

Relativistic Quantum Mechanics

**David J. Miller
University of Glasgow**

**SUPA Graduate School
October/November 2008**

http://www.physics.gla.ac.uk/~dmiller/lectures/RQM_2008.pdf

Recommended Text: **Quarks & Leptons** by F. Halzen and A. Martin
(though this is not really necessary)

Outline of topics

1. The Schrödinger equation
 - Non-relativistic quantum mechanics
2. The Klein Gordon Equation
 - A relativistic wave equation for bosons
3. The Dirac Equation
 - A relativistic wave equation for fermions
4. Quantum Electrodynamics
 - The Dirac equation in an electromagnetic potential
5. Scattering and Perturbation Theory
 - Feynman rules, cross-sections and widths
6. Quantum Chromodynamics
 - Quarks, gluons and color, renormalisation, running couplings

1. The Schrödinger Equation

- Consider a plane wave with energy $E = \hbar\omega$ and momentum $\vec{p} = \hbar\vec{k}$:

$$\psi(t, \vec{x}) = N e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

These values can be extracted using the energy and momentum **operators**:

$$\hat{E} = i\hbar \frac{\partial}{\partial t} \quad \hat{\vec{p}} = -i\hbar \vec{\nabla}$$

[see later for what I really mean by “=”]

- It should not be surprising that these operators do not commute with time and position respectively, and indeed obey the usual commutation relations:

$$[\hat{x}_i, \hat{p}_j] \psi(t, \vec{x}) = -i\hbar x_i \frac{\partial}{\partial x_j} \psi(t, \vec{x}) + i\hbar \frac{\partial}{\partial x_j} (x_i \psi(t, \vec{x})) = i\hbar \psi(t, \vec{x}) \delta_{ij}$$

- Indeed, we could have started with the commutation relation as a **postulate** and worked the other way.

Recall that the wavefunction is just the coefficient when we write the state vector in terms of the position eigenbasis $|x\rangle$,

$$|\psi\rangle = \int d^3x \psi(x)|x\rangle , \quad \text{i.e. } \psi(x) \equiv \langle x|\psi\rangle$$

-  **Exercise:** working in one space dimension only and assuming the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ show that

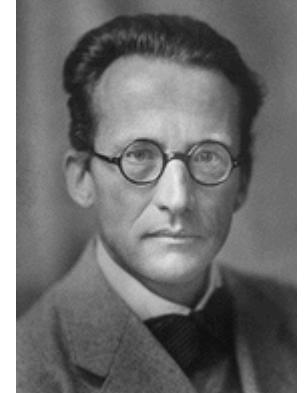
$$\langle \phi | \hat{p} | \psi \rangle = \int dx \phi^*(x) \left(-i\hbar \frac{\partial}{\partial x} \right) \psi(x)$$

Hint: First consider $\langle x | [\hat{x}, \hat{p}] | y \rangle$ and use $\delta(x - y) = (x - y) \frac{\partial}{\partial y} \delta(x - y)$

- This demonstrates that $-i\hbar \vec{\nabla}$ and $-i\hbar \frac{\partial}{\partial x}$ are strictly speaking only **position space representations** of the momentum operator. Can you work out (or guess) the *momentum space representations* of position and momentum?

- Classically we know $E = \frac{p^2}{2m} + V$, and writing this in terms of operators gives us the **Schrödinger equation**:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi$$



Erwin Schrödinger

- But how do we interpret the Schrödinger equation and the associated wavefunction?

The best way is to see what it conserves. What are its conserved currents and density?

$$\psi^* \times \text{S.E.} : \quad \psi^* i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \psi^* \nabla^2 \psi + V \psi^* \psi \quad (1)$$

$$\psi \times \text{S.E.}^* : \quad -\psi i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \psi \nabla^2 \psi^* + V \psi^* \psi \quad (2)$$

Now subtract (1) – (2)

$$\Rightarrow i\hbar \frac{\partial [\psi^* \psi]}{\partial t} = \frac{\hbar^2}{2m} \left[-\psi^* \nabla^2 \psi + \psi \nabla^2 \psi^* \right] = \frac{\hbar^2}{2m} \vec{\nabla} \cdot \left[\psi^* \vec{\nabla} \psi + \psi \vec{\nabla} \psi^* \right]$$

$\vec{\nabla} \cdot [\psi^* \vec{\nabla} \psi] = \vec{\nabla} \psi^* \cdot \vec{\nabla} \psi + \psi^* \nabla^2 \psi$

conserved density

We have shown that the quantity $\rho = \psi^* \psi$ satisfies a **continuity equation**

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{with} \quad \vec{J} = \frac{\hbar}{2im} \left[\psi^* (\vec{\nabla} \psi) - (\vec{\nabla} \psi^*) \psi \right]$$

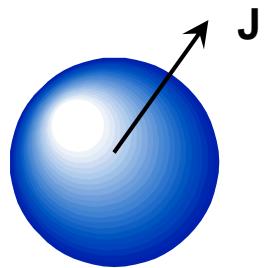
conserved current

Now, integrating over a volume V :

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_V \vec{\nabla} \cdot \vec{J} dV$$

and using Gauss' Theorem

$$\frac{\partial}{\partial t} \int_V \rho dV = - \int_A \vec{J} \cdot d\vec{A}$$



Any change in the total ρ in the volume must come about through a current \mathbf{J} through the surface of the volume.

Volume V enclosed
by Area A

- $\rho = \psi^* \psi$ is a conserved density and we interpret it as the **probability density** for finding a particle at a particular position.

Notice that ρ is positive definite, as required for a probability.

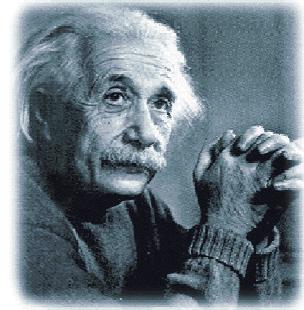
2. Klein-Gordon Equation

The Schrödinger Equation only describes particles in the non-relativistic limit. To describe the particles at particle colliders we need to incorporate special relativity.

A quick review of special relativity

We construct a position *four-vector* as

$$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (ct, \vec{x}) \quad (\mu = \{0, 1, 2, 3\})$$



An observer in a frame S' will instead observe a four-vector $x'^\mu = \Lambda^\mu_\nu x^\nu$ where Λ denotes a Lorentz transformation.

e.g. under a Lorentz boost by v in the positive x direction:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

- The quantity $x^\mu x_\mu$ is **invariant** under a Lorentz transformation

$$x^\mu x_\mu \equiv g_{\mu\nu} x^\mu x^\nu = (ct)^2 - |\vec{x}|^2$$

↑
 note the definition of
 a **covector** $x_\mu \equiv g_{\mu\nu} x^\nu$

Here $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the **metric tensor** of Minkowski space-time.

- This invariance implies that the Lorentz transformation is **orthogonal**:

$$\left. \begin{aligned} x'^\mu x'_\mu &= g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha x^\alpha \Lambda^\nu{}_\beta x^\beta \\ x^\mu x_\mu &= g_{\mu\nu} x^\mu x^\nu \end{aligned} \right\}$$

$$x'^\mu x'_\mu = x^\mu x_\mu \Leftrightarrow g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta} \Leftrightarrow [\Lambda^{-1}]_{\mu\nu} = \Lambda_{\nu\mu}$$

(Notice that I could have started with orthogonality and proven the invariance.)

- A particle's **four-momentum** is defined by $p^\mu = m \frac{dx^\mu}{d\tau}$

τ is **proper time**, the time in the particle's own rest frame.

It is related to an observer's time via $t = \gamma\tau$

Its four-momentum's time component is the particle's energy, while the space components are its three-momentum

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right)$$

and its length is an invariant, its **mass²** (times c^2):

$$p^\mu p_\mu = \frac{E^2}{c^2} - |\vec{p}|^2 = m^2 c^2$$

Finally, I define the derivative

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right)$$

This is a covector (index down).

You will sometimes use the **vector** expression

$$\partial^\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right) \quad \text{← Watch the minus sign!}$$

∂_μ transforms as $\partial_\mu \rightarrow \partial'_\mu = [\Lambda^{-1}]^\nu_\mu \partial_\nu$

$$\left[\partial x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \partial x^\nu = \Lambda^\mu_\nu \partial x^\nu \quad \text{so} \quad \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = [\Lambda^{-1}]^\nu_\mu \frac{\partial}{\partial x^\nu} \right]$$

- For simplicity, from now on I will use **natural units**.

Instead of writing quantities in terms of kg, m and s, we could write them in terms of c, \hbar and eV:

$$\begin{aligned}c &= 299792458 \text{ m s}^{-1} \\ \hbar &= 6.58211889(26) \times 10^{-16} \text{ eV s} \\ 1\text{eV} &= 1.782661731(70) c^2 \text{ kg}\end{aligned}$$

So any quantity with dimensions $\text{kg}^a \text{ m}^b \text{ s}^c$ can be written in units of $c^\alpha \hbar^\beta \text{ eV}^\gamma$, with

$$\begin{aligned}\alpha &= b + c - 2a \\ \beta &= b + c \\ \gamma &= a - b - c\end{aligned}$$

Then we **omit** \hbar and c in our quantities (you can work them out from the dimensions) – we don't just “set them to be one”.

The Klein-Gordon Equation

The invariance of the four-momentum's length provides us with a relation between energy, momentum and mass:



$$p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2$$

Oskar Klein

- Replacing energy and momentum with $E \rightarrow i\frac{\partial}{\partial t}$, $\vec{p} \rightarrow -i\vec{\nabla}$ gives the **Klein-Gordon** equation:

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \phi = m^2 \phi$$

(I have set $V=0$ for simplicity.)

Alternatively, in covariant notation: $p^\mu p_\mu = m^2$ with $p^\mu \rightarrow i\partial^\mu$ gives

$$(\partial^2 + m^2) \phi = 0$$

($\partial^2 \equiv \partial^\mu \partial_\mu$ is sometimes written as \square or \square^2)

This has plane-wave solutions $\phi(t, \vec{x}) = N e^{-i(Et - \vec{p} \cdot \vec{x})}$

↑
normalization

This is the relativistic wave equation for a **spin zero** particle, which conventionally is denoted ϕ .

- Is the Klein-Gordon equation the same in all reference frames?

Under a Lorentz transformation the **Klein-Gordon operator is invariant**, so:

$$\left(\frac{\partial}{\partial x'^\mu} \frac{\partial}{\partial x'_\mu} + m^2 \right) \phi'(x') = \left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} + m^2 \right) \phi'(\Lambda x) = 0$$

To give the **same physics** in the new frame, we need:

$$\phi'(x') = S\phi(x) \quad \text{with} \quad \begin{cases} S \text{ real} & (\text{since } \Lambda \text{ is real}) \\ |S|^2 = 1 & (\text{Lorentz trans. preserve the norm}) \end{cases}$$

Under continuous Lorentz transformations, S must be the same as for the identity, ie. S = 1

But for a parity inversion $(t, \vec{x}) \rightarrow (t, -\vec{x})$ it can take either sign

If $S = 1$, then ϕ is a **scalar**

$$\phi'(t', \vec{x}') = \phi'(t, -\vec{x}) = \phi(t, \vec{x})$$

If $S = -1$, then ϕ is a **pseudoscalar**

$$\phi'(t', \vec{x}') = \phi'(t, -\vec{x}) = -\phi(t, \vec{x})$$

Since $|\phi|$ is invariant, then $|\phi|^2$ **does not change** with a Lorentz transformation.

This sounds good – the probability doesn't change with your reference frame!

- Unfortunately, the probability **should** change with reference frame!

Remember that $|\phi|^2$ is a probability **density**:

Length contraction changes volumes $V' = \frac{1}{\gamma}V$



The probability $P = \rho V$ so for P to be invariant we need $\rho' = \gamma \rho$

- We need new definitions for the density ρ and current \mathbf{J} which satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad \text{or, covariantly,} \quad \partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = (\rho, \vec{J})$$

One possible choice is:

$$\rho = i \left[\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t} \phi \right] \quad \vec{J} = -i \left[\phi^* (\vec{\nabla} \phi) - (\vec{\nabla} \phi^*) \phi \right]$$

As a four-vector,

$$j^\mu = i [\phi^* (\partial^\mu \phi) - (\partial^\mu \phi^*) \phi]$$

(this is the same current as before, just with a different normalisation)



Exercise: Derive the continuity equation above, in a non-covariant notation (just as we did for the Schrödinger equation). Now derive it using a covariant notation.

Consider our plane-wave solution: $\phi(t, \vec{x}) = Ne^{-i(Et - \vec{p} \cdot \vec{x})}$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\phi &= (\nabla^2 - m^2)\phi \quad \Rightarrow \quad E^2 = m^2 + |\vec{p}|^2 \\ &\Rightarrow \quad E = \pm\sqrt{m^2 + |\vec{p}|^2}\end{aligned}$$

We have **solutions with negative energy**, and even worse,

$$\rho = i \left[\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t} \phi \right] = 2|N|^2 E$$



so these negative energy states have **negative probability distributions!**

We can't just ignore these solutions since they will crop up in any Fourier decomposition.



This is why Schrödinger abandoned this equation and developed the non-relativistic Schrödinger equation instead – he (implicitly) took the positive sign of the square root so that he could ignore the negative energy solutions.

Feynman-Stuckelberg Interpretation

Quantum Field Theory tells us that positive energy states must propagate forwards in time in order to preserve causality.

- Feynman and Stuckelberg suggested that negative energy states propagate **backwards in time**.

Our negative energy ($E < 0$) plane wave solutions are

$$\phi_{E, \vec{p}}(t, \vec{x}) = N e^{-i(Et - \vec{p} \cdot \vec{x})} = N e^{-i(|E|(-t) - \vec{p} \cdot \vec{x})} = \phi_{|E|, -\vec{p}}(-t, \vec{x})$$

moved the minus sign
over to the time

remember

$$\vec{p} = m \frac{d\vec{x}}{d\tau} = -m \frac{d\vec{x}}{d(-\tau)}$$

Particles flowing backwards in time are then reinterpreted as **anti-particles** flowing forwards in time.

If the field is charged, we may reinterpret j^μ as a charge density, instead of a probability density:

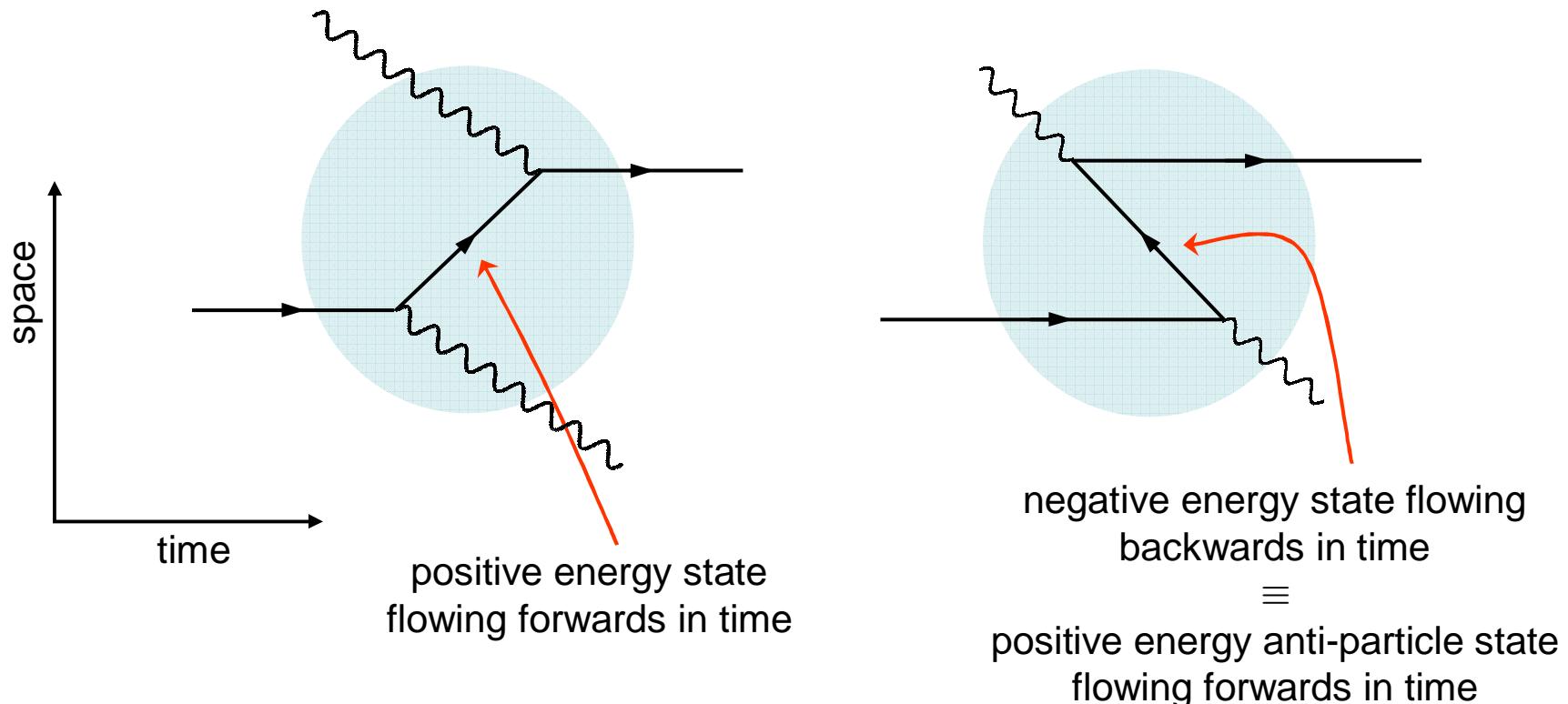
$$j^\mu = -ie [\phi^*(\partial^\mu \phi) - (\partial^\mu \phi^*)\phi]$$

Now $\rho = j^0$, so for a particle of energy E : $j^0 = -2e|N|^2 E$

while for an anti-particle of energy E : $j^0 = +2e|N|^2 E = -2e|N|^2(-E)$

which is the same as the charge density for an electron of energy $-E$

In reality, we only ever see the final state particles, so we must include these anti-particles anyway.



