Extremal minimal bipartite matching covered graphs

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Dedicated to the late Ajit A. Diwan

Abstract

A connected graph, on four or more vertices, is *matching covered* (aka 1-*extendable*) if every edge is present in some perfect matching. An ear decomposition theorem (similar to the one for 2-connected graphs) exists for bipartite matching covered graphs due to Hetyei. From the results and proofs of Lovász and Plummer [*Matching Theory*, Annals of Discrete Math. 29, 1986], that rely on Hetyei's Theorem, one may deduce that any minimal bipartite matching covered graph has at least $2(m-n+2)$ vertices of degree two (where *minimal* means that deleting any edge results in a graph that is not matching covered); such a graph is *extremal* if it attains the stated bound.

In this talk, we provide a complete characterization of the class of extremal minimal bipartite matching covered graphs. In particular, we prove that every such graph G is obtained from two copies of a tree devoid of degree two vertices, say T and T', by adding edges — each of which joins a leaf of T with the corresponding leaf of T'.

Apart from the aforementioned bound, there are four other bounds that appear in, or may be deduced from, the work of Lovász and Plummer. Each of these bounds leads to a notion of extremality. In this paper, we obtain a complete characterization of all of these extremal classes and also establish relationships between them. Two of our characterizations are in the same spirit as the one stated above. For the remaining two extremal classes, we reduce each of them to one of the already characterized extremal classes using standard matching theoretic operations.

A connected graph is k-*extendable* if it has a matching of cardinality k and each such matching extends to a perfect matching. We also discuss bounds proved by Lou [*On the structure of minimally n-extendable bipartite graphs*, Discrete Math. 202 (1), 1999] for minimal k-extendable bipartite graphs (where *minimal* means that deleting any edge results in a graph that is not k-extendable). We conjecture stronger bounds and provide evidence for our conjectures by constructing tight examples that are straightforward generalizations of the ones that appear in the 1-extendable case.

This is joint work with our friend, collaborator and mentor — the late Ajit A. Diwan — and it has been accepted for publication in the Journal of Combinatorics. Apart from the technical details, we shall also delve into specific instances of our collaboration and camaraderie with Ajit to highlight his immense contributions and brilliant insights that helped us whenever we were stuck.