P410M: Relativistic Quantum Fields

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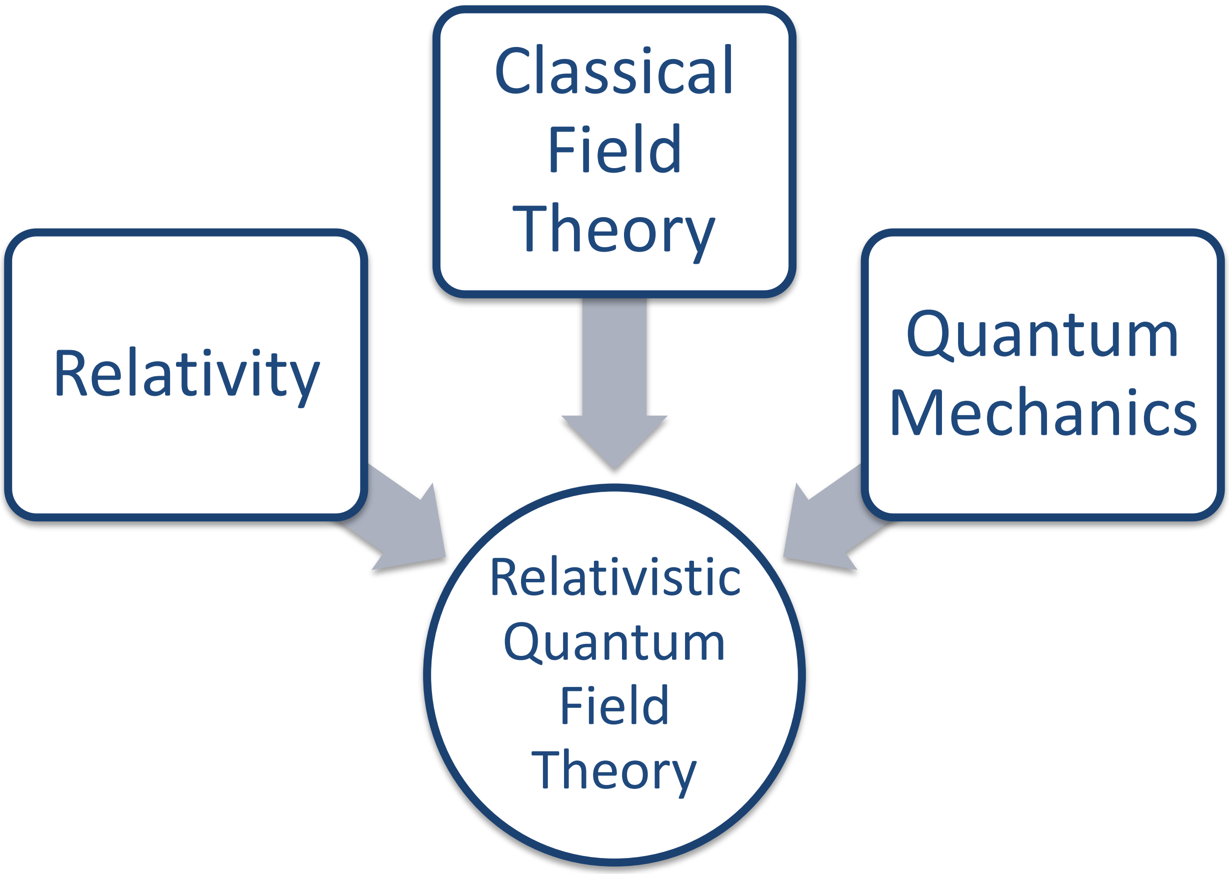
Recommended Texts:

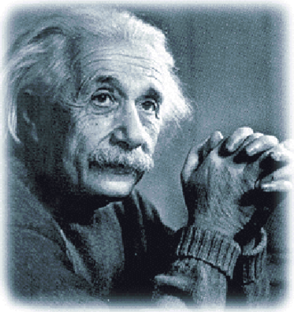
An Introduction to Quantum Field Theory
Peskin and Schroeder
Perseus Books (2008)

Quantum Field Theory
Mandl and Shaw
Wiley Blackwell (1993)

1. Introduction



Review: Special Relativity



Albert Einstein
1879-1955

The position of any event in space-time can be described by four numbers:

- the time it occurs t
- its three-dimensional coordinates x, y and z.

Each of these must be specified relative to an origin O, and in a specific frame of reference S.

We write these coordinates as a single object – a position *four-vector* as

$$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (c\,t, x, y, z) \qquad (\mu = 0, 1, 2, 3)$$

Where *c* is the speed of light.

The quantity $x^{\mu}x_{\mu}$ is the squared space-time *length* of the four vector

$$x^{\mu}x_{\mu} \equiv g_{\mu\nu}x^{\mu}x^{\nu} = (ct)^2 - |\vec{x}|^2$$

In the above:

- $g_{\mu\nu}$ is the *metric tensor* of *Minkowski space-time*. This provides our definition of length.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

[Note: Conventions differ – you may have used this with a different sign.]

- We have used the *Einstein Summation convention*. Repeated indices are summed over. (Greek indices are summed from 0...3, while Roman indices are summed from 1...3.)

e.g. $$g_{\mu\nu}x^{\mu}x^{\nu} \equiv \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} x^{\mu}g_{\mu\nu}x^{\nu} = (x^0\ x^1\ x^2\ x^3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

- Note the definition of a *covector* $x_{\mu} \equiv g_{\mu\nu}x^{\nu} = (x^0, -x^1, -x^2, -x^3)$

- Repeated Greek indices should always be written with one index up and one down.

An observer in a different reference frame S′ will instead observe a four-vector $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$ where Λ denotes a *Lorentz transformation*.

e.g. under a Lorentz boost to a frame moving with relative velocity *v*

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad \text{with} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

The *length $x^{\mu}x_{\mu}$ is conserved* under this transformation, which implies that the Lorentz transformation is *orthogonal*:

$$\left.\begin{aligned} x'^{\mu}x'_{\mu} &= g_{\mu\nu}x'^{\mu}x'^{\nu} = g_{\mu\nu}\,\Lambda^{\mu}{}_{\alpha}x^{\alpha}\,\Lambda^{\mu}{}_{\beta}\,x^{\beta} \\ x^{\mu}x_{\mu} &= g_{\mu\nu}x^{\mu}x^{\nu} \end{aligned}\right\}$$

$$x'^{\mu}x'_{\mu} = x^{\mu}x_{\mu} \Leftrightarrow g_{\mu\nu}\,\Lambda^{\mu}{}_{\alpha}\,\Lambda^{\nu}{}_{\beta} = g_{\alpha\beta} \Leftrightarrow \left[\Lambda^{-1}\right]_{\mu\nu} = \Lambda_{\nu\mu}$$

(Notice that I could have started with orthogonality and proven the invariance of $x^{\mu}x_{\mu}$.)

A particle's *four-momentum* is defined by $p^\mu = m\frac{dx^\mu}{d\tau}$ where τ is *proper time*, the time in the particle's own rest frame.

Proper time is related to an observer's time via $t = \gamma\tau$ so $p^\mu = \gamma m\frac{dx^\mu}{dt} = \gamma m(1, \vec{v})$

Its four-momentum's time component is the particle's *energy*, while the space components are its *three-momentum*

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

Its length is an invariant, its *rest mass* squared (times c^2):

$$p^\mu p_\mu = \frac{E^2}{c^2} - |\vec{p}|^2 = m^2c^2$$

Just like the position, this maps as a four-vector under a Lorentz Transformation:

$$p'^{\,\mu} = m\frac{dx'^{\,\mu}}{d\tau} = m\frac{d\left[\Lambda^\mu{}_\nu\, x^\nu\right]}{d\tau} = \Lambda^\mu{}_\nu\, m\frac{dx^\nu}{d\tau} = \Lambda^\mu{}_\nu\, p^\nu$$

Finally, the four dimensional gradient is a *covector* (index down):

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

You will sometimes use the vector expression

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$$ ← Watch the minus sign!

∂_μ transforms via the inverse Lorentz transformation (as do all covectors).

$$\partial x'^{\,\mu} = \frac{\partial x'^{\,\mu}}{\partial x^\nu}\partial x^\nu = \Lambda^\mu{}_\nu\, \partial x^\nu \quad \text{so} \quad \frac{\partial}{\partial x'^{\,\mu}} = \frac{\partial x^\nu}{\partial x'^{\,\mu}}\frac{\partial}{\partial x^\nu} = \left[\Lambda^{-1}\right]^\nu{}_\mu \frac{\partial}{\partial x^\nu}$$

Thus $\quad \partial_\mu \to \partial'_\mu = \left[\Lambda^{-1}\right]^\nu{}_\mu \partial_\nu$

For simplicity, from now on I will use *natural units*.

Instead of writing quantities in terms of kg, m and s, we could write them in terms of c,$\hbar$ and eV:

$$\begin{aligned} c &= 299792458\,\mathrm{m\,s^{-1}} \\ \hbar &= 6.58211889(26)\times 10^{-16}\,\mathrm{eV\,s} \\ 1\mathrm{eV} &= 1.782661731(70)\,\mathrm{c^2\,kg} \end{aligned}$$

So any quantity with dimensions $\mathrm{kg}^a\,\mathrm{m}^b\,\mathrm{s}^c$ can be written in units of $c^\alpha \hbar^\beta \mathrm{eV}^\gamma$, with

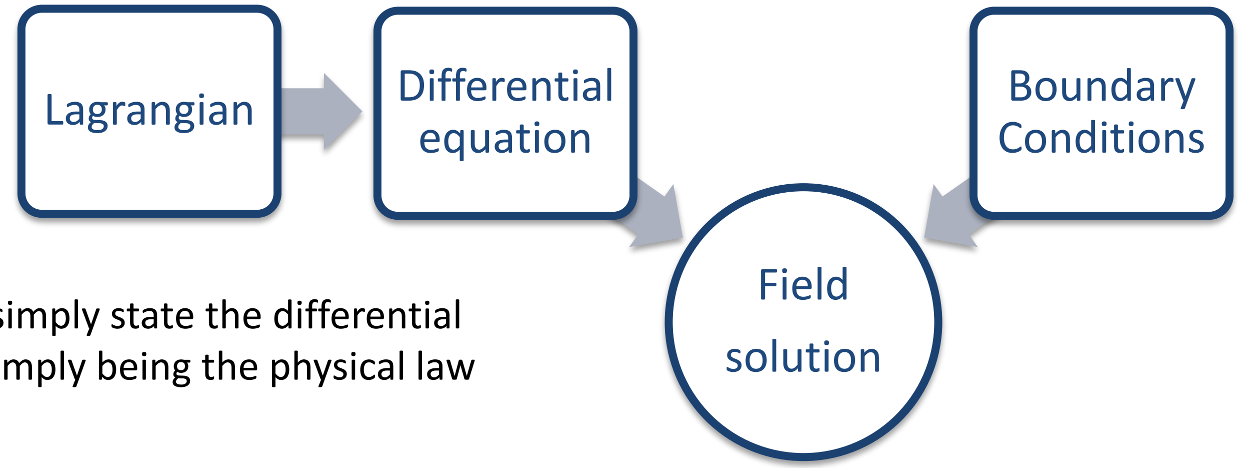
$$\begin{aligned} \alpha &= b+c-2a \\ \beta &= b+c \\ \gamma &= a-b-c \end{aligned}$$

Then we *omit* $\hbar$ and c in our quantities (you can work them out from the dimensions) – we don't just "set them to be one".

Review: Classical Field Theory

A field is a set of numbers assigned to each space-time point. For example, the temperature in a room, water velocity in a river and gravitational potential energy are all fields.

The values of field in physics are usually governed by two restrictions: a differential equation and boundary conditions.



Typically, we simply state the differential equation as simply being the physical law - it just is!

However, we can describe the dynamics of classical fields (how they change with time) using the *Lagrangian* and the *principle of least action*.

The evolution of a system progresses along the path of least action, where the action *S* is defined in terms of the Lagrangian.

Euler-Lagrange Equations and the Principle of Least Action

The *action* is

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

where $\mathcal{L}$ is the *Lagrangian* and the integration is over space-time.

A few technicalities:

- Technically this is the Lagrange density, since the Lagrangian is $L = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^3x$
- The Lagrangian depends on the gradient $\partial_\mu \phi$ and this is regarded as a separate variable.
- Often ϕ is complex, so we then have $\mathcal{L} = \mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*)$

The field will adopt the configuration which reduces the action to its minimum. To find this configuration, we look for a field configuration such that an infinitesimally small variation of the field leaves the action unchanged.

$$\phi(x) \rightarrow \phi(x) + \delta\phi(x) \quad \Rightarrow \quad S \rightarrow S + \delta S \quad \text{with} \quad \delta S = 0$$

$$\delta S = \delta\phi \frac{\partial}{\partial \phi} \int \mathcal{L}(\phi, \partial_\nu \phi)\, d^4x + \delta\left(\partial_\mu \phi\right) \frac{\partial}{\partial\left(\partial_\mu \phi\right)} \int \mathcal{L}(\phi, \partial_\nu \phi)\, d^4x$$

$$= \int \left(\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \left(\delta\phi\right) \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \right) d^4x$$

Convince yourself that $\delta\left(\partial_\mu \phi\right) = \partial_\mu \left(\delta\phi\right)$ for a well behaved field.

Now,

$$\partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \right) = \partial_\mu \left(\delta\phi\right) \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} + \delta\phi\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \right)$$

so

$$\delta S = \int \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \cancel{\partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \right)} - \delta\phi\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \right) \right] d^4x$$

total derivative is zero
since $\delta\phi$ vanishes at boundaries

True for all $\delta\phi$ so finally we have the *Euler-Lagrange Equations*:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \right) = 0}$$

For example, consider a Lagrangian $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi$

The terms in the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} = \partial^\mu \phi \quad \Rightarrow \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \right) = \partial_\mu \partial^\mu \phi$$

So the Euler-Lagrange equation turns into the wave equation

$$\partial_\mu \partial^\mu \phi = \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) \phi = 0$$

Noether's Theorem

Emmy Noether
1882-1935

Noether's Theorem says that if the action is *unchanged under a transformation*, then there exists a *conserved current* associated with the symmetry.

Consider the infinitesimal transformation of both coordinates and the fields,

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu = x^\mu + X^\mu{}_\nu \, \omega^\nu$$
$$\phi \to \phi' = \phi + \delta\phi = \phi + \Phi_\nu \, \omega^\nu$$

parameterised by the infinitesimal parameter ω^ν

The change in the field is due to the change in the functional form of the fields $\delta_\phi \phi$ but also the change in the coordinates:

$$\delta\phi = \delta_\phi \phi + \partial_\mu \phi \delta x^\mu$$

[In the derivation of the Euler-Lagrange equations, this term was absent, $\delta\phi = \delta_0\phi$]

The change in the Lagrangian is

$$\delta\mathcal{L} = \partial_\mu \mathcal{L} \, \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\phi \phi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi\right)} \delta_\phi \left(\partial_\nu \phi\right)$$
$$= \partial_\mu \mathcal{L} \, \delta x^\mu + \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi\right)} \right) \delta_\phi \phi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi\right)} \partial_\nu \left(\delta_\phi \phi\right)$$

So
$$\delta\mathcal{L} = \partial_\mu \mathcal{L}\, \delta x^\mu + \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)} \delta_\phi \phi \right)$$

So far, this is just the change in the *Lagrangian*. We need the change in the *action*:

$$\delta S = \delta \left(\int d^4x\, \mathcal{L} \right)$$

The integration measure also changes,

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x \quad = \left(1 + \partial_\mu \delta x^\mu\right)\, d^4x$$

[Note that $\frac{\partial x'}{\partial x}$ is a 4x4 matrix and $\left|\frac{\partial x'}{\partial x}\right|$ is a determinant]

So the change in the action is

$$\begin{aligned}
\delta S &= \int d^4x\, \left(\delta\mathcal{L} + \partial_\mu \delta x^\mu\right) \\
&= \int d^4x\, \left(\partial_\mu \mathcal{L}\, \delta x^\mu + \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)} \delta_\phi \phi \right) + \mathcal{L}\, \partial_\mu \delta x^\mu \right) \\
&= \int d^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)} \delta_\phi \phi + \mathcal{L}\, \delta x^\mu \right)
\end{aligned}$$

Writing $\delta x^\mu = X^\mu{}_\nu\, \omega^\nu$ and $\delta\phi = \Phi_\nu\, \omega^\nu$, we have

$$\delta_\phi \phi = \delta\phi - \partial_\mu\phi\, \delta x^\mu = \left(\Phi_\nu - \partial_\mu \phi X^\mu{}_\nu\right) \omega^\nu$$

We can write the change in the action in terms of the *divergence of a current*,

$$\delta S = -\int d^4x\, \partial_\mu\, j^\mu{}_\nu\, \omega^\nu$$

where

$$j^\mu{}_\nu = -\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu\phi\right)} \left(\Phi_\nu - \partial_\rho\phi X^\rho{}_\nu\right) - \mathcal{L}\, X^\mu{}_\nu$$

Rearranging

$$\boxed{j^\mu{}_\nu = \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\nu\phi - g^\mu{}_\rho\,\mathcal{L}\right) X^\rho{}_\nu - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\Phi_\nu}$$

To ensure that the action is invariant under this transformation, this must be a *conserved current*

$$\boxed{\partial_\mu j^\mu{}_\nu = 0}$$

This is *Noether's Theorem*.

The Energy-Momentum Tensor

The most classic example of Noether's Theorem is its application to *space-time translations*.

$$x^\mu \to x'^\mu = x^\mu + \omega^\nu \qquad \Rightarrow \qquad X^\mu{}_\nu = g^\mu{}_\nu$$

$$\phi \to \phi' = \phi \qquad \Rightarrow \qquad \Phi_\nu = 0$$

This transformation must be a symmetry of any model because it is saying that the laws of physics are the same everywhere and don't change with time!

Sticking these in, the *conserved current* is

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial^\nu \phi - g^{\mu\nu}\,\mathcal{L}}$$

This is the *energy-momentum tensor* for the field ϕ .

[If you have done a course on General Relativity, you have probably met this before.]

What does this mean, physically?

Let’s consider only a part of this 4x4 matrix, $J^{\mu} = T^{0\mu} = \dfrac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)}\, \partial^{\mu}\phi - g^{0\mu}\,\mathcal{L}$

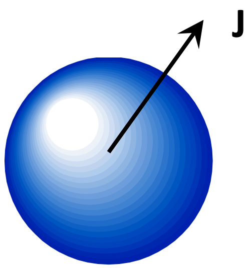
and integrate the divergence over a three-dimensional volume V:

$$\int_V \partial_\mu J^\mu \, dV = \int_V \left(\frac{\partial J^0}{\partial t} - \vec{\nabla} \cdot \vec{J} \right) dV$$

Since the divergence is zero,

Gauss’ Theorem

$$\int_V \frac{\partial J^0}{\partial t}\, dV = -\int_V \vec{\nabla} \cdot \vec{J}\, dV = -\int_A \vec{J} \cdot d\vec{A}$$



Volume V enclosed by Area A

Any *change* in the total J^0 *in the volume* must come about by a *current* **J** flowing *through the surface* of the volume.

The conserved quantity associated with time translations is the Hamiltonian (energy operator):

$$\boxed{H = \int T^{00} \, d^3x}$$

The conserved quantity associated with space translations is the (three) momentum operator:

$$\boxed{P^i = \int T^{0i} \, d^3x}$$

Green's Functions and the Dirac Delta Function

The *Dirac delta function* is (non-rigourously) defined as :

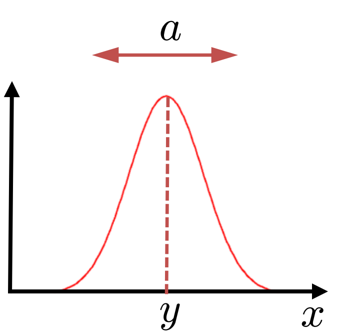
$$\delta(x-y) = \begin{cases} \infty & \text{for} \quad x = y \\ 0 & \text{for} \quad x \neq y \end{cases}$$

such that its integral is 1:

$$\int \delta(x-y)\, dx = 1$$

It can be visualised as a limit of the Gaussian distribution,

$$\delta(x-y) = \lim_{a\to 0} \frac{1}{a\sqrt{\pi}} \exp^{-(x-y)^2/a^2}$$



and can be written as an integration of a complex exponential,

$$\delta(x-y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-y)k}\, dk$$

[This is actually a consequence of the orthogonality of the exponential function. If $f(x) = e^{ixk}$ then

$$\int f(x) f^*(y) dk = 2\pi\, \delta(x-y)$$]

It is very useful because of its property under integration,

$$\int \delta(x-y)\, f(x)\, dx = f(y)$$

Other useful properties to know are:

$$\delta(ax) = \frac{\delta(x)}{|a|}$$

$$\delta\left(g(x)\right) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$$ where x_i are the roots of $g(x)$ i.e. $g(x_i) = 0$

$$\delta\left(x^2 - a^2\right) = \frac{1}{2|a|}\left(\delta(x-a) + \delta(x+a)\right)$$

$$(x-y)\frac{\partial}{\partial y}\delta(x-y) = \delta(x-y)$$

We will also use the multidimensional form where,

$$\delta^4\left(x-y\right) = \delta\left(x^0 - y^0\right)\,\delta\left(x^1 - y^1\right)\,\delta\left(x^2 - y^2\right)\,\delta\left(x^3 - y^3\right)$$

Imagine a field satisfies a differential equation of the form $\hat{D}\phi(x) = \rho(x)$,

for example, Poisson's equation $\vec{\nabla}^2\phi(x) = \rho(x)$

some differential operator

a source

Now, let $G(x, y)$ be the solution of the same equation, but with a *point source* at $x = y$.

So

$$\hat{D}G(x, y) = \delta(x - y)$$

then

$$\hat{D}\int G(x, y)\rho(y)\,dy = \int \delta(x - y)\rho(y)\,dy = \rho(x)$$

In other words, $\phi(x) = \int G(x, y)\rho(y)\,dy$ is a solution to the original equation.

The function $G(x, y)$ is a *Green's Function*. Green's Functions let us convert the problem of solving a differential equation into the problem of doing an integral.

We will see that Green's Functions are very important in Quantum Field Theory.

Review: Quantum Mechanics

The state vector

A quantum mechanical state can be completely described by a *state vector* in a (possibly infinite dimensional) *complex vector space* known as a *Hilbert space*.

(Don't confuse this vector with the four-vector in Minkowski space that we defined earlier.)

We use Dirac's "*bra*" and "*ket*" notation:

- a vector is written as $|\psi\rangle$
- its complex conjugate is written $\langle\psi| \equiv |\psi\rangle^*$

Any and all information about the state is contained in the vector $|\psi\rangle$

Observables

$\hat{A} = \hat{A}^\dagger$

Every *observable* A has a corresponding linear *Hermitian operator* $\hat{A}$ acting on the Hilbert space, for which there is complete set of orthonormal *eigenvectors* $|a\rangle$ with *eigenvalue* a .

$$\hat{A}|a\rangle = a|a\rangle$$

Since these eigenvectors span the space (they are complete), we can write $\int |a\rangle\langle a|\, da = 1$

and any state vector can be written as

$$|\psi\rangle = \int \psi_A(a)|a\rangle\, da$$

[This could be a discrete sum if the Hilbert space is finite dimensional, but for the rest of this course I will assume it is infinite dimensional.]

The function $\psi_A(a)$ is the *wavefunction* in the eigenspace of $\hat{A}$ and can be obtained via

$$\langle a|\psi\rangle = \int \psi_A(b)\langle a|b\rangle\, db = \int \psi_A(b)\delta(a-b)\, db = \psi_A(a)$$

where we have used the orthonormality relation $\langle a|b\rangle = \delta(a-b)$

A measurement of the observable A will return a result a with probability $|\langle a|\psi\rangle|^2 = |\psi_A(a)|^2$

After the measurement, the state will no longer be $|\psi\rangle$ but will have *collapsed* onto the corresponding eigenvector $|a\rangle$.

The position and momentum eigenbases

The most common bases used are the *position* and *momentum* bases, corresponding to the position $\hat{x}$ and momentum $\hat{p}$ operators.

For example, the position-space wavefunction is $\langle x|\psi\rangle = \psi_x(x)$ and $|\langle x|\psi\rangle|^2 = |\psi_x(x)|^2$ is the probability of finding the particle at position x.

Since $|x\rangle$ and $|p\rangle$ are not aligned bases, the state cannot be an eigenvector of position and momentum simultaneously. Also, since the measurements change the state of the system, the order of measurements is important:

$$\hat{x}\hat{p}|\psi\rangle \neq \hat{p}\hat{x}|\psi\rangle$$

We *postulate* a *commutation relation* $[\hat{x},\hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$

We can use this to determine representations for these operators:

$$i\hbar\langle x|y\rangle = \langle x|[\hat{x},\hat{p}]|y\rangle = \langle x|\hat{x}\hat{p}|y\rangle - \langle x|\hat{p}\hat{x}|y\rangle = (x-y)\langle x|\hat{p}|y\rangle$$

$$\Rightarrow \qquad \langle x|\hat{p}|y\rangle = i\hbar\frac{\delta(x-y)}{x-y} = i\hbar\frac{\partial}{\partial y}\delta(x-y)$$

But

$$\langle x|\hat{p}|\psi\rangle = \int \langle x|\hat{p}|y\rangle\langle y|\psi\rangle\, dy = \int \left(i\hbar\frac{\partial}{\partial y}\delta(x-y)\right)\langle y|\psi\rangle\, dy = -i\hbar\frac{\partial}{\partial x}\psi_x(x)\,,$$

so we see that $\hat{p} = -i\hbar\dfrac{\partial}{\partial x}$ is a suitable *representation* for the (one-dimensional) momentum operator.

To see *how the position and momentum bases are related*, consider

$$p\langle x|p\rangle = \langle x|\hat{p}|p\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|p\rangle$$

Solving this differential equation gives

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}.$$

In other words, the bases are related by a *Fourier transformation*:

$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int e^{ipx/\hbar}|x\rangle\, dx, \qquad |x\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}|p\rangle\, dp.$$

This embodies the *Heisenberg Uncertainty principle*.

