Chapter 4(1) Probability

4(1).01 Introduction

In our daily life very often we use the term 'Probability'. Probability or a tendency 'uncertainity' or 'chance' refers to the probable movements or to occuring an event. Everyday, we express our thinkings using the sentences like :

(i) Everyone who lives will must die.

(ii) We cannot live without breathing.

(iii) It may rain today.

The idea of probability is expressed. In the above three sentences. In the first sentence we see a must, that is a certain incident or event. Every certain event is an obvious proclaimation of probability. Here value of the probability is 1.

In the second sentence, the probability of living without breathing is zero. Now if we consider the third sentence, we realize that there lies a probability to rain today. Here we do not know the exact probability but the probability lies between 0 to 1.

Example : If we throw a coin or a die then it will fall down. After it's falling down it will show one of its two sides for coin. And for die, it will show one of it's six sides. For coin, the two sides are head and tail. Therefore, the probability of head (or tail) is $\frac{1}{2}$.

For die the six sides are one dot, two dot, three dot, four dot, five dot and six dot. Therefore, the probability of having one-dot or two dot or three dot, or four dot or five dot or six dot is $\frac{1}{6}$.

4(1).02 Set Theory

The concept of set theory is verity needed for the development and understanding of probability theory. It is so important that now -a-days set theory is considered as a fundamental tool of probability theory. Here we are providing a brief concept of set theory.

Definition of set : A set is a well defined collection of objects or elements having certain specified properties. Each object comprising a set is called element (or point or member) of the set.

Usually a set is denoted by capital (or upper case) letter and the elements are placed within a braces or second bracket.

Example : The vowels of English alphabet.

Description of a set : A set may be described by three different approaches. These approaches are described below :

(a) **Tabular Form :**

We may list the elements of the set within a braces or second bracket,

e.g., $A = \{1, 3, 5, 7, 9\}$

(b) **By description :** We may describe all the elements of any set A in words, e.g., A = {All the odd numbers less than 10}

(c) **By Notation** : By using natation we may also describe a set,

e.g., $A = \{x \mid x \text{ is an odd number} < 10]$

Finite set : A set is called a finite set if it contains a finite number of elements.

Examples (1) : Letters in Bengali alphabet.

(2) The vowels of English alphabet.

Infinite set : A set is known as an infinite set if it contains an infinite number of elements.

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Example : $A = \{x \mid x > 100\}$
It is read as : A is a set of all the number greater than 100 and '|' is read as such that.

Universal set (space Universe) :
The arrangement of all possible objects under contain consideration in a given experiment (or discussion) is called universal set or space universe. A universal set is denoted by U or Ω (Omega).

Example : In the experiment of throwing a die our possible outcomes are 1, 2, 3, 4, 5 and 6. Therefore universal set in this experiment, $U = \{1, 2, 3, 4, 5, 6\}$.

Subset : If every element of a set A is also an element of the set B then A is defined to be a subset of B and is denoted by $A \subset B$ or $B \supset A$ and we read as 'A is contained in B' or 'B contains A'.

Example : Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4, 5, 6\}$

then A is called a subset of B i.e., $A \subset B$

Proper Subset : A set 'A' is said to be a proper subset of another set B if A is a subset of B but B is not a subset of A. Generally, we write $A \subseteq B$ to indicate A is a proper subset of B,

Example : If $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Then A is a proper subset of B because A is a subset of B but B is not that of A. Therefore, $A \subseteq B$.

Equivalent or Equal sets :
Two sets A and B are said to be equivalent or equal sets or equility of sets if they contain exactly the same elements,i.e.,$A \subset B$ and $B \subset A$. They may be written as $A = B$.

Example : If $A = \{1, 4, 7\}$ and $B = \{4, 7, 1\}$ then A and B are called equal sets and be written as $A = B$.

Empty (or Null) set :
A set which contains no element is called an empty set or null set. It is denoted by ϕ read as 'phi') and is also known as ϕ -set. The empty set is a sub set of any set.

Example : $A = \{x|x \text{ is a man having 3 eyes}\}$

Family of sets : A set whose members are also set themselves individually is called a class or family of sets. Usually, it is denoted by script letters A, B, C etc.

Example : $A = \{\{1\}, \{3\}, \{1,3\}\}$

Class of subsets : The set of all possible subsets of a given set is known as the class of subsets or power set of the given set.

Example : Let $S = \{2, 4\}$ then the power set $P(S) = [\{\}, \{2\}, \{4\}, \{2, 4\}]$

If a set contains n elements then there shall be 2^n number of subsets in the power set.

Venn Diagram : The diagram by which we can show the relationships between subsets and the corresponding universal set is called venn-diagram. It was introduced first by an English logician John Venn.

A Venn diagram is a device which represents a set as a portion of a plane. It provides visual portrayed of various relationships among the sets. It is drawn as where the rectangle represents the universal set and the circles represent the other sets.

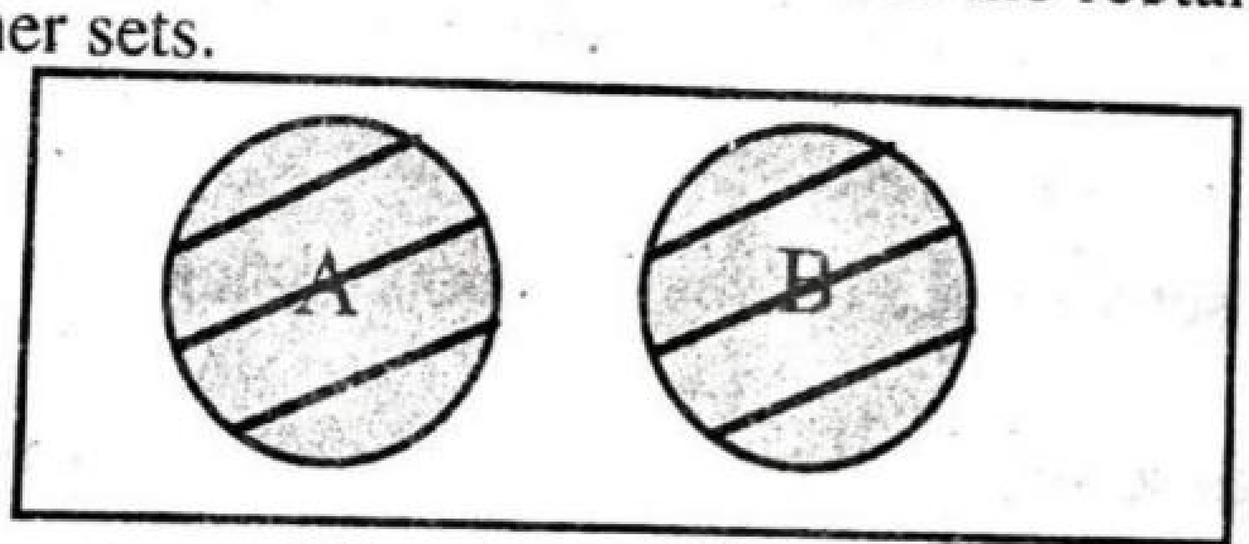

Fig: Venn diagram.

Complement of a set : If a set A is a subset of the universal set U then the complement of A with respect to U is the all elements that are in U set not in A. The complement of a set A is denoted by $\overline{A}$ or A^c or A'.

Example : If $A = \{1, 2, 3, 4, 5\}$ and $U = \{1, 2, 3, 4, 5, 6, 7\}$ then $A^c = \{6, 7\}$.

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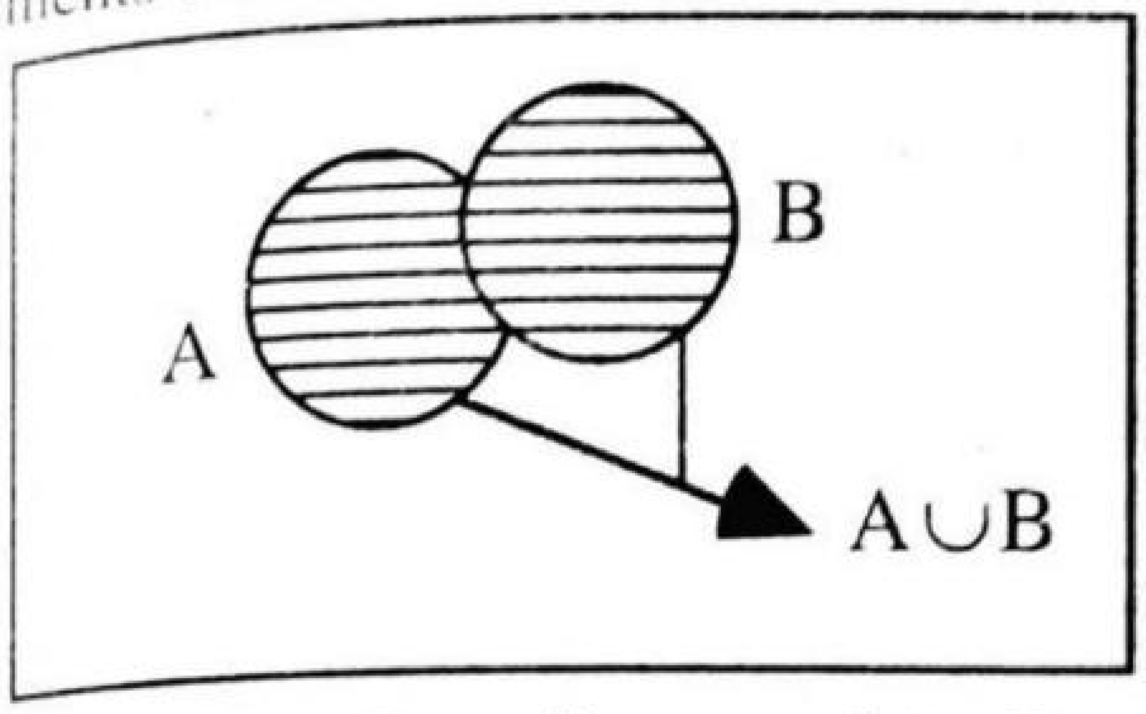
4(1).03 Set Operations :

There are a number of set operation by which we may have new sets with two or more given sets. But here, our discussion is only on binary operations, i.e., the operations in where we deal with only two given sets. The operations we shall consider here, are union, intersection and difference of two given sets.

Union of sets : If A and B be two sets then the union of A and B is a set that contains all the elements which are in A or B or in both. The union of A and B is denoted as $A \cup B$.

