CME241: Assignment 7: Problem 1 : Pablo Veyrat

Let us derive the solution to Merton's Portfolio problem for the case of the log(.) Utility function.

At any time t, we want to determine optimal $[\pi(t,W_t), c(t,W_t)]$ to maximize:

$$\mathbb{E}_t\left(\int_t^T e^{-\rho(s-t)}\log(c_s) + e^{-\rho(T-t)}B(T)\log(W_T) \mid W_t\right)$$: we will assume that $B(T) = \varepsilon^\gamma$ for $0<\varepsilon \ll 1$

we have $dW_t = ((\pi_t.(\mu-r)+r).W_t - c_t).dt + \pi_t\sigma W_t\, dz_t$

$V^*(t,W_t)$ satisfies a simple recursive formulation for $0\le t<t_1<T$: $V^*(t,W_t) = \max_{\pi,c} \mathbb{E}_t\left(\int_t^{t_1} e^{-\rho(s-t)}\log(c_s).ds + e^{-\rho(t_1-t)}V^*(t_1,W_{t_1})\right)$

The HJB formulation is:

$$\max_{\pi_t,c_t} \mathbb{E}_t\left(d(e^{-\rho t}.V^*(t,W_t)) + e^{-\rho t}\log(c_t)dt\right] = 0$$ which gives us:

$$\max_{\pi_t,c_t} \mathbb{E}_t\left(dV^*(t,W_t) + \log c_t\, dt\right) = \rho V^*(t,W_t)dt$$

With Ito's lemma on dV^*, we remove the dz_t term since it's a martingale and divide throughout by dt to produce the HJB Equation in PDE form:

$$\max_{\pi_t,c_t}\underbrace{\left(\frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W}((\pi_t(\mu-r)+r)W_t - c_t) + \frac{\partial^2 V^*}{\partial W^2}\cdot\frac{\pi_t^2\sigma^2 W_t^2}{2} + \log c_t\right]}_{\phi(t,W_t,\pi_t,c_t)} = \rho V^*(t,W_t)$$

Let us take the partial derivatives of ϕ with respect to π_t and c_t and equate to 0

- With respect to π_t: $\frac{\partial\phi}{\partial\pi_t} = 0 \Rightarrow \pi_t^* = -\frac{\partial V^*}{\partial W_t}.(\mu-r) \times \frac{1}{\frac{\partial^2 V^*}{\partial W_t^2}\sigma^2.W_t}$
- With respect to c_t: $\frac{\partial\phi}{\partial c_t} = 0 \Rightarrow c_t^* = \left(\frac{\partial V^*}{\partial W_t}\right)^{-1}$

We can now substitute the optimal values π_t^* and c_t^* in ϕ and equate to ρV^*. It gives us:

$$\frac{\partial V^*}{\partial t} - \frac{(\mu-r)^2}{2\sigma^2}\cdot\left(\frac{\partial V^*}{\partial W_t}\right)^2 \times \frac{1}{\frac{\partial^2 V^*}{\partial W_t^2}} + \frac{\partial V^*}{\partial W_t} r W_t - 1 + \log\left(\left(\frac{\partial V^*}{\partial W_t}\right)^{-1}\right) = \rho V^*$$

The boundary condition is $V^*(T,W_T) = \varepsilon^\gamma \log W_T$, and the second order conditions for ϕ are satisfied under some assumptions.

Let us assume the guess solution: $V^*(t,W_t) = f(t)\log W_t$, hence $\frac{\partial V^*}{\partial t} = f'(t)\log W_t$ and $\frac{\partial V^*}{\partial W_t} = \frac{f(t)}{W_t}$ and

$$\frac{\partial^2 V^*}{\partial W_t^2} = \frac{-f(t)}{W_t^2}$$

We can substitute the guess solution in the PDE we obtained:

$$f'(t)\log W_t - \frac{(\mu-r)^2}{2\sigma^2}\left(\frac{f(t)}{W_t}\right)^2 \times \frac{W_t^2}{-f(t)} + \frac{f(t)}{W_t} r W_t - 1 + \log\left(\frac{W_t}{f(t)}\right) = \rho f(t)\log W_t$$

this gives us:

$$f'(t)\log W_t + \frac{(\mu-r)^2}{2\sigma^2} f(t) + f(t) r - 1 + \log W_t - \log f(t) = \rho f(t) \log W_t$$

$$f'(t)\log W_t = f(t)\left[\rho \log W_t - \frac{(\mu-r)^2}{2\sigma^2} - r\right] + 1 - \log W_t + \log f(t)$$

We need to solve this differential equation to get an expression for f.

