CHAPTER—11

DISCRETE DISTRIBUTIONS

11.1 Introduction

In this Chapter, we shall discuss some important discrete distributions which are found a number of applications in real life. The distributions will be studied are Bernoulli, binomial, Poisson, geometric, negative binomial, hypergeometric, uniform and multinomial distributions.

11.2 Bernoulli Distribution

Most of the discrete distributions are related with Bernoulli trials. An experiment is called Bernoulli trial if it has two possible outcomes namely success and failure. The probability of success as well as the probability of failure of a Bernoulli trial remains the same from trial to trial. If p is the probability of success, then q = 1 - p is the probability of failure. The name Bernoulli trial was given in honour of James Bernoulli, who wrote in Latin, about the year 1700 an important work on probability titled "Ars Conjectandi."

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Definition 11.2.1. Bernoulli trial. A random experiment whose outcomes have been classified into two categories, called "success" and "failure", represented by letters s and f, respectively, is called a Bernoulli trial.

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Definition 11.2.2. Bernoulli random variable. A discrete random variable X is called a Bernoulli random variable if it takes the value 1 when the Bernoulli trial is a success and the value 0 when the same Bernoulli trial is a failure.

If p is the proability of success in a Bernoulli trial and q = 1 - p is the probability of failure of the same Bernoulli trial, then the probability of X is

$$f(x) = \begin{cases} p, & \text{if } X = 1 \\ q, & \text{if } X = 0 \end{cases}$$

This probability function is called Bernoulli probability function or the Bernoulli probability distribution.

Definition 11.2.3. Bernoulli Distribution. A discrete random variable X is said to have a Bernoulli distribution if its probability function is given by

$$f(x\,;p) = \begin{cases} p^x q^{1-x} & \text{for } x = 0, 1, \\ 0, & \text{otherwise} \end{cases} \qquad (11.2.1))$$

where p is the parameter of the distribution satisfying $0 \le p \le 1$ and $p + q = 1$.

The bar chart of Bernoulli distribution is shown in fig. 11.2.1

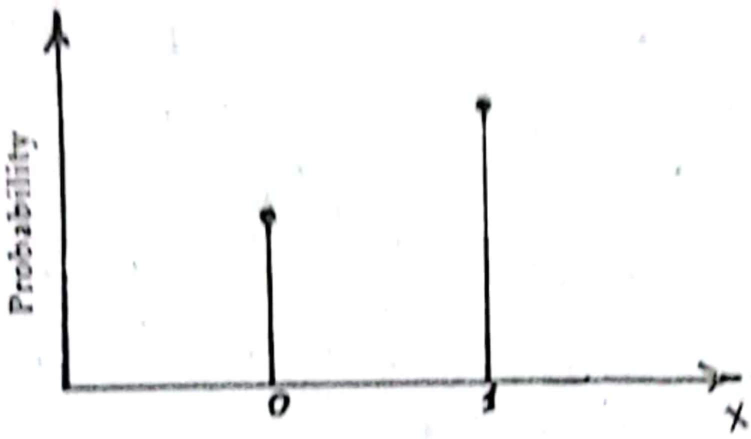

Fig. 11.2.1

Bernoulli Function

Example 11.2.1. A coin is tossed in which the outcome 'head' is a success and the probability of head is p. Then $q = 1 - p$ is the probability of failure or tail. If the number of heads or successes is a random variable X, then X can take values 0 or 1 according to the outcome is a tail (failure) or head (success). Then the probability function of X is

$$f(x\,;p) = \begin{cases} p^x q^{1-x} & \text{for } x = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

Theorem 11.2.1. If X is a Bernoulli variate with parameter p then

$$E[X] = p \quad \text{and} \quad var[X] = pq$$

Proof. According to definition,

$$E[X] = \sum_x x f(x) = \sum_{x=0}^{1} x.\, p^x q^{1-x} = 0.q + 1.\,p = p.$$

Now, $var[X] = E[X^2] - \{E[X]\}^2$

Thus, $E[X^2] = \sum_x x^2 f(x) = \sum_{x=0}^{1} x^2 p^x q^{1-x} = 0.q + 1.p = p.$

Therefore, $var[X] = p - p^2 = p(1 - p) = pq$

11.3 Binomial Distribution

Introduction. Binomial distribution was first derived by Swiss mathematician James Bernoulli (1654 - 1705) in this treatise Ars canjectandi in the year 1700 and was first published posthumously in 1913, eight years after his death.