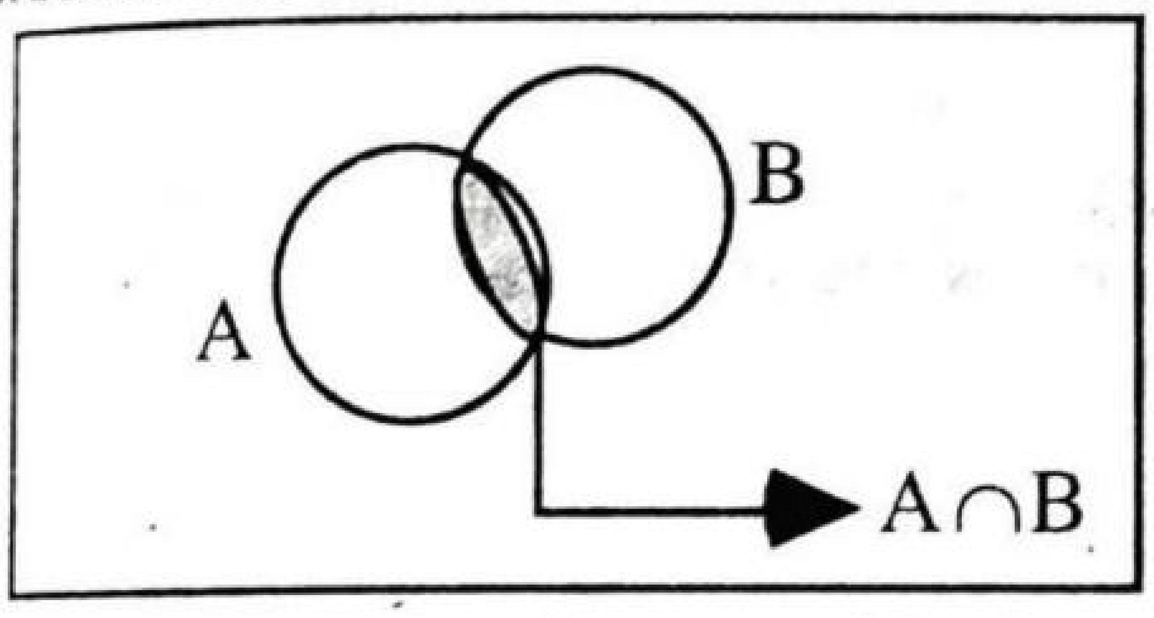


Here, the shaded area is $A \cup B$.

Fig. Venn diagram of $A \cup B$.

Example : If $A = \{1, 3, 5\}$, $B = \{2, 4\}$ then $A \cup B = \{1, 2, 3, 4, 5\}$

Intersection of sets : If A and B be any two sets then the intersection of A and B is another set whose elements belong to both A and B. That is, the common elements of A and B constitute the intersection of A and B. The intersection of two sets A and B is expressed as $A \cap B$ or AB.

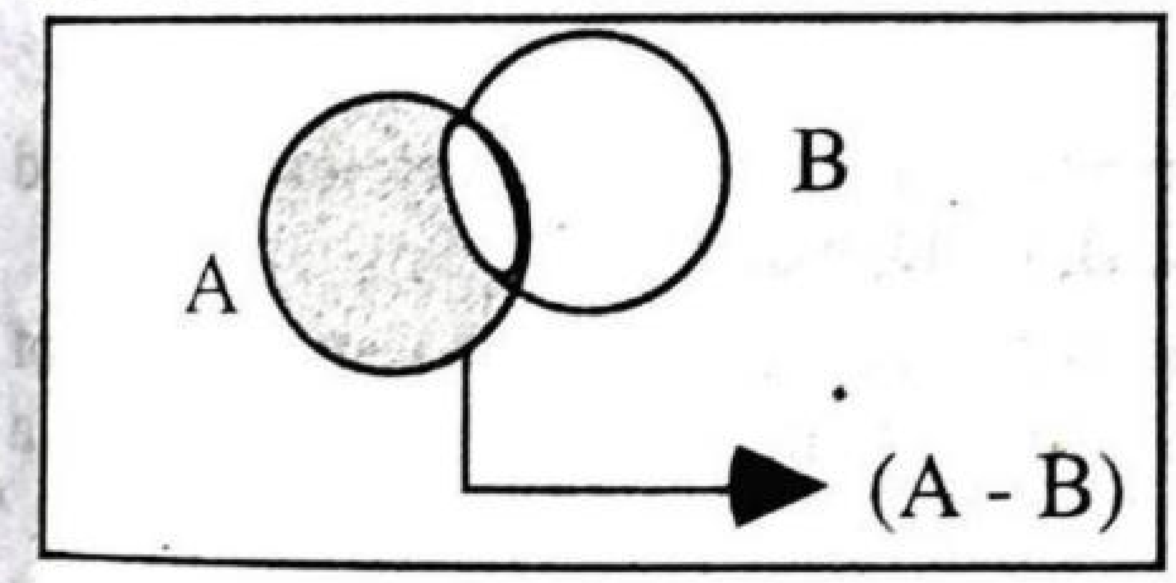


Here, the shaded region is AB.

Fig: Venn diagram of $A \cap B$.

Example : Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6\}$ $\therefore$ $A \cap B = \{2, 4\}$

Set difference : The difference between two sets A and B is such a set that contains all elements that are in A but not in B. The set difference between A and B is denoted by A-B.



Here, the shaded area is A-B.

Fig: Venn diagram of A-B.

Example : Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6\}$ $\therefore$ $A\text{-}B = \{1, 3, 5\}$

Disjoint (Mutually Exclusive) Sets : When two sets A and B have no element in common then the sets A and B are called disjoint or mutually exclusive sets. In this case, $A \cap B = \phi$ always.

Example : If $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$ then A and B are disjoint sets.

Event space : The class of all events associated with a given experiment is defined to be the event space. An event space is generally denoted by script letters **A, B** etc.

Example : Let us consider the tossing of a fair coin:

Here, sample space, $\Omega = \{H, T\}$ 'H' for head and 'T' for tail

Therefore, the event space, $A = \{H, T, \Omega, \phi\}$

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4(1).04 Rules (Laws) for Set Operations :

(i) Commulative laws : (i) $A \cup B = B \cup A$, (ii) $A \cap B = B \cap A$

(2) Associative laws : (i) $A \cup (B \cup C) = (A \cup B) \cup C$, (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

(3) Distribution Laws :

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4) Demorgan's law : The laws state that if $A \cup B$ is equal to the intersection of the complement of A with the complement of B and that the complement of $A \cap B$ is equal to the union of the complement of A and the complement of B.

Symbolically, (i) $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ (ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(5) Idempotent laws : (i) $A \cup A = A$ (ii) $A \cap A = A$

(6) Identity laws : (i) $A \cup \phi = A$, $A \cap \phi = \phi$ (ii) $A \cup \Omega = \Omega$, $A \cap \Omega = A$

(7) Complementary laws :

(i) $A \cup A' = \Omega$, $A \cap A' = \phi$, (ii) $(A')' = A$, $\Omega' = \phi$, $\phi' = \Omega$

4(1).05 Some Topics Related with Probability

It is useful and necessary to have the concept or idea of some topics for a clear-cut understanding of probability. The related topics in this field are discussed below :

(a) Experiment : An experiment is an act that can be repeated under some given conditions. For example, throwing a fair coin or a fair die.

(b) Random Experiment : A random experiment is an experiment that can be repeated any number of times under some identical conditions. In any random experiment the outcome of any particular trial should not be known beforehand. But all possible outcomes should be known in advance.

Examples :

(i) Tossing a fair coin or throwing a die and observe what the top shows.

(ii) The number of road accidents per day in Dhaka City.

(c) Trial : Consider an experiment which though repeated under essentially indentical conditions, does not give unique results but may result in any one of the several possible outcomes. The experiment is known as a trial.

Example : Throwing of a die in a trial.

(d) Event : Any possible outcome or a set of possible outcomes of a random experiment is called an event. Generally, events be denoted by capital letter A, B, C, D, etc.

Example : If the sample space of drawn an unbiased die is S : {1, 2, 3, 4, 5, 6} and the set of odd number is denoted by A : {1, 3, 5}. Then A is an event of the obtained odd number in sample space.

(e) Simple event : When an event corresponds to a single possible outcome then it is called simple (or elementary) event. For example, in case of rolling a die, to have two-dot is a simple event.

(f) Compound (Composite) event : When an event corresponds to a set of possible outcomes then it is known as composite or compound event. For example, in case of rolling a die, to have two-dot, four dot and six dot is a composite event.

(g) Certain (Sure) event : An event whose occurance is a must in any random experiment is known as a certain (or sure) event.

Example : To die for every living is a certain event.

(h) Impossible event : An event whose occurance is quite impossible in a random experiment is called an impossible event.

Example : To live without breathing is an impossible event.

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(i) **Equally likely Events :** The outcomes of a trial or experiment are said to be equally likely if each of them have equal chance to be occured.

Example : In case of tossing a fair coin head and tail are equally likely events.

(j) **Mutually Exclusive Events :** If the happening of any of the events excludes the happening of all the other then the events (or cases) would be termed as mutually exclusive events.

Examples : (i) If a die is thrown up then any of the six possible outcomes will appear. In this case, more than one outcome cannot appear at the same time.

(ii) The six of a newborn.

(k) **Non-mutually exclusive events :** When two or more events have common elements in a random experiment then these are called non mutually exclusive events. In other words, the two events A and B are called non mutually exclusive if $A \cap B \neq \phi$. In this case, $P(A \cap B) \neq 0$.

Example : If $A = \{2, 4, 6\}$ and $B = \{3, 6\}$, then A and B are non mutually exclusive events.

(l) **Exhaustive Events :** The total number of all possible outcomes of a random experiment is known as exhaustive events. For example, the number of all possible outcomes, in case of throwing a die, 1, 2, 3, 4, 5 and 6 create an exhaustive event.

(m) **Independent Events :** If the occaurance of a set of events is not affected by any other set of events in any way, then the set of events is known as independent events. For example, if we throw a die three times, the results of the 1st draw, 2nd draw and 3rd would be independent of each other.

(n) **Dependent Events :** If the occurance or non-occurance of an event in a trial is affected by the other subsequent trials then the events are said to be dependent events. For example, if we consider 10 balls in a box where 5 are red, then the probability of drawing a red ball in the first draw is $\frac{5}{10}$. If we do not return the ball back then the probability of drawing a red ball in the second draw is $\frac{4}{9}$.