In the meantime, we get that: $\pi_t^* = -\frac{\partial V^*}{\partial W}(\mu - r) \cdot \frac{1}{\frac{\partial^2 V^*}{\partial W^2}\sigma^2 W_t} = -\frac{f(t)}{W_t}(\mu-r)\frac{W_t^2}{-f(t)} \times \frac{1}{\sigma^2 W_t} = \frac{\mu - r}{\sigma^2}$

And $c_t^* = \frac{W_t}{f(t)}$ where $f(t)$ is solution to the complex equation above

Let us find the change we need to make to the distribution to get the probability distribution of states at any time step. We observe that:

$$W_{t+1} = W_t(1 + \texttt{riskless\_rate}_t) + \texttt{alloc}_t(\texttt{risky\_rate}_t - \texttt{riskless\_rate}_t)$$

The random variables $Z_t = \texttt{alloc}_t(\texttt{risky\_rate}_t - \texttt{riskless\_rate}_t)$ are iid for each t.

Let us now consider the case where `riskless_rate` is a constant in time. Hence W_t is an autoregressive sequence of order 1.

This gives us that:

$$W_t = (1 + \texttt{riskless\_rate})^t W_0 + \sum_{i=0}^{t-1} (1 + \texttt{riskless\_rate})^i Z_{n-i}$$

This still does not give us a way to directly get the probability distribution of wealth at any time step, and hence to speed up the code. We would still have to process everything (and fetch the sample of the `risky_rate` and of the `alloc` at each time step) sequentially over all time steps till the final time step of interest.

CME241: Assignment 7: Problem 3:

Let us sketch out the states, actions, rewards and transition function for this problem.

Here a person is characterized by its employment state and its skill level:

If we decide to factor in the fact that we can consume earnings, we can also consider that the wealth at a given time is part of the state for the individual.

Here we will consider that wealth is not part of the state and that earnings cannot be consumed: the reward per unit of time is thus the utility of the number of dollars earned and the return at time t is the accumulated discounted reward.

Hence . state at time t is (t, s_t, K_t) where . s_t is skill level at time t

. K_t: is employment status at time t: $K_t = (U, E)$ where E stands for employed and U for unemployed.

. The action at time t is: . if the person is employed: α_t: the person must choose the fraction of time it spends learning or working. If the person is unemployed, it has no choice to make

. The reward per unit of time: . if the person is unemployed: the reward is null, the skill level evolves as $ds_t = -\lambda s dt$ and you can get a job with probability $h(s)$

. if the person is employed: it earns in a unit of time:

. $dW_t = \alpha_t f(s) dt$, so per minute worked the reward is $U(\alpha_t f(s))$

. The skill level evolves as: $ds_t = (1-\alpha_t) g(s) dt$.

. You can lose your job with probability p.

. The return at time t is thus: $\int_t^T e^{-\rho(k-t)} \left(U(\alpha_k f(s_k)) \mathbb{1}(K_k = E) \right) dk$: we're considering here the finite horizon case. There may be an additional bequest function.

With an infinite horizon the return at time t is $\int_t^{+\infty} e^{-\rho(k-t)} U(s_k f(s_k)) \mathbb{1}(K_k = E)\, dk$

Every day you can then transition from E to U with probability p and from U to E with probability $h(s)$

If you are employed, you can transition from s_t to s_{t+1} (where t is in minutes according to):

$$s_{t+1} = s_t + (1-\alpha_t\, g(s))$$

If you're unemployed $ds_t = -\lambda s dt$.

If there are multiple jobs, we're back to a case similar to the one of problem 3 of assignment 4 except that we need to take into account the skill level.