P1 $\{\cos x, \cos 3x, \cos 5x \ldots\}$ $[0, \pi/2]$.

For orthogonal:

$$\int f_m(x)\, f_n(x)\, dx = 0$$

$f_m(x) = \cos(2\overset{n}{[\text{redacted}]}+1)x$

$f_n(x) = \cos(2m+1)x$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$\int_0^{\pi/2} \cos(2n+1)\cos(2m+1)$$

$$\frac{1}{2}\int_0^{\pi/2} \cos(2n+2m+2)+\cos(2n-2m)$$

$$\frac{1}{2}\int_0^{\pi/2} \cos(2n+2m+2) + \cos(2n-2m)$$

$$\frac{1}{2}\int_0^{\pi/2} \cos 2(n+m+1) + \cos 2(n-m)$$

$$\frac{1}{2}\left[\frac{\sin 2(n+m+1)x}{2(n+m+1)} + \frac{\sin 2(n-m)x}{2(n-m)}\right]_0^{\pi/2}$$

$$\sin \pi = 0$$

$$\frac{1}{2}[0+0-0-0] = 0$$

// Hence proved
→ orthogonal

For orthonormal:

$$\| f_n(x) \| = \sqrt{\int_a^b f_n^2(x)}$$

$$= \int_0^{\pi/2} \cos^2(2n+1)x = \int_0^{\pi/2} \frac{1+\cos 2(2n+1)}{2}\,dx$$

$$= \left[\frac{1}{2}x + \frac{\sin 2(2n+1)x}{4(2n+1)}\right]_0^{\pi/2}$$

$$= \sqrt{\frac{\pi}{4}} = \boxed{\frac{\sqrt{\pi}}{2}}$$

(b) $\{\sin x, \sin 3x, \sin 5x \ldots\}$ $[0, \pi/2]$

$f_n(x) = \sin(2n+1)x$

$f_m(x) = \sin(2m+1)x$

$\int_a^b f_n(x) f_m(x) = 0$

for orthogonal

$$\int_0^{\pi/2} \sin(2n+1) \sin(2m+1)$$

$2\sin A \sin B = \cos(A+B) - \cos(A-B)$

$$\frac{1}{2}\int_0^{\pi/2} \cos(2n+2m+2)x - \cos(2n-2m)x \, dx$$

$$\frac{1}{2}\int_0^{\pi/2} \cos 2(n+m+1)x - \cos 2(n-m)x$$

$$\frac{1}{2}\int_0^{\pi/2} \cos 2(n+m+1)x - \cos 2(n-m)x$$

$$\frac{1}{2}\left[\frac{\sin 2(n+m+1)x}{2(n+m+1)} - \frac{\sin 2(n-m)x}{2(n-m)}\right]_0^{\pi/2}$$

$$\frac{1}{2}[0 - 0 - 0 + 0] \Rightarrow 0$$

// orthogonal proved.

For orthornormal.

$$\sqrt{\int_a^b [\sin(2n+1)]^2}$$

$\cos 2x = \cancel{\cos^2 - \sin^2}$

$\cancel{\sin^2 - \cos^2}$

$1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$$\int_0^{\pi/2} \sin^2 (2n+1)$$

$$\frac{1}{2}\int_0^{\pi/2} 1 - \cos 2x$$

$$\frac{1}{2}\int_0^{\pi/2} 1 - \cos 2(2n+1)x$$

$$\frac{1}{2}\int_0^{\pi/2} 1 - \cos 2(2n+1)x = \left[x - \frac{\sin 2(2n+1)x}{4(2n+1)}\right]_0^{\pi/2}$$

$$\Rightarrow \frac{\pi}{2} - 0 - 0 + 0 = \boxed{\sqrt{\frac{\pi}{2}}}$$

$\| f_n(x) \|$

P2

$$f(x) = \begin{cases} \pi - x & 0 < x < \pi \\ 0 \,, & \pi < x < 2\pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right]$$

$$a_0 = \frac{1}{2\pi} \int_0^{\pi} (\pi - x)\, dx = \frac{1}{2\pi} \left[\pi x - \frac{x^2}{2} \right]_0^{\pi} = \frac{\pi}{4}$$