(o) **Complementary Events :** The complement of an event implies the non-occurance of that event. Therefore, the complement of an event E contains those points of the sample space which are not in E. The complement of event E is denoted by $\overline{E}$. Both E and $\overline{E}$ are complement of each other.

On other words, A event "A occurs" and the event "A does not occur" are called complementary events. The "event A does not occur" is denoted by $\overline{A}$ (read as A bar) or A' (read as A prime) or A^c (read as A complement). It is important to note.

That $P(A) + P(\overline{A}) = 1 \Rightarrow P(\overline{A}) = 1 - P(A)$.

(p) **Sample space :** The set or collection of all possible outcomes of a random experiment is known as sample space. It is usually denoted by capital letters. And each and every possible outcome in a sample space is called sample point.

Example : If we consider the experiment with throwing a die, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$ and each of 1, 2, 3, 4, 5, 6 is known as sample point.

(q) **Favourable outcomes (or cases) :** The number of outcomes that results the happening of a desired event are known as favourable outcomes of that event. For example, if we have 10 balls in a box of which 5 are red then the favourable outcomes of getting a red ball is 5.

(r) **Null event :** An event having no sample point is called a null event and is denoted by ϕ.

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4(1).06 Different Approaches of Probability

The theory and definition related to probability were started to be introduced from more than 300 years ago. Probability is a number that describes the chance that something will happen. It's value lies between zero and one, inclusive, describing the relative possibility (chance) an event will occur. There are various approaches to define probability. The principal approaches are as follows :

(a) Classical (or priori) approach.

(b) Relative frequency (or Emperical approach)

(c) Axiomatic approach.

(a) Classical or Priori Approach :

The classical approach is the earlist approach for defining probability. By this approach probability is defined as the ratio of favourable outcomes to the total number of possible outcomes. That is, the probability of an event.

$$P = \frac{\text{Number of favourable outcome}}{\text{Total number of possible outcomes}}$$

Consider that in an experiment the event A contains n(A) of these (i.e., favourable) outcomes, then the probability of A is given by $P(A) = \frac{n(A)}{n(S)}$, where n(S) is the total number of outcomes. This is also known as priori definition as it is based on prior consideration.

Example : If we want to know the probability of getting a king in a draw from a pack of 52 cards, then total number of cases (or outcomes), n(S) = 52

Total number of kings, n(A) = 4

Where A is the event of getting a king.

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

Drawbacks (limitations) : There are also some limitations of classical approach of probability. The limitations are as follows :

(i) If the number of trials is infinity then classical definition cannot be used.

(ii) When the possible outcomes are not equally likely then this definition can provide nothing e.g., considering a bias coin.

(iii) The concept of equally likely outcomes is essentially based on the concept of probability that we are trying to define.

(iv) In case of some practical situations this definition cannot be applicable. For example, if we want to know the probability of giving birth of an infant girl per day in Dhaka City then this definition cannot be used.

(b) Relative Frequency or Emperical Approach : From classical approach of probability we have an overall idea about probability, but that may not be true in practical situations. In relative frequency approach we take a decision or get the value of probability from past experience. The relative frequency approach is also known as emperical or posterior approach as the measure of probability is calculated from emperical or (Previous) statistical findings.

Interms of a formula :

$$\text{Probability of happening an event} = \frac{\text{Number of times events occured in past}}{\text{Total number of observations}}$$

If an experiment be repeated n(S) times and the favourable outcome of event A is repeated n(A) times, then the limiting value of the ratio of the number of times A happens to the total number of repeatations of the experiment is called the probability of the occurance of A.

Mathematically, $P(A) = \lim_{n(S)\to\infty} \frac{n(A)}{n(S)}$

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$\therefore P(A\cup B) = \frac{n(A\cup B)}{n(S)} = \frac{n(A)+n(B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = P(A) + P(B)$

$\Rightarrow P(A\cup B) = P(A) + P(B)$ [Proved]

Theorem (2) : The additive law of probability for non-mutually exclusive events.

Statement : The probability of occurance of any one event of two non-mutually exclusive events is equal to the sum of their individual probabilities minus the probability that both events occur.

Symbolically, if A and B are two non mutually exclusive events, then $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

Proof : Let A and B are two non mutually exclusive events of the obtained sample space S from a random experiment E. The following venn diagram is given by -

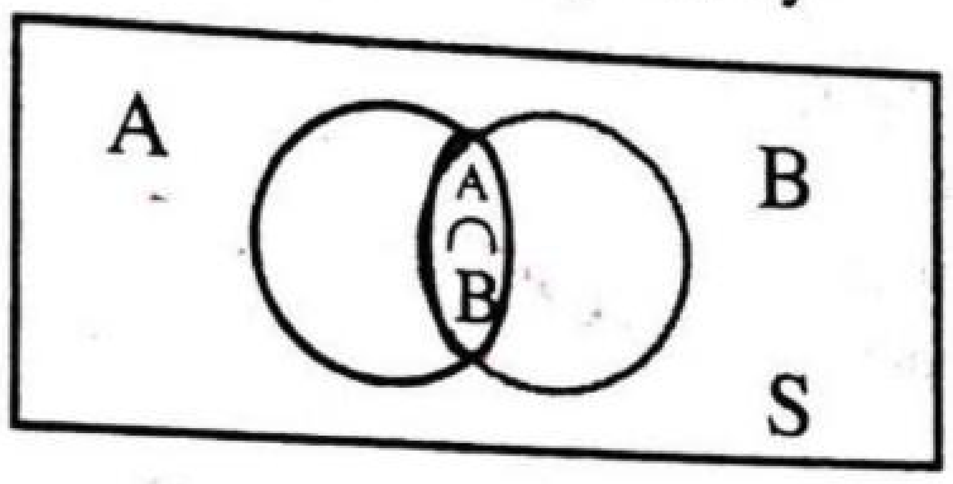

Fig. Venn diagram.

Let us consider,

The total number of elements in the sample space is n(S)

The number of elements belong to the event A is n(A)

The number of elements belong to the event B is n(B)

The number of occurances favourable to the compound event $A\cap B$ is $n(A\cap B)$

According to the definition of priori probability, we have,

$$P(A) = \frac{n(A)}{n(S)},\ P(B) = \frac{n(B)}{n(S)},\ P(A\cap B) = \frac{n(A\cap B)}{n(S)}$$

Since A and B are two non mutually exclusive events. So, the number of elements belong to the event $(A\cup B)$ is

$$\begin{aligned} n(A\cup B) &= \{n(A) - n(A\cap B)\} + n(A\cap B) + \{n(B) - n(A\cap B)\} \\ &= n(A) - n(A\cap B) + n(A\cap B) + n(B) - n(A\cap B) \\ &= n(A) + n(B) - n(A\cap B). \end{aligned}$$

$$\therefore P(A\cup B) = \frac{n(A\cup B)}{n(S)} = \frac{n(A) + n(B) - n(A\cap B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A\cap B)}{n(S)} = P(A) + P(B) - P(A\cap B)$$

$\therefore P(A\cup B) = P(A) + P(B) - P(A\cap B)$ [Proved]

Theorem (3) : The multiplicative law of probability for two dependent events.

Statement : The probability of the joint occurance of the two dependent events is equal to the product of the unconditional probability of it's any one event and the conditional probability of another event.

Symbolically, if A and B are two dependent events, then

$$P(A\cap B) = P(A)\,P(B|A),\quad P(A\cap B) = P(B)\,P(A|B)$$

Proof : Let, A and B are two dependent events in the obtained sample space S from the random experiment E.

Venn diagram has been shown in the following figure.

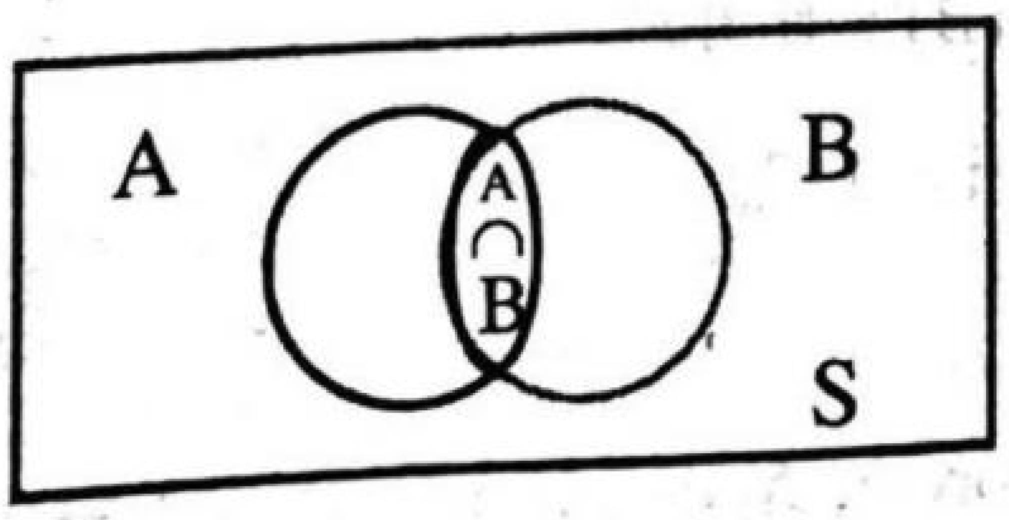

Fig. Venn diagram.

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Suppose, the sample space S contains n(S) occurrences of which n(A) occurrences belong to the event A and n(B) occurrences belong to the event B. Let $n(A \cap B)$ be the number of occurences favourable to the compound event $A \cap B$.

Now, the conditional probability $P(A|B)$ refers to the sample space of n(B) occurances. Out of which $n(A \cap B)$ occurrences certain to the occurrence of A, i.e., when B has already happened.

$$P(A|B) = \frac{n(A \cap B)}{n(B)}$$

Similarly we have, $P(B|A) = \dfrac{n(A \cap B)}{n(A)}$

Now, $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{n(A \cap B)}{n(A)} \cdot \dfrac{n(A)}{n(S)} = P(B|A)\, P(A)$

$$\Rightarrow P(A \cap B) = P(A)\, P(B|A)$$

and $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{n(A \cap B)}{n(B)} \cdot \dfrac{n(B)}{n(S)} = P(A|B)\, P(B)$

$$\Rightarrow P(A \cap B) = P(B)\, P(A|B)$$

Theorem (4) : The multiplicative law of probability for two independent events.

Statement : The probability of the joint occurrence of the two independent events is equal to the product of their individual probabilities.

Symbolcally, if A and B are two independent events, then $P(A \cap B) = P(A)\, P(B)$

Proof : Let, A and B are the two independent events of the obtained two sample space S_1 and S_2 from the two independent random experiment E_1 and E_2.