Quantum mechanics does not adequately handle the creation of particle—anti-particle pairs out of the vacuum. For that you will need **Quantum Field Theory**.

Normalization of KG solutions

The particle (or charge) density allows us to normalize the KG solutions in a box.

$\rho = 2|N|^2 E$ so in a box of volume V the number of particles is:

$$\int_V \rho dV = \int_V 2|N|^2 E dV = 2|N|^2 EV$$

So if we normalize to **2E particles per unit volume**, then $N = 1$

Notice that this is a covariant choice. Since the number of particles in a box should be independent of reference frame, but the volume of the box changes with a Lorentz boost, the density must also change with a boost. In fact, the density is the time component of a four-vector j^0 .

The Klein-Gordon Equation from a Lagrangian

In classical mechanics we can use **Lagrangians** to describe dynamical systems, via the **principle of least action**.

The evolution of a system progresses along the path of least action, where the action is defined in terms of a Lagrangian

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x$$

Technically \mathcal{L} is a Lagrange density.
The Lagrangian is
$$L = \int \mathcal{L}(\phi, \partial_\mu \phi) d^3x$$

Is it clear why this must also depend on $\partial_\mu \phi$?

Actually, more correctly $\mathcal{L} = \mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*)$

We want to know the field configuration such that an infinitesimally small variation of the field leaves the action unchanged.

i.e. $\phi(x) \rightarrow \phi(x) + \delta\phi(x)$ \Rightarrow $S \rightarrow S + \delta S$ with $\delta S = 0$

$$\delta S = \delta\phi \frac{\partial}{\partial\phi} \int \mathcal{L}(\phi, \partial_\nu\phi) d^4x + \cancel{\delta(\partial_\mu\phi)} \frac{\partial}{\partial(\partial_\mu\phi)} \int \mathcal{L}(\phi, \partial_\nu\phi) d^4x$$

$$= \int \left(\delta\phi \frac{\partial\mathcal{L}}{\partial\phi} + \cancel{\partial_\mu(\delta\phi)} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) d^4x$$

But $\partial_\mu(\delta\phi) \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu \left(\delta\phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) - \delta\phi \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right)$

So $\delta S = \int \left[\delta\phi \frac{\partial\mathcal{L}}{\partial\phi} + \cancel{\partial_\mu \left(\delta\phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right)} - \delta\phi \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) \right] d^4x$

~~$\partial_\mu \left(\delta\phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right)$~~ total derivative is zero
since ϕ vanishes at ∞

True for all ϕ so,

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) = 0}$$

This is the **Euler-Lagrange Equation**.

- Consider the Lagrangian for a free scalar field:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi$$

We have two Euler-Lagrange Equations. One for ϕ and one for ϕ^* :

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0 \quad \frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} \right) = 0$$

Let's use the one for ϕ^* :

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = m^2 \phi \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} = \partial^\mu \phi \quad \Rightarrow \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} \right) = \partial^2 \phi$$

So $(\partial^2 - m^2) \phi = 0$ which is the Klein-Gordon equation!

3. The Dirac Equation

The problems with the Klein-Gordon equation all came about because of the square root required to get the energy:



Paul Dirac

$$E = \pm \sqrt{m^2 + |\vec{p}|^2}$$

Dirac tried to get round this by finding a field equation which was linear in the operators.

$$E = \vec{\alpha} \cdot \vec{p} + \beta m \longrightarrow i \frac{\partial \psi}{\partial t} = (-i \vec{\alpha} \cdot \vec{\nabla} + \beta m) \psi$$

All we need to do is work out $\vec{\alpha}$ and β

Now, we have $E = \alpha_i p_i + \beta m$ where i and j are summed over 1,2,3

$$\begin{aligned}\Rightarrow E^2 &= \alpha_i \alpha_j p_i p_j + (\alpha_i \beta + \beta \alpha_i) m p_i + \beta^2 m^2 \\ &= \frac{1}{2} (\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + (\alpha_i \beta + \beta \alpha_i) m p_i + \beta^2 m^2\end{aligned}$$

relabel $i \leftrightarrow j$, i.e. $\sum_{i,j} \alpha_i \alpha_j p_i p_j = \sum_{j,i} \alpha_j \alpha_i p_j p_i = \sum_{i,j} \alpha_j \alpha_i p_i p_j$

So, comparing with $E^2 = m^2 + |\vec{p}|^2$ we must have:

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}$$

$$\alpha_i \beta + \beta \alpha_i = 0$$

$$\beta^2 = 1$$

$\vec{\alpha}$ and β are anti-commuting objects – not just numbers!

These commutation relations **define** α and β . Anything which obeys these relations will do. One possibility, called the **Dirac representation**, is the 4×4 matrices:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2x2 matrices

where σ_i are the usual Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since these act on the field ψ , ψ itself must now be a 4 component vector, known as a **spinor**.

$$(-i\vec{\alpha} \cdot \vec{\nabla} + \beta m) \psi \sim \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

[Strictly speaking
this is also just a
representation.]

We can write this equation in a four-vector form by defining a new quantity γ^μ :

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{\gamma} \equiv \beta \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

The anti-commutation relations become:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

And the Dirac Equation is: (with $p^\mu \rightarrow i\partial^\mu$)

$$(i\gamma^\mu \partial_\mu - m) \psi = 0$$

Often $\gamma^\mu \partial_\mu$ is written as \not{d}



Exercise: Show that the above anticommutation relation reproduces the required anticommutation relations for α and β .



Exercise: Show that the matrices α and β in the Dirac equation are Hermitian, traceless, have even dimension and have eigenvalues ± 1 .
(Hint: showing they are Hermitian is a bit of a cheat!)



Exercise: Prove that $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$, $i = 1, 2, 3$ and therefore $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$

Does the Dirac Equation have the right properties?

Is the probability density positive definite?

$$\left(\psi^\dagger = (\psi^*)^T \right)$$

A appropriate conserved quantity is now $\rho = \psi^\dagger \psi$ with $\vec{J} = \psi^\dagger \vec{\alpha} \psi$

In four-vector notation,

$$j^\mu = \bar{\psi} \gamma^\mu \psi \quad \text{with} \quad \bar{\psi} \equiv \psi^\dagger \gamma^0 \quad (\text{Note } \gamma^0 \gamma^0 = \beta^2 = 1)$$

Clearly $\rho = \psi^\dagger \psi > 0$ always! 



Exercise: Starting from the Dirac equation derive the continuity equation for the above density and current (you can stick to covariant notation this time if you like).

Does the Dirac Equation only have positive energy solutions?

Look for plane wave solutions:

$$\psi(t, \vec{x}) = u(\vec{p}) e^{-i(Et - \vec{p} \cdot \vec{x})} = \begin{pmatrix} \chi \\ \phi \end{pmatrix} e^{-i(Et - \vec{p} \cdot \vec{x})}$$

4 component spinor

2 component spinors

$$u \rightarrow \begin{pmatrix} (\cdot) & \xleftarrow{\chi} \\ (\cdot) & \xleftarrow{\phi} \end{pmatrix}$$

Since we want the energy, it is easier to work without four-vector notation:

$$i \frac{\partial \psi}{\partial t} = (-i \vec{\alpha} \cdot \vec{\nabla} + \beta m) \psi \Rightarrow E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

For a particle **at rest**, $\vec{p} = 0$

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

Solutions:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ with } E = m$$

OR

positive energy
solutions



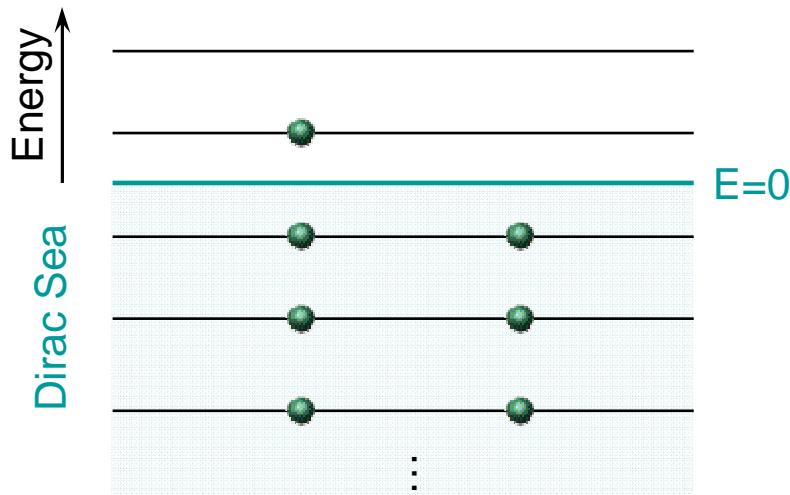
negative energy
solutions

$$u = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ with } E = -m$$

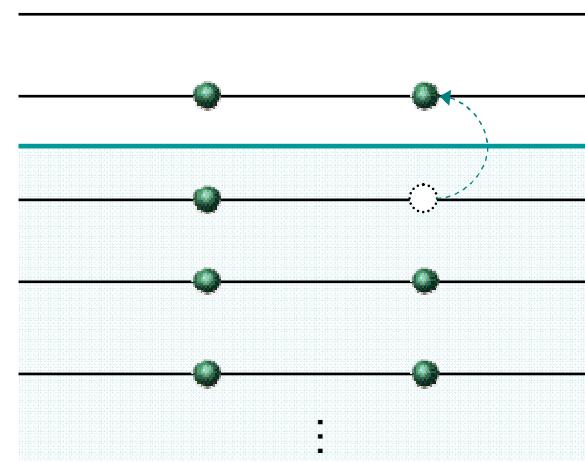
Oops! We still have negative energy solutions! 

Dirac got round this by using the **Pauli Exclusion principle**.

He reasoned that his equation described particles with spin (e.g. electrons) so only two particles can occupy any particular energy level (one spin-up, the other spin-down).



If all the energy states with $E < 0$ are already filled, the electron can't fall into a negative energy state.



Moving an electron from a negative energy state to a positive one leaves a hole which we interpret as an anti-particle.

Note that we couldn't have used this argument for bosons (no exclusion principle) so the **Feynman-Stuckelberg interpretation** is more useful.

A General Solution

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} \Rightarrow \begin{cases} \chi = \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \phi \\ \phi = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi \end{cases}$$

Check these are compatible:

$$\phi = \left(\frac{\vec{\sigma} \cdot \vec{p}}{E + m} \right) \left(\frac{\vec{\sigma} \cdot \vec{p}}{E - m} \right) \phi = \frac{|\vec{p}|^2}{E^2 - m^2} \phi = \phi \quad \text{since} \quad E^2 = m^2 + |\vec{p}|^2$$

$$\left. \left(\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k \Rightarrow (\vec{\sigma} \cdot \vec{p})^2 = |\vec{p}|^2 + i (\vec{p} \times \vec{p}) \cdot \vec{\sigma} = |\vec{p}|^2 \right) \right\}$$

↑
property of Pauli matrices, e.g. $\sigma_1 \sigma_2 = i \sigma_3$

We need to choose a basis for our solutions. Choose,

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Positive Energy Solutions, $E > 0$, are $\psi^{(1)}, \psi^{(2)}$

$$\phi = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi \quad \Rightarrow \quad \begin{cases} \chi = \xi^{(s)} \\ \phi = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \xi^{(s)} \end{cases} \quad (s = 1, 2)$$

$$\psi^{(s)}(x) = u^{(s)} e^{-ip \cdot x} = \sqrt{E + m} \left(\begin{array}{c} \xi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \xi^{(s)} \end{array} \right) e^{-ip \cdot x}$$

↑
[Normalization choice - see later]

Negative Energy Solutions, $E < 0$, are $\psi^{(3)}, \psi^{(4)}$

$$\chi = \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \phi \quad \Rightarrow \quad \begin{cases} \chi = \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \xi^{(s)} \\ \phi = \xi^{(s)} \end{cases} \quad (s = 1, 2)$$

$$\psi^{(s+2)}(x) = u^{(s+2)} e^{-ip \cdot x} = \sqrt{-E + m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E - m} \xi^{(s)} \\ \xi^{(s)} \end{pmatrix} e^{-ip \cdot x}$$

Typically, we write this in terms of the antiparticle's energy and momentum:

$$u^{(3,4)}(-\vec{p}) e^{-i(-p) \cdot x} \equiv v^{(1,2)}(\vec{p}) e^{ip \cdot x}$$



Conventions differ
here: sometimes the
order is inverted

Orthogonality and completeness

With the normalization of $2E$ particles per unit volume, it is rather obvious that:

$$u^{(r)\dagger} u^{(s)} = 2E\delta^{rs} \quad v^{(r)\dagger} v^{(s)} = 2E\delta^{rs}$$

This is a statement of orthogonality.

[positive energy]

Less obvious, but easy to show, are the completeness relations:

$$\sum_{s=1,2} u^{(s)}(p) \bar{u}^{(s)}(p) = \not{p} + m$$

$$\sum_{s=1,2} v^{(s)}(p) \bar{v}^{(s)}(p) = \not{p} - m$$

Note that this is a matrix equation:

$$\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} (\dots) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



Exercise: Prove the above completeness relations

The Dirac Equation from a Lagrangian

Consider the Lagrangian for a free Dirac field:

$$\mathcal{L} = \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi = \bar{\psi}_i ([\gamma^\mu]_{ij} \partial_\mu - m \delta_{ij}) \psi_j$$

spinor indices

We have two sets of Euler-Lagrange Equations. One for ψ and one for $\bar{\psi}$:

$$\frac{\partial \mathcal{L}}{\partial \psi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_i)} \right) = 0 \quad \frac{\partial \mathcal{L}}{\partial \bar{\psi}_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi}_i)} \right) = 0$$

Let's use the one for $\bar{\psi}$:

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_i} = ([\gamma^\mu]_{ij} \partial_\mu - m \delta_{ij}) \psi_j$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi}_i)} = 0$$

So $(\gamma^\mu \partial_\mu - m) \psi = 0$ which is the Dirac equation!

I could have used the other set of Euler-Lagrange Equations to give an equation for the antiparticle:

$$(\partial_\mu \bar{\psi}) \gamma^\mu + m \bar{\psi} = 0 \quad \text{sometimes written} \quad \bar{\psi} (\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0$$

↑
arrow denotes
acting to the left

The Dirac Lagrangian looks rather asymmetric in its treatment of ψ and $\bar{\psi}$.

