

On Circuit Description Languages, Indexed Monads, and Resource Analysis

KEN SAKAYORI, The University of Tokyo, Japan

ANDREA COLLEDAN, University of Bologna, Italy and Centre Inria d'Université Côte d'Azur, France

UGO DAL LAGO, University of Bologna, Italy and Centre Inria d'Université Côte d'Azur, France

In this paper, a monad-based denotational model is introduced and shown adequate for the Proto-Quipper family of calculi, themselves being idealized versions of the Quipper programming language. The use of a monadic approach allows us to separate the *value* to which a term reduces from the *circuit* that the term itself produces as a side effect. In turn, this enables the denotational interpretation and validation of rich type systems in which the size of the produced circuit can be controlled. Notably, the proposed semantic framework, through the novel concept of circuit algebra, suggests forms of effect typing guaranteeing quantitative properties about the resulting circuit, even in presence of optimizations.

CCS Concepts: • **Theory of computation** → Denotational semantics; Program analysis; • **Software and its engineering** → Domain specific languages; • **Hardware** → Quantum computation.

1 Introduction

Quantum computing promises to revolutionize various sub-fields within computer science by solving complex problems exponentially faster than classical computing [Shor 1994]. This technology leverages the concept of quantum bits (or qubits), a unit of information whose dynamics is governed by the rules of quantum mechanics, thus enabling superposition and entanglement, keys to the aforementioned speedup. To harness this potential, several programming languages have been developed specifically for quantum computing, such as Q# [Svore et al. 2018], Qiskit [Javadi-Abhari et al. 2024], and Cirq [2025]. In turn, fields like program verification have adapted well-known techniques like abstract interpretation [Perdrix 2008; Yu and Palsberg 2021], type systems [Amy 2019; Colledan and Dal Lago 2025], and Hoare logic [Liu et al. 2019; Ying et al. 2017; Zhou et al. 2019] to these languages.

We can identify at least two ways to design a quantum programming language. On the one hand, we could simply allow programs written in traditional programming languages to access not only *classical* data but also *quantum* data. The latter cannot be used the same way as the former, and is rather supported by specific *initialization*, *modification*, and *reading* operations. As an example, the reading of a qubit value, often called a *measurement*, can alter its value and has, in general, a probabilistic outcome, thus being substantially different from the corresponding classical operation. In programming languages of this kind, the quantum data are assumed to be stored in an external device accessible interactively through the aforementioned operations. This model, often indicated with the acronym QRAM [Knill 2022], is adopted by a multitude of proposals in the literature (see e.g. [Bettelli et al. 2003; Sanders and Zuliani 2000; Selinger 2004; Selinger and Valiron 2005]).

In theory, QRAM languages are the natural adaptation of classical programming languages to the quantum world. In practice, however, quantum hardware architectures can hardly be programmed *interactively*: not only is the number of qubits available very small, but the *time* within which computation must be completed should itself be minimized, given that the useful lifespan of a qubit is short. Consequently, quantum architectures typically take as input a whole quantum circuit, i.e. a precise description of *all* the necessary qubits and the operations to be performed on them. This circuit must therefore be available in its entirety, preferably already subjected to an

Authors' Contact Information: Ken Sakayori, The University of Tokyo, Japan; Andrea Colledan, University of Bologna, Italy and Centre Inria d'Université Côte d'Azur, France; Ugo Dal Lago, University of Bologna, Italy and Centre Inria d'Université Côte d'Azur, France.

aggressive optimization process. In such a context, so-called *circuit description languages* (CDLs for short) are to be preferred, and most mainstream languages in the field, including Qiskit and Cirq, are of this nature. CDLs are high-level languages used to describe and generate circuits, a quintessential example being the quantum circuit. Circuits are typically seen like any other ordinary data structure, with specific operations on them available through, e.g., methods or subroutines. Directly manipulating circuits from within a classical program offers the advantage of having more direct control over their shape and size. This is crucial given the state of quantum hardware architectures today, which provide a *limited number of error-prone* qubits, and for which *not all operations* can be implemented at the same cost.