Definition 11.3.1. A discrete random variable X is said to have a binomial distribution if its probability function is defined by

$$f(x\,;n,p)=\begin{cases}\binom{n}{x}p^{x}q^{n-x} & \text{for } x=0,1,2,\ldots\ldots,n\\ 0, & \text{otherwise}\end{cases} \qquad (11.3.1)$$

where the two parameters n and p satisfy $0 \le p \le 1$ and n is a positive integer and $q = 1 - p$. A distribution defined by the probability function of X given in (11.3.1) is called binomial distribution. The random variable X is called binomial variate with parameters n and p.

If p is the probability of success in a Bernoulli trial and p remains the same from trial to trial, then the probability of x successes in n trials is given by

$$P[X=x]=f(x\,;n,p)=\binom{n}{x}p^{x}(1-p)^{n-x}$$

$$=\binom{n}{x}p^{x}q^{n-x}.$$

where $q = 1 - p$ which is the probability of failure.

Here X can take values 0, 1, 2,......, n. The probability distribution of the number of successes, so obtained is called binomial probability distribution, for the obvious reason that the probabilities of 0, 1, 2,......., n successes, are q^n, $\binom{n}{1}q^{n-1}p$, $\binom{n}{2}q^{n-2}p^2$,......., p^n which are the successive terms of the bionomial expansion $(q+p)^n$.

Derivation of Binomial distribution

Consider a series of n independent Bernoulli trials with probability of success p, then the probability of failure is $q = 1 - p$. The probability of x successes and consequently $(n - x)$ failures in n independent trials in a specified order (say)

$$\underbrace{SSS\ldots\ldots S}_{x\text{ times}}\;\underbrace{FFF\ldots\ldots F}_{(n-x)\text{ times}}$$

[where S and F represent success and failure] is given by the compound probability theorem by the expression

$$P\left[\underbrace{SS\ldots\ldots S}_{x\text{ times}}\;\underbrace{FFF\ldots\ldots F}_{(n-x)\text{ times}}\right]=p^{x}q^{n-x}.$$

But x successes in n trials can occur in $\binom{n}{x}$ ways and the probability of each of these ways is $p^x q^{n-x}$. Hence the probability of x successes in n trials in any order is given by

$$f(x\,;n,p)=\binom{n}{x}p^{x}q^{n-x}\quad;\quad x=0,1,\ldots\ldots\ldots,n.$$

The probability distribution of the number of successes so obtained is called the binomial distribution. We can state a theorem as follows :

Theorem 11.2.2. If p is the probability of success in a single Bernoulli trial and if p remains the same from trial to trial, then the probability of x successes in n independent trials is given by

$$f(x\,;n,p) = P[\,X = x\,] = \binom{n}{x} p^x q^{n-x} \quad ; \quad x = 0, 1, \ldots\ldots, n.$$

Binomial distribution is a discrete distribution as X can take only the integral values, viz, 0, 1, 2,......n.

A binomial variate with parameters n and p is usually denoted by $X \sim B(x\,;n, p)$.

Remarks.

1. Binomial distribution is the generalisation of the Bernoulli distribution. If we put n = 1 in (11.3.1), we get Bernoulli distribution with parameter p. That is why, Bernoulli distribution is sometimes known as point binomial.
2. It is easy to varify that
$$\Sigma f(x\,;n,p) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p+q)^n = 1.$$
3. If $p = q = \frac{1}{2}$, then the distribution becomes symmetrical.
4. The distribution function of the binomial variate is
$$F(x) = P[\,X \le x\,] = \sum_{t=0}^{x} \binom{n}{t} p^t q^{n-t}.$$

The bar chart of some binomial distributions with different values of the parameters n and p are given in fig. 11.3.1.