The Schrödinger and Heisenberg Pictures

In the *Schrödinger Picture* of Quantum Mechanics, the state vector $|\psi\rangle$ depends on time, but the operators do not. The *time evolution* is governed by the Hamiltonian:

$$\langle x|\hat{H}|\psi\rangle = \langle x|i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

We can write a new operator $\hat{U}$ which relates the state vector at time t to that at time t_0,

$$|\psi, t\rangle = \hat{U}(t, t_0)|\psi, t_o\rangle$$

then

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t, t_0)|\psi, t_0\rangle = \hat{H}\,\hat{U}(t, t_0)|\psi, t_0\rangle$$

$$\Rightarrow \qquad i\hbar\frac{\partial}{\partial t}\hat{U}(t, t_0) = \hat{H}\,\hat{U}(t, t_0)$$

If H is constant, this simplifies to

$$\hat{U}(t, t_0) = e^{-i(t-t_0)\hat{H}/\hbar}$$

In the *Heisenberg picture*, the *state vector remains fixed* and the *operators change with time*.

The Heisenberg state vector is just $|\psi\rangle_H = |\psi, t_o\rangle = \hat{U}^\dagger(t, t_0)|\psi\rangle_S$

this subscript H is for Heisenberg, and this S is for Schrödinger

and the operators are

$$\hat{O}_H(t) = \hat{U}^\dagger(t, t_0)\,\hat{O}_S\,\hat{U}(t, t_0)$$

Note that any expectation value will remain unchanged,

$$\begin{aligned}\langle\phi|_H\,\hat{O}_H(t)|\psi\rangle_H &= \langle\phi, t|_S\,\hat{U}(t, t_0)\hat{U}^\dagger(t, t_0)\,\hat{O}_S\,\hat{U}(t, t_0)\hat{U}^\dagger(t, t_0)|\psi\rangle_S \\ &= \langle\phi, t|_S\,\hat{O}_S\,|\psi, t\rangle_S\end{aligned}$$

The equivalent of the Schrödinger Equation for the Heisenberg Picture is

$$\boxed{i\hbar\frac{d\hat{O}_H}{dt} = \left[\hat{O}_H, \hat{H}\right]}$$

The Heisenberg picture is rather useful in Quantum Field Theory.

The Schrödinger Equation

Erwin Schrödinger
1887-1961

The Schrödinger equation is simply a statement about conservation of energy.

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi}$$

Total energy $\widehat{H}$

Kinetic energy $\frac{\widehat{p}^2}{2m}$

Potential energy

Let's examine its *conserved currents*:

$$\psi^* \times \text{S.E.}: \quad \psi^*\, i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V\psi^*\psi$$

$$\psi \times \text{S.E.}^*: \quad -\psi\, i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\psi\nabla^2\psi^* + V\psi^*\psi$$

Now subtract one equation from the other

$$\Rightarrow i\hbar\frac{\partial\left[\psi^*\psi\right]}{\partial t} = \frac{\hbar^2}{2m}\left[-\psi^*\nabla^2\psi + \psi\nabla^2\psi^*\right] = \frac{\hbar^2}{2m}\vec{\nabla}.\left[-\psi^*\vec{\nabla}\psi + \psi\vec{\nabla}\psi^*\right]$$

$$\vec{\nabla}.\left[\psi^*\vec{\nabla}\psi\right] = \vec{\nabla}\psi^*.\vec{\nabla}\psi + \psi^*\nabla^2\psi$$

We see that we have a *continuity equation*

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

with $\rho = |\psi|^2$ and $\vec{J} = \frac{\hbar}{2im}\left[\psi^*(\vec{\nabla}\psi) - (\vec{\nabla}\psi^*)\psi\right]$

The probability is conserved.

The Harmonic Oscillator and Ladder Operators

The quantum harmonic oscillator is defined by the Hamiltonian $\hat{H} = \dfrac{\hat{p}^2}{2m} + \dfrac{1}{2} m\omega^2 \hat{x}^2$.

We could just put this into the Schrödinger Equation and solve it, but it is instructive to use *ladder operators* instead.

Define,

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + i\sqrt{\frac{\hbar}{m\omega}}\hat{p}\right) \quad \text{and} \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - i\sqrt{\frac{\hbar}{m\omega}}\hat{p}\right)$$

It is easy to see that

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1\,, \quad \left[\hat{H}, \hat{a}\right] = -\hbar\omega\hat{a} \;\text{ and }\; \left[\hat{H}, \hat{a}^\dagger\right] = \hbar\omega\hat{a}^\dagger\,,$$

and we can write the Hamiltonian as

$$\hat{H} = \frac{1}{2}\hbar\omega\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

Consider an eigenstate of energy $|n\rangle$ with eigenvalue E_n so that $\hat{H}|n\rangle = E_n|n\rangle$

Now, $\hat{H}\hat{a}^\dagger|n\rangle = \left(\left[\hat{H}, \hat{a}^\dagger\right] + \hat{a}^\dagger \hat{H}\right)|n\rangle = \left(\hbar\omega\hat{a}^\dagger + \hat{a}^\dagger\hat{H}\right)|n\rangle = \left(\hbar\omega\hat{a}^\dagger + \hat{a}^\dagger E_n\right)|n\rangle$

$$= (E_n + \hbar\omega)\,\hat{a}^\dagger|n\rangle$$

In other words, $\hat{a}^\dagger|n\rangle$ is also an eigenstate, but now with eigenvalue $E_{n+1} = E_n + \hbar\omega$

Similarly $\hat{H}\hat{a}|n\rangle = (E_n - \hbar\omega)\,\hat{a}|n\rangle$

We say that $\hat{a}^\dagger$ is a *creation operator* – it creates one quantum on energy $\hbar\omega$ – while $\hat{a}$ is an *annihilation operator*.

By definition, the ground state $|0\rangle$ has the lowest energy, so we must have $\hat{a}|0\rangle = 0$.

Let's work out the ground state energy:

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$$

The ground state has non-zero energy!

2. Free Scalar Field Theory

Oskar Klein
1894 - 1997

The Free Real Scalar Field

The simplest field theory that we can write down is that of a *free real scalar field* with Lagrangian

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2}$$

The Euler-Lagrange equations give us the wave-equation:

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2 \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} = \partial^\mu\phi \qquad \Rightarrow \qquad \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right) = \partial_\mu\partial^\mu\phi$$

$$\boxed{\left(\partial^2 + m^2\right)\phi = 0}$$

$\partial^2 \equiv \partial^\mu\partial_\mu$ is sometimes written as $\Box$ or $\Box^2$

This is the *Klein-Gordon Equation*.

Notice that, just like the Schrödinger Equation, this is just a relation between energy and momentum.

Writing it in a non-covariant notation, we can see the motivation of this equation:

$$\left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right)\phi = m^2\phi$$

mass

Energy $\widehat{H}^2$

three-momentum $\widehat{p}^2$

In other words, taking a plane wave solution $\sim e^{ik\cdot x} = e^{i\left(k^0 t - \vec{k}\cdot\vec{x}\right)}$ we require that

$$\left(k^0\right)^2 - \vec{k}^2 = m^2$$

This is just the *relativistic* relation between energy and momentum.

For this reason, the Klein-Gordon Equation is sometimes referred to as the *relativistic Schrödinger Equation*.

$$\left(k^0\right)^2 - \vec{k}^2 = m^2 \quad \Rightarrow \quad k^0 = \pm\sqrt{\mathbf{k}^2 + m^2}$$

Notice that *both* + and – are equally good solutions. Also, this is a solution for any $\vec{k}$.

Therefore the general solution is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} \left(a(-\vec{k})\, e^{i\left(-E(\vec{k})t - \vec{k}\cdot\vec{x}\right)} + a^*(\vec{k})\, e^{i\left(E(\vec{k})t - \vec{k}\cdot\vec{x}\right)} \right)$$

just a normalization choice (next slide)

coefficients related since ϕ is real

$$E(\vec{k}) \equiv +\sqrt{\vec{k}^2 + m^2}$$

In the first integral we are free to switch $\vec{k} \longrightarrow -\vec{k}$ so that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} \left(a(\vec{k})\, e^{-ik\cdot x} + a^*(\vec{k})\, e^{ik\cdot x} \right)$$

where $k \cdot x \equiv E(\vec{k})t - \vec{k} \cdot \vec{x}$. This is just a Fourier decomposition in terms of plane waves.

The normalisation of the measure

On the previous slide we used an integration measure

$$\frac{d^3k}{(2\pi)^3\,2E}$$

Although this looks non-covariant, it is actually the covariant choice. To see this, consider,

$$\begin{aligned}\frac{d^3k}{(2\pi)^3\,2E} &= \frac{d^3k}{(2\pi)^3\,2E}\,dE^2\,\delta(E^2-\vec{k}^2-m^2)\\ &= \frac{d^3k}{(2\pi)^3\,2E}\,2E\,dE\,\delta(E^2-\vec{k}^2-m^2)\\ &= \frac{d^4k}{(2\pi)^4}\,2\pi\delta(k^2-m^2)\end{aligned}$$

This result is clearly covariant.

condition insisting the
particle is *on-shell*

The energy momentum tenor revisited

The *energy momentum tensor* is

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

$$= \partial^\mu \phi \, \partial^\nu \phi - g^{\mu\nu} \left(\frac{1}{2} \partial_\rho \phi \, \partial^\rho \phi - \frac{1}{2} m^2 \phi \right),$$

So

$$T^{00} = \partial^0 \phi \, \partial^0 \phi - \left(\frac{1}{2} \partial_0 \phi \, \partial^0 \phi - \frac{1}{2} \vec{\nabla} \phi \cdot \vec{\nabla} \phi - \frac{1}{2} m^2 \phi \right)$$

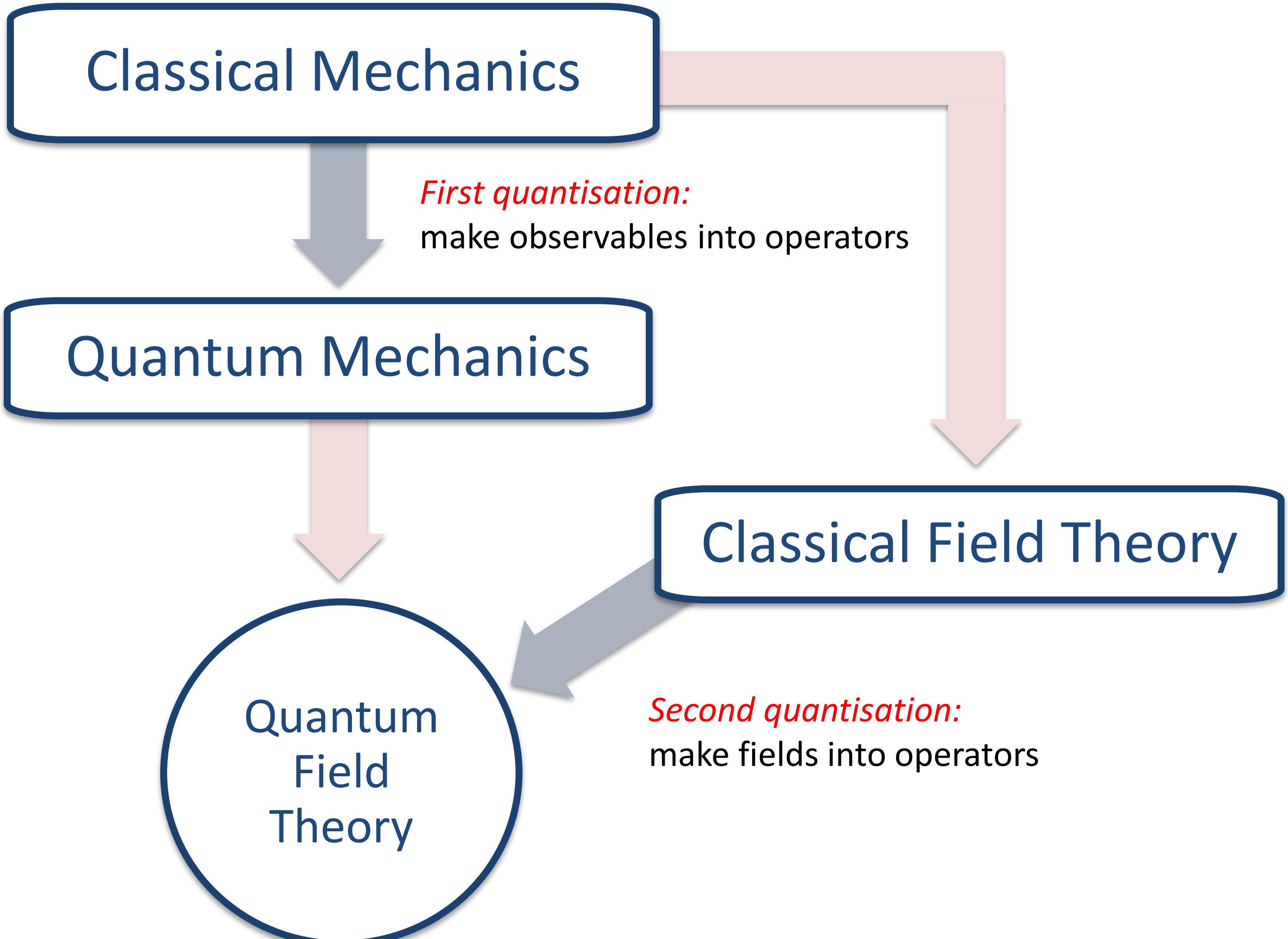
and the *Hamiltonian* is

$$H = \int T^{00} \, d^3 x = \frac{1}{2} \int \left((\partial_0 \phi)^2 + \left(\vec{\nabla} \phi\right)^2 + m^2 \phi^2 \right) \, d^3 x$$

This is true in a classical field theory, but what does this mean in Quantum Mechanics? Isn't H supposed to be the energy *operator*?

When thinking about field theories, it doesn't make sense to just turn the Hamiltonian into an operator – *the fields themselves have to become operators too*!

A schematic depiction of first and second quantisation



Quantising the scalar field

We define the *canonically conjugate momentum* $\pi(x) = \dfrac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)} = \partial_0 \phi(x) = \dot{\phi}(x)$

We postulate that ϕ and π are operators who obey the *equal-time commutation relations*:

$$\left[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)\right] = i\delta^3(\vec{x}-\vec{y})$$

$$\left[\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{y},t)\right] = \left[\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{y},t)\right] = 0$$

The Klein-Gordon Equation doesn't change (though it now acts on an operator) so we still have,

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} \left(\hat{a}(\vec{k})\, e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$

Note that now $\hat{a}(x)$ and $\hat{a}^\dagger(x)$ are operators too.

The corresponding equation for π is

$$\hat{\pi}(x) = \partial_0 \hat{\phi}(x) = \frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \left(-\hat{a}(\vec{k})\, e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$

We can *invert* the Fourier transformation to find $\widehat{a}(x)$ and $\widehat{a}^\dagger(x)$.

From the previous slide,

$$\widehat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} \left(\widehat{a}(\vec{k})\, e^{-ik\cdot x} + \widehat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$

Multiplying by a complex exponential and integrating, $e^{i(k'-k)\cdot x} = e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} e^{i(E(\vec{k}')-E(\vec{k}))t}$

$$\int d^3x\, \widehat{\phi}(x)\, e^{-ik\cdot x} = \int \frac{d^3k'}{(2\pi)^3\, 2E(\vec{k}')} \Bigg(\widehat{a}(\vec{k}') \int d^3x\, e^{-i(k+k')\cdot x} + \widehat{a}^\dagger(\vec{k}') \int d^3x\, e^{i(k'-k)\cdot x}\Bigg)$$

$$= \int \frac{d^3k'}{2E(\vec{k})} \Bigg(\widehat{a}(\vec{k}')\, \delta^3(\vec{k}+\vec{k}')e^{-i(E(\vec{k})+E(\vec{k}'))t} + \widehat{a}^\dagger(\vec{k}')\, \delta^3(\vec{k}'-\vec{k})e^{i(E(\vec{k}')-E(\vec{k}))t}\Bigg)$$

$$= \frac{1}{2E(\vec{k})} \left(\widehat{a}(-\vec{k})e^{-i2E(\vec{k})t} + \widehat{a}^\dagger(\vec{k})\right)$$

Similarly for π

$$\widehat{\pi}(x) = \frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\left(-\widehat{a}(\vec{k})\, e^{-ik\cdot x} + \widehat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$

we find

$$\begin{aligned}
\int d^3x\,\widehat{\pi}(x)\, e^{-ik\cdot x} &= \frac{i}{2}\int \frac{d^3k'}{(2\pi)^3}\left(-\widehat{a}(\vec{k}')\int d^3x\, e^{-i(k+k')\cdot x}\right.\\
&\qquad\left.+\widehat{a}^\dagger(\vec{k}')\int d^3x\, e^{i(k'-k)\cdot x}\right)\\
&= \frac{i}{2}\int d^3k'\left(-\widehat{a}(\vec{k}')\,\delta^3(\vec{k}+\vec{k}')e^{-i(E(\vec{k})+E(\vec{k}'))t}\right.\\
&\qquad\left.+\widehat{a}^\dagger(\vec{k}')\,\delta^3(\vec{k}'-\vec{k})e^{i(E(\vec{k}')-E(\vec{k}))t}\right)\\
&= \frac{i}{2}\left(-\widehat{a}(-\vec{k})e^{-i2E(\vec{k})t} + \widehat{a}^\dagger(\vec{k})\right)
\end{aligned}$$

Putting these together

$$\int d^3x \left[E(\vec{k})\,\hat{\phi}(x) - i\hat{\pi}(x)\right] e^{-ik\cdot x} = \hat{a}^\dagger(\vec{k})$$

$$\int d^3x \left[E(\vec{k})\,\hat{\phi}(x) + i\hat{\pi}(x)\right] e^{-ik\cdot x} = \hat{a}(-\vec{k})e^{-2iE(\vec{k})t}$$

The second one needs a little massaging. First split the time and space parts of the exponential

$$\int d^3x \left[E(\vec{k})\,\hat{\phi}(x) + i\hat{\pi}(x)\right]\ e^{-iE(\vec{k})t}\,e^{i\vec{k}\cdot\vec{x}} = \hat{a}(-\vec{k})e^{-2iE(\vec{k})t}$$

Now, multiply by $e^{2iE(\vec{k})t}$

$$\int d^3x \left[E(\vec{k})\,\hat{\phi}(x) + i\hat{\pi}(x)\right]\ e^{iE(\vec{k})t}\,e^{i\vec{k}\cdot\vec{x}} = \hat{a}(-\vec{k})$$

Replace $\vec{k} \to -\vec{k}$

$$\int d^3x \left[E(\vec{k})\,\hat{\phi}(x) + i\hat{\pi}(x)\right]\ e^{iE(\vec{k})t}\,e^{-i\vec{k}\cdot\vec{x}} = \hat{a}(\vec{k})$$

So

$$\int d^3x \left[E(\vec{k})\,\hat{\phi}(x) + i\hat{\pi}(x)\right]\ e^{ik\cdot x} = \hat{a}(\vec{k})$$

[Alternatively we could have just taken the Hermitean conjugate of the first equation!]

The operators $\widehat{a}(x)$ and $\widehat{a}^\dagger(x)$ also obey *commutation relations*: $\left(\left[\widehat{\phi}(\vec{x},t),\widehat{\pi}(\vec{y},t)\right] = i\delta^3(\vec{x}-\vec{y})\right)$

$$\left[\widehat{a}(\vec{k}),\widehat{a}^\dagger(\vec{p})\right] = \int d^3x \int d^3y \left(-iE(\vec{k})\,\left[\widehat{\phi}(\vec{x},t),\widehat{\pi}(\vec{y},t)\right]\right.$$

$$\left. +iE(\vec{p})\left[\widehat{\pi}(\vec{x},t),\widehat{\phi}(\vec{y},t)\right]\right) e^{i(k\cdot x - p\cdot y)}$$

$$= \int d^3x \int d^3y \left(E(\vec{k})\delta^3(\vec{x}-\vec{y}) + E(\vec{p})\delta^3(\vec{x}-\vec{y})\right) e^{i(k\cdot x - p\cdot y)}$$

$$= \int d^3x \left(E(\vec{k}) + E(\vec{p})\right) e^{i(k-p)\cdot x}$$

$$= (2\pi)^3\, 2E(\vec{k})\delta^3(\vec{k}-\vec{p})$$

It is also fairly easy to show

$$\left[\widehat{a}(\vec{k}),\widehat{a}(\vec{p})\right] \;=\; \left[\widehat{a}^\dagger(\vec{k}),\widehat{a}^\dagger(\vec{p})\right] \;=\; 0.$$

Note that we could have started with these as the postulates and derived the commutation relations for ϕ and π.

Energy and Momentum

Lets return to our definition of the Hamiltonian (as a conserved quantity)

$$\hat{H} = \frac{1}{2}\int \left(\hat{\pi}^2 + \left(\vec{\nabla}\hat{\phi}\right)^2 + m^2\hat{\phi}^2\right)\, d^3x$$

$$\hat{\pi}(x) = \frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\left(-\hat{a}(\vec{k})\, e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})}\left(\hat{a}(\vec{k})\, e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$

$$\vec{\nabla}\hat{\phi}(x) = i\int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})}\,\vec{k}\,\left(\hat{a}(\vec{k})\, e^{-ik\cdot x} - \hat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$

Sticking these in

$$\hat{H} = \frac{1}{2}\int d^3x \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})}\frac{d^3p}{(2\pi)^3\, 2E(\vec{p})}$$

$$\times\Big[\quad \left(-E(\vec{k})E(\vec{p}) - \vec{k}\cdot\vec{p} + m^2\right)\hat{a}(\vec{k})\hat{a}(\vec{p})\, e^{-i(k+p)\cdot x}$$

$$+\left(-E(\vec{k})E(\vec{p}) - \vec{k}\cdot\vec{p} + m^2\right)\hat{a}^\dagger(\vec{k})\hat{a}^\dagger(\vec{p})\, e^{i(k+p)\cdot x}$$

$$+\left(E(\vec{k})E(\vec{p}) + \vec{k}\cdot\vec{p} + m^2\right)\left(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{p})\, e^{-i(k-p)\cdot x} + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{p})\, e^{i(k-p)\cdot x}\right)\Big]$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})}\frac{d^3p}{2E(\vec{p})}$$

First 2 terms zero using $E(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$

$$\times\Big[\quad \left(\cancel{-E(\vec{k})^2 + \vec{k}^2 + m^2}\right)\hat{a}(\vec{k})\hat{a}(-\vec{k})e^{-2iE(\vec{p})t}\,\delta^3(\vec{k}+\vec{p})$$

$$+\left(\cancel{-E(\vec{k})^2 + \vec{k}^2 + m^2}\right)\hat{a}^\dagger(\vec{k})\hat{a}^\dagger(-\vec{k})e^{2iE(\vec{k})t}\,\delta^3(\vec{k}+\vec{p})$$

$$+\left(E(\vec{k})^2 + \vec{k}^2 + m^2\right)\left(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})\right)\delta^3(\vec{k}-\vec{p})\Big]$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\vec{k})}\, E(\vec{k})\, \left(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})\right)$$

From the previous slide $\widehat{H} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} E(\vec{k}) \left(\widehat{a}(\vec{k})\widehat{a}^\dagger(\vec{k}) + \widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k})\right)$

Does this look familiar? Recall our earlier discussion of the *quantum harmonic oscillator*. There we had,

$$\widehat{H} = \frac{1}{2}\hbar\omega\left(\widehat{a}\widehat{a}^\dagger + \widehat{a}^\dagger\widehat{a}\right),$$

where the $\widehat{a}$ and $\widehat{a}^\dagger$ were *creation and annihilation operators*. We could generate the entire spectrum of states by acting on the ground state $|0\rangle$ with $\widehat{a}^\dagger$.

Let us *postulate* a lowest energy state $|0\rangle$ called *the vacuum*, for which $\widehat{a}(\vec{k})|0\rangle = 0$.

Then the energy of this vacuum state is,

$$E_0 = \langle 0|\widehat{H}|0\rangle = \frac{1}{4}\int \frac{d^3k}{(2\pi)^3} \left(\langle 0|\widehat{a}(\vec{k})\widehat{a}^\dagger(\vec{k})|0\rangle + \langle 0|\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k})|0\rangle\right) .$$

The second term vanishes since $\widehat{a}(\vec{k})|0\rangle = 0$ and,

$$\langle 0|\widehat{a}(\vec{k})\widehat{a}^\dagger(\vec{k})|0\rangle = \langle 0|\left((2\pi)^3\, 2E(\vec{k})\delta^3(\vec{k}-\vec{k}) + \widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k})\right)|0\rangle$$
$$= (2\pi)^3\, 2E(\vec{k})\,\delta^3(0)$$

The energy of the vacuum is then,

$$E_0 = \frac{1}{2}\delta^3(0) \int d^3k \, E(\vec{k}) = \infty$$

In hindsight this is not so surprising. We regarded our field as an infinite of harmonic oscillators. Since the ground state of the harmonic oscillator is non-zero, we expect that our vacuum energy picks up infinitely many such contributions and becomes infinite.

However, notice that observables can never measure an absolute energy scale – we only ever measure energies relative to some reference scale. Therefore, we can just *subtract* this vacuum energy from any energy we calculate and not worry about it.

Caveat:

There is a fairly major caveat to this. Gravity (General Relativity) has the energy-momentum tensor as a source term. So this vacuum energy should show up as a cosmological constant. This is known as the *cosmological constant problem*.

Normal ordering

To automate this subtraction, we define *normal ordering*. In any product of creation and annihilation operators which are normal ordered, the *annihilation operators appear to the right* of the creation operators. The notation is to surround normal ordered operators with colons.

So $:\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) := \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$,

and,

$$: \hat{H} := \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} E(\vec{k}) \left(: \hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) : + : \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) : \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} E(\vec{k})\, \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$$

The vacuum energy is then

$$E_0 = \langle 0 | : \hat{H} : | 0 \rangle = \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} E(\vec{k}) \, \langle 0 | \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) | 0 \rangle = 0$$

The energy of states

Consider the state $|\vec{k}\rangle \equiv \widehat{a}(\vec{k})^{\dagger}|0\rangle$.

We can calculate its energy using the Hamiltonian:

$$: \widehat{H} := \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} E(\vec{k})\, \widehat{a}^{\dagger}(\vec{k})\widehat{a}(\vec{k})$$

$$\begin{aligned}
: \widehat{H} : |\vec{k}\rangle &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \widehat{a}^{\dagger}(\vec{p})\, \widehat{a}(\vec{p})\, \widehat{a}^{\dagger}(\vec{k})|0\rangle \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \widehat{a}^{\dagger}(\vec{p})\, \left[\widehat{a}(\vec{p}), \widehat{a}^{\dagger}(\vec{k})\right] |0\rangle \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \widehat{a}^{\dagger}(\vec{p})\, (2\pi)^3\, 2E(\vec{p})\delta^3(\vec{k}-\vec{p})|0\rangle \\
&= \int d^3p\, E(\vec{p})\delta^3(\vec{k}-\vec{p})a^{\dagger}(\vec{p})|0\rangle \\
&= E(\vec{k})|\vec{k}\rangle
\end{aligned}$$

As we hoped, the state $|\vec{k}\rangle$ is an energy eigenstate with energy $E(\vec{k})$.

