**Workshop Specification and Testing, Week 4**

This workshop is about working with sets and relations. Sets and lists are treated in chapter 4 of “The Haskell Road”. Ordered sets of pairs (HR, 4.5) are the basic ingredients of relations. Relations are treated in chapter 5 of “The Haskell Road”.

If the following exercises are difficult for you, you should study the relevant material (again).

If you still have difficulty with the set theoretic notation: read or reread chapter 4 of "The Haskell Road".

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Consider the following relations on the natural numbers. Check their properties. The *successor* relation on $\mathbb{N}$ is the relation given by $\{(n,m) \mid n+1=m\}$. The divisor relation on $\mathbb{N}$ is $\{(n,m) \mid n \text{ divides } m\}$. The *coprime* relation $C$ on $\mathbb{N}$ is given by $nCm :\equiv \mathrm{GCD}(n,m)=1$, i.e., the only factor of $n$ that divides $m$ is 1, and vice versa.

| | **$<$** | **$\leq$** | **successor** | **diviso r** | **coprime** |
|---|---|---|---|---|---|
| irreflexive | | | | | |
| reflexive | | | | | |
| asymmetric | | | | | |
| antisymmetric | | | | | |
| symmetric | | | | | |
| transitive | | | | | |
| linear | | | | | |

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Consider the relation

$$R = \{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$$

on the set $A = \{0,1,2,3,4\}$.

1. Determine $R^2$, $R^3$ and $R^4$.
2. Give a relation $S$ on $A$ such that $R \cup (S \circ R) = S$.

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The transitive closure of a relation $R$ is by definition the smallest transitive relation $S$ such that $R \subseteq S$. Notation: $R^+$

Consider again the relation

$$R = \{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$$

on the set $A = \{0,1,2,3,4\}$. What is the transitive closure of $R$?

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A binary relation $R$ is transitive iff $R \circ R \subseteq R$. You should check this!

Give an example of a transitive relation $R$ for which $R \circ R = R$ is false.

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The reflexive transitive closure of a relation $R$ is by definition the smallest transitive and reflexive relation $S$ such that $R \subseteq S$. Notation: $R^*$.

Give the reflexive transitive closure of the following relation:

$$R = \{(n, n+1) \mid n \in \mathbb{N}\}.$$

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The inverse of a relation $R$ is the relation $\{(y,x) \mid (x,y) \in R\}$. Notation $R^{-1}$ or $R^{\smile}$.

1. if $S$ is the successor relation on the natural numbers, what is $S^{\smile}$?
2. if $S$ is the successor relation on the natural numbers, what is $S \cup S^{\smile}$?
3. if $S$ is the successor relation on the natural numbers, what is $(S \cup S^{\smile})^*$?

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Suppose a relation $R$ satisfies $R^{\smile} \subseteq R$.

1. Does it follow from this that $R$ is reflexive?
2. Does it follow from this that $R$ is symmetric?

3. Does it follow from this that $R$ is transitive?

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1. Is $R \cup R^{\smile}$ symmetric for all relations $R$? Give a counterexample if your answer is negative.
2. Is $R^* \cup R^{\smile *}$ symmetric for all relations $R$? Give a counterexample if your answer is negative.
3. Is $R^* \cup R^{\smile *}$ transitive for all relations $R$? Give a counterexample if your answer is negative.
4. Is $(R \cup R^{\smile})^*$ an equivalence relation (reflexive, transitive and symmetric) for all relations $R$?