately, these studies cannot be carried out for distances of order 10^{-13}cm or larger, due to infrared divergences. Thus it is not known whether there is confinement in the color quark gauge theory.

Finally a discussion of the important diagrams for small g_0 will be given. We shall show that very large orders in K are important, of order K^{16} to K^{30}. Diagrams which are high order in K will tend also to have many gauge loop squares. At present these diagrams are too complex to be computed and summed.

For large g_o the π meson mass is 0 for $K \simeq 1/4$, as shown earlier. This means the π meson propagator at 0 momentum has a singularity at $K \simeq 1/4$: the sum of the diagrams for the propagator is divergent for $K = 1/4$.

When g_o is 0 the sum of diagrams for the meson propagator diverges for $K > 1/8$ due to a branch point at $K = 1/8$. The reason for this is that for $g_o = 0$ the meson propagator becomes the composite meson propagator of the free quark theory. Free quarks have mass zero for $K = 1/8$. Hence the π meson propagator has a quark-antiquark threshold singularity at zero mass for $K = 1/8$. The threshold singularity is very mild: it can be determined from the Feynman graph of Fig. 16. There are two quark propagators both behaving as $1/\not{p}$ when p is small and an integral $\int d^4p$. The singularity comes only from the region $p \sim$ quark mass; the quark mass is of order 1/8-K. The contribution of $p \sim (1/8-K)$ to the integral comes out to be $(1/8-K)^2 \ln(1/8-K)$.

The π meson propagator is finite for $g_o = 0$ and $K = 1/8$, but the expansion in K diverges for $K > 1/8$ due to the branch point.

For g_o very small but non-zero, the terms of not too high order in K will change very little, thus the series should still diverge for same value of K near 1/8. If there is confinement, the divergence should be due to a π meson pole rather than a branch cut. However, a consequence of the branch cut is that terms of intermediate order in K, for example K^2 to K^8, do not look at all like a geometric series giving a pole near $K = 1/8$. Instead, these terms are much smaller than a geometric series as one expects for a singularity like $(1/8-K)^2 \ln (1/8-K)$. To be specific, for $g_o = 0$ the series to order K^8 for the π meson propagator for zero momentum is

$$D(p) = 12 \{ 1 + .25(8K)^2 + .086\,(8K)^4 + .0332(8K)^6 + .0126(8K)^8 + \ldots \} \qquad (59)$$

Even increasing K a bit, say from .125 to .15, has little effect on the orders K^2, K^4, K^6 and K^8; these orders are still too small to appear to be part of a diverging series. For much higher orders in K the series is divergent for <u>any</u> K greater than 1/8, and the changes due to g_0 not being 0 can easily convert the high order terms into a geometric series.

I am grateful to theory groups of Lawrence Berkeley Laboratory and CERN for making computer time available for the calculations mentioned earlier.

Appendix A

Classical Continuum Limit of the Lattice Quark-String Theory

We will show in this Appendix that the lattice action A of Eq. (42) has a continuum limit, namely the continuum action of quarks coupled to a color gauge field. The continuum action will be in a Euclidean metric, since the lattice action is on a Euclidean lattice. To obtain the continuum limit one must make specific assumptions about the continuum limit of ψ_n and $U_{n\mu}$, namely

$$\psi_n \simeq \left(\frac{a^3}{2K}\right)^{1/2} \psi(na) \qquad \text{(for small a)} \tag{60}$$

where $\psi(x)$ is the continuum quark field and a is the lattice spacing, and

$$U_{n\mu} \simeq \exp\left\{ isag_o \sum_{b=1}^{8} A_\mu^b(na)\, T^b \right\} \tag{61}$$

where s is a numerical factor related to the color group, namely

$$\mathrm{Tr}(T^b T^c) = s^{-2}\,\delta_{bc} \tag{62}$$

and the $A_\mu^b(x)$ ($1 \leq b \leq 8$) are the octet of color gauge fields. The functions $\psi(x)$ and $A_\mu^b(x)$ are assumed to be continuous and differentiable functions. It will be assumed below that a is small enough so that $\psi(na + \hat{\mu}a)$ has a rapidly convergent Taylor series expansion about $\psi(na)$, and likewise for $A_\mu^b(na + \hat{\mu}a)$.

These assumptions are reasonable for a classical calculation where it is expected that a finite, continuous, and differentiable continuum form exists for $\psi(x)$ and $A_\mu^b(x)$. The assumptions are <u>not true</u> for quantum field theory: see later.

The action of Eq. (42) must now be computed to order a^4. The reason for this is that $a^4 \sum_n$ becomes a continuum integral $\int d^4x$ with $x = na$, for $a \to 0$.

For small a we also let K be near 1/8, namely (from Eq. (30))

$$K = \frac{1}{8+2ma} \tag{63}$$

The various terms of Eq. (42) will now be treated in turn. First,

$$-\bar{\psi}_n \psi_n \simeq \frac{-a^3}{2K} \bar{\psi}(x)\, \psi(x) \simeq -(4a^3 + ma^4)\bar{\psi}(x)\, \psi(x) \tag{64}$$

where x = na.

Next

$$K \bar{\psi}_n(1 - \gamma_\mu)\, U_{n\mu}\, \psi_{n+\hat{\mu}}$$
$$\simeq \frac{Ka^3}{2K}\bar{\psi}(x)\, (1 - \gamma_\mu) \left\{ 1 + isag_o\, A^b_\mu(x)\, T^b \right\} \left\{ \psi(x) + a\frac{\partial \psi(x)}{\partial x_\mu} \right\} \tag{65}$$

where a sum over b is assumed.

(The terms $U_{n\mu}$ and ψ (na + $\hat{\mu}$a) have been expanded to order a. The terms of order a in these expansions generate terms of order a^4 overall. Higher order terms in the expansions can be neglected since they only generate terms of overall order a^5 or higher.)

$$K \bar{\psi}_n\, (1 - \gamma_\mu)\, U_{n\mu}\, \psi_{n+\hat{\mu}} \simeq$$
$$\frac{a^3}{2}\, \bar{\psi}(x)\, (1 - \gamma_\mu)\, \psi(x) + \frac{a^4}{2}\, \bar{\psi}(x)\, (1 - \gamma_\mu) \frac{\partial \psi}{\partial x_\mu}(x)$$
$$+ \frac{ia^4}{2} sg_o\, \bar{\psi}(x)\, (1 - \gamma_\mu)\, A^b_\mu(x)\, T^b\, \psi(x) \tag{66}$$

It is convenient to rewrite the term with $\sum_n \bar{\psi}_{n+\hat{\mu}} \ldots \psi_n$ in Eq. (42) as $\sum_n \bar{\psi}_n\, (1 + \gamma_\mu)\, U^+_{n-\hat{\mu},\mu}\, \psi_{n-\hat{\mu}}$. Approximately (to order a),

$$(U_{n-\hat{\mu},\mu})^+ \simeq 1 - isag_o\, A^b_\mu(x)\, T^b \tag{67}$$

Hence

$$\overline{\psi}_n\ (1 + \gamma_\mu)\ U^+_{n-\hat{\mu},\mu}\ \psi_{n-\hat{\mu}} \simeq \frac{a^3}{2} \overline{\psi}(x)\ (1 + \gamma_\mu)\ \psi(x)$$

$$- \frac{a^4}{2} \overline{\psi}(x)\ (1 + \gamma_\mu) \frac{\partial \psi}{\partial x_\mu} (x) - \frac{ia^4}{2} sg_o\ \overline{\psi}(x)\ (1 + \gamma_\mu)\ A^b_\mu(x)\ \psi(x) \tag{68}$$

The sum of the terms involving ψ give (before summing over n)

$$-ma^4\ \overline{\psi}(x)\ \psi(x) - a^4\ \overline{\psi}(x)\gamma_\mu \frac{\partial \psi(x)}{\partial x_\mu} - ia^4 s\ g_o\ \overline{\psi}(x)\gamma_\mu\ A^b_\mu(x)\ T^b\ \psi(x) \tag{69}$$

All the terms of order a^3 cancel.