The momentum operator

We can follow a similar approach for the momentum operator.

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \hat{\phi}\right)}\, \partial^i \hat{\phi} - g^{0i}\, \mathcal{L} \;\; = \hat{\pi}\, \partial^i \hat{\phi}$$

watch the sign, $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$

So,

$$\vec{\hat{P}} = -\int \hat{\pi} \vec{\nabla} \hat{\phi}\, d^3x$$

$$= \frac{1}{2}\int d^3x \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3\, 2E(\vec{p})} \left(-\hat{a}(\vec{k})\, e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$
$$\times \vec{p}\, \left(\hat{a}(\vec{p})\, e^{-ip\cdot x} - \hat{a}^\dagger(\vec{p})\, e^{ip\cdot x}\right)$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \frac{d^3p}{2E(\vec{p})}\, \vec{p} \Big(-\hat{a}(\vec{k})\, \hat{a}(\vec{p})\, \delta^3(\vec{k}+\vec{p}) e^{-i(E(\vec{k})+E(\vec{p}))t}$$
$$+\hat{a}(\vec{k})\, \hat{a}^\dagger(\vec{p})\, \delta^3(\vec{k}-\vec{p}) e^{-i(E(\vec{k})-E(\vec{p}))t}$$
$$+\hat{a}^\dagger(\vec{k})\, \hat{a}(\vec{p})\, \delta^3(\vec{k}-\vec{p}) e^{i(E(\vec{k})-E(\vec{p}))t}$$
$$-\hat{a}^\dagger(\vec{k})\, \hat{a}^\dagger(\vec{p})\, \delta^3(\vec{k}+\vec{p}) e^{i(E(\vec{k})+E(\vec{p}))t} \Big)$$

$$\hat{\vec{P}} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\vec{k})}\,\vec{k}\left(\hat{a}(\vec{k})\,\hat{a}(-\vec{k})\,e^{-2iE(\vec{k})t} + \hat{a}(\vec{k})\,\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\,\hat{a}(\vec{k})\right.$$
$$\left. + \hat{a}^\dagger(\vec{k})\,\hat{a}^\dagger(-\vec{k})\,e^{2iE(\vec{k})t}\right)$$

Notice that in the first and last terms the integrands are *antisymmetric* under $\vec{k} \rightarrow -\vec{k}$, so these terms vanish, leaving us with,

$$\hat{\vec{P}} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\vec{k})}\,\vec{k}\left(\hat{a}(\vec{k})\,\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\,\hat{a}(\vec{k})\right).$$

We could have guessed this form from covariance. Together with the Hamiltonian,

$$\hat{H} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\vec{k})}\,E(\vec{k})\,\left(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})\right)$$

we have,

$$\hat{P}^\mu = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\vec{k})}\,k^\mu\left(\hat{a}(\vec{k})\,\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\,\hat{a}(\vec{k})\right)$$

States with multiple particles

One of the biggest advantages of Quantum Field Theory is that we are now able to describe systems with *many particles*.

For example, a two particle state is $|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle$

Now we have,

$$
\begin{aligned}
:\hat{H}: |\vec{k}_1, \vec{k}_2\rangle &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \hat{a}^\dagger(\vec{p})\, \hat{a}(\vec{p})\, \hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \hat{a}^\dagger(\vec{p})\, \left(\left[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{k}_2)\right] + \hat{a}^\dagger(\vec{k}_2)\hat{a}(\vec{p})\right)\, \hat{a}^\dagger(\vec{k}_1)|0\rangle \\
&= E(\vec{k}_2)\, \hat{a}^\dagger(\vec{k}_2)\, \hat{a}^\dagger(\vec{k}_1)|0\rangle + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \hat{a}^\dagger(\vec{p})\, \hat{a}^\dagger(\vec{k}_2)\, \left[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{k}_1)\right] |0\rangle \\
&= E(\vec{k}_2)\, \hat{a}^\dagger(\vec{k}_2)\, \hat{a}^\dagger(\vec{k}_1)|0\rangle + E(\vec{k}_1)\, \hat{a}^\dagger(\vec{k}_1)\, \hat{a}^\dagger(\vec{k}_2)|0\rangle \\
&= \left(E(\vec{k}_1) + E(\vec{k}_2)\right)\, |\vec{k}_1, \vec{k}_2\rangle
\end{aligned}
$$

as expected.

Some additional comments

- Note that

$$|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^\dagger(\vec{k}_2)\hat{a}(\vec{k_1})^\dagger|0\rangle = \hat{a}^\dagger(\vec{k}_1)\hat{a}(\vec{k_2})^\dagger|0\rangle = |\vec{k}_2, \vec{k}_1\rangle$$

so the state is *symmetric* in interchange of particles and these are *bosons*.

- The operator $\hat{N} = \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})}\, \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$ measures the *number of particles*,

so $$\hat{N}|\vec{k}_1 \ldots \vec{k}_n\rangle = n\,|\vec{k}_1 \ldots \vec{k}_n\rangle$$

- Normalisation of these fields is tricky.

$$\langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k})|0\rangle = \langle 0|\left[\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k})\right]|0\rangle = (2\pi)^3\,\delta^3(0)E(\vec{k}\langle 0|0\rangle = \infty$$

Need to put the field in a box with periodic boundary conditions. This discretizes the momenta, making the states normalisable. Then take the box size to infinity.

The complex scalar field

More generally, the field $\widehat{\phi}(x)$ could be complex. The Lagrangian for a *complex scalar field* is,

$$\boxed{\mathcal{L} = \partial_\mu \widehat{\phi}^\dagger \, \partial^\mu \widehat{\phi} - m^2 \widehat{\phi}^\dagger \widehat{\phi}}$$

We have two Euler-Lagrange Equation, one for $\widehat{\phi}(x)$ and one for $\widehat{\phi}^\dagger(x)$. The one for $\widehat{\phi}^\dagger(x)$ gives

$$\frac{\partial \mathcal{L}}{\partial \widehat{\phi}^\dagger} = -m^2 \widehat{\phi} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \widehat{\phi}^\dagger\right)} = \partial^\mu \widehat{\phi} \quad \Rightarrow \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \widehat{\phi}^\dagger\right)} \right) = \partial_\mu \partial^\mu \widehat{\phi}$$

resulting in the *Klein-Gordon Equation* just as before:

$$\boxed{\left(\partial^2 + m^2\right) \widehat{\phi} = 0}$$

[The other Euler-Lagrange Equation gives the complex conjugate of this.]

Similarly to the real scalar field, this has a general solution of the form,

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} \left(\hat{a}(\vec{k})\, e^{-ik\cdot x} + \hat{b}^\dagger(\vec{k})\, e^{ik\cdot x} \right)$$

$$\hat{\phi}(x)^\dagger = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} \left(\hat{a}^\dagger(\vec{k})\, e^{-ik\cdot x} + \hat{b}(\vec{k})\, e^{ik\cdot x} \right)$$

Notice that we now have *two sets of creation and annihilation operators*.

The commutation relations are,

$$\left[\hat{a}(\vec{k}), \hat{a}(\vec{p})\right] = \left[\hat{a}(\vec{k}), \hat{b}(\vec{p})\right] = \left[\hat{b}(\vec{k}), \hat{b}(\vec{p})\right] = 0.$$

$$\left[\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{p})\right] = \left[\hat{a}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{p})\right] = \left[\hat{b}^\dagger(\vec{k}), \hat{b}^\dagger(\vec{p})\right] = 0.$$

$$\left[\hat{a}(\vec{k}), \hat{b}^\dagger(\vec{p})\right] = \left[\hat{a}^\dagger(\vec{k}), \hat{b}(\vec{p})\right] = 0$$

$$\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{p})\right] = (2\pi)^3\, 2E(\vec{k})\delta^3(\vec{k} - \vec{p})$$

$$\left[\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{p})\right] = (2\pi)^3\, 2E(\vec{k})\delta^3(\vec{k} - \vec{p})$$

We have two "species", created by $\widehat{a}^\dagger,\ \widehat{b}^\dagger$ and annihilated by $\widehat{a},\ \widehat{b}$.

So the state $\widehat{a}^\dagger\widehat{b}^\dagger|0\rangle$, for example, contains one of "particle" of each type.

The four-momentum operator is now,

$$\widehat{P}^\mu = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E(\vec{k})}\, k^\mu \left(\widehat{a}(\vec{k})\,\widehat{a}^\dagger(\vec{k}) + \widehat{a}^\dagger(\vec{k})\,\widehat{a}(\vec{k}) + \widehat{b}(\vec{k})\,\widehat{b}^\dagger(\vec{k}) + \widehat{b}^\dagger(\vec{k})\,\widehat{b}(\vec{k})\right)$$

so,

$$:\widehat{P}^\mu: = \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})}\, k^\mu \left(\widehat{a}^\dagger(\vec{k})\,\widehat{a}(\vec{k}) + \widehat{b}^\dagger(\vec{k})\,\widehat{b}(\vec{k})\right)$$

and if we operate this on our example state,

$$:\widehat{P}^\mu:\widehat{a}^\dagger(\vec{k}_2)\widehat{a}^\dagger(\vec{k}_1)|0\rangle = \left(k_1^\mu + k_2^\mu\right)|0\rangle$$

as expected, but now also we have,

$$:\widehat{P}^\mu:\widehat{a}^\dagger(\vec{k}_2)\widehat{b}^\dagger(\vec{k}_1)|0\rangle = \left(k_1^\mu + k_2^\mu\right)|0\rangle$$

Note: This looks more complicated than the real scalar field. This is true, but it is not as different as it looks. Write,

$$\hat{\phi}(x) = \frac{1}{\sqrt{2}}\left(\hat{\phi}_1(x) + i\hat{\phi}_2(x)\right)$$

Normalization to make all fields have $\int \hat{\phi}^*\hat{\phi}\,dx = \int \hat{\phi}_1^2\,dx = \int \hat{\phi}_2^2\,dx = 1$

then

$$\mathcal{L} = \frac{1}{2}\partial_\mu\hat{\phi}_1\,\partial^\mu\hat{\phi}_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}\partial_\mu\hat{\phi}_2\,\partial^\mu\hat{\phi}_2 - \frac{1}{2}m^2\phi_2^2 \,.$$

So, at the moment, this new model is just *two real scalar fields*. The momentum is just the sum

$$\hat{P}^\mu = \hat{P}_1^\mu + \hat{P}_2^\mu$$

where

$$\hat{P}_i^\mu = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3 2E(\vec{k})}\,k^\mu\left(\hat{a}_i(\vec{k})\,\hat{a}_i^\dagger(\vec{k}) + \hat{a}_i^\dagger(\vec{k})\,\hat{a}_i(\vec{k})\right) \qquad (i = 1, 2)$$

The connection with the previous expression is made by noting

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2)\,, \qquad \hat{b} = \frac{1}{\sqrt{2}}(\hat{a}_1 - i\hat{a}_2)\,.$$

Charge conservation

The complex scalar Lagrangian has a *symmetry* under $\widehat{\phi}(x) \to e^{i\theta}\widehat{\phi}(x)$.

$$\mathcal{L} = \partial_\mu \widehat{\phi}^\dagger \, \partial^\mu \widehat{\phi} - \frac{1}{2} m^2 \widehat{\phi}^\dagger \widehat{\phi}$$
$$\to \partial_\mu \widehat{\phi}^\dagger e^{-i\theta} \, e^{i\theta} \partial^\mu \widehat{\phi} - \frac{1}{2} m^2 \widehat{\phi}^\dagger e^{-i\theta} \, e^{i\theta} \widehat{\phi} = \mathcal{L}$$

as long as $\theta^* = \theta$ and $\partial_\mu \theta = 0$.

Recall Noether's theorem tells us that there must be an associated *conserved current* $\widehat{j}^\mu(x)$.

The infinitesimal transformation is $\widehat{\phi}(x) \to \widehat{\phi}(x) + i\theta\widehat{\phi}(x)$ so we have

$$\widehat{j}^\mu = -i \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \widehat{\phi} \right)} \widehat{\phi} + i \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \widehat{\phi}^\dagger \right)} \widehat{\phi}^\dagger = i \left(\widehat{\phi} \, \partial^\mu \widehat{\phi}^\dagger - \widehat{\phi}^\dagger \, \partial^\mu \widehat{\phi} \right)$$

$\widehat{\phi}^\dagger(x)$ also transforms via
$\widehat{\phi}^\dagger(x) \to \widehat{\phi}^\dagger(x) - i\theta\widehat{\phi}^\dagger(x)$

Since we have a conserved current, we have a *conserved charge*, the time-component.

$$\widehat{Q} = \int i\left(\widehat{\phi}\,\widehat{\pi}^{\dagger} - \widehat{\phi}^{\dagger}\,\widehat{\pi}\right)\, d^3x\,.$$

The *normal ordered* operator is,

$$:\widehat{Q}: = \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} \left(\widehat{a}^{\dagger}(\vec{k})\,\widehat{a}(\vec{k}) - \widehat{b}^{\dagger}(\vec{k})\,\widehat{b}(\vec{k})\right),$$

so we have,

$$\left[:\hat{Q}:, \hat{a}^{\dagger}\right] = \hat{a}^{\dagger} \quad \text{and} \quad \left[:\hat{Q}:, \hat{b}^{\dagger}\right] = -\hat{b}^{\dagger}\,.$$

These species are distinguishable by eigenvalues of $:\hat{Q}:$,

$:\hat{Q}:\hat{a}^{\dagger}|0\rangle = \hat{a}^{\dagger}|0\rangle$ particle "a" has *positive charge*

$:\hat{Q}:\hat{b}^{\dagger}|0\rangle = -\hat{b}^{\dagger}|0\rangle$ particle "b" has *negative charge*

We interpret particle b as being the *antiparticle* of particle a.

The Heisenberg picture

Notice that everything we have done so far has been in the *Heisenberg picture*. The *operators* such as $\widehat{\phi}(x)$ and $\widehat{\pi}(x)$ all *depend on time*.

It is fairly straightforward to show that,

$$\dot{\widehat{\phi}}(x) = i\left[\widehat{H}, \widehat{\phi}(x)\right] = \widehat{\pi}(x)$$

and also

$$\dot{\widehat{\pi}}(x) = i\left[\widehat{H}, \widehat{\pi}(x)\right] = \nabla^2\widehat{\pi}(x) - m^2\widehat{\phi}^2,$$

which is the equation of motion – the *Klein-Gordon Equation*.

Similarly, the *states*, e.g. $\hat{a}^\dagger(\vec{k})|0\rangle$ are *time independent*.

In principle, we could have done all this in the Schrödinger picture, but things become more complicated (and look less covariant).

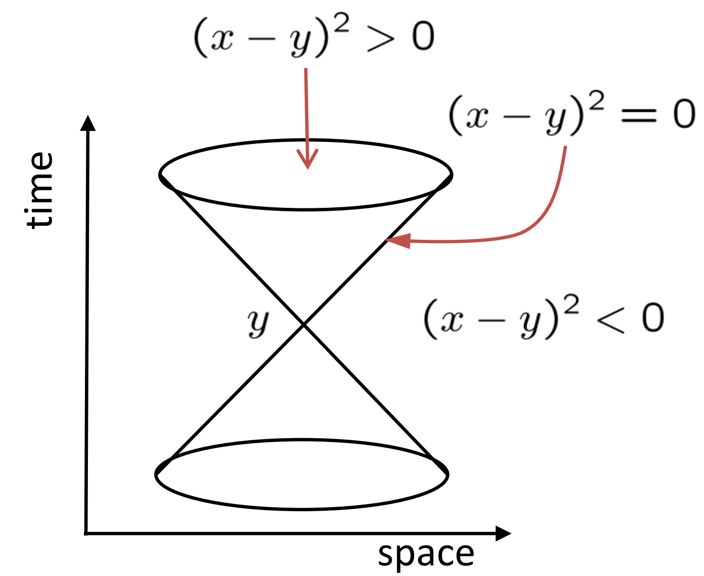
Causality

Our original postulate was that the field (operators) satisfy *same-time* commutation relations.

e.g. for the real scalar field

$$\left[\hat{\phi}(\vec{x},t),\hat{\pi}(\vec{y},t)\right] = i\delta^3(\vec{x}-\vec{y})$$
$$\left[\hat{\phi}(\vec{x},t),\hat{\phi}(\vec{y},t)\right] = \left[\hat{\pi}(\vec{x},t),\hat{\pi}(\vec{y},t)\right] = 0$$

But what do we have if we allow the *times to be different*?



When two space-time points are at the same time, their separation is *space-like*, i.e.$(x-y)^2<0$

Under a Lorentz transformation, space-like points remain space-like since $(x-y)^2$ is Lorentz invariant.

The commutator $\left[\hat{\phi}(\vec{x}),\hat{\phi}(\vec{y})\right]$ is also Lorentz invariant since the field operators are, so if it is zero for one frame In which the separation is space-like, it is zero in all frames.

$$\Rightarrow \quad \left[\hat{\phi}(\vec{x}),\hat{\phi}(\vec{y})\right] = 0 \quad \text{for} \quad (x-y)^2<0$$

A similar argument holds for $\left[\hat{\phi}(\vec{x}),\hat{\pi}(\vec{y})\right]$

The Propagator

What is the amplitude for a particle to "*propagate*" from a point y to a point x?

To calculate this we need to project the initial state $\hat{\phi}(y)|0\rangle$ onto the final state $\hat{\phi}(x)|0\rangle$.

For the *real* scalar field, this is,

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} \left(\hat{a}(\vec{k})\, e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$

$$\begin{aligned} D(x-y) &= \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} \frac{d^3p}{(2\pi)^3 2E(\vec{p})} \langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(\vec{p})|0\rangle\, e^{-i(k\cdot x - p\cdot y)} \\ &= \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})}\, e^{-ik\cdot(x-y)} \end{aligned}$$

This amplitude is called the *propagator* of the theory.

One can show (the integral is a little tricky) that for a *space-like* x-y with x^0 - y^0 = 0 this becomes

$$D(x-y) \sim e^{-m|\vec{x}-\vec{y}|}$$

The amplitude is non-zero even outside the light-cone!

What went "wrong"? (Actually nothing, as we will see.)

Let's go back to the commutator:

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} \frac{d^3p}{(2\pi)^3 2E(\vec{p})} \left(\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{p})\right] e^{-i(k\cdot x - p\cdot y)} + \left[\hat{a}^\dagger(\vec{k}), \hat{a}(\vec{p})\right] e^{i(k\cdot x - p\cdot y)}\right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right) \quad = D(x-y) - D(y-x)$$

This gives us a clue as to how to interpret our result.

If x-y is *space-like* then there is no well defined sense in which x or y happened "*first*". I can always chose a frame of reference where x happens before y, and equally, I could chose a frame where y happens before x.

So we have to add together the amplitude for travelling from x to y, with that from y to x. *The two contributions cancel.*

For a *complex scalar field*, the amplitude for a particle travelling from x to y, cancels with that of the *antiparticle* travelling from y to x. (For the real scalar field, the particle is its own antiparticle.)

The Feynman propagator

Let's define a new quantity, the *Feynman propagator*, by

$$\Delta_F(x-y) = \begin{cases} \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle & \text{for} \quad x^0 > y^0 \\ \langle 0|\hat{\phi}(y)\hat{\phi}(x)|0\rangle & \text{for} \quad x^0 < y^0 \end{cases}$$

This can be more compactly written by introducing (yet another!) new notation. We define the *Time Ordering operator*,

$$T\,\hat{\phi}(x)\hat{\phi}(y) = \begin{cases} \hat{\phi}(x)\hat{\phi}(y) & \text{for} \quad x^0 > y^0 \\ \hat{\phi}(y)\hat{\phi}(x) & \text{for} \quad x^0 < y^0 \end{cases}$$

So the Feynman propagator becomes $\Delta_F(x-y) = \langle 0|T\,\hat{\phi}(x)\hat{\phi}(y)|0\rangle$

Using *contour integration in the complex plane*, one can show that this can be written

$$i\Delta_F(x-y) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}$$

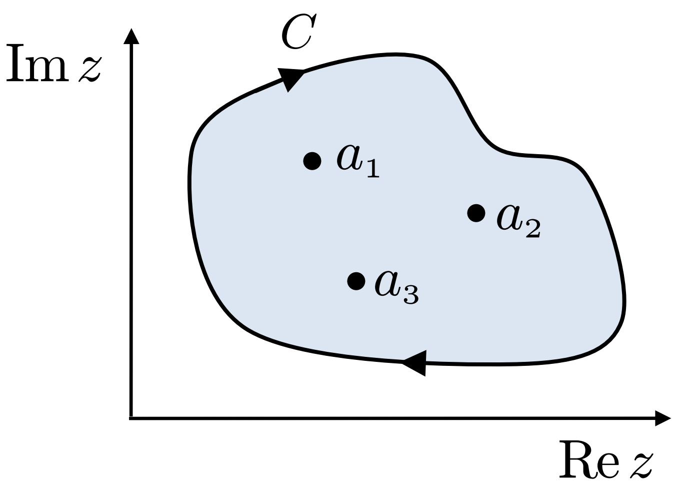
It is quite easy to show that this is a *Green's function for the Klein Gordon Equation*:

$$\left(\partial^2 + m^2 - i\epsilon\right) i\Delta_F(x-y) = -\int \frac{d^4k}{(2\pi)^4} \frac{\left(-k^2 + m^2 - i\epsilon\right)\, e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} = \delta^4(x-y)$$

An Aside: Cauchy's Residue Theorem

The integration of a function over a *closed curve in the complex plane*, is equal to the *sum of the residues* of the function in the area enclosed, multiplied by 2πi.

$$\oint_C f(z)dz = 2\pi i \sum_k \mathrm{Res}(f, a_k)$$



For a function of the form

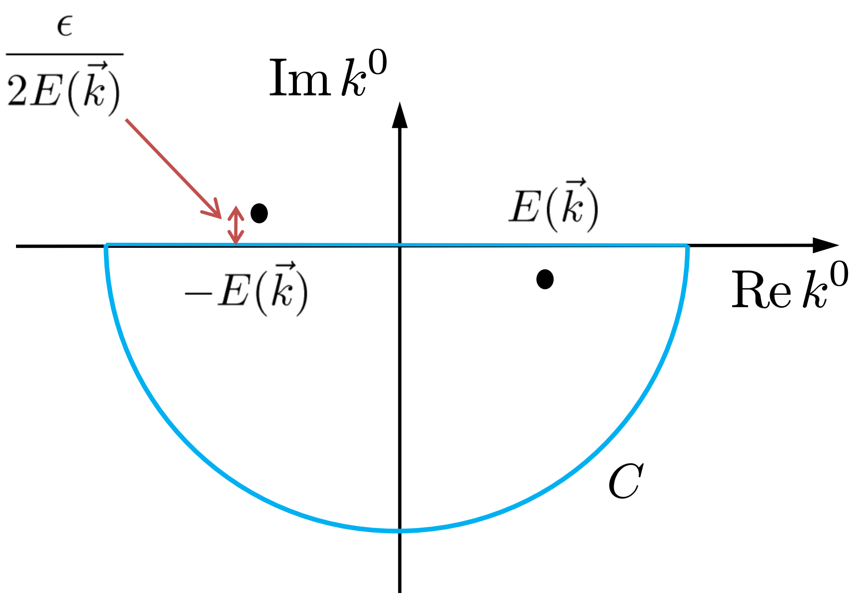
$$f(z) = \sum_k \frac{g_k(z)}{z - a_k}$$

the residues are $g_k(a_k)$.

Now the Feynman propagator is

$$i\Delta_F(x-y) = -\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon} \quad = -\int dk^0 \frac{d^3k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{\left(k^0\right)^2 - E(\vec{k})^2 + i\epsilon}$$

$$= -\int dk^0 \frac{d^3k}{(2\pi)^4}\frac{e^{-ik^0(x^0-y^0)}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\left[k^0 - E(\vec{k}) + i\epsilon/2E(\vec{k})\right]\left[k^0 + E(\vec{k}) - i\epsilon/2E(\vec{k})\right]}$$



Now let us write

$$e^{-ik^0(x^0-y^0)} = e^{-i\mathrm{Re}k^0(x^0-y^0)}e^{\mathrm{Im}k^0(x^0-y^0)}$$

If $x^0 > y^0$ then we can *complete the contour in the lower half plane*. In the limit where the radius of the half-sphere goes to infinity, the extra contribution will vanish and we have

$$i\Delta_F(x-y) = -\oint_C dk^0 \frac{d^3k}{(2\pi)^4}\frac{e^{-ik^0(x^0-y^0)}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\left[k^0 - E(\vec{k}) + i\epsilon/2E(\vec{k})\right]\left[k^0 + E(\vec{k}) - i\epsilon/2E(\vec{k})\right]}$$

The $i\epsilon$ prescription causes only one pole to be enclosed. We apply the residue theorem:

$$i\Delta_F(x-y) = -2\pi i \int \frac{d^3k}{(2\pi)^4} \frac{e^{-iE(\vec{k})(x^0-y^0)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{2E(\vec{k})}$$

$$= i \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} e^{-ik\cdot(x-y)}$$

$$= iD(x-y)$$

residue at
$k^0 = E(\vec{k})$

If $x^0 < y^0$ then I need to *complete the curve in the upper half plane* if I want the extra contribution to vanish. This picks out the pole at $k^0 = -E(\vec{k})$.

Then

$$i\Delta_F(x-y) = -2\pi i \int \frac{d^3k}{(2\pi)^4} \frac{e^{iE(\vec{k})(x^0-y^0)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{2E(\vec{k})}$$

$$= i \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} e^{ik\cdot(x-y)}$$

$$= iD(y-x)$$

mapped
$\vec{k} \rightarrow -\vec{k}$

3. Interacting Scalar Fields

We will consider an interacting field theory based on the Lagrangian,

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \hat{\phi}\, \partial^\mu \hat{\phi} - \frac{1}{2} m^2 \hat{\phi}^2 - \frac{\lambda}{4!}\hat{\phi}^4}$$

kinetic term

mass term

interaction

- This is a *real scalar field*

- The interaction is the only interaction term we can add that has a *dimensionless* coupling λ

- This is known as ϕ^4 *theory* (for obvious reasons).

