21K-3278-D
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Problem 1 :

(a) $(D^4 + 6D^3 + 15D^2 + 20D + 12)y = 0$

$\because$ Homogenous higher order diff eq:

$m^4 + 6m^3 + 15m^2 + 20m + 12 = 0$

$m=-2$ $\quad m=-2$ $\quad m=-1+\sqrt{2}i$ $\quad m=-1-\sqrt{2}i$

$y_c = c_1 e^{-2x} + c_2 x e^{-2x} + e^{-x}(c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x)$

$y = y_c + \cancel{y_p}^{\,0} \rightarrow y = y_c$

$y = c_1 e^{-2x} + c_2 x e^{-2x} + e^{-x}(c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x)$

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(b) $(D^3 - 27)y = 0$

$\therefore$ Auxillary eq

$m^3 - 27 = 0$

$m = 3 \qquad m = -\frac{3}{2} + \frac{3\sqrt{3}}{2} \qquad m = -\frac{3}{2} - \frac{3\sqrt{3}}{2}$

$$y_c = c_1 e^{3x} + e^{-3/2 x}\left[c_2 \cos \frac{3\sqrt{3}}{2}x + c_3 \sin \frac{3\sqrt{3}}{2}x\right]$$

$$y = y_c + \cancel{y_p}^{\,0}$$

$$y = c_1 e^{3x} + e^{-3/2 x}\left[c_2 \cos \frac{3\sqrt{3}}{2}x + c_3 \sin \frac{3\sqrt{3}}{2}x\right]$$

Problem 2 :

(a) $(D^2 - 7D + 12)y = e^{2x}(x^3 - 5x^2)$

Auxillary eq.

$m^2 - 7m + 12 = 0$ $\qquad y_p = e^{2x}(Ax^3 + Bx^2 + Cx + D)$

$m = 3 \qquad m = 4$

$y_c = c_1 e^{3x} + c_2 e^{4x}$

$y_p' = e^{2x}(3Ax^2 + 2Bx + C) + 2e^{2x}(Ax^3 + Bx^2 + Cx + D)$

$y_p'' = e^{2x}(6Ax + 2B) + 2e^{2x}(3Ax^2 + 2Bx + C) + 2e^{2x}(3Ax^2 + 2Bx + C) + 4y_p$

$y_p'' = e^{2x}(6Ax + 2B) + 4e^{2x}(3Ax^2 + 2Bx + C) + 4y_p$

$e^{2x}(6Ax + 2B) + 4e^{2x}(3Ax^2 + 2Bx + C) - 7e^{2x}(3Ax^2 + 2Bx + C) - 14e^{2x}(Ax^3 + Bx^2 + Cx + D)$

$+ 4y_p + 12y_p$

$= e^{2x}(x^3 - 5x^2)$

$e^{2x}(6Ax + 2B) - 3e^{2x}(3Ax^2 + 2Bx + C) + 2e^{2x}(Ax^3 + Bx^2 + Cx + D) = e^{2x}(x^3 - 5x^2)$

$x^3 : \quad +2A = 1$ $\qquad x^2 : \quad -9A + 2B = -5$

$A = +1/2$ $\qquad 2B = -5 + 9/2$

$B = -1/4$

$x : \quad 6A - 6Bx + 2Cx = 0$

$3 + 6/4 + 2C = 0$ $\qquad c : \quad 2B - 3C + 2D = 0$

$C = -9/4$ $\qquad -1/2 + 27/4 + 2D = 0$

$D = -25/8$

$y_p = e^{2x}\left(\frac{x^3}{2} - \frac{x^2}{4} - \frac{9x}{4} - \frac{25}{8}\right)$ $\qquad y = y_c + y_p$

$y = c_1 e^{3x} + c_2 e^{4x} + e^{2x}\left(\frac{x^3}{2} - \frac{x^2}{4} - \frac{9x}{4} - \frac{25}{8}\right)$

(b) $y'' + y' - 2y = x^2 + 2\sin x - e^{3x}$

Auxilary eq. $(D^2 + D - 2)y = 0$

$D = -2 \qquad D = 1$

$y_c = c_1 e^{-2x} + c_2 e^{x}$

form of $y_p = (Ax^2 + Bx + C) + D\cos x + E\sin x + Fe^{3x}$

$y_p' = 2Ax + B - D\sin x + E\cos x + 3Fe^{3x}$

$y_p'' = 2A - D\cos x - E\sin x + 9Fe^{3x}$

$y'' + y' - 2y = x^2 + 2\sin x - e^{3x}$

$2A - D\cos x - E\sin x + 9Fe^{3x} + 2Ax + B - D\sin x + E\cos x + 3Fe^{3x} - 2Ax^2 - 2Bx$