$$b_n = \frac{1}{2\pi} \int_0^{\pi} (\pi - x) \sin\left(\frac{nx}{2}\right) = \frac{-2\sin\left(\frac{n\pi}{2}\right)}{\pi n^2} + \frac{1}{n}$$

$$a_n = \frac{1}{2\pi} \int_0^{\pi} (\pi - x) \cos\left(\frac{nx}{2}\right) = \frac{-2}{\pi n^2} \left[\cos\left(\frac{n\pi}{2}\right) + 1 \right]$$

$$f(x) = \frac{\pi}{8} + \sum_{n=1}^{\infty} \left\{ \frac{-2}{\pi n^2} \left(\cos\left(\frac{n\pi}{2}\right) + 1 \right) \right\} \cos\left(\frac{nx}{2}\right) \Big\}$$
$$+ \sum_{n=1}^{\infty} \left\{ \frac{-2\sin\left(\frac{n\pi}{2}\right)}{\pi n^2} + \frac{1}{n} \right\} \sin\left(\frac{nx}{2}\right)$$

(Ans).

P3 (a)

$$f(x) = \begin{cases} 1 & -2 < x < -1 \\ 0 & -1 < x < 1 \\ 1 & 1 < x < 2. \end{cases}$$

Graph: mirror
so even: $b_n = 0$

cosine series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{2}{2} \int_0^2 f(x) = \frac{1}{2} \int_1^2 1 = 1$$

$$a_n = \frac{\cancel{2}}{\cancel{2}} \int_1^2 1 \cos\left(\frac{n\pi x}{2}\right) = \frac{\cancel{2}}{\cancel{2}} \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} = \left[\frac{2 \sin\left(\frac{n\pi x}{2}\right)}{n\pi}\right]_1^2$$

$$a_n = \frac{-2 \sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$\Rightarrow \quad f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2 \sin\left(\frac{n\pi}{2}\right)}{n\pi} \cos\left(\frac{n\pi x}{2}\right)$$

(Ans)

P3 (b) $f(x) = \begin{cases} x+1 & -1 < x < 0 \\ x-1 & 0 \leq x < 1 \end{cases}$

Graph : $180°$ appart or in opp. quadrant $\rightarrow$ odd. func.

sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{1}{L}\int_{-1}^{1} f(x) \sin \frac{n\pi x}{L} = \int_{-1}^{1} f(x) \sin(n\pi x)$$

$$b_n = \int_{-1}^{0} (x+1) \sin n\pi x \, dx + \int_{0}^{1} (x-1) \sin(n\pi x) \, dx$$

$$b_n = \int_{-1}^{0} x \sin n\pi x + \sin n\pi x + \int_{0}^{1} x \sin n\pi x - \sin n\pi x$$

$$b_n = \left[\frac{-x \cos n\pi x}{n\pi} + \frac{\sin n\pi x}{(n\pi)^2} \right]_{-1}^{0} + \left[-\frac{\cos n\pi x}{n\pi} \right]_{-1}^{0} + \left[\frac{-x \cos n\pi x}{n\pi} + \frac{\sin n\pi x}{(n\pi)^2} + \frac{\cos n\pi x}{n\pi} \right]_{0}^{1}$$

$$b_n = 0 + \frac{\cos \cancel{n\pi}}{\cancel{n\pi}} - \frac{\sin \cancel{n\pi}}{(\cancel{n\pi})^2} - \frac{1}{n\pi} + \frac{\cos -\cancel{n\pi}}{\cancel{n\pi}} - \frac{\cos \cancel{n\pi}}{\cancel{n\pi}} + \frac{\sin \cancel{n\pi}}{(\cancel{n\pi})^2} + \frac{\cos \cancel{n\pi}}{\cancel{n\pi}} - \frac{1}{n\pi}$$

$$b_n = -\frac{2}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n\pi} \sin(n\pi x) \quad \text{(Ans)}$$