Our earlier result for the 00 component of the energy-momentum tensor,

$$\hat{T}^{00} = \frac{\partial \hat{\mathcal{L}}}{\partial \left(\partial_0 \hat{\phi} \right)} \partial^0 \hat{\phi} - \hat{\mathcal{L}}$$

leads to *a new Hamiltonian*,

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

where $\hat{H}_0$ is the Hamiltonian of our *free scalar field theory* and

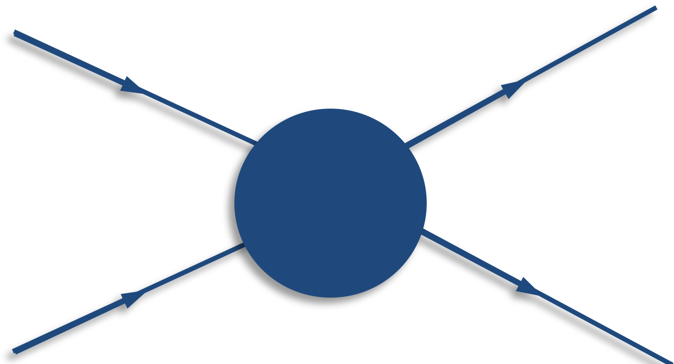
$$\hat{H}_{\text{int}} = -\int \hat{\mathcal{L}}_{\text{int}} \, d^3x = \int \frac{\lambda}{4!} \hat{\phi}^4(x) \, d^3x$$

Ultimately, we want to make predictions for *particle scattering*, that we can test in experiment.

We will see that, if λ is small enough, we can use perturbation theory in order to work out *scattering cross-sections*.

The S Matrix

Let us imagine that we have some *initial state* of particles $|i\rangle$ at time $t = -\infty$ which *interact* with one another before resulting in a *final state* $|f\rangle$ at time $t = \infty$.



- We *assume* the initial and final states are eigenstates of the free Hamiltonian. (This is not true! - see next slide!)

- The state undergoes the interaction and becomes a state $\widehat{S}|i\rangle$.

- This operator $\widehat{S}$ is known as the *S-Matrix*

- After the interaction we measure the energy and momentum of the final states and it "collapses" to a new eigenstate $|f\rangle$ with probability $|\langle f|\widehat{S}|i\rangle|^2$.

- There can be any number of particles in initial and final state (the diagram is just an example)

- We can't handle bound states in this way.

Our task then is to calculate $\langle f|\widehat{S}|i\rangle$.

The vacuum

In the free theory we had a lowest energy state, the vacuum $|0\rangle$, from which we generate particles via creation operators. It is defined by $\widehat{a}|0\rangle = 0$.

However, recall that the $\widehat{a}$ are just coefficients in the Fourier expansion of $\widehat{\phi}$ which is itself a solution of the equation of motion, as derived from the Lagrangian.

So, for an interacting theory, we, in principle, have a *different* $\widehat{\phi}$ as a solution of our new equation of motion, and we have a new $\widehat{a}$ and a *different vacuum* $|\Omega\rangle$.

Similarly, any state we represent with $|i\rangle$ or $|f\rangle$ are not eigenstates of the free theory, since they will interact with a cloud of *virtual particles* from the surrounding vacuum.

For now, we are going to ignore this issue. We will see some consequences of this later.

The Interaction (Dirac) Picture

We have already seen the Schrödinger and Heisenberg pictures – now it is time to introduce another one, the *interaction (or Dirac) picture*.

Previously we had

$$\hat{\phi}_H(\vec{x},t) = e^{i\hat{H}(t-t_0)}\hat{\phi}_S(\vec{x})e^{-i\hat{H}(t-t_0)}$$

The Heisenberg operators change with time but the Schrödinger operators do not.

In principle, this is all we need. $\hat{\phi}_H(\vec{x},t_0) = \hat{\phi}_S(\vec{x})$ is the operator at the start and we know how this changes with time.

In practice, this is going to be too hard to solve, so we need to be a bit cleverer.

We *define operators in the interaction picture* according to

$$\hat{\mathcal{O}}_I(\vec{x},t) = e^{i\hat{H}_0(t-t_0)}\hat{\mathcal{O}}_S(\vec{x})e^{-i\hat{H}_0(t-t_0)}$$

and in particular,

$$\hat{\phi}_I(\vec{x},t) = e^{i\hat{H}_0(t-t_0)}\hat{\phi}_S(\vec{x})e^{-i\hat{H}_0(t-t_0)}$$

where $\hat{H}_0$ is the *Hamiltonian of the free theory*.

We have already solved this for $\hat{\phi}_I(\vec{x},t)$ since this is just the Heisenberg field operator for the non-interacting theory.

$$\hat{\phi}_I(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} \left(\hat{a}(\vec{k})\, e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})\, e^{ik\cdot x}\right)$$

This is related to the field operator for the interacting theory in the Heisenberg picture by

$$\begin{aligned}
\hat{\phi}_H(\vec{x},t) &= e^{i\hat{H}(t-t_0)}\hat{\phi}_S(\vec{x})e^{-i\hat{H}(t-t_0)} \\
&= e^{i\hat{H}(t-t_0)}\left[e^{-i\hat{H}_0(t-t_0)}\hat{\phi}_I(\vec{x})e^{i\hat{H}_0(t-t_0)}\right]e^{-i\hat{H}(t-t_0)} \\
&= \hat{U}^\dagger(t,t_0)\hat{\phi}_I(\vec{x})\hat{U}(t,t_0)
\end{aligned}$$

where we have defined the *time evolution operator*

$$\hat{U}(t,t_0) \equiv e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)}$$

All we need to do is write $\hat{U}(t,t_0)$ in terms of the interaction picture fields $\hat{\phi}_I(\vec{x},t)$.

Note that $\hat{U}(t, t_0)$ is *not* just equal to $e^{-i\hat{H}_{\text{int}}(t-t_0)}$ since the interacting and free Hamiltonians don't commute!

For non-commuting objects, one must use the Campbell-Baker-Hausdorff expansion,

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$$

It is easiest to differentiate $\hat{U}(t, t_0)$ to construct a differential equation:

$$i\frac{\partial}{\partial t}\hat{U}(t,t_0) = i\left(\frac{\partial}{\partial t}e^{i\hat{H}_0(t-t_0)}\right)e^{-i\hat{H}(t-t_0)} + ie^{i\hat{H}_0(t-t_0)}\left(\frac{\partial}{\partial t}e^{-i\hat{H}(t-t_0)}\right)$$

$$= -e^{i\hat{H}_0(t-t_0)}\hat{H}_0 e^{-i\hat{H}(t-t_0)} + e^{i\hat{H}_0(t-t_0)}\hat{H}e^{-i\hat{H}(t-t_0)}$$

$$= e^{i\hat{H}_0(t-t_0)}\hat{H}_{\text{int}}e^{-i\hat{H}(t-t_0)}$$

$$= \underbrace{e^{i\hat{H}_0(t-t_0)}\hat{H}_{\text{int}}e^{-i\hat{H}_0(t-t_0)}}_{\hat{H}_{\text{int},\,I}}\underbrace{e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)}}_{\hat{U}(t,t_0)}$$

Beware: Peskin and Schröder call this $\hat{H}_I$

We need to solve $i\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_{\mathrm{int},\,I}\,\hat{U}(t,t_0)$ with boundary condition $\hat{U}(t_0,t_0) = 1$.

In *integral form* this is

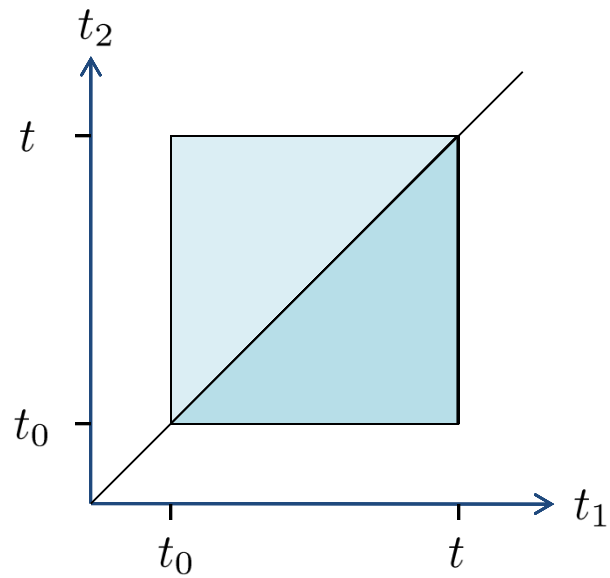
$$\hat{U}(t,t_0) = 1 - i\int_{t_0}^{t} dt_1\,\hat{H}_{\mathrm{int},\,I}(t_1)\,\hat{U}(t_1,t_0),$$

which we can solve *iteratively*,

$$\hat{U}(t,t_0) = 1 - i\int_{t_0}^{t} dt_1\,\hat{H}_{\mathrm{int},\,I}(t_1)\left(1 - i\int_{t_0}^{t_1} dt_2\,\hat{H}_{\mathrm{int},\,I}(t_2)\,\hat{U}(t_2,t_0)\right)$$

Repeating this ad infinitum,

$$\hat{U}(t,t_0) = \sum_{n=0}^{\infty}(-i)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\ldots\int_{t_0}^{t_{n-1}} dt_n\,\hat{H}_{\mathrm{int},\,I}(t_1)\,\hat{H}_{\mathrm{int},\,I}(t_2)\ldots\hat{H}_{\mathrm{int},\,I}(t_n)$$



We can make this simpler by changing the area of integration. For example, when $n = 2$,

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \hat{H}_{\text{int},\,I}(t_1)\, \hat{H}_{\text{int},\,I}(t_2)$$

$$= \frac{1}{2}\int_{t_0}^{t} dt_1 dt_2\, T\left\{\hat{H}_{\text{int},\,I}(t_1)\, \hat{H}_{\text{int},\,I}(t_2)\right\}$$

time ordering

symmetric under $t_1 \leftrightarrow t_2$

Then we can write,

$$\hat{U}(t, t_0) = \sum_{n=0}^{\infty}(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n\, \hat{H}_{\text{int},\,I}(t_1)\, \hat{H}_{\text{int},\,I}(t_2) \ldots \hat{H}_{\text{int},\,I}(t_n)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 dt_2 \ldots dt_n\, T\left\{\hat{H}_{\text{int},\,I}(t_1)\, \hat{H}_{\text{int},\,I}(t_2) \ldots \hat{H}_{\text{int},\,I}(t_n)\right\}$$

$$\equiv T e^{\left[-i \int_{t_0}^{t} dt'\, \hat{H}_{\text{int},\,I}(t')\right]}$$

notation: the exponential is defined by the Taylor series, so this is the expansion with each term time ordered

Now we return to our S-matrix,

$$\langle f|\hat{S}|i\rangle = \lim_{t_\pm \to \pm\infty} \langle f|\hat{U}(t_+, t_-)|i\rangle$$
$$= \langle f|Te^{\left[-i\int_{-\infty}^{\infty} dt'\, \hat{H}_{\mathrm{int},\, I}(t')\right]}|i\rangle$$
$$= \langle f|Te^{\left[-i\int \frac{\lambda}{4!}\hat{\phi}_I^4(x)\, d^4x\right]}|i\rangle$$

using $\hat{H}_{\mathrm{int},I} = \int \frac{\lambda}{4!}\hat{\phi}_I^4(x)\, d^3x$

This is still very difficult to calculate, but we can make an approximation if λ is small – we expand in powers of λ. This is *perturbation theory.*

$$\langle f|\hat{S}|i\rangle = \langle f|i\rangle - i\frac{\lambda}{4!}\int d^4x\, \langle f|T\hat{\phi}_I^4(x)\,|i\rangle + \left(-i\frac{\lambda}{4!}\right)^2 \int d^4x\, d^4x'\, \langle f|T\hat{\phi}_I^4(x)\hat{\phi}_I^4(x')\,|i\rangle + \cdots$$

Potentially the time-ordering could be a nuisance in calculating this further, but thankfully, we are able to replace time-ordering by *normal-ordered products and propagators*, using *Wick's Theorem*.

(From now on I am going to drop the *I* subscript on my fields – they will always be in the interaction picture.)

Wick's Theorem

Let's consider the simplest case first, $T\hat{\phi}(x)\hat{\phi}(y)$, and for ease of notation, let's split the field into positive and negative frequency parts,

$$\hat{\phi}(x) = \hat{\phi}^{+} + \hat{\phi}^{-}$$

$$\hat{\phi}^{+}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})}\, \hat{a}(\vec{k})\, e^{-ik\cdot x} \qquad \hat{\phi}^{-}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})}\, \hat{a}^{\dagger}(\vec{k})\, e^{ik\cdot x}$$

(This stupid notation is allegedly Pauli's fault.)

For $x^0 > y^0$,

$$\begin{aligned} T\hat{\phi}(x)\hat{\phi}(y) &= \left(\hat{\phi}^{+}(x) + \hat{\phi}^{-}(x)\right)\left(\hat{\phi}^{+}(y) + \hat{\phi}^{-}(y)\right) \\ &= \hat{\phi}^{+}(x)\hat{\phi}^{+}(y) + \hat{\phi}^{+}(x)\hat{\phi}^{-}(y) + \hat{\phi}^{-}(x)\hat{\phi}^{+}(y) + \hat{\phi}^{-}(x)\hat{\phi}^{-}(y) \\ &= \hat{\phi}^{+}(x)\hat{\phi}^{+}(y) + \hat{\phi}^{-}(y)\hat{\phi}^{+}(x) + \hat{\phi}^{-}(x)\hat{\phi}^{+}(y) + \hat{\phi}^{-}(x)\hat{\phi}^{-}(y) + \left[\hat{\phi}^{+}(x), \hat{\phi}^{-}(y)\right] \\ &=: \hat{\phi}(x)\hat{\phi}(y) : + D(x-y) \end{aligned}$$

In the last step we have used,

$$D(x-y) = \langle 0|\widehat{\phi}(x)\widehat{\phi}(y)|0\rangle = \langle 0|\widehat{\phi}^{+}(x)\widehat{\phi}^{-}(y)|0\rangle$$
$$= \langle 0|\left[\widehat{\phi}^{+}(x),\widehat{\phi}^{-}(y)\right]|0\rangle = \left[\widehat{\phi}^{+}(x),\widehat{\phi}^{-}(y)\right]$$

For $x^0 < y^0$ we have instead,

$$T\widehat{\phi}(x)\widehat{\phi}(y) = \left(\widehat{\phi}^{+}(y)+\widehat{\phi}^{-}(y)\right)\left(\widehat{\phi}^{+}(x)+\widehat{\phi}^{-}(x)\right)$$
$$= \widehat{\phi}^{+}(y)\widehat{\phi}^{+}(x)+\widehat{\phi}^{+}(y)\widehat{\phi}^{-}(x)+\widehat{\phi}^{-}(y)\widehat{\phi}^{+}(x)+\widehat{\phi}^{-}(y)\widehat{\phi}^{-}(x)$$
$$= \widehat{\phi}^{+}(y)\widehat{\phi}^{+}(x)+\widehat{\phi}^{-}(x)\widehat{\phi}^{+}(y)+\widehat{\phi}^{-}(y)\widehat{\phi}^{+}(x)+\widehat{\phi}^{-}(y)\widehat{\phi}^{-}(x)+\left[\widehat{\phi}^{+}(y),\widehat{\phi}^{-}(x)\right]$$
$$=: \widehat{\phi}(x)\widehat{\phi}(y) : + D(y-x)$$

We can put these together to read (for any $x^0,\ y^0$),

$$T\widehat{\phi}(x)\widehat{\phi}(y) =: \widehat{\phi}(x)\widehat{\phi}(y) : + \Delta_F(x-y)$$

Notice that this meshes well with our earlier definition of the Feynman propagator,

$$\Delta_F(x-y) = \langle 0|T\widehat{\phi}(x)\widehat{\phi}(y)|0\rangle$$

This result can be generalised to include more fields. The proof is by induction, but we won't go through it here.

$$T\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n) =: \hat{\phi}(x)\hat{\phi}(y)\cdots\hat{\phi}(x_n) : + \text{all possible contractions}$$

This is known as *Wick's Theorem*.

For example,

$$\begin{aligned}
T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) \; &=: \hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) : \\
&+ :\hat{\phi}(x_3)\hat{\phi}(x_4): \Delta_F(x_1 - x_2) \quad + :\hat{\phi}(x_2)\hat{\phi}(x_4): \Delta_F(x_1 - x_3) \\
&+ :\hat{\phi}(x_2)\hat{\phi}(x_3): \Delta_F(x_1 - x_4) \quad + :\hat{\phi}(x_1)\hat{\phi}(x_4): \Delta_F(x_2 - x_3) \\
&+ :\hat{\phi}(x_1)\hat{\phi}(x_3): \Delta_F(x_2 - x_4) \quad + :\hat{\phi}(x_1)\hat{\phi}(x_2): \Delta_F(x_3 - x_4) \\
&+ \Delta_F(x_1 - x_2)\,\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\,\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\,\Delta_F(x_2 - x_3)
\end{aligned}$$

Notice that when one takes the vacuum-expectation value, only the last three terms survive.

$$\begin{aligned}
&\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)|0\rangle \\
&\quad = \Delta_F(x_1 - x_2)\,\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\,\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\,\Delta_F(x_2 - x_3)
\end{aligned}$$

Notation:

A common notation is to write $\wick{\c{\hat{\phi}}(x)\c{\hat{\phi}}(y)} = \Delta_F(x-y)$ even when surrounded by other operators. So,

$$\wick{\hat{\phi}(x_1)\ldots\c{\hat{\phi}}(x_i)\ldots\c{\hat{\phi}}(x_j)\ldots\hat{\phi}(x_n)} = \hat{\phi}(x_1)\ldots\hat{\phi}(x_{i-1})\hat{\phi}(x_{i+1})\ldots\hat{\phi}(x_{j-1})\hat{\phi}(x_{j+1})\ldots\hat{\phi}(x_n)\,\Delta_F(x_i - x_j)$$

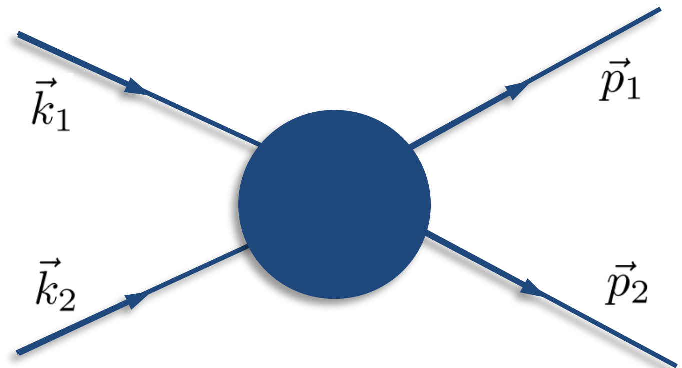
and our Wick's theorem example becomes,

$$\begin{aligned}
T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) \; =& :\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4): \\
&+ :\wick{\c{\hat{\phi}}(x_1)\c{\hat{\phi}}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)}: \; + :\wick{\c{\hat{\phi}}(x_1)\hat{\phi}(x_2)\c{\hat{\phi}}(x_3)\hat{\phi}(x_4)}: \\
&+ :\wick{\c{\hat{\phi}}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\c{\hat{\phi}}(x_4)}: \; + :\wick{\hat{\phi}(x_1)\c{\hat{\phi}}(x_2)\c{\hat{\phi}}(x_3)\hat{\phi}(x_4)}: \\
&+ :\wick{\hat{\phi}(x_1)\c{\hat{\phi}}(x_2)\hat{\phi}(x_3)\c{\hat{\phi}}(x_4)}: \; + :\wick{\hat{\phi}(x_1)\hat{\phi}(x_2)\c{\hat{\phi}}(x_3)\c{\hat{\phi}}(x_4)}:
\end{aligned}$$

$$+ :\wick{\c1{\hat{\phi}}(x_1)\c1{\hat{\phi}}(x_2)\c2{\hat{\phi}}(x_3)\c2{\hat{\phi}}(x_4)}: + :\wick{\c1{\hat{\phi}}(x_1)\c2{\hat{\phi}}(x_2)\c1{\hat{\phi}}(x_3)\c2{\hat{\phi}}(x_4)}: + :\wick{\c1{\hat{\phi}}(x_1)\c2{\hat{\phi}}(x_2)\c2{\hat{\phi}}(x_3)\c1{\hat{\phi}}(x_4)}:$$

An example: 2 $\rightarrow$ 2 scattering

Let's consider a scattering process where the initial state has 2 particles with momenta $\vec{k}_1$ and $\vec{k}_2$, and the final state is also 2 particles, now with momenta $\vec{p}_1$ and $\vec{p}_2$.



The initial state is,

$$|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle\,,$$

And the final state is,

$$|\vec{p}_1, \vec{p}_2\rangle = \hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_1)|0\rangle.$$

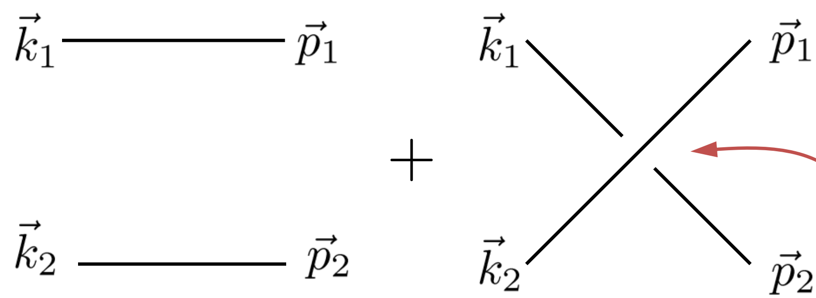
The first term in the expansion of the S-Matrix is just,

$$\begin{aligned}
\langle\vec{p}_1, \vec{p}_2|\vec{k}_1, \vec{k}_2\rangle &= \langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle \\
&= \langle 0|\hat{a}(\vec{p}_2)\,\left(\left[\hat{a}(\vec{p}_1), \hat{a}^\dagger(\vec{k}_2)\right] + \hat{a}^\dagger(\vec{k}_2)\hat{a}(\vec{p}_1)\right)\,\hat{a}^\dagger(\vec{k}_1)|0\rangle \\
&= \left[\hat{a}(\vec{p}_1), \hat{a}^\dagger(\vec{k}_2)\right]\langle 0|\hat{a}(\vec{p}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle + \langle 0|\hat{a}(\vec{p}_2)\hat{a}^\dagger(\vec{k}_2)\left[\hat{a}(\vec{p}_1), \hat{a}^\dagger(\vec{k}_1)\right]|0\rangle \\
&= \left[\hat{a}(\vec{p}_1), \hat{a}^\dagger(\vec{k}_2)\right]\left[\hat{a}(\vec{p}_2), \hat{a}^\dagger(\vec{k}_1)\right] + \left[\hat{a}(\vec{p}_1), \hat{a}^\dagger(\vec{k}_1)\right]\left[\hat{a}(\vec{p}_2), \hat{a}^\dagger(\vec{k}_2)\right]
\end{aligned}$$

Using $\left[a(\vec{p}_1), \hat{a}^{\dagger}(\vec{k}_1)\right] = (2\pi)^3\, 2E(\vec{p}_1)\delta^3(\vec{p}_1 - \vec{k}_1)$ this gives,

$$(2\pi)^6 E(\vec{k}_1)E(\vec{k}_2)\left(\delta^3\left(\vec{p}_1 - \vec{k}_1\right)\delta^3\left(\vec{p}_2 - \vec{k}_2\right) + \delta^3\left(\vec{p}_1 - \vec{k}_2\right)\delta^3\left(\vec{p}_2 - \vec{k}_1\right)\right)$$

We can view this situation diagramatically as follows:



This is not actually scattering, so we shouldn't include this in our scattering calculation.

Note that these lines don't "meet"

The second term is,

$$-i\frac{\lambda}{4!}\int d^4x\, \langle\vec{p}_1, \vec{p}_2| T\hat{\phi}^4(x)\, |\vec{k}_1, \vec{k}_2\rangle$$

In this case, the time ordering is trivial, since they are all at the same time, but it is still useful to use Wick's Theorem

Wick's theorem gives,

$$T\hat{\phi}^4(x) = :\hat{\phi}^4(x): + 6 :\hat{\phi}^2(x): \Delta_F(x-x) + 3\Delta_F(x-x)\,\Delta_F(x-x)$$

The normal ordered product gives,

$$-i\frac{\lambda}{4!}\int d^4x\,\langle\vec{p}_1,\vec{p}_2|:\hat{\phi}^4(x):|\vec{k}_1,\vec{k}_2\rangle$$

$$=-i\frac{\lambda}{4}\int d^4x\,\frac{d^3q_1}{(2\pi)^3 2E(\vec{q}_1)}\,\frac{d^3q_2}{(2\pi)^3 2E(\vec{q}_2)}\,\frac{d^3q_3}{(2\pi)^3 2E(\vec{q}_3)}\,\frac{d^3q_4}{(2\pi)^3 2E(\vec{q}_4)}$$

$$\times\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)a(\vec{q}_3)a(\vec{q}_4)\,\hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle\,e^{i(q_1+q_2-q_3-q_4)\cdot x}$$

$\frac{4!}{2!2!}=6$ cross-terms

since this is a $2\leftrightarrow 2$ process, I must have the same number of annihilation operators as creation operators!

Now,

$$\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)a(\vec{q}_3)a(\vec{q}_4)\,\hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle$$

$$=\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)a(\vec{q}_3)\left(\left[a(\vec{q}_4),\hat{a}^\dagger(\vec{k}_2)\right]+\hat{a}^\dagger(\vec{k}_2)a(\vec{q}_4)\right)\hat{a}^\dagger(\vec{k}_1)|0\rangle$$

$$=\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)a(\vec{q}_3)\hat{a}^\dagger(\vec{k}_1)|0\rangle\,\left[a(\vec{q}_4),\hat{a}^\dagger(\vec{k}_2)\right]$$
$$+\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)a(\vec{q}_3)\hat{a}^\dagger(\vec{k}_2)|0\rangle\,\left[a(\vec{q}_4),\hat{a}^\dagger(\vec{k}_1)\right]$$

$$=\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)|0\rangle\,\left[a(\vec{q}_3),\hat{a}^\dagger(\vec{k}_1)\right]\,\left[a(\vec{q}_4),\hat{a}^\dagger(\vec{k}_2)\right]$$
$$+\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)|0\rangle\,\left[a(\vec{q}_3),\hat{a}^\dagger(\vec{k}_2)\right]\,\left[a(\vec{q}_4),\hat{a}^\dagger(\vec{k}_1)\right]$$

These two terms are identical except for $\vec{q}_3 \leftrightarrow \vec{q}_4$, but since we are integrating over $\vec{q}_3,\ \vec{q}_4$ we can just re-label and will find they give the same contribution.

