**Lecture note**

**On**

# Random Variable and Probability Distribution


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### ❖ Introduction or Concept of random variable

Before tossing a fair coin we don't know what will appear in the toss. It may happen that we have head (or tail) from the toss. It may happen that we have tail (or head) from the toss. If we toss some coin also then we cannot tell in advance that

(i) 0 head will appear or
(ii) 1 head will appear or
(iii) 2 heads will appear

We know that one of three of the above events will occur but we cannot determine in advance the exact event. That is, the occurrence of head $(H)$ or tail $(T)$ in case of tossing a fair coin occurs at random or depends on chance.

Now, if we define

$x =$ Number of heads in two successive tossed of a fair coin.

Then $x = 0, 1, 2$

As $x$ assumes 3 definite but different values which depend on chance so $x$ be termed as a random variable.

### ❖ Random Variable

When a variable may have the values which can not be known in advance; rather the values depend on chance then the variable might be termed as random variable. The concept of 'chance' is verily related to the term 'probability' and the random variable is also known as stochastic or chance variable. We generally use upper-case letter such as X, Y, Z etc., to denote random variable.

### ❖ Types of Random Variable

Depending on the specific numerical values a random variable can be classified as-

- Discrete random variable
- Continuous random variable

- **Discrete Random Variable**

A random variable that can take a countable number of possible values is called discrete random variable.
**Example:**

(i) The number of heads in two successive tossed of a fair coin.
(ii) The number of women died each year from breast cancer in Bangladesh.

- **Continuous Random Variable**

A random variable which can takes the values within in a range is called continuous random variable.
**Example:**

(i) The height of students in a certain class.
(ii) The life time of a Bangladeshi.

- **Probability Distribution**

The set of all possible values of a random variable together with associated probabilities is called a probability distribution.
If X be a random variable that may have the values $X_1, X_2, \dots \dots, X_n$ with respective probabilities $P(X_1), P(X_2), \dots \dots, P(X_n)$ such that $\sum P(X) = 1$. Then the probability distribution is written as

| $X$ | $X_1$ | $X_2$ | …… | $X_n$ |
|---|---|---|---|---|
| $P(X)$ | $P(X_1)$ | $P(X_2)$ | …… | $P(X_n)$ |

For example, if an unbiased die is thrown once and X be a random variable of the obtained number on the top face, then the probability distribution of X is given below:

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(X)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

- **Types of Probability Function**

There are two types of probability function such as

- Discrete probability function or probability mass function
- Continuous probability function or probability density function

- **Discrete probability function or probability mass function**

A probability function $P(X)$ is called probability function or probability mass function of the discrete random variable $X$ for each possible outcome $X$ it satisfies the following properties:

(i) $P(X) > 0;\ P(X)$is non negative.
(ii) $\sum P(X) = 1$ i.e. the sum of probability function $P(X)$ is equal to 1.

- **Continuous probability function or probability density function**

The function f(X) is called the probability density function (p.d.f) or density function of a continuous random variable $X$ if $f(X)$ satisfies the following properties:

(i) $f(X) > 0;$ i. e. $f(X)$is non negative.
(ii) $\int_{-\infty}^{\infty} f(X)dX = 1.$

(iii) For any interval $(a, b)$

$$P(a \leq X \leq b) = P(a < X < b) = \int_{a}^{b} f(X)dX$$

❖ **Problem-1**

If a fair coin is tossed 3 times, establish the probability distribution of the number of heads $(X)$ of a fair coin. From the probability distribution find $(i) P(X = 2), (ii) P(X \geq 1)$ and $(iii) P(X \leq 2)$.