In principle, $\bar{\psi}$ is just as fundamental as ψ and we can rewrite the Lagrangian:

$$\mathcal{L} = \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi = \partial_\mu \cancel{(\bar{\psi} \gamma^\mu \psi)} - (\partial_\mu \bar{\psi}) \gamma^\mu \psi - m \bar{\psi} \psi$$

total derivative

$$\begin{aligned} \text{or even } \mathcal{L} &= \frac{1}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{1}{2} (\partial_\mu \bar{\psi}) \gamma^\mu \psi - m \bar{\psi} \psi \\ &= \frac{1}{2} \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi - \frac{1}{2} \bar{\psi} (\gamma^\mu \overleftarrow{\partial}_\mu + m) \psi \end{aligned}$$

Angular Momentum and Spin

The angular momentum of a particle is given by $\vec{L} = \vec{r} \times \vec{p}$.

If this commutes with the Hamiltonian then angular momentum is conserved.

$$[H, \vec{L}] = [\vec{\alpha} \cdot \vec{p}, \vec{r} \times \vec{p}] = -i\vec{\alpha} \times \vec{p}$$

This is not zero, so $\vec{L} = \vec{r} \times \vec{p}$ is **not** conserved!



But, if we define $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\vec{\alpha}$ ($= -i\gamma_1\gamma_2\gamma_3\vec{\gamma}$)

$$\text{then } [H, \vec{\Sigma}] = [\vec{\alpha} \cdot \vec{p}, -i\alpha_1\alpha_2\alpha_3\vec{\alpha}] = 2i\vec{\alpha} \times \vec{p}$$

So the quantity $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ is conserved! $[H, \vec{J}] = 0$



Exercise: Demonstrate the above commutation relations.

(Hint: show that $\Sigma_i = -\frac{i}{2}\epsilon_{ijk}\alpha_j\alpha_k$ first)

\vec{L} is the orbital angular momentum, whereas $\frac{1}{2}\vec{\Sigma}$ is an **intrinsic** angular momentum

Notice that our basis spinors are eigenvectors of $\frac{1}{2}\Sigma^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

with eigenvalues $\pm\frac{1}{2}$

So now we know why the spinor contains four degrees of freedom:

- positive energy solution, with spin up
- positive energy solution, with spin down
- negative energy solution, with spin up
- negative energy solution, with spin down

Helicity of massless fermions

If the mass is zero, our wave equation becomes

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} \quad \Rightarrow \begin{cases} E\phi = \vec{\sigma} \cdot \vec{p} \chi \\ E\chi = \vec{\sigma} \cdot \vec{p} \phi \end{cases}$$

Writing $\Psi_{R,L} \equiv \frac{1}{2}(\chi \pm \phi)$ then we find the equations decouple

$$E\Psi_R = \vec{\sigma} \cdot \vec{p} \Psi_R \quad \text{and} \quad E\Psi_L = -\vec{\sigma} \cdot \vec{p} \Psi_L$$

These two component spinors, called **Weyl spinors**, are completely independent, and can even be considered as separate particles!

Notice that each is an eigenstate of the operator $\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$ with eigenvalues ± 1

\nearrow

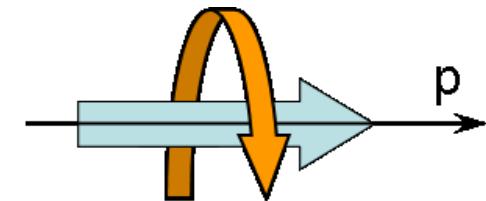
$\left[|\vec{p}| = E \text{ for massless state} \right]$

For the full Dirac spinor, we define the **Helicity** operator as

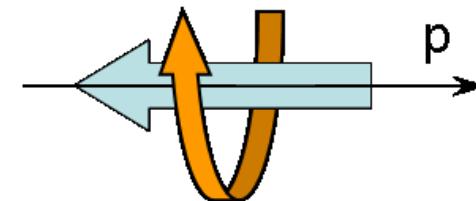
$$\frac{\vec{\Sigma} \cdot \vec{p}}{2|\vec{p}|}$$

This is the component of spin in
the direction of motion.

A particle with a helicity eigenvalue $+\frac{1}{2}$ is **right handed**



A particle with a helicity eigenvalue $-\frac{1}{2}$ is **left handed**



Since an antiparticle has opposite momentum it will have opposite helicity.

left handed particle \longrightarrow right handed antiparticle

[this is why the labelling of solutions in the antiparticle spinor ψ is sometimes reversed]

We can project out a particular helicity from a Dirac spinor using γ matrices.

Define $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

This is the Dirac representation.

and **projection operators** $P_{R/L} \equiv \frac{1}{2}(1 \pm \gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$

Then a spinor $P_L u$ will be left handed, while $P_R u$ will be right handed.

e.g. $P_R u = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \chi + \phi \\ \chi + \phi \end{pmatrix}$

but $\frac{\vec{\Sigma} \cdot \vec{p}}{2|\vec{p}|} \frac{1}{2}\begin{pmatrix} \chi + \phi \\ \chi + \phi \end{pmatrix} = \frac{1}{2} \frac{1}{2}\begin{pmatrix} \chi + \phi \\ \chi + \phi \end{pmatrix}$ so $P_R u$ is right handed.

We can make this more explicit by using a different representation of the γ matrices.

The **chiral representation** (sometimes called the **Weyl representation**) is:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Now } P_L \equiv \frac{1}{2}(1 - \gamma^5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad P_R \equiv \frac{1}{2}(1 + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The left-handed Weyl spinor sits in the upper part of the Dirac spinor, while the right handed Weyl spinor sits in the lower part.

e.g.

$$P_R u = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_R \end{pmatrix}$$

The **weak interaction** acts only on left handed particles.

Parity transforms $\vec{r} \rightarrow -\vec{r}$ but leaves spin unchanged (it doesn't change which of the solutions you have). Therefore parity changes helicity - it transforms left handed particles onto right handed ones (and vice versa),

$$\text{i.e. } P\Psi_L = \Psi_R \quad \text{and} \quad P\Psi_R = \Psi_L$$

So the weak interactions are parity violating.

Also, helicity is only a **good quantum number** for **massless** particles.

If a particle has a mass, I can always move to a reference frame where I am going faster than it, causing the momentum to reverse direction. This causes the helicity to change sign.

For a massless particle there is no such frame and helicity is a good quantum number.

We saw earlier that the mass term in the Dirac Lagrangian looks like

$$m\bar{\psi}\psi = m (\Psi_L^\dagger \Psi_R^\dagger) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = m (\Psi_L^\dagger \Psi_R + \Psi_R^\dagger \Psi_L)$$

(chiral representation)

Mass terms mix left and right handed states.

Therefore massive particle are not compatible with the weak interaction!

The solution to this problem is to introduce a new field called the **Higgs field**. This couples left handed particles, to right handed ones, mixing them up and giving them an **effective mass**.

$$\mathcal{L}_{\text{Higgs}} \supset Y \bar{\psi}_L \cdot \phi \psi_R$$

If the vacuum (lowest energy state) of the system contains a non-zero amount of this new field $\langle \phi \rangle \neq 0$, we generate a mass $Y\langle \phi \rangle$

The theory also predicts a new particle, the **Higgs boson**, which we hope to find soon!

Symmetries of the Dirac Equation

The Lorentz Transformation

This notation differs in different texts.
e.g. Peskin and Schroeder would write

$$\psi(x) \rightarrow \psi'(x) = S\psi(\Lambda^{-1}x)$$



How does the field $\psi(x)$ behave under a Lorentz transformation?

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \partial_\mu \rightarrow \partial'_\mu = [\Lambda^{-1}]^\nu{}_\mu \partial_\nu \quad \psi(x) \rightarrow \psi'(x') = S\psi(x)$$

$$(i\gamma^\mu \partial_\mu - m) \psi(x) = 0 \rightarrow (i\gamma^\mu [\Lambda^{-1}]^\nu{}_\mu \partial_\nu - m) S\psi(x) = 0$$

(γ^μ and m are just numbers and don't transform)

Premultiply by S^{-1} : $(iS^{-1}\gamma^\mu S[\Lambda^{-1}]^\nu{}_\mu \partial_\nu - m) \psi(x) = 0$

\Rightarrow

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$$

We can find S for an **infinitesimal proper** transformation $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$

$$\Lambda^\mu{}_\nu \gamma^\nu = S^{-1} \gamma^\mu S$$

↑
[antisymmetric]

write $S = 1 + \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}$ (just a parameterisation)

$$\gamma^\mu + \omega^\mu{}_\nu \gamma^\nu = \left(1 - \frac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta}\right) \gamma^\mu \left(1 + \frac{i}{4}\sigma_{\sigma\rho}\omega^{\sigma\rho}\right)$$

$$\Rightarrow 2i\omega^{\alpha\beta} \left(\delta^\mu{}_\alpha \gamma_\beta - \delta^\mu{}_\beta \gamma_\alpha\right) = [\gamma^\mu, \sigma_{\alpha\beta}] \omega^{\alpha\beta} \quad [\text{ignoring terms } \mathcal{O}(\omega^2)]$$

$$\Rightarrow \boxed{\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]} \quad [\text{I jumped a few steps here}]$$

This tells us how a fermion field transforms under a Lorentz boost.



Exercise: Demonstrate that this choice of $\sigma_{\mu\nu}$ satisfies the transformation equation (ignoring terms $\mathcal{O}(\omega^2)$)

The adjoint transforms as $\bar{\psi} \equiv \psi^\dagger \gamma^0 \rightarrow \psi^\dagger S^\dagger \gamma^0 = \psi^\dagger \gamma^0 S^{-1} = \bar{\psi} S^{-1}$
 [since $S^\dagger \gamma^0 = \gamma^0 S^{-1}$ for the explicit form of S derived above]

So $\bar{\psi}\psi$ is invariant.

And $j^\mu = \bar{\psi} \gamma^\mu \psi \rightarrow \bar{\psi} S^{-1} \gamma^\mu S \psi = \Lambda^\mu{}_\nu \bar{\psi} \gamma^\nu \psi$ so our current is a four-vector.

Common fermion bilinears:

$$\bar{\psi}\psi \rightarrow \bar{\psi}\psi \quad \text{scalar}$$

$$\bar{\psi} \gamma^5 \psi \rightarrow \text{Det}(\Lambda) \bar{\psi} \gamma^5 \psi \quad \text{pseudoscalar}$$

$$\bar{\psi} \gamma^\mu \psi \rightarrow \Lambda^\mu{}_\nu \bar{\psi} \gamma^\nu \psi \quad \text{vector}$$

$$\bar{\psi} \gamma^\mu \gamma^5 \psi \rightarrow \text{Det}(\Lambda) \Lambda^\mu{}_\nu \bar{\psi} \gamma^\nu \gamma^5 \psi \quad \text{axial vector}$$

$$\bar{\psi} \sigma^{\mu\nu} \gamma^5 \psi \rightarrow \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \bar{\psi} \sigma^{\alpha\beta} \gamma^5 \psi \quad \text{tensor}$$

Parity $t \rightarrow t, \quad \vec{x} \rightarrow -\vec{x}$

A **parity transformation** is an improper Lorentz transformation $t \rightarrow t, \vec{x} \rightarrow -\vec{x}$ described by

$$[\Lambda^\mu{}_\nu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Again $\Lambda^\mu{}_\nu \gamma^\nu = P^{-1} \gamma^\mu P$, so $P \gamma^0 = \gamma^0 P$ and $P \gamma^i = -\gamma^i P$ $[i = 1, 2, 3]$

Since γ^0 commutes with itself (trivially) and anticommutes with γ^i , a suitable choice is

$$P = \eta \gamma^0, \quad |\eta| = 1$$

$$P : \psi(t, \vec{x}) \rightarrow \psi_P(t, -\vec{x}) = P\psi(t, \vec{x}) = \gamma^0 \psi(t, \vec{x})$$

Can you derive the parity transformations of the bilinears given on the last slide?

You should see that η drops out, so there is no loss of generality setting $\eta = 1$

Charge Conjugation

Another discrete symmetry of the Dirac equation is the interchange of particle and anti-particle.

$$\psi \rightarrow \psi_c \equiv C\bar{\psi}^T$$

Take the complex conjugate of the Dirac equation:

$$\begin{aligned}(i\gamma^\mu \partial_\mu - m)^* \psi^*(x) &= \left(-i(\gamma^{\mu\dagger})^T \partial_\mu - m\right)(\psi^\dagger)^T \\ &= \gamma^{0T} (-i\gamma^{\mu T} \partial_\mu - m) \bar{\psi}^T \quad (\text{used } \gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 \text{ and } \bar{\psi} \equiv \psi^\dagger \gamma^0)\end{aligned}$$

Premultiply by $C\gamma^{0T}$ and the Dirac equation becomes:

$$(-iC\gamma^{\mu T}C^{-1}\partial_\mu - m)\psi_c = 0$$

Therefore we need C such that

$$C\gamma^{\mu T}C^{-1} = -\gamma^\mu$$

The form of C changes with the representation of the γ -matrices. For the Dirac representation a suitable choice is

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$C : \psi(t, \vec{x}) \rightarrow \psi_C(t, \vec{x}) = C\bar{\psi}^T(t, \vec{x}) = i\gamma^2\gamma^0\bar{\psi}^T(t, \vec{x})$$

How does this transformation affect the stationary solutions?

$$\psi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \longrightarrow \psi_c = C \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{imt} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imt}$$

$$\psi = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \longrightarrow \psi_c = C \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{imt} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{imt}$$

etc

We have mapped particle states onto antiparticle states, as desired.

Time Reversal

A naive transformation of the wavefunction $t \rightarrow -t$, $\vec{x} \rightarrow \vec{x}$ is not sufficient for time reversal. Since the momentum of a particle, is a **rate of change**, it too must change sign.

Changing the momentum direction and time for a plane wave gives:

$$e^{-i(E\vec{t}-\vec{p}\cdot\vec{x})} \longrightarrow e^{-i(E(-t)-(-\vec{p})\cdot\vec{x})} = e^{i(Et-\vec{p}\cdot\vec{x})} = \left(e^{-i(Et-\vec{p}\cdot\vec{x})}\right)^*$$

We must (again!) make a complex conjugation: $\psi(t, \vec{x}) \longrightarrow \psi_T(-t, \vec{x}) = T\psi^*(t, \vec{x})$

Take complex conjugation of Dirac Equation, switch $t \rightarrow -t$ and pre-multiply by T:

$$\begin{aligned} \left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma} \cdot \vec{\nabla} - m\right) \psi(t, \vec{x}) &\longrightarrow T \left(-i\gamma^{0*} \frac{\partial}{\partial(-t)} - i\vec{\gamma}^* \cdot \vec{\nabla} - m\right) T^{-1} T\psi^*(-t, \vec{x}) \\ &= \left(i[T\gamma^{0*}T^{-1}] \frac{\partial}{\partial t} + i[-T\vec{\gamma}^*T^{-1}] \cdot \vec{\nabla} - m\right) \psi_T(t, \vec{x}) \end{aligned}$$

Need:

$$T\gamma^0 * T^{-1} = \gamma^0, \quad T\vec{\gamma}^* T^{-1} = -\vec{\gamma}$$

A suitable choice is:

$$T = i\gamma^1\gamma^3 = \begin{pmatrix} -i\sigma_1\sigma_3 & 0 \\ 0 & -i\sigma_1\sigma_3 \end{pmatrix} = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$T : \psi(t, \vec{x}) \rightarrow \psi_T(-t, \vec{x}) = T\psi^*(t, \vec{x}) = i\gamma^1\gamma^3\psi^*(t, \vec{x})$$

CPT

For the discrete symmetries, we have shown:

$$\begin{aligned} C : \psi(t, \vec{x}) &\rightarrow \psi_C(t, \vec{x}) = C\bar{\psi}^T(t, \vec{x}) = i\gamma^2\gamma^0\bar{\psi}^T(t, \vec{x}) \\ P : \psi(t, \vec{x}) &\rightarrow \psi_P(t, -\vec{x}) = P\psi(t, \vec{x}) = \gamma^0\psi(t, \vec{x}) \\ T : \psi(t, \vec{x}) &\rightarrow \psi_T(-t, \vec{x}) = T\psi^*(t, \vec{x}) = i\gamma^1\gamma^3\psi^*(t, \vec{x}) \end{aligned}$$

Doing all of these transformations gives us

$$\begin{aligned} CPT : \psi(t, \vec{x}) \rightarrow \psi_{CPT}(-t, -\vec{x}) &= i\gamma^2\gamma^0\gamma^{0T} [\gamma^0 i\gamma^1\gamma^3\psi^*(t, \vec{x})]^* \\ &= i\gamma^2\gamma^0\gamma^0\gamma^0 (-i)\gamma^1\gamma^3\psi(t, \vec{x}) \\ &= \gamma^0\gamma^1\gamma^2\gamma^3\psi(t, \vec{x}) \\ &= -i\gamma^5\psi(t, \vec{x}) \end{aligned}$$

So if $\psi(x)$ is an electron, $\psi_{CPT}(-x)$ is a positron travelling backwards in space-time multiplied by a factor $-i\gamma^5$.