A peculiar circuit description language is Quipper [Green et al. 2013]. In Quipper, circuits are not just like any other data structure. Rather, they are seen as the by-product of certain effect-producing computations that modify some underlying quantum circuit when executed. This, combined with the presence of higher-order functions and operations meant to turn any term (of an appropriate type) into a circuit, makes Quipper a very powerful and flexible idiom. Its metatheory has been the subject of quite some investigations by the programming language community in the last years, with contributions ranging from advanced type systems [Fu et al. 2020, 2022a] to fancy features like dynamic lifting [Colledan and Dal Lago 2023; Fu et al. 2022b, 2023] to denotational semantics [Fu et al. 2022b, 2023, 2022a; Lindenholvius et al. 2018; Rios and Selinger 2017]. This last aspect of Quipper, in particular, has been studied by providing semantic models for some of the languages of the so-called Proto-Quipper family, which includes various calculi, such as Proto-Quipper-M [Rios and Selinger 2017], Proto-Quipper-D [Fu et al. 2022a], Proto-Quipper-Dyn [Fu et al. 2023], etc. In these cases, such semantics are built around concepts such as that of a presheaf and turn out to take the shape of a LNL model [Benton 1994]. A by-product of the use of presheaves is that the interpretation of the term and the underlying circuit are somehow *merged* into a single mathematical object. As a result, it is difficult to read interesting features of the underlying circuit from the interpretation of a term or closure: what the circuit does and what the program does to produce the circuit are inextricably coupled.

This coupling, in turn, prevents those models from adequately accounting for variants of the Proto-Quipper family that are specifically designed to control the shape of the produced circuits, and more specifically to derive upper bounds on the size of the latter [Colledan and Dal Lago 2024, 2025]. The correctness of these systems has been proved by purely operational means, and a denotational semantics for them is still missing. This ultimately makes such systems somewhat rigid and complicates their definition.

The aim of this paper is precisely to give a denotational semantics to languages in the Proto-Quipper family in which the interpretation of terms is *kept separate* from that of the produced circuit. This is achieved by seeing circuit building as an indexed monad [Atkey 2009].¹ Remarkably, this new point of view allows us to give semantics to languages such as Colledan and Dal Lago’s Proto-Quipper-R, and even allows us to justify some of their peculiarities. The introduced semantic framework suggests a natural way to unify so-called *local* and *global* circuit metrics, at the same time allowing the definition of metrics that go substantially beyond those proposed by Colledan and Dal Lago, in particular accounting for simple forms of circuit optimization.

The contributions of this paper can be thus summarized as follows:

- First, we give a simple type system for Proto-Quipper. The introduced system is a slight variation on the theme of Proto-Quipper-M [Rios and Selinger 2017], whereas the input to the circuit produced by each effectful functional term needs to be exposed in its type

¹Indexed monads are also known as and were originally called parameterized monads. We avoid this name since other “parameters”, such as parameters of circuits or grades of a monad, appear in this work.

and thus becomes an integral part of the arrow type, turning it into a *closure type*. This change, as we will explain in the next section, seems inevitable since, without it, it would not be possible to know even the nature, i.e. the type, of the circuit produced by the term in question. Noticeably, closure types are present in Colledan and Dal Lago’s [2025] most recent contribution. We call this calculus with closure type Proto-Quipper-C.

- We then show that Atkey’s indexed monad is an appropriate framework for giving denotational semantics to Proto-Quipper-C. By treating circuits as (pre)monoidal morphisms and considering the category-action indexed monad—a many-sorted generalization of the writer monad—we maintain a clear separation between the value a term evaluates to and the circuit produced alongside the evaluation. Proto-Quipper-C is interpreted in the parameterized Freyd category induced by this indexed monad, making explicit how “parameters”, computations, and circuits interact as these three notions are interpreted in different categories. The semantics is proved both sound and computationally adequate.
- We then move to richer type systems, where simple types are enriched by a form of effect typing. The model based on indexed monads remains adequate, and suggests an abstract notion of circuit algebra, through which it is possible to capture various circuit metrics (including all those considered by Dal Lago and Colledan), but also new forms of metrics induced by assertion-based circuit optimization schemes [Häner et al. 2020].
- We briefly discuss how dependent types might be incorporated into both the syntax and semantics of the variant of Proto-Quipper-C (without effect typing) introduced earlier. On the semantic side, this is achieved by applying the families construction to the denotational model of Proto-Quipper-C in the spirit of fibered adjunction models [Ahman et al. 2016].