A series of approximations are used to compute the pure gauge field term. A direct expansion in a is not used because there are too many terms. This occurs because the trace in Eq. (42) is of order 1 and all terms to order a^4 must be computed.

First, write

$$U = U_{n\mu}\ U_{n+\hat{\mu},\nu}\ U_{n+\hat{\nu},\mu}^{\ +}\ U_{n\nu}^{+} \tag{70}$$

Since U is a unitary SU(3) matrix, one can write

$$U = e^{i \sum\limits_b F^b_{n\mu\nu} T^b} \tag{71}$$

For the SU(3) group the matrices T^b are all traceless; hence one can write

$$\mathrm{Tr}U = \mathrm{Tr}\ 1 - \frac{1}{2} \sum_b \sum_c \mathrm{Tr}\ F_{n\mu\nu}^{\ b}\ F_{n\mu\nu}^{\ c}\ \ T^b T^c + \text{higher orders} \tag{72}$$

We will show that $F_{n\mu\nu}^{\ b}$ is of order a^2, so terms of higher order than F^2 are not needed.

Thus (using Eq. (62))

$$\mathrm{Tr}\ U = 3 - \frac{1}{2s^2} \sum_b (\ F_{n\mu\nu}^{\ b}\ )^2 \tag{73}$$

The constant term (namely, 3) is not of order a^4, but since it is a constant it can be removed from the action without changing the physical consequences of the action.

The matrix U has to be computed only to order a^2 in order to determine $F_{n\mu\nu}^{\ b}$. The sequence of approximations determining U are as follows. First compute

$$U_{n+\hat{\mu},\nu} - U_{n\nu} \simeq isg_o\, a \sum_b \left\{ A_\nu^b(na + \hat{\mu}a) - A_\nu^b(na) \right\} T^b$$
$$\simeq isg_o\, a^2 \sum_b \frac{\partial A_\nu^b}{\partial x_\mu} T^b \tag{74}$$

Note that $U_{n+\hat{\mu},\nu}$ and $U_{n\nu}$ were only expanded to order a although the final result is of order a^2. The reason for this is that the terms of order a^2 in $U_{n+\hat{\mu},\nu}$ and $U_{n\nu}$ separately combine to form a term of order a^3 in the difference. A second auxiliary computation that will be needed is for the commutator

$$[U_{n+\hat{\mu},\nu}, U_{n+\hat{\nu},\mu}^{+}] \simeq (isg_o a)\,(-\,isg_o a) \left[\sum_b A_\nu^b(na + \hat{\mu}a)\, T^b, \sum_c A_\mu^c(na + \hat{\nu}a)\, T^c \right] \tag{75}$$

This is correct to order a^2.

The SU(3) commutation rules are

$$[T^b, T^c] = i \sum_d f^{bcd}\, T^d \tag{76}$$

Thus

$$\left[U_{n+\hat{\mu},\nu} \, , \, U_{n+\hat{\nu},\mu}^{+} \right] = is^2 g_o^2 a^2 \sum_b \sum_c \sum_d f^{bcd}\, A_\nu^b(x)\, A_\mu^c(x)\, T^d \tag{77}$$

Now U can be written

$$\begin{aligned} U = {} & U_{n\mu}\, [U_{n+\hat{\mu},\nu} \, , \, U_{n+\hat{\nu},\mu}^{+}]\, U_{n\nu}^{+} \\ & + U_{n\mu}\, (U_{n+\hat{\nu},\mu}^{+} - U_{n\mu}^{+})\, U_{n+\hat{\mu},\nu}\, U_{n\nu}^{+} \\ & + U_{n\mu}\, U_{n\mu}^{+}\, (U_{n+\hat{\mu},\nu} - U_{n\nu})\, U_{n\nu}^{+} \\ & + U_{n\mu}\, U_{n\mu}^{+}\, U_{n\nu}\, U_{n\nu}^{+} \end{aligned} \tag{78}$$

From unitarity, $U_{n\mu} U_{n\mu}^{+}$ and $U_{n\nu} U_{n\nu}^{+}$ are 1. The first three terms in the above equation are all of order a^2, and can be computed to order a^2 from the auxiliary calculations. The result determines $F_{n\mu\nu}{}^{b}$ to be

$$F_{n\mu\nu}{}^{b} = sg_o \ a^2 \left\{ \frac{\partial A_\nu^b(x)}{\partial x_\mu} - \frac{\partial A_\mu^b(x)}{\partial x_\nu} \right\}$$

$$+ s^2 g_o^2 \ a^2 \sum_c \sum_d f^{cdb} A_\nu^c(x) A_\mu^d(x)$$

$$= sg_o \ a^2 \ F_{\mu\nu}{}^{b}(x) \tag{79}$$

where

$$F_{\mu\nu}{}^{b}(x) = \frac{\partial A_\nu(x)}{\partial x_\mu} - \frac{\partial A_\mu(x)}{\partial x_\nu} + sg_o \sum_d f^{bcd} A_\nu^c(x) A_\mu^d(x) \tag{80}$$

(using the symmetry of the group factors: $f^{bcd} = f^{cdb}$.)

Thus, after removing a constant, the total action in the continuum limit is

$$A = \int d^4x \ \{ - m \overline{\psi}(x) \psi(x) - \overline{\psi}(x) \gamma_\mu (\nabla_\mu + isg_o A_\mu^b(x) T^b) \psi(x)$$

$$- \frac{1}{4} \sum_b [F_{\mu\nu}{}^{b}(x)]^2 \} \tag{81}$$

This is the continuum Yang-Mills type action for the colored quark gauge theory with a coupling constant sg_o.

The above calculation is valid only for classical physics. In classical physics the principle of least action ensures that the assumptions made above about the continuum limit are correct, namely it leads to equations of motion whose solutions are continuous and differentiable. In quantum mechanics, however, there can be large quantum fluctuations of $\psi(x)$ and $A_\nu^b(x)$ in a small volume. Consider first the case that the fluctuation in $\psi(x)$ and $A_\nu^b(x)$ is of a fixed magnitude but confined to a region of roughly size a. The change in the action due to such a fluctuation is of order a^2 at most. The reason for this is that the action involves a volume factor which is a^4 for the fluctuation. There are also inverse powers of a due to derivatives: $F_{\mu\nu}^{b}(x)$ and $\frac{\partial\psi(x)}{\partial x_\mu}$ are both of order 1/a because of the short wavelength of the fluctuation. However, the largest term is $(F_{\mu\nu}^{b})^2$ which is of order a^{-2}; multiplied by a^4, one obtains a^2. Now a quantum fluctuation is limited in size only by the requirement that the change in the action be of order 1 or less. The reason for this is that quantum mechanics involves a path integral over e^A (in the Euclidean metric there is no factor i in the exponent). The integration will only be cutoff when A itself becomes large (and negative). Since a bounded fluctuation causes only a miniscule change in A, it follows that the fluctuation will be large: namely, a fluctuation in ψ can be of order $a^{-3/2}$ while a fluctuation in $A_\mu^{b}(x)$ can be of order a^{-1}, both for small a. Since these are fluctuations with wavelength a, they cause violations of all the assumptions we made about ψ_n and $U_{n\mu}$ for small a, except one. If g_o is small, then $U_{n\mu}$ will be close to 1 despite fluctuations in $A_\mu^b(x)$ of order 1/a. However, even this assumption is false due to the symmetry of A to gauge transformations.

One can force $U_{n\mu}$ to be close to 1 by adding a gauge-fixing term to the quantum action. This can be done by Faddeev-Popov methods [8] and will not be discussed here. But beyond this one cannot go, quantum mechanically. One must study the lattice theory for small g_o without taking the continuum limit, due to the presence of short wavelength fluctuations. The lattice provides a finite ultraviolet cutoff for the theory. The problem caused by short wavelength fluctuations is the problem of cutoff dependence, i.e., the problem of renormalization. This will not be discussed here.