$-2C - 2D\cos x - 2E\sin x - 2Fe^{3x} = x^2 + 2\sin x - e^{3x}$

$x^2 : -2A = 1$

$A = -\frac{1}{2}$

$\sin x : -E - D - 2E = 2$

$-3(3D) - D = 2$

$-9D - D = 2$

$D = -\frac{1}{5}$

$-D + E - 2D = 0$

$E = 3D.$

$E = -\frac{3}{5}$

$e^{3x} : 9F + 3F - 2F = -1$

$F = -\frac{1}{10}$

$2A - 2B = 0$

$B = -\frac{1}{2}$

$2A + B - 2C = 0$

$-1 - \frac{1}{2} - 2C = 0$

$C = -\frac{3}{4}$

$y_p = \left(-\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}\right) - \frac{1}{5}\cos x - \frac{3}{5}\sin x - \frac{e^{3x}}{10}$

$y = y_c + y_p$

$y = c_1 e^{-2x} + c_2 e^{x} - \frac{x^2}{2} - \frac{x}{2} - \frac{3}{4} - \frac{\cos x}{5} - \frac{3\sin x}{5} - \frac{e^{3x}}{10}$

Problem 3.

(a) $(D^2+1)y = \csc x$.

$\therefore$ Auxillary eq.

$D^2+1 = 0$

$D = \pm i$ $\qquad y_c = c_1 \cos x + c_2 \sin x$.

$y_1 = \cos x \qquad y_2 = \sin x$

$y_1' = -\sin x \qquad y_2' = \cos x$.

$W = \cos^2 x - (-\sin^2 x)$

$W = \cos^2 x + \sin^2 x = 1$

$$u = \int -\frac{y_2 X}{W}$$

$u = \int -\sin x \csc x \Rightarrow \int -1$

$u = -x$

$u = \sin x$

$du = \cos x \, dx$.

$$v = \int \frac{y_1 X}{W} \Rightarrow \int \cos x \csc x \Rightarrow \int \cot x. \Rightarrow \int \frac{\cos x}{\sin x}$$

$v = \int \frac{1}{u} \Rightarrow \ln u \Rightarrow \ln|\sin x|$

$y_p = u y_1 + v y_2$

$y_p = -x\cos x + \sin x \ln|\sin x|$

$y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \ln|\sin x|$

(b) $(D^2-1)y = \frac{2}{1+e^x}$

Auxillary eq.

$D^2-1 = 0$

$D = \pm 1$ $\quad y_c = c_1 e^x + c_2 e^{-x}$

$y_1 = e^x \quad y_2 = e^{-x}$

$u = \int -\frac{X\, y_2}{W} = \int \frac{\left(\frac{-2}{1+e^x}\right)(e^{-x})}{-2}$ $\quad y_1' = e^x \quad y_2' = -e^{-x}$

$W = -1 - 1 = -2$

$u = \frac{+1}{2}\int \frac{-2e^x}{1+e^x} = \int \frac{e^{-x}}{1+e^x}$ $\quad u = e^x$, $du = e^x\,dx$.

$\int \frac{1}{u^2(1+u)} \Rightarrow \int\left(-\frac{1}{u}+\frac{1}{u^2}+\frac{1}{1+u}\right) \Rightarrow -\ln u - u^{-1} + \ln(1+u) \Rightarrow -\frac{1}{u} + \ln\left[\frac{(1+u)}{(u)}\right]$

$u \Rightarrow \frac{1}{e^x} + \ln\frac{(1+e^x)}{e^x}.$

$v = \int \frac{X\, y_1}{W} \Rightarrow \int \frac{\frac{2}{1+e^x}(e^x)}{-2} \Rightarrow -\int \frac{e^x}{1+e^x} \Rightarrow -\ln|1+e^x|$

$y_p = u y_1 + v y_2 = \left[\frac{-1}{e^x} + \ln\left|\frac{(1+e^x)}{e^x}\right|\right]e^x - e^{-x}\ln|1+e^x|$

$y = y_c + y_p.$

$y = c_1 e^x + c_2 e^{-x} - 1 + e^x \ln\left|\frac{(1+e^x)}{e^x}\right| - e^{-x}\ln|1+e^x|$