Suppose, $n(S_1)$ and $n(S_2)$ are the total number of elements in the sample space S_1 and S_2 respectively and n(A) and n(B) are occurrences belong to the events A and B respectively.

According to the definition of priori probability, we have,

$$P(A) = \frac{n(A)}{n(S_1)} \text{ and } P(B) = \frac{n(B)}{n(S_2)}$$

Since E_1 and E_2 are the two independent experiments. So, the elements of the sample space S_1 and S_2 are join indepndently with each other and construct a combined sample space.

$\therefore$ The total number of elements in the combined sample space, $n(S_1 \cap S_2) = n(S_1)\, n(S_2)$ and the number of elements belong to the event $(A \cap B)$ is $n(A \cap B) = n(A)\, n(B)$

$$\therefore P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{n(A)\, n(B)}{n(S_1)\, n(S_2)} = \frac{n(A)}{n(S_1)} \times \frac{n(B)}{n(S_2)} = P(A)\, P(B)$$

$$\Rightarrow P(A \cap B) = P(A)\, P(B) \text{ [Proved]}$$

Theorem (5) : Bayes' Theorem :

Bayes' theorem (according to it's founder Reverend Thomas Bayes, 1763) is a particular application of conditional probability. It is also known as the inverse probability theorem. This theorem is widely used in business, medicine, industry and so forth.

Statement : Let $\{A_1, A_2, ---, A_i, --- A_k\}$ be a set of mutually exclusive and exhustive events forming a partition of the sample space S such that -

(i) $A_1 \cup A_2 \cup --- \cup A_k = S$

(ii) $P(A_i) > 0,$ $[i = 1, 2, ---------, k]$

Let B be the event of S such that $P(B) > 0$, then

$$P(A_i \mid B) = \frac{P(A_i)\, P(B|A_i)}{\sum_{i=1}^{k} P(A_i)\, P(B \mid A_i)} \; ; \qquad i = 1, 2, -----, k$$

Which is Baye's theorem.

Proof : According to the given theorem, A_i and B are dependent. Then by using multiplication rule of probability for dependent events, we have,

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$P(A_i \cap B) = P(B)\,P(A_i \mid B)$ -------------------------- (i)

$P(A_i \cap B) = P(A_i)\,P(B|A_i)$ -------------------------- (ii)

From equation (i), we have, $P(B)\,P(A_i \mid B) = P(A_i \cap B)$

$$\Rightarrow P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)\,P(B|A_i)}{P(B)} \quad \text{[From equation (ii)]}$$

$$\therefore\ P(A_i \mid B) = \frac{P(A_i)\,P(B|A_i)}{P(B)} \quad \text{---------------------- (iii)}$$

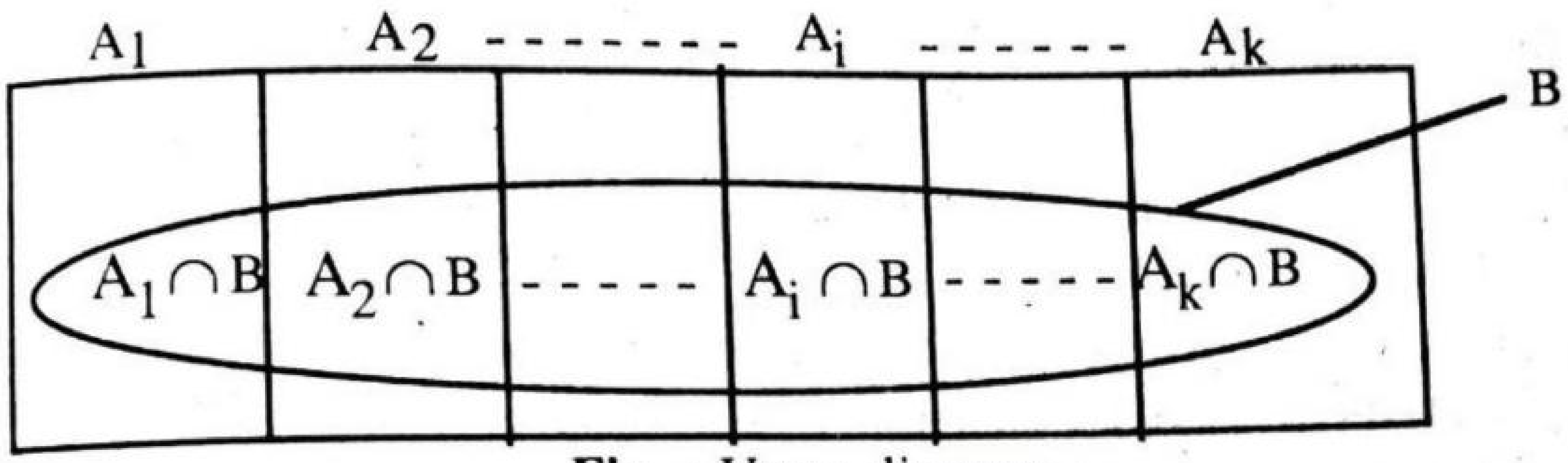

Fig. : Venn diagram

We know, $S = A_1 \cup A_2 \cup \text{--------} \cup A_k$

and $B = S \cap B$

$= (A_1 \cup A_2 \cup \text{--------} \cup A_k) \cap B$

$= (A_1 \cap B) \cup (A_2 \cap B) \cup \text{-----------} \cup (A_k \cap B)$

$\therefore P(B) = P[(A_1 \cap B) \cup (A_2 \cap B) \cup \text{-----------} \cup (A_k \cap B)]$

$= P(A_1 \cap B) + P(A_2 \cap B) + \text{-----------} + P(A_k \cap B)$

[Since, $(A_1 \cap B), (A_2 \cap B), \text{----}, (A_k \cap B)$ Are mutually exclusive]

$$= \sum_{i=1}^{k} P(A_i \cap B) = \sum_{i=1}^{k} P(A_i)\,P(B|A_i) \quad \text{[From equation (ii)]}$$

Now, putting the value of P(B) in equation (iii), We have,

$$P(A_i|B) = \frac{P(A_i)\,P(B|A_i)}{\sum_{i=1}^{k} P(A_i)\,P(B|A_i)} \quad \text{(Proved)}$$

Theorem (6) : The sum of the probabilities of happening and non-happening of an event is one.

Symbolically, $P(A) + P(\overline{A}) = 1$ or $p + q = 1$.

Proof : Let A be an event belong to the obtained sample space S from random experiment E in which complement event $\overline{A}$.

Normally, A and $\overline{A}$ are disjoint events.

$\therefore A \cup \overline{A} = S$

$\Rightarrow P(A \cup \overline{A}) = P(S)$

$\Rightarrow P(A) + P(\overline{A}) = 1$ [By using axiomatic probability]

Theorem (7) : The value of probability lies between 0 to 1. i.e., $0 \le P(A) \le 1$.

Proof: Let, A be an event belong to the sample space S from a random experiment E.

According to the Axiomatic definition of probability, we have,

$P(S) = 1$ and $P(A) \ge 0$

$\Rightarrow 0 \le P(A)$ ---------------------- (i)

Here, $A \subset S$

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$\Rightarrow P(A) \le P(S)$

$\Rightarrow P(A) \le 1$ -------------------------- (ii)

From equation (i) and (ii), we get, $0 \le P(A) \le 1$. [Proved]

Theorem (8) : If A and B are two independent events then prove that $\overline{A}$ and $\overline{B}$ independent.

Proof : Since A and B are two independent events.

$\therefore P(A \cap B) = P(A)\, P(B)$ -------- (i)

Now, $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$ [By using De-Morgan law]

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - P(A) - P(B) + P(A)\, P(B)$ [From equation (i)]

$= 1[1-P(A)] - P(B)\, [1-P(A)]$

$= [1-P(A)]\, [1-P(B)\,]$

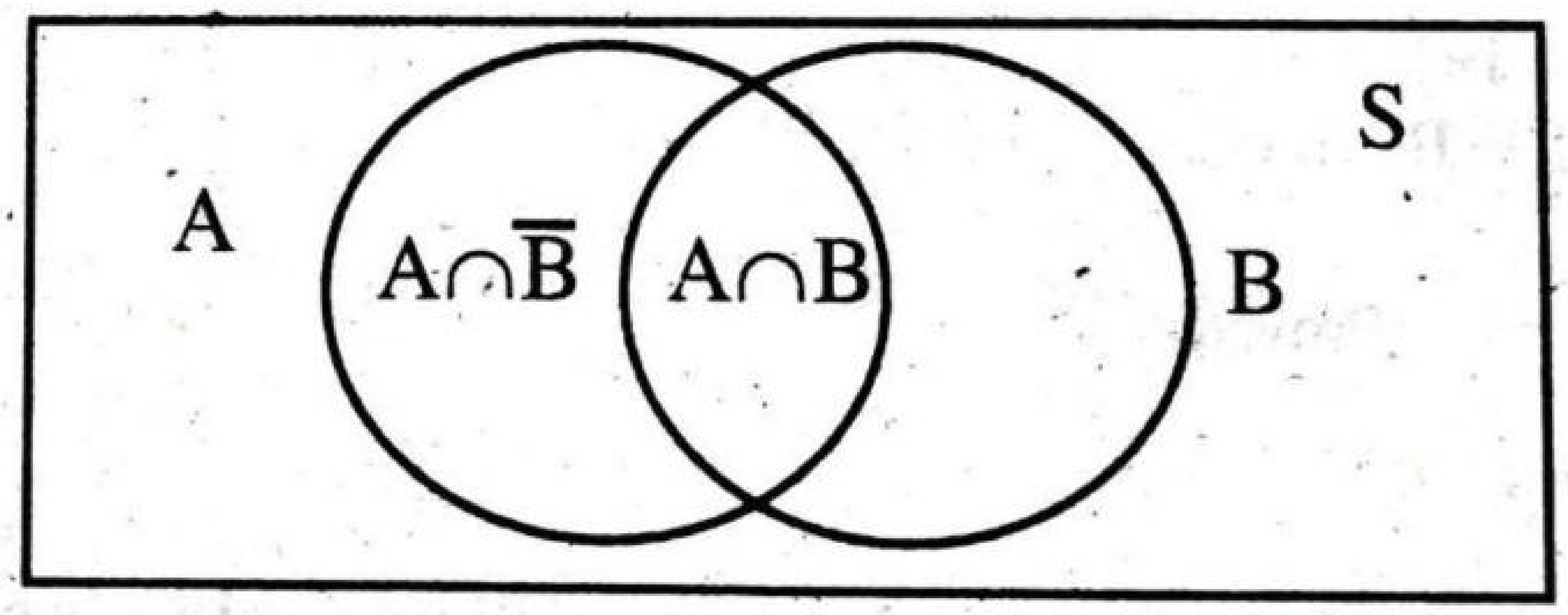
$= P(\overline{A})\, P(\overline{B})$

$\therefore \quad P(\overline{A} \cap \overline{B}) = P(\overline{A})\, P(\overline{B})$

Hence, if A and B are two independent events then $\overline{A}$ and $\overline{B}$ independent.