**Solution:**

If a fair coin is tossed 3 times; then the sample space will be

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Since X denotes the number of heads in three successive tossed of a fair coin.

$$\therefore X = 0, 1, 2, 3.$$

$$P(H) = P(T) = \frac{1}{2}$$

$$P(HHH) = P(H).P(H).P(H) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

$$P(HHT) = P(H).P(H).P(T) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

$$P(HTH) = P(H).P(T).P(H) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

$$P(HTT) = P(H).P(T).P(T) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

$$P(THH) = P(T).P(H).P(H) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

$$P(THT) = P(T).P(H).P(T) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

$$P(TTH) = P(T).P(T).P(H) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

$$P(TTT) = P(T).P(T).P(T) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

Therefore

$$P(X = 0) = P(TTT) = \frac{1}{8}$$

$$P(X = 1) = P(HTT \text{ or } THT \text{ or } TTH) = P(HTT) + P(THT) + P(TTH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P(X = 2) = P(HHT \text{ or } HTH \text{ or } THH) = P(HHT) + P(HTH) + P(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P(X = 3) = P(HHH) = \frac{1}{8}$$

The probability distribution of $X$ is

| $X$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

(i) $P(X = 2) = \frac{3}{8}$

(ii) $P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{3+3+1}{8} = \frac{7}{8}$

(iii) $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{1+3}{8} = \frac{4}{8} = \frac{1}{2}$

- **Problem-2**

If $x$ be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{1}{12}(x+2); 1 < X < 3 \\ 0 \qquad ; Otherwise \end{cases}$$

Then find (i)$\mathrm{P}(\mathrm{x} > 2)$and (ii)$\mathrm{P}(\mathrm{x} < 2)$.

**Solution:**

(i) $\mathrm{P}(\mathrm{x} > 2) = \int_2^3 \frac{1}{12}(x+2)dx = \frac{1}{12}\left[\frac{x^2}{2} + 2x\right]_2^3 = \frac{1}{12}\left[\left(\frac{3^2}{2} + 2.3\right) - \left(\frac{2^2}{2} + 2.2\right)\right]$

$$= \frac{1}{12}\left[\left(\frac{9}{2} + 6\right) - (2+4)\right] = \frac{1}{12}.\frac{9}{2} = \frac{3}{8}$$

(ii) $\mathrm{P}(\mathrm{x} < 2) = \int_1^2 \frac{1}{12}(x+2)dx = \frac{1}{12}\left[\frac{x^2}{2} + 2x\right]_1^2 = \frac{1}{12}\left[\left(\frac{2^2}{2} + 2.2\right) - \left(\frac{1^2}{2} + 2.1\right)\right]$

$$= \frac{1}{12}\left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} + 2\right)\right] = \frac{1}{12}.\frac{7}{2} = \frac{7}{24}$$

- **Problem-3**

The probability density function of a random variable $x$ is given below:

$$f(x) = \begin{cases} cx^2 \ ; 0 < x < 3 \\ 0 \quad ; otherwise \end{cases}$$

(i) Determine the value of c.

(ii) Compute (a) $P(1 < x < 2)$and (b)$P(X < 2)$.

**Solution:**

(i) For $x$ to be density function , we have

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_0^3 cx^2 dx = 1$$

$$\Rightarrow c\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$\Rightarrow c\left[\left(\frac{3^3}{3}\right) - 0\right] = 1$$

$$\Rightarrow 9c = 1$$

$$\therefore c = \frac{1}{9}$$

(ii) (a)

$$P(1 < x < 2) = \int_1^2 \frac{1}{9}x^2 dx = \frac{1}{9}\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{9}\left[\left(\frac{2^3}{3}\right) - \left(\frac{1^3}{3}\right)\right] = \frac{1}{9}\left[\frac{8}{3} - \frac{1}{3}\right] = \frac{1}{9}.\frac{7}{3} = \frac{7}{27}$$

(b)

$$P(X < 2) = \int_0^2 \frac{1}{9}x^2 dx = \frac{1}{9}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{9}\left[\left(\frac{2^3}{3}\right) - 0\right] = \frac{1}{9}.\frac{8}{3} = \frac{8}{27}$$