P4

(a) $$y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = 0$$

$$U = XY \qquad \frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$y X'Y + x XY' = 0$$

$$y X'Y = -x XY'$$

$$\frac{X'}{xX} = -\frac{Y'}{yY} = -\lambda$$

$$X' + \lambda x X = 0 \qquad Y' - \lambda y Y = 0$$

If $\lambda = 0$

$$X' = 0 \qquad Y' = 0 \qquad U = XY$$

$$X = C_1 \qquad Y = C_2 \qquad U = C_1 C_2$$

$$U = A_1$$

If $\lambda \neq 0$ $\quad X' + \lambda x X = 0$

$$X' = -\lambda x X \qquad Y' - \lambda y Y = 0$$

$$\int \frac{X'}{X} dx = \int \lambda x \, dx \qquad \int \frac{Y'}{Y} = \int \lambda y$$

$$\ln X = \frac{\lambda x^2}{2} + C_1 \qquad \ln Y = \frac{\lambda y^2}{2} + C_2$$

$$X = C_1 e^{\frac{\lambda x^2}{2}} \qquad Y = C_2 e^{\frac{\lambda y^2}{2}}$$

$$U = XY$$

$$U = C_1 e^{\frac{\lambda x^2}{2}} C_2 e^{\frac{\lambda y^2}{2}} \text{ (Ans).}$$

P4 (b) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

$U = XT \qquad \dfrac{\partial^2 U}{\partial x^2} = X''T \qquad \dfrac{\partial^2 U}{\partial t^2} = XT''$

$a^2 X''T = XT''$

$\dfrac{X''}{X} = \dfrac{T''}{a^2 T} = -\lambda$

$\dfrac{X''}{X} = \dfrac{T''}{aT} = -\lambda$

$X'' + X\lambda = 0 \qquad T'' + \lambda a^2 T = 0$

if $\lambda = 0$

$X'' = 0 \qquad T'' = 0$

$X = c_1 x + c_2 \qquad T = c_3 t + c_4$

$U = XT$

$U = (c_1 x + c_2)(c_3 T + c_4)$

$U = (A_1 x + B_1)(C_1 T + D_1)$

if $\lambda = -\alpha^2$

$X'' - \alpha^2 X = 0 \qquad T'' - \alpha^2 a^2 T = 0$

$m^2 - \alpha^2 = 0 \qquad m^2 - \alpha^2 a^2 = 0$

$m = \pm\alpha \qquad m = \pm\alpha a$

$X = c_1 \cosh \alpha x + c_2 \sinh \alpha x \qquad T = c_3 \cosh \alpha a t + c_4 \sinh \alpha a t$

$U = XT$

$U = (c_1 \cosh \alpha x + c_2 \sinh \alpha x)(c_3 \cosh \alpha a t + c_4 \sinh \alpha a t)$

(Ans).

P5

$u(0,t) = 0 \qquad u(L,t) = 0 \qquad t > 0$

$u(x,0) = x(L-x) \qquad 0 < x < L$

$$X'' + \lambda^2 X = 0$$

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$$

$X(0) = 0 \qquad c_1 = 0$

$X(1) = 0 \qquad \lambda = \frac{n\pi}{L}$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

$$= -\frac{4}{L}\left(\frac{L}{n\pi}\right)^3 \left((-1)^n - 1\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4L^2(1-(-1)^n)}{n^2\pi^3} e^{-\frac{kn^2\pi^2}{L^2}} \frac{\sin(n\pi x)}{L}$$

P6

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = 0 \qquad \frac{\partial u}{\partial t} \quad t_s = x(L-x)$$

wave eq. $\dfrac{\partial^2 u(x,t)}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^3}(x,t)$

(i) $x(x) \quad T(t) = u(x,-1)$

$X(x) \quad T''(t) = c^2 T(t) x''(x)$

$$T''(t) - c^2 6 T(t) = 0$$

$$(r^2-6)e^{nx} = 0$$

$$r = \pm\sqrt{6}$$

$$T''(t) - c^2 6 T(t) = 0$$

$$s = c\sqrt{6}$$

$$= -\frac{1}{\pi}\int \frac{d}{dx}\left(\cos\left(\frac{\pi x}{L}\right)\right)$$

$$= \frac{k^2}{k\pi}\cos k\pi$$

$$u(x,t) = \sum_{n=1}^{\infty} in \sin \pi 4x \sin \frac{\pi n c t}{L}$$

$$\sum_{n=1}^{\infty} \frac{8L^3}{cn^4\pi^4} \sin\left(\frac{\pi n x}{2}\right) \cos\left(\frac{\pi 4 c t}{L}\right)$$