Theorem (9) : For any two non-mutually exclusive events A and B prove th $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Proof : Let, A and B are two non mutually exclusive events belongs to sample space S. The Ve diagram is given below :



We have, $A = (A \cap \overline{B}) \cup (A \cap B)$

$\Rightarrow P(A) = P[(A \cap \overline{B}) \cup (A \cap B)]$

$\Rightarrow P(A) = P(A \cap \overline{B}) + P(A \cap B)$ [$\because (A \cap \overline{B})$ and $(A \cap B)$ mutually exclusive]

$\Rightarrow P(A) - P(A \cap B) = P(A \cap \overline{B})$

$\Rightarrow P(A \cap \overline{B}) = P(A) - P(A \cap B)$ [Proved)]

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Probability 4(1)-16

Problem-(1) : A fair coin is tossed two times. Construct the sample space of the experiment. What is the probability of getting (i) all head (ii) at least one head (iii) at best one head (iv) a head and a tail

Solution : A fair coin is tossed two times. The sample space of the experiment S : {HH, HT, TH, TT}

∴ Total number of eqally likely cases of sample points is, n (S) = 4

(i) Let, the event A : All head

∴ The set of favourable cases of event A : {HH}, n (A) = 1

So, the required probability, $P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}$ (Ans)

(ii) Let the event B : At least one head.

∴ The set of favourable cases of event B : {HT, TH, HH}, n (B) = 3

$P(B) = \frac{n(B)}{n(S)} = \frac{3}{4}$ (Ans)

(iii) Let the event C : At best one head.

∴ The set of favourable cases of event C : {HT, TH, TT}, n (C) = 3

The required probability $P(C) = \frac{n(C)}{n(S)} = \frac{3}{4}$

(iv) Let the event D: A head and a tail

∴ The set of favourable cases of event D : {HT, TH}, n (D) = 2

The desired probability, $P(D) = \frac{n(D)}{n(S)} = \frac{2}{4} = \frac{1}{2}$ (Ans)

Problem (2) : An unbiased coin is tossed four times. What is the probability of getting (i) at least 3 heads (ii) at best 1 head.

Solution : If an unbiased coin is tossed four times, then the sample space is given below :

S	HH	HT	TH	TT
HH	HHHH	HHHT	HHTH	HHTT
HT	HTHH	HTHT	HTTH	HTTT
TH	THHH	THHT	THTH	THTT
TT	TTHH	TTHT	TTTH	TTTT

∴ Total number of possible outcome of sample space, n(S) = 16

(i) Let, the event A : at least 3 heads

∴ The set of fovourable cases of event

A : {HHHH, HHHT, HHTH, HTHH, THHH}, n(A) = 5

So, the required probability, $P(A) = \frac{n(A)}{n(S)} = \frac{5}{16}$.

(ii) Let event B : at best 1 head

∴ The set of favourable cases of event B : {HTTT, THTT, TTHT, TTTH, TTTT}

∴ The number of favourable cases of event B, n(B) = 5.

So, the required probability, $P(B) = \frac{n(A)}{n(S)} = \frac{5}{16}$

Problem-(3) : An unbiased coin is tossed 10 times. Find the probability of (i) just 3 heads (ii) at least one head (iii) at least two heads and (iv) at least three heads (v) at best one head.

Solution : If an unbiased coin is tossed 10 times, then the total number of sample points is n (S) = $2^{10} = 1024$

(i) The probability of just 3 heads = P (just 3 heads) $= \frac{^{10}C_3}{1024} = \frac{\frac{10 \times 9 \times 8}{3 \times 2 \times 1}}{1024} = 0.1172$

(ii) The probability of at least one head

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$= 1 - P\text{ (0 head)} = 1 - \frac{^{10}C_0}{1024} = 1 - \frac{1}{1024} = \frac{1023}{1024} = 0.999$

(iii) The probability of at least two heads $= 1 - P\text{ (0 head)} - P\text{ (1 head)}$

$= 1 - \frac{^{10}C_0}{1024} - \frac{^{10}C_1}{1024} = 1 - \frac{1}{1024} - \frac{10}{1024} = \frac{1013}{1024} = 0.9893$

(iv) The probability of at least three heads

$= 1 - P\text{ (0 head)} - P\text{ (1 head)} - P\text{ (2 head)} = 1 - \frac{^{10}C_0}{1024} - \frac{^{10}C_1}{1024} - \frac{^{10}C_2}{1024}$

$= 1 - \frac{1}{1024} - \frac{10}{1024} - \frac{45}{1024} = \frac{968}{1024} = 0.9453$ (Ans)

(v) The probability of at best one head

$= P\text{ (0 head)} + P\text{ (1 head)}$

$= \frac{^{10}C_0}{1024} + \frac{^{10}C_1}{1024} = \frac{1}{1024} + \frac{10}{1024} = \frac{11}{1024} = 0.0107$ (Ans)

Problem- (4) : Two dice are thrown at random. Find the probability that :

(i) the total of the numbers on the dice is 8

(ii) the first die shows 6

(iii) the total of the numbers on the dice is greater than 8

(iv) the total of the numbers on the dice is 13

(v) both the dice show the same number

(vi) the sum of the numbers shown by the dice less than 5

(vii) the sum of the numbers shown by the dice is exactly 6

Solution : When two dice are thrown then the sample space S are listed below :

S	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

∴ Total number of possible outcome of sample space, n (S) = 36.

(i) Let event A : The total of the numbers on the dice is 8.

∴ The set of favourable cases of event A : {(2,6), (6,2), (4,4), (3,5), (5,3)}

The number of favourable cases of event A, n (A) = 5.

Therefore, the probability, $P(A) = \frac{n(A)}{n(S)} = \frac{5}{36}$

(ii) Let event B: The first die shows 6

∴ The set of favourable 6 cases of event B:{(6,1),(6,2),(6,3),(6,4),(6,5), (6,6)}

∴ The number of favourable cases of event B, n (B) = 6.

The required probability, $P(B) = \frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

(iii) Let event C : The total of the numbers on the dice is greater than 8.

∴ The set of favourable cases of event C : {(3,6), (6,3), (4,5), (5,4), (4,6), (6,4), (5,5), (5,6), (6,5), (6,6)}, n (C) = 10

The probability, $P(C) = \frac{n(C)}{n(S)} = \frac{10}{36} = \frac{5}{18}$

(iv) Let event D : The total of numbers on the dice is 13.

∴ The set of favourable cases of event D : ϕ, n (D) = 0

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The probability, $P(D) = \frac{n(D)}{n(S)} = \frac{0}{36} = 0$

(v) Let event E : Both the dice show the same number,

∴ The set of favourable cases of event E : {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)}

∴ n (E) = 6

The required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$ (Ans.)

(vi) Let event F : The sum of the numbers less than 5.

∴ The set of favourable cases of event F : {(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)}

∴ n (F) = 6

The required probability, $P(F) = \frac{n(F)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

(vii) Let, event G : The sum of the numbers show exactly 6.

∴ The set of favourable cases of event G:{(1,5),(5,1),(2,4),(4,2),(3,3)},n (G)=5

The desired probability, $P(G) = \frac{n(G)}{n(S)} = \frac{5}{36}$ (Ans.)

Problem (5) : Two unbiased coins and an unbiased die are tossed once. Write down the sample space and find the probability of the following events- (i) Opposite face on the coin and odd number on the die. (ii) even number on the die.

Solution : If two unbiased coins and an unbiased die are tossed once, then the sample space is given below :

S	1	2	3	4	5	6
HH	HH1	HH2	HH3	HH4	HH5	HH6
HT	HT1	HT2	HT3	HT4	HT5	HT5
TH	TH1	TH2	TH3	TH4	TH5	TH6
TT	TT1	TT2	TT3	TT4	TT5	TT6

∴ Total number of possible outcome of sample space, n (S) = 24

(i) Let, event A : Opposite face on the coin and odd number on the die.

∴ The set of favourable cases of event A : {HT1, HT3, HT5, TH1, TH3, TH5}

n (A) = 6

The required probability, $P(A) = \frac{n(A)}{n(S)} = \frac{6}{24} = \frac{1}{4}$

(ii) Let the event, B : Even number on the die.

∴ The set of favourable cases of event B : {HH2, HH4, HH6, HT2, HT4, HT6, TH2, TH4, TH6, TT2, TT4, TT6}, n (B) = 12

The required probability, $P(B) = \frac{n(B)}{n(S)} = \frac{12}{24} = \frac{1}{2}$

Problem- (6) : Tickets are numbered from 1 to 100. They are well shuffled and a ticket is drawn at random. What is the probability that the drawn ticket has:

(i) an even number

(ii) a number 5 or a multiple of 5

(iii) a number which is greater than 75

(iv) a number which is a square?

Solution :

If tickets are numberd from 1 to 100, then the sample space,

S : {1, 2, 3, --------, 100}

∴ Total number of equally likely cases of sample points, n (S) = 100

(i) Let, event A : The drawn ticket has an even number,

∴ The set of favourable cases of event A : {2, 4, 6, ----------, 100}, n(A) = 50,

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Required probability in the question	To find out	Helping formula
1. At least one / any one / A or B/ either A or B/ problem solve	$P(A \cup B)$	● $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [For non mutually exclusive) [**N.B.** : In this format, all events will be non-mutually exclusive.]
2. Both A and B	$P(A \cap B)$	● $P(A \cap B) = P(A)\, P(B)$ [For independent]
3. None of the two / neither A nor B	$P(A \cup B)'$	● $P(A \cup B)' = 1 - P(A \cup B)$
4. Only A / Just A / A but not B	$P(A \cap B')$	● $P(A \cap B') = P(A) - P(A \cap B)$
5. Only B/ just B/ B but not A	$P(A' \cap B)$	● $P(A' \cap B) = P(B) - P(A \cap B)$
6. Only one/ Exactly one	$P[(A \cap \overline{B}) \cup (\overline{A} \cap B)]$	● $P[(A \cap \overline{B}) \cup (\overline{A} \cap B)] = P(A \cap \overline{B}) + P(\overline{A} \cap B)$
7. At best or at most one	$P[(A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})]$	● $P[(A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})] = P(A \cap \overline{B}) + P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B})$
8. The conditional probability of B given that A / If A is given, the probability of B.	$P(B \mid A)$	$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ That is, P(the required event | the given event) $= \frac{\text{P(the given event amd the required event)}}{\text{P(the given event)}}$

Problem (12) : In a survey of 100 readers, it was found 40 read The Daily Ittefaq and 15 read The Daily Star and 10 read both. What is the probability of a person reading at least one of the newspapers?