Problem 4 :

(a) $y'' + y' + \frac{1}{4}y = e^{x}(\sin 3x + \cos 3x)$

$D^2y + Dy + \frac{1}{4}y = 0$

$D^2 + D + \frac{1}{4} = 0 \qquad D = -\frac{1}{2} \qquad D = -\frac{1}{2}$

$y_c = (c_1 + c_2 x)\, e^{-x/2}$

$y_c = c_1 e^{-x/2} + c_2 x e^{-x/2}$

$\text{Ann} = (D^2 - 2D + 10)$

$(D^2 - 2D + 10)(D^2 + D + \frac{1}{4})y = (D^2 - 2D + 10)\, e^{x}(\sin 3x + \cos 3x)$

$(D^2 - 2D + 10)(D + \frac{1}{2})^2 y = 0 \qquad (D^2 + 3^2)$

$\downarrow$

$D = 1 \pm 3i \qquad y_p = e^{x}(c_3 \cos 3x + c_4 \sin 3x)$

$y_p' = e^{x}(-3c_3 \sin 3x + 3c_4 \cos 3x) + e^{x}(c_3 \cos 3x + c_4 \sin 3x)$

$y_p'' = e^{x}(-9c_3 \cos 3x - 9c_4 \sin 3x) + e^{x}(-3c_3 \sin 3x + 3c_4 \cos 3x)$

$+ e^{x}(-3c_3 \sin 3x + 3c_4 \cos 3x) + e^{x}(c_3 \cos 3x + c_4 \sin 3x)$

$\left(9B - \frac{27}{4}A\right) e^{x}\cos 3x - \left(9 + \frac{27}{4}B\right) e^{x}\sin x = e^{x}(\sin 3x + \cos 3x)$

$A = -4/225 \qquad B = -20/225$

$y = c_1 e^{-x/2} + c_2 x e^{-x/2} - \frac{4}{225} e^{x}\cos 3x - \frac{20}{225} e^{x}\sin 3x$

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(b) $y'' + 2y' + y = x^2 e^{-x}$

$D^2 y + 2Dy + y = 0$

$D^2 + 2D + 1 = 0$

$D = -1 \qquad D = -1 \qquad y_c = c_1 e^{-x} + c_2 x e^{-x}$

$\text{Ann}(x) = (D+1)^3$

$(D+1)^3 (D+1)^2 = 0$

$(D+1)^5 = 0 \qquad D = -1, -1, -1, -1, -1$

$y_0 = (\underbrace{c_1 + c_2 x}_{y_c} + c_3 x^2 + c_4 x^3 + c_5 x^4) e^{-x}$

$y_p = c_3 x^2 e^{-x} + c_4 x^3 e^{-x} + c_5 x^4 e^{-x}$

$y_p' = 2x c_3 e^{-x} - c_3 x^2 e^{-x} + 3c_4 x^2 e^{-x} - c_4 x^3 e^{-x} + 4c_5 x^3 e^{-x} - c_5 x^4 e^{-x}$

$y_p'' = 2c_3 e^{-x} - 2x c_3 e^{-x} - 2c_3 x e^{-x} + c_3 x^2 e^{-x} + 6c_4 x e^{-x} - 3c_4 x^2 e^{-x} - 3c_4 x^2 e^{-x}$

$+ c_4 x^3 e^{-x} + 12 c_5 x^2 e^{-x} - 4c_5 x^3 e^{-x} - 4c_5 x^3 e^{-x} + c_5 x^4 e^{-x}$

$x^2 : \qquad 1 = -2A \qquad e^{2x} :$

$A = -1/2 \qquad 10F = -1$

$F = \frac{-1}{10} \qquad D = \frac{-1}{5}$

$\left[\begin{array}{l} 2 = -3E - D \\ D = -3E - 2 \end{array}\right] \qquad y'' + 2y' + y = xe^{-x} \qquad E = -3/5$

$y_p = -\frac{1}{2}x^2 - \frac{1}{2}x \qquad x^3 : \quad 8C_5 = 0 \qquad C_5 = 0$

$2C_3 + 20C_5 = 1 \qquad x : \quad 6C_4 = 0 \qquad C_4 = 0$

$c_3 = 1/12 \qquad \Rightarrow \quad y_p = \frac{1}{12} x^4 e^{-x}$

$y = c_1 e^{-x} + c_2 e^{-x} x + \frac{1}{12} x^4 e^{-x}$

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Problem 5:

$$x^2y'' + xy' - y = x^3e^x$$

Auxillary eq.

