CME241: Assignment 8: Problem 1

Let us model an MDP so that we can run the bank in the most optimal manner, maximizing the expected utility of assets less liabilities at the end of a T day horizon.

There are three sources of randomness in this problem:

- each day, customers deposit a random amount of money
- each day, customers withdraw a random amount of money
- The returns of the investment in the risky asset are random.

Since we assumed that the deposit rate customers earn can be ignored, at each time step the value of assets less liabilities in this problem is going to be equal to the sum of the returns of the investments on the risky asset minus the sum of the penalties paid to the regulator minus the sum of the returns paid on the borrowings to other banks.

Let

- W_t be the value of assets less liabilities at the end of period t.
- α_t be the amount in risky asset at the beginning of day t, and r_t the random return provided by α_t during day t. Let c_t be the amount available to customers at the start of day t and β_t the cash amount borrowed.

We have $\underline{W_t = \sum_{i=1}^{t} r_i \alpha_i - \sum_{i=1}^{t} P(c_t) - \sum_{i=1}^{t} R \beta_t}$ where $P(c_t) = \begin{cases} K\cot\left(\frac{\pi \cdot c_t}{2C}\right) & \text{if } c_t \leq C \\ 0 & \text{otherwise.} \end{cases}$

The states in this problem are (t, W_t, d_t) where d_t is the money available to the bank at the end of day t.

Basically $d_t = c_t - P(c_t) + (1+r_t)\alpha_t + \text{deposit}_t - \text{withdrawal}_t - (1+R)\beta_t$ where deposit_t is the amount of money deposited during day t.

We have: $c_{t+1} = \underbrace{d_t}_{\text{money from customers}} + \underbrace{\beta_{t+1}}_{\text{money borrowed next time step}} - \underbrace{\alpha_{t+1}}_{\text{money on risky investment}}$ where $c_{t+1} > 0$

We could even restrict the state space to (t, d_t).

The actions at each time step are the choice of (α_t, β_t) depending on the total money we have available. The choice of α_t, β_t according to d_{t-1} then fixes the choice of c_t.

The transition equation is then as follows:

$\underline{d_t = c_t - P(c_t) + \text{deposit}_t - \text{withdrawal}_t - (1+R)\beta_t + (1+r_t)\alpha_t}$: r_t, deposit_t and withdrawal_t are random quantities.

And $c_{t+1} = d_t + \beta_{t+1} - \alpha_{t+1}$

The expression for the rewards (assuming $\gamma = 1$) is that of W_t: the return is $\mathbb{E}(U(W_T))$.

Given all the simplifying hypothesis that we have made, if we suppose in addition a normal distribution of asset return and of the amounts withdrawn and deposited, and a constant relative risk-aversion: then

this problem might be tractable using dynamic programming.
However, as these assumptions are never really satisfied in real life. Besides, the state space is here continuous: we couldn't use the tabular representation of MDP or of the Value Function: we would need to resort to approximation of the value function and thus to ADP.

CME241: Assignment 8: Problem 2:

Let us find the optimal supply that minimizes the expected cost.

$$g(S) = p\int_S^{+\infty}(x-S)\,f(x)dx + h\int_{-\infty}^{S}(S-x)\,f(x)dx$$

$$g(S) = -p\int_{+\infty}^{S} x f(x)dx + pS\int_{+\infty}^{S} f(x)dx + hS\int_{-\infty}^{S} f(x)dx - h\int_{-\infty}^{S} x f(x)dx$$

To find S that minimizes g, we compute $g'(S)$:

$$g'(S) = -pSf(S) + pSf(S) + p\int_{+\infty}^{S} f(x)dx + hSf(S) + h\int_{-\infty}^{S} f(x)dx - hSf(S)$$

$$= p\int_{+\infty}^{S} f(x)dx + h\int_{-\infty}^{S} f(x)dx$$ and $\cdot \int_{-\infty}^{S} f(x)dx = 1 + \int_{+\infty}^{S} f(x)dx$ as f is a density of probability

$$= p\,(F(S)-1) + h\,F(S)$$ $\cdot \int_{-\infty}^{S} f(x)dx = F(S)$ where F is the cdf of x

We get that $g'(S) = 0$ when: $F(S) = \frac{p}{p+h}$: <u>S should be the $\frac{p}{p+h}$'th quantile of the distribution of x</u>

This problem is equivalent to a call/put options portfolio problem.

Indeed $g_1(S)$ is the expected payoff of a put option with strike price S where x denotes the price of the underlying asset at the expiration of the option.

$g_2(S)$ is the expected payoff of a call option with strike price S.

g_1 is thus the cost at which the option should be sold if we consider that the price should be equal to the expected payoff of the option. The same goes for g_2.

Hence, finding the optimal S to minimize the expected cost is equivalent to finding the optimal strike price for someone issuing p puts and h calls associated to the same strike price to minimize the expected payoffs to people buying these options: <u>it is a call/put options portfolio problem.</u>

This is also equivalent to the problem of class of portfolio optimization where we have a position in 2 derivatives that expire after one time step, and g(S) is the portfolio-aggregated contingent cashflow after one step. p and h correspond to hedge positions we need to adapt at each time step yielding transaction costs and PnL at each time step.

Solving this problem like we did in class for a general S could help us find the optimal actions (choice of p and h) at each time step, and we could choose for the S (when buying the options in beginning) yielding the optimal value function.

The code for this problem can be found in the `assignment8_code.py` file.

As written in the comments of this exercise, we tested two different solutions for this problem.