The rest of this paper is structured as follows. After the next section, which serves to frame the problem without going into the technical details, we move on to Section 3, in which Proto-Quipper-C, i.e. a slight variation of the Proto-Quipper-M calculus, is introduced and endowed with an operational semantics. In Section 4, then, a monadic denotational semantics for Proto-Quipper-C is introduced and proved adequate. In Section 5, an extension of these results to a calculus with effect typing, along the lines of Proto-Quipper-R, is presented. Section 6 discusses the possibility of extending our framework to incorporate dependent types. Section 7 discusses related work, and Section 8 concludes the paper.

2 A Monadic Semantics for CDLs: Why and How?

In this section, we will describe in a little more detail the problem of giving a monadic denotational semantics to CDLs, focusing on *how* this can be done, but also on *why* such an effort might be worth it.

Typically, a program written in a CDL is just a program in a mainstream programming language (e.g. Python, Haskell, or dialects thereof) whose purpose is that of facilitating the construction of (quantum) circuits which, once built, can then be sent to quantum hardware for their evaluation, or merely simulated through high-performance classical hardware. As already mentioned, the CDL we are mainly concerned with is Quipper, which is embedded in Haskell.

A program written in any CDL, and particularly in Quipper, does not describe a *single* circuit but a *family* of circuits, depending on some parameters, e.g. a number n representing the number of input qubits, or, in the case of Shor’s algorithm, the size of the natural number to be factored. The ability to describe families of circuits enables Quipper to succinctly and elegantly describe quantum algorithms, thus having a pragmatic impact and attracting the attention of the programming language community [Fu et al. 2023, 2022a; Lindenholvius et al. 2018; Rios and Selinger 2017], which

proposed idealized languages capturing the essence of Quipper. Such formal calculi come equipped with an operational semantics, type system, and often with a denotational semantics.

Most forms of denotational semantics for the Proto-Quipper family are based on presheaves, and enjoy a *constructive property*, which states that the interpretation of a judgment in a certain form is indeed a parameterized family of circuits. For example, in the categorical semantics of Proto-Quipper- \mathcal{M} [Rios and Selinger 2017], the judgment $n : \text{nat}, x : \text{Qubit} \vdash M : \text{Qubit}$ is interpreted as a function $\llbracket M \rrbracket : \mathbb{N} \rightarrow \mathcal{M}(\text{Qubit}, \text{Qubit})$, where $\mathcal{M}(\text{Qubit}, \text{Qubit})$ is the set of circuits whose input and output interfaces both consist of a single qubit. In the above, the type nat can be replaced by any “classical data type” and Qubit can be replaced by any “wire type”. However, some judgments cannot be interpreted as a family of circuits in the same way, including terms with free variables with a function type. As an example, a term of type

$$n : \text{nat}, x : (\text{Qubit} \multimap \text{Qubit}) \vdash N : \text{Qubit} \quad (1)$$

which evaluates to a qubit type value while generating a circuit, would be interpreted as $\llbracket N \rrbracket : \mathbb{N} \rightarrow \text{Nat}(\llbracket \text{Qubit} \multimap \text{Qubit} \rrbracket, \llbracket \text{Qubit} \rrbracket)$ where $\alpha \in \text{Nat}(\llbracket \text{Qubit} \multimap \text{Qubit} \rrbracket, \llbracket \text{Qubit} \rrbracket)$ is a natural transformation (i.e. a morphism in the presheaf category). Each component α_T has a type $\llbracket \text{Qubit} \multimap \text{Qubit} \rrbracket(T) \rightarrow \mathcal{M}(T, \text{Qubit})$ because $\llbracket \text{Qubit} \rrbracket$ is defined via the Yoneda embedding $\mathfrak{J} : \mathcal{M} \rightarrow [\mathcal{M}^{\text{op}}, \text{Set}]$ and $\llbracket \text{Qubit} \rrbracket(T) = \mathfrak{J}(\text{Qubit})(T) = \mathcal{M}(T, \text{Qubit})$. In other words, and more informally, N is interpreted as a family of polymorphic functions $\{f_i\}_{i \in \mathbb{N}}$, each of them having type

