21k-3278 Sohaib Sarosh Shamsi

$$\curvearrowright \theta = \sin^{-1}(2x)$$

Q1. $\int \frac{1}{\sqrt{1-4x^2}} \, dx$ $\qquad 2x = 1\sin\theta \rightarrow x = \frac{1}{2}\sin\theta.$

$dx = \frac{1}{2}\cos\theta \, . \, d\theta$

$\int \frac{1}{\sqrt{1^2-(2x)^2}} \, . \, dx.$

$\int \frac{1}{\sqrt{1-(2x)^2}} \Rightarrow \int \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{1}{2}\cos\theta \, . \, d\theta.$

$\int \frac{1}{\cos\theta} \cdot \frac{1}{2}\cos\theta \, . \, d\theta \Rightarrow \frac{1}{2}\theta + c$

$\Rightarrow \frac{1}{2}\sin^{-1}(2x) + c$ (Ans)

Q2. $\int \frac{1}{x^2+25} \, dx \Rightarrow \int \frac{1}{x^2+5^2} \, dx$ $\qquad x = 5\tan\theta \qquad \theta = \tan^{-1}(x/5)$

$dx = 5\sec^2\theta \, . \, d\theta.$

$\int \frac{1}{25(\tan^2\theta+1)} \, dx \Rightarrow \int \frac{1}{25\cancel{\sec^2\theta}} \, . \, 5\cancel{\sec^2\theta} . d\theta$

$\frac{1}{5}\int d\theta \Rightarrow \frac{1}{5}\theta + c$

$\Rightarrow \frac{1}{5}\tan^{-1}(x/5) + c$ (Ans)

Q3. $\int \frac{x \, dx}{x^4+16} \Rightarrow \int \frac{x}{(x^2)^2+4^2} \, dx$ $\qquad x^2 = 4\tan\theta. \qquad \theta = \tan^{-1}(x^2/4)$

$dx = \frac{4\sec^2\theta}{2x}$

$\frac{4}{2}\int \frac{\cancel{x}}{16(\cancel{\tan^2\theta+1})} \cdot \frac{\cancel{\sec^2\theta}}{\cancel{x}}$

$\frac{1}{8}\int d\theta$

$\Rightarrow \frac{1}{8}\tan^{-1}(x^2/4) + c$ (Ans)

(4) $\int \frac{1}{\sqrt{2-5x^2}} \, dX$ $\qquad$ $U = \sqrt{\frac{5}{2}}\, x$

$dU . \sqrt{\frac{2}{5}} = dx$

$\frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{1-(\sqrt{\frac{5}{2}}x)^2}} \, dX$

$\frac{1}{\cancel{\sqrt{2}}} \int \frac{1}{\sqrt{1-U^2}} \cdot \left(\frac{\cancel{\sqrt{2}}}{\sqrt{5}} \right) dU$

$\Rightarrow \frac{1}{\sqrt{5}} \sin^{-1} U + c \Rightarrow \frac{1}{\sqrt{5}} \sin^{-1}\left(\sqrt{\frac{5}{2}}\, x\right) + c$ (Ans)

(5) $\int \frac{3}{x\sqrt{x^2-9}} \, dx$ $\qquad$ $x = 3\sec\theta.$ $\qquad$ $\theta = \sec^{-1}(x/3)$

$dx = 3\sec\theta\tan\theta \, . \, d\theta.$

$\int \frac{3}{\cancel{\sec\theta} \sqrt{9\sec^2\theta - 9}} \, . \, 3\cancel{\sec\theta}\tan\theta \, . \, d\theta.$

$\int \frac{\tan\theta}{\sqrt{\sec^2\theta - 1}} \, d\theta \Rightarrow \int \frac{1}{3}$ [illegible] $. \, d\theta \Rightarrow \frac{1}{3}\theta + c$

$\Rightarrow \frac{1}{3} \sec^{-1}(x/3) + c$ (Ans).

(6) $\int \frac{x}{\sqrt{16-9x^4}} \, dx$ $\qquad$ $3x^2 = 4\sin\theta$ $\qquad$ $\theta = \left(\frac{3x^2}{4}\right) \sin^{-1}$

$dx . 6x = 4\cos\theta \, . \, d\theta.$

$\int \frac{x}{\sqrt{4^2-(3x^2)^2}} \, dx$ $\qquad$ $dx = \frac{4\cos\theta}{6x} \, . \, d\theta.$

$\frac{1}{6} \int \frac{4\cos\theta \cdot d\theta}{\sqrt{16(1-\sin^2\theta)}}$

$\frac{1}{6} \int d\theta \cdot = \frac{1}{6}\theta + c$

$\Rightarrow \frac{1}{6} \sin^{-1}\left(\frac{3x^2}{4}\right) + c$ (Ans)