P7 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\left|\frac{\partial u}{\partial x}\right|_{x=0} = u(0,y)$, $u(\pi, y) = 1$

$u(x,0) = 0$, $u(x,\pi) = 0$

$u = X(x)\,Y(y)$

$\frac{\partial u^2}{\partial x^2} = X''Y$

$\frac{\partial u^2}{\partial y^2} = XY''$

case 1 $\lambda = 0$

$X = c_1 x + m_1$ $\quad Y = c_2 y + m_2$

$X = 0$ $\quad X'(0) = X(0)$ $\quad m_1 = c_1$

$c_1 = 0$

$u = (0)(c_2 y + m_2)$

$\lambda = -n^2 < 0$

$Y'' + n^2 T = 0$

$Y = c_5 \cos ny + c_6 \sin ny$

$c_4 = \frac{1}{n\cosh n\pi + \sinh n\pi}$ $\quad \frac{n\cosh nx + \sinh nx}{n\cosh n\pi + \sinh n\pi}$

$c_3 = \frac{n}{n\cosh n\pi + \sinh n\pi}$ $\quad y = c_6 \sin ny$

$u = \sum_{n=1}^{\infty} A_n \sin ny$

$A_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{2}{\pi}\left(\frac{1-(-1)^n}{n}\right)$

$u = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\frac{n\cosh nx + \sinh nx}{n\cosh nx + \sinh nx}\sin ny$

P8

wave eq by separable variable

$$c^2 \frac{\partial u^2}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$u(x,t) = v(x)\, w(t)$$

$$\frac{w''(t)}{wt} = \frac{c^2 v''(x)}{v(x)} = \lambda$$

$k > 0$

$k = p^2$

$T'' - kc^2 T = 0 \quad \& \quad X'' - kX = 0$

$T^2 - p^2 c^2 T = 0$

$T = c_1 e^{pct} + c_2 e^{-pct}$

$u(x,t) = [c_1 e^{pct} + c_2 e^{-pct}]$
$[c_3 e^{px} + c_4 e^{-px}]$

$T = 0$

$\& \ X'' = 0$

$u(x,t) = (c_1 t + c_2)(c_3 x + c_4)$

$k < 0$

$X'' + p^2 X = 0 \qquad (D^2 + p^2) x = 0$

$m^2 + p^2 c^2 = 0$

$m = \pm ipc$

$m^2 + p^2 = 0$

$m = \pm ip$

$T = (c_1 \cos pct + c_2 \sin pct)\, e^0$

$u(x,t) = (c_1 \cos pct + c_2\, pct)\,(c_3 \cos px + c_4 \sin px)$

Heat eq.

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad k = 0$$

$$u(0,t) = 0 \qquad X(0)T(t) = 0$$

$$X'' - kX = 0$$

$$k = \lambda^2 > 0 \qquad \lambda > 0$$

$$X'' - \lambda^2 X = 0$$

$$Ae^{\lambda L} - Ae^{-\lambda L} = 0 \qquad A = 0, \; B = 0$$

$$X(x) = B_n \sin \frac{n\pi}{L} x$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\lambda n^2 \pi^2}{L^2} t}$$

$$u(x,0) = f(x)$$

Laplace eq.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$X - kX = 0$ $\qquad$ $Y'' + kY = 0$

$k = 0$

$$u = (c_1 x + c_2)(c_3 y + c_4)$$

case 2 : $k > 0$ $\quad k = p^2$

$X'' - p^2 X = 0$

$m = \pm p$ $\qquad$ $X = c_1 e^{px} + c_2 e^{-px}$

$$u(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

case 3 : $k < 0$

$X'' + p^2 X = 0$ $\qquad$ $m = \pm p$

$m = \pm pi$ $\qquad$ $y = c_3 e^{py} + c_4 e^{-py}$

$X = (c_1 \cos px + c_2 \sin px)$

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$