LA-A03 21K-3278 Sohaib Shamsi

Q1.

$u_1 = (1, 1, 0, 2)$

$u_2 = (1, 0, 1, -1)$

$u_3 = (1, 0, 0, 1)$

$u_4 = (1, 1, 2, 1)$

$v_1 = u_1 = (1, 1, 0, 2)$

$$v_2 = u_2 - \frac{u_2 v_1}{\|v_1\|^2} v_1 = u_2 - \frac{(-1)}{(\sqrt{6})^2} = u_2 + \frac{1}{6} v_1$$

$v_2 = (7/6, 1/6, 1, -2/3)$

$$v_3 = u_3 - \frac{u_3 . v_1}{\|v_1\|^2} v_1 - \frac{u_3 v_2}{\|v_2\|^2} v_2 = u_3 - \frac{1}{2} v_1 - \frac{3}{17} v_2$$

$v_3 = (5/17, -9/17, -3/17, 2/17)$

$$v_4 = u_4 - \frac{u_4 v_1}{\|v_1\|^2} v_1 - \frac{u_4 v_2}{\|v_2\|^2} v_2 - \frac{u_4 v_3}{\|v_3\|^2} v_3$$

$v_4 = (-3/7, -3/7, 6/7, 3/7)$

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, 0\ \frac{\sqrt{6}}{3}\right)$$

$$q_2 = \left(\frac{7\sqrt{102}}{102}, \frac{\sqrt{102}}{102}, \frac{\sqrt{102}}{17}, \frac{-2\sqrt{102}}{51}\right)$$

$$q_3 = \left(\frac{5\sqrt{119}}{119}, \frac{-9\sqrt{119}}{119}, \frac{-3\sqrt{119}}{119}, \frac{2\sqrt{119}}{119}\right)$$

$$q_4 = \left(\frac{-\sqrt{7}}{7}, \frac{-\sqrt{7}}{7}, \frac{2\sqrt{7}}{7}, \frac{\sqrt{7}}{7}\right)$$

$$Q = \begin{bmatrix} \sqrt{6}/6 & 7\sqrt{102}/102 & 5\sqrt{119}/119 & -\sqrt{7}/7 \\ \sqrt{6}/6 & \sqrt{102}/102 & -9\sqrt{119}/119 & -\sqrt{7}/7 \\ 0 & \cancel{-9\sqrt{119}/119}\ \frac{\sqrt{102}}{17} & -3\sqrt{119}/119 & 2\sqrt{7}/7 \\ \sqrt{6}/3 & \cancel{4\sqrt{17}/17}\ -2\sqrt{102}/51 & 2\sqrt{119}/119 & \sqrt{7}/7 \end{bmatrix}$$

$$Q^T = \begin{bmatrix} \sqrt{6}/6 & \sqrt{6}/6 & 0 & \sqrt{6}/3 \\ 7\sqrt{102}/102 & \sqrt{102}/102 & \sqrt{102}/17 & -2\sqrt{102}/51 \\ 5\sqrt{119}/119 & -9\sqrt{119}/119 & -3\sqrt{119}/119 & 2\sqrt{119}/119 \\ -\sqrt{7}/7 & -\sqrt{7}/7 & 2\sqrt{7}/7 & \sqrt{7}/7 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} -\sqrt{6} & -\sqrt{6}/6 & \sqrt{6}/2 & 2\sqrt{6}/3 \\ 0 & \sqrt{102}/6 & \sqrt{102}/34 & 8\sqrt{102}/51 \\ 0 & 0 & \sqrt{119}/17 & -8\sqrt{119}/119 \\ 0 & 0 & 0 & 3\sqrt{7}/7 \end{bmatrix}$$

Q2. $Q = 5x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2 = x^TAx$

$$Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-5 & -2 & 0 \\ -2 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix}$$

$\lambda = 4, 1, 6$

$\lambda = 1$: $\begin{bmatrix} -4 & -2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = -1/2\, r$, $x_2 = r$, $x_3 = 0$ $\Rightarrow r(-1/2, 1, 0)$

