CME 241: Assignment 6 : Pablo Veyrat

Problem 1: • Let us assume the utility function is $U(x) = x - \frac{ax^2}{2}$ and $x \sim N(\mu, \sigma^2)$

• We want to compute $E(U(x)) = E(x) - \frac{a}{2} E(x^2)$ by linearity. And $V(x) = E(x^2) - E(x)^2$

So $E(x^2) = \sigma^2 + \mu^2$

Hence $E(U(x)) = \mu - a\frac{\sigma^2+\mu^2}{2}$

• Let us compute the certainty equivalent value x_{CE}

We have $U(x_{CE}) = E(U(x))$ by definition, so: $x_{CE} - \frac{a}{2} x_{CE}^2 = \mu - \frac{a}{2}(\sigma^2+\mu^2)$

We write this as a second degree polynomial:

$$-\frac{a}{2} x_{CE}^2 + x_{CE} + \frac{a}{2}(\sigma^2+\mu^2) - \mu = 0$$

$$\Delta = 1 + 4\frac{a}{2}\times\left(\frac{a}{2}(\sigma^2+\mu^2) - \mu\right) = 1 + a^2(\sigma^2+\mu^2) - 2a\mu > 0$$

The roots of this equation are therefore:

$$x_{CE,1} = \frac{-1-\sqrt{\Delta}}{-a} = \frac{1+\sqrt{\Delta}}{a} = \frac{1+\sqrt{1+a^2(\sigma^2+\mu^2) - 2a\mu}}{a}$$

The other root is most likely negative (depending on the values of a and μ)

So $x_{CE} = \frac{1+\sqrt{1+a^2(\sigma^2+\mu^2) - 2a\mu}}{a}$

• The absolute risk premium Π_A is defined by: $\Pi_A = \mu - x_{CE} = \mu - \frac{1+\sqrt{1+a^2(\sigma^2+\mu^2) - 2a\mu}}{a}$

• If we have to choose between a risky and a non risky asset, and denote by W the portfolio wealth, we have $W \sim N(1+(1-z)r + z\mu, z^2\sigma^2) = N(1+r+z(\mu-r), z^2\sigma^2)$

In this problem, we are seeking to maximize x_{CE} which is now a function of z.

$$x_{CE}(z) = \frac{1+\sqrt{1+a^2(z^2\sigma^2 + (1+(1-z)r+z\mu)^2) - 2a(1+(1-z)r+z\mu)}}{a}$$

This is thus equivalent to maximizing: $f(z) = a(z^2\sigma^2 + (1+(1-z)r+z\mu)^2) - 2 + 2(z-1)r - 2\mu$

$$f(z) = az^2\sigma^2 + a\left[z^2(\mu-r)^2 + (1+r)^2 + 2z(\mu-r)(1+r)\right] - 2(1+r) - 2z(\mu-r)$$

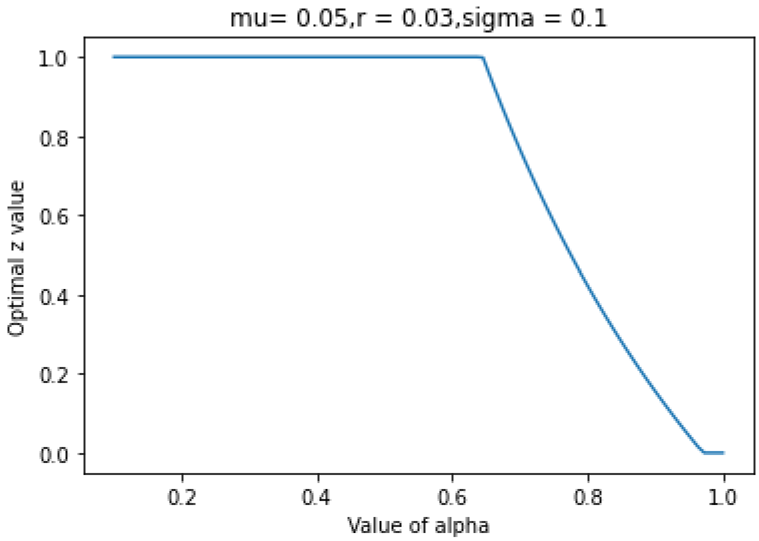
$$f'(z) = 2a\sigma^2 z + a\left[2z(\mu-r)^2 + 2(\mu-r)(1+r)\right] + 2(r-\mu)$$

We get that $f'(z) = 0$ when: $z(2a\sigma^2 + 2a(\mu-r)^2) = 2(\mu-r)(1-(1+r)a)$

The optimal value of z is then: $z = \frac{2(\mu-r)(1-(1+r)a)}{2a\sigma^2 + 2a(\mu-r)^2}$

The code for the plot can be found in the `assignment6_code.py` file

This gives us the following evolution for the optimal z as a function of α.