$$\forall(T : \text{WireType}). \text{Clos}(\text{Qubit}, \text{Qubit})[T] \rightarrow \mathcal{M}(T, \text{Qubit})$$

The type $\text{Clos}(\text{Qubit}, \text{Qubit})[T]$ represents a *closure type* disclosing the type T of the data that the closure captures. Given a “closure” as input, f_i returns a circuit whose input and output interfaces are T and Qubit , respectively. In a sense, then, not even the *input type* of the generated circuit can be read from $\llbracket N \rrbracket(n)$ because we need to know the *actual data* that will be passed to x to determine the interface of the circuit. This implies that modular reasoning about the produced circuits cannot be easily performed *within* the model, in which it would be hard to interpret type systems specifically built for intensional analysis [Colledan and Dal Lago 2024].

This paper introduces a denotational semantics for a CDL in which *every* judgment is interpreted as a family of circuits whose types are uniquely defined. More specifically, any judgment $\Gamma \vdash M : A$ is interpreted as a function

$$\llbracket M \rrbracket : \llbracket \flat \Gamma \rrbracket \rightarrow \llbracket \flat A \rrbracket \times \mathcal{M}(\llbracket \sharp \Gamma \rrbracket, \llbracket \sharp A \rrbracket), \quad (2)$$

where \sharp and \flat are operations extracting the “circuit part” and “parametric part” of any type, respectively. The mathematical object $\llbracket M \rrbracket$, in other words, is a family of pairs $\{(v_i, C_i)\}_{i \in \llbracket \flat \Gamma \rrbracket}$ indexed by $\llbracket \flat \Gamma \rrbracket$ where v_i is a “value” in $\llbracket \flat A \rrbracket$ and C_i is a circuit in $\mathcal{M}(\llbracket \sharp \Gamma \rrbracket, \llbracket \sharp A \rrbracket)$; a family of circuits is a special case where v_i is a unit value. The fact that *every* judgment is interpreted as a family of circuits is important since it allows us to *compositionally* reason about the family of circuits generated by a program. For example, upper bounds to the width of the circuits generated by any program can be computed by looking at the interpretation of the subprograms.

The just sketched construction evidently has the form of a monad [Moggi 1991], structurally very similar to a writer monad. There is, however, one important difference: the type of circuit produced during the execution of a term is *not* fixed a priori. As a consequence, the classical notion of monad, being somehow monomorphic, cannot be applied directly. Instead, indexed monads [Atkey 2009] can be fruitfully employed to model the circuit generated by the underlying program. In fact, the interpretation (2) can be rewritten as

$$\llbracket M \rrbracket : \llbracket \flat \Gamma \rrbracket \rightarrow \mathcal{T}(\llbracket \sharp \Gamma \rrbracket, \llbracket \sharp A \rrbracket, \llbracket \flat A \rrbracket), \quad (3)$$

where $\mathcal{T}(T, U, X)$ is an indexed monad defined as $X \times \mathcal{M}(T, U)$. Using such a monadic approach allows us to structure the interpretation of languages in the Proto-Quipper family in a new way, fundamentally different from that considered in the literature on the subject: every term is interpreted as a mathematical object in which the value produced and the underlying circuit are kept separate. Technically, we interpret terms in a parameterized Freyd category obtained by applying an indexed version of the well-known Kleisli construction to the indexed monad above.