This justifies the Feynman-Stuckelberg interpretation!

4. Quantum Electrodynamics [without using QFT]

Classical Electromagnetism

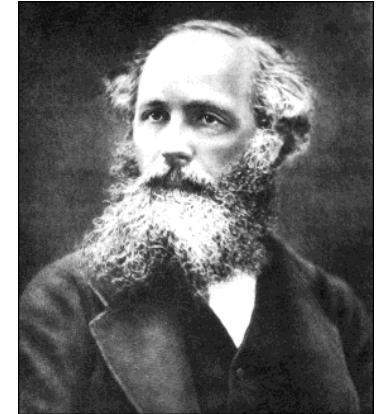
Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \rho$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}$$



James Clerk Maxwell

Maxwell wrote these down in **1864**, but amazingly they are covariant!

Writing $F^{\mu\nu} \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$ and $j^\mu \equiv (\rho, \vec{J})$ they are

[Note: $F^{\mu\nu} = -F^{\nu\mu}$]

$$\partial_\mu F^{\mu\nu} = j^\nu$$

[Note: the ability to write
Maxwell's Equations in this form
is **not** a proof of covariance!]

Maxwell's equations can also be written in terms of a **potential** A^μ

Writing $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

we have $\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$

Now, notice that I can change A_μ by a derivative of a scalar and leave $F_{\mu\nu}$ unchanged

$$A^\mu \rightarrow A^\mu + \lambda \partial^\mu \phi \quad \Rightarrow \quad F^{\mu\nu} \rightarrow F^{\mu\nu} + \lambda \partial^\mu \partial^\nu \phi - \lambda \partial^\nu \partial^\mu \phi = F^{\mu\nu}$$

Choose λ such that

$$\partial_\mu A^\mu = 0$$

This is a **gauge transformation**, and the choice $\partial_\mu A^\mu = 0$ is known as the **Lorentz gauge**.

In this gauge:

$$\partial^2 A^\mu = j^\mu$$

← does this look familiar?

The wave equation with no source, $\partial^2 A^\nu = 0$ has solutions

$$A^\mu = \epsilon^\mu e^{iq \cdot x} \quad \text{with} \quad q^2 = 0$$

↑
polarisation vector with 4
degrees of freedom

The Lorentz condition ($\partial_\mu A^\mu = 0$) $\Rightarrow q_\mu \epsilon^\mu = 0$

So ϵ^μ has only 3 degrees of freedom (two transverse d.o.f. and one longitudinal d.o.f.)

We still have some freedom to change A^μ , even after our Lorentz gauge choice:

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \quad \text{is OK, as long as } \partial^2 \chi = 0$$

Usually we choose χ such that $\vec{\nabla} \cdot \vec{A} = 0$. This is known as the **Coulomb gauge**.

$$\vec{\nabla} \cdot \vec{A} = 0 \quad \Rightarrow \quad \vec{q} \cdot \vec{\epsilon} = 0$$

So only two polarisation states remain (both transverse).

A Lagrangian for the free photon field

We want a Lagrangian which will give us $\partial_\mu F^{\mu\nu} = 0$ (Maxwell's equations with no sources)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\text{Recall } F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu), \text{ so } \frac{\partial F_{\mu\nu}}{\partial A_\rho} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial A_\rho} = 0$$

$$\begin{aligned} \frac{\partial F_{\mu\nu}}{\partial (\partial_\sigma A_\rho)} &= g_\mu^\sigma g_\nu^\rho - g_\mu^\rho g_\nu^\sigma \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial (\partial_\sigma A_\rho)} = -\frac{1}{2}F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial (\partial_\sigma A_\rho)} \\ &= -\frac{1}{2}F^{\mu\nu} (g_\mu^\sigma g_\nu^\rho - g_\mu^\rho g_\nu^\sigma) \\ &= -\frac{1}{2}(F^{\sigma\rho} - F^{\rho\sigma}) = -F^{\sigma\rho} \end{aligned}$$

$$\left. \begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\rho} - \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial (\partial_\sigma A_\rho)} \right) &= 0 \\ \Rightarrow \quad \partial_\sigma F^{\sigma\rho} &= 0 \end{aligned} \right| \quad \begin{array}{l} \text{F}^{\rho\sigma} \text{ antisymmetric} \\ \uparrow \end{array}$$

The Dirac Equation in an Electromagnetic Field

So far, this has been entirely classical. So how do we incorporate electromagnetism into the quantum Dirac equation?

We have one more **symmetry** of the Dirac Lagrangian which we haven't looked at yet.

Consider phase shifting the electron field by $\psi \rightarrow e^{i\theta}\psi$.

The adjoint field transforms as $\bar{\psi} \rightarrow \bar{\psi} e^{-i\theta}$ and the Lagrangian transforms as

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi \rightarrow \bar{\psi} e^{-i\theta} (i\gamma^\mu \partial_\mu - m) e^{i\theta} \psi = \mathcal{L}$$

The Lagrangian doesn't change so the physics stays the same.

This is known as a **global U(1) symmetry**

(since $e^{i\theta}$ doesn't vary with space-time coordinate)

(since $e^{i\theta}$ is a **unitary** 1×1 matrix)

What happens if we make our transformation **local**, i.e. depend on space-time point?

$$\psi \longrightarrow e^{i\theta(x)}\psi \quad \bar{\psi} \longrightarrow \bar{\psi} e^{-i\theta(x)}$$

$$\mathcal{L} \longrightarrow \bar{\psi} e^{-i\theta(x)} (i\gamma^\mu \partial_\mu - m) e^{i\theta(x)} \psi = \mathcal{L} - \bar{\psi} \gamma^\mu (\partial_\mu \theta(x)) \psi$$

The free Dirac Lagrangian is no longer invariant. If we really want this to be a symmetry of the theory, we will have to add in something new.

Let's postulate a new field A^μ which couples to the electron according to

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - e\gamma_\mu A^\mu - m) \psi$$

charge of the electron = -e

Often this is written in terms of a “**covariant derivative**”

$$D^\mu \equiv \partial^\mu + ieA^\mu$$

Beware: conventions differ,
e.g. Halzen and Martin have

$$+e\gamma_\mu A^\mu$$

while Peskin & Schroeder
have as above

Now $\mathcal{L} = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi \longrightarrow \bar{\psi} (i\gamma^\mu e^{-i\theta(x)} D'_\mu e^{i\theta(x)} - m) \psi$

So, to preserve the Lagrangian, we need D_μ to transform too:

$$D_\mu \longrightarrow D'_\mu = e^{i\theta(x)} D_\mu e^{-i\theta(x)}$$

$$\Rightarrow \partial_\mu + ieA'_\mu = e^{i\theta(x)} (\partial_\mu + ieA_\mu) e^{-i\theta(x)} = \partial_\mu - i\partial_\mu\theta(x) + ieA_\mu$$

therefore we need A_μ to transform as

$$A_\mu \longrightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta(x)$$

This is the gauge transformation we saw for the (classical) photon earlier!

Coupling the **electron** to a **photon** makes the theory **locally U(1) symmetric**

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi} (i\gamma^\mu D_\mu - m) \psi$$

$$(i\gamma^\mu D_\mu - m) \psi = 0,$$

$$\partial^2 A^\mu = j^\mu$$

$$j^\mu = e\bar{\psi}\gamma^\mu\psi$$

The Magnetic Moment of the Electron

We saw that the interaction of an electron with an electromagnetic field is given by

$$(i\gamma^\mu \partial_\mu - e\gamma_\mu A^\mu - m)\psi = 0$$

Writing $u \propto \begin{pmatrix} \chi \\ \phi \end{pmatrix}$ as before, $E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A}) \\ \vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A}) & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}$

(Coulomb gauge $\Rightarrow A^0=0$)

So $\phi = \frac{\vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A})}{E + m}\chi$ and $\left(E - m + \frac{[\vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A})]^2}{E + m}\right)\chi = 0$

Also,

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \Rightarrow [\vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A})]^2 = | -i\vec{\nabla} - e\vec{A} |^2 - e(\vec{\nabla} \times \vec{A} + \vec{A} \times \vec{\nabla}) \cdot \vec{\sigma}$$

$$\vec{\nabla} \times \vec{A} \psi + \vec{A} \times \vec{\nabla} \psi = (\vec{\nabla} \times \vec{A})\psi - \vec{A} \times (\vec{\nabla} \psi) + \vec{A} \times (\vec{\nabla} \psi) = (\vec{\nabla} \times \vec{A})\psi = \vec{B}\psi$$

So we have,
$$\left(E - m + \frac{|\vec{p} - e\vec{A}|^2 - e\vec{B} \cdot \vec{\sigma}}{E + m} \right) \chi = 0$$

In the non-relativistic limit, $E \approx m$ and $\phi \approx \frac{\vec{\sigma} \cdot (\vec{p} - e\vec{A})}{2m} \chi \ll \chi$, so we can write the Dirac equation as approximately:

$$\boxed{\frac{1}{2m} |\vec{p} - e\vec{A}|^2 \psi - \frac{e\vec{B} \cdot \vec{\Sigma}}{2m} \psi = 0}$$

This is an **magnetic moment** interaction $-\vec{\mu} \cdot \vec{B}$ with $\vec{\mu} = -\frac{e}{2m} \vec{\Sigma}$

The magnetic moment $\vec{\mu}$ is composed of a contribution from the orbital angular momentum,

$$\vec{\mu}_L = -\frac{e}{2m} \vec{L}$$

and the intrinsic spin angular momentum

$$\vec{\mu}_s = -g \frac{e}{2m} \vec{s}$$

$\vec{s} = \frac{1}{2} \vec{\Sigma}$

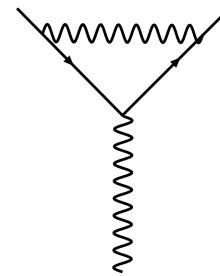
gyromagnetic ratio

The Dirac equation predicts a gyromagnetic ratio **$g = 2$**

We can compare this with experiment: **$g_{\text{exp}} = 2.0023193043738 \pm 0.0000000000082$**

The discrepancy of $g-2$ from zero is due to **radiative corrections**

The electron can emit a photon, interact, and reabsorb the photon.



If one does a more careful calculation, including these effects, QED predicts:

$$\frac{g - 2}{2} = \frac{\alpha}{2\pi} - 0.328 \left(\frac{\alpha}{\pi}\right)^2 + 1.181 \left(\frac{\alpha}{\pi}\right)^3 - 1.510 \left(\frac{\alpha}{\pi}\right)^4 + \dots + 4.393 \times 10^{-12}$$

Theory:

$$\frac{g_{\text{th}} - 2}{2} = 1159652140(28) \times 10^{-12}$$

Experiment:

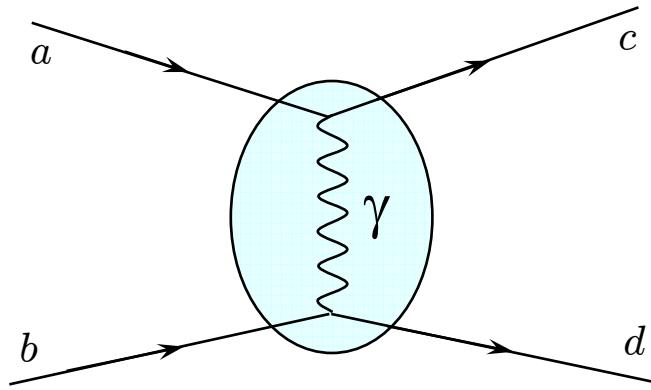
$$\frac{g_{\text{exp}} - 2}{2} = 1159652186.9(4.1) \times 10^{-12}$$

} excellent
agreement!

The muon's magnetic moment is more interesting because it is more sensitive to new physics.

5. Scattering and Perturbation theory

Now we have the Dirac equation in an Electromagnetic field we can calculate the scattering of electrons (via electromagnetism) $a + b \rightarrow c + d$



We will assume that the coupling e is small, and that far away from the interaction, i.e. outside the shaded area, the electrons are free particles.

- Then the initial state is a solution of the **free** Dirac Equation.
- It will evolve in time according to the Hamiltonian of the **interacting** Dirac Equation.
- The probability of finding a particular final state (also a solution of the **free** Dirac Equation) is the projection of the evolved state onto this particular final state.

The Dirac Equation (in a field) can be written:

$$i\frac{\partial\psi}{\partial t} = -i\gamma^0\gamma^i\nabla_i\psi + m\gamma^0\psi + eV\psi \quad \text{with} \quad V = \gamma^0\gamma^\mu A_\mu$$

c.f. the Schrödinger equation in a potential V [Remember $\gamma^0\gamma^0 = 1$]

Let's assume that the state at time $t = -\infty$ is an momentum eigenstate $\Psi_{\vec{p}}$ of the free Dirac equation ($V=0$) with energy $E = \sqrt{\vec{p}^2 + m^2}$

$$\text{i.e. } H_0\Psi_{\vec{p}} = i\frac{\partial}{\partial t}\Psi_{\vec{p}} \quad \text{with} \quad H_0 = -i\gamma^0\gamma^i\nabla_i + m\gamma^0$$

Our Dirac equation in an external field is $(H_0 + eV)\psi = i\frac{\partial\psi}{\partial t}$

We need to solve this equation for ψ .

Now, since $\Psi_{\vec{p}}$ form a complete set, any solution must be of the form

$$\psi = \int d^3p \underbrace{\frac{1}{2E_{\vec{p}}}}_{\text{(this normalisation choice to ensure } |\kappa|^2 \text{ can be interpreted as a probability)}} \kappa_{\vec{p}}(t) \Psi_{\vec{p}}(x) \quad \text{with} \quad \int d^3x \Psi_{\vec{q}}^\dagger \Psi_{\vec{p}} = 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{q})$$

Let's expand κ in powers of e : i.e. $\kappa_{\vec{p}} = \sum_n \tilde{\kappa}_{\vec{p}}^{(n)} e^n$

$$\psi = \sum_n e^n \int d^3p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(n)}(t) \Psi_{\vec{p}}(x)$$

Let's stick this in and see what we get:

$$(H_0 + eV)\psi = i \frac{\partial \psi}{\partial t}$$

$$\Rightarrow \sum_n e^n \int d^3p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(n)} (\textcolor{red}{H_0} \Psi_{\vec{p}} + eV \Psi_{\vec{p}}) = \sum_n e^n \int d^3p \frac{1}{2E_{\vec{p}}} \left(i \frac{\partial \tilde{\kappa}_{\vec{p}}^{(n)}(t)}{\partial t} \Psi_{\vec{p}} + \tilde{\kappa}_{\vec{p}}^{(n)} i \frac{\partial}{\partial t} \Psi_{\vec{p}} \right)$$



$$\sum_n e^n \int d^3 p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(n)} e V \Psi_{\vec{p}} = \sum_n e^n \int d^3 p \frac{1}{2E_{\vec{p}}} i \frac{\partial \tilde{\kappa}_{\vec{p}}^{(n)}(t)}{\partial t} \Psi_{\vec{p}}$$

Equate order by order in e^n

To order e^0 : $\int d^3 p \frac{1}{2E_{\vec{p}}} i \frac{\partial \tilde{\kappa}_{\vec{p}}^{(0)}(t)}{\partial t} \Psi_{\vec{p}} = 0$ i.e. $\tilde{\kappa}_{\vec{p}}^{(0)}(t) = \tilde{\kappa}_{\vec{p}}^{(0)}(-\infty)$

To order e^1 : $\int d^3 p \frac{1}{2E_{\vec{p}}} i \frac{\partial \tilde{\kappa}_{\vec{p}}^{(1)}(t)}{\partial t} \Psi_{\vec{p}} = \int d^3 p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(0)} V \Psi_{\vec{p}}$

We can now extract $\frac{\partial \tilde{\kappa}_{\vec{p}}^{(1)}}{\partial t}$ using the orthogonality of $\Psi_{\vec{p}}$: $\int d^3 x \Psi_{\vec{q}}^\dagger \Psi_{\vec{p}} = 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{q})$

$$\Rightarrow i \frac{\partial \tilde{\kappa}_{\vec{q}}^{(1)}(t)}{\partial t} = \int d^3 p \frac{1}{2E_{\vec{p}}} \tilde{\kappa}_{\vec{p}}^{(0)} \int \Psi_{\vec{q}}^\dagger V \Psi_{\vec{p}} d^3 x$$