$r(-1, 2, 0)$ v_1

$\lambda = 4$: $\begin{bmatrix} -1 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = 0$, $x_2 = s$, $x_3 = s$ $\Rightarrow s(0, 0, 1)$

v_2

$\lambda = 6$: $\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = 2t$, $x_2 = t$, $x_3 = 0$ $\Rightarrow t(2, 1, 0)$

v_3

$q_1 = \left(-\frac{1}{\sqrt5}, \frac{2}{\sqrt5}, 0\right)$ $q_2 = (0, 0, 1)$ $q_3 = \left(2/\sqrt5, 1/\sqrt5, 0\right)$

$Q \Rightarrow x = qy$

$Q = x^TAx = y^T(P^TAP)y = (y_1\ y_2\ y_3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$Q = y_1^2 + 4y_2^2 + 6y_3^2$

(Ans)

Q3.

sec 5.5

Q11 (a) The entry 0.2 represents the probability that the system will stay in state 1 when it is in state 1.

(b) The entry 0.1 represents the probability that the system will move to state 1 when it is in state 2.

(c) $\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$

Therefore $P(\text{state } 1 \to \text{state } 2) = \underline{0.8}$

(d) $\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.85 \end{bmatrix}$

If 50% chance of state 1 then probability that it will be in state 2 in next observation $= \underline{0.85}$

Q17. For stochastic, each column vector must be probability vector

$P(X) = 1$

column1 $\Rightarrow$ $7/10 + P_{21} + \frac{1}{10} = 1$

$P_{21} = 2/10 \Rightarrow 1/5$

column 2 $\Rightarrow$ $P_{12} + 3/10 + 3/5 = 1$

$P_{12} = 1/10$

column 3 $\Rightarrow$ $1/5 + P_{23} + 3/10 = 1$

$P_{23} = 5/10 \Rightarrow 1/2$

$P \Rightarrow \begin{bmatrix} 7/10 & 1/10 & 1/5 \\ 1/5 & 3/10 & 1/2 \\ 1/10 & 3/5 & 3/10 \end{bmatrix}$ (Ans)

$(I-P)q = 0$

RREF $\to$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

therefore steady state vector $q = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

$t = 1/3$

Q15.

(a) Transition matrix (P) $=$

$$\begin{array}{c c} & \begin{array}{cc} \text{city} & \text{suburbs} \end{array} \\ \begin{array}{c} \text{city} \\ \text{suburbs} \end{array} & \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \end{array}$$

$$\text{(initial) } X_0 = \begin{bmatrix} \frac{100,000}{125,000} \\ \frac{25,000}{125,000} \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

$$\text{After one year } (X_1) = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.766 \\ 0.234 \end{bmatrix}$$

$$\text{After two year } (X_2) = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.766 \\ 0.234 \end{bmatrix} = \begin{bmatrix} 0.73472 \\ 0.26528 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0.705942 \\ 0.294058 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 0.679467 \\ 0.320533 \end{bmatrix}$$

$$X_5 = \begin{bmatrix} 0.65510 \\ 0.344890 \end{bmatrix}$$

	K = 0	1 year	2 year	3 year	4 year	5 year
city population	100,000	95,750	91,840	88,243	84,933	81,889
suburb population	25,000	29,250	33,160	36,757	40,067	43,111

(b) $(I-P)\,q = 0$

$$\begin{bmatrix} 0.05 & -0.03 \\ -0.05 & 0.03 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3/5 \\ 0 & 0 \end{bmatrix} \qquad q_2 = t \qquad q_1 = \tfrac{3}{5}t \qquad t\begin{bmatrix} 1 \\ 3/5 \end{bmatrix}$$

$$t + 3/5\,t = 1 \qquad t = 5/8 \qquad = \begin{bmatrix} 5/8 \\ 3/8 \end{bmatrix}$$

city $= 125,000 \times 5/8 = 46,875$

suburb $= 125,000 \times 3/8 = 78,125.$

(Ans)