Solution : Let us define the events A and B as follows.

A : The person read The Daily Ittefaq

B : The person read The Daily Star

From the given information, we have

$P(A) = \frac{40}{100} = 0.4$, $P(B) = \frac{15}{100} = 0.15$ and $P(A \cap B) = \frac{10}{100} = 0.10$

Hence, the required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.4 + 0.15 - 0.10 = 0.45$ (Ans.)

Problem - (13) : A husband and wife appear in an interview for two vacnacies in the same post. The probability of husband's selection is 1/7 and that of wife's is 1/5 what is the probability that

(i) both of them will be selected

(ii) only one of them will be selected

(iii) none of them will be selected

Solution : Let us define to events A and B as follows :

A : Husband will be selected

B : Wife will be selected

It is given that, $P(A) = \frac{1}{7}$ and $P(B) = \frac{1}{5}$

Normally, the events A and B are independent as they are not affected by each other.

(i) The probability that both of them will be selected is

$P(A \cap B)$

$= P(A)\, P(B)$ [By using the multiplication law of independent events]

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$$= \frac{1}{7} \times \frac{1}{5} = \frac{1}{35} \text{ (Ans)}$$

(ii) The probability that only one of them will be selected is

$$P[(A \cap \overline{B}) \cup (\overline{A} \cap B)]$$

$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$ [Since $(A \cap \overline{B})$ and $(\overline{A} \cap B)$ mutually exclusive events]

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$$= \frac{1}{7} - \frac{1}{35} + \frac{1}{5} - \frac{1}{35} = \frac{5 - 1 + 7 - 1}{35} = \frac{12 - 2}{35} = \frac{10}{35} = \frac{2}{7} \text{ (Ans)}$$

(iii) The probability that none of them will be selected

$$= P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$$= 1 - [\frac{1}{7} + \frac{1}{5} - \frac{1}{35}]$$

$$= 1 - \frac{1}{7} - \frac{1}{5} + \frac{1}{35} = \frac{35 - 5 - 7 + 1}{35} = \frac{36 - 12}{35} = \frac{24}{35} \text{ (Ans)}$$

Problem (14): Suppose 35% of the students failed English, 25% of the studen failed Statistics and 15% of the students failed both English and Statistics in certain college. A student is selected at random.

(i) If he failed statistics, what is the probability that he failed english?

(ii) If the failed english, what is the probability that he failed statistics?

(iii) What is the probability that he failed english or statistics?

Solution : Let us define two events E and S as follows :

E = The students who failed english,

S= The students who failed statistics.

It is given that,

$P(E) = 35\% = 0.35$

$P(S) = 25\% = 0.25$

$P(E \cap S) = 15\% = 0.15$

(i) The probability that a students failed english, given that he was failed statistics is $P(E \mid S) = $ \F(P $\cap S)/P(S)) = \frac{0.15}{0.25} = 0.6$

(ii) The probability that a student failed statistics, given that he was failed english is

$$P(S \mid E) = \frac{P(S \cap E)}{P(E)} = \frac{0.15}{0.35} = 0.4286$$

(iii) The probability that a student failed english or statistics is

$$P(E \cup S) = P(E) + P(S) - P(E \cap S) = 0.35 + 0.25 - 0.15 = 0.45$$

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$= 0.5 + 0.45 + 0.4 - 0.25 - 0.1 - 0.16 + 0.08 = 0.92$

(ii) The required probability = $P\,\overline{(A \cup B \cup C)}$ $= 1 - P(A \cup B \cup C) = 1 - 0.92 = 0.08$

(iii) The required probability = $P(A \cap \overline{B} \cap \overline{C})$

$= P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

$= 0.5 - 0.25 - 0.16 + 0.08 = 0.17$

Format (4) : Problems of Baye's Theorem

$$P(A_i|B) = \frac{P(A_i)\,P(B|A_i)}{P(A_1)P(B|A_i) + P(A_2)P(B|A_2) + \text{----} + P(A_k)\,P(B|A_k)} \quad [i = 1, 2, \text{---}, k]$$

Workings structure :

Let us define events as follows :

A_1 : ------------------------------------

A_2 : ------------------------------------

A_3 : ------------------------------------

and event B : ------------------------------------

According to the question, we have,

$P(A_1) = *$, $P(A_2) = *$, $P(A_3) = *$

$P(B|A_1) = *$, $P(B|A_2) = *$, $P(B|A_3) = *$

By using Baye's Theorem,

the required probability = $P(A_1|B) = \dfrac{P(A_1)\,P(B\backslash A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)}$

Similarly, we can find the value of $P(A_2 \mid B)$ and $P(A_3 \mid B)$.

Problem (54) : In Rajshahi city, male and female each form 50% of the population. It is known that 20% of the males and 5% of the females are unemployed. A reserch student studying the employment situation selects an unemployed person at random. Which is the probability that the person so selected is (i) male (ii) female?

Solution : Let us define the events as follows :

A_1 : Male

A_2 : Female and B : Unemployed.

From the given information, we have,

$P(A_1) = 50\% = 0.50$, $P(A_2) = 50\% = 0.50$

$P(B|A_1) = 20\% = 0.20$, $P(B|A_2) = 5\% = 0.05$

By using Baye's theorem, we have

(i) The probability that the unemployed person selected being a male

$$= P(A_1 \mid B) = \frac{(A_1)\,P(B|A_1)}{P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2)}$$

$$= \frac{0.50 \times 0.20}{0.50 \times 0.20 + 0.50 \times 0.05} = \frac{0.10}{0.125} = 0.8 \text{ (Ans.)}$$

(ii) The probability that the unemployed person selected being a female

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$$= P(A_2|B) = \frac{(A_2)\ P(B|A_2)}{P(A_1)\ P(B|A_1) + P(A_2)\ P(B|A_2)}$$

$$= \frac{0.50 \times 0.05}{0.50 \times 0.20 + 0.50 \times 0.05} = \frac{0.025}{0.125} = 0.2 \text{ (Ans.)}$$

Problem (55) : Given the probability of three events A, B and C are 0.35, 0.45 and 0.2 respectively. Assuming that A, B and C have occured, the conditional probabilities of another event X occuring are 0.8, 0.65 and 0.3 respectively. Find P(A|X).

Solution : From the given information, we have,

$P(A) = 0.35$, $P(B) = 0.45$, $P(C) = 0.2$

$P(X|A) = 0.8$, $P(X|B) = 0.65$, $P(X|C) = 0.3$

By using Baye's theorem, we have,

$$P(A|X) = \frac{P(A)\ P(X|A)}{P(A)\ P(X|A) + P(B)\ P(X|B) + P(C)\ P(X|C)}$$

$$= \frac{0.35 \times 0.8}{0.35 \times 0.8 + 0.45 \times 0.65 + 0.2 \times 0.3}$$

$$= \frac{0.28}{0.28 + 0.2925 + 0.06} = \frac{0.28}{0.6325} = 0.4427$$

Problem (56) : 10% of the employees of a certain company have been to public school. Out of these, 30% hold administrative positions. Of these that have not have to public school, 30% hold administrative positions. If an employee is selected at random from the administrative staff, what is the probability that he was educated in a public school?

Solution : Let us define the events as follows :

A_1 : The company have been to public school.

A_2 : The company have not been to public school.

and B : The administrative staff.

From the given information, we have.

$P(A_1) = 10\% = 0.10$, $P(A_2) = (100 - 10)\% = 90\% = 0.90$

$P(B/A_1) = 30\% = 0.30$, $P(B/A_2) = 30\% = 0.30$

By using Baye's therem, the required probability.

$$P(A_1/B) = \frac{P(A_1)P(B/A_1)}{P(A_1)P(B/A_1)+P(A_2)P(B/A_2)} = \frac{0.10\times0.30}{0.10\times0.30+0.90\times0.30} = \frac{0.03}{0.03+0.27} = \frac{0.03}{0.30} = 0.10$$

Problem- (57) : A company has four production sections S_1, S_2, S_3, S_4 Which contribute 30%, 20%, 22% and 28% respectively to the total output. It was observed that of their production 1%, 2%, 3% and 4% are defective units. If a unit is selected at random and found to be defective, what is the probability that the unit so selected has come from (i) section one, (ii) section four, (iii) either section one or four

Solution : Let, event A_1: production section S_1 contributes the output

event A_2: production section S_2 contributes the output

event A_3: production section S_3 contributes the output

event A_4: production section S_4 contributes the output

And, event B : Defective unit

Given that,

$P(A_1) = 30\% = 0.30$ $P(B|A_1) = 1\% = 0.01$

$P(A_2) = 20\% = 0.20$ $P(B|A_2) = 2\% = 0.02$

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Problem- (65) : An urn contains 8 white and 3 red balls. If two balls are drawn at random, find the probability that (i) both are white (ii) both are red and (iii) one is of each colour.

Solution : The total number of balls in the urn = 8 + 3 = 11.

(i) Let, the event A: Both balls are white

The probability that both balls are white, $P(A) = \frac{^8C_2}{^{11}C_2} = \frac{28}{55}$

(ii) Let, the event B : Both balls are red.

$\therefore$ The probability, $P(B) = \frac{^3C_2}{^{11}C_2} = \frac{3}{55}$

(iii) Let, the event C : One white and one red ball.

$\therefore$ The required probability, $P(C) = \frac{^8C_1 \times {^3C_1}}{^{11}C_2} = \frac{24}{55}$

Problem (66) : A box contains 24 identical balls of which 8 are black, 10 are red and the remaining balls are white, 3 balls are drawn successively at random without replacement. Find the probability that the first ball is black, the second ball is red and the third ball is white. What will be the result if it is drawn successively with replacement? [BBA, Third Semester, N. U. 2007]

Solution :

In a box, 8 black, 10 red balls and white balls = 24 – (8+10) = 6

Let the event A : The first ball is black, the second ball is red and the third ball is white.