If we look at Equation (3) in more detail, we soon realize that a monadic interpretation like the one we are discussing requires knowing $\llbracket \# \Gamma \rrbracket$ whenever the effectful term M is a value $\lambda x.M$ (where x is any of the variables in Γ). In other words, the “circuit portion” of Γ must become part of the (functional) type of $\lambda x.M$. This last observation justifies the small discrepancies between Proto-Quipper-C (the language we present in Section 3 below) and Proto-Quipper-M and provides a denotational reading to some of the type-theoretical tricks in [Colledan and Dal Lago 2024]. As an example, the typing judgment (1) becomes $n : \text{nat}, x : (\text{Qubit} \multimap_T \text{Qubit}) \vdash N : \text{Qubit}$, where T is the type of circuit variables the argument function captures.

The advantage of moving to a monadic view like the one just described is that we have now exposed the space \mathcal{M} of circuits in the interpretation. As we will see in Section 5, in fact, this naturally suggests a way to control the size of the generated circuits through a form of effect typing: any well-behaved functor from \mathcal{M} to a category \mathcal{E} which captures the relevant characteristics of the underlying circuit, e.g. its size, induces a form of effect typing which is sound by construction. This is what denotational semantics is good for: not only is the programming language in question interpreted compositionally, but the interpretation naturally suggests what in the language might be subject to modification or adaptation while, at the same time, indicating what the underlying axiomatics should be, and factoring out most proofs.

3 Simple Types

In this section, we introduce the syntax and operational semantics of Proto-Quipper-C, our dialect of Proto-Quipper-M [Rios and Selinger 2017]. We will point out the differences with the language Proto-Quipper-M, still trying to keep the presentation as self-contained as possible.

3.1 Type and Syntax

We first introduce the simple type system of Proto-Quipper-C. The grammar for types is as follows:

$$\begin{aligned} \text{Types } A, B &::= P \mid T \mid A \otimes B \mid A \multimap_T B \\ \text{Parameter Types } P, R &::= \mathbb{1} \mid \text{Nat} \mid !A \mid P \otimes R \mid \text{Circ}(T, U) \\ \text{Bundle Types } T, U &::= I \mid w \mid T \otimes U \end{aligned}$$

There are three kinds of types. In addition to generic types, which are intended for (not necessarily duplicable) terms, there are *parameter* types whose inhabitants are freely duplicable and which include circuits and values of type $!A$. There are the unit and natural number type as base types, but the specific choice of base types is not that important. We also need *bundle* types, which are used to give a type to circuit wires. Observe that the tensor product operator \otimes is available in the three kinds of types, while the construction of functions is not available in parameter and bundle types.

Typing is almost the same as in Proto-Quipper-M as defined in [Rios and Selinger 2017]. There is a significant difference in the definition of the arrow type, however. We annotate the arrow type with a bundle type T , which describes the types of the free variables captured by the function, effectively turning it into a form of *closure type*. Note that we only care about the bundle type variables that are captured by the function and drop the information of variables with parameter types. A similar kind of annotation was used by Colledan and Dal Lago [2024], although in their

work the label was rather a natural number abstracting the type T . We have already argued about the need for this change to the type system, and will come back to that in the next section. We might write $A \multimap B$ for $A \multimap_I B$ for the sake of simplifying the notation.

Another difference compared to Proto-Quipper-M (or Proto-Quipper-R, as defined in [Colledan and Dal Lago 2024]) is that we do not have a type for lists. We removed lists as they cannot be directly interpreted in our semantic model. However, as we shall see in Section 6, we can extend our language with a vector type (i.e. a list with specified length).

The syntax of Proto-Quipper-C terms is defined as follows. We use a fine-grained call-by-value style syntax [Levy et al. 2003] as done by Colledan and Dal Lago [2024].²

Terms $M, N ::= V W \mid \text{let } \langle x, y \rangle = V \text{ in } M \mid \text{ifz } V \text{ then } M \text{ else } N$

$\mid \text{force } V \mid \text{box}_T V \mid \text{apply}(V, W)$

$\mid \text{return } V \mid \text{let } x = M \text{ in } N$

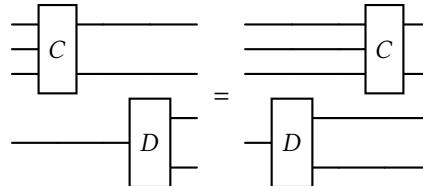
Values $V, W ::= * \mid n \mid x \mid \ell \mid \lambda x_A. M \mid \text{lift } M \mid (\bar{\ell}, C, \bar{k}) \mid \langle V, W \rangle$

Wire Bundles $\bar{\ell}, \bar{k} ::= * \mid \ell \mid \langle \bar{\ell}, \bar{k} \rangle$.