Problem 2:

Let us repeat the calculations for the portfolio application of CRRA with $U(x) = \log(x)$

If the random outcome is log normal, with $\log(x) \sim N(\mu, \sigma^2)$, we have:

$\cdot\ \mathbb{E}(U(x)) = \mu$

As x_{CE} is such that $U(x_{CE}) = \mathbb{E}(U(x))$

We get: $\log(x_{CE}) = \mu$, hence $x_{CE} = e^{\mu}$

And the relative risk premium is : $\pi_R = 1 - \frac{x_{CE}}{\bar{x}} = 1 - e^{-\frac{\sigma^2}{2}}$

In the portfolio application of lecture, we showed that $\log W \sim N(r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}, \pi^2\sigma^2)$

In this case, we hence get: $x_{CE} = e^{r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}}$

We are just trying to maximize : $f(\pi) = r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}$

We have $f'(\pi) = \mu - r - \sigma^2\pi$, hence $f'(\pi) = 0 \Rightarrow \pi^* = \frac{\mu - r}{\sigma^2}$: this is the optimal investment fraction in risky asset.

CME 241: Assignment 6: Problem 3:

- The two outcomes for wealth W at the end of our single bet of $f \cdot W_0$ are:
  - $W_0(1-f) + W_0 f(1+\alpha) = W_0(1+f\alpha)$
  - $W_0(1-f) + W_0 f(1-\beta) = W_0(1-f\beta)$

In terms of utility, it gives:
- $\log(W) = \log W_0 + \log(1+f\alpha)$ in the first case
- $\log(W) = \log W_0 + \log(1-f\beta)$ in the second case.

- Hence $E(\log(W)) = \log W_0 + p\log(1+f\alpha) + (1-p)\log(1-f\beta) = g(f)$

The derivative of $E(\log(W))$ with respect to f is:

$$g'(f) = p\frac{\alpha}{1+f\alpha} + (1-p)\frac{-\beta}{1-f\beta}$$

We get that: $g'(f) = 0 \Rightarrow$

$$p\alpha(1-f\beta) = (1-p)\beta(1+f\alpha)$$

$$p\alpha - fp\alpha\beta = (1-p)\beta + f\alpha(1-p)\beta$$

Hence

$$f[\alpha(1-p)\beta + p\alpha\beta] = p\alpha - (1-p)\beta$$

$$f\alpha\beta = p\alpha - (1-p)\beta$$

This gives us that $f^* = \frac{p}{\beta} - \frac{1-p}{\alpha}$: we need to verify if this f^* is a maximum.

Second part : Problem 3:

We have $g''(f) = -(1-p)\dfrac{\beta^2}{(1-f\beta)^2} - p\dfrac{\alpha^2}{(1+f\alpha)^2}$

This expression is negative for all f, especially for $f = f^*$

This is hence a maximum $f^* = \dfrac{p}{\beta} - \dfrac{(1-p)}{\alpha}$

This formula makes complete sense.

If we're in the case where $\alpha = \beta$, then $f^* = 2p - 1$: if $p = \frac{1}{2}$: there is no incentive to bet.

: if $p < \frac{1}{2}$: $f^* < 0$: then the game is meant to make us lose, we shouldn't play at all.

: if $p > \frac{1}{2}$, then $f^* > 0$ and we should bet a fraction of our gains at each step.

Note that if $p = 1$, then we're sure to win and we should bet everything we can in the first bet.

when $\alpha \neq \beta$: · Then the smaller α, the bigger p has to be to arrive in a situation where we can have $f^* > 0$.

· The smaller β, the smaller p "

It corresponds to common sense of how one would bet in the game.