$y = x^m$
$y' = mx^{m-1}$
$y'' = m(m-1)x^{m-2}$

$$m(m-1)x^m + mx^m - x^m = 0$$

$$x^m(m^2 - m + m - 1) = 0$$

$$m^2 - 1 = 0 \quad \rightarrow \quad m = \pm 1 \qquad y_c = c_1x^{-1} + c_2x$$

$y_1 = x^{-1} \qquad y_2 = x$

$$y'' + \left(\frac{1}{x}\right)y' - \left(\frac{1}{x^2}\right)y = xe^x \quad \therefore \text{standard form.}$$

$$u = \int -\frac{X y_2}{W} = \int -\frac{xe^x(x)}{2x^{-1}} = -\frac{1}{2}\int x^3e^x$$

x^{-1}	x
$-x^{-2}$	1

$$u = -\frac{1}{2}\left[x^3e^x - 3x^2e^x + 6xe^x - 6e^x\right] \qquad W = x^{-1} + x^{-1} \Rightarrow 2x^{-1}$$

$$v = \int \frac{X y_1}{W} = \int \frac{xe^x(x^{-1})}{2x^{-1}} = \frac{1}{2}\int xe^x = \frac{1}{2}\left[xe^x - e^x\right]$$

$$y_p = uy_1 + vy_2$$

$$y_p = -\frac{1}{2}x^{-1}\left[x^3e^x - 3x^2e^x + 6xe^x - 6e^x\right] + \frac{1}{2}x\left[xe^x - e^x\right]$$

$$y_p = \cancel{-\frac{1}{2}x^2e^x} + \frac{3}{2}xe^x - 3e^x + \frac{3e^x}{x} + \cancel{\frac{x^2e^x}{2}} - \frac{xe^x}{2}$$

$$y_p = xe^x - 3e^x + 3x^{-1}e^x$$

$$y = y_c + y_p$$

$$y = c_1x^{-1} + c_2x + xe^x - 3e^x + 3x^{-1}e^x$$

$\cos 2x = 2\cos^2 x - 1$

$\frac{d}{dx}\frac{\cos 2x + 1}{2} = \frac{d\cos^2 x}{dx}$.

$\frac{d}{dx}\left(\frac{\cos 2x}{2} + \frac{1}{2}\right) = -\sin 2x$

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Problem 6.

(i) $y_1 = \cos 2x$ $\quad y_2 = 1$ $\quad y_3 = \cos^2 x$

$$W = \begin{vmatrix} \cos 2x & 1 & \cos^2 x \\ -2\sin 2x & 0 & -\sin 2x \\ -4\cos 2x & 0 & -2\cos 2x \end{vmatrix}$$

$W = \cos 2x(0) - 1(4\sin 2x\cos 2x - 4\sin 2x\cos 2x) + \cos^2 x(0)$

$W = -1(0) \Rightarrow 0$

$\therefore$ so it is dependent.

(ii) $y_1 = x$ $\quad y_2 = x^{-2}$ $\quad y_3 = x^2\ln x$

$$W = \begin{vmatrix} x & x^{-2} & x^2\ln x \\ 1 & -2x^{-3} & x + 2x\ln x \\ 0 & 6x^{-4} & 1 + 2\ln x + 2 \end{vmatrix}$$

$W = x(-2x^{-3} - 4x^{-3}\ln x - 4x^{-3} - 6x^{-3} - 12x^{-3}\ln x) - x^{-2}$

$(1 + 2\ln x + 2 - 0) + x^2\ln x(6x^{-4} - 0)$

$W = -2x^{-2} - 4x^{-2}\ln x - 4x^{-2} - 6x^{-2} - 12x^{-2}\ln x - x^{-2} - 2x^{-2}\ln x + 6x^{-2}\ln x - 2x^{-2}$

$W = -15x^{-2} - 12x^{-2}\ln x \neq 0$

$\therefore$ so it is linearly independent.

$$y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx.$$

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(b) (i) $9y'' - 12y' + 4y = 0 \qquad y_1 = e^{2x/3}$

$y = uy_1 \rightarrow y = ue^{2x/3}$

$y' = \frac{2}{3}ue^{2/3x} + u'e^{2x/3}$

$y'' = \frac{2}{3}\left(\frac{2}{3}ue^{2/3x} + u'e^{2/3x}\right) + \frac{2}{3}u'e^{2/3x} + u''e^{2x/3}.$