The informal behavior for terms is in line with that of Proto-Quipper-M, which adds to the usual constructs of a call-by-value linear lambda-calculus (abstractions, applications, let bindings, pairs, etc.) specific operators for circuit manipulation:

- ℓ is a *label*, that is, a pointer to a wire in the underlying circuit.
- $(\bar{\ell}, C, \bar{k})$ is a *boxed circuit* and represents the circuit C as a value in the language. Wire bundles $\bar{\ell}$ and \bar{k} represent the input and output interfaces of C , respectively.
- $\text{apply}(V, W)$ appends a boxed circuit V to the wires identified by W among the outputs of the underlying circuit.
- $\text{box}_T V$ evaluates a circuit building function V in isolation and returns the result as a boxed circuit.

At this point, we should make it clear what we mean by a “circuit”. We do not fix what a circuit is (except when considering concrete examples), as is often the case for Proto-Quipper calculi. Usually, when giving a semantics to Proto-Quipper, circuits are seen as morphisms in a *monoidal category* \mathcal{M} and the semantics is parametric to the choice of \mathcal{M} . In this work, we assume that the category \mathcal{M} of circuits is a *premonoidal* category. Roughly, a premonoidal category is a monoidal category without the interchange law



which is too strong in a cost sensitive scenario. (For example, we may say that the two circuits above are different because the width of the circuit on the left-hand-side is 4 whereas the width of the circuit on the right-hand-side is 5.)

Definition 1 (Premonoidal Category [Power and Robinson 1997]). A *binoidal category* is a category \mathcal{A} equipped with, for each $a \in \text{Obj}(\mathcal{A})$, endofunctors $a \multimap -$ and $- \multimap a$ from \mathcal{A} to \mathcal{A} such that

²It is easy to extend the language with effect-free constants such as arithmetic operations or meta-operations on circuits as in [Rios and Selinger 2017], but we omit these, since they are immaterial to our main semantic results.

$a \bowtie b = a \bowtie b \stackrel{\text{def}}{=} a \otimes b$ for every pair of objects (a, b) of \mathcal{A} . A morphism $f: a \rightarrow b$ in \mathcal{A} is *central* if it interchanges with any morphism $g: a' \rightarrow b': (f \bowtie a') \circ (b \bowtie g) = (a \bowtie g) \circ (f \bowtie b')$ and $(a' \bowtie f) \circ (g \bowtie b) = (g \bowtie a) \circ (b' \bowtie f)$.³ In case two composites agree, we write $f \otimes g$ and $g \otimes f$, respectively. A *premonoidal category* is a binoidal category equipped with an object I , the “monoidal unit”, central natural (separately at each given component) isomorphisms for associativity and right and left units satisfying the standard pentagon and triangle equations. A premonoidal category is *strict* if all the coherence morphisms are identities, and is *symmetric* if it has a central isomorphism $a \otimes b \cong b \otimes a$ that is natural (at each component) and satisfies the usual axioms of a symmetry. \triangleleft

Typically, the category of circuits \mathcal{M} is defined by giving a syntactic description of circuits. In other words, it is the free category generated by some collection of base types and gates. While we do not fix the category \mathcal{M} , we assume that \mathcal{M} has some distinguished objects Q and B that are used to interpret qubits and classical bits, respectively. We also assume that \mathcal{M} is small and a *strict symmetric premonoidal category*.⁴

A typing judgment for terms is of the form $\Gamma \vdash_c M : A$, and intuitively means that M is well-typed under the typing context Γ ; similarly, we have a typing judgment for values of the form $\Gamma \vdash_v V : A$. A *typing context* Γ is a finite sequence of bindings each of which is either of the form $x : A$ or $\ell : w$. The reason for using finite sequences rather than finite sets is to simplify the definition of the semantics; if two contexts Γ_1 and Γ_2 are equal up to permutation, then we write $\Gamma_1 \cong_\sigma \Gamma_2$. We use metavariables a, b, \dots to denote variables or labels. A typing context is a *label context* if it is of the form $\ell_1 : w_1, \dots, \ell_n : w_n$. Label contexts are denoted by Q, L . A *parameter context*, written Φ , is a typing context that only contains variables with parameter types.