But at time $t = -\infty$ the initial state is $\psi_i = \int d^3p \frac{1}{2E_{\vec{p}}} \kappa_{\vec{p}}(-\infty) \Psi_{\vec{p}} = \Psi_{\vec{p}_i}$,

$$\text{ie. } \tilde{\kappa}_{\vec{p}}^{(0)} \approx \kappa_{\vec{p}}(-\infty) = 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{p}_i)$$

$$\Rightarrow i \frac{\partial \tilde{\kappa}_{\vec{q}}^{(1)}}{\partial t} = \int \Psi_{\vec{q}}^\dagger V \Psi_{\vec{p}_i} d^3x$$

Integrate over t:

$$\tilde{\kappa}_{\vec{q}}^{(1)}(t) = \cancel{\tilde{\kappa}_{\vec{q}}^{(1)}(-\infty)} - i \int_{-\infty}^t dt' \int d^3x \Psi_{\vec{q}}^\dagger V \Psi_{\vec{p}_i}$$

zero

By time $t = \infty$ the interaction has stopped. The probability of finding the system in a state

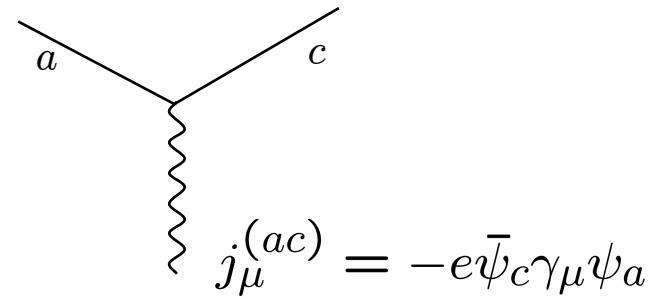
$$\psi_f = \Psi_{\vec{p}_f}(x)$$

is given by $|\kappa_f|^2 \approx |e\tilde{\kappa}_{\vec{p}_f}^{(1)}|^2$ to order e^1 with:

$$\kappa_f \approx -ie \int d^4x \psi_f^\dagger V \psi_i$$

Explicitly putting in our $V = \gamma^0 \gamma^\mu A_\mu$ gives

$$\begin{aligned}\kappa_{ca} &= -ie \int d^4x \psi_c^\dagger (\gamma^0 \gamma^\mu A_\mu) \psi_a \\ &= i \int j_\mu^{(ac)} A^\mu d^4x\end{aligned}$$

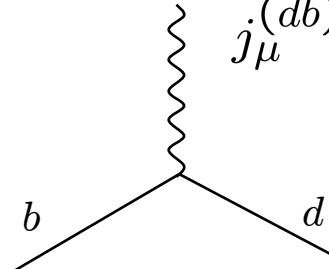


OK, so now we know the effect of the field A^μ on the electron, but what A^μ does the other electron produce to cause this effect?

$$\partial^2 A^\mu = j_{(db)}^\mu = -e \bar{u}(p_d) \gamma^\mu u(p_b) e^{i(p_d - p_b) \cdot x}$$

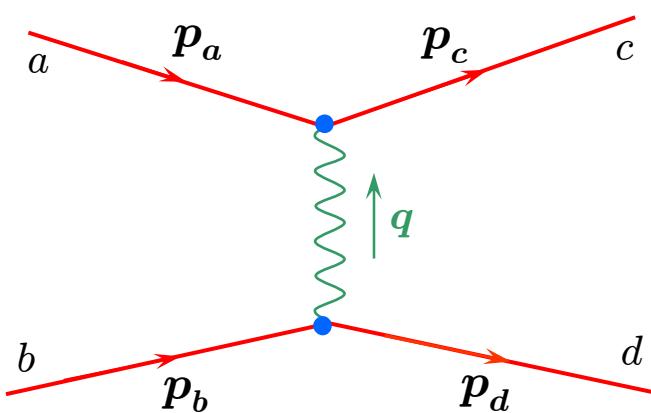
$$j_\mu^{(db)} = -e \bar{\psi}_d \gamma_\mu \psi_b$$

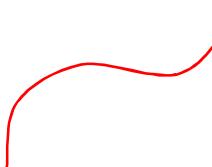
$$\Rightarrow A^\mu = -\frac{1}{q^2} j_{(db)}^\mu \quad (q = p_d - p_b)$$



Putting this all together:

$$\begin{aligned}
 \kappa_{fi} &= -i \int j_\mu^{(ac)} \frac{1}{q^2} j^\mu_{db} d^4x \\
 &= i \bar{u}(p_c) (-e\gamma_\mu) u(p_a) \left(-\frac{1}{q^2} \right) \bar{u}(p_d) (-e\gamma^\mu) u(p_b) \underbrace{\int e^{i(-p_a - p_b + p_c + p_d) \cdot x} d^4x}_{(2\pi)^4 \delta^4(p_a + p_b - p_c - p_d)}
 \end{aligned}$$




 forces momentum conservation

Note: This is not just a pretty picture, or a graphical aid. This diagram is a **mathematical notation** for the expression above!

Feynman Diagrams: The QED Feynman Rules

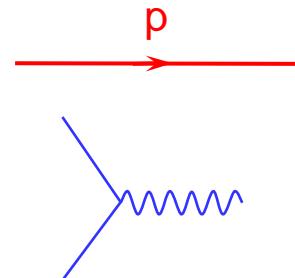
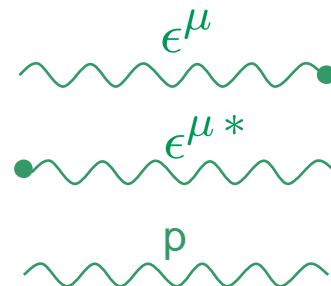
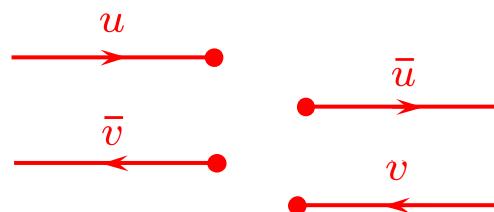
We can construct transition amplitudes simply by associating a mathematical expression with the diagram describing the interaction.



Richard Feynman

For each diagram, write:

- u for each incoming electron
- \bar{u} for each outgoing electron
- \bar{v} for each incoming positron
- v for each outgoing positron
- ϵ^μ for each incoming photon
- $\epsilon^{\mu*}$ for each outgoing photon
- $-ig_{\mu\nu} \frac{1}{p^2}$ for each internal photon
- $i \frac{p + m}{p^2 - m^2}$ for each internal electron
- $-ieQ\gamma^\mu$ for each vertex

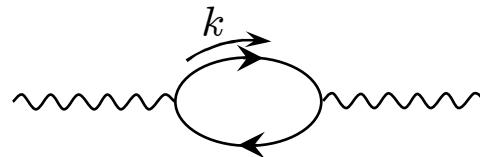


(fermion charge Q)

Remember that γ -matrices and spinors do not commute, so be careful with the order in spin lines. Write left to right, **against** the fermion flow.

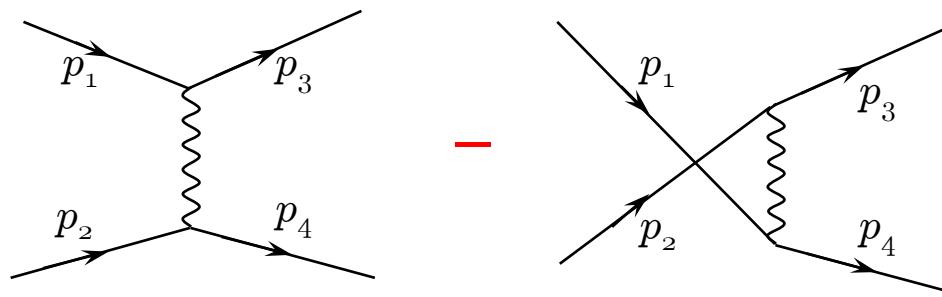
2 details:

- **Closed loops:**



Integrate over loop momentum $\int \frac{d^4k}{(2\pi)^4}$ and include an extra factor of -1 if it is a fermion loop.

- **Fermi Statistics:** If diagrams are identical except for an exchange of electrons, include a relative – sign.

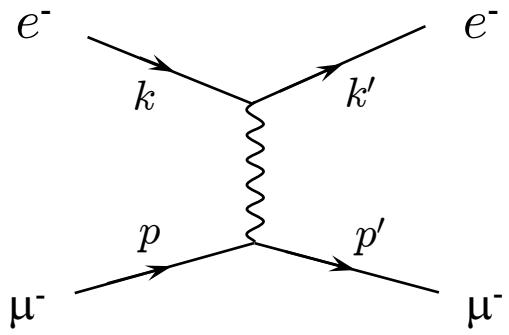


These rules provide $i\mathcal{M}$, and the transition amplitude is

$$\kappa_{fi} = -i(2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}$$

The probability of transition from initial to final state is $|\kappa_{fi}|^2$

An example calculation: $e^- \mu^- \rightarrow e^- \mu^-$



$$i\mathcal{M} = -e^2 \bar{u}(k') \gamma^\mu u(k) \frac{g_{\mu\nu}}{q^2} \bar{u}(p') \gamma^\nu u(p)$$

This is what we had before.

To get the total probability we must **square** this, **average over initial spins**, and **sum over final spins**.

$$\begin{aligned} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{e^4}{q^4} \frac{1}{4} \sum_{\text{spins}} \left\{ [\bar{u}(k') \gamma^\mu u(k)] [\bar{u}(k') \gamma^\nu u(k)]^* \right\} \left\{ [\bar{u}(p') \gamma_\mu u(p)] [\bar{u}(p') \gamma_\nu u(p)]^* \right\} \\ &= \frac{e^4}{q^4} \mathcal{L}^{\mu\nu}(k, k') \mathcal{L}_{\mu\nu}(p, p') \end{aligned}$$

$$\begin{aligned} \text{But } [\bar{u}(k') \gamma^\mu u(k)]^* &= [u^\dagger(k') \gamma^0 \gamma^\mu u(k)]^\dagger && (\bar{u} \equiv u^\dagger \gamma^0) \\ &= u^\dagger(k) \gamma^\mu \gamma^0 u(k') \\ &= u^\dagger(k) \gamma^0 \gamma^\mu \gamma^0 u(k') && (\gamma^\mu \dagger = \gamma^0 \gamma^\mu \gamma^0, \gamma^0 \dagger = \gamma^0) \\ &= \bar{u}(k) \gamma^\mu u(k') && (\gamma^0 \gamma^0 = 1) \end{aligned}$$

But don't forget that the \mathbf{u} are 4-component spinors and the γ are 4×4 matrices:

$$\text{So } \mathcal{L}^{\mu\nu}(k, k') = \frac{1}{2} \sum_{s=1,2} \sum_{s'=1,2} \bar{u}_i^{(s')}(k') [\gamma^\mu]_{ij} u_j^{(s)}(k) \bar{u}_m^{(s)}(k) [\gamma^\nu]_{mn} u_n^{(s')}(k')$$

↑
spinor indices

(summation over $i,j,m,n = 1,..,4$)

We can simplify this using the completeness relation for spinors:

$$\sum_{s=1,2} u_i^{(s)}(k) \bar{u}_j^{(s)}(k) = k_\mu [\gamma^\mu]_{ij} + m \delta_{ij} \quad \left(\begin{array}{l} \text{beware normalization here - this is only} \\ \text{true for 2E particles per unit volume} \end{array} \right)$$

$$\begin{aligned} \text{Then } \mathcal{L}^{\mu\nu}(k, k') &= \frac{1}{2} \sum_{s'=1,2} u_n^{(s')}(k') \bar{u}_i^{(s')}(k') [\gamma^\mu]_{ij} \sum_{s=1,2} u_j^{(s)}(k) \bar{u}_m^{(s)}(k) [\gamma^\nu]_{mn} \\ &= \frac{1}{2} \underbrace{\left(k'_\rho [\gamma^\rho]_{ni} + m \delta_{ni} \right)}_{\uparrow} [\gamma^\mu]_{ij} \underbrace{\left(k_\sigma [\gamma^\sigma]_{jm} + m \delta_{jm} \right)}_{\uparrow} [\gamma^\nu]_{mn} \\ &= \frac{1}{2} \text{Tr} \left[(\not{k'} + m) \gamma^\mu (\not{k} + m) \gamma^\nu \right] \end{aligned}$$

We need some trace identities!

Trace Identities

$$\text{Tr } \mathbf{1} = 4$$

$$\text{Tr } \gamma^\mu \gamma^\nu = \frac{1}{2} \text{Tr} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = \frac{1}{2} \text{Tr} (2g^{\mu\nu}) = g^{\mu\nu} \quad \text{Tr } \mathbf{1} = 4 g^{\mu\nu}$$

$$\text{Tr } \gamma^\mu \gamma^\nu \gamma^\lambda = \text{Tr } \gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda = \text{Tr } \gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^5 = -\text{Tr } \gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda = 0$$

$$\text{Tr } \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa = 4 (g^{\mu\nu} g^{\lambda\kappa} - g^{\mu\lambda} g^{\nu\kappa} + g^{\mu\kappa} g^{\nu\lambda})$$

$$\text{Tr } \gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa = -4i \epsilon^{\mu\nu\lambda\kappa}$$

[This is true for any **odd** number of γ -matrices]

be careful with this one!



Exercise: Show that $\text{Tr } \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\kappa = 4 (g^{\mu\nu} g^{\lambda\kappa} - g^{\mu\lambda} g^{\nu\kappa} + g^{\mu\kappa} g^{\nu\lambda})$

Using this identity and $\text{Tr } \gamma^\mu \gamma^\nu = 4 g^{\mu\nu}$, show:

$$\mathcal{L}^{\mu\nu}(k, k') = 2 \left(k^\mu k'^\nu + k^\nu k'^\mu - (k \cdot k' - m^2) g^{\mu\nu} \right)$$

$$\begin{aligned}
\text{So } \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{e^4}{q^4} 4 \left(k^\mu k'^\nu + k^\nu k'^\mu - (k \cdot k' - m_e^2) g^{\mu\nu} \right) \left(p_\mu p'_\nu + p_\nu p'_\mu - (p \cdot p' - m_\mu^2) g_{\mu\nu} \right) \\
&= 8 \frac{e^4}{q^4} \left((k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p') - m_e^2(p' \cdot p) - m_\mu^2(k' \cdot k) + 2m_e^2 m_\mu^2 \right)
\end{aligned}$$

If we are working at sufficiently high energies, then $p^2 \gg m_e^2, m_\mu^2$ and we may ignore the masses.

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \approx 8 \frac{e^4}{(k - k')^4} \left((k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p') \right)$$

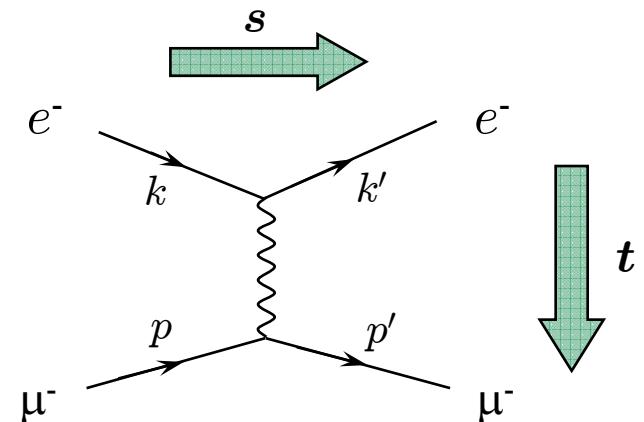
Often this is written in terms of **Mandelstam Variables**, which are defined:

$$\begin{aligned}s &= (k + p)^2 \approx 2k \cdot p \approx 2k' \cdot p' \\t &= (k - k')^2 \approx -2k \cdot k' \approx -2p \cdot p' \\u &= (k - p')^2 \approx -2k \cdot p' \approx -2k' \cdot p\end{aligned}$$

[Note that $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \approx 0$]

Then

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \frac{s^2 + u^2}{t^2}$$



- s is the square of the momentum flowing in the time direction
- t is the square of the momentum flowing in the spacee direction

Cross-sections

So we have $|\mathcal{M}|^2$ but we are not quite there yet – we need to turn this into a cross-section.