$a = 3$

(7) $\int \frac{1}{x\sqrt{(4x)^2-9}} \, dx$ $\quad 4x = 3\sec\theta \quad \theta = \sec^{-1}(4x/3)$

$dx = \frac{3\sec\theta\tan\theta}{4}$

$$\int \frac{1\,(\cancel{4})}{3\sec\theta\sqrt{(3\sec\theta)^2-3^2}} \cdot \frac{3\sec\theta\tan\theta}{\cancel{4}} \, d\theta$$

$$\int \frac{\cancel{3\sec\theta\tan\theta}}{\cancel{3\sec\theta\tan\theta}\cdot 3} \cdot d\theta$$

$$\frac{1}{3}\int d\theta \Rightarrow \frac{1}{3}\theta.$$

$\Rightarrow \frac{1}{3}\sec^{-1}(4x/3) + c$ (Ans).

(8) $\int \frac{e^x}{7+e^{2x}} \cdot dx$ $\quad e^x = \sqrt{7}\tan\theta. \quad \theta = \tan^{-1}(e^x/\sqrt{7})$

$dx = \frac{\sqrt{7}\sec^2\theta}{e^x} \, d\theta.$

$$\int \frac{e^x}{(\sqrt{7})^2+(e^x)^2} \cdot dx$$

$$\int \frac{1}{(\sqrt{7})^2+(\sqrt{7})^2\tan^2\theta} \cdot d\theta \cdot \sqrt{7}\sec^2\theta.$$

$$\int \frac{\sqrt{7}\,\cancel{\sec^2\theta}}{(\sqrt{7})^2\cancel{(1+\tan^2\theta)}} \cdot d\theta.$$

$$\frac{1}{\sqrt{7}}\int d\theta. \Rightarrow \frac{1}{\sqrt{7}}\theta.$$

$\Rightarrow \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{e^x}{\sqrt{7}}\right) + c$ (Ans)

(9) $\int \frac{\sin x}{\sqrt{2-\cos^2 x}} \, dx$ $\quad \frac{\cos x}{\sqrt{2}} = 1\sin\theta. \quad \sin^{-1}\left(\frac{\cos x}{\sqrt{2}}\right)$

$$\frac{1}{\sqrt{2}}\int \frac{\sin x}{\sqrt{1^2-\left(\frac{\cos x}{\sqrt{2}}\right)^2}} \, dx$$

$-\sin x.\,dx = \sqrt{2}\cos\theta.\,d\theta.$

$dx = \frac{\sqrt{2}\cos\theta \, d\theta}{-\sin x}.$

$$-\int \frac{\cos\theta}{\cancel{\sqrt{1-\sin^2\theta}}} \cdot d\theta.$$

$$-\int d\theta. \Rightarrow -\theta + c$$

$\Rightarrow -\sin^{-1}\left(\frac{\cos x}{\sqrt{2}}\right) + c$ (Ans)

(10) $\int \frac{1}{\sqrt{x}(1+x)} dx$

$u = \sqrt{x}$

$u^2 = x$

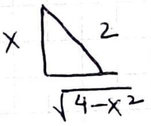
$du = \frac{1}{2\sqrt{x}} dx$

$2du = dx$

$\int \frac{1}{\cancel{u}(1+u^2)} \cdot 2\cancel{u}$

$2\int \frac{1}{1+u^2}$

$\Rightarrow 2\tan^{-1}(\sqrt{x}) + c$ (Ans).



(11) $\int \frac{1}{x^2\sqrt{4-x^2}} dx$

$x = 2\sin\theta$

$dx = 2\cos\theta \cdot d\theta$.

$\tan\theta = \frac{x}{\sqrt{4-x^2}}$

$\int \frac{1}{4\sin^2\theta\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta \, d\theta$

$\int \frac{1 \cdot \cancel{2\cos\theta}}{4\sin^2\theta \cdot \cancel{2\cos\theta}} \cdot d\theta$

$\frac{1}{4}\int \frac{1}{\sin^2\theta} d\theta$.

$\frac{1}{4}\int \operatorname{cosec}^2\theta \cdot d\theta = -\frac{1}{4}\cot\theta + c$

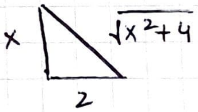
$\Rightarrow -\frac{\sqrt{4-x^2}}{4} + C$ (Ans)

(12) $\int \frac{1}{x\sqrt{x^2+4}} dx$

$x = 2\tan\theta$.