We can do a similar procedure on the other operators to give,

$$-i\frac{\lambda}{4!}\int d^4x\,\langle\vec{p}_1,\vec{p}_2|\hat{\phi}^4(x)\,|\vec{k}_1,\vec{k}_2\rangle$$

$$= -i\frac{\lambda}{4}\int d^4x\,\frac{d^3q_1}{(2\pi)^3 2E(\vec{q}_1)}\,\frac{d^3q_2}{(2\pi)^3 2E(\vec{q}_2)}\,\frac{d^3q_3}{(2\pi)^3 2E(\vec{q}_3)}\,\frac{d^3q_4}{(2\pi)^3 2E(\vec{q}_4)}$$

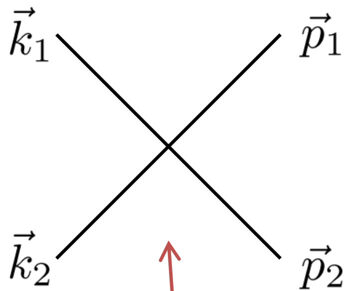
$$\times 4\,\left[a(\vec{p}_1),\hat{a}^\dagger(\vec{q}_1)\right]\,\left[a(\vec{p}_2),\hat{a}^\dagger(\vec{q}_2)\right]\,\left[a(\vec{q}_3),\hat{a}^\dagger(\vec{k}_2)\right]\,\left[a(\vec{q}_4),\hat{a}^\dagger(\vec{k}_1)\right]\,e^{i(q_1+q_2-q_3-q_4)\cdot x}$$

Now use $\left[a(\vec{p}_1),\hat{a}^\dagger(\vec{q}_1)\right] = (2\pi)^3\,2E(\vec{p}_1)\delta^3(\vec{p}_1-\vec{q}_1)$ etc.,

$$= -i\lambda\int d^4x\,e^{i(p_1+p_2-k_1-k_2)\cdot x}$$

$$= -i\lambda\,(2\pi)^4\,\delta^4\left(p_1+p_2-k_1-k_2\right)$$

forces momentum conservation



Now the lines meet at a point

The next piece is,

$$-i\frac{\lambda}{4!}\int d^4x\,\langle\vec{p}_1,\vec{p}_2|:\hat{\phi}^2(x):|\vec{k}_1,\vec{k}_2\rangle\,\Delta_F(x-x)$$

Now the only non-zero contribution is from one annihilation operator and one creation operator

$$=-i2\frac{\lambda}{4!}\int d^4x\,\frac{d^3q_1}{(2\pi)^3 2E(\vec{q}_1)}\,\frac{d^3q_2}{(2\pi)^3 2E(\vec{q}_2)}$$

$$\times\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a(\vec{q}_2)\,\hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle\,e^{i(q_1-q_2)\cdot x}\;\Delta_F(x-x)$$

2 cross-terms

$$=i\frac{\lambda}{12}\int\frac{d^4k}{(2\pi)^4}\,\frac{d^3q_1}{(2\pi)^3 2E(\vec{q}_1)}\,\frac{d^3q_2}{(2\pi)^3 2E(\vec{q}_2)}$$

$$\times\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a(\vec{q}_2)\,\hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle\,\frac{\delta^4(q_1-q_2)}{k^2-m^2+i\epsilon}$$

After a little algebra,

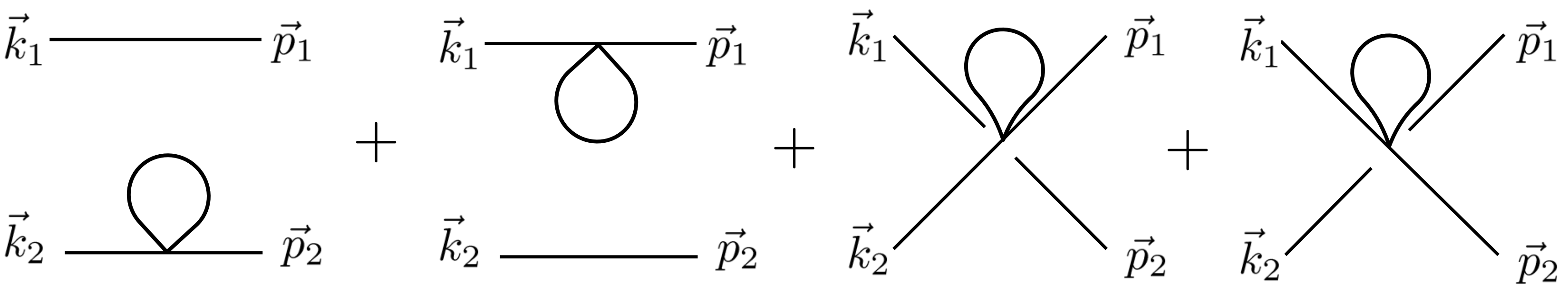
$$\langle 0|\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_1)\,\hat{a}^\dagger(\vec{q}_1)a(\vec{q}_2)\,\hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle/\left((2\pi)^6\,2E(\vec{q}_1)\,2E(\vec{q}_2)\right)$$

$$\begin{aligned}
&=\;2\,(2\pi)^3\,E(\vec{k}_1)\,\delta^3(\vec{p}_2-\vec{q}_1)\,\delta^3(\vec{p}_1-\vec{k}_1)\,\delta^3(\vec{q}_2-\vec{k}_2)\\
&\;+2\,(2\pi)^3\,E(\vec{k}_1)\,\delta^3(\vec{p}_2-\vec{k}_1)\,\delta^3(\vec{p}_1-\vec{q}_1)\,\delta^3(\vec{q}_2-\vec{k}_2)\\
&\;+2\,(2\pi)^3\,E(\vec{k}_2)\,\delta^3(\vec{p}_2-\vec{q}_1)\,\delta^3(\vec{p}_1-\vec{k}_2)\,\delta^3(\vec{q}_2-\vec{k}_1)\\
&\;+2\,(2\pi)^3\,E(\vec{k}_2)\,\delta^3(\vec{p}_2-\vec{k}_2)\,\delta^3(\vec{p}_1-\vec{q}_1)\,\delta^3(\vec{q}_2-\vec{k}_1)
\end{aligned}$$

So,

$$-i\frac{\lambda}{4!}\int d^4x\,\langle\vec{p}_1,\vec{p}_2|:\hat{\phi}^2(x):|\vec{k}_1,\vec{k}_2\rangle\,\Delta_F(x-x)$$

$$= i\frac{\lambda}{6}\int\frac{d^4k}{(2\pi)^4}\Big[E(\vec{k}_1)\,\Big(\delta^3(\vec{p}_1-\vec{k}_1)\,\delta^4(p_2-k_2)\,+\,\delta^3(\vec{p}_2-\vec{k}_1)\,\delta^4(p_1-k_2)\Big)$$
$$+E(\vec{k}_2)\,\Big(+\delta^3(\vec{p}_1-\vec{k}_2)\,\delta^4(p_2-k_1)\,+\,\delta^3(\vec{p}_2-\vec{k}_2)\,\delta^4(p_1-k_1)\Big)\Big]\,\frac{1}{k^2-m^2+i\epsilon}$$

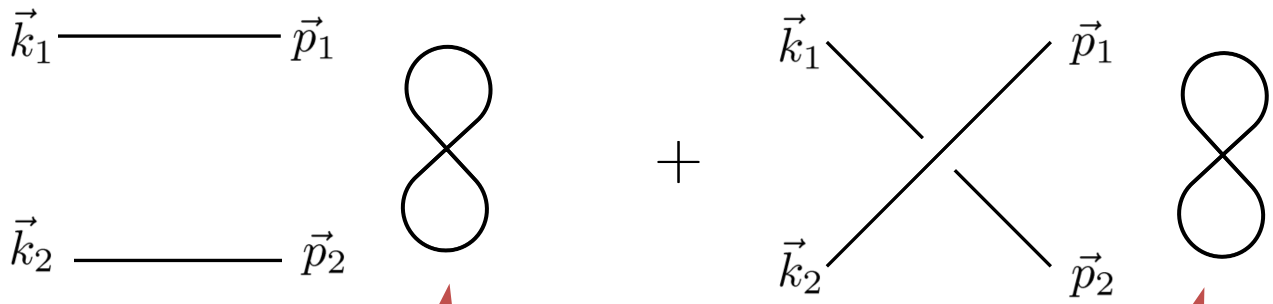


Once again, the lines don't transfer momentum, so *this is not scattering* either.

Also notice that the integral $\int\frac{d^4k}{k^2-m^2+i\epsilon}$ is divergent (oops!) – we will come back to this later.

The last piece is also clearly *not scattering*,

$$-i\frac{\lambda}{6}\int d^4x\,\langle\vec{p}_1,\vec{p}_2|\vec{k}_1,\vec{k}_2\rangle\Delta_F^2(x-x)$$



These pieces are the $\Delta_F^2(x-x)$. They are also present for vacuum to vacuum transitions, $\langle 0|S|0\rangle$ so are really a consequence of using the free vacuum rather than the interacting vacuum.

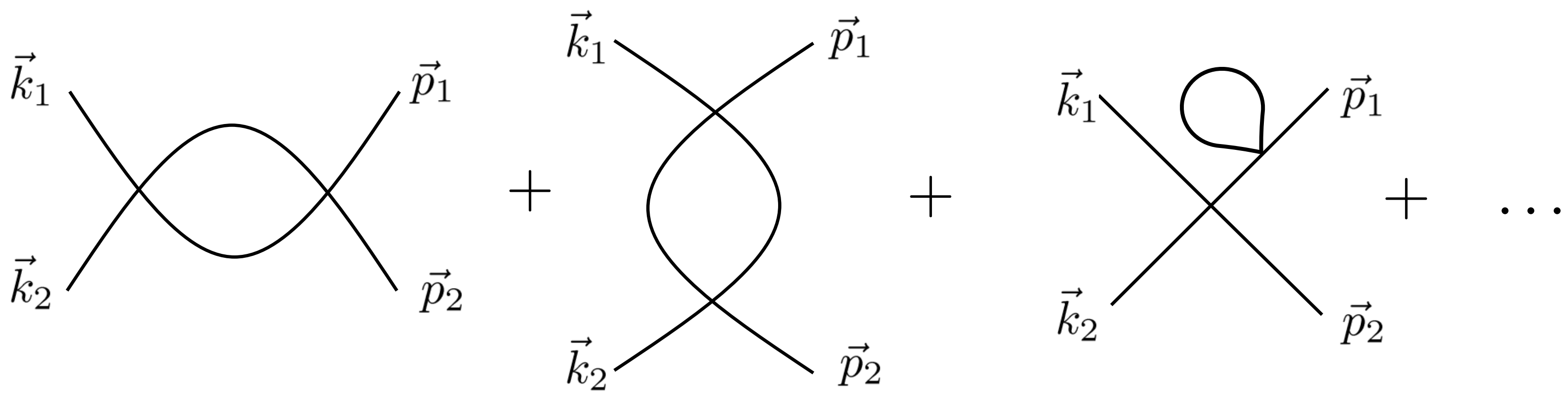
Using the interacting vacuum, these *disconnected diagrams* are removed (see later).

The next term in the perturbative expansion is,

$$\left(-i\frac{\lambda}{4!}\right)^2 \int d^4x\, d^4y\, \langle \vec{p}_1, \vec{p}_2 | T\hat{\phi}^4(x)\hat{\phi}^4(y)\, | \vec{k}_1, \vec{k}_2 \rangle$$

Now the time ordering is important, since x and y are different events.

The *connected scattering* contributions are of the form:



Again, the momentum circulating in the loop is unconstrained, so must be integrated over. These integrals are again *divergent*.

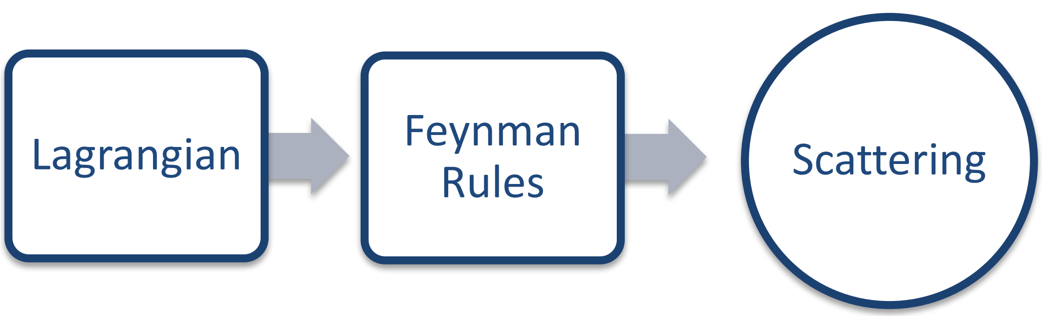
Feynman Diagrams and Feynman Rules

In the last calculation, we drew diagrams to allow us to visualize the processes being described by each term in the perturbative expansion.

These diagrams can be thought of more formally as a notation for the mathematical expressions themselves. We draw all the possible *connected scattering* diagrams for a process (to the appropriate order in perturbation theory) and then translate these into a mathematical expression using a set of notational rules.

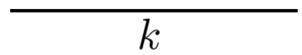
The diagrams are known as *Feynman Diagrams*.

The rules for converting them to a mathematical expression are the *Feynman Rules*.

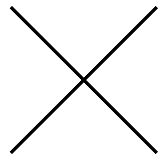


Thankfully, once we have the Feynman Rules for a theory we don't need to go back to calculating the S-matrix in terms of creation and annihilation operators every time.

The Feynman Rules for ϕ^4 theory

1. For each propagator 

$$= \frac{i}{k^2 - m^2 + i\epsilon}$$

2. For each vertex 
$$= -i\lambda$$

3. Impose momentum conservation at vertices, e.g. $(2\pi)^4 \, \delta^4 \left(p_1 + p_2 - k_1 - k_2\right)$

4. Integrate over every unconstrained momentum $\displaystyle\int \frac{d^4k}{(2\pi)^4}$

5. Include symmetry factors – count the number of ways you can swap lines around while leaving the diagram unchanged.

These are what we call the *Momentum Space Feynman Rules* since the objects like the propagator are the momentum-spce versions (i.e. the Fourier Transform of $\Delta_F(x-y)$).

Incorporating the true vacuum

We waved hands a little by ignoring disconnected graphs, using the vacuum as an excuse. The *true vacuum* is the vacuum of the full interacting Hamiltonian (or Lagrangian), whereas we only considered the vacuum of the free theory.

Let's call the true vacuum $|\Omega\rangle$, defined by $\hat{H}|\Omega\rangle = 0$. This should not be confused with $|0\rangle$ which is the vacuum of the free theory, defined by $\hat{H}_0|0\rangle = 0$.

Notice that these are Heisenberg operators

The object

$$G_n(x_1, x_2, \ldots x_n) \equiv \langle\Omega|T\widehat{\phi}_H(x_1)\widehat{\phi}_H(x_2)\cdots\widehat{\phi}_H(x_n)|\Omega\rangle$$

is known as the *n-point Green's Function*, or sometimes, the *n-point correlator*.

These are really the objects that we need for our scatterings – the initial state is at times much earlier than the interaction, and the final state is much later, so we can include their creation and annihilation operators in the time ordered product (though the above gives a transition between position eigenstates).

Now we will try and relate this to the expressions we have just been looking at.

Let's first assume that $x_1^0 > x_2^0 > \ldots > x_n^0$, i.e. the product is already time ordered.

We can convert the Heisenberg fields to Interaction fields using our expression,

$$\hat{\phi}_H(\vec{x}) = \hat{U}^\dagger(t, t_0)\hat{\phi}_I(\vec{x})\hat{U}(t, t_0)$$

where

$$\hat{U}(t, t_0) \equiv e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)} = Te^{\left[-i\int_{t_0}^{t} dt'\,\hat{H}_{\text{int},\,I}(t')\right]}.$$

Also notice that $\hat{U}(t_1, t_2)\hat{U}(t_2, t_3) = \hat{U}(t_1, t_3) \qquad (t_1 > t_2 > t_3)$,

and $\hat{U}(t_1, t_3)\hat{U}^\dagger(t_2, t_3) = \hat{U}(t_1, t_2)$.

So,

$$\langle\Omega|\hat{\phi}_H(x_1)\hat{\phi}_H(x_2)\cdots\hat{\phi}_H(x_n)|\Omega\rangle$$

$$= \langle\Omega|\hat{U}^\dagger(t_1, t_0)\hat{\phi}_I(x_1)\hat{U}(t_1, t_0)\hat{U}^\dagger(t_2, t_0)\hat{\phi}_I(x_2)\hat{U}(t_2, t_0)\cdots\hat{U}^\dagger(t_n, t_0)\hat{\phi}_I(x_n)\hat{U}(t_n, t_0)|\Omega\rangle$$

$$= \langle\Omega|\hat{U}^\dagger(t_1, t_0)\hat{\phi}_I(x_1)\hat{U}(t_1, t_2)\hat{\phi}_I(x_2)\hat{U}(t_2, t_3)\cdots\hat{U}(t_{n-1}, t_n)\hat{\phi}_I(x_n)\hat{U}(t_n, t_0)|\Omega\rangle$$

Now we need to figure out how to handle the *true vacuum* $|\Omega\rangle$.

Consider the quantity $e^{-i\hat{H}t}|0\rangle$. Insert a complete set of Energy eigenstates and we have,

$$e^{-i\hat{H}t}|0\rangle = e^{-i\hat{H}t}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-i\hat{H}t}|n\rangle\langle n|0\rangle = e^{-iE_0 t}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n t}|n\rangle\langle n|0\rangle$$

Now if we take t $\rightarrow \infty$ then all but the first term vanish (this is the Riemann-Lesbesgue lemma). So,

$$\lim_{t\to\infty} e^{-i\hat{H}t}|0\rangle = \lim_{t\to\infty} e^{-iE_0 t}|\Omega\rangle\langle\Omega|0\rangle$$

which gives us *an expression for the true vacuum in terms of the free vacuum*:

$$|\Omega\rangle = \lim_{t\to\infty} \left(e^{-iE_0 t}\langle\Omega|0\rangle\right)^{-1} e^{-iHt}|0\rangle = \lim_{t\to\infty} \left(e^{-iE_0(t+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t+t_0)}|0\rangle$$

$$= \lim_{t\to\infty} \left(e^{-iE_0(t+t_0)}\langle\Omega|0\rangle\right)^{-1} \underbrace{e^{-i\hat{H}(t+t_0)}e^{i\hat{H}_0(t+t_0)}}_{\hat{U}^\dagger(-t,t_0)=\hat{U}(t_0,-t)}|0\rangle \longleftarrow \hat{H}_0|0\rangle = 0$$

Similarly,

$$\langle\Omega| = \lim_{t\to\infty} \langle 0|\hat{U}(t,t_0)\left(e^{-iE_0(t-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

Sticking these in our expression,

$$\begin{aligned}
&\langle\Omega|\widehat{\phi}_H(x_1)\widehat{\phi}_H(x_2)\cdots\widehat{\phi}_H(x_n)|\Omega\rangle \\
&= \lim_{t\to\infty}\left(|\langle 0|\Omega\rangle|^2 e^{-i2E_0 t}\right)^{-1}\langle 0|\widehat{U}(t,t_0)\widehat{U}(t_0,t_1)\widehat{\phi}_I(x_1)\widehat{U}(t_1,t_2)\widehat{\phi}_I(x_2)\widehat{U}(t_2,t_3)\cdots \\
&\qquad \times\widehat{U}(t_{n-1},t_n)\widehat{\phi}_I(x_n)\widehat{U}(t_n,t_0)\widehat{U}(t_0,-t)|0\rangle \\
&= \lim_{t\to\infty}\left(|\langle 0|\Omega\rangle|^2 e^{-i2E_0 t}\right)^{-1}\langle 0|\widehat{U}(t,t_1)\widehat{\phi}_I(x_1)\widehat{U}(t_1,t_2)\widehat{\phi}_I(x_2)\widehat{U}(t_2,t_3)\cdots \\
&\qquad \times\widehat{U}(t_{n-1},t_n)\widehat{\phi}_I(x_n)\widehat{U}(t_n,-t)|0\rangle \\
&= \lim_{t\to\infty}\left(|\langle 0|\Omega\rangle|^2 e^{-i2E_0 t}\right)^{-1}\langle 0|T\widehat{\phi}_I(x_1)\widehat{\phi}_I(x_2)\cdots\widehat{\phi}_I(x_n)\widehat{U}(t,-t)|0\rangle
\end{aligned}$$

In this last expression, the *time ordering* makes sure all the fields (including those in the $\hat{U}$) are placed in the correct order.

But this was true for any n, so for n=0,

$$\langle\Omega|\Omega\rangle = \lim_{t\to\infty}\left(|\langle 0|\Omega\rangle|^2 e^{-i2E_0 t}\right)^{-1}\langle 0|\widehat{U}(t,-t)|0\rangle$$

$$\Rightarrow \lim_{t\to\infty}\left(|\langle 0|\Omega\rangle|^2 e^{-i2E_0 t}\right)^{-1} = \lim_{t\to\infty}\frac{1}{\langle 0|\widehat{U}(t,-t)|0\rangle}$$

The *n-point Green's Function* becomes,

$$G_n(x_1, x_2, \ldots x_n) = \frac{\langle 0|T\widehat{\phi}_I(x_1)\widehat{\phi}_I(x_2)\cdots\widehat{\phi}_I(x_n)\widehat{S}|0\rangle}{\langle 0|\widehat{S}|0\rangle}$$

This division is the mathematical justification of our *removal of disconnected diagrams*. Any disconnected parts of diagrams are also a part of $\langle 0|\widehat{S}|0\rangle$, and cancel.

The Lehmann-Symanzik-Zimmermann Reduction Formula

The *n-point Green's Function* (sometimes called the *n-point correlator*) is related to S-Matrix expectation values via the *Lehmann-Symanzik-Zimmermann (LSZ) Reduction Formula*.

We won't prove the LSZ reduction formula here – we will just state it.

The S matrix for scattering of m particles with momenta $\vec{k}_i$ to n particles with momenta $\vec{p}_i$ is

$$\langle \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n | \hat{S} | \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_m \rangle$$

$$= i^{m+n} \int d^4x_1 \ldots d^4x_m \, d^4y_1 \ldots d^4y_n e^{-i(k_1 \cdot x_1 + \ldots + k_m \cdot x_m - p_1 \cdot y_1 - \ldots - p_n \cdot y_n)}$$

$$\times \left(\partial^2_{x_1} + m^2\right) \ldots \left(\partial^2_{x_m} + m^2\right) \left(\partial^2_{y_1} + m^2\right) \ldots \left(\partial^2_{y_n} + m^2\right) G_n(x_1, x_2, \ldots, x_n)$$

4. The Free Dirac Field

The problem of negative energies

Historically, before the advent of Quantum Field Theory, the Klein-Gordon Equation was regarded as a problem since it has *negative energy solutions*.

Consider a plane-wave solution: $\phi(t, \vec{x}) = Ne^{-i(Et - \vec{p}\cdot\vec{x})}$

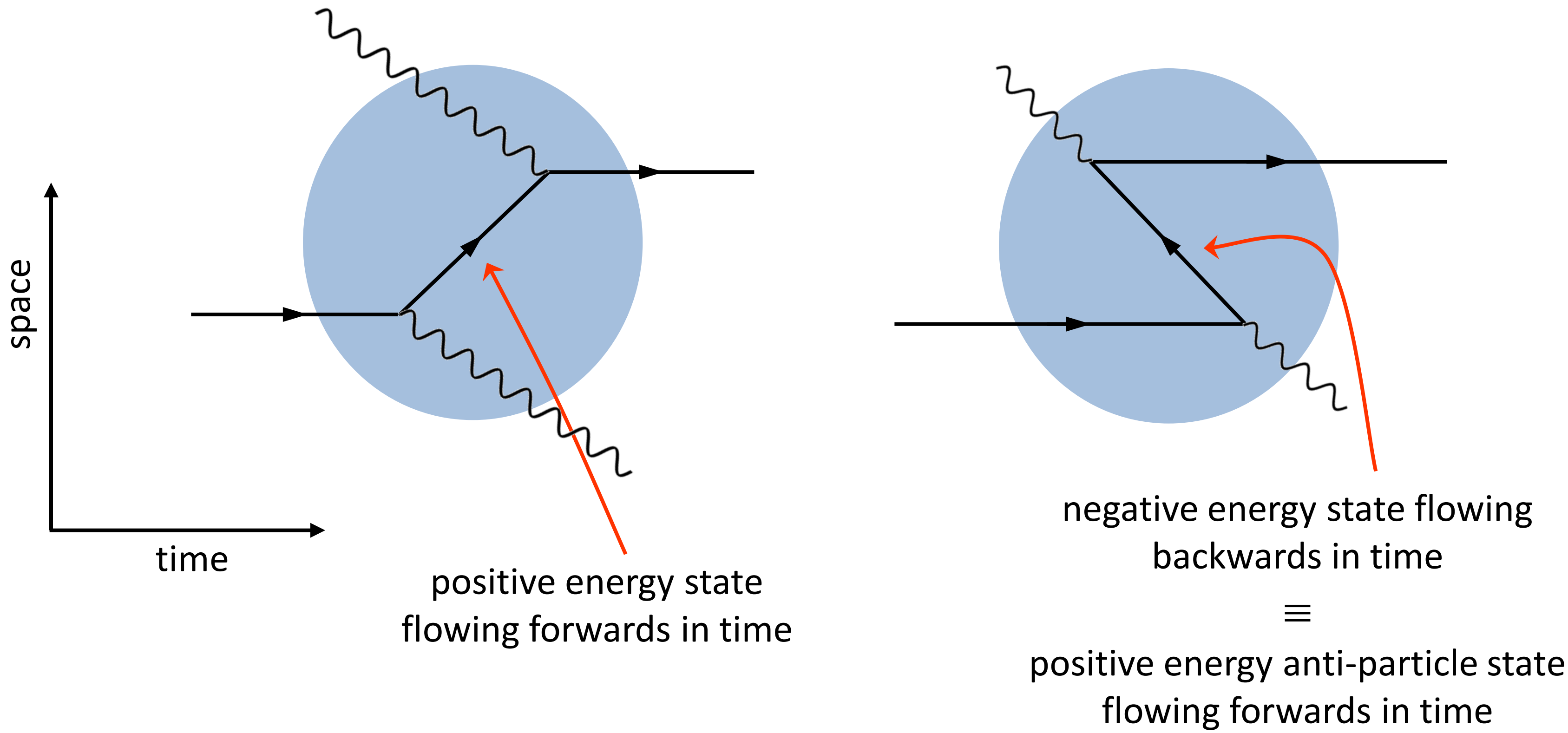
$$\frac{\partial^2}{\partial t^2}\phi = (\nabla^2 - m^2)\phi \quad \Rightarrow \quad E^2 = m^2 + |\vec{p}|^2$$

$$\Rightarrow \quad E = \pm\sqrt{m^2 + |\vec{p}|^2}$$

Schrödinger actually wrote down the Klein-Gordon Equation first, but abandoned it and developed the non-relativistic Schrödinger equation instead – he (implicitly) took the positive sign of the square root so that he could ignore the negative energy solutions.