Appendix B

Higher Order Gauge Field Contractions

To complete the rules for diagrams one must specify higher order contractions. These have been computed for products of up to seven U's or U^+'s. However, a complication arises. In order to justify the elimination of disconnected graphs one must define "connected" and "disconnected" pieces of a gauge field integration. For background on this problem see Jasnow and Wortis [9]. The connected part will be called the "contraction", and denoted $\langle \ldots \rangle$.

For example, one must define a contraction of two U's and two U^+'s:

$$\langle U_{ia}\, U_{jb}\, U^+_{ck}\, U^+_{dl} \rangle = \int_U U_{ia}\, U_{jb}\, U^+_{ck}\, U^+_{dl} - \langle U_{ia}\, U^+_{ck} \rangle \langle U_{jb}\, U^+_{dl} \rangle - \langle U_{ia}\, U^+_{dl} \rangle \langle U_{jb}\, U^+_{ck} \rangle \tag{82}$$

where

$$\langle U_{ia}\, U^+_{ck} \rangle = \int_U U_{ia}\, U^+_{ck} = \frac{1}{3}\, \delta_{ik}\, \delta_{ac} \tag{83}$$

In general the disconnected piece of a gauge field integral consists of all possible products of contractions of subproducts of U's and U^+'s from the original integrand.

The results for the contractions of up to seven U's are shown in Figs. 17 - 20.

Appendix C

Definition of J

The simplified rules for diagrams discussed in the middle of Lecture III involve a parameter J with the range $0 \leq J \leq 1$, in place of the original parameter g_o with a range $0 \leq g_o \leq \infty$. A simple diagrammatic summation procedure leads to a natural definition for J such that $0 \leq J \leq 1$. The summation procedure involves summing all diagrams with a given planar structure but with arbitrarily many gauge loop squares at any location in the structure. A planar structure is a surface containing no cubes or, i.e., a surface with no totally enclosed volume. For example, Fig. 21 shows a planar diagram. The simplification which one obtains for a planar diagram is that one can make a change of variables from individual string variables such as $U_{o\mu}$ to square variables such as $U = U_{o\mu} U_{\hat{\mu},\pi} U^{+}_{\hat{\pi},\mu} U^{+}_{o\pi}$, and all the square variables corresponding to squares in the planar structure are independent. For a non-planar structure, such as a cube, the square variables are not independent. Since the action contains a sum of terms, one for each square, one can write independent integrations for each square in the planar structure.

A further simplification results if one starts with a planar diagram containing only two-link contractions (as in Fig. 21) and then adds all diagrams with extra superposed squares. The simplification is that the integration for each square has the specific form

$$J = \frac{1}{3} \frac{1}{Z} \int_U (\mathrm{Tr}\ U) \exp\left\{ \frac{1}{2g_o^2} (\mathrm{Tr}\ U + \mathrm{Tr}\ U^+) \right\} \tag{84}$$

where

$$Z = \int_U \exp\left\{ \frac{1}{2g_o^2} (\mathrm{Tr}\ U + \mathrm{Tr}\ U^+) \right\} \tag{85}$$

The factor 1/3 in the definition of J is for normalization. Each square in the planar structure contributes a factor of this form. Since $|\mathrm{Tr}\,U| \leq 3$, J cannot be larger than 1. For very large g_0, one finds

$$J \simeq \frac{1}{3}\,\frac{1}{2g_0^{\,2}} \tag{86}$$

The simple planar diagrams are computed by calculating using the quark rules as before and replacing the original rules for squares and contractions by the rule that there is a factor J for each square (before adding any superposed squares) and a factor 3 for each quark loop. These rules automatically sum the diagrams with superposed squares. Note that these rules ensure the unimportance of planar diagrams of not too large size since these diagrams are equivalent to free quark diagrams for J = 1 and become smaller for J < 1. As shown at the end of Lecture IV the low order free quark diagrams (of order K^2 to K^8, at least) are very small.

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Figure Captions

Fig. 1 A plane section of a cubic lattice.

Fig. 2 A string bit on the lattice. The lattice sites n and $n+\hat{\mu}$ are the endpoints of the string bit. Unit vectors on the lattice are denoted $\hat{\mu}$ (μ = 1, 2, or 3).

Fig. 3 A gauge invariant state of four string bits.

Fig. 4 Gauge invariant states of two or three superposed string bits. Both are absent in the lattice theory due to nonlinear constraints on the string bit operators.

Fig. 5 Gauge invariant quark transition. Initially (5a) the quark is on site n. After the transition (5b) the quark is at site $n+\hat{\mu}$ and connected to a string bit between sites n and $n+\hat{\mu}$. The color end of the string is at site n. The quark and the anticolor string end form a color singlet state.

Fig. 6 Dominant graphs for free meson propagator for small K. The contributions to D(p) from graphs a, b, and c are 1, $K\,e^{Ea}$ and K^2e^{2Ea} respectively.

Fig. 7 Two forms of quark lines for meson propagator diagrams.

Fig. 8 Graphical form of the simplest construction: (a) representation for an SU(3) group integration involving $(U_{n\mu})_{ij}\ (U_{n\mu}^{+})_{kl}$;(b) the representation of $\frac{1}{3}\,\delta_{il}\,\delta_{jk}$, the result of the integration.

Fig. 9 Example of a diagram. The external points are 0 and $\hat{\nu}$. The outer square represents the two quark lines between 0 and $n=\hat{\nu}$. The inner square is a gauge loop square.

Fig. 10 Dominant diagrams for the meson propagator near the static limit in the quark-string theory. The contributions to D(p) from graphs 10a, 10b, and 10c are 12, 24 $K_u^2 e^{Ea}$, 96 $K_u^4 e^{2Ea}$ respectively.

Fig. 11 Diagrams for the propagator for an excited state. The external operators in the propagator are $\overline{\psi}_n \gamma_5 U_{n\mu} \psi_{n+\hat{\mu}}$ and $\overline{\psi}_{\hat{\mu}} \gamma_5 U_{o\mu}^+ \psi_o$ (μ = 1, 2, or 3). The links marked e refer to the external string operators and γ_5 in the external operator. The remaining outer links are quark lines. The inside squares are gauge loop squares.

Fig. 12 Some dominant graphs for the baryon propagator (except for very large $1/2g_o^2$).

Fig. 13 Diagram for the meson propagator indicating final segment from $n-\hat{\mu}$ to n.

Fig. 14 Example of a quark diagram before including gauge loop squares.

Fig. 15 Configuration of $U_{n\pi}$'s maximizing exponent of Eq. (56) for a periodic lattice (shown for two dimensions). The matrices $U_{n\pi}$ are 1 or W_o or W_1, as shown: W_o and W_1 must be commuting SU(3) unitary matrices.

Fig. 16 Feynman diagram for composite meson propagator with momentum p. The propagators are quark propagators (Eq. (32)).

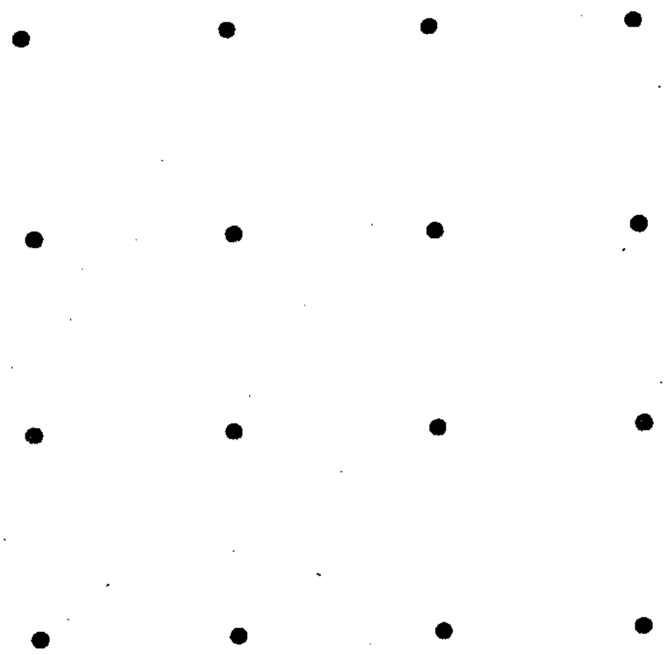
Fig. 17 Contraction $\langle U_{ia}\, U_{jb}\, U_{ck}^+\, U_{dl}^+ \rangle$. Each line on the right hand side denotes a Kronecker delta.