$dx = 2\sec^2\theta \cdot d\theta$.

$\int \frac{2\sec^2\theta}{2\tan\theta \cdot 2\sqrt{\tan^2\theta - 1}} \cdot d\theta$



$\frac{1}{2}\int \frac{\sec\theta}{\tan\theta}$.

$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

$\frac{1}{2}\int \frac{d\theta}{\sin\theta}$.

$\cos\theta = \frac{2}{\sqrt{x^2+4}}$

$\frac{1}{2}\int \operatorname{cosec}\theta \cdot d\theta$.

$\frac{1}{2}\ln\left(\frac{1-\cos\theta}{\sin\theta}\right) + c$

$\frac{1}{2}\ln\left(\frac{\sqrt{x^2+4}-2}{x}\right) + c$

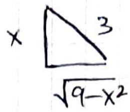
(13) $\int \frac{\sqrt{9-x^2}}{x^2} . dx$ $\qquad x = 3\sin\theta \qquad \theta = \sin^{-1}(x/3)$

$dx = 3\cos\theta . d\theta .$

$\int \frac{3(\sqrt{1-\sin^2\theta})}{9\sin^2\theta} . 3\cos\theta . d\theta .$

$\cot^2 = cosec^2 - 1$

$\int \cot^2\theta . d\theta .$



$-\int d\theta + \int cosec^2\theta . d\theta .$

$-\theta + \cot\theta$

$\Rightarrow -\sin^{-1}(x/3) - \frac{\sqrt{9-x^2}}{x} + c$ (Ans).

(14) $\int \frac{1}{x\sqrt{25-x^2}} . dx$ $\qquad x = 5\sin\theta .$

$dx = 5\cos\theta . d\theta .$

$\int \frac{\cancel{\cos\theta}}{5\sin\theta\cancel{\sqrt{1-\sin^2\theta}}} . d\theta .$ $\qquad cosec^2\theta = \frac{25}{x^2}$

$\frac{1}{5}\int cosec\theta .$ $\qquad 1 + \cot^2\theta = \frac{25}{x^2}$

$\frac{1}{5}\ln(cosec\theta - \cot\theta) + c$ $\qquad \cot\theta = \frac{\sqrt{25-x^2}}{x}$

$\Rightarrow \frac{1}{5}\ln\left(\frac{5-\sqrt{25-x^2}}{x}\right) + c$

(15) $\int \frac{1}{\sqrt{x^2-a^2}} dx$ $\qquad x = a\sec\theta$

$dx = a\sec\theta\tan\theta .$

$\int \frac{1}{\cancel{a}\cancel{\sqrt{\sec^2-1}}} . \cancel{a}\sec\theta\cancel{\tan\theta} . d\theta .$ $\qquad \tan^2 - 1 = \frac{x^2}{a^2}$

$\int \sec\theta d\theta .$ $\qquad \tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

$\ln(\sec\theta + \tan\theta) + c .$

$\Rightarrow \ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right) + c$

Q2. Find indefinite integral (partial fractions)

(1) $\int \frac{6x+5}{x+2}\, dx$

$$\begin{array}{r} 6 \\ x+2\,\overline{)\,6x+5} \\ \underline{-6x \mp 12} \\ -7 \end{array}$$

$\int 6 . dx + \int \frac{-7}{x+2} dx$

$\Rightarrow 6x - 7\ln|x+2| + c$ (Ans)

(2) $\int \frac{4x^2-12x-25}{x-5}\, dx$

$$\begin{array}{r} 4x+8 \\ x-5\,\overline{)\,4x^2-12x-25} \\ \underline{-4x^2 \mp 20x} \\ 8x-25 \\ \underline{-8x \mp 40} \\ 15 \end{array}$$

$\int 4x+8 + \int \frac{15}{x-5}$

$\frac{4x^2}{2} + 8x + 15\ln|x-5| + c$

$\Rightarrow 2x^2 + 8x + 15\ln|x-5| + c$ (Ans)

(3) $\int \frac{5x^3+3x-2}{x-1}\, dx$

$$\begin{array}{r} 5x^2+5x+8 \\ x-1\,\overline{)\,5x^3+3x-2} \\ \underline{-5x^3 \mp 5x^2} \\ 5x^2+3x-2 \\ \underline{-5x^2 \mp 5x} \\ 8x-2 \\ \underline{-8x \mp 8} \\ 6 \end{array}$$

$\int 5x^2+5x+8 + \int \frac{6}{x-1}$

$\Rightarrow \frac{5x^3}{3} + \frac{5x^2}{2} + 8x + 6\ln|x-1| + c$ (Ans)