We now know that this isn't a problem after all. In QFT, we reinterpreted these negative energy states as positive energy antiparticle states. This reinterpretation is called the *Feynman-Stückelberg interpretation*.

In reality, we only ever see the final state particles, so we must include these anti-particles anyway.



Note that quantum mechanics does not adequately handle the creation of particle—anti-particle pairs out of the vacuum. For that we need *Quantum Field Theory* and its creation and annihilation operators.

The Dirac Equation

Paul Dirac
1902 –1984

Dirac tried to get round the problem of negative energy states by finding a field equation which was *linear* in the operators.

This can only take the form:

$$E = \vec{\alpha} \cdot \vec{p} + \beta m \longrightarrow i\frac{\partial \psi}{\partial t} = \left(-i\vec{\alpha} \cdot \vec{\nabla} + \beta m\right) \psi$$

So all we need to do is work out $\vec{\alpha}$ and β.

We can do this by insisting that we also satisfy $E^2 = m^2 + |\vec{p}|^2$.

Writing $E = \alpha_i p_i + \beta m$ with an implicit summation, we have,

$$E^2 = \alpha_i \alpha_j p_i p_j + (\alpha_i \beta + \beta \alpha_i) m p_i + \beta^2 m^2$$

We are free to re-label $i \leftrightarrow j$ in the summation, so

$$\sum_{i,j} \alpha_i \alpha_j \, p_i p_j = \sum_{j,i} \alpha_j \alpha_i \, p_j p_i = \sum_{i,j} \alpha_j \alpha_i \, p_i p_j$$

$$E^2 = \frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i)p_ip_j + (\alpha_i\beta + \beta\alpha_i)mp_i + \beta^2m^2$$

Enforcing $E^2 = m^2 + |\vec{p}|^2$ gives us conditions on $\vec{\alpha}$ and β.

$$\boxed{\begin{gathered} \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \\ \alpha_i\beta + \beta\alpha_i = 0 \\ \beta^2 = 1 \end{gathered}}$$

$\vec{\alpha}$ and β are anti-commuting objects – not just numbers!

These commutation relations *define* α and β. Anything which obeys these relations will do. One possibility, called the *Dirac representation*, is the 4×4 matrices:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

2×2 matrices

where σ_i are the usual Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since these act on the field, the field itself must now have 4 components – this is a *spinor*.

$$\left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi \quad \sim \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

[This is also just a representation.]

We can write this equation in a four-vector form by defining a new quantity γ^μ:

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\gamma} \equiv \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

The anti-commutation relations become:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

And the Dirac Equation is: (with $p^\mu \to i\partial^\mu$)

$$\boxed{(i\gamma^\mu\partial_\mu - m)\,\psi = 0}$$

Often $\gamma^\mu\partial_\mu$ is written as $\not{\partial}$, and $\gamma^\mu p_\mu$ is written as $\not{p}$.

This equation describes the (free) dynamics of a *fermion* field, e.g. the electron.

Some useful properties of Dirac matrices

- $\left(\gamma^0\right)^\dagger = \gamma^0$ and $\left(\gamma^i\right)^\dagger = -\gamma^i, \; i = 1, 2, 3$

  ($\vec{\alpha}$ and β are *Hermitean* since the energy and momentum operators are Hermitean.)

- $\left(\gamma^0\right)^2 = 1$ and $\left(\gamma^i\right)^2 = -1$ (follows from commutation relations).

- The γ matrices must be of even dimension (can you show this?)

- $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$ (Put the first two properties together)

- Note that these are just fixed matrices. Despite the notation γ^μ they are *not four-vectors* and don't transform under a Lorentz boost.

Does the Dirac Equation only have positive energy solutions?

Look for plane wave solutions:

$$\psi(t,\vec{x}) = u(\vec{p})e^{-i(Et-\vec{p}\cdot\vec{x})} = \begin{pmatrix} \chi \\ \phi \end{pmatrix} e^{-i(Et-\vec{p}\cdot\vec{x})}$$

4 component spinor

2 component spinors

$$u \rightarrow \begin{pmatrix} \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \leftarrow \chi \\ \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \leftarrow \phi \end{pmatrix}$$

Since we want the energy, it is easier to work without four-vector notation:

$$i\frac{\partial\psi}{\partial t} = \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi \Rightarrow E\begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

For a particle *at rest*, $\vec{p} = 0$

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

Solutions:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{with } E = m$$

positive energy solutions

OR

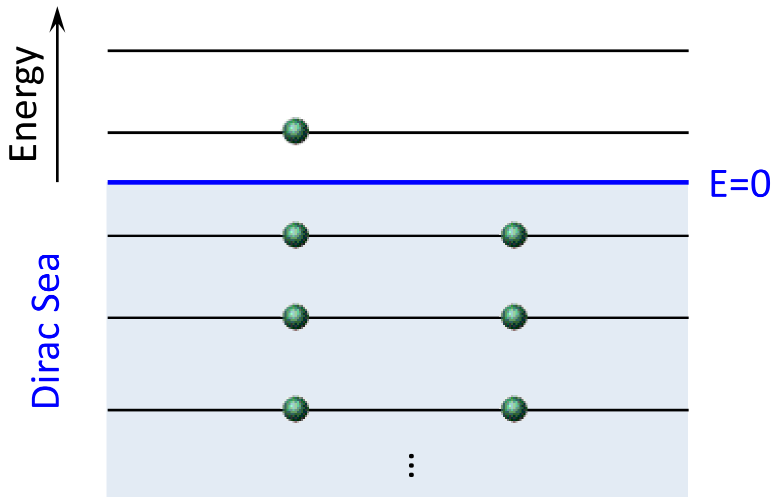
negative energy solutions

$$u = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{with } E = -m$$

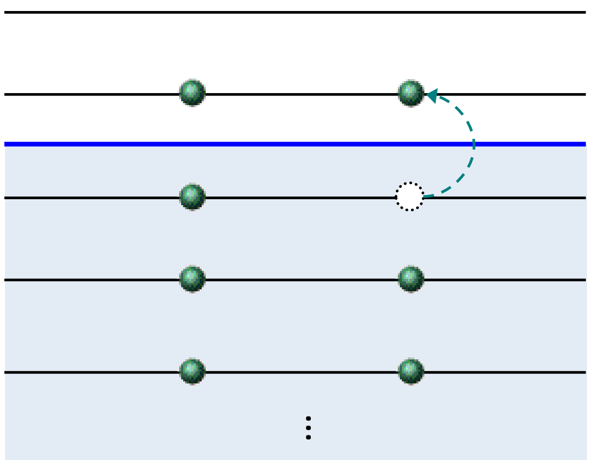
We still have negative energy solutions!

Dirac got round this by using the *Pauli Exclusion principle*.

He reasoned that his equation described particles with spin (e.g. electrons) so only two particles can occupy any particular energy level (one spin-up, the other spin-down).



If all the energy states with E<0 are already filled, the electron can't fall into a negative energy state.



Moving an electron from a negative energy state to a positive one leaves a hole which we interpret as an anti-particle.

Note that we couldn't have used this argument for bosons (no exclusion principle) so the *Feynman-Stückelberg interpretation* is more useful.

A General Solution

$$E\begin{pmatrix}\chi\\ \phi\end{pmatrix} = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m\end{pmatrix}\begin{pmatrix}\chi\\ \phi\end{pmatrix} \qquad \Rightarrow \begin{cases} \chi = \dfrac{\vec{\sigma}\cdot\vec{p}}{E-m}\phi \\[2ex] \phi = \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \end{cases}$$

Check these are compatible:

$$\phi = \left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\left(\frac{\vec{\sigma}\cdot\vec{p}}{E-m}\right)\phi = \frac{|\vec{p}|^2}{E^2-m^2}\phi = \phi \quad \text{since} \qquad E^2 = m^2 + |\vec{p}|^2$$

$$\left(\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \;\Rightarrow\; (\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2 + i\,(\vec{p}\times\vec{p})\cdot\vec{\sigma} = |\vec{p}|^2 \right)$$

property of Pauli matrices, e.g. $\sigma_1\sigma_2 = i\sigma_3$

We need to choose a basis for our solutions. Choose,

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Positive Energy Solutions: $\psi^{(1)}, \psi^{(2)}$

$$\phi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \quad \Rightarrow \begin{cases} \chi = \xi^{(s)} \\ \phi = \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi^{(s)} \end{cases} \qquad (s = 1, 2)$$

$E > 0$

$$\psi^{(s)}(x) = u^{(s)}e^{-ip\cdot x} = \sqrt{E+m}\begin{pmatrix} \xi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi^{(s)} \end{pmatrix} e^{-ip\cdot x}$$

[Normalization choice - see later]

Negative Energy Solutions: $\psi^{(3)},\ \psi^{(4)}$

$$\chi = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\phi \qquad \Rightarrow \begin{cases} \chi = \dfrac{\vec{\sigma}\cdot\vec{p}}{E-m}\xi^{(s)} \\ \phi = \xi^{(s)} \end{cases} \qquad (s = 1,2)$$

$E < 0$

$$\psi^{(s+2)}(x) = u^{(s+2)}e^{-ip\cdot x} = \sqrt{-E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\xi^{(s)} \\ \xi^{(s)} \end{pmatrix} e^{-ip\cdot x}$$

Typically, we write this in terms of the antiparticle's energy and momentum:

$$u^{(3,4)}(-\vec{p})e^{-i(-p)\cdot x} \equiv v^{(1,2)}(\vec{p})e^{ip\cdot x}$$

Conventions differ here: sometimes the order is inverted

Orthogonality and completeness

With the normalization of $2E$ particles per unit volume, it is rather obvious that:

$$u^{(r)\,\dagger}u^{(s)} = 2E\delta^{rs} \qquad v^{(r)\,\dagger}v^{(s)} = 2E\delta^{rs}$$

[positive energy]

This is a statement of orthogonality.

Less obvious, but easy to show, are the completeness relations:

$$\boxed{\begin{aligned}\sum_{s=1,2} u^{(s)}(p)\bar{u}^{(s)}(p) &= \not{p} + m \\ \sum_{s=1,2} v^{(s)}(p)\bar{v}^{(s)}(p) &= \not{p} - m\end{aligned}}$$

Note that this is a matrix equation:

$$\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

The Dirac Equation from a Lagrangian

Consider the Lagrangian for a free Dirac field:

spinor indices

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi = \bar{\psi}_i\left(i\left[\gamma^\mu\right]_{ij}\partial_\mu - m\delta_{ij}\right)\psi_j$$

We have two sets of Euler-Lagrange Equations. One for ψ and one for $\bar{\psi}$:

$$\frac{\partial \mathcal{L}}{\partial \psi_i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi_i\right)}\right) = 0 \qquad \frac{\partial \mathcal{L}}{\partial \bar{\psi}_i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \bar{\psi}_i\right)}\right) = 0$$

Let's use the one for $\bar{\psi}$:

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_i} = \left(i\left[\gamma^\mu\right]_{ij}\partial_\mu - m\delta_{ij}\right)\psi_j$$

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \bar{\psi}_i\right)} = 0$$

So $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$ which is the Dirac equation!

I could have used the other set of Euler-Lagrange Equations to give an equation for the antiparticle:

$$i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu + m\bar{\psi} = 0 \qquad \text{sometimes written} \qquad \bar{\psi}\left(i\gamma^\mu\overleftarrow{\partial_\mu} + m\right) = 0$$

arrow denotes acting to the left

The Dirac Lagrangian looks rather asymmetric in its treatment of ψ and $\bar{\psi}$.

In principle, $\bar{\psi}$ is just as fundamental as ψ and we can rewrite the Lagrangian:

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi = \underbrace{\partial_\mu\left(\cancel{i\bar{\psi}\gamma^\mu\psi}\right)}_{\text{total derivative}} - i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi - m\bar{\psi}\psi$$

or even $$\mathcal{L} = \frac{1}{2}\bar{\psi}i\gamma^\mu\partial_\mu\psi - \frac{1}{2}i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi - m\bar{\psi}\psi$$

$$= \frac{1}{2}\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{1}{2}\bar{\psi}\left(i\gamma^\mu\overleftarrow{\partial_\mu} + m\right)\psi$$

Angular Momentum and Spin

The angular momentum of a particle is given by $\vec{L} = \vec{r} \times \vec{p}$

If this commutes with the Hamiltonian then angular momentum is conserved.

$$[H, \vec{L}] = [\vec{\alpha} \cdot \vec{p}, \vec{r} \times \vec{p}] = -i\vec{\alpha} \times \vec{p}$$

This is not zero, so $\vec{L} = \vec{r} \times \vec{p}$ is *not* conserved!

But, if we define $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\vec{\alpha} \quad (= -i\gamma_1\gamma_2\gamma_3\vec{\gamma})$

then $[H, \vec{\Sigma}] = [\vec{\alpha} \cdot \vec{p}, -i\alpha_1\alpha_2\alpha_3\vec{\alpha}] = 2i\vec{\alpha} \times \vec{p}$

So the quantity $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ **is** conserved! $\quad [H, \vec{J}] = 0$

$\vec{L}$ is the orbital angular momentum, whereas $\frac{1}{2}\vec{\Sigma}$ is an *intrinsic* angular momentum

Notice that our basis spinors are eigenvectors of $\frac{1}{2}\Sigma^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

with eigenvalues $\pm\frac{1}{2}$

So now we know why the spinor contains four degrees of freedom:

- positive energy solution, with spin up
- positive energy solution, with spin down
- negative energy solution, with spin up
- negative energy solution, with spin down

Helicity of massless fermions

If the mass is zero, our wave equation becomes

$$E\begin{pmatrix}\chi\\ \phi\end{pmatrix} = \begin{pmatrix}0 & \vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & 0\end{pmatrix}\begin{pmatrix}\chi\\ \phi\end{pmatrix} \quad \Rightarrow \begin{cases} E\phi = \vec{\sigma}\cdot\vec{p}\chi \\ E\chi = \vec{\sigma}\cdot\vec{p}\phi \end{cases}$$

Writing $\Psi_{R,L} \equiv \frac{1}{2}(\chi \pm \phi)$ then we find the equations decouple

$$E\Psi_R = \vec{\sigma}\cdot\vec{p}\Psi_R \quad \text{and} \quad E\Psi_L = -\vec{\sigma}\cdot\vec{p}\Psi_L$$

These two component spinors, called *Weyl spinors*, are completely independent, and can even be considered as separate particles!

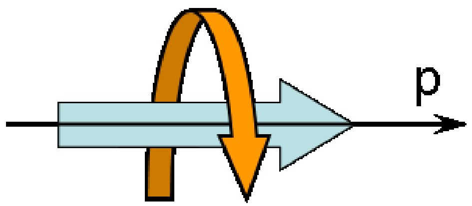
Notice that each is an eigenstate of the operator $\dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$ with eigenvalues ± 1

$|\vec{p}| = E$ for massless state

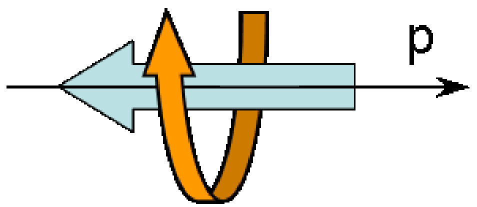
For the full Dirac spinor, we define the *Helicity* operator as $\dfrac{\vec{\Sigma}\cdot\vec{p}}{2|\vec{p}|}$

This is the component of spin
in the direction of motion.

A particle with a helicity eigenvalue $+\frac{1}{2}$ is **right handed**



A particle with a helicity eigenvalue $-\frac{1}{2}$ is **left handed**



Since an antiparticle has opposite momentum it will have opposite helicity.

left handed particle $\longrightarrow$ right handed antiparticle

[this is why the labelling of solutions in the antiparticle spinor v is sometimes reversed]

We can project out a particular helicity from a Dirac spinor using γ matrices.

This is the Dirac representation.

Define $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$

and *projection operators* $P_{R/L} \equiv \frac{1}{2}\left(1 \pm \gamma^5\right) = \frac{1}{2}\begin{pmatrix} \mathbf{1} & \pm\mathbf{1} \\ \pm\mathbf{1} & \mathbf{1} \end{pmatrix}$

Then a spinor $P_L u$ will be left handed, while $P_R u$ will be right handed.

e.g.

$$P_R\, u = \tfrac{1}{2}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \chi \\ \phi \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} \chi + \phi \\ \chi + \phi \end{pmatrix}$$

but $\dfrac{\vec{\Sigma}\cdot\vec{p}}{2|\vec{p}|}\tfrac{1}{2}\begin{pmatrix} \chi + \phi \\ \chi + \phi \end{pmatrix} = \dfrac{1}{2}\tfrac{1}{2}\begin{pmatrix} \chi + \phi \\ \chi + \phi \end{pmatrix}$ so $P_R\, u$ is right handed.

We can make this more explicit by using a different representation of the γ matrices.

The *chiral representation* (sometimes called the *Weyl representation*) is:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

Now $P_L \equiv \frac{1}{2}\left(1 - \gamma^5\right) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \qquad P_R \equiv \frac{1}{2}\left(1 + \gamma^5\right) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$

The left-handed Weyl spinor sits in the upper part of the Dirac spinor, while the right handed Weyl spinor sits in the lower part.

e.g.

$$P_R\, u = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_R \end{pmatrix}$$

The *weak interaction* acts only on left handed particles.

Parity transforms $\vec{r} \rightarrow -\vec{r}$ but leaves spin unchanged (it doesn't change which of the solutions you have). Therefore parity changes helicity - it transforms left handed particles onto right handed ones (and vice versa),

i.e. $P\Psi_L = \Psi_R$ and $P\Psi_R = \Psi_L$

So the weak interactions are *parity violating*.

Also, helicity is only a *good quantum number* for *massless* particles.

If a particle has a mass, I can always move to a reference frame where I am going faster than it, causing the momentum to reverse direction. This causes the helicity to change sign.

For a massless particle there is no such frame and helicity is a good quantum number.

We saw earlier that the mass term in the Dirac Lagrangian looks like

$$m\bar{\psi}\psi = m\,(\Psi_L^\dagger\,\Psi_R^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = m\left(\Psi_L^\dagger\Psi_R + \Psi_R^\dagger\Psi_L\right)$$

chiral representation

Mass terms mix left and right handed states.

Therefore massive particle are not compatible with the weak interaction!

The solution to this problem is to introduce a new field called the *Higgs field*. This couples left handed particles, to right handed ones, mixing them up and giving them an *effective mass*.

$$\mathcal{L}_{\text{Higgs}} \supset Y\,\bar{\psi}_L \cdot \phi\,\psi_R$$

If the vacuum (lowest energy state) of the system contains a non-zero amount of this new field ($\langle\phi\rangle \neq 0$) we generate a mass $Y\langle\phi\rangle$

The theory also predicts a new particle, the *Higgs boson*, which we hope to find soon!

Symmetries of the Dirac Equation

This notation differs in different texts.
e.g. Peskin and Schroeder would write
$\psi(x) \to \psi'(x) = S\psi(\Lambda^{-1}x)$

The Lorentz Transformation

How does the field $\psi(x)$ behave under a Lorentz transformation?

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \qquad \partial_\mu \to \partial'_\mu = [\Lambda^{-1}]_{\mu}{}^{\nu}\partial_\nu \qquad \psi(x) \to \psi'(x') = S\psi(x)$$

$$(i\gamma^\mu\partial_\mu - m)\,\psi(x) = 0 \to \left(i\gamma^\mu[\Lambda^{-1}]^{\nu}{}_{\mu}\partial_\nu - m\right) S\psi(x) = 0$$

(γ^μ and m are just numbers and don't transform)

Don't confuse this S with the S-matrix!

Premultiply by S^{-1}

$$\left(iS^{-1}\gamma^\mu S[\Lambda^{-1}]^{\nu}{}_{\mu}\partial_\nu - m\right)\psi(x) = 0$$

$$\Rightarrow \qquad S^{-1}\gamma^\mu S = \Lambda^{\mu}{}_{\nu}\gamma^\nu$$

We can find S for an *infinitesimal proper* transformation $\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$

[antisymmetric]

$$\Lambda^{\mu}{}_{\nu}\gamma^{\nu} = S^{-1}\gamma^{\mu}S$$

write $S = 1 + \frac{i}{4}\sigma_{\mu\nu}\,\omega^{\mu\nu}$ (just a parameterisation)

$$\gamma^{\mu} + \omega^{\mu}{}_{\nu}\gamma^{\nu} = \left(1 - \tfrac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta}\right)\gamma^{\mu}\left(1 + \tfrac{i}{4}\sigma_{\sigma\rho}\omega^{\sigma\rho}\right)$$

$$\Rightarrow\ 2i\omega^{\alpha\beta}\left(\delta^{\mu}{}_{\alpha}\gamma_{\beta} - \delta^{\mu}{}_{\beta}\gamma_{\alpha}\right) = [\gamma^{\mu}, \sigma_{\alpha\beta}]\omega^{\alpha\beta}$$

[ignoring terms $\mathcal{O}(\omega^2)$]

$$\Rightarrow \quad \sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu}, \gamma_{\nu}]$$

[I jumped a few steps here]

This tells us how a fermion field transforms under a Lorentz boost.

The adjoint transforms as $\bar{\psi} \equiv \psi^\dagger\gamma^0 \to \psi^\dagger S^\dagger \gamma^0 = \psi^\dagger\gamma^0 S^{-1} = \bar{\psi}S^{-1}$

[since $S^\dagger\gamma^0 = \gamma^0 S^{-1}$ for the explicit form of S derived above]

So $\bar{\psi}\psi$ is invariant.

And $j^\mu = \bar{\psi}\gamma^\mu\psi \to \bar{\psi}S^{-1}\gamma^\mu S\psi = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$ so our current is a four-vector.

Common fermion bilinears:

$\bar{\psi}\psi \to \bar{\psi}\psi$	*scalar*
$\bar{\psi}\gamma^5\psi \to \text{Det}\,(\Lambda)\bar{\psi}\gamma^5\psi$	*pseudoscalar*
$\bar{\psi}\gamma^\mu\psi \to \Lambda^\mu{}_\nu\,\bar{\psi}\gamma^\nu\psi$	*vector*
$\bar{\psi}\gamma^\mu\gamma^5\psi \to \text{Det}(\Lambda)\,\Lambda^\mu{}_\nu\,\bar{\psi}\gamma^\nu\gamma^5\psi$	*axial vector*
$\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi \to \Lambda^\mu{}_\alpha\,\Lambda^\nu{}_\beta\,\bar{\psi}\sigma^{\alpha\beta}\gamma^5\psi$	*tensor*

Charge conjugation, Parity and Time Reversal

The Dirac Equation has additional symmetries:

Parity $t \longrightarrow t, \quad \vec{x} \longrightarrow -\vec{x}$

Charge Conjugation $\psi \rightarrow \psi_c \equiv C\bar{\psi}^T$

Time Reversal $t \rightarrow -t,\ \vec{x} \rightarrow \vec{x}$ and $\psi(t,\vec{x}) \longrightarrow \psi_T(-t,\vec{x}) = T\psi^*(t,\vec{x})$

It is not hard to show that the appropriate field transformations are:

$$\begin{array}{llllllll} C: & \psi(t,\vec{x}) & \rightarrow & \psi_C(t,\vec{x}) & = & C\bar{\psi}^T(t,\vec{x}) & = & i\gamma^2\gamma^0\bar{\psi}^T(t,\vec{x}) \\ P: & \psi(t,\vec{x}) & \rightarrow & \psi_P(t,-\vec{x}) & = & P\psi\ (t,\vec{x}) & = & \gamma^0\psi\ (t,\vec{x}) \\ T: & \psi(t,\vec{x}) & \rightarrow & \psi_T(-t,\vec{x}) & = & T\psi^*(t,\vec{x}) & = & i\gamma^1\gamma^3\psi^*(t,\vec{x}) \end{array}$$

Doing all of these transformations maps a particle moving forwards in time onto an antiparticle moving backwards in time.

We find that physics is invariant under CPT.

This is further justification of the *Feynman-Stückelburg* interpretation.

Second Quantisation

So far our discussion of the Dirac Equation has just been quantum mechanics. Now we need to turn this into a *Quantum Field Theory* by converting the fields to operators.

We proceed as we did for the scalar field. The fermion fields are,

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3\, 2E(\vec{p})} \sum_s \left(\hat{a}^{(s)}(\vec{p})\, u^{(s)}(p)\, e^{-ip\cdot x} + \hat{b}^{(s)\,\dagger}(\vec{p})\, v^{(s)}(p)\, e^{ip\cdot x} \right)$$

sum over spins

annihilates a particle

normalisation spinors

creates an anti-particle

$$\hat{\bar{\psi}}(x) = \int \frac{d^3p}{(2\pi)^3\, 2E(\vec{p})} \sum_s \left(\hat{a}^{(s)\,\dagger}(\vec{p})\, \bar{u}^{(s)}(p)\, e^{ip\cdot x} + \hat{b}^{(s)}(\vec{p})\, \bar{v}^{(s)}(p)\, e^{-ip\cdot x} \right)$$

From now on, I am going to stop putting hats on my operators, since the notation is becoming cumbersome.

