You will find the code for this question in the `RL-book/Assignment2/assignment9_code.py` file.

CME 241: Assignment 9: Problem 2:

Let us derive the expressions for the Optimal Value Function and Optimal Policy for the LPT price impact model.

We have: $V_t^\pi((P_t, R_t)) = \mathbb{E}_\pi\left(\sum_{i=t}^{T} N_i Q_i \mid (P_t, R_t)\right) = \mathbb{E}_\pi\left(\sum_{i=t}^{T} N_i (P_i (1 - \beta N_i - \theta X_i)) \mid (P_i, R_i)\right)$

With the Bellman Equation, we get that: $V_t^*((P_t, R_t)) = \max_{N_t} \{ N_t Q_t + \mathbb{E}(V_{t+1}^*((P_{t+1}, R_{t+1}))) \}$ assuming $\gamma = 1$

And $V_{T-1}^*((P_{T-1}, R_{T-1})) = N_{T-1} P_{T-1} (1 - \beta N_{T-1} - \theta X_{T-1}) = R_{T-1} Q_{T-1}$

From this, we can infer $V_{T-2}^*((P_{T-2}, R_{T-2})) = \max_{N_{T-2}} \left(N_{T-2} P_{T-2}(1 - \beta N_{T-2} - \theta X_{T-2}) + \mathbb{E}(R_{T-1} P_{T-1}(1 - \beta N_{T-1} - \theta X_{T-1})\right)$

Let us compute: $a = \mathbb{E}(R_{T-1} P_{T-1} (1 - \beta N_{T-1} - \theta X_{T-1}) \mid (P_{T-2}, R_{T-2}))$

$R_{T-1} = R_{T-2} - N_{T-2}$ and $P_{T-1} = P_{T-2} e^{Z_{T-2}}$, also $X_{T-1} = \rho X_{T-2} + \eta_{T-2}$

Hence $a = \mathbb{E}\left((R_{T-2} - N_{T-2}) P_{T-2} e^{Z_{T-2}} (1 - \beta R_{T-2} + \beta N_{T-2} - \theta \rho X_{T-2} + \theta \eta_{T-2})\right)$

$$= (R_{T-2} - N_{T-2}) P_{T-2} \left[(1 - \beta R_{T-2} + \beta N_{T-2} - \theta \rho X_{T-2}) \underbrace{\mathbb{E}(e^{Z_{T-2}})}_{= \exp(\mu_Z + \frac{\sigma_Z^2}{2})} + \theta \underbrace{\mathbb{E}(e^{Z_{T-2}} \eta_{T-2})}_{= \exp(\mu_Z + \frac{\sigma_Z^2}{2}) \times 0 \text{ by independence}}\right]$$

This gives us:

$$a = (R_{T-2} - N_{T-2}) P_{T-2} (1 - \beta(R_{T-2} - N_{T-2}) - \theta \rho X_{T-2}) \exp\left(\mu_Z + \frac{\sigma_Z^2}{2}\right)$$

If we go back to: $V_{T-2}^*((P_{T-2}, R_{T-2})) = \max_{N_{T-2}} \left(N_{T-2} P_{T-2}(1 - \beta N_{T-2} - \theta X_{T-2}) + (R_{T-2} - N_{T-2}) P_{T-2}(1 - \beta(R_{T-2} - N_{T-2}) - \theta\rho X_{T-2}) e^{\mu_Z + \frac{\sigma_Z^2}{2}}\right)$

Let us derive this expression with respect to N_{T-2} and set it to 0

$$\cancel{P_{T-2}}(1 - \beta N_{T-2} - \theta X_{T-2}) - \beta N_{T-2} \cancel{P_{T-2}} - \cancel{P_{T-2}}(1 - \beta(R_{T-2} - N_{T-2}) - \theta \rho X_{T-2}) e^{\mu_Z + \frac{\sigma_Z^2}{2}} + \beta (R_{T-2} - N_{T-2}) \cancel{P_{T-2}} e^{\mu_Z + \frac{\sigma_Z^2}{2}} = 0$$

$$1 - 2\beta N_{T-2} - \theta X_{T-2} + (-1 + \beta R_{T-2} - \beta N_{T-2} - \theta \rho X_{T-2}) e^{\mu_Z + \frac{\sigma_Z^2}{2}} + \beta R_{T-2} e^{\mu_Z + \frac{\sigma_Z^2}{2}} - \beta N_{T-2} e^{\mu_Z + \frac{\sigma_Z^2}{2}} = 0$$

$$1 - e^{\mu_Z + \frac{\sigma_Z^2}{2}} + N_{T-2}\left(-2\beta - 2\beta e^{\mu_Z + \frac{\sigma_Z^2}{2}}\right) + X_{T-2}\left(-\theta - \theta \rho e^{\mu_Z + \frac{\sigma_Z^2}{2}}\right) + R_{T-2} \cdot 2\beta e^{\mu_Z + \frac{\sigma_Z^2}{2}} = 0$$

$$1 - e^{\mu_Z + \frac{\sigma_Z^2}{2}} - 2\beta N_{T-2}\left(1 + e^{\mu_Z + \frac{\sigma_Z^2}{2}}\right) - \theta X_{T-2}\left(1 + \rho e^{\mu_Z + \frac{\sigma_Z^2}{2}}\right) + R_{T-2} 2\beta e^{\mu_Z + \frac{\sigma_Z^2}{2}} = 0$$

This gives us that:

$$N_{T-2}^* = \frac{R_{T-2} e^{\mu_Z + \frac{\sigma_Z^2}{2}}}{1 + e^{\mu_Z + \frac{\sigma_Z^2}{2}}} + \frac{e^{\mu_Z - \frac{\sigma_Z^2}{2}} - 1}{2\beta(1 + e^{\mu_Z + \frac{\sigma_Z^2}{2}})} - \frac{\theta X_{T-2}}{2\beta} \times \frac{1 + \rho e^{\mu_Z + \frac{\sigma_Z^2}{2}}}{1 + e^{\mu_Z + \frac{\sigma_Z^2}{2}}}$$

Continuing backwards gives us a similar solution for all t in $[0, T]$

$$N_t^* = \frac{R_t e^{\mu_Z + \frac{\sigma_Z^2}{2}}}{1 + e^{\mu_Z + \frac{\sigma_Z^2}{2}}} + \frac{e^{\mu_Z - \frac{\sigma_Z^2}{2}} - 1}{2\beta(1 + e^{\mu_Z + \frac{\sigma_Z^2}{2}})} - \frac{\theta X_t}{2\beta} \frac{1 + \rho e^{\mu_Z + \frac{\sigma_Z^2}{2}}}{1 + e^{\mu_Z + \frac{\sigma_Z^2}{2}}}$$

The solution is of the form: $N_t^* = c_t^{(1)} + c_t^{(2)} R_t + c_t^{(3)} X_t$, and plugging this in the optimal value function, we get: $V^*(t, P_t, R_t, X_t) = e^{\mu_Z + \frac{\sigma_Z^2}{2}} \cdot P_t \cdot (c_t^{(4)} + c_t^{(5)} R_t + c_t^{(6)} X_t + c_t^{(7)} R_t^2 + c_t^{(8)} X_t^2 + c_t^{(9)} R_t X_t)$

You will find the corresponding code in the `RL-book/Assignment2/assignment9_code.py` file.