$y'' = \frac{4}{9}ue^{2/3x} + \frac{4}{3}u'e^{2/3x} + u''e^{2/3x}$

$9y'' - 12y' + 4y = 0$

$9\left(\frac{4}{9}ue^{2/3x} + \frac{4}{3}u'e^{2/3x} + u''e^{2/3x}\right) - 12\left(\frac{2}{3}ue^{2/3x} + u'e^{2x/3}\right) + 4ue^{2x/3} = 0$

$4ue^{2/3x} - 8ue^{2/3x} + 4ue^{2/3x} + 12u'e^{2/3x} - 12u'e^{2/3x} + 9u''e^{2/3x} = 0$

$9e^{2/3x}u'' = 0 \qquad \int u'' = \int 0$

$\int u' = \int c_1$

$y_2 = (c_1x + c_2)e^{2x/3} \quad - (1) \qquad u = c_1x + c_2$

$\uparrow$

$y_2 = c_1xe^{2/3x} \qquad y = uy_1$

By formula : $y'' - \frac{4}{3}y' + \frac{4}{9}y = 0 \qquad p = -\frac{4}{3}$

$y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2} = e^{2x/3} \int \frac{e^{-\int -\frac{4}{3}dx}}{e^{4x/3}}$

$y_2 = e^{2x/3} \int \frac{\cancel{e^{4/3x}}}{\cancel{e^{4/3x}}} = e^{2x/3} \int 1\,dx$

$y_2 = xe^{2x/3} \quad - (2)$

By comparing eq (1) and (2) we get

$\Rightarrow y_2 = xe^{2x/3}$

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(ii) $y'' - 3y' + 2y = 5e^{3x}$ $\qquad y_1 = e^x$

$y = uy_1$

$y = ue^x$

$y' = ue^x + u'e^x$

$y'' = ue^x + u'e^x + u'e^x + u''e^x$

$y'' = ue^x + 2u'e^x + u''e^x$

$ue^x + 2u'e^x + u''e^x - 3ue^x - 3u'e^x + 2ue^x = 5e^{3x}$

$u''e^x - e^x u' = 5e^{3x}$

$v = u' \qquad v' = u''$

$v'e^x - ve^x = 5e^{3x}$

$v' - v = 5e^{2x}$ (Linear) $\qquad IF = e^{-x}$

$\int e^{-x}(v' - v) = \int 5e^x$

$ve^{-x} = 5e^x + c_1$

$y_2 = uy_1$

$v = 5e^{2x} + c_1 e^{+x}$

$\int u' = \int 5e^{2x} + c_1 e^x$

$u = \frac{5}{2}e^{2x} + c_1 e^x + c_2$

$y_2 = e^x \int \frac{e^{\int 3dx}}{(e^x)^2}$

$Y = \left(\frac{5}{2}e^{2x} + c_1 e^x + c_2\right)e^x$

$y_2 = e^x \int \frac{e^{3x}}{e^{2x}}$

$Y = \frac{5}{3}e^{3x} + c_1 e^{2x} + c_2 e^x$

$\uparrow y_P \qquad \uparrow y_2 \qquad \uparrow y_1$ (given)

$y_2 = e^x \int e^x$

$\boxed{y_2 = e^{2x}}$

// proved //

Problem 7 :

(a) $L = 4000$; $P = \frac{4000}{1 + be^{-kt}}$; since $P(0) = 40$

$40 = \frac{4000}{1 + be^{-k(0)}}$; $1 + b = \frac{400}{4}$; $b = 99$

$P(5) = 104$; $104 = \frac{4000}{1 + 99e^{-5k}}$

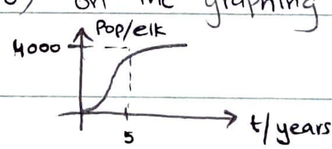
$99e^{-5k} = \frac{4000}{104} - 1$

$e^{-5k} = 0.378$

$k = 0.194$

$P = \frac{4000}{1 + 99e^{-0.1947}}$

(b) solution passes through $(0, 40)$ on the graphing utility.



(c) $t = 15$

$P = \frac{4000}{1 + 99e^{-0.197(15)}} \Rightarrow 626$

(d) $\lim_{t \to \infty} \frac{4000}{1 + 99e^{-0.197t}}$

$= \frac{4000}{1 + 99e^{-0.197(\infty)}}$

$= \frac{4000}{1}$

$\Rightarrow 4000$