Recall

$$\kappa_{fi} = -i(2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}$$

$$\begin{aligned}\Rightarrow |\kappa_{fi}|^2 &= [(2\pi)^4 \delta^4(p_f - p_i)]^2 |\mathcal{M}|^2 \\ &= (2\pi)^4 \delta^4(p_f - p_i) VT |\mathcal{M}|^2\end{aligned}$$

$$\text{since } (2\pi)^4 \delta^4(p_f - p_i) = \int d^4x e^{i(p_f - p_i) \cdot x} = \int_{-T/2}^{T/2} dt \int_V d^3x = VT$$

But we need the transition probability per unit time and per unit volume is:

$$\frac{|\kappa_{fi}|^2}{VT} = (2\pi)^4 \delta^4(p_f - p_i) |\mathcal{M}|^2$$

The **cross-section** is the probability of transition per unit volume, per unit time \times the number of final states / initial flux.

$$d\sigma = \frac{|\kappa_{fi}|^2}{VT} \times \frac{\# \text{ final states}}{\text{initial flux}}$$

Initial Flux

In the lab frame, particle A, moving with velocity \vec{v}_A , hits particle B, which is stationary.



The number of particles like A
in the beam, passing through
volume V per unit time is

The number of particles like B
per volume V in the target is

$$\left. \begin{aligned} & | \vec{v}_A | \frac{2E}{V} \\ \text{So the initial flux in a volume } V \text{ is } & \frac{1}{V^2} | \vec{v}_A | 2E_A 2E_B \end{aligned} \right\}$$

$$\text{But we can write this in a covariant form: } \frac{1}{V^2} | \vec{v}_A | 2E_A 2E_B = \frac{4}{V^2} \sqrt{[(p_A \cdot p_B)^2 - m_A^2 m_B^2]}$$

This must also be true for a **collider**, where A and B are both moving, since the lab frame and centre-of-mass frame are related by a Lorentz boost.

final states

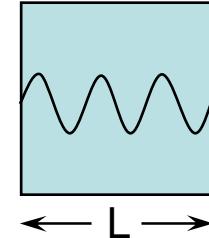
How many states of momentum \vec{p} can we fit in a volume V ?

In order to not have any particle flow through the boundaries of the box, we must impose **periodic boundary conditions**.

$Lp_x = 2\pi n$ so the number of states between p_x and $p_x + dp_x$ is $\frac{L}{2\pi} dp_x$

So in a volume V we have

$$\left(\frac{L}{2\pi} dp_x\right) \left(\frac{L}{2\pi} dp_y\right) \left(\frac{L}{2\pi} dp_z\right) = \frac{V}{(2\pi)^3} d^3 p$$



But there are $2EV$ particles per volume V , so

$$\text{\# final states per particle} = \frac{1}{(2\pi)^3} \frac{d^3 p}{2E}$$

Note that $\int \frac{1}{(2\pi)^3} \frac{d^3 p}{2E} = \int \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2)$ so this is covariant!

Putting all this together, the **differential cross-section** is:

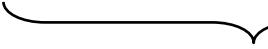
$$d\sigma = \frac{1}{F} |\mathcal{M}|^2 d\text{Lips}$$

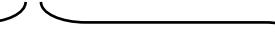
where the **Flux** F is given by,

$$F = 4\sqrt{[(p_A \cdot p_B)^2 - m_A^2 m_B^2]}$$

and the **Lorentz invariant phase space** is,

$$d\text{Lips} = (2\pi)^4 \delta^4(p_c + p_d - p_a - p_b) 2\pi\delta(p_c^2 - m_c^2) 2\pi\delta(p_d^2 - m_d^2) \frac{d^4 p_c}{(2\pi)^4} \frac{d^4 p_d}{(2\pi)^4}$$

 momentum conservation

 on-shell conditions

 integration measure

In the **centre-of-mass**, this becomes much simpler

This frame is defined by $\vec{p}_a = -\vec{p}_b$ and $E_a + E_b = \sqrt{s}$

Remember
 $s \equiv (p_a + p_b)^2$

So $p_a = (E_a, \vec{p}_a)$ and $p_b = (E_b, -\vec{p}_a)$

with $|\vec{p}_a| = \frac{1}{4s} \sqrt{\lambda(s, m_a^2, m_b^2)}$ $\lambda(\alpha, \beta, \gamma) \equiv \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma$

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \quad E_b = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}$$

Then the **Flux** becomes

$$\begin{aligned} F &= 4\sqrt{[(p_a \cdot p_b)^2 - m_a^2 m_b^2]} \\ &= 4(|\vec{p}_a|E_b + |\vec{p}_b|E_a) \end{aligned}$$

$F = 4|\vec{p}_a|\sqrt{s}$

Also $\vec{p}_c = -\vec{p}_d$ and $E_c + E_d = \sqrt{s}$ with relations analogous to those for p_a and p_b

The phase space measure becomes:

$$\begin{aligned}
 d\text{Lips} &= (2\pi)^4 \delta^4(p_c + p_d - p_a - p_b) \frac{1}{2E_c} \frac{d^3\vec{p}_c}{(2\pi)^3} \frac{1}{2E_d} \frac{d^3\vec{p}_d}{(2\pi)^3} \\
 &= \frac{1}{4\pi^2} \delta(E_c + E_d - \sqrt{s}) \frac{1}{4E_c E_d} d^3\vec{p}_c \\
 &= \frac{1}{4\pi^2} \delta(E_c + E_d - \sqrt{s}) \frac{1}{4E_c E_d} |\vec{p}_c|^2 d|\vec{p}_c| d\Omega \quad \left(\text{since } \frac{d|\vec{p}_c|}{d\sqrt{s}} = \frac{E_c E_d}{|\vec{p}_c| \sqrt{s}} \right) \\
 &= \frac{1}{16\pi^2} \delta(\sqrt{s} - E_c - E_d) \frac{|\vec{p}_c|}{\sqrt{s}} d\sqrt{s} d\Omega
 \end{aligned}$$

$$d\text{Lips} = \frac{1}{16\pi^2} \frac{|\vec{p}_c|}{\sqrt{s}} d\Omega$$

Putting this together:

$$\left. \frac{d\sigma}{d\Omega} \right|_{CM} = \frac{1}{64\pi^2} \frac{|\vec{p}_c|}{|\vec{p}_a|} \frac{1}{s} |\mathcal{M}|^2$$

Returning to our process $e^- \mu^- \rightarrow e^- \mu^-$

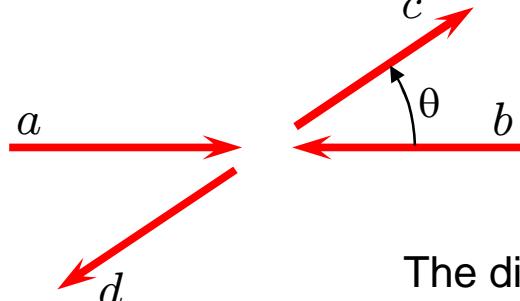
(With $m_e = m_\mu = 0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \sum_{\text{spins}} \frac{1}{64\pi^2} \frac{|\vec{p}_c|}{|\vec{p}_a|} \frac{1}{s} |\mathcal{M}|^2 = \frac{1}{64\pi^2} \frac{1}{s} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{32\pi^2 s} \frac{s^2 + u^2}{t^2} = \frac{\alpha^2}{2s} \frac{s^2 + u^2}{t^2}$$

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \frac{s^2 + u^2}{t^2}$$

The fine structure constant $\alpha \equiv \frac{e^2}{4\pi}$

In terms of the angle between a and c



$$t \equiv (p_a - p_c)^2 = -2p_a \cdot p_c = -\frac{s}{2}(1 - \cos\theta)$$

$$u \equiv (p_a - p_d)^2 = -2p_a \cdot p_d = -\frac{s}{2}(1 + \cos\theta)$$

The differential cross-section is:

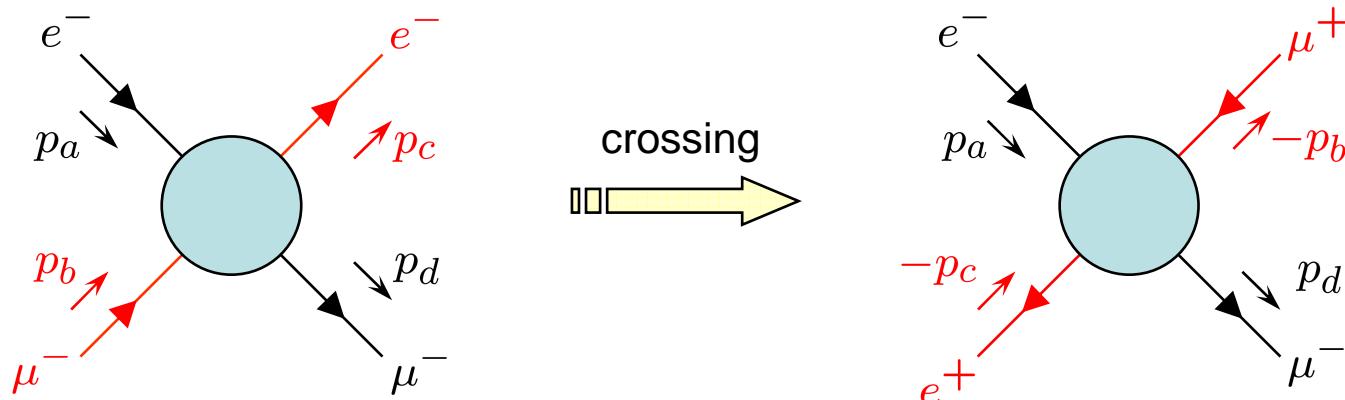
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8s} \frac{4 + (1 + \cos\theta)^2}{(1 - \cos\theta)^2}}$$

Notice that this is divergent for small angles: $\frac{d\sigma}{d\Omega} \sim \frac{4\alpha^2}{s} \frac{1}{\theta^4}$ as $\theta \rightarrow 0$

This is exactly the same divergence as is in the Rutherford scattering formula.

Crossing symmetry

Generally, in a Feynman diagram, any incoming particle with momentum \mathbf{p} is equivalent to an outgoing antiparticle with momentum $-\mathbf{p}$.



This lets us use our result for $e^- \mu^- \rightarrow e^- \mu^-$ to easily calculate the differential cross-section for $e^+ e^- \rightarrow \mu^+ \mu^-$.

$$s = (p_a + p_b)^2$$

$$t = (p_a - p_c)^2$$

$$u = (p_a - p_d)^2$$

crossing
➡

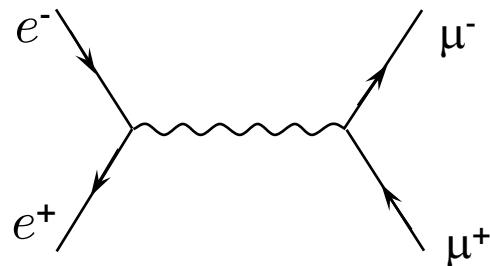
$$s = (p_a - p_c)^2$$

$$t = (p_a + p_d)^2$$

$$u = (p_a - p_d)^2$$

i.e.

$$s \leftrightarrow t$$



$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \frac{t^2 + u^2}{s^2} \Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \frac{t^2 + u^2}{s^2}$$

(Be careful not to change the s
 from flux and phase space!)

Writing θ as the angle between the e^- and μ^- ,

$$t = -\frac{s}{2}(1 - \cos \theta), \quad u = -\frac{s}{2}(1 + \cos \theta) \quad (\text{as before})$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta)}$$

(Notice the singularity
 is gone!)

The total cross-section is

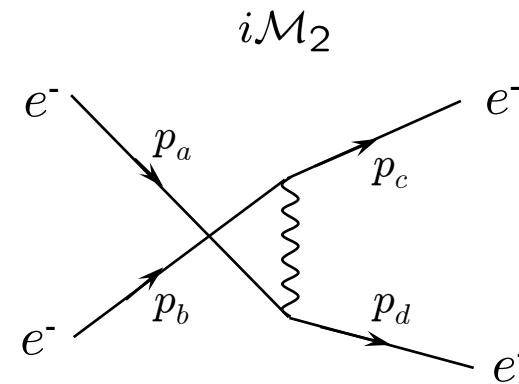
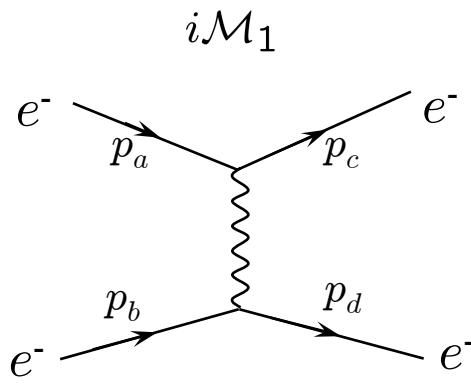
$$\sigma_{\text{Tot}} = \frac{\alpha^2}{4s} 2\pi \int_{-1}^{+1} (1 + \cos^2 \theta) d(\cos \theta) = \frac{4\pi\alpha^2}{3s}$$

Identical particles in initial or final state

So far, in the reactions we have looked at, the final state particles have all been **distinguishable**. If the final state particles are **identical**, we have additional Feynman diagrams.

e.g. $e^- e^- \rightarrow e^- e^-$

[See Feynman rules]



$$ie^2 \frac{1}{t} \bar{u}(p_c)\gamma^\mu u(p_a) \bar{u}(p_d)\gamma_\mu u(p_b)$$

interchange of identical fermions \Rightarrow minus sign

$$-ie^2 \frac{1}{u} \bar{u}(p_d)\gamma^\mu u(p_a) \bar{u}(p_c)\gamma_\mu u(p_b)$$

p_c and p_d interchanged

Since the final state particles are identical, these diagrams are **indistinguishable** and must be summed **coherently**.

$$|\mathcal{M}|^2 = |\mathcal{M}_1 + \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2 \operatorname{Re} \mathcal{M}_1 \mathcal{M}_2^*$$

We have interference between the two contributions.

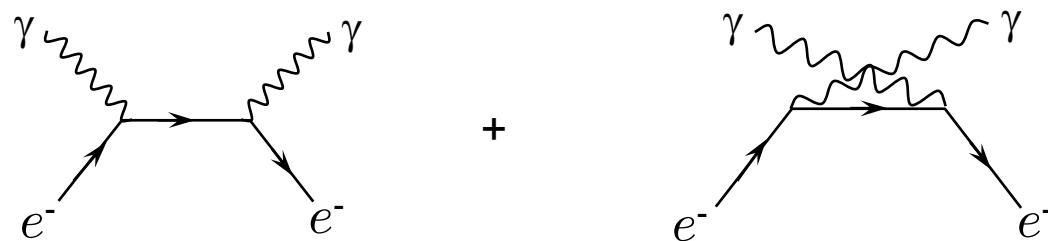
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left(\frac{s^2 + u^2}{t^2} + \frac{s^2 + t^2}{u^2} + 2 \frac{s^2}{tu} \right)$$



Exercise: Show that the spin summed/averaged differential cross-section for $e^-e^- \rightarrow e^-e^-$ in QED is given by the above equation, neglecting the electron mass.

Compton Scattering and the fermion propagator

Compton scattering is the scattering of a photon with an electron.



I just quoted the Feynman rule for the **fermion propagator**, but where did it come from?

Let's go back to the photon propagator first.