If 3 balls are drawn successively at random without replacement,

then the required probability, $P(A) = \frac{^8C_1 \times {^{10}C_1} \times {^6C_1}}{^{24}C_1 \times {^{23}C_1} \times {^{22}C_1}} = \frac{8 \times 10 \times 6}{24 \times 23 \times 22} = \frac{480}{12144} = 0.0395$

Again, if it is drawn successively with replacement.

Then the required probability, $P(A) = \frac{^8C_1 \times {^{10}C_1} \times {^6C_1}}{^{24}C_1 \times {^{24}C_1} \times {^{24}C_1}} = \frac{480}{13824} = 0.0347$

Problem (67) : A box contains 8 red, 3 blue, and 9 green balls. If 3 balls are drawn at random, determine the probability that :

(i) all 3 are red

(ii) all 3 are blue

(iii) at least 1 is blue

(iv) 2 are red, and 1 green

(v) 1 of each color

(vi) the balls are drawn in the order red, blue, and green colors.

Solution : Total number of balls in the box = 8 + 3 + 9 = 20

If 3 balls are drawn at random, then

(i) $P(\text{all 3 are red}) = \frac{^8C_3}{^{20}C_3} = \frac{56}{1140} = 0.0491$

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(ii) P (all 3 are blue) $= \frac{^3C_3}{^{20}C_3} = \frac{1}{1140} = 0.000877$

(iii) P(at least 1 is blue)
= P (1 is blue and 2 another colour except blue) + P (2 is blue and 1 another colour except blue) + P (3 is blue)

$= \frac{^3C_1 \times {^{17}C_2}}{^{20}C_3} + \frac{^3C_2 \times {^{17}C_1}}{^{20}C_3} + \frac{^3C_3}{^{20}C_3}$

$= \frac{3 \times 136}{1140} + \frac{3 \times 17}{1140} + \frac{1}{1140} = \frac{408}{1140} + \frac{51}{1140} + \frac{1}{1140} = \frac{460}{1140} = 0.4035$

(iv) P (2 are red and 1 green) $= \frac{^8C_2 \times {^9C_1}}{^{20}C_3} = \frac{28 \times 9}{1140} = \frac{252}{1140} = 0.2211$

(v) P(1 of each colour) $= \frac{^8C_1 \times {^3C_1} \times {^9C_1}}{^{20}C_3} = \frac{216}{1140} = 0.1895$

(vi) Since 3 balls are drawn in the orderwise. So, 3 balls are drawn at without replacement (say).
The required probability
= P(The first ball is red, the second ball is blue and the third ball is green)

$= \frac{^8C_1 \times {^3C_1} \times {^9C_1}}{^{20}C_1 \times {^{19}C_1} \times {^8C_1}} = \frac{216}{6840} = 0.0316$

Problem (68) : A box contains 10 white, 7 black, and 3 green balls. 2 balls are drawn at random. Find out the probability that :
(i) Both are white
(ii) One is white, and one is green
(iii) One is black, and one is green
Also find the probabilities in case of without replacement.

Solution : Total number of balls in the box = 10 + 7 + 3 = 20
In the case of 2 balls are drawn at random, then

(i) P(Both balls white) $= \frac{^{10}C_2}{^{20}C_2} = \frac{45}{190} = 0.2368$

(ii) P (One is white and one is green) $= \frac{^{10}C_1 \times {^3C_1}}{^{20}C_2} = \frac{30}{190} = 0.1579$

(iii) P(One is black and one is green) $= \frac{^7C_1 \times {^3C_1}}{^{20}C_2} = \frac{21}{190} = 0.1105$

Again, in the case of 2 balls are drawn at without placement, then

(i) P(1st ball is white and 2nd ball is white) $= \frac{^{10}C_1 \times {^9C_1}}{^{20}C_1 \times {^{19}C_1}} = \frac{90}{380} = 0.2368$

(ii) P{(First ball is white and second ball is green) or (First ball is green and second ball is white)}
= P (First ball is white and second ball is green) + P(First ball is green and second ball is white)

$= \frac{^{10}C_1 \times {^3C_1}}{^{20}C_1 \times {^{19}C_1}} = \frac{^3C_1 \times {^{10}C_1}}{^{20}C_1 \times {^{19}C_1}} = \frac{30}{380} + \frac{30}{380} = \frac{60}{380} = 0.1579$

(iii) The required probability
= P{(First ball is black and second ball is green) or (First ball is green and second ball is black)}
= P(First ball is black and second ball is green) + P(First ball is green and second ball is block)

$= \frac{^7C_1 \times {^3C_1}}{^{20}C_1 \times {^{19}C_1}} + \frac{^3C_1 \times {^7C_1}}{^{20}C_1 \times {^{19}C_1}} = \frac{21}{380} + \frac{21}{380} = \frac{42}{380} = 0.1105$

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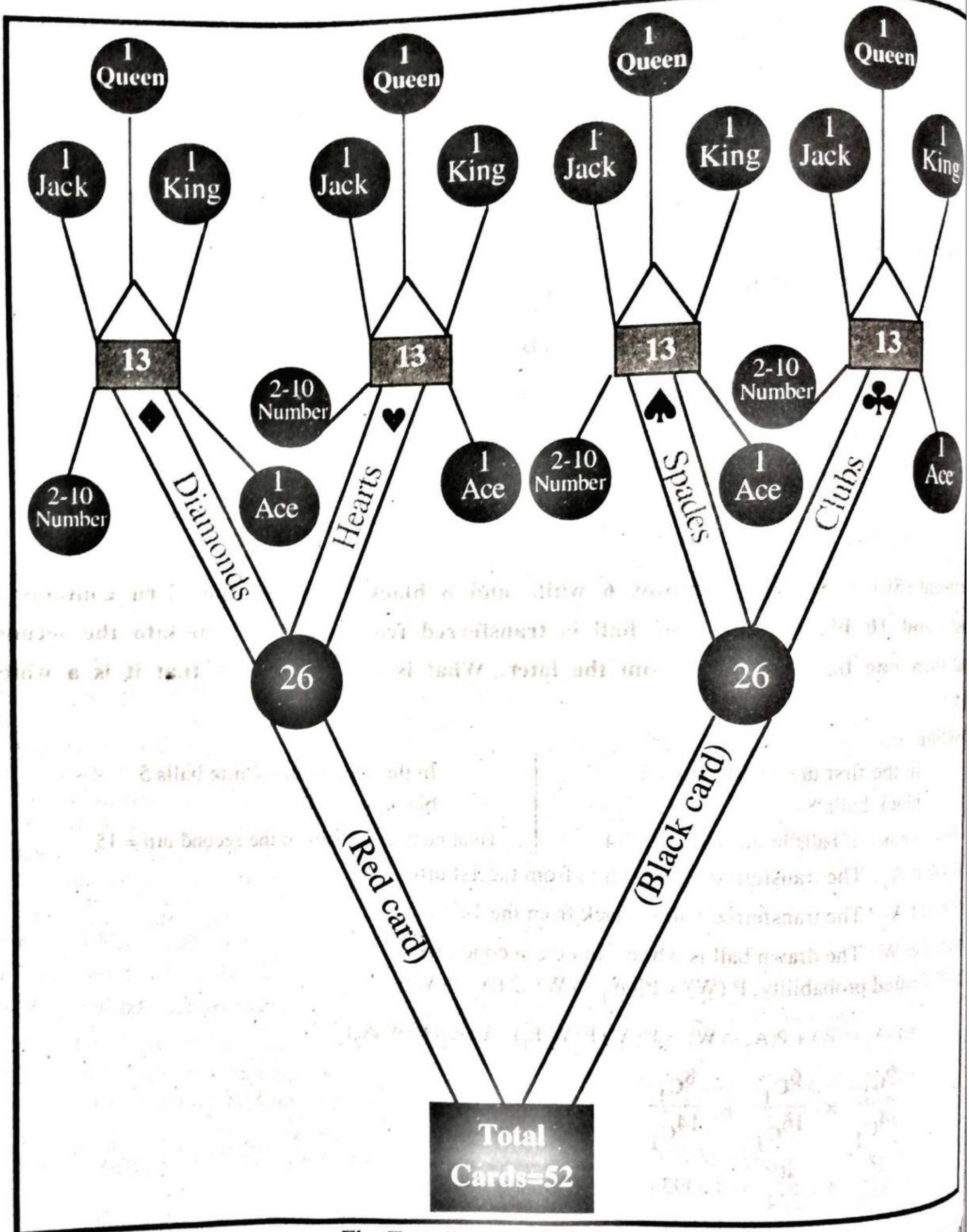

Fig. Tree diagram of playing cards.

Problem- (81) : A card is drawn from a pack of 52 cards. Find the probability of getting (i) an ace (ii) spade (iii) hearts or king

Solution :

(i) Let, event A : The card is an ace.

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$\therefore$ The required probability, $P(A) = \frac{^{4}C_{1}}{^{52}C_{1}} = \frac{4}{52} = \frac{1}{13}$

(ii) Let, event B : The drawn card is spade.

$\therefore$ The dersired probability $P(B) = \frac{^{13}C_{1}}{^{52}C_{1}} = \frac{13}{52} = \frac{1}{4}$

(iii) Let, us define the following events

C : The drawn card is hearts

D : The drawn card is king

Normally, events C and D are not mutually exclusive events. Since the set of hearts contain a king.

$\therefore$ The probability of getting hearts or king is

$$P(C \cup D) = P(C) + P(D) - P(C \cap D)$$

$$= \frac{^{13}C_{1}}{^{52}C_{1}} + \frac{^{4}C_{1}}{^{52}C_{1}} - \frac{^{1}C_{1}}{^{52}C_{1}} = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$ (Ans)

Problem- (82) : 4 cards are drawn from a pack of 52 cards. Find the probability of getting (i) 4 aces (ii) no ace (iii) at least one ace

Solution : Let, 4 cards are drawn at random.

If 4 cards are drawn from a pack of 52 cards, then the total number of equally likely cases of sample points, $n(S) = {^{52}C_{4}} = 270725$

(i) Let, event A : 4 cards are ace

$\therefore$ The number of favourable cases of event A is, $n(A) = {^{4}C_{4}} = 1$

$\therefore$ The required probability, $P(A) = \frac{n(A)}{n(S)} = \frac{1}{270725}$

(ii) Let, event B : event of getting no ace.

