Workshop Specification and Testing, Week 5

Further Exercises with Induction

---

1. The *gossip problem* is the following. N people each have a secret. How many phone calls are needed to ensure that everyone knows all secrets? Assume that during any call, the two persons in the call share all secrets they know. Clearly, if there are two people, one call is enough. Similarly, if there are three people a, b, c, the following procedure works: a calls b, so both know the secrets of a and b. Next b calls c, so b knows all secrets and c knows the secrets of b, c. Finally c calls a. As a result a, c also know all secrets. So 3 calls are enough. Prove the following by induction: for $N \geq 4$ it holds that $2(N-2)$ calls are enough to ensure that everyone knows all secrets.

---

2. Prove by induction that the following definition of the Extended Euclidean algorithn is correct:

```
> fctGcd :: Integer -> Integer -> (Integer,Integer)
> fctGcd a b =
>   if b == 0
>   then (1,0)
>   else
>      let
>        (q,r) = quotRem a b
>        (s,t) = fctGcd b r
>      in (t, s - q*t)
```

Explanation: the extended algorithm is supposed to compute two integers x, y with the property that $xa + yb = \gcd(a, b)$, where a, b are the input to the function.

---

3. Prove with induction that $(1-\frac{1}{4})(1-\frac{1}{9})\cdots(1-\frac{1}{n^2}) = \frac{n+1}{2n}$.

---

4. In an earlier workshop we have seen the match removal game for two players. Start situation: a number of matches is on a stack. The players take turns. A move consists in removing 1, 2 or 3 matches from the stack. The player who can make the last move (the move that leaves the stack empty) has won the game. Suppose there are $4N$ matches on the stack, and the other player moves. Then if the player takes x matches, you should take $4-x$ matches for a sure win. Now consider the following variation: you can take 1, 3 or 4 matches. The player who takes the last match(es) wins. Prove by induction that now the configurations with $7N$ or $7N+2$ matches, with the other player moving, are sure wins. Explain what the winning strategy is.

---

5. Prove that every integer $N \geq 12$ can be written as $4X + 5Y$, with X, Y integers ≥ 0. (Application: if you have supplies of 4-packs and 5-packs of some item, then you can sell any number of items ≥ 12 without ever having to open a pack.)

---

6. Prove that every integer $N \geq 60$ can be written as $6X + 11Y$, with X, Y integers ≥ 0.

---

7. The Fibonacci numbers are given by the following recursion:

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0.$$

Prove with induction that for all $n > 1$: $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

---

8. The Tower of Hanoi is a tower of 8 disks of different sizes, stacked in order of decreasing size on a peg. Next to the tower, there are two more pegs. The task is to transfer the whole stack of disks to one of the other pegs (using the third peg as an auxiliary) while keeping to the following rules:

i. move only one disk at a time,

ii. never place a larger disk on top of a smaller one.

Answer the following questions.

- How many moves does it take to completely transfer a tower consisting of n disks?
- Prove by mathematical induction that your answer to the previous question is correct.
- How many moves does it take to completely transfer the tower of Hanoi?