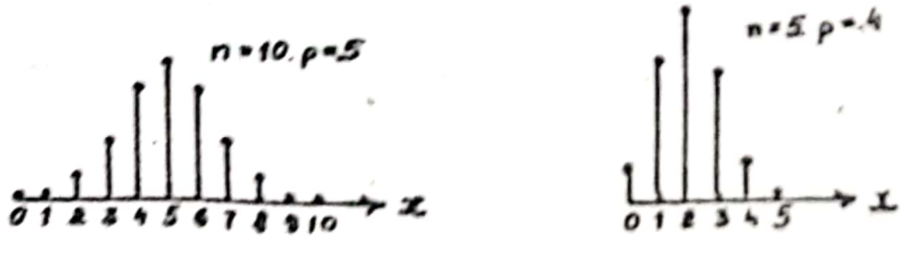

Fig. 11.3.1
Binomial Function

It is obvious from the above bar charts that the distribution becomes symmetrical as n increases. But if $p = q = \frac{1}{2}$, the distribution is always symmetrical for any n. Tables for $f(x ; n, p)$ are available for various values of n and p.

Example 11.3.1. In a community, the probability that a newly born child will be boy is 2/5. Among the 4 newly born children in that community, what is the probability that (a) all the four boys, (b) at least two boys, (c) no boys, (d) exactly one boy and (e) at most two boys.

Solution. Let us consider the event that a newly born child is a boy as success in Bernoulli trial with probability of success 2/5. Let the number of boys be a random variable X. Then X can take values 0, 1, 2, 3 and 4. According to binomial law, the probability function of X is

$$f(x ; 4, 2/5) = \binom{4}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{4-x} \quad \text{for } x = 0, 1, 2, 3, 4.$$

(a) $P[\text{all boys}] = P[X = 4] = \left(\frac{2}{5}\right)^4 = \frac{16}{625}$.

(b) $P[\text{at least 2 boys}] = P[X \geq 2] = 1 - P[X < 2]$

$$= 1 - P[X = 0] - P[X = 1]$$

$$= 1 - \left[\left(\frac{3}{5}\right)^4 + 4\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^3\right]$$

$$= 1 - \left[\frac{81}{625} + \frac{216}{625}\right] = 1 - \frac{297}{625} = \frac{328}{625}.$$

(c) $P[\text{no boys}] = P[X = 0] = (3/5)^4 = 81/625$,

(d) $P[\text{exactly one boy}] = P[X = 1] = 4(2/5)(3/5)^3 = 216/625$

(e) $P[\text{at most two boys}] = P[X = 0] + P[X = 1] + P[X = 2]$

$= (3/5)^4 + 4(2/5)(3/5)^3 + 6(2/5)^2(3/5)^2$

$= 81/625 + 216/625 + 216/625 = 513/625.$

Example 11.3.2. A fair coin is tossed 5 times. Find the probability of (i) exactly two heads, (ii) at least 3 heads, (iii) no heads and (iv) at most 2 heads.

Solution. Let the number of heads be a random variable X which can take values 0, 1, 2, 3, 4, and 5. Then X is a binomial variate with $p = \frac{1}{2}$ and $n = 5$. The probability function of X is

$$f\left(x; 5, \frac{1}{2}\right) = \binom{5}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{5-x} \text{ for } x = 0, 1, 2, 3, 4, 5.$$

(i) P [exactly two heads] $= P[X=2] = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$

$$= 10 \cdot \frac{1}{32} = \frac{5}{16},$$

(ii) P [at least 3 heads] $= P[X \geq 3]$

$$= P[X=3] + P[X=4] + P[X=5]$$

$$= \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 + \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5$$

$$= \frac{10}{32} + \frac{5}{32} + \frac{1}{32} = \frac{16}{32} = \frac{1}{2},$$

(iii) P [no heads] $= P[X=0] = \binom{5}{0}\left(\frac{1}{2}\right)^5 = \frac{1}{32}$ and

(iv) P [at most two heads] $= P[X \leq 2]$

$$= P[X=2] + P[X=1] + P[X=0]$$

$$= \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + \binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5$$

$$= \frac{10}{32} + \frac{5}{32} + \frac{1}{32} = \frac{1}{2}.$$

Moments of the Distribution

The first moment of the distribution about origin is

$$\mu_1' = E[X] = \Sigma x\, f(x)$$

$$= \sum_{x=0}^{n} x\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x}$$

Let $x - 1 = y$, when $x = 1$, $y = 0$ and when $x = n$, $y = n - 1$

Now, $\mu_1' = np \sum_{y=0}^{n-1} \frac{(n-1)!}{y!\,(n-1-y)!} p^y q^{n-1-y}$

$$= np\,(p+q)^{n-1} = np, \text{ since } p + q = 1$$

Second moment about origin,

$\mu_2' = E[X^2] = \Sigma x^2 f(x; n, p)$

$= E[X(X-1) + X]$

$= E[X(X-1)] + E[X]$

$$= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n} x(x-1)\frac{n!}{x!\,(n-x)!} p^x q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{r=0}^{n} \frac{m!}{r!\,(m-r)!} p^r q^{m-r} + np \quad (\text{Put } r = x-2,\ m = n-2)$$