Recall the photon propagator is $-ig^{\mu\nu} \frac{1}{p^2}$

$$\epsilon^{\mu*} \quad -\frac{1}{p^2} \quad \epsilon^\nu$$

The $-\frac{1}{p^2}$ is the inverse of the photon's wave equation:

$$\partial^2 A^\mu = -p^2 A^\mu = j^\mu \quad \Rightarrow \quad A^\mu = -\frac{1}{p^2} j^\mu$$

The $g^{\mu\nu}$ is coming from summing the photon polarization vectors over spins:

$$\sum_{\lambda=1}^4 \epsilon_{(\lambda)}^{\mu*} \epsilon_{(\lambda)}^{\nu} = -g^{\mu\nu} \quad \longleftarrow \text{this is for virtual photons}$$

So, the photon propagator is then

$$-i \left(-\frac{1}{p^2} \right) \sum_{\lambda=1}^4 \epsilon_{(\lambda)}^{\mu*} \epsilon_{(\lambda)}^{\nu} = -ig^{\mu\nu} \frac{1}{p^2}$$

For a fermion propagator we follow the same procedure

$$\bar{u} \quad \overline{-\frac{1}{p^2 - m^2}} \quad u$$

The massless fermion spin sum is

$$\sum_{s=1}^2 u_{(s)} \bar{u}_{(s)} = \not{p} + m$$

Remember that ψ also obeys the KG equation.

$$(\not{p} - m) \psi = 0$$

$$\Rightarrow (\not{p} + m)(\not{p} - m)\psi = (p^2 - m^2)\psi = 0$$

so the fermion propagator is

$$-i \left(-\frac{1}{p^2 - m^2} \right) \sum_{s=1}^2 u_{(s)} \bar{u}_{(s)} = i \frac{\not{p} + m}{p^2 - m^2} \quad \left(\text{sometimes written } \frac{i}{\not{p} - m} \right)$$

More precisely, the propagator is the momentum space Fourier transform of the wave equation's **Greens function**

$$\text{Green's function } S \text{ obeys: } (i\cancel{\partial} - m)S(x - y) = i\delta^{(4)}(x - y)$$

$$\left. \begin{aligned} & \text{This is a definition of a Green's function. They are very useful to know since we can use them to build up solutions for any source.} \\ & (i\cancel{\partial} - m)\psi(x) = \rho(x) \quad \Rightarrow \quad \psi(x) = -i \int \rho(y) S(x-y) dy \end{aligned} \right\}$$

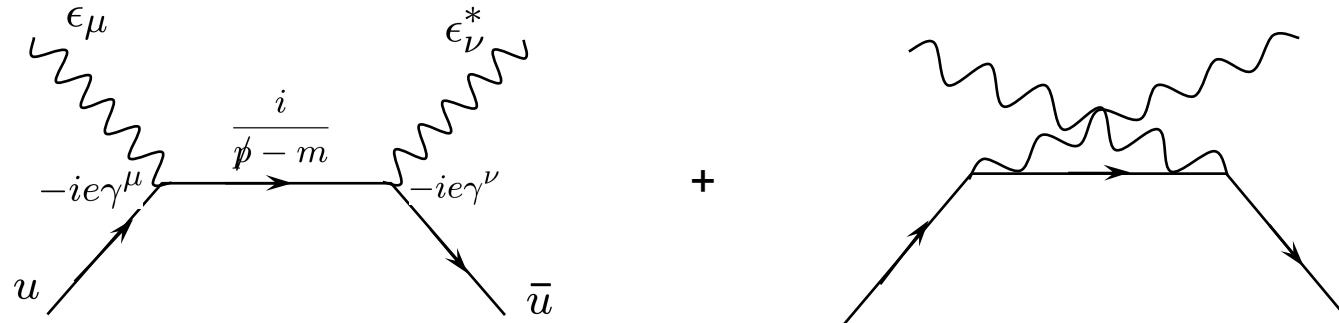
Writing $S(x - y) = \int \frac{d^4 p}{(2\pi)^4} \tilde{S}(p) e^{-ip \cdot (x-y)}$, and pre-multiplying by $(i\cancel{\partial} - m)$

gives $\int \frac{d^4 p}{(2\pi)^4} (\cancel{\partial} - m) \tilde{S}(p) e^{-ip \cdot (x-y)} = i\delta^{(4)}(x - y)$

$$\Rightarrow \boxed{\tilde{S}(p) = \frac{i}{\cancel{p} - m} = \frac{i(\cancel{p} + m)}{p^2 - m^2}}$$

Remember
 $\int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} = \delta^{(4)}(x - y)$

So now we are armed with enough information to calculate **Compton Scattering**



Putting in the Feynman rules, and following through, with $m_e = 0$

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = -2e^4 \left(\frac{u}{s} + \frac{s}{u} \right)$$



Exercise: Reproduce the above equation (starting from the Feynman Rules).

You will need to use $\sum_{\text{spins}} \epsilon_{(T)}^\mu \epsilon_{(T)}^\nu \rightarrow -g^{\mu\nu}$

[sum over transverse
polarizations]

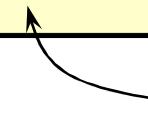
What happened to the interference?

Decay Rates

So far we have only looked at $2 \rightarrow 2$ processes, but what about decays?

A decay width is given by:

$$d\Gamma = \frac{|\kappa_{fi}|^2}{VT} \times \frac{\# \text{ final states}}{\# \text{ of decaying particles per unit volume}}$$

 This replaces the Flux.

For a decay $a \rightarrow b + c$ we have

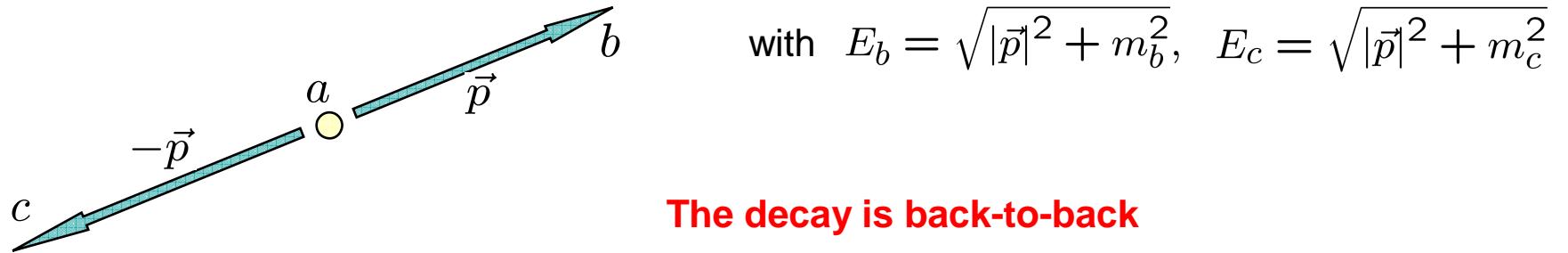
$$\frac{|\kappa_{fi}|^2}{VT} = (2\pi)^2 \delta^4(p_a - p_b - p_c) |\mathcal{M}|^2$$

$$\# \text{ final states} = \frac{1}{2E_b} \frac{d^3 \vec{p}_b}{(2\pi)^3} \frac{1}{2E_c} \frac{d^3 \vec{p}_c}{(2\pi)^3} = \frac{d^4 p_b}{(2\pi)^4} \frac{d^4 p_c}{(2\pi)^4} (2\pi) \delta(p_b^2 - m_b^2) (2\pi) \delta(p_c^2 - m_c^2)$$

$$\# \text{ of decay particles per unit volume} = \frac{1}{2E_a}$$

$$\Rightarrow d\Gamma = \frac{1}{2E_a} |\mathcal{M}|^2 d\text{Lips}$$

In the rest frame of particle a : $p_a = (m_a, \vec{0})$, $p_b = (E_b, \vec{p})$, $p_c = (E_c, -\vec{p})$,



$$d\Gamma = \frac{1}{2m_a} |\mathcal{M}|^2 \frac{1}{4E_b E_c} \frac{1}{(2\pi)^2} \delta(m_a - E_b - E_c) |\vec{p}|^2 d|\vec{p}| d\Omega$$

$$\text{But } |\vec{p}| d|\vec{p}| = E_b dE_b = E_c dE_c = \frac{E_b E_c}{E_b + E_c} d(E_b + E_c)$$

$$\Gamma = \int \frac{1}{8m_a} |\mathcal{M}|^2 \frac{1}{(2\pi)^2} \delta(m_a - E_b - E_c) \frac{|\vec{p}|}{E_b + E_c} d(E_b + E_c) d\Omega$$

$$\boxed{\Gamma = \int \frac{1}{32m_a^2 \pi^2} |\mathcal{M}|^2 |\vec{p}| d\Omega}$$

Remember, to get the total decay rate, you need to sum over all possible decay processes.

$$\Gamma_{\text{Tot}} = \sum_i \Gamma_i$$

The inverse of the total width Γ_{Tot}^{-1} will give the **lifetime** of the particle:

If the number of particles = N_a then,

$$\Gamma_{\text{Tot}} = -\frac{1}{N_a} \frac{dN_a}{dt} \quad \Rightarrow \quad N_a(t) = N_a(0)e^{-\Gamma_{\text{Tot}} t}$$

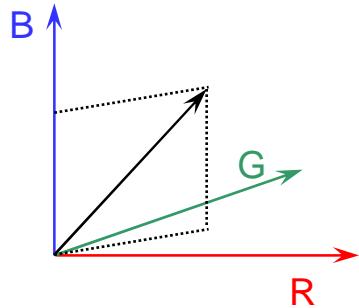
6. Quantum Chromo Dynamics (QCD)

Quarks, Gluons and Color

QCD describes the interaction of **quarks** and **gluons**.

It is very similar to QED, except we have 3 types of ‘charge’ instead of just one.

Conventionally we call these charges **red**, **green** and **blue**, and each quark can be written as a vector in “color space”:



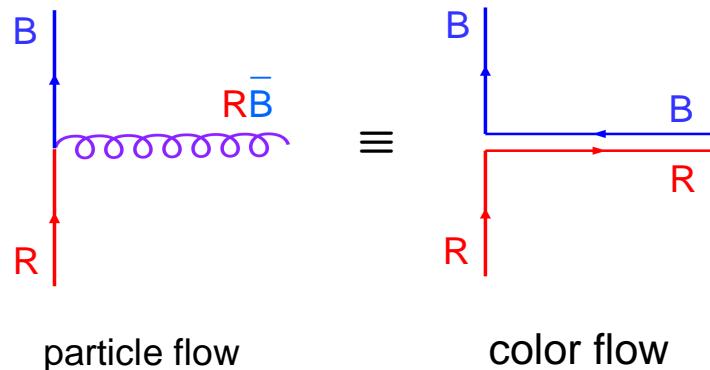
$$q = \begin{pmatrix} q^R \\ q^G \\ q^B \end{pmatrix}$$

However, QCD is symmetric under rotations in this color-space, so we can always rotate the quarks to pure color states and say they are either red, green or blue.

This symmetry is known as $SU(3)_{\text{color}}$, and parallels the $U(1)_{\text{QED}}$ symmetry of QED.

The force between the quarks is mediated by **gluons** which can also change the color of the quarks.

Since we have **3** different sorts of quark (red, green and blue), to connect them all together we naïvely need **$3 \times 3 = 9$** different gluons.



Since we are connecting together quarks of different color, the gluons must be colored too.

So, for example, we could have gluons:

$$R\bar{B}, R\bar{G}, G\bar{R}, G\bar{B}, B\bar{R}, B\bar{G} + \text{three orthogonal combinations of } R\bar{R}, G\bar{G}, B\bar{B}$$

Conventionally these last 3 are

$$\frac{1}{\sqrt{2}}(R\bar{R} - B\bar{B}), \frac{1}{\sqrt{6}}(R\bar{R} + B\bar{B} - 2G\bar{G}), \frac{1}{\sqrt{3}}(R\bar{R} + B\bar{B} + G\bar{G}).$$

Since QCD is symmetric to rotations in color-space, the first 8 of these must have related couplings. However, the last one is a color singlet, so in principle can have an arbitrary coupling. In QCD, its coupling is zero.

⇒ We have **8** gluons

In order to transform one quark color-vector onto another, we need eight 3×3 matrices.

For example to turn a blue quark into a red quark we need a $R\bar{B}$ gluon represented by

$$T^{R\bar{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

The above matrix is not a very convenient choice (it is actually a ladder operator). Instead we normally write T^A in terms of the **Gell-Mann λ matrices**.

$$T^A = \frac{1}{2} \lambda^A$$

These matrices are generators of the SU(3) group and obey the **SU(3) algebra**,

$$[T^A, T^B] = i f^{ABC} T^C$$

SU(3) structure constants

are conventionally normalised by $\text{Tr}(T^A T^B) = \frac{1}{2} \delta^{AB}$ and are **traceless**.

(Hence the removal of $\frac{1}{\sqrt{3}} (R\bar{R} + B\bar{B} + G\bar{G})$)

The Gell-Mann matrices are:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda^8 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Notice there are only 2 diagonal Gell-Mann matrices.

QCD from a Lagrangian

Just like QED, we can describe the physics of QCD with a **Lagrangian**:

$$\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}^A F_A^{\mu\nu}}_{\text{kinetic term for the gluon}} + \sum_{\text{flavours}} \bar{q}_a (i\gamma^\mu D_\mu - m)_{ab} q_b$$

↑
up, down etc.

↑
covariant derivative
↑
quark field

The covariant derivative is now a 3×3 matrix in color space:

$$[D_\mu]_{ab} = \delta_{ab}\partial_\mu + ig_s [T^A A_\mu^A]_{ab}$$

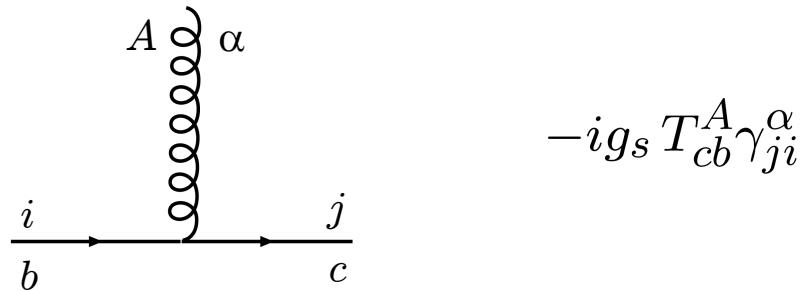
↑
coupling constant
↑
gluon field (with color A)

The gluon field strength has an extra term compared to the photon's:

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A - g f^{ABC} A_\mu^B A_\nu^C$$

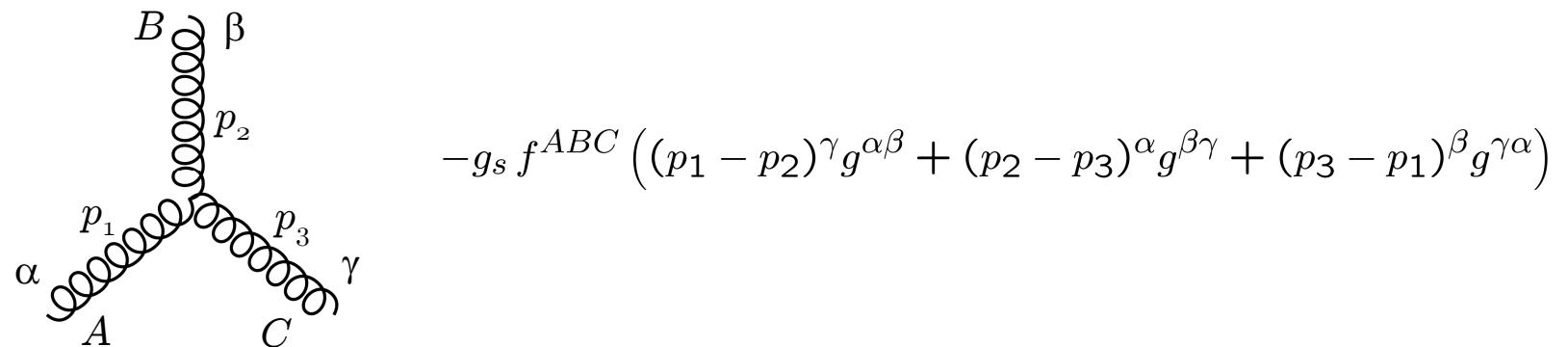
←
gluon self interaction

At a vertex between quark and gluons we need to include a factor



$$-ig_s T_{cb}^A \gamma_{ji}^\alpha$$

The **gluons also carry color**, so we must also include a gluon-gluon interaction. This is given by



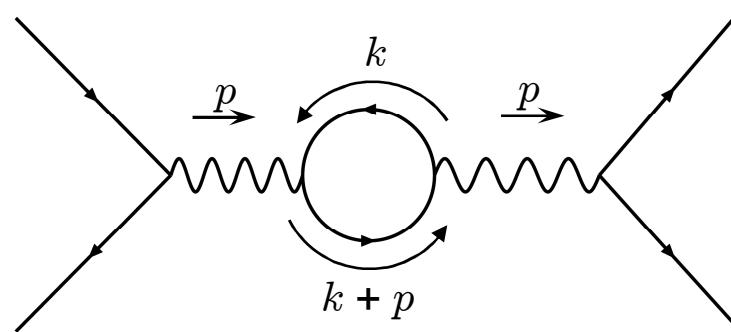
$$-g_s f^{ABC} ((p_1 - p_2)^\gamma g^{\alpha\beta} + (p_2 - p_3)^\alpha g^{\beta\gamma} + (p_3 - p_1)^\beta g^{\gamma\alpha})$$

The full QCD Feynman rules will be given to you in the Standard Model course.