The energy momentum tenor for a fermion field

The *energy momentum tensor* from Noether's Theorem is

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)} \partial^\nu \psi + \partial^\nu \bar{\psi} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \bar{\psi}\right)} - g^{\mu\nu} \mathcal{L} \quad = \bar{\psi} i \gamma^\mu \partial^\nu \psi$$

where we used the Dirac equation to remove the last term, i.e. $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$

So

$$T^{00} = i\psi^\dagger \partial^0 \psi$$

and the *Hamiltonian* is

$$H = \int T^{00}\, d^3x = \int i\psi^\dagger \partial^0 \psi \, d^3x$$

$$= \int d^3x \int \frac{d^3p}{(2\pi)^3\, 2E(\vec{p})} \frac{d^3q}{(2\pi)^3\, 2} \sum_{s,s'} \left(\hat{a}^{(s)\,\dagger}(\vec{p})\, u^{(s)\,\dagger}(p)\, e^{ip\cdot x} + \hat{b}^{(s)}(\vec{p})\, v^{(s)\,\dagger}(p)\, e^{-ip\cdot x}\right)$$

$$\times \left(\hat{a}^{(s')}(\vec{q})\, u^{(s')}(q)\, e^{-iq\cdot x} - \hat{b}^{(s')\,\dagger}(\vec{q})\, v^{(s')}(q)\, e^{iq\cdot x}\right)$$

Using our orthogonality relations, $v^{(r)\,\dagger} v^{(s)} = 2E\delta^{rs}$, $u^{(r)\,\dagger} u^{(s)} = 2E\delta^{rs}$ and $u^{(r)\,\dagger} v^{(s)} = 0$ this becomes,

$$H = \int \frac{d^3p}{(2\pi)^3 2E(\vec{p})} E(\vec{p}) \sum_s \left(\hat{a}^{(s)\,\dagger}(\vec{p}) \hat{a}^{(s)}(\vec{p}) - \hat{b}^{(s)}(\vec{p})\, \hat{b}^{(s)\,\dagger}(\vec{p}) \right)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E(\vec{p})} E(\vec{p}) \sum_s \left(\hat{a}^{(s)\,\dagger}(\vec{p}) \hat{a}^{(s)}(\vec{p}) - \hat{b}^{(s)\,\dagger}(\vec{p})\, \hat{b}^{(s)}(\vec{p}) - \left[\hat{b}^{(s)}(\vec{p}), \hat{b}^{(s)\,\dagger}(\vec{p}) \right] \right)$$

Postulating a commutation relation for the creation and annihilation operators doesn't work. Even after subtraction of an infinite c-number, the energy contribution from the antiparticles is negative. So we can *reduce the energy by creating more antiparticles*. Clearly this system would not be stable.

To make the contribution from antiparticles positive, we need *anticommutation relations*.

We postulate:

$$\left\{\hat{a}^{(r)}(\vec{k}), \hat{a}^{(s)\,\dagger}(\vec{p})\right\} = \left\{\hat{b}^{(r)}(\vec{k}), \hat{b}^{(s)\,\dagger}(\vec{p})\right\} = \delta^{rs}\,(2\pi)^3\,2E(\vec{k})\delta^3(\vec{k}-\vec{p})$$

$$\left\{\hat{a}^{(r)}(\vec{k}), \hat{a}^{(s)}(\vec{p})\right\} = \left\{\hat{a}^{(r)}(\vec{k}), \hat{b}^{(s)}(\vec{p})\right\} = \cdots = 0.$$

Which leads to anti-commutation relations for the fields too,

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0 = i\psi^\dagger$$

$$\left\{\hat{\psi}^{(r)}(\vec{x}, t), \hat{\pi}^{(s)}(\vec{y}, t)\right\} = i\delta^{rs}\delta^3(\vec{x}-\vec{y})$$

$$\left\{\hat{\psi}^{(r)}(\vec{x}, t), \hat{\psi}^{(s)}(\vec{y}, t)\right\} = \left\{\hat{\pi}^{(r)}(\vec{x}, t), \hat{\pi}^{(s)}(\vec{y}, t)\right\} = 0$$

Notice that this means that the state with two fermions is *antisymmetric* under their interchange,

$$|\vec{p}_1, r_1; \vec{p}_2, r_2\rangle = \hat{a}^{(r_1)\,\dagger}(\vec{p}_1)\hat{a}^{(r_2)\,\dagger}(\vec{p}_2)|0\rangle = -\hat{a}^{(r_2)\,\dagger}(\vec{p}_2), \hat{a}^{(r_1)\,\dagger}(\vec{p}_1)|0\rangle = -|\vec{p}_2, r_2; \vec{p}_1, r_1\rangle,$$

which is of course *the origin of Fermi-Dirac statistics and the Pauli exclusion principle itself*.

The Dirac propagator

The *Greens function* for the Dirac Equation satisfies, $(S_F(x-y) \equiv G_2(x,y))$,

$$(i\partial\!\!\!/ - m)S_F(x-y) = i\delta^{(4)}(x-y)$$

Writing $S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\tilde{S}(p)\,e^{-ip\cdot(x-y)}$ and pre-multiplying by $(i\partial\!\!\!/ - m)$ gives,

$$\int \frac{d^4p}{(2\pi)^4}\,(p\!\!\!/ - m)\,\tilde{S}_F(p)\,e^{-ip\cdot(x-y)} = i\delta^{(4)}(x-y)$$

Remember that $\int \frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot(x-y)} = \delta^{(4)}(x-y)$ so we must have,

$$\boxed{\tilde{S}_F(p) = \frac{i}{p\!\!\!/ - m} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2}}$$

Just as for the scalar fields, this propagator is a time ordered product of fields, but now we need to be a bit *careful with our signs*.

For a fermion field,

$$T\,\psi^{(r)}(x)\bar{\psi}^{(s)}(y) = \begin{cases} \psi^{(r)}(x)\bar{\psi}^{(s)}(y) & \text{for} \quad x^0 > y^0 \\ -\bar{\psi}^{(s)}(y)\psi^{(r)}(x) & \text{for} \quad x^0 < y^0 \end{cases}$$

With this definition, one can show,

$$S_F^{rs}(x-y) = \langle 0|T\psi^{(r)}(x)\bar{\psi}^{(s)}(y)|0\rangle$$

5. Quantum Electrodynamics

James Clerk Maxwell
1831 –1879

Classical Electromagnetism

Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}$$

Maxwell wrote these down in *1864*, but amazingly they are covariant!

Writing $F^{\mu\nu} \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$ $\quad j^\mu \equiv \left(\rho, \vec{J}\right)$ they are

[Note: $F^{\mu\nu} = -F^{\nu\mu}$]

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu}$$

[Note: the ability to write Maxwell's Equations in this form is **not** a proof of covariance!]

Maxwell's equations can also be written in terms of a *potential* A^μ

Writing $\quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

we have $\quad \partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$

Now, notice that I can change A_μ by a derivative of a scalar and leave $F_{\mu\nu}$ unchanged

$$A^\mu \to A^\mu + \lambda \partial^\mu \phi \qquad \Rightarrow \qquad F^{\mu\nu} \to F^{\mu\nu} + \lambda \partial^\mu \partial^\nu \phi - \lambda \partial^\nu \partial^\mu \phi = F^{\mu\nu}$$

Choose λ such that

$$\boxed{\partial_\mu A^\mu = 0}$$

This is a *gauge transformation*, and the choice $\partial_\mu A^\mu = 0$ is know as the *Lorentz gauge*.

In this gauge:

$$\boxed{\partial^2 A^\mu = j^\mu}$$

$\longleftarrow$ does this look familiar?

The wave equation with no source, $\partial^2 A^\nu = 0$ has solutions

$$A^\mu = \epsilon^\mu e^{iq\cdot x} \quad \text{with} \quad q^2 = 0$$

polarisation vector with
4 degrees of freedom

The Lorentz condition $\quad (\,\partial_\mu A^\mu = 0\,) \quad \Rightarrow \quad q_\mu \epsilon^\mu = 0$

So ϵ^μ has only 3 degrees of freedom (two transverse d.o.f. and one longitudinal d.o.f.)

We still have some freedom to change A^μ, even after our Lorentz gauge choice:

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \quad \text{is OK, as long as} \quad \partial^2 \chi = 0$$

Usually we choose χ such that $\vec{\nabla} \cdot \vec{A} = 0$. This is known as the *Coulomb gauge*.

$$\vec{\nabla} \cdot \vec{A} = 0 \quad \Rightarrow \quad \vec{q} \cdot \vec{\epsilon} = 0$$

So only two polarisation states remain (both transverse).

A Lagrangian for the free photon field

We want a Lagrangian which will give us $\partial_\mu F^{\mu\nu} = 0$ (Maxwell's equations with no sources)

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

Recall $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$

$$\frac{\partial F_{\mu\nu}}{\partial A_\rho} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial A_\rho} = 0$$

$$\frac{\partial F_{\mu\nu}}{\partial\left(\partial_\sigma A_\rho\right)} = g_\mu^\sigma g_\nu^\rho - g_\mu^\rho g_\nu^\sigma \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial\left(\partial_\sigma A_\rho\right)} = -\frac{1}{2} F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial\left(\partial_\sigma A_\rho\right)}$$

$$= -\frac{1}{2} F^{\mu\nu}\left(g_\mu^\sigma g_\nu^\rho - g_\mu^\rho g_\nu^\sigma\right)$$

$$= -\frac{1}{2}\left(F^{\sigma\rho} - F^{\rho\sigma}\right) = -F^{\sigma\rho}$$

$F^{\rho\sigma}$ antisymmetric

$$\frac{\partial \mathcal{L}}{\partial A_\rho} - \partial_\sigma\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\sigma A_\rho\right)}\right) = 0$$

$$\Rightarrow \quad \partial_\sigma F^{\sigma\rho} = 0$$

Second Quantisation of the photon field

The photon is a boson, but now has vector structure. Our decomposition in terms of creation and annihilation vectors is now,

$$\hat{A}_\mu(x) = \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} \left(\hat{a}_\mu(\vec{k})\, e^{-ik\cdot x} + \hat{a}^\dagger_\mu(\vec{k})\, e^{ik\cdot x} \right)$$

now these are four-vectors

We need to be careful with our canonically conjugate states though, since $\dfrac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu}$.

So $\pi^0 = 0$ and $\pi^i = -F^{0i} = E_i$ the electric field.

This leads to all sorts of complications which we will not go into here.

The Hamiltonian is,

$$\hat{H} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3\, 2E(\vec{k})} E(\vec{k}) \left(-\hat{a}^\dagger_\mu(\vec{k})\, \hat{a}^\mu(\vec{k}) - \hat{a}^\mu(\vec{k})\, \hat{a}^\dagger_\mu(\vec{k}) \right)$$

We have commutation relations,

$$\left[\hat{a}_\mu(\vec{k}), \hat{a}^\dagger_\mu(\vec{p})\right] = -g_{\mu\nu}\,(2\pi)^3\,2E(\vec{k})\delta^3(\vec{k}-\vec{p})$$

$$\left[\hat{a}_\mu(\vec{k}), \hat{a}_\mu(\vec{p})\right] = \left[\hat{a}^\dagger_\mu(\vec{k}), \hat{a}^\dagger_\mu(\vec{p})\right] = 0.$$

To do this properly, we should *impose our gauge conditions via the Lagrangian*,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\,(\partial_\mu A^\mu)^2.$$

The λ is a *Lagrange multiplier* which forces the gauge condition $\partial_\mu A^\mu = 0$. For general λ the propagator is,

$$D_F(x-y) = \langle 0|T A_\mu(x) A_\nu(y)|0\rangle = i\int \frac{d^4k}{(2\pi)^4}\left(-g^{\mu\nu} + \left(1-\frac{1}{\lambda}\right)\frac{k^\mu k^\nu}{k^2}\right)\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon}$$

The photon propagator *in the Feynman Gauge* for which λ = 1 is

$$\langle 0|T A_\mu(x) A_\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}}{k^2+i\epsilon}e^{-ik\cdot(x-y)}$$

The Dirac Equation in an Electromagnetic Field

How do we incorporate electromagnetism into our theory of fermions?

We have one more *symmetry* of the Dirac Lagrangian which we haven't looked at yet.

Consider phase shifting the electron field by $\psi \longrightarrow e^{i\theta}\psi$ and $\bar{\psi} \longrightarrow \bar{\psi}\, e^{-i\theta}$.

$$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi \longrightarrow \bar{\psi}\,e^{-i\theta}\,(i\gamma^\mu\partial_\mu - m)\,e^{i\theta}\psi = \mathcal{L}$$

The Lagrangian doesn't change so the physics stays the same.

The conserved current associated with this symmetry via Noether's theorem is $j^\mu = e\bar{\psi}\gamma^\mu\psi$.

This is known as a *global U(1) symmetry*

(since $e^{i\theta}$ doesn't vary with space-time coordinate)

(since $e^{i\theta}$ is a unitary 1×1 matrix)

What happens if we make our transformation *local*, i.e. depend on space-time point?

$$\psi \longrightarrow e^{i\theta(x)}\psi \qquad\qquad \bar{\psi} \longrightarrow \bar{\psi}\, e^{-i\theta(x)}$$

$$\mathcal{L} \longrightarrow \bar{\psi}\, e^{-i\theta(x)}\,(i\gamma^\mu\partial_\mu - m)\, e^{i\theta(x)}\psi \;= \mathcal{L} - \bar{\psi}\gamma^\mu\,(\partial_\mu\theta(x))\,\psi$$

The free Dirac Lagrangian is no longer invariant. If we really want this to be a symmetry of the theory, we will have to add in something new.

Let's postulate a new field A^μ which couples to the electron according to

$$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu\partial_\mu - e\gamma_\mu A^\mu - m)\,\psi$$

charge of the electron = -e

Beware: conventions differ, Peskin & Schroeder have as above, but some other texts define e<0.

Often this is written in terms of a *"covariant derivative"*

$$\boxed{D^\mu \equiv \partial^\mu + ieA^\mu}$$

Now $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \longrightarrow \bar{\psi}\left(i\gamma^\mu e^{-i\theta(x)} D'_\mu e^{i\theta(x)} - m\right)\psi$

So, to preserve the Lagrangian, we need D_μ to transform too:

$$D_\mu \longrightarrow D'_\mu = e^{i\theta(x)} D_\mu e^{-i\theta(x)}$$

$$\Rightarrow \quad \partial_\mu + ieA'_\mu = e^{i\theta(x)}\left(\partial_\mu + ieA_\mu\right)e^{-i\theta(x)} = \partial_\mu - i\partial_\mu\theta(x) + ieA_\mu$$

therefore we need A_μ to transform as

$$A_\mu \longrightarrow A'_\mu = A_\mu - \tfrac{1}{e}\partial_\mu\theta(x)$$

This is the gauge transformation we saw for the (classical) photon earlier!

Coupling the *electron* to a *photon* makes the theory *locally U(1) symmetric*

$$\boxed{\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu}$$

$$\boxed{\left(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu\right)\psi = 0, \qquad \partial^2 A^\mu = e\bar{\psi}\gamma^\mu\psi}$$

6. Scattering in QED

Now we have the QED Lagrangian, we can calculate some scattering cross-sections.

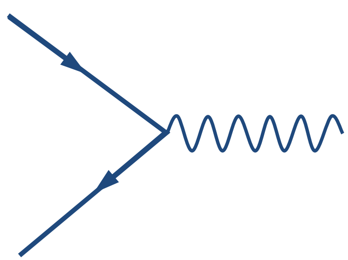
We can calculate the transition amplitudes as we did for the scalar fields. Now our interaction is,

$$\mathcal{L}_{\text{int}} = -e\bar{\psi}\gamma_\mu A^\mu \psi\,,$$

and our S-matrix is

$$\begin{aligned}
\langle f|\hat{S}|i\rangle &= \langle f|Te^{i\int d^4x \mathcal{L}_{\text{int}}}|i\rangle = \langle f|Te^{-ie\int d^4x \bar{\psi}\gamma_\mu A^\mu \psi}|i\rangle \\
&= \langle f|i\rangle - ie\int d^4x\, \langle f|T\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,|i\rangle \\
&\quad + (-ie)^2 \int d^4x\, d^4y\, \langle f|T\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y)|i\rangle + \cdots
\end{aligned}$$

As in the case with scalars, the first term is not scattering. The second term O(e) are processes of the form,



Note that this cannot conserve momentum if all three particles are on-shell.

$$p_\gamma^2 = (p_{e^-} + p_{e^+})^2 = 2m^2 + 2p_{e^-}\cdot p_{e^+} \neq 0$$

For scattering, we must go to the O(e²) term, and use Wick's Theorem,

$$T\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y)$$

$$= :\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y): \qquad \text{Term 1}$$

$$+ :\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y): \qquad \text{Term 2a}$$

$$+ :\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y): \qquad \text{Term 2b}$$

$$+ :\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y): \qquad \text{Term 2c}$$

$$+ :\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y): \qquad \text{Term 3a}$$

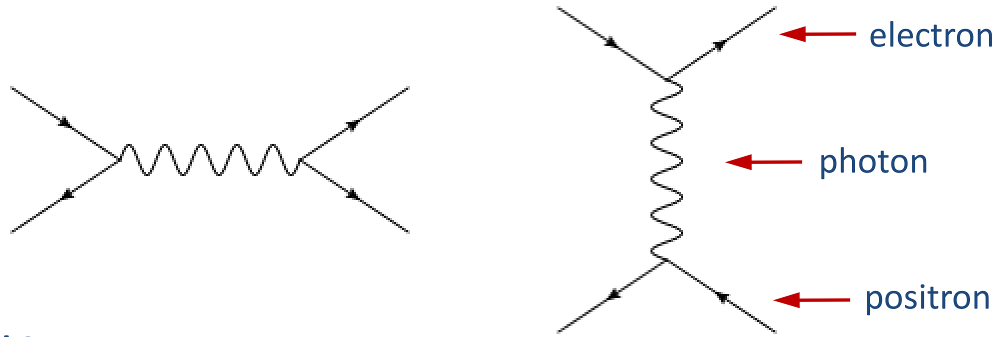
$$+ :\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y): \qquad \text{Term 3b}$$

$$+ :\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y): \qquad \text{Term 3c}$$

$$+ :\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y): \qquad \text{Term 4}$$

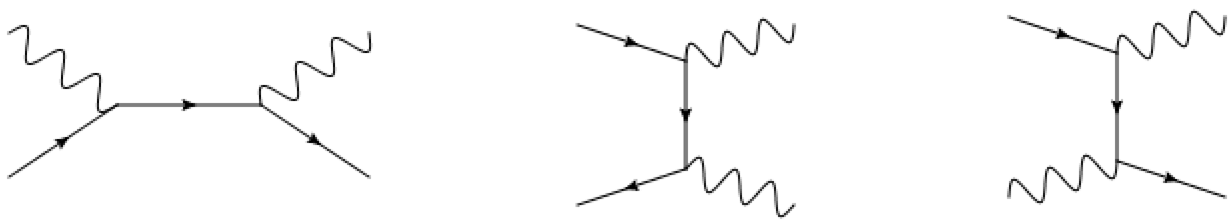
Term 1: Never contributes to scattering, since it is two disconnected vertices.

Term 2a:



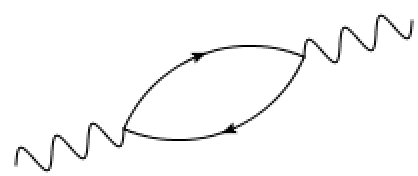
Note: These are only a sample subset of possible diagrams.

Term 2b and 2c:



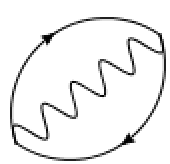
Term 3a and 3b:

Term 3c:



Not scattering

Term 4:



Disconnected graph

Let's consider the *scattering of two electrons* $e^- e^- \to e^- e^-$

Initial state: $|\vec{p}_1, \vec{p}_2\rangle$ Final state: $|\vec{p}_3, \vec{p}_4\rangle$

In principle, these states have particular spins, but I will suppress these in my notation for now.

Only *term 2a* contributes to scattering, so we have,

$$-e^2 \int d^4x d^4y \langle 0| a(\vec{p}_4) a(\vec{p}_3) :\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\,\bar{\psi}(y)\gamma_\nu A^\nu(y)\psi(y): a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2)|0\rangle$$

Use $\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3\, 2E(\vec{p})} \sum_s \left(\hat{a}^{(s)}(\vec{p})\, u^{(s)}(p)\, e^{-ip\cdot x} + \hat{b}^{(s)\,\dagger}(\vec{p})\, v^{(s)}(p)\, e^{ip\cdot x}\right)$ and notice that all the antiparticle operators will just vanish.

After quite a bit of algebra, we get,

$$e^2 \left(\bar{u}(p_3)\gamma_\mu u(p_1)\, \frac{g^{\mu\nu}}{(p_3 - p_1)^2}\, \bar{u}(p_4)\gamma_\nu u(p_2) - \bar{u}(p_4)\gamma_\mu u(p_1)\, \frac{g^{\mu\nu}}{(p_4 - p_1)^2}\, \bar{u}(p_3)\gamma_\nu u(p_2)\right)$$
$$\times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

Note the two terms have a relative – sign.

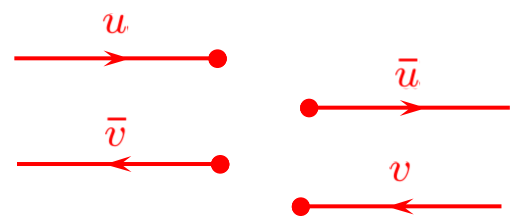
Feynman Diagrams: The QED Feynman Rules (Feynman Gauge)

Richard Feynman
1918 - 1988

Thankfully, we can construct the transition amplitudes simply by associating mathematical expressions with the diagrams describing the interaction.

For each diagram, write:

- u for each incoming electron
- $\bar{u}$ for each outgoing electron
- $\bar{v}$ for each incoming positron
- v for each outgoing positron

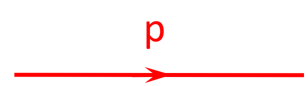


- ϵ^{μ} for each incoming photon

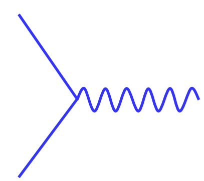
- $\epsilon^{\mu\,*}$ for each outgoing photon

- $-ig_{\mu\nu}\dfrac{1}{p^2}$ for each internal photon

- $i\dfrac{\not{p}+m}{p^2-m^2}$ for each internal electron



- $ieQ\gamma^{\mu}$ for each vertex

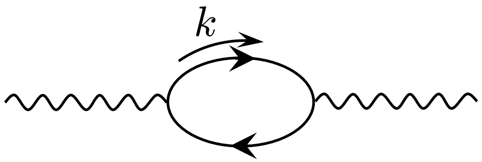


(fermion charge Q)

Remember that γ-matrices and spinors do not commute, so be careful with the order in spin lines. Write left to right, *against* the fermion flow.

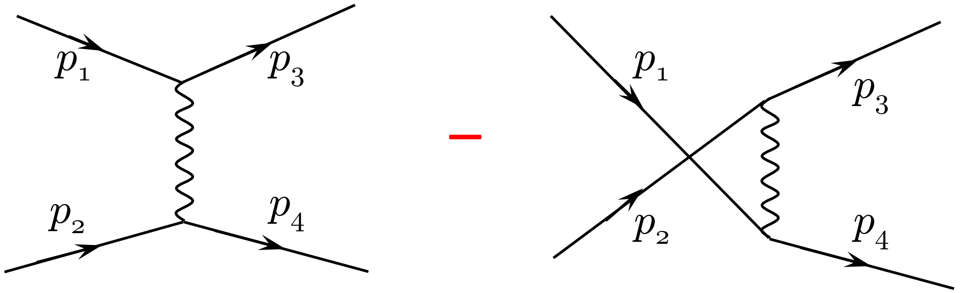
2 details:

- *Closed loops*:



Integrate over loop momentum $\int \frac{d^4k}{(2\pi)^4}$ and include an extra factor of -1 if it is a fermion loop.

- *Fermi Statistics*: If diagrams are identical except for an exchange of electrons, include a relative – sign.

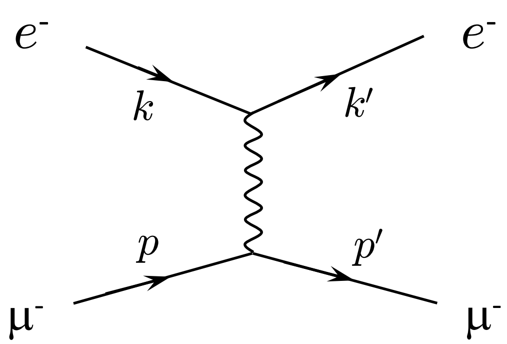


These rules provide $i\mathcal{M}$, and the transition amplitude is

$$S_{fi} = \langle f|\widehat{S}|i\rangle = \langle f|i\rangle + (2\pi)^4\delta^4(p_f - p_i)\mathcal{M} = 1 + iT_{fi}$$

We write $\widehat{S} = 1 + i\widehat{T}$ to remove the non-scattering piece.

An example calculation: $e^- \mu^- \to e^- \mu^-$ (non-identical particles allows us to consider only one diagram)



$$i\mathcal{M} = -e^2\, \bar{u}(k')\gamma^\mu u(k)\, \frac{g_{\mu\nu}}{q^2}\, \bar{u}(p')\gamma^\nu u(p)$$

This is what we had before.

To get the total probability we must *square* this, *average over initial spins*, and *sum over final spins*.

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{e^4}{q^4}\frac{1}{4}\sum_{\text{spins}} \left\{\left[\bar{u}(k')\gamma^\mu u(k)\right]\, \left[\bar{u}(k')\gamma^\nu u(k)\right]^*\right\} \left\{\left[\bar{u}(p')\gamma_\mu u(p)\right]\, \left[\bar{u}(p')\gamma_\nu u(p)\right]^*\right\} \\ &= \frac{e^4}{q^4}\mathcal{L}^{\mu\nu}(k,k')\,\mathcal{L}_{\mu\nu}(p,p')\end{aligned}$$

But

$$\begin{aligned}\left[\bar{u}(k')\gamma^\mu u(k)\right]^* &= \left[u^\dagger(k')\gamma^0\gamma^\mu u(k)\right]^\dagger && \left(\bar{u} \equiv u^\dagger\gamma^0\right) \\ &= u^\dagger(k)\gamma^{\mu\,\dagger}\gamma^{0\,\dagger}u(k') \\ &= u^\dagger(k)\gamma^0\gamma^\mu\gamma^0\gamma^0 u(k') && \left(\gamma^{\mu\,\dagger} = \gamma^0\gamma^\mu\gamma^0,\ \gamma^{0\,\dagger} = \gamma^0\right) \\ &= \bar{u}(k)\gamma^\mu u(k') && \left(\gamma^0\gamma^0 = \mathbf{1}\right)\end{aligned}$$

But don't forget that the u are 4-component spinors and the γ are 4×4 matrices:

So $$\mathcal{L}^{\mu\nu}(k,k') = \frac{1}{2}\sum_{s=1,2}\sum_{s'=1,2} \bar{u}_i^{(s')}(k')\,[\gamma^\mu]_{ij}\,u_j^{(s)}(k)\,\bar{u}_m^{(s)}(k)\,[\gamma^\nu]_{mn}\,u_n^{(s')}(k')$$

spinor indices

(summation over i,j,m,n = 1,..,4)

We can simplify this using the completeness relation for spinors:

$$\sum_{s=1,2} u_i^{(s)}(k)\,\bar{u}_j^{(s)}(k) = k_\mu[\gamma^\mu]_{ij} + m\,\delta_{ij}$$

(beware normalization here – this is only true for 2E particles per unit volume)

Then $$\begin{aligned}\mathcal{L}^{\mu\nu}(k,k') &= \tfrac{1}{2} \sum_{s'=1,2} u_n^{(s')}(k')\bar{u}_i^{(s')}(k')\,[\gamma^\mu]_{ij} \sum_{s=1,2} u_j^{(s)}(k)\,\bar{u}_m^{(s)}(k)\,[\gamma^\nu]_{mn} \\ &= \tfrac{1}{2} \left(k'_\rho[\gamma^\rho]_{ni} + m\delta_{ni}\right)\;[\gamma^\mu]_{ij}\;\left(k_\sigma[\gamma^\sigma]_{jm} + m\delta_{jm}\right)\;[\gamma^\nu]_{mn} \\ &= \tfrac{1}{2}\;\mathrm{Tr}\left[\left(\not{k}' + m\right)\gamma^\mu\,(\not{k} + m)\,\gamma^\nu\right]\end{aligned}$$

We need some trace identities!

