CME 241 : Assignment 4 : Problem 1

Let us determine the optimal deterministic policy with the value iteration algorithm.

We have $V_0 \in \mathbb{R}^2$ and $V_0 = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$

k=1 : Let us compute $q_1(s_1,a_1)$, $q_1(s_1,a_2)$, $q_1(s_2,a_1)$ and $q_1(s_2,a_2)$ for $\gamma = 1$.

We have $q_1(s_1,a_1) = R(s_1,a_1) + P(s_1,a_1,s_1)V_0(s_1) + P(s_1,a_1,s_2)V_0(s_2)$
$= 8 + 0.2 \times 10 + 0.6 \times 1 = 10.6$.

Similarly $q_1(s_1,a_2) = 10 + 0.1 \times 10 + 0.2 \times 1 = 11.2$.

As $V_1(s_1) = \max_{a \in A} \{q_1(s_1,a)\}$, we get $V_1(s_1) = 11.2$ and $\pi_1(s_1) = a_2$

Besides: $q_1(s_2,a_1) = 1 + 0.3 \times 10 + 0.3 \times 1 = 4.3$
$q_1(s_2,a_2) = -1 + 0.5 \times 10 + 0.3 \times 1 = 4.3$
, this gives us $V_1(s_2) = 4.3$ and the optimal action to take is either a_1 or a_2 in this case.

k=2 : We have $V_1 = \begin{pmatrix} 11.2 \\ 4.3 \end{pmatrix}$

$q_2(s_1,a_1) = 8 + 0.2 \times 11.2 + 0.6 \times 4.3 = 12.82$
$q_2(s_1,a_2) = 10 + 0.1 \times 11.2 + 0.2 \times 4.3 = 11.98$
$q_2(s_2,a_1) = 1 + 0.3 \times 11.2 + 0.3 \times 4.3 = 5.65$
$q_2(s_2,a_2) = -1 + 0.5 \times 11.2 + 0.3 \times 4.3 = 5.89$

$\rightarrow$ This gives us that $V_2(s_1) = 12.82$ and $V_2(s_2) = 5.89$ and $\pi_2(s_1) = a_1$ and $\pi_2(s_2) = a_2$

Let us show that $\pi_k(\cdot)$ for $k > 2$ will be the same as π_2

For $k > 2$, we indeed have: $q_k(s_1,a_1) = 8 + 0.2\, V_{k-1}(s_1) + 0.6\, V_{k-1}(s_2)$
$q_k(s_1,a_2) = 10 + 0.1\, V_{k-1}(s_1) + 0.2\, V_{k-1}(s_2)$
$q_k(s_2,a_1) = 1 + 0.3\, V_{k-1}(s_1) + 0.3\, V_{k-1}(s_2)$
$q_k(s_2,a_2) = -1 + 0.5\, V_{k-1}(s_1) + 0.3\, V_{k-1}(s_2)$

In particular: $q_k(s_1,a_1) - q_k(s_1,a_2) = -2 + 0.1\, V_{k-1}(s_1) + 0.4\, V_{k-1}(s_2)$

As $V_k(s_1) = \max_{a \in A} q_k(s_1,a)$, we have $V_k(s_1) \geq 8 + 0.2\, V_{k-1}(s_1) + 0.6\, V_{k-1}(s_2)$

Similarly: $V_k(s_2) \geq -1 + 0.5\, V_{k-1}(s_1) + 0.3\, V_{k-1}(s_2)$

In particular for $k=3$: $V_3(s_1) \geq 8 + 0.2 \times 12.82 + 0.6 \times 5.89 = 14.098$
$V_3(s_2) \geq -1 + 0.5 \times 12.82 + 0.3 \times 5.89 = 7.177$

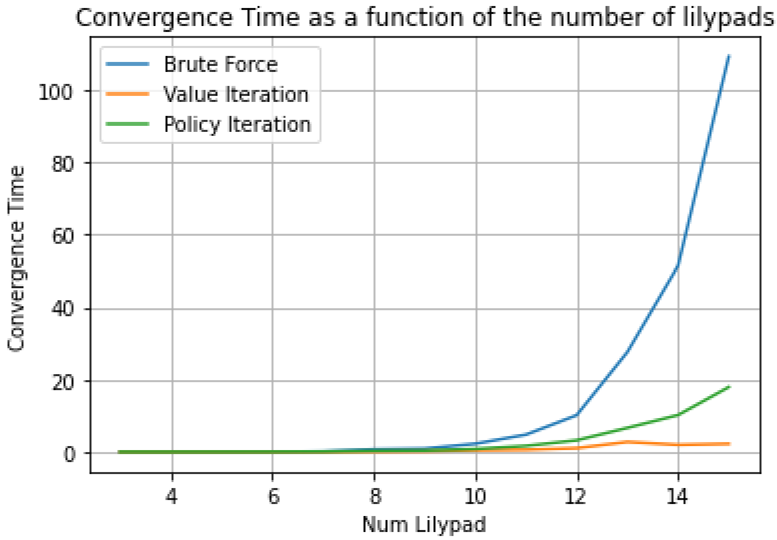
By induction, we can easily get: $\forall k \geq 2$: $V_k(s_1) \geq 10$ and $V_k(s_2) \geq 5$.