Fig. 18 Contraction $\langle U_{ia}\, U_{jb}\, U_{kc}\, U_{ld}\, U_{em}^+ \rangle$. The term indicated explicitly is $-\frac{1}{72}\, e(ijk)\, e(abc)\, \delta_{lm}\, \delta_{de}$.

Fig. 19 Contraction $\langle U\ U\ U\ U^+\ U^+\ U^+ \rangle$.

Fig. 20 Contractions $\langle U\ U\ U\ U\ U\ U \rangle$ and $\langle U\ U\ U\ U\ U\ U^+\ U^+ \rangle$

Fig. 21 Example of a "planar diagram" in the sense defined in Appendix C.



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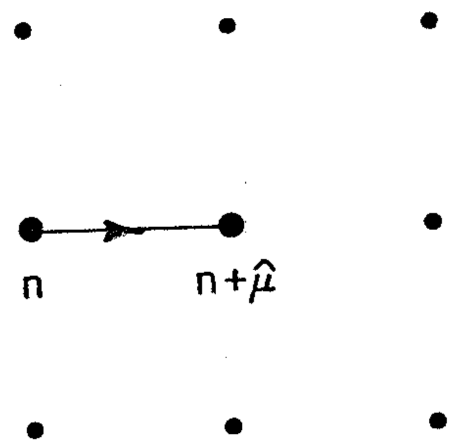

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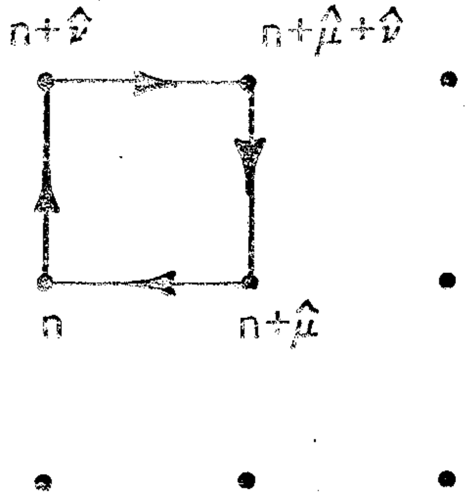
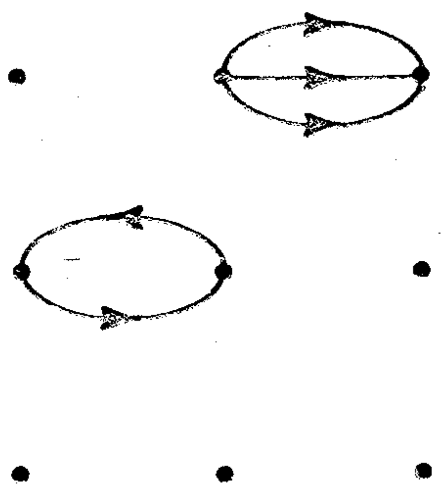


Fig. 3



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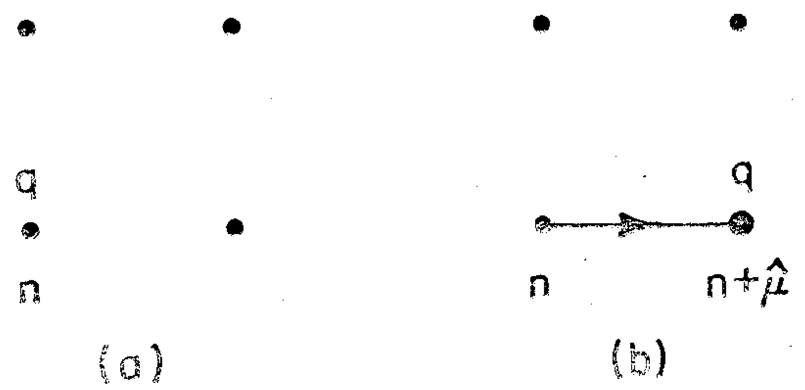

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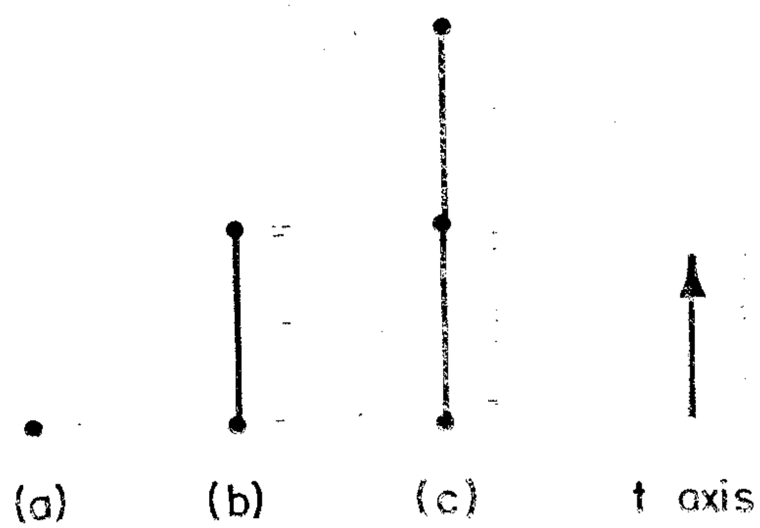

Fig. 6

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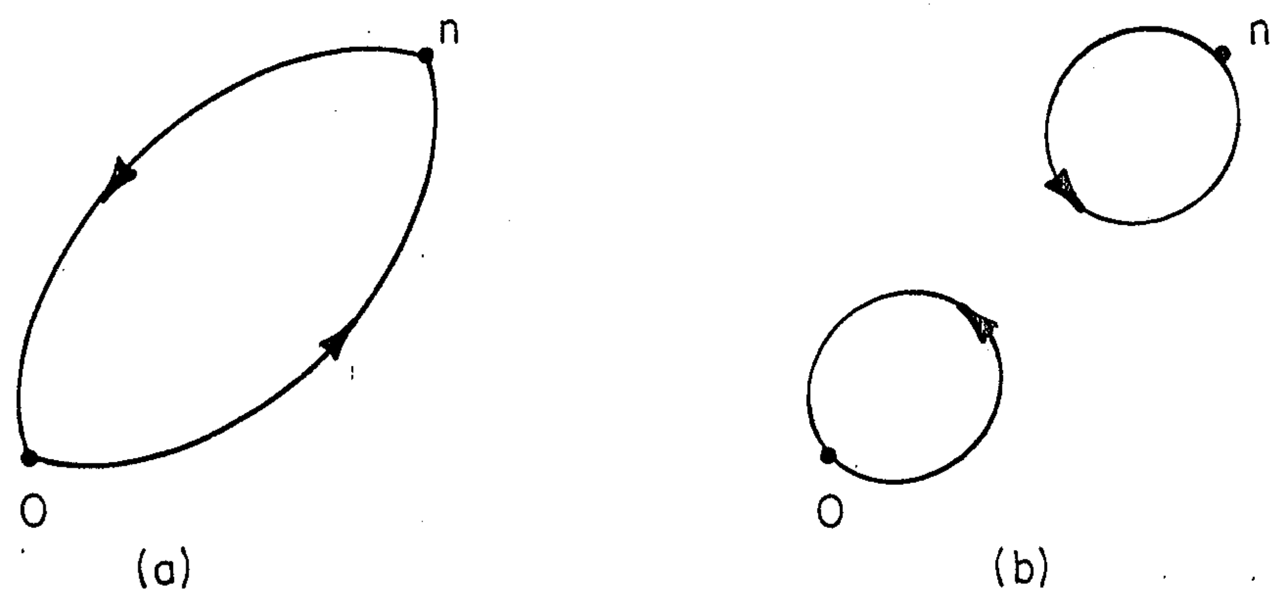