$$= n(n-1)p^2(p+q)^m + np = n(n-1)p^2 + np.$$

The third moment about origin is

$\mu_3' = E[X^3] = E[X(X-1)(X-2) + 3X(X-1) + X]$

$= E[X(X-1)(X-2)] + 3E[X(X-1) + E[X].$

$$= \sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x} + 3\sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x} + np$$

$$= n(n-1)(n-2)p^3 \sum_{x=3}^{n} \binom{n-3}{x-3} p^{x-3} q^{n-x} + 3n(n-1)p^2 + np$$

$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$ [On simplification]

The fourth moment about origin is

$\mu_4' = E[X^4]$

$= E[X(X-1)(X-2)(X-3) + 6X(X-1)(X-2) + 7X(X-1) + X]$

$= E[X(X-1)(X-2)(X-3) + 6E[X(X-1)(X-2) + 7E[X(X-1) + E[X]$

$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

(On simplification)

Corrected or Central moments of Binomial distribution

The first moment about origin is the mean of the distribution. That is

$$\text{mean} = \mu = \mu_1' = np.$$

Variance is the second central moment of the distribution.

$\text{var}[X] = \mu_2 = \mu_2' - (\mu_1')^2$

$= n(n-1)p^2 + np - n^2p^2$

$$= \frac{n!}{k!\,(n-k-1)!} q^{n-k} \sum_{x=0}^{k} \binom{k}{x} p^x q^{k-x} B(n-k, k-x+1)$$

$$= \frac{n!}{k!\,(n-k-1)!} q^{n-k} \sum_{x=0}^{k} \binom{k}{x} p^x q^{k-x} \int_0^1 y^{k-x} (1-y)^{n-k-1}\, dy$$

$$= \frac{n!}{k!\,(n-k-1)!} q^{n-k} \int_0^1 \sum_{x=0}^{k} \binom{k}{x} p^x (qy)^{k-x} (1-y)^{n-k-1}\, dy$$

$$= \frac{n!}{k!\,(n-k-1)!} q^{n-k} \int_0^1 (1-y)^{n-k-1} (p+qy)^k\, dy$$

Put $y = 1 - \frac{t}{q}$, the above expression becomes

$$F(k; n, p) = \frac{n!}{k!\,(n-k-1)!} \int_0^q t^{n-k-1} (1-t)^k\, dt$$

$$= \frac{n!}{k!\,(n-k)!} \frac{(n-k)!}{(n-k-1)!} \int_0^q t^{n-k-1} (1-t)^k\, dt$$

$$= \binom{n}{k} (n-k) \int_0^q t^{n-k-1} (1-t)^k\, dt$$

This result is of great practical utility. It enables us to represent the cumulative binomial probabilities in terms of incomplete Beta functions which are tabulated in Karl Pearson's Tables of the Incomplete Beta Functions.

Example 11.3.5. Determine the binomial distribution for which mean is 4 and variance is 3.

Solution. Let X be a binomial variate with parameters n and p. Here, we have

$np = 4$ and $npq = 3$

Thus, $\frac{npq}{np} = \frac{3}{4} \Rightarrow q = \frac{3}{4}$ and $p = 1 - q = \frac{1}{4}$

Then, $n = \frac{4}{p} = 4 \times 4 = 16.$

Hence the required binomial distribution is

$$f\left(x; 16, \frac{1}{4}\right) = \begin{cases} \binom{16}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{16-x} & \text{for } x = 0, 1, 2, \ldots\ldots, 16 \\ 0, \text{ otherwise} \end{cases}$$

Table 11.3.2.