(4) $\int \frac{x^3+3x^2-4x-6}{x^2+2x-15}\, dx$

$$\begin{array}{r} x+1 \\ x^2+2x-15\,\overline{)\,x^3+3x^2-4x-6} \\ \underline{-x^3 \pm 2x^2 \mp 15x} \\ x^2+11x-6 \\ \underline{-x^2 \pm 2x \mp 15} \\ 9x+9 \end{array}$$

$\int x+1 + \int \frac{9x+9}{x^2+2x-15}$

$\int x+1 + \int \frac{9}{2(x-3)} + \frac{9}{2(x+5)}$

$\frac{x^2}{2} + x + \frac{9}{2}\ln|x-3| + \frac{9}{2}\ln|x+5| + c$ (Ans)

$$\frac{A(x+5)+B(x-3)}{(x-3)(x+5)} = \frac{9x+9}{(x-3)(x+5)}$$

$A = 9 \qquad B = 9$

(5) $\frac{4x^2-8x+3}{x^2-3x-4}$ dx

$$x^2-3x-4 \overset{4}{\overline{\big)\, 4x^2-8x+3}}$$

$$\underline{\overset{}{4x^2 \overset{+}{-} 12x \overset{+}{-} 16}}$$

$$4x+19$$

$4\int 1.dx + \int \frac{4x+19}{(x+1)(x-4)} dx$

$4x + \int \frac{-3}{x+1} + \frac{7}{x-4}$

$\Rightarrow 4x - 3\ln|x+1| + 7|x-4| + c$ (Ans)

$4x+19 = A(x-4) + B(x+1)$

$A=-3 \quad B=7$

$\frac{4x+19}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$

$A = -3 \quad B = 7$

(6) $\int \frac{x^3-3x^2}{x^2-3x-10} dx$

$$x^2-3x-10 \overset{x}{\overline{\big)\, x^3-3x^2}}$$

$$\underline{-x^3 \mp 3x^2 \mp 10x}$$

$$10x$$

$\int x + \int \frac{10x}{x^2-3x-10}$

$\int x + \int \frac{20}{7(x+2)} + \frac{50}{7(x-5)}$

$\frac{10x}{(x+2)(x-5)} = \frac{A}{x+2} + \frac{B}{x-5}$

$\Rightarrow \frac{x^2}{2} + \frac{20}{7}\ln|x+2| + \frac{50}{7}\ln|x-5| + c$ (Ans).

$A = 20/7 \quad B = 50/7$

Q3. Find indefinite integrals:

(1) $\int \frac{3-4x}{x(x+1)} dx$

$x=-1$: $B=-7$

$x=0$: $A=3$

$\frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)}$

$\Rightarrow \int \frac{3}{x} - \frac{7}{x+1} = 3\ln|x| - 7\ln|x+1| + C$ (Ans)

(2) $\int \frac{x}{x^2+7x+10} = \int \frac{x}{(x+2)(x+5)}$

$\frac{A}{x+2} + \frac{B}{x+5}$

$\int \frac{-2}{3(x+2)} + \frac{5}{3(x+5)}$

$x=-5$: $B = 5/3$

$x=-2$: $A = -2/3$

$\Rightarrow -\frac{2}{3}\ln|x+2| + \frac{5}{3}\ln|x+5| + c$ (Ans)

(3) $\int \frac{6}{3x^2-14x+8}\,dx = \int \frac{6}{(x-4)(x-2/3)}\,dx$

$\int \frac{9}{5(x-4)} - \frac{9}{5(x-2/3)}$

$\frac{9}{5}\ln|x-4| - \frac{9}{5}\ln|x-2/3| + c$

$\Rightarrow \frac{9}{5}\ln\left|\frac{x-4}{x-2/3}\right| + c$ (Ans)

$6 = A(x-2/3) + B(x-4)$

$x = 2/3 \quad : \quad B = -9/5$

$x = 4 \quad : \quad A = 9/5$

(4) $\int \frac{3x^2+8x-7}{(x+4)(x+3)(x+1)}\,dx$

$\int \frac{3}{x+4} + \frac{2}{x+3} - \frac{2}{x+1}\,dx$

$3\ln|x+4| + 2\ln|x+3| - 2\ln|x+1| + c$

$\Rightarrow \ln\left|\frac{(x+4)^3 . (x+3)^2}{(x+1)^2}\right| + c$ (Ans)