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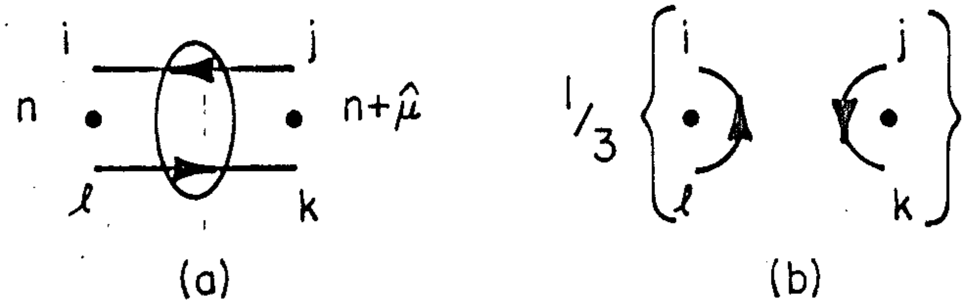

Fig. 8

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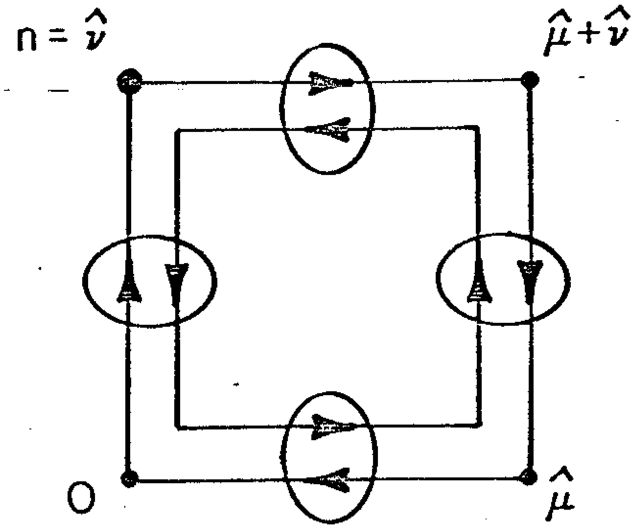

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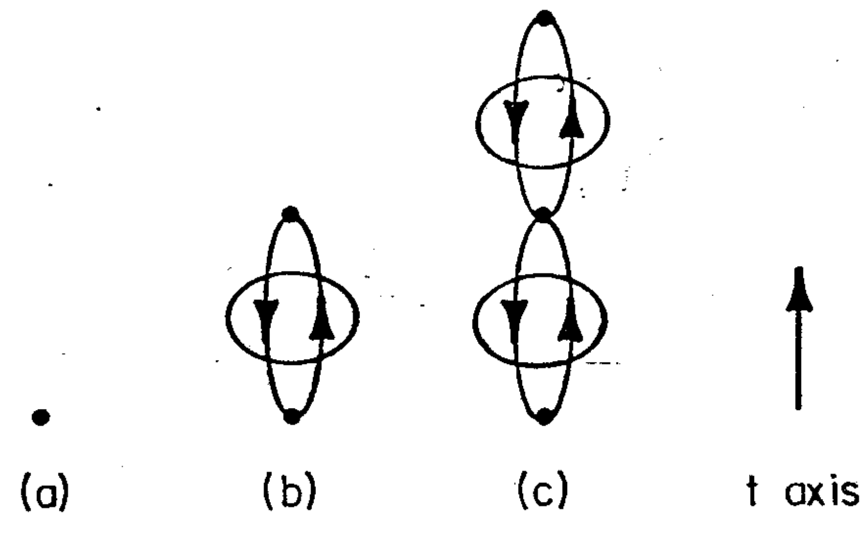

Fig. 10

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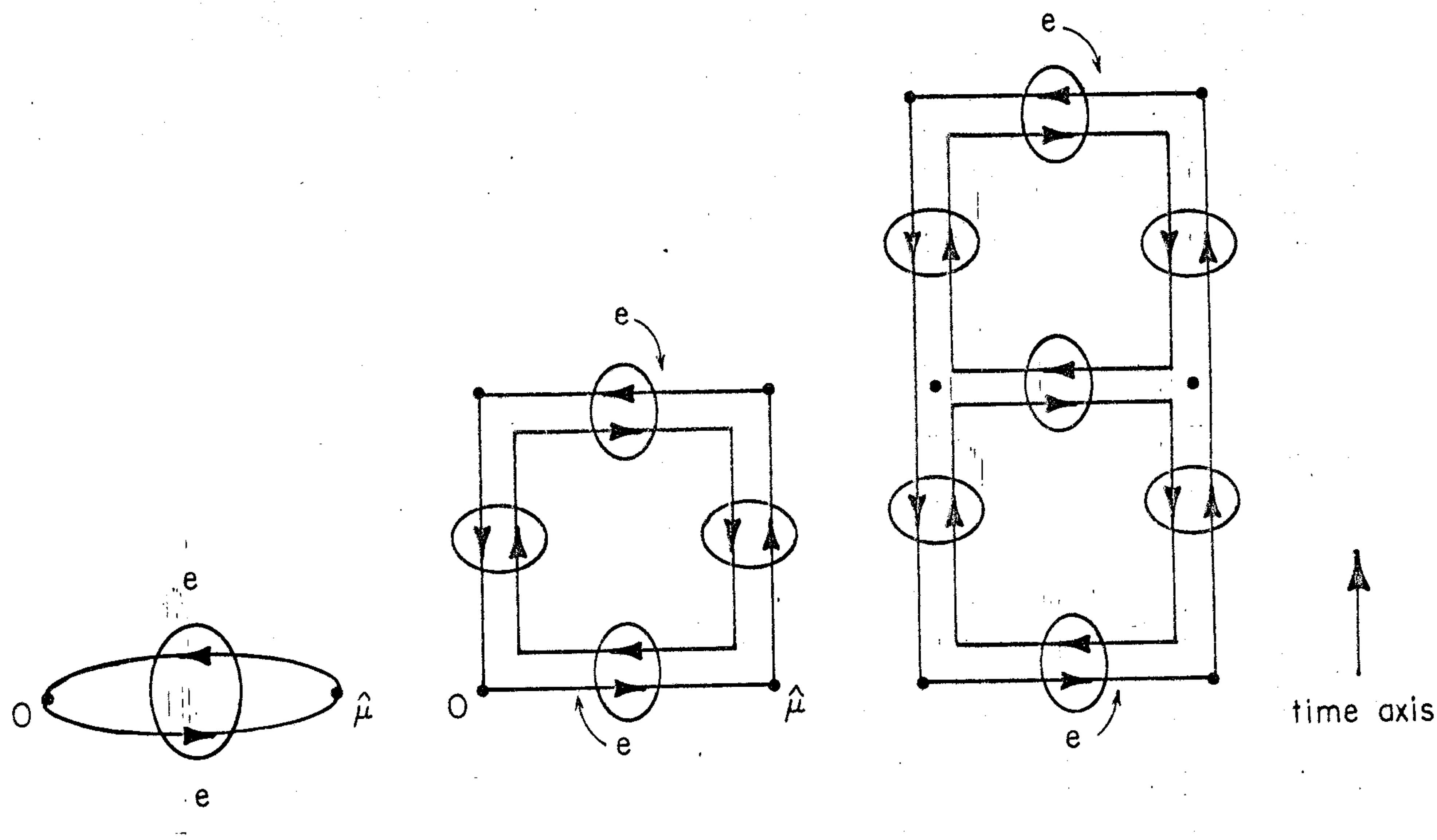