Typing rules are in Figure 1, and most of them are self-explanatory. In the typing rules, when we write Φ, Γ , we stipulate that Γ does not contain any variable with a parameter type. Rules *circ*, *box* and *apply* are specific to a CDL, and thus warrant discussion. A boxed circuit $(\bar{\ell}, C, \bar{k})$ is well-typed if the labels $\bar{\ell}$ and \bar{k} acting as language-level interfaces to C have types that match with the (co)domain of C . The notation $C: Q \rightarrow L$ means that C is a morphism from $\llbracket Q \rrbracket$ to $\llbracket L \rrbracket$ in \mathcal{M} ; $\llbracket Q \rrbracket$ is the obvious interpretation, which we formally define in Section 4. The *box* rule says that if V is a circuit building function that, once applied to an input of type T , builds a circuit of output type U , then V can be turned into a circuit whose interface has types T and U . Note that the *box* rule requires the typing context to be Φ so as to ensure that the function is not capturing any variable with a bundle type. The rule *apply*, on the other hand, can be read as a special version of the typing rule for function application.

Once again, the main novelty with respect to Proto-Quipper- \mathcal{M} has to do with the arrow type. The operation \sharp used in the *abs* rule extracts a bundle type from any type A .⁵ Formally, the operation \sharp is inductively defined as follows:

$$\begin{aligned} \sharp(P) &\stackrel{\text{def}}{=} I & \sharp(I) &\stackrel{\text{def}}{=} I & \sharp(w) &\stackrel{\text{def}}{=} w & \sharp(A \multimap_T B) &\stackrel{\text{def}}{=} T \\ \sharp(P \otimes R) &\stackrel{\text{def}}{=} \sharp(P) \otimes \sharp(R) & \sharp(T \otimes U) &\stackrel{\text{def}}{=} \sharp(T) \otimes \sharp(U). \end{aligned}$$

We let the operator \sharp act on typing contexts by $\sharp(a_1 : A_1, \dots, a_n : A_n) \stackrel{\text{def}}{=} \sharp(A_1) \otimes \dots \otimes \sharp(A_n)$. It should now be clear that the rule *abs* does nothing more than inserting the *bundle* type of the variables free in the abstraction we are typing into its arrow type.

³We use diagrammatic order for compositions in this paper.

⁴Every premonoidal category is equivalent to a strict one since the coherence theorem holds [Power and Robinson 1997].

⁵A similarly looking operation was called *wire count* in [Colledan and Dal Lago 2024]. We do not use this name since we extract the whole type and not just a natural number counting “how many wires are used”.