Renormalisation

When we calculate beyond leading order in our perturbative expansion, we will find that we have diagrams with loops in them.

For example, the corrections to our $e^+e^- \rightarrow \mu^+\mu^-$ would include the diagram



But momentum conservation at all vertices leaves the momentum flowing around the loop unconstrained! We need to integrate over this loop momentum, and find a result containing

$$\sim \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - m^2]} \frac{1}{[(k-p)^2 - m^2]}$$

This integral is infinite!

To see that it is infinite, lets look at this integral in the limit as $k \rightarrow \infty$. Then we can neglect the momentum p and the mass m . The integral becomes

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \sim \int_0^\infty k^3 dk \frac{1}{k^4} = \int_0^\infty dk \frac{1}{k} = \log \infty - \log 0$$

Ultra-Violet (UV) singularity

this is a fake, because our approximation doesn't work for $k \rightarrow 0$

This is not really that surprising. Even in classical electromagnetism we have singularities when we go to small distances/high energies.

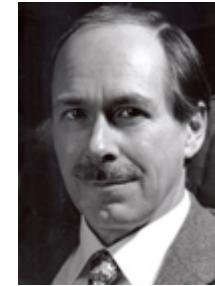
For example, in classical electromagnetism, the energy associated with a charged sphere of radius R is:

$$\frac{3}{54\pi\epsilon_0} \frac{Q^2}{R} \rightarrow \infty \quad \text{as} \quad R \rightarrow 0$$

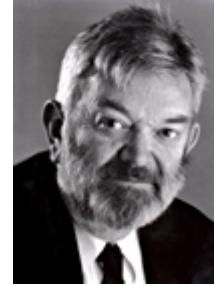
So classically, a point charge should have infinite energy!

Are infinities really a problem?

Our theories such as QED and QCD make **predictions of physical quantities**. While infinities may make the theory difficult to work with, there is no real problem as long as our **predictions** of physical quantities are finite and match experiment.



Gerardus
't Hooft



Martinus
Veltman

We find, that in both QED and QCD, that our physical observables are finite: they are **renormalizable** theories.

To understand this, lets think about the one-loop calculation of the electron mass

$$\text{---} \bullet \text{---} = \text{---} + \text{---} + \mathcal{O}(e^4)$$
$$m = m_0 + e^2 m_1 + \mathcal{O}(e^4)$$

finite ∞ ∞

In order for the physically measured mass to be **finite**, the '**bare mass**' must be **infinite** and cancel the divergence from the loop. But this is OK, since m_0 is not measurable, only m is.

We absorb infinities into unmeasurable bare quantities.

In reality, what we are doing is measuring differences between quantities.

$$Q^2 \quad \text{---} \bullet \text{---} = \quad \text{---} \rightarrow + \quad \text{---} \nearrow \swarrow + \mathcal{O}(e^4)$$

Since the loop contains a dependence on the momentum scale, Q , the mass changes with probed energy. The **difference** between two masses at different scales is:

$$\begin{aligned} m(Q_1^2) - m(Q_2^2) &= (m_0 + e^2 m_1(Q_1^2) + \mathcal{O}(e^4)) - (m_0 + e^2 m_1(Q_2^2) + \mathcal{O}(e^4)) \\ &= \cancel{(m_0 - m_0)} + e^2 \underbrace{(m_1(Q_1^2) - m_1(Q_2^2))}_{\text{infinities are the same in both } m_1\text{'s}} + \mathcal{O}(e^4) \end{aligned}$$

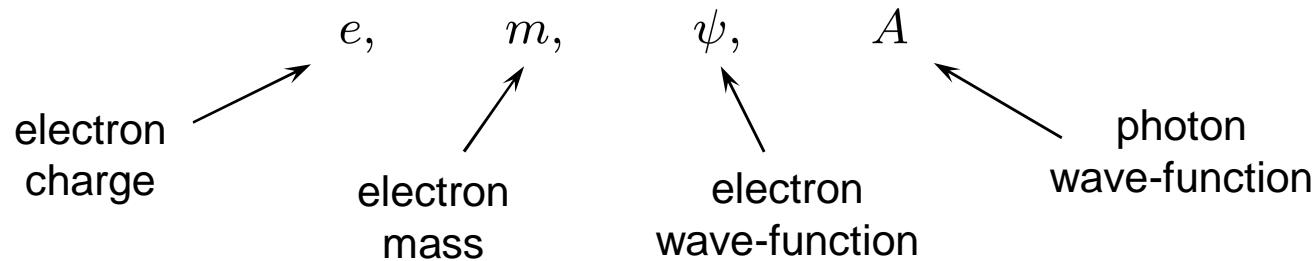
infinities are the same in both m_1 's
 \Rightarrow finite

The difference between the masses is **finite**.

Both philosophies, absorption or subtraction of singularities, are doing the same thing. We replace the **infinite bare quantities** in the Lagrangian with **finite physical ones**. This is called **renormalization**.

The beauty of QED (and QCD) is that we don't need to do this for every observable (which would be rather useless). Once we have done it for certain observables, everything is finite! This is a very non-trivial statement. We say that QED and QCD are **renormalizable**.

In QED we choose to absorb the divergences into:



Instead of writing observables in terms of the infinite bare quantities e_0, m_0, ψ_0, A_0 , we write them in terms of the measurable 'renormalized' quantities e_R, m_R, ψ_R, A_R .

In order to do this, we must first **regularize** the divergences in our integrals.

Regularization by a Momentum cut-off

The most obvious regularization is to simply forbid any momenta above an scale Λ . Then, the integral becomes

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4} \sim \int_0^\Lambda dk \frac{1}{k} = \log \Lambda - \log 0$$

The UV divergence has been **regularized** (remember the infra-red divergence here, $\log 0$, is fake). This isn't very satisfactory though, since this breaks gauge invariance.

Dimensional Regularization

The most usual way to regulate the integrals is to work in $d = 4 - 2\epsilon$ dimensions rather than 4 dimensions.

$$\int \frac{d^{4-2\epsilon} k}{(2\pi)^{4-2\epsilon}} \frac{1}{k^4} \sim \int_0^\infty k^{3-2\epsilon} dk \frac{1}{k^4} = \int_0^\infty dk \frac{1}{k^{1+2\epsilon}} = \left[-\frac{k^{-2\epsilon}}{2\epsilon} \right]_0^\infty$$



we have increased the power of k in the denominator, making the integral finite

More precisely, our original integral (ignoring masses for simplicity) gives:

$$\int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \frac{1}{k^2} \frac{1}{(k+p)^2} = i(4\pi)^{\epsilon-2} \underbrace{\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}}_{\text{finite}} \underbrace{\frac{(-p^2)^{-\epsilon}}{\epsilon(1-2\epsilon)}}_{\text{divergent as } \epsilon \rightarrow 0}$$

Notice that it is rather arbitrary which bit one wants to absorb or subtract off.

One could subtract off only the pole in ϵ , i.e. $\frac{i}{(4\pi)^2} \frac{1}{\epsilon}$ for the above integral.

This is known as the **Minimal Subtraction**, denoted **MS**.

Alternatively we could have removed some of the finite terms too,

$$\text{e.g. } i(4\pi)^{\epsilon-2} \Gamma(1+\epsilon) \frac{(p^2/Q^2)^{-\epsilon}}{\epsilon} = \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} + \gamma_E - \log(p^2/Q^2) \right)$$

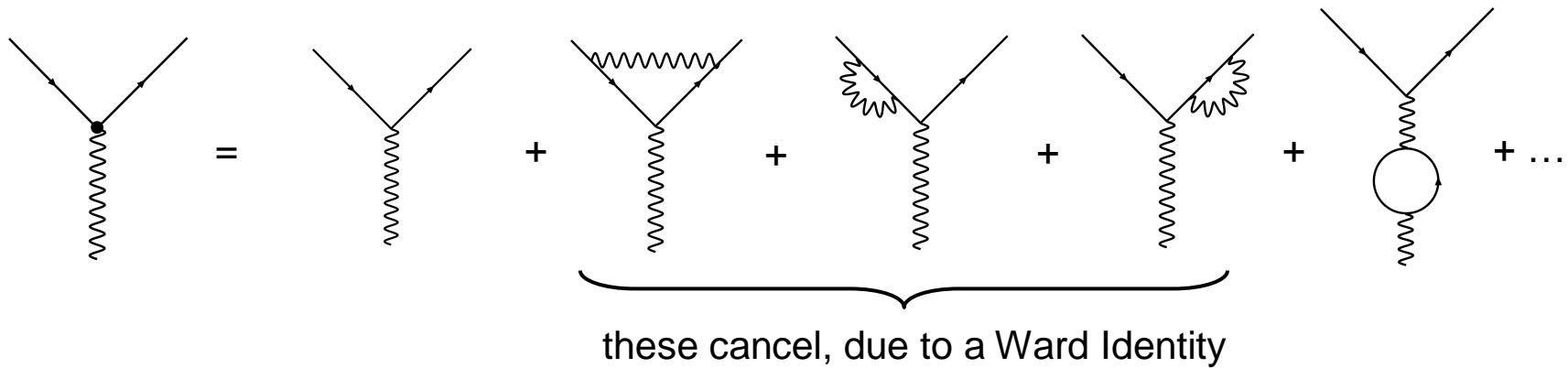
↑ [Euler-Mascheroni Constant]

This choice is known as **$\overline{\text{MS}}$**

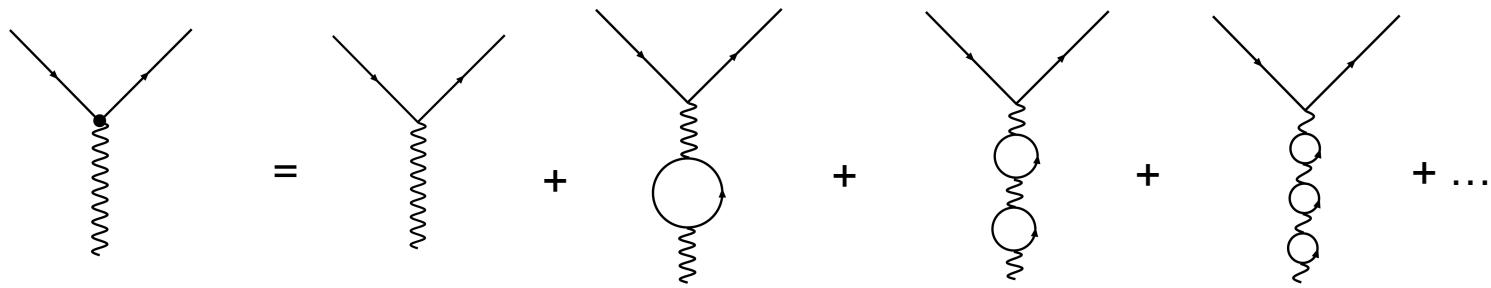
Also notice the **renormalization scale Q**.

Running couplings

How does the QED coupling e change with quantum corrections?



I can include some extra loops by....



Writing $I(Q) = \text{loop}$ this is $e(Q) = e_0 (1 + I(Q) + I^2(Q) + I^3(Q) + \dots)$

$$= e_0 \left(\frac{1}{1 - I(Q)} \right)$$

In terms of $\alpha \equiv \frac{e^2}{4\pi}$, we find

$$\alpha(Q) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \log(Q^2/\Lambda^2)}$$

cut-off

but since this was general, I could have chosen to evaluate my coupling at a different scale

e.g.

$$\alpha(\mu) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \log(\mu^2/\Lambda^2)}$$

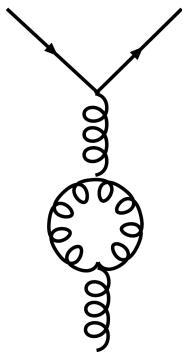
I can use this second equation to eliminate α_0 (which is infinite) from my first equation.

$$\alpha(Q) = \frac{\alpha(\mu)}{1 - \frac{\alpha(\mu)}{3\pi} \log(Q^2/\mu^2)}$$

The QED coupling changes with energy.

We can do the same thing for QCD, except we have some extra diagrams

e.g.



We find,

$$\alpha_s(Q) = \frac{\alpha(\mu)}{1 - \frac{\alpha_s(\mu)}{4\pi} \beta_0 \log(Q^2/\mu^2)}$$

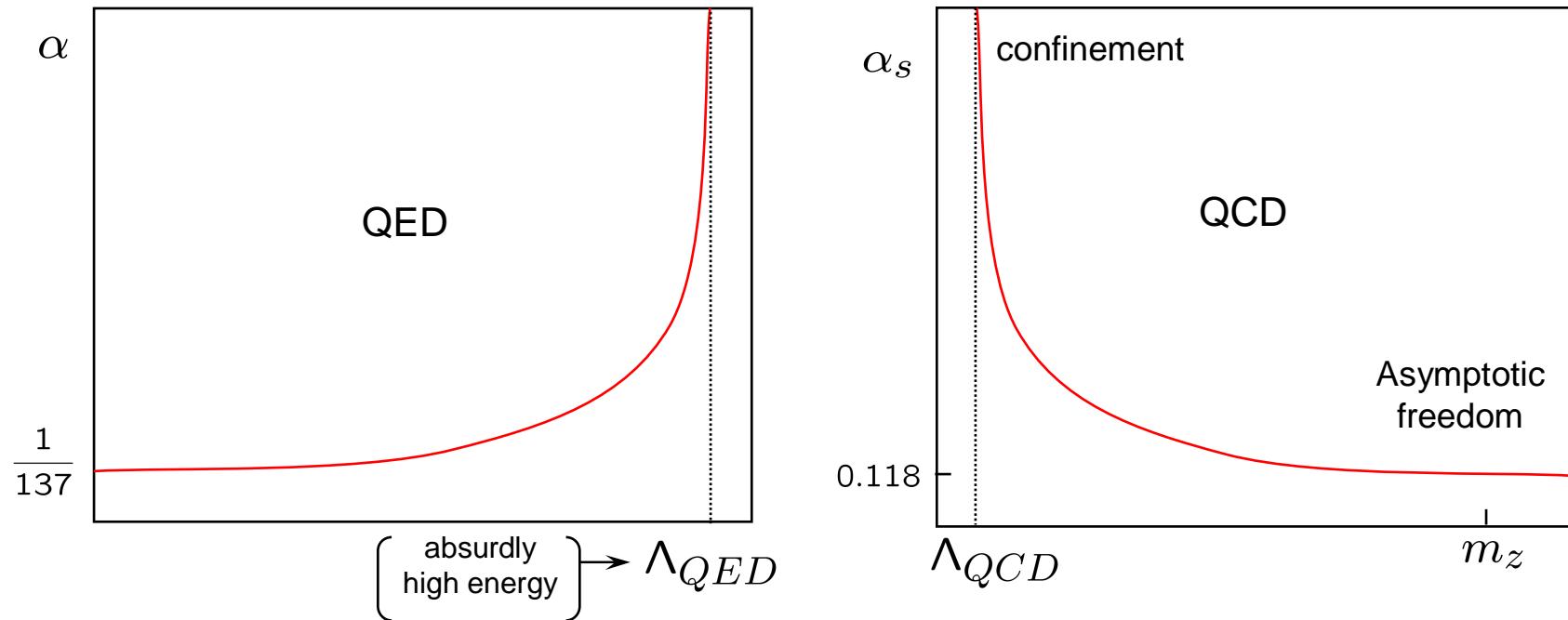
where $\beta_0 = \frac{11N_c - 2N_f}{3}$ $\left[\begin{array}{l} N_c = \# \text{ of colors} = 3 \\ N_f = \# \text{ of active flavors} \end{array} \right]$

At higher orders in perturbation theory we will have more contributions. The complete evolution of the coupling is described by the **beta function**

$$\beta(\alpha_s(Q^2)) = Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2}$$

$$\beta(\alpha_s) = -\beta_0 \alpha^2 + \mathcal{O}(\alpha^3)$$

For $N_f \leq 16$, the QCD and QED couplings run in the opposite direction.



At low energies QCD becomes strong enough to **confine** quarks inside hadrons.
 (The β function is not proof of this!)

At high energies QCD is **asymptotically free**, so we can use perturbation theory.