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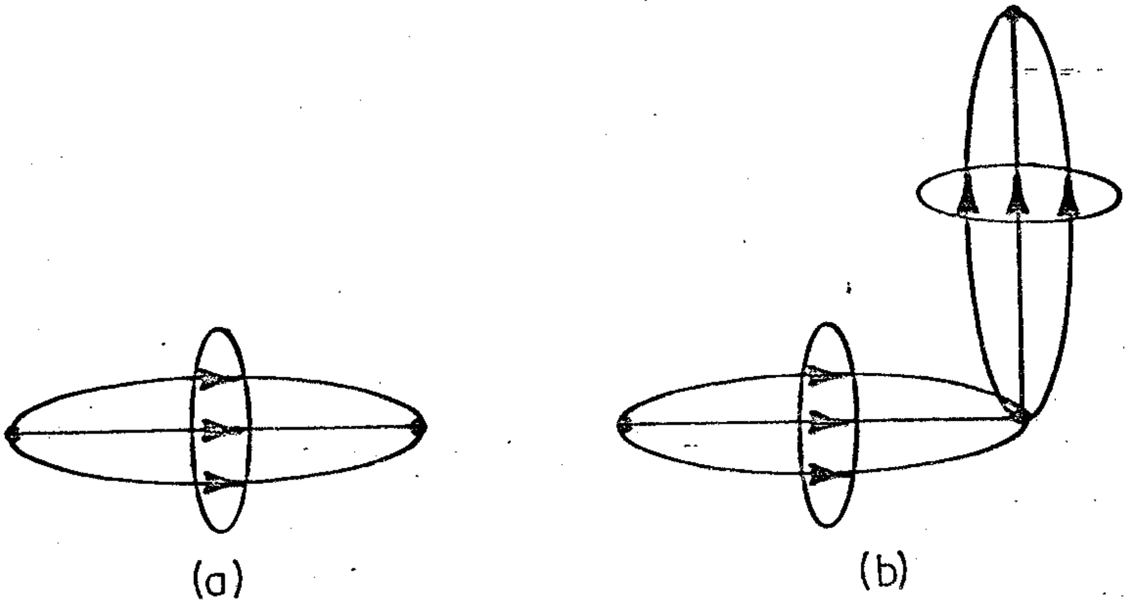
Fig. 11



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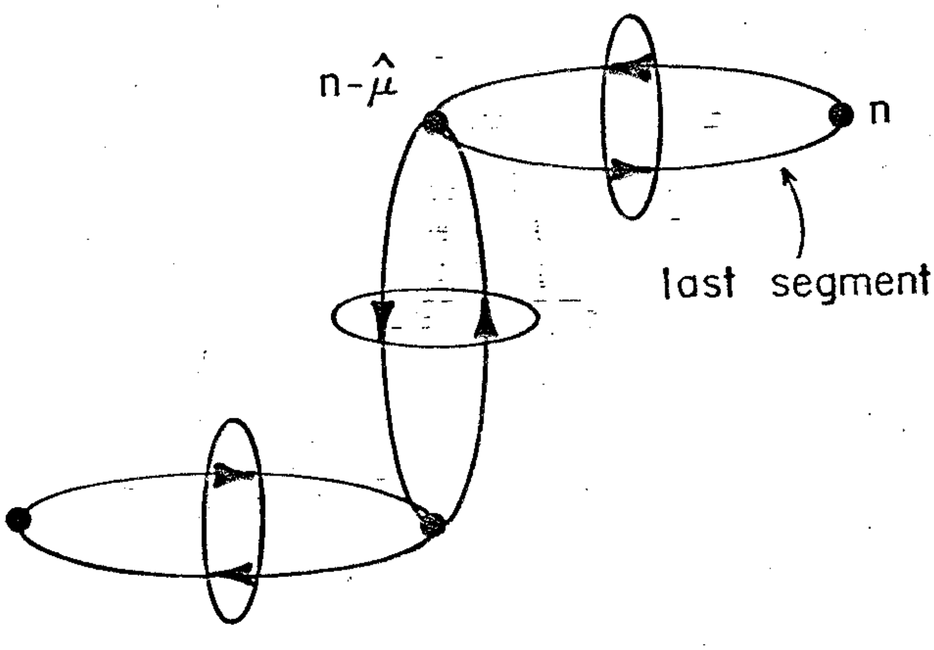

Fig. 13

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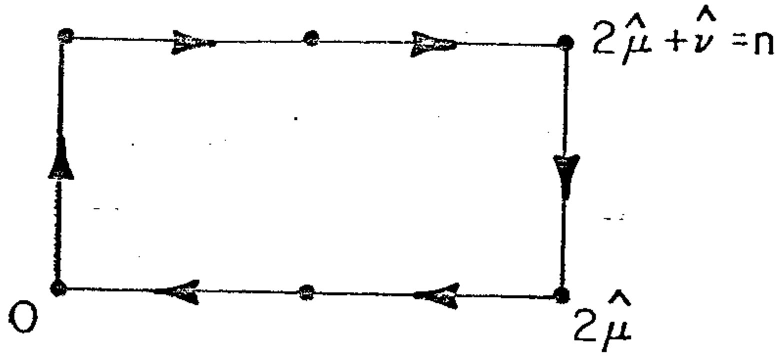

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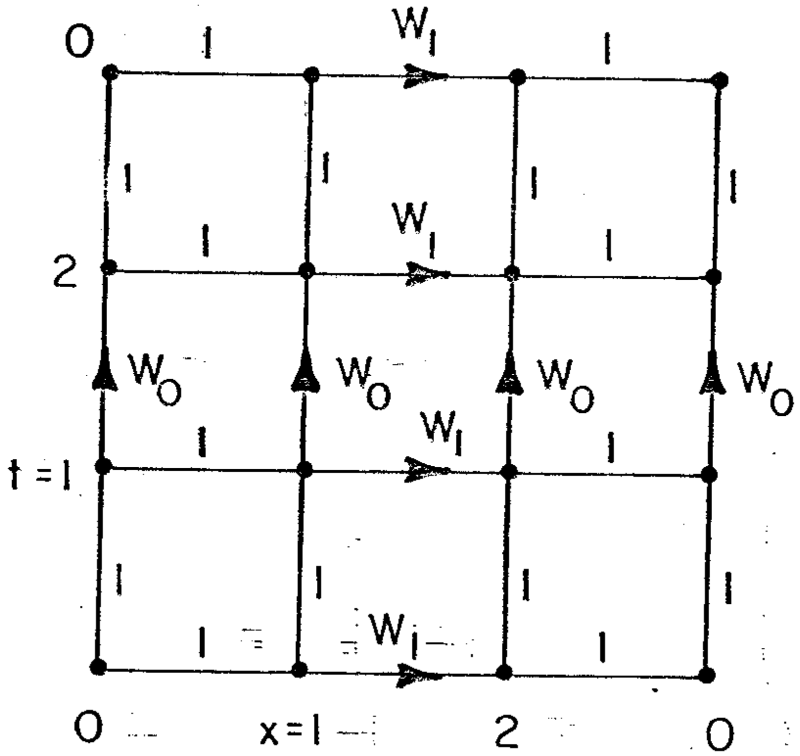