$A(x+3)(x+1) + B(x+4)(x+1) + C(x+4)(x+3) = 3x^2+8x-$

$x = -3 \quad : \quad B = 2$

$x = -4 \quad : \quad A = 3$

$x = -1 \quad : \quad C = -2$

(5) $\int \frac{2-4x^2}{(x+2)(x-2)(x-5)}\,dx$

$\int \frac{-1}{2(x+2)} + \frac{7}{6(x-2)} - \frac{14}{3(x-5)}$

$-\frac{1}{2}\ln|x+2| + \frac{7}{6}|x-2| - \frac{14}{3}\ln|x-5|$

$\Rightarrow \ln\left(\frac{(x-2)^{7/6}}{(x+2)^{1/2}(x-5)^{14/3}}\right) + c$ (Ans)

$A(x-2)(x-5) + B(x+2)(x-5) + C(x+2)(x-2) = 2-4x^2$

$x = 2 \quad : \quad B = 7/6$

$x = -2 \quad : \quad A = -1/2$

$x = 5 \quad : \quad C = -14/3$

(6) $\int \frac{3x}{(x+4)(x-1)(x-3)}\,dx$

$\int \frac{-12}{35(x+4)} - \frac{3}{10(x-1)} + \frac{9}{14(x-3)}$

$\frac{-12}{35}\ln|x+4| - \frac{3}{10}\ln|x-1| + \frac{9}{14}\ln|x-3|$

$\Rightarrow \ln\left|\frac{(x+4)^{-12/35} . (x-3)^{9/4}}{(x-1)^{3/10}}\right| + c$ (Ans)

$A(x-1)(x-3) + B(x+4)(x-3) + C(x+4)(x-1) = 3x$

$x = 1 \quad : \quad B = -3/10$

$x = 3 \quad : \quad C = 9/14$

$x = -4 \quad : \quad A = -12/35$

(7) $\int \frac{3-2x}{x^2+6x+9}$

$\int \frac{-2}{x+3} + \frac{9}{(x+3)^2}$

$\Rightarrow -2\ln|x+3| - \frac{9}{(x+3)} + C$ (Ans)

$\frac{A}{x+3} + \frac{B}{(x+3)^2} = \frac{3-2x}{(x+3)^2}$

$A(x+3) + B = 3-2x$

$x=-3 \; : \; B=9$

$x=1 \; : \; A=-2$

(8) $\int \frac{3x-1}{x^3-2x^2} \, dx$

$\int \frac{-5}{4x} + \frac{1}{2x^2} + \frac{5}{4(x-2)} \, .dx$

$\Rightarrow -\frac{5}{4}\ln|x| - \frac{1}{2x} + \frac{5}{4}\ln|x-2| + C$ (Ans)

$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = Ax^2(x-2)+B(x-2) + C(x^2)$

$x=2 \; : \; C=5/4$

$x=0 \; : \; B=1/2$

$x=1 \; : \; A=-5/4$

(9) $\int \frac{2x^2+x+4}{(x+1)(x-4)^2} \, dx$

$\int \frac{1}{5(x+1)} + \frac{9}{5(x-4)} + \frac{8}{(x-4)^2}$

$\Rightarrow \frac{1}{5}\ln|x+1| + \frac{9}{5}\ln|x-4| - \frac{8}{x-4} + C$ (Ans)

$A(x-4)^2 + B(x+1)(x-4) + C(x+1) = 2x^2+x+4$

$x=4 \; : \; C=8$

$x=-1 \; : \; A=1/5$

$x=1 \; : \; B=9/5$

(10) $\int \frac{5x^2+8x+6}{(x+4)(x^2+2)} \, dx = \frac{A}{x+4} + \frac{Bx+C}{x^2+2}$

$\int \frac{3}{x+4} + \frac{2x}{x^2+2} \longrightarrow$

$u = x^2$

$du = 2x \, .dx$

$\Rightarrow 3\ln|x+4| + \ln|x^2+2| + C$ (Ans)

$A(x^2+2) + (Bx+C)(x+4) = 5x^2+8x+6$

$x=4 \; : \; A=3$

$x=1 \quad 10 = 5B+5C \quad B=2$

$x=0 \quad 6 = 6+4C \quad C=0$

(11) $\int \frac{12x+8}{(x-3)(2x^2+8x+9)} \, .dx$

$\frac{A}{x-3} + \frac{Bx+C}{2x^2+8x+9}$

$\int \frac{18}{17(x-3)} - \frac{(36x+48)}{17(2x^2+8x+9)}$

$\Rightarrow \frac{18}{17}\ln|x-3| + 3\ln|2x^2+8x+9| + C$ (Ans)

$x=3 \; : \; A = 18/17$

$x=0 \; : \; C = -48/17$

$x=1 \; : \; B = -36/17$