Hence $q_k(s_1,a_1) - q_k(s_1,a_2) \geq -2 + 0.1 \times 10 + 0.3 \times 5 > 0$

Similarly $q_k(s_2,a_2) - q_k(s_2,a_1) \geq -2 + 0.2\, V_{k-1}(s_1) = -2 + 0.2 \cdot 10 = 0$

. This gives us: $\forall k > 2$ $\underline{\pi_k(s_1) = a_1}$ and $\underline{\pi_k(s_2) = a_2}$

We plotted the graph of the convergence speed for different values of n the number of lilypads using the code in `assignment4_code.py`. This gave us the following plot:



Problem 3:

Let us write the spaces and functions for this problem.

Here a person is characterized by its employment state and the job offered or employed which can be referred to by an index.

Hence $S = \{E, U\} \times \{i \mid 1 \leq i \leq n\}$ where i is the index of the job, E stands for employed and U for unemployed

When someone is employed, it has no choice to make, and when someone is unemployed it has two actions to choose : Accept (A) or Decline (D)

The action space is $A = \{A, D\}$ but D is impossible when $s = (E, i)$ for any i.

Let us write the transition function. Let $s = (J, i)$ be the current state and $s' = (J', i')$ be the transition state:

$$\cdot\ \mathbb{P}(s' \mid s, A) = \begin{cases} 1-\alpha & \text{when } i' = i,\ J \in \{U, E\} \text{ and } J' = E \\ \alpha p_{i'} & \text{for each } 1 \leq i' \leq n,\ J \in \{U, E\} \text{ and } J' = U \\ 0 & \text{otherwise} \end{cases}$$

(situation where you lose your job but receive offer i' after that)

$$\text{And}: \mathbb{P}(s' \mid s, D) = \begin{cases} p_{i'} & \text{for each } 1 \leq i' \leq n \text{ and } J' = U \\ 0 & \text{otherwise} \end{cases}$$

: in this case s is of the form (U, i): you can only choose to decline when you are already unemployed.

Let us define the reward function: it is the function that defines the expected reward when taking action $a \in A$ in state $s \in S$. We call it $R(s, a) = R((J, i), a)$

We have: $\underline{\cdot\ R(s, A) = U(w_i) = \log(w_i)}$ for each s of the form (J, i) where $1 \leq i \leq n$ and $J \in \{U, E\}$

$\underline{\cdot\ R(s, D) = U(w_0) = \log(w_0)}$ for each s of the form (U, i) where $1 \leq i \leq n$.

Let us write the Bellman Optimality Equation customized for this MDP.

When you are employed, you can only take one action which is: remain employed.

$$\text{Hence } \cdot\ V^*((E, i)) = \underbrace{R((E, i), A)}_{U(w_i)} + \gamma\left[(1-\alpha) V^*((E, i)) + \alpha \sum_{i'=1}^{n} p_{i'} V^*((U, i'))\right]$$

When you are unemployed, you can choose to accept your offer or decline it.

$$: \text{Hence}: V^*((U, i)) = \max\left(\underbrace{V^*((E, i))}_{\text{you progress like the employed day}},\ U(w_0) + \gamma \sum_{i'=1}^{n} p_{i'} V^*((U, i'))\right)$$

This means that job offer i will be accepted only if: $\underline{V^*((E, i)) \geq U(w_0) + \gamma \sum_{i'=1}^{n} p_{i'} V^*((U, i'))}$

We can simplify this problem. For the moment $V^* \in \mathbb{R}^{2n}$, but note that if we let $V_0 = \sum_{i'=1}^{n} p_{i'} V^*((U, i'))$, we get: $V_0 = \sum_{i=1}^{n} p_{i'} \max(U(w_0) + \gamma V_0, V_i)$ and $\forall i \in [\![1, n]\!] : V_i = U(w_i) + \gamma(\alpha V_0 + (1-\alpha) V_i)$ where $V_i = V^*((E, i))$

The code to solve this Bellman Optimality Equation with a numerical iterative algorithm has been written in the `assignment4_code.py` file.

In our expressions of the value function, we used the simplification derived above (our value function is thus of dimension $n+1$ instead of $2 \times n$, and the optimal policy only involves n choices to make: the choices you need to make when you are unemployed and receive a job offer of one of the n jobs.