Trace Identities

$$\mathrm{Tr}\,\mathbb{1} = 4$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \frac{1}{2}\mathrm{Tr}\,(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \frac{1}{2}\mathrm{Tr}\,(2g^{\mu\nu}) = g^{\mu\nu}\,\mathrm{Tr}\,\mathbb{1} = 4\,g^{\mu\nu}$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = \mathrm{Tr}\,\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda = \mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5 = -\mathrm{Tr}\,\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda = 0$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa = 4\left(g^{\mu\nu}g^{\lambda\kappa} - g^{\mu\lambda}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\lambda}\right)$$

This is true for any **odd** number of γ-matrices

$$\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa = -4i\epsilon^{\mu\nu\lambda\kappa}$$

be careful with this one!

Using the identities for $\mathrm{Tr}\,\gamma^\mu\gamma^\nu$ and $\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\kappa$, we find

$$\mathcal{L}^{\mu\nu}(k,k') = 2\left(k^\mu k'^{\,\nu} + k^\nu k'^{\,\mu} - \left(k\cdot k' - m^2\right)g^{\mu\nu}\right)$$

So $\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$

$$= \frac{e^4}{q^4} 4 \left(k^\mu k'^\nu + k^\nu k'^\mu - \left(k \cdot k' - m_e^2\right) g^{\mu\nu}\right) \left(p_\mu p'_\nu + p_\nu p'_\mu - \left(p \cdot p' - m_\mu^2\right) g_{\mu\nu}\right)$$

$$= 8\frac{e^4}{q^4} \left((k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p') - m_e^2(p' \cdot p) - m_\mu^2(k' \cdot k) + 2m_e^2 m_\mu^2\right)$$

If we are working at sufficiently high energies, then $p^2 \gg m_e^2,\, m_\mu^2$ and we may ignore the masses.

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 \approx 8\frac{e^4}{(k-k')^4} \left((k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p')\right)$$

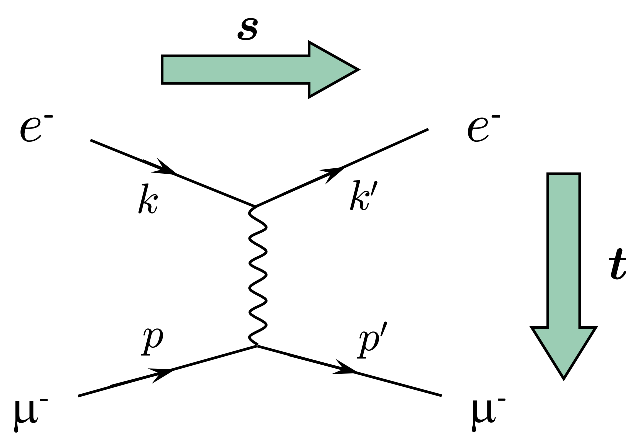
Often this is written in terms of *Mandlestam Variables*, which are defined:

$$
\begin{aligned}
s &= (k+p)^2 &&\approx \quad 2k\cdot p &&\approx \quad 2k'\cdot p' \\
t &= (k-k')^2 &&\approx -2k\cdot k' &&\approx -2p\cdot p' \\
u &= (k-p')^2 &&\approx -2k\cdot p' &&\approx -2k'\cdot p
\end{aligned}
$$

[Note that $s+t+u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \approx 0$

Then

$$\boxed{\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4\,\frac{s^2+u^2}{t^2}}$$



- s is the square of the momentum flowing in the time direction
- t is the square of the momentum flowing in the space direction

Cross-sections

So we have $|\mathcal{M}|^2$ but we are not quite there yet – we need to turn this into a *cross-section*.

Recall $$T_{fi} = -i(2\pi)^4\delta^4(p_f - p_i)\mathcal{M}$$

$$\Rightarrow \qquad \begin{aligned} |T_{fi}|^2 &= \left[(2\pi)^4\delta^4(p_f - p_i)\right]^2 |\mathcal{M}|^2 \\ &= (2\pi)^4\delta^4(p_f - p_i)VT\,|\mathcal{M}|^2 \end{aligned}$$

since $(2\pi)^4\delta^4(p_f - p_i) = \int d^4x e^{i(p_f - p_i)\cdot x} = \int_{-T/2}^{T/2} dt \int_V d^3x = VT$

But we need the transition probability per unit time and per unit volume is:

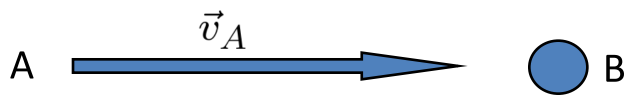
$$\boxed{\frac{|T_{fi}|^2}{VT} = (2\pi)^4\delta^4(p_f - p_i)\,|\mathcal{M}|^2}$$

The *cross-section* is the probability of transition per unit volume, per unit time $\times$ the number of final states / initial flux.

$$\boxed{d\sigma = \frac{|T_{fi}|^2}{VT} \times \frac{\#\text{ final states}}{\text{initial flux}}}$$

Initial Flux

In the lab frame, particle A, moving with velocity $\vec{v}_A$, hits particle B, which is stationary.



The number of particles like A in the beam, passing through unit volume per unit time is $|\vec{v}_A| 2E$

The number of particles like B per volume V in the target is $2E$

So the initial flux in a volume V is $|\vec{v}_A|\, 2E_A\, 2E_B$

But we can write this in a covariant form: $|\vec{v}_A|\, 2E_A\, 2E_B = \sqrt{\left[(p_A \cdot p_B)^2 - m_A^2 m_B^2\right]}$

This must also be true for a *collider*, where A and B are both moving, since the lab frame and centre-of-mass frame are related by a Lorentz boost.

 # final states

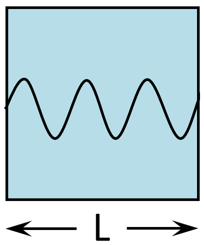
How many states of momentum $\vec{p}$ can we fit in a volume V?

In order to not have any particle flow through the boundaries of the box, we must impose *periodic boundary conditions*.

$Lp_x = 2\pi\, n$ so the number of states between p_x and p_x+dp_x is $\dfrac{L}{2\pi}\, dp_x$

So in a volume V we have

$$\left(\frac{L}{2\pi}dp_x\right)\left(\frac{L}{2\pi}dp_y\right)\left(\frac{L}{2\pi}dp_z\right) = \frac{V}{(2\pi)^3}d^3p$$



But there are $2EV$ particles per volume V, so

\# final states per particle = $\dfrac{1}{(2\pi)^3}\dfrac{d^3p}{2E}$

Remember that $\displaystyle\int \frac{1}{(2\pi)^3}\frac{d^3p}{2E} = \int \frac{d^4p}{(2\pi)^4}\, 2\pi\delta(p^2 - m^2)$ so this is covariant

Putting all this together, the *differential cross-section* is:

$$\boxed{d\sigma = \frac{1}{F}|\mathcal{M}|^2 d\text{Lips}}$$

where the *Flux* F is given by,

$$\boxed{F = 4\sqrt{\left[(p_A \cdot p_B)^2 - m_A^2 m_B^2\right]}}$$

and the **L**orentz **i**nvariant **p**hase **s**pace is,

$$\boxed{d\text{Lips} = \underbrace{(2\pi)^4 \delta^4(p_c + p_d - p_a - p_b)}_{\text{momentum conservation}} \underbrace{2\pi\delta(p_c^2 - m_c^2)\, 2\pi\delta(p_d^2 - m_d^2)}_{\text{on-shell conditions}} \underbrace{\frac{d^4 p_c}{(2\pi)^4}\frac{d^4 p_d}{(2\pi)^4}}_{\text{integration measure}}}$$

In the *centre-of-mass*, this becomes much simpler

This frame is defined by $\vec{p}_a = -\vec{p}_b$ and $E_a + E_b = \sqrt{s}$

Remember $s \equiv (p_a + p_b)^2$

So $p_a = (E_a, \vec{p}_a)$ and $p_b = (E_b, -\vec{p}_a)$

with $|\vec{p}_a| = \frac{1}{4s}\sqrt{\lambda(s, m_a^2, m_b^2)}$ $\qquad \lambda(\alpha,\beta,\gamma) \equiv \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma$

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \qquad E_b = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}$$

Then the *Flux* becomes

$$\begin{aligned} F &= 4\sqrt{\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]} \\ &= 4\left(|\vec{p}_a| E_b + |\vec{p}_b| E_a\right) \end{aligned}$$

$$\boxed{F = 4|\vec{p}_a|\sqrt{s}}$$

Also $\vec{p}_c = -\vec{p}_d$ and $E_c + E_d = \sqrt{s}$ with relations analogous to those for p_a and p_b

The phase space measure becomes:

$$
\begin{aligned}
d\text{Lips} &= (2\pi)^4 \delta^4(p_c + p_d - p_a - p_b)\, \frac{1}{2E_c} \frac{d^3\vec{p}_c}{(2\pi)^3} \frac{1}{2E_d} \frac{d^3\vec{p}_d}{(2\pi)^3} \\
&= \frac{1}{4\pi^2} \delta(E_c + E_d - \sqrt{s}) \frac{1}{4E_cE_d}\, d^3\vec{p}_c \\
&= \frac{1}{4\pi^2} \delta(E_c + E_d - \sqrt{s}) \frac{1}{4E_cE_d} |\vec{p}_c|^2\, d|\vec{p}_c|\, d\Omega \\
&= \frac{1}{16\pi^2} \delta(\sqrt{s} - E_c - E_d)\, \frac{|\vec{p}_c|}{\sqrt{s}}\, d\sqrt{s}\, d\Omega
\end{aligned}
$$

$$\left(\text{since} \quad \frac{d|\vec{p}_c|}{d\sqrt{s}} = \frac{E_cE_d}{|\vec{p}_c|\sqrt{s}} \right)$$

$$\boxed{d\text{Lips} = \frac{1}{16\pi^2} \frac{|\vec{p}_c|}{\sqrt{s}}\, d\Omega}$$

Putting this together:

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{\text{CM}} = \frac{1}{64\pi^2} \frac{|\vec{p}_c|}{|\vec{p}_a|} \frac{1}{s} |\mathcal{M}|^2}$$

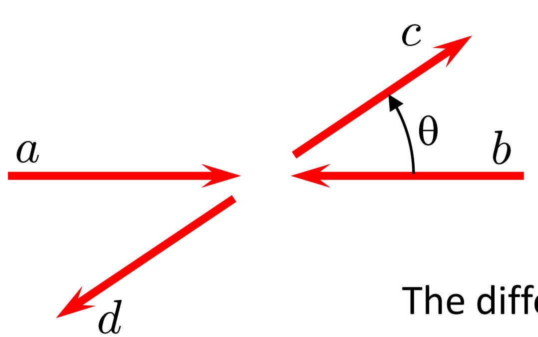
Returning to our process $e^-\mu^- \to e^-\mu^-$ (With $m_e = m_\mu = 0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\sum_{\text{spins}} \frac{1}{64\pi^2}\frac{|\vec{p}_c|}{|\vec{p}_a|}\frac{1}{s}|\mathcal{M}|^2 = \frac{1}{64\pi^2}\frac{1}{s}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{32\pi^2}\frac{1}{s}\frac{s^2+u^2}{t^2} = \frac{\alpha^2}{2s}\frac{s^2+u^2}{t^2}$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = 2e^4\frac{s^2+u^2}{t^2}$$

The fine structure constant $\alpha \equiv \frac{e^2}{4\pi}$

In terms of the angle between a and c



$$t \equiv (p_a - p_c)^2 = -2p_a\cdot p_c = -\frac{s}{2}(1-\cos\theta)$$

$$u \equiv (p_a - p_d)^2 = -2p_a\cdot p_d = -\frac{s}{2}(1+\cos\theta)$$

The differential cross-section is:

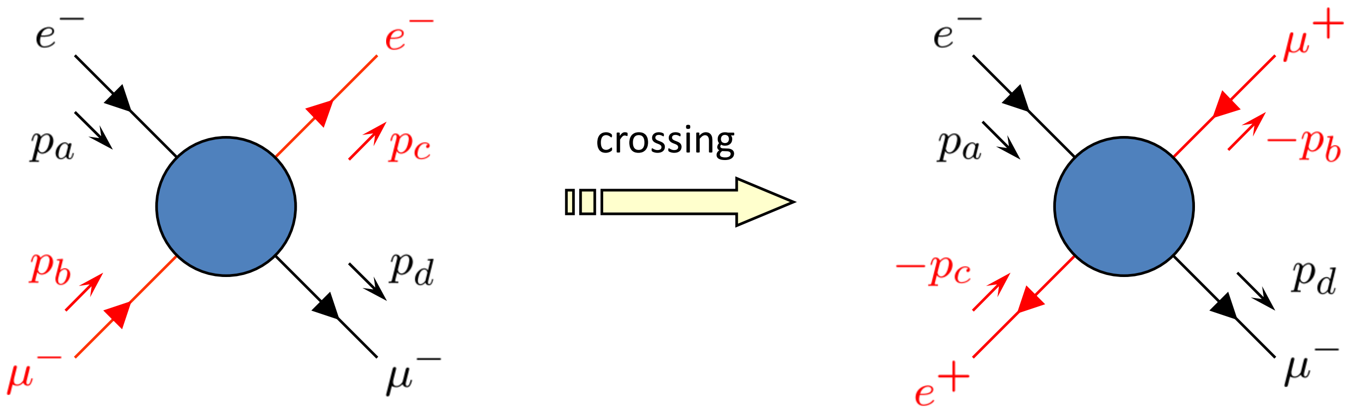
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8s}\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}}$$

Notice that this is divergent for small angles: $$\frac{d\sigma}{d\Omega} \sim \frac{4\alpha^2}{s}\frac{1}{\theta^4} \qquad \theta \to 0$$

This is exactly the same divergence as is in the Rutherford scattering formula.

Crossing symmetry

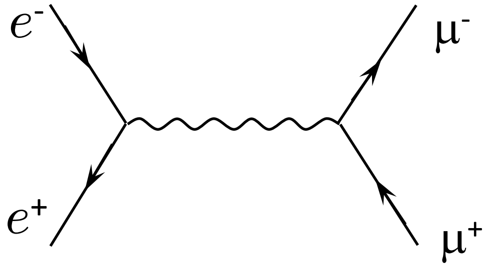
Generally, in a Feynman diagram, any incoming particle with momentum p is equivalent to an outgoing antiparticle with momentum $-p$.



We can use our result for $e^- \mu^- \to e^- \mu^-$ to calculate the differential cross-section for $e^+e^- \to \mu^+\mu^-$.

$$\begin{aligned} s &= (p_a + p_b)^2 \\ t &= (p_a - p_c)^2 \\ u &= (p_a - p_d)^2 \end{aligned} \quad \xrightarrow{\text{crossing}} \quad \begin{aligned} s &= (p_a - p_c)^2 \\ t &= (p_a + p_d)^2 \\ u &= (p_a - p_d)^2 \end{aligned}$$

i.e. $\boxed{s \leftrightarrow t}$



$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = 2e^4\,\frac{t^2+u^2}{s^2} \quad\Rightarrow\quad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\,\frac{t^2+u^2}{s^2}$$

Be careful not to change the s from flux and phase space!

Writing θ as the angle between the e^- and μ^-,

$$t = -\frac{s}{2}(1-\cos\theta)\,, \qquad u = -\frac{s}{2}(1+\cos\theta)$$

as before

$$\Rightarrow \quad \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)}$$

Notice the singularity is gone!

The total cross-section is

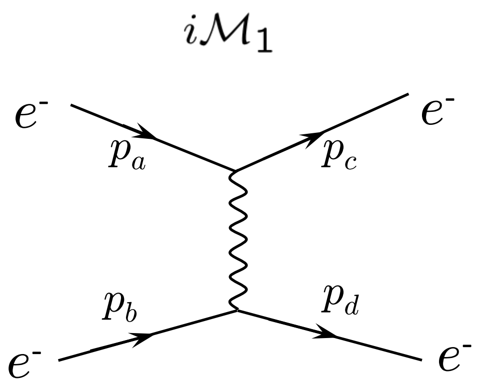
$$\sigma_{\text{Tot}} = \frac{\alpha^2}{4s}2\pi\int_{-1}^{+1}\left(1+\cos^2\theta\right)d(\cos\theta) = \frac{4\pi}{3}\frac{\alpha^2}{s}$$

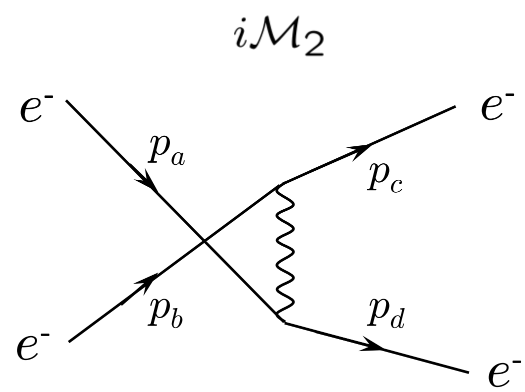
Identical particles in initial or final state

So far, in our example, the final state particles have all been *distinguishable* form one another. If the final state particles are *identical*, we have additional Feynman diagrams.

e.g. $e^- e^- \to e^- e^-$

[See Feynman rules]





$$ie^2\,\frac{1}{t}\,\bar{u}(p_c)\gamma^\mu u(p_a)\;\bar{u}(p_d)\gamma_\mu u(p_b)$$

$$ie^2\,\frac{1}{u}\,\bar{u}(p_d)\gamma^\mu u(p_a)\;\bar{u}(p_c)\gamma_\mu u(p_b)$$

p_c and p_d interchanged

These two diagrams must be added with a relative – sign

$$\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$$

Since the final state particles are identical, these diagrams are *indistinguishable* and must be summed *coherently*.

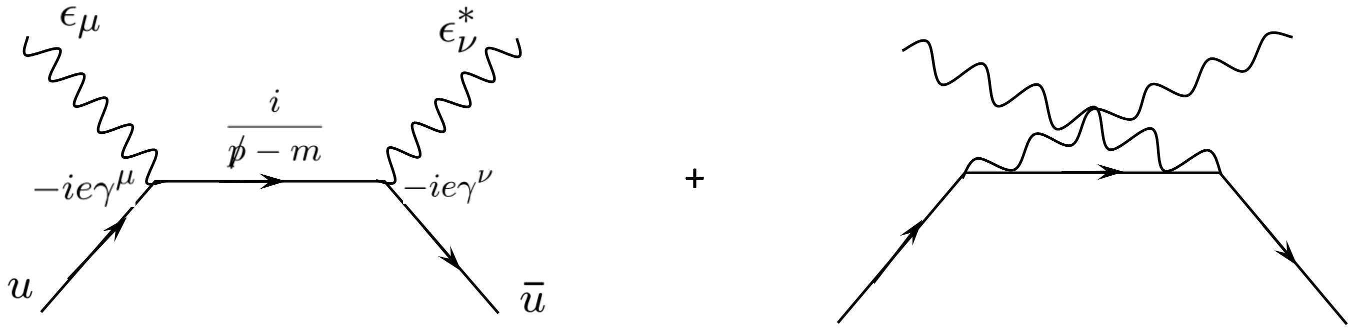
$$|\mathcal{M}|^2 = |\mathcal{M}_1 - \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 - 2\,\Re e\, \mathcal{M}_1 \mathcal{M}_2^*$$

We have interference between the two contributions.

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + 2\frac{s^2}{tu}\right)}$$

Compton Scattering

Compton scattering is the scattering of a photon with an electron.



Putting in the Feynman rules, and following through, with m_e = 0

$$\boxed{\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = -2e^4\left(\frac{u}{s} + \frac{s}{u}\right)}$$

Notice that the interference term has vanished. These two processes are *distinguishable* from the spins/polarizations of the external states.

The Ward Identity

To reproduce the result for Compton scattering on the last slide we need to perform a sum over the polarisation states of the photon. That is,

$$\sum_{\text{spins}} \epsilon^{\mu\,*}_{(T)} \epsilon^{\nu}_{(T)} \rightarrow -g^{\mu\nu}$$

This *sum is over transverse states* only, but the right-hand-side is really true for *virtual photons* only, i.e. a sum over all four polarizations.

For example, if photon momentum is $k^{\mu} = (k, 0, 0, k)$ then suitable transverse polarization vectors are $\epsilon_1^{\mu} = (0, 1, 0, 0),\ \epsilon_2^{\mu} = (0, 0, 1, 0).$ We really have a separate matrix element for each photon polarisation. Writing,

$$\mathcal{M}_{(\lambda)} = \epsilon^{\mu}_{(\lambda)} \tilde{\mathcal{M}}_{\mu} \qquad \text{then} \quad \sum_{\lambda=1,2} |\mathcal{M}_{(\lambda)}|^2 = |\tilde{\mathcal{M}}_1|^2 + |\tilde{\mathcal{M}}_2|^2$$

However, the matrix element has another property called the Ward Identity: $k^{\mu} \tilde{\mathcal{M}}_{\mu} = 0$

This is actually a result of gauge invariance, and it implies that $\tilde{\mathcal{M}}_0 = \tilde{\mathcal{M}}_3$, letting us write

$$\sum_{\lambda=1,2} |\mathcal{M}_{(\lambda)}|^2 = -|\tilde{\mathcal{M}}_0|^2 + |\tilde{\mathcal{M}}_1|^2 + |\tilde{\mathcal{M}}_2|^2 + |\tilde{\mathcal{M}}_3|^2 = -g_{\mu\nu} \tilde{\mathcal{M}}^{\mu} \tilde{\mathcal{M}}^{\nu\,*}$$

Decay Rates

So far we have only looked at $2 \to 2$ processes, but what about decays?

A decay width is given by:

$$\boxed{d\Gamma = \frac{|\kappa_{fi}|^2}{VT} \times \frac{\text{\# final states}}{\text{\# of decaying particles per unit volume}}}$$

This replaces the Flux.

For a decay $a \to b + c$ we have

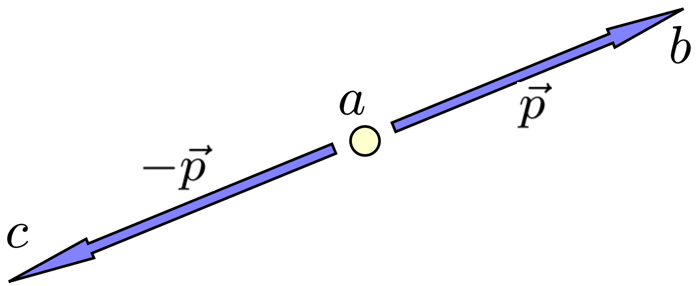
$$\frac{|\kappa_{fi}|^2}{VT} = (2\pi)^2 \delta^4(p_a - p_b - p_c)\, |\mathcal{M}|^2$$

$$\text{\# final states} = \frac{1}{2E_b}\frac{d^3\vec{p}_b}{(2\pi)^3}\frac{1}{2E_c}\frac{d^3\vec{p}_c}{(2\pi)^3} = \frac{d^4p_b}{(2\pi)^4}\frac{d^4p_c}{(2\pi)^4}\,(2\pi)\,\delta(p_b^2 - m_b^2)\,(2\pi)\,\delta(p_c^2 - m_c^2)$$

$$\text{\# of decay particles per unit volume} = 2E_a$$

$$\Rightarrow \qquad \boxed{d\Gamma = \frac{1}{2E_a}\,|\mathcal{M}|^2\, d\text{Lips}}$$

In the rest frame of particle a: $p_a = (m_a, \vec{0}), \quad p_b = (E_b, \vec{p}), \quad p_c = (E_c, -\vec{p}),$



with $E_b = \sqrt{|\vec{p}|^2 + m_b^2}, \quad E_c = \sqrt{|\vec{p}|^2 + m_c^2}$

The decay is back-to-back

$$d\Gamma = \frac{1}{2m_a} |\mathcal{M}|^2 \frac{1}{4E_b E_c} \frac{1}{(2\pi)^2} \delta(m_a - E_b - E_c) \, |\vec{p}|^2 d|\vec{p}| \, d\Omega$$

But $|\vec{p}| d|\vec{p}| = E_b dE_b = E_c dE_c = \dfrac{E_b E_c}{E_b + E_c} d(E_b + E_c)$

$$\Gamma = \int \frac{1}{8m_a} |\mathcal{M}|^2 \frac{1}{(2\pi)^2} \delta(m_a - E_b - E_c) \, \frac{|\vec{p}|}{E_b + E_c} d(E_b + E_c) \, d\Omega$$

$$\boxed{\Gamma = \int \frac{1}{32 m_a^2 \pi^2} |\mathcal{M}|^2 \, |\vec{p}| \, d\Omega}$$

Remember, to get the total decay rate, you need to sum over all possible decay processes.

$$\Gamma_{\text{Tot}} = \sum_i \Gamma_i$$

The inverse of the total width Γ_{Tot}^{-1} will give the *lifetime* of the particle:

If the number of particles = N_a then,

$$\Gamma_{\text{Tot}} = -\frac{1}{N_a}\frac{dN_a}{dt} \qquad \Rightarrow \qquad N_a(t) = N_a(0)e^{-\Gamma_{\text{Tot}}t}$$