Fitting of binomial distribution to the frequency distribution of number of dice, showing 5 or 6 in 2,630 throws of 12 dice.

x	$\frac{n-x+1}{x}\frac{p}{q}$	$f(x)=\frac{n-x+1}{x}\frac{p}{q}f(x-1)$	Expected frequency N . f(x)	observed frequency
0	—	0.0077073	20.27=20	18
1	6.0000	0.0462438	121.62=122	115
2	2.7500	0.1271704	334.46=334	326
3	1.66667	0.2119511	557.43=557	518
4	1.12500	0.2384450	627.11=622	611
5	0.80000	0.1907560	501.69=502	519
6	0.58333	0.7112737	292.65=293	307
7	042857	0.0476886	125.42=125	133
8	0.31250	0.0149027	39.19=39	40
9	0.22222	0.0033117	8.71=9	11
10	0.15000	0.0004968	1.31=1	2
11 and 12	—	0.0000529	0.14=0	0
Total		1.0000000	2.630.00	2630

Here also a comparison of the last two columns shows that the fit has been fairly satisfactory, although case-I is better then case-II.

11.4. Poisson Distribution

Poisson distribution was discovered by the French mathematician and physicist Simeon Denis Poisson [1781 - 1840], who published it in 1837. Simon Denis Poisson was an eminant French mathematician and physicist, an academic administrator of some note and according to an 1826 letter from the mathematician Abel to a friend, a man who knew "how to behave with a great deal of dignity". One of Poisson's many interest was the aplication of probability to the law and in 1837 he wrote "Recherches sur la probabilite de Judgements." This text contained a good deal of mathematics including a limit theorem for the binomial distribution. Although credit for this theorem is given to Poisson, there is some evidence that De-Moivre may have discovered it almost a century earlier. Although initially viewed as little more than a welcome approximation for hand to complete binomial probabilities, this particular result was destined for bigger things. It was the analytical seed out of which grew what is now one of the most important of all probability models, the Poisson distribution.

Poisson distribution is a limiting case of the binomial distribution under the following conditions : which condition binomial turns to poisson di

i) The probability of success or failure in Bernoulli trial is very small. That is $p \to 0$ or $q \to 0$.

ii) n, the number of trials is very large

iii) $np = \lambda$ (say) is a finite constant.

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Definition 11.4.1. A discrete random variable X is said to have a Poisson distribution if its probability function is given by

$$f(x;\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & ;\ x = 0, 1, 2, \ldots\ldots, \infty \\ 0, \text{ otherwise} \end{cases} \tag{11.4.1}$$

where e = 2.71828 and λ is the parameter of the distribution which is the mean number of success.

It can be easily shown that

i) $f(x;\lambda) \geq 0$

ii) $\sum\limits_{x=0}^{\infty} f(x;\lambda) = 1$

$$\sum_{x=0}^{\infty} f(x;\lambda) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}e^{+\lambda} = 1.$$

11.4.1. Derivation of Poisson distribution from binomial distribution

Poisson distribution can be derived from the binomial distribution under the following conditions :

i) p, the probability of success in a Bernoulli trial is very small, that is $p \to 0$.

ii) n, the number of trials is very large, that is $n \to \infty$.

iii) $np = \lambda$ is finite constant, that is average number of success is finite.

We have $np = \lambda$ $\therefore p = \lambda/n$ and $q = 1 - \frac{\lambda}{n} = 1 - p$,

The probability function of binomial variate X with parameters n and p is

$$f(x;n,p) = \binom{n}{x} p^x q^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^{x}}{x!}\left(1-\frac{\lambda}{n}\right)^{-x}\left(1-\frac{\lambda}{n}\right)^{n}\frac{n!}{n^{x}\,(n-x)!}$$

Now, for fixed x ; $\lim\limits_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1$

n^x

and

n

$$\lim_{n\to\infty}\frac{n!}{n^{x}\,(n-x)!} = \lim_{n\to\infty}\frac{n\,(n-1)\ldots\ldots[n-(x-1)]\,(n-x)!}{n^{x}\,(n-x)!}$$

$$= \lim_{n\to\infty}\frac{n^{x}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{x-1}{n}\right)}{n^{x}}$$

$$= 1$$

while, $\lim\limits_{n\to\infty}(1-\lambda/n)^{n} = e^{\frac{-\lambda}{n}\cdot n} = e^{-\lambda}$

Hence $\lim\limits_{n\to\infty} f(x\,;n,p) = f(x\,;\lambda) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$; $x = 0, 1, 2, \ldots\ldots, \infty$

which is the probability function of Poisson distribution with parameters λ.