### ❖ Distribution Function or Cumulative Distribution Function

The cumulative distribution function or briefly distribution function for a random variable $X$ is defined by

$$F(X) = P(X \le x) = \begin{cases} \sum^{x} P(X); if\ X\ is\ discrete\ random\ variable \\ \int_{-\infty}^{x} f(X)dX; if\ X\ is\ continuous\ random\ variable \end{cases}$$

### ❖ Expected value of a random variable

Sum of the value of a random variable multiplying with associated probability is known as the expected value of a random variable. Let the random variable $x$ having n values $x_1, x_2, \dots \dots, x_n$ with their respective probabilities $P(x_1), P(x_2), \dots \dots, P(x_n)$. Then the expected value for discrete random variable $x$ is defined by

$$E(x) = \sum_{i=1}^{n} x_i P(x_i)$$

If $x$ is a continuous random variable with the probability density function f(x), then

$$E(x) = \int_{-\infty}^{\infty} xf(x)dx$$

## ❖ Variance of a random variable

The expected value of the squared deviation of the value of $x$ from their mean is known as variance of the random variable.

If $x$ be a random variable with the expected value $E(x)$, then the variance of the random variable $x$ is defined by

$$V(x) = E[x - E(x)]^2$$

$$= E[x^2 - 2xE(x) + \{E(x)\}^2]$$

$$= E(x^2) - 2E(x)E(x) + \{E(x)\}^2$$

$$= E(x^2) - 2\{E(x)\}^2 + \{E(x)\}^2$$

$$\therefore V(x) = E(x^2) - \{E(x)\}^2$$

**Note.**

(i) Expected value of a constant is equal to constant i.e. $E(c) = c$.
(ii) $E(ax + b) = aE(x) + b$; where a and b are arbitrary constants.
(iii) $E(ax) = aE(x)$
(iv) Variance of a constant is zero i.e. $V(c) = 0$.
(v) $V(ax + b) = a^2V(x)$; where a and b are arbitrary constants.
(vi) $V(ax) = a^2V(x)$

## ❖ Problem-4

Let x be a random variable with the probability distribution

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(x)$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

(i) Compute the expected value of $x, x^2\ and\ 2x$.
(ii) Find the variance of x.

**Solution:**

(i)

$$E(x) = \sum_{x=0}^{2} xP(x) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

$$E(x^2) = \sum_{x=0}^{2} x^2P(x) = 0^2 \times \frac{1}{4} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4} = 0 + \frac{1}{2} + 1 = \frac{3}{2}$$

$$E(2x) = 2E(x) = 2 \times 1 = 2$$

(ii) $V(x) = E(x^2) - \{E(x)\}^2 = \frac{3}{2} - 1^2 = \frac{1}{2}$

❖ **Problem-5**

If $X$ be a random variable with probability density function

$$f(x) = \begin{cases} 2x; & 0 < x < 1 \\ 0; & otherwise. \end{cases}$$

Find the mean of $(i) X, (ii) X^2, (iii) \sqrt{X},\ (iv) E(X-1),$ and $(v)\ V(X)$.

**Solution:**

(i) We know that

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x2xdx = \int_0^1 2x^2dx = 2\left[\frac{x^3}{3}\right]_0^1 = 2\left[\frac{1^3}{3} - 0\right] = \frac{2}{3}$$

(ii) $E(X^2) = \int_0^1 x^2 2xdx = \int_0^1 2x^3dx = 2\left[\frac{x^4}{4}\right]_0^1 = 2\left[\frac{1^4}{4} - 0\right] = \frac{1}{2}$

(iii) $E(\sqrt{X}) = \int_0^1 \sqrt{x}2xdx = \int_0^1 2x^{3/2}dx = 2\left[\frac{x^{5/2}}{5/2}\right]_0^1 = 2\left[\frac{1^{5/2}}{5/2} - 0\right] = \frac{4}{5}$

(iv) $E(X+1) = E(X) - 1 = \frac{2}{3} + 1 = \frac{5}{3}$

(v) $V(X) = E(x^2) - \{E(x)\}^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$