Fig. 15

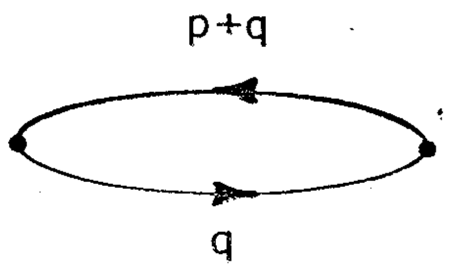

Fig. 16

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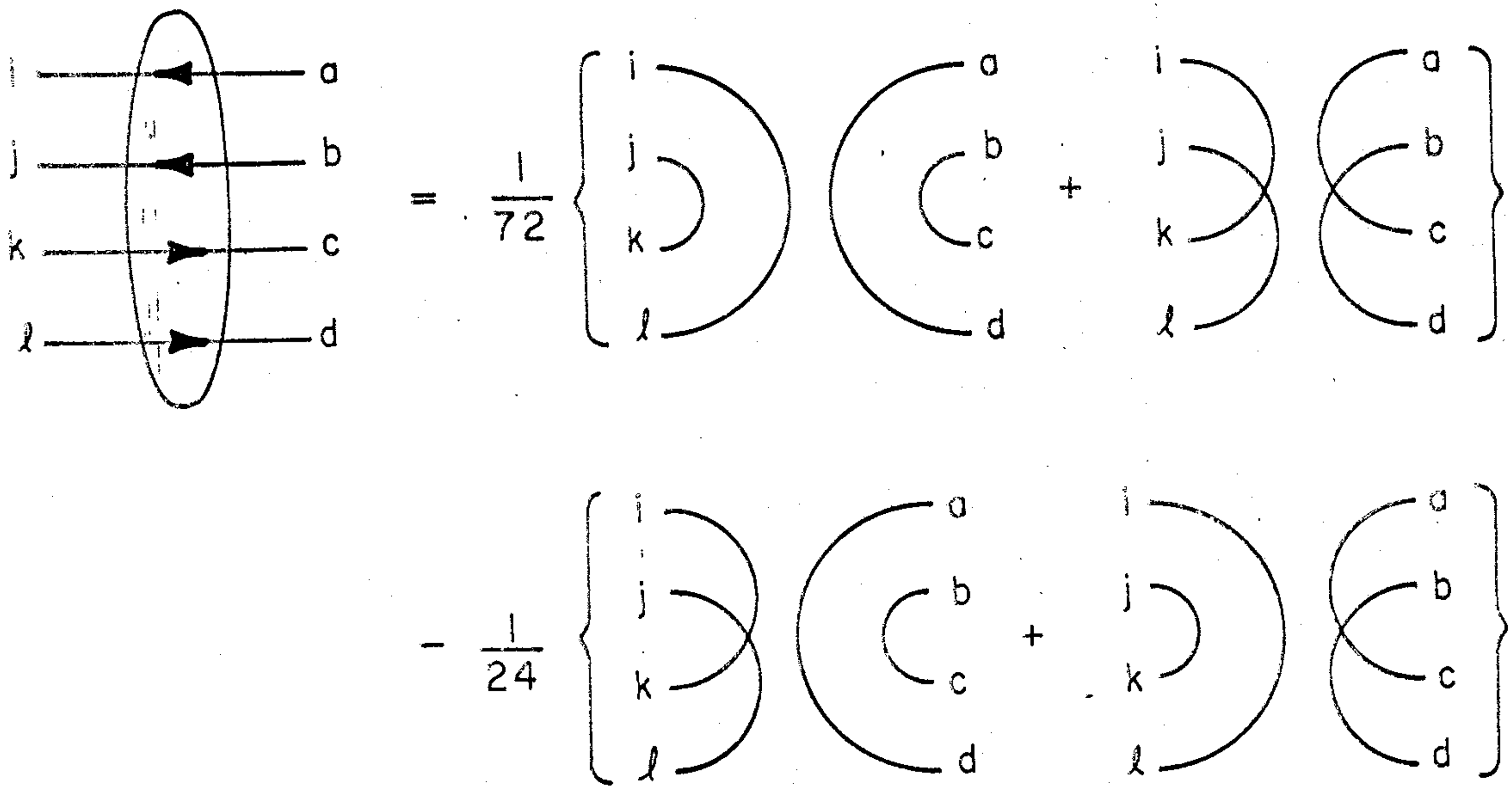

Fig. 17

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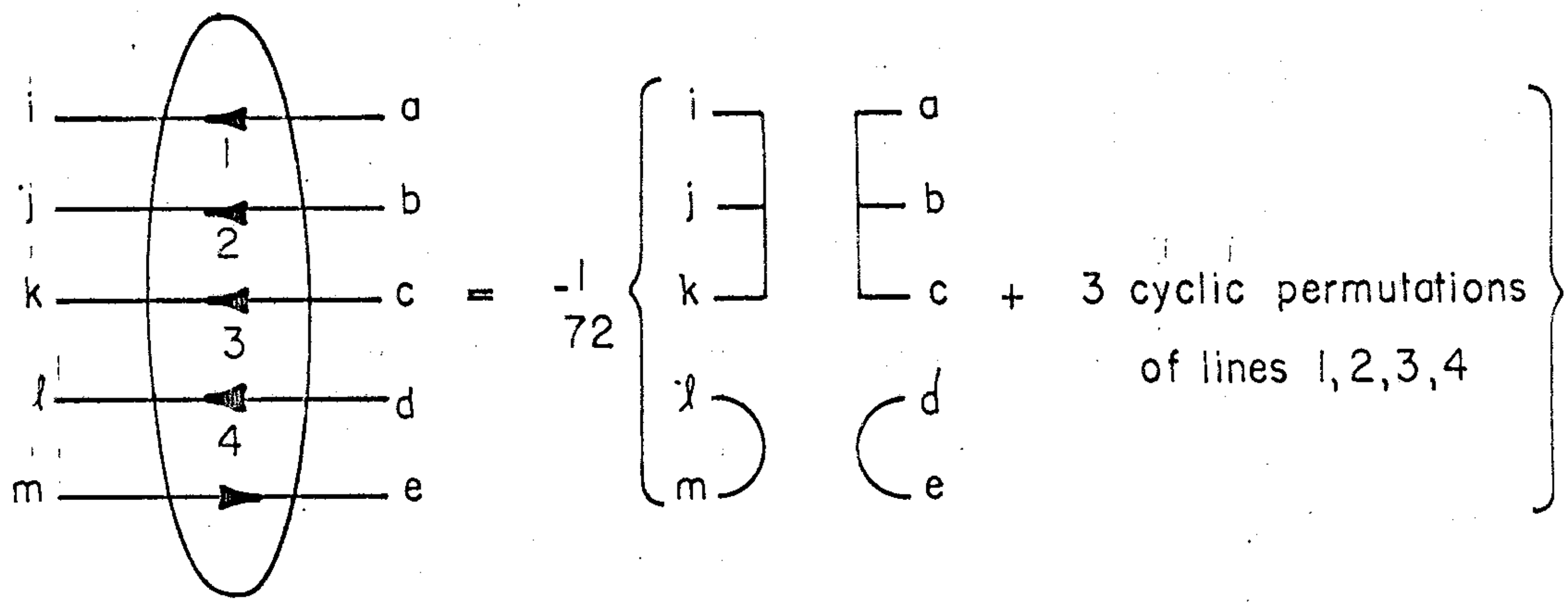