The bar chart of some Possion probability functions are given below for different values of λ.

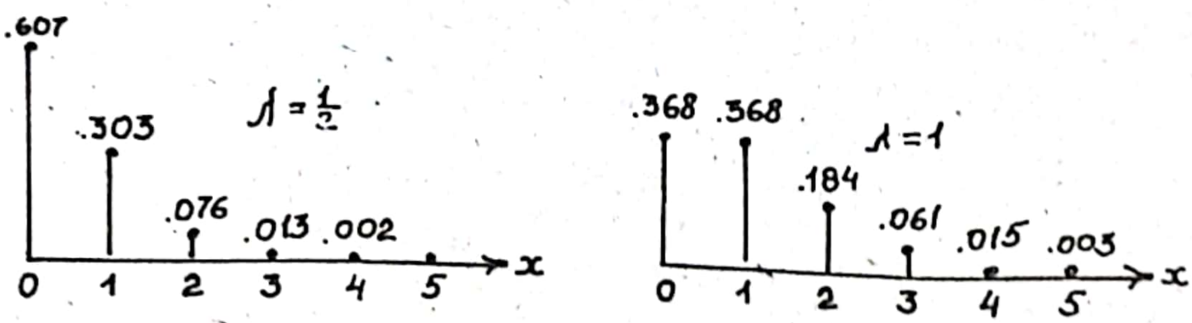

fig. 11.4.1
Poisson functions

Theorem 11.4.1. If X is a Poisson variate with parameter λ, then mean = $\mu_1' = \lambda$ and Variance = $\mu_2 = \lambda$.

Proof. The probability function of Poisson variate with parameter λ is

$$f(x\,;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}\,;\; x = 0, 1, 2, \ldots\ldots \infty.$$

By definition

$$\text{mean} = E[X] = \sum_{x=0}^{\infty} x\,\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!} = e^{-\lambda}\lambda\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\lambda\sum_{y=0}^{\infty}\frac{\lambda^y}{y!} \quad \text{when } y = x-1$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

Again, $\text{var}[X] = E[X^2] - [E(X)]^2$

Now, $E[X^2] = E[X(X-1) + X]$

$$= E[X(X-1)] + E[X]$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \lambda$$

$$= \sum_{x=2}^{\infty}\frac{e^{-\lambda}\lambda^x}{(x-2)!} + \lambda = \lambda^2 e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda}\sum_{y=0}^{\infty}\frac{\lambda^y}{y!} + \lambda \quad \text{where } y = x-2$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda.$$

Thus, $\text{var}[X] = \lambda^2 + \lambda - \lambda^2 = \lambda.$

Hence mean and variance of Poisson distribution are equal.

Theorem 11.4.2. If X is a Poisson variate with parameter λ then,

$$\beta_1 = \frac{1}{\lambda} \text{ and } \beta_2 = 3 + \frac{1}{\lambda}$$

where β_1 and β_2 are the measures of skewness and kurtosis.

Proof. By definition,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2}$$

taken as mode of the Poisson distribution with mean λ. The distribution in this case can be regarded as a bimodal one.

Mean deviation about mean of the Poisson Distribution

Theorem 11.4.9. Let X be a Poisson variate with unit mean, then mean deviation about mean of the distribution is 2/e times the standard deviation.

Proof. Here we are given $\lambda = 1$.

The probability function of X is

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$$f(x;\lambda) = f(x;1) = \frac{e^{-1}}{x!} \;;\; x = 0, 1, 2, \ldots\ldots$$

By definition, mean deviation η about mean $\lambda = 1$ is

$$\eta = E[\,|X-1|\,] = \sum_{x=0}^{\infty} |x-1|\, f(x;\lambda) = e^{-1} \sum_{x=0}^{\infty} \frac{|x-1|}{x!}$$

$$\eta = e^{-1}\left[1 + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots\right]$$

We know, $\dfrac{n}{(n+1)!} = \dfrac{(n+1)-1}{(n+1)!} = \dfrac{1}{n!} - \dfrac{1}{(n+1)!}$

$$\text{Hence } \eta = e^{-1}\left[1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \ldots\ldots\right.$$

$$= e^{-1}(1+1) = \frac{2}{e} = \frac{2}{e} \text{ x standard deviation}$$

Since mean and variance of Poisson distribution are equal. Here it is 1. Hence the proof.