The cards other than aces is (52 - 4) or 48

$\therefore$ The number of favourable cases of event B is $n(B) = {^{48}C_{4}} = 194580$

$\therefore$ The required probability, $P(B) = \frac{n(B)}{n(S)} = \frac{194580}{270725} = 0.72$

(iii) The probability of getting at least one ace

$= P(\text{at least one ace}) = 1 - P(\text{no ace}) = 1 - P(B) = 1 - 0.72 = 0.28$

Problem- (83) : 2 cards are drawn from a pack of 52 cards without replacement. Find the probability of getting (i) king of same colour (ii) no king (iii) cards of same colour (iv) cards of different colour.

Solution : If 2 cards are drawn from a pack of 52 cards without replacement then the total number of equally likely cases of sample points is

$$n(S) = {^{52}C_{1}} \times {^{51}C_{1}} = 2652$$

(i) Let event A : The drawn cards are red king

And event B : The drawn cards are black king.

The number of favourable cases of event A is $n(A) = {^{2}C_{1}} \times {^{1}C_{1}} = 2$

The number of favourable cases of event B is $n(B) = {^{2}C_{1}} \times {^{1}C_{1}} = 2$

Normally, events A and B are mutually exclusive.

So, the required probability, $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = \frac{2}{2652} + \frac{2}{2652} = \frac{4}{2652} = \frac{1}{663}$$

(ii) Let, event B : The cards are no king

The cards other than king is (52 - 4) or 48

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$\therefore$ The number of favourable cases of event B is $n(B) = 48_{C_1} \times 47_{C_1} = 2256$

$\therefore$ The probability of getting no king, $P(B) = \frac{n(B)}{n(S)} = \frac{2256}{2652} = 0.8507$

(iii) Let, event C : The drawn cards are red, and
event D : The drawn cards are black

$\therefore$ The number of favourble cases of event C is $n(C) = 26_{C_1} \times 25_{C_1} = 650$

and the number of favourble cases of event D is $n(D) = 26_{C_1} \times 25_{C_1} = 650$

Normally, events C and D are mutually exclusive events.

$\therefore$ The probability of getting cards of same colour

$= P(C \cup D) = P(C) + P(D)$

$= \frac{n(C)}{n(S)} + \frac{n(D)}{n(S)} = \frac{650}{2652} + \frac{650}{2652} = \frac{1300}{2652} = 0.4902$ (Ans)

(iv) Let event E : 1st drawn card is red and 2nd drawn is black
and event F : 1st drawn card is black and 2nd drawn is red

$\therefore$ The number of favourable cases of event E is $n(E) = 26_{C_1} \times 26_{C_1} = 676$

Similarly, $n(F) = 26_{C_1} \times 26_{C_1} = 676$

So, the probability of getting cards of different colour

$= P(E \cup F) = P(E) + P(F)$

$= \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} = \frac{676}{2652} + \frac{676}{2652} = \frac{1352}{2652} = 0.5098$ (Ans)

Problem- (84) : 3 cards are drawn from a pack of 52 cards with replacement Fin the probability of getting (i) kings (ii) hearts.

Solution : In a pack of cards,

The number of hearts = 13

The number of kings = 4

If 3 cards is drawn from a pack of 52 cards with replacement, the total number of equally likely cases o sample points, $n(S) = 52_{C_1} \times 52_{C_1} \times 52_{C_1} = 140608$

(i) Let event A : The drawn cards are kings

$\therefore$ The number of favourable cases of event A is $n(A) = 4_{C_1} \times 4_{C_1} \times 4_{C_1} = 64$

$\therefore$ The required probability, $P(A) = \frac{n(A)}{n(S)} = \frac{64}{140608} = \frac{1}{2197}$

(ii) Let event B : The cards are hearts

$\therefore$ The number of. favourable cases of event B is $n(B) = 13_{C_1} \times 13_{C_1} \times 13_{C_1} = 2197$

$\therefore$ Probability, $P(B) = \frac{n(B)}{n(S)} = \frac{2197}{140608} = \frac{1}{64}$

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MATHEMATICAL PROBLEMS :

The mathematical problems of this chapter can be divided into the following several formats.

Format (1) : Incase of tossed or rolled or drawn of coins and dice
Format (2) : Application of Additive and Multiplicative formula
Format (3) : Incase of general events
Format (4) : Problems of Baye's Theorem
Format (5) : Problems about different characteristics
Format (6) : Problems about playing cards

Now, we are describing the followcharts with related problems :

Format (1) : Incase of tossed or rolled or drawn of coins and dice

Working Rules :

Rule (1) : If an unbiased or fair die tossed

(i) at once, then the sample space, S : {1, 2, 3, 4, 5, 6}

(ii) at two times, then the sample space is given below :

S	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

After these working structure :

The total number of sample points n(S) = *

Let, the event A : .... the required question......................

$\therefore$ The set of favourable cases of event A : {------------}, n (A) = *

So, the required probability, $P(A) = \frac{n(A)}{n(S)}$

Rule (2) : If an unbiased or fair coin tossed

(i) at once, then the sample space, S : {H, T}, n(S) = 2

(ii) at two times, then the sample space, s = {H, T} × {H, T}

= {HH, HT, TH, TT}, n(S) = 4

Or,

S	H	T
H	HH	HT
T	TH	TT

(iii) at three times, then the sample space, S = {H, T} × {H, T} × {H, T}

= {H, T} × {HH, HT, TH, TT}

= {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, n(S) = 8

Or,

S.	HH	HT	TH	TT
H	HHH	HHT	HTH	HTT
T	THH	THT	TTH	TTT

Example : Out of 5 olds and 3 youngs, a committee of 5 members is to be formed. In how many ways the committee can be formed where we have to include 3 olds?

Solution : We are given that the committee would be formed with 5 members and at least 3 of them are old, i.e., in the committee the number of olds would be either 3 or 4 or 5. Therefore, the committee might be formed in the following ways.

(i) With 3 olds and 2 youngs

$$^5C_3 \times {}^3C_2 = \frac{5!}{(5-3)!\,3!} \times \frac{3!}{(3-2)!(2!)} = \frac{5!}{2!\times 3!} \times \frac{3!}{2!2!} = \frac{5\times4\times3!}{3!2!} \times \frac{3\times2!}{2!} = \frac{5\times4}{2\times1} \times 3 = 30$$

(ii) With 4 olds and 1 young, $^5C_4 \times {}^3C_1 = \frac{5!}{(5-4)!4!} \times \frac{3!}{(3-1)!1!} = 5 \times 3 = 15$

(iii) With 5 olds and 0 young, $^5C_5 \times {}^3C_0 = \frac{5!}{(5-5)!5!} \times \frac{3!}{(3-0)!0!} = 1 \times 1 = 1 \quad [\because 0! = 1]$

$\therefore$ The total number of ways in which the committee can be formed = 30 + 15 + 1 = 46.

4(1).11 Conditional Probability :

If two events A and B are dependent, i.e., the occurance of B depends on the occurance of A or B occures only when A has already been occured then the probability is known as conditional probability. Symbolically P(B|A) is known as conditional probability of B given that A has already been occured. And we express P(B|A) as follows :

$$P(B|A) = \frac{P(AB)}{P(A)}; \quad P(A) > 0 \qquad \text{i.e., } P(AB) = P(A)\,P(B|A)$$

In the same way, $P(A|B) = \frac{P(AB)}{P(B)}$: $P(B) > 0$ i.e., $P(AB) = P(B)\,P(A|B)$

This is known as multiplication rule of probability for two dependent events.

4(1).12 Theorem on probability :

On the basis of the definition of priori and axiomatic probability, the following theorems can be stated :

Theorem (1): The additive law of probability for two mutually exclusive events.

Statement : The probability of occurrance of any one event of two mutually exclusive (or disjoint) events is equal to the sum of their individual probabilities.

Symbolically, if A and B are two mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

Proof : Let, A and B are two mutually exclusive events of the obtained sample space S from a random experiment E. The venn diagram is given below :

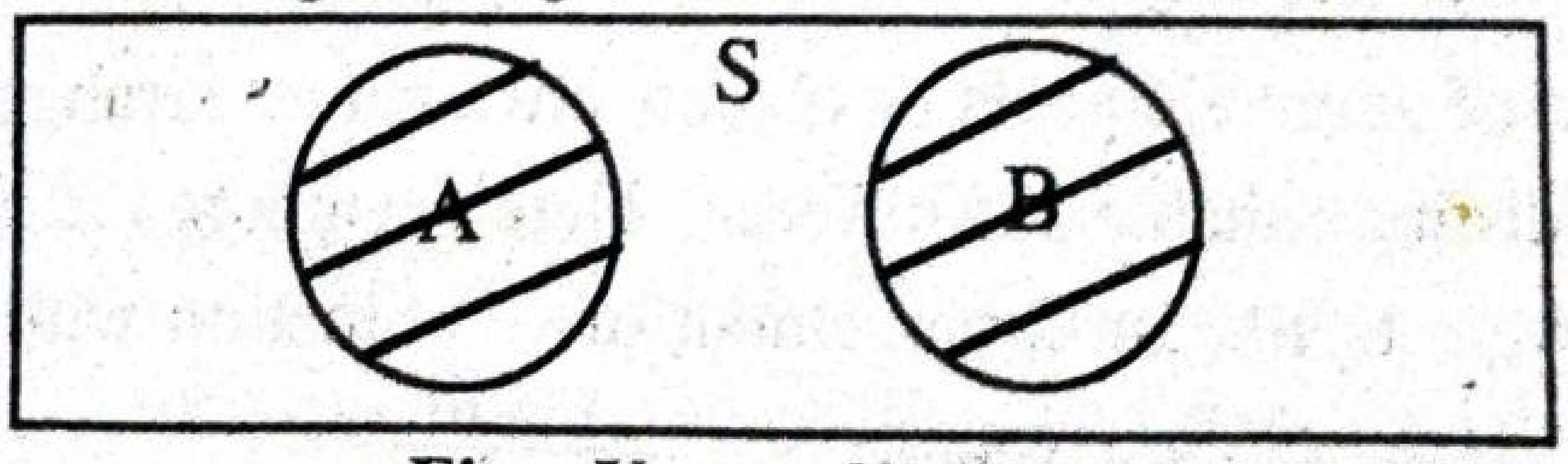

Fig. Venn diagram.

Let us consider, the total number of elements in the sample space is n(S)
The number of elements belong to the event A is n(A)
The number of elements belong to the event B is n(B)
According to the definition of priori probability, we have

$$P(A) = \frac{n(A)}{n(S)}, \quad P(B) = \frac{n(B)}{n(S)}$$

Since the events A and B are mutually exclusive.
So, the number of elements belong to the event $A \cup B$ is $n(A \cup B) = n(A) + n(B)$