Fig. 18

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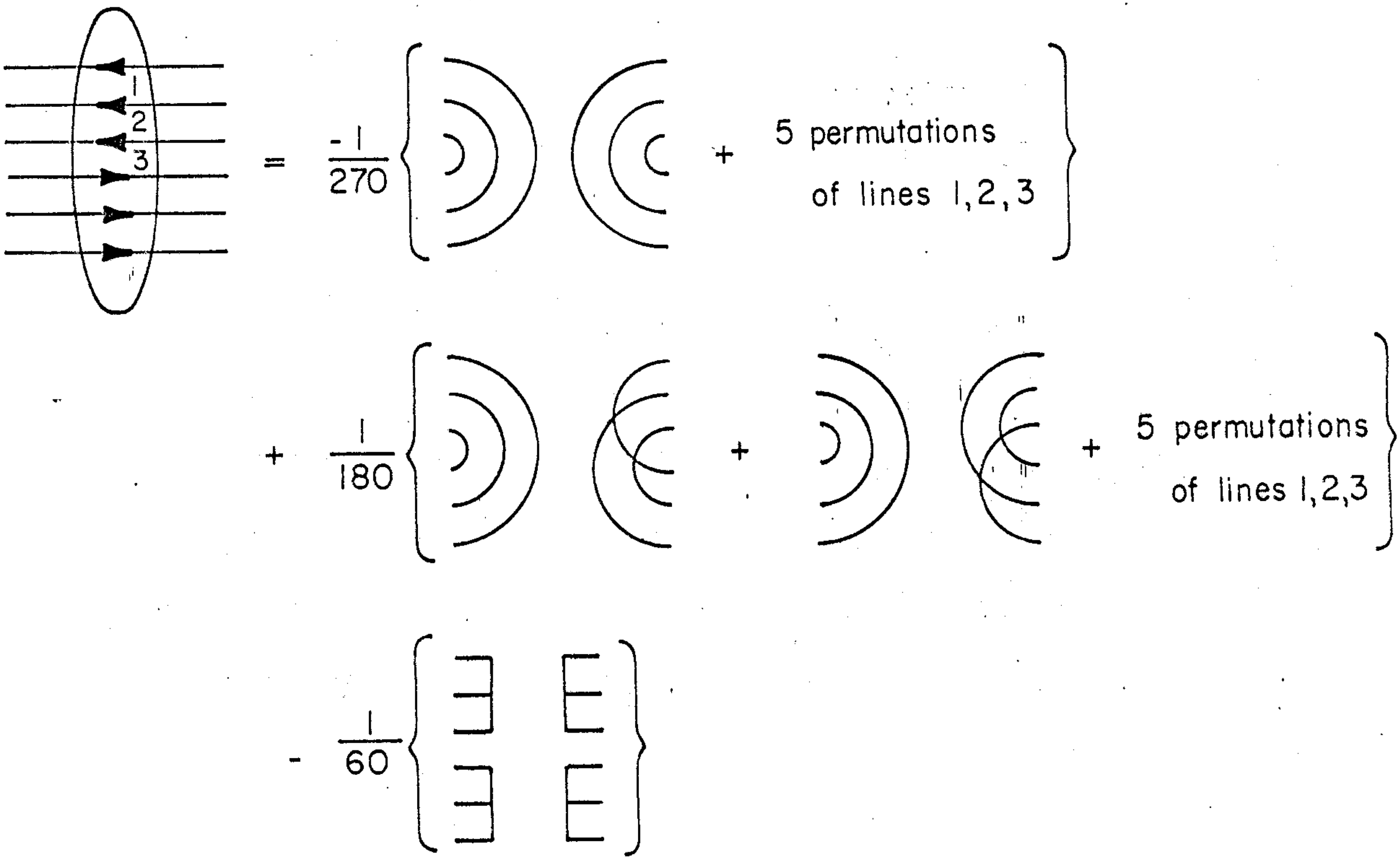

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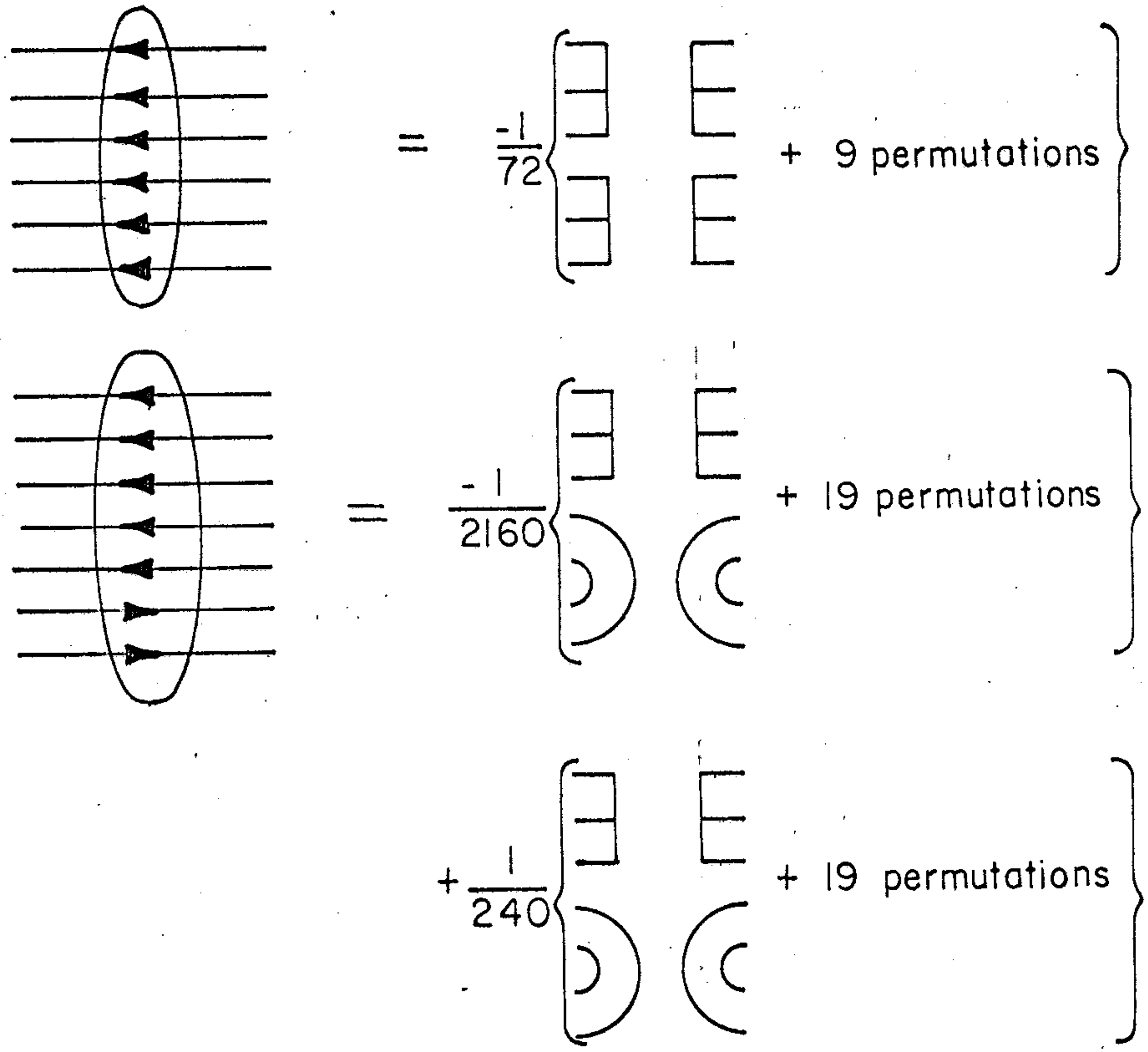
Fig. 19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+ 9 permutations
=
$-\frac{1}{2160}$
+ 19 permutations
$+\frac{1}{240}$
+ 19 permutations

Fig. 20

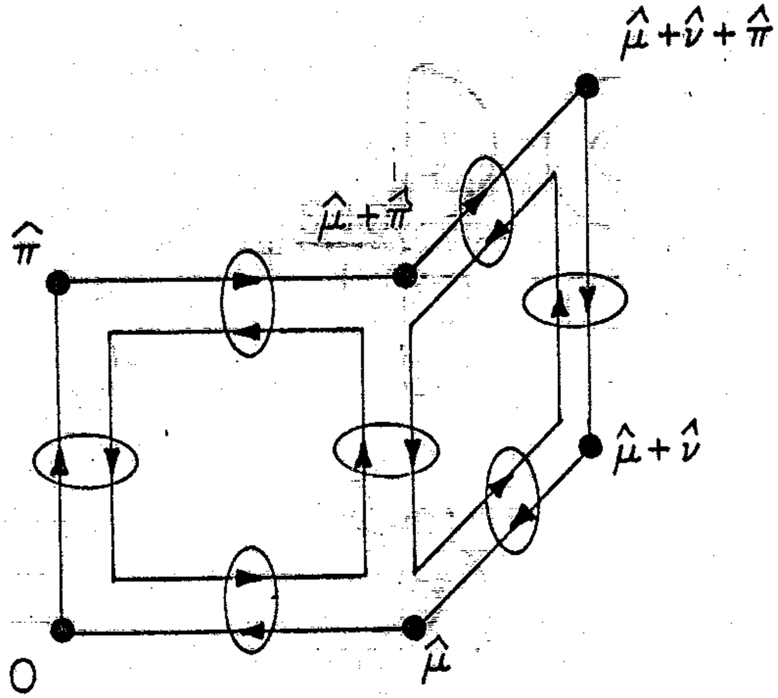

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Fig. 21