Some examples where Poisson distribution may be successively applied.

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(1) The number of cars passing through a certain street in time t.

(2) Number of suicides reported in a particular day.

(3) Number of faulty blades in a packet of 100.

(4) Number of printing mistakes at each page of a book.

(5) Number of air accidents in some unit of time.

(6) Number of deaths from a disease such as heart attack or cancer or due to snake bite.

(7) Number of telephone calls received at a particular telephone exchange in some unit of time.

(8) The number of fragments receivd by a surface area t from fragmentation bomb.

(9) The number of defective materials in a packing manufactured by a good concern.

(10) The number of letters lost in a mail on a given day in a certain big city.

(11) The number of fishes caught in a day in a certain city.

(12) The number of robbers caught on a given day in a certain city.

Example 11.4.1. Suppose that the number of emergency patients in a given day at a certain hospital is a Poisson variable X with parameter $\lambda = 20$. What is the probability that in a given day there will be (a) 15 emergency patients, (b) at least 3 emergency patients and (c) more than 20 but less than 25 patients.

Solution. (a) Here we have $\lambda = 20$, $x = 15$.

The random variable X follows Poisson distribution with $\lambda = 20$, therefore

$$P[X = 15; \lambda = 20] = \frac{e^{-20}(20)^{15}}{15!} = 0.0516$$

(From the table of Poisson distribution)

(b) $P[\text{at least 3 patients}] = P[X \geq 3] = 1 - P[X \leq 2]$

$= 1 - P[X = 0] - P[X = 1] - P[X = 2]$

$= 1 - .0000 - .0000 - .0000 = 1.$

(c) $P[20 < X < 25] = P[X = 21] + P[X = 22] + P[X = 23] + P[X = 24]$

$= .0846 + .0769 + .669 + .0537 = 0.2441.$

(From the table of Poisson distribution)

Example 11.4.2. If the probability that a car accident happens in a very busy road in an hour is .001. If 2000 cars passed in one hour by that road, what is the probability that (i) exactly 3, (ii) more than 2 car accidents happended on that hour of the road.

Solution. Let X be the number of car accident which follows Poisson distribution with $\lambda = 2000 \times .0001 = 2$, as the probability of accident is very small.

(i) $P[\text{exactly 3 accidents}] = P[X = 3] = \frac{e^{-2}(2)^3}{3!} = \frac{4}{3e^2} = 0.180,$

(ii) $P[\text{more than 2 accidents}] = P[X > 2] = 1 - P[X \leq 2]$

$= 1 - P[X=0] - P[X=1] - P[X=2]$

$= 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}\right] = 1 - \frac{5}{e^2} = 0.323.$

Example 11.4.3. A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality.

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Solution. We are given n = 100

Let p be the probability of defective pin = 5% = .05

$\therefore \lambda$ = mean number of defective pins in a box of 100

$= np = 100 \times .05 = 5$

Since p is small, we may use Poisson distribution.

Probability of x defective pins in a box of 100 is

$$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-5}(5)^x}{x!};\ x = 0, 1, 2.....$$

Probability that a box will fail to meet the guranteed quality is

$$P[X>10] = 1 - P[X \le 10] = 1 - \sum_{x=0}^{10} \frac{e^{-5}5^x}{x!} = 1 - e^{-5}\sum\frac{5^x}{x!}.$$

Example 11.4.4. If X is a Poisson variate such that

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$$P[X=2] = 9P[X=4] + 90P[X=6]$$

Find (i) λ, the mean of X, (ii) β_1, the measure of skewness.

Solution. If X is a Poisson variate with parameter λ, then

$$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!};\ x = 0, 1, 2, ...;\ \lambda > 0$$

We are given,

$P[X=2] = 9P[X=4] + 90P[X=6]$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9.\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$= e^{-\lambda}\left[\frac{9\lambda^4}{4!} + \frac{90\lambda^6}{6!}\right] = e^{-\lambda}\left[3\frac{\lambda^4}{8} + \frac{\lambda^6}{8}\right]$$

$$= \frac{e^{-\lambda}\lambda^2}{8}[3\lambda^2 + \lambda^4]$$