unit	$\frac{}{\Phi \vdash_v * : \mathbb{1}}$	nat	$\frac{}{\Phi \vdash_v n : \text{Nat}}$	lab	$\frac{}{\Phi, \ell : w \vdash_v \ell : w}$	var	$\frac{}{\Phi, x : A \vdash_v x : A}$
abs	$\frac{\Gamma, x : A \vdash_c M : B}{\Gamma \vdash_v \lambda x_A.M : A \multimap_{(\Gamma)} B}$			app	$\frac{\Phi, \Gamma_1 \vdash_v V : A \multimap_T B \quad \Phi, \Gamma_2 \vdash_v W : A}{\Phi, \Gamma_1, \Gamma_2 \vdash_c V W : B}$		
lift	$\frac{\Phi \vdash_c M : A}{\Phi \vdash_v \text{lift } M : !A}$			force	$\frac{\Phi \vdash_v V : !A}{\Phi \vdash_c \text{force } V : A}$		
circ	$\frac{C : Q \rightarrow L \quad Q \cong_\sigma Q' \quad L \cong_\sigma L'}{Q' \vdash_v \bar{\ell} : T \quad L' \vdash_v \bar{k} : U}$			box	$\frac{\Phi \vdash_v V : T \multimap_I U}{\Phi \vdash_c \text{box}_T V : \text{Circ}(T, U)}$		
apply	$\frac{\Phi, \Gamma_1 \vdash_v V : \text{Circ}(T, U) \quad \Phi, \Gamma_2 \vdash_v W : T}{\Phi, \Gamma_1, \Gamma_2 \vdash_c \text{apply}(V, W) : U}$			dest	$\frac{\Phi, \Gamma_1 \vdash_v V : A \otimes B \quad \Phi, \Gamma_2, x : A, y : B \vdash_c M : C}{\Phi, \Gamma_2, \Gamma_1 \vdash_c \text{let } \langle x, y \rangle = V \text{ in } M : C}$		
ifz	$\frac{\Phi \vdash_v V : \text{Nat} \quad \Phi, \Gamma \vdash_c M : A \quad \Phi, \Gamma \vdash_c N : A}{\Phi, \Gamma \vdash_c \text{ifz } V \text{ then } M \text{ else } N : A}$			pair	$\frac{\Phi, \Gamma_1 \vdash_v V : A \quad \Phi, \Gamma_2 \vdash_v W : B}{\Phi, \Gamma_1, \Gamma_2 \vdash_v \langle V, W \rangle : A \otimes B}$		
return	$\frac{\Gamma \vdash_v V : A}{\Gamma \vdash_c \text{return } V : A}$			let	$\frac{\Phi, \Gamma_1 \vdash_c M : A \quad \Phi, \Gamma_2, x : A \vdash_c N : B}{\Phi, \Gamma_2, \Gamma_1 \vdash_c \text{let } x = M \text{ in } N : B}$		
ex		$\frac{\Gamma_1, a : A, b : B, \Gamma_2 \vdash_c M : C}{\Gamma_1, b : B, a : A, \Gamma_2 \vdash_c M : C}$					

Fig. 1. Typing Rules for Proto-Quipper-C.

The *box* rule is slightly different from the one commonly seen in the Proto-Quipper family, in which *box* is a coercion from $!(T \multimap U)$ to $\text{Circ}(T, U)$. In our rule, instead, we drop the $!$ operator. Intuitively, $!$ is needed to ensure that the function with type $T \multimap U$ does not capture any variable with bundle type. But this information is already explicit in our type system by the subscript I , and there is no reason to additionally require the of-course modality. We will later also give a semantic explanation against this design choice (see Proposition 1 and the remark after it).

3.2 Operational Semantics

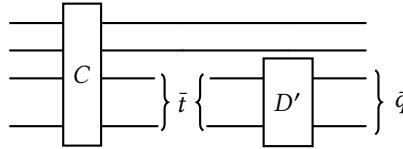
The operational semantics is defined as a big-step evaluation relation on *configurations*. A configuration is a pair (C, M) , where C is a circuit being generated and M is the term being evaluated. The definition of the big-step evaluation relation \Downarrow is in Figure 2. (The rule for evaluating the else branch of ifz is omitted.)

The *box* rule relies on the *freshlabels* function, which is used to produce a fresh label context Q and a wire bundle $\bar{\ell}$ such that $Q \vdash_v \bar{\ell} : T$. On the other hand, the *apply* rule relies on the *append* function, which attaches the circuit D to the wires identified by $\bar{\ell}$ among the outputs of C . This operation often requires a renaming of the labels in D , so that its input interface $\bar{\ell}$ matches $\bar{\ell}$. More formally, we say that two boxed circuits $(\bar{\ell}, D, \bar{k})$ and $(\bar{\ell}', D', \bar{k}')$ are *equivalent*, and we write $(\bar{\ell}, D, \bar{k}) \cong (\bar{\ell}', D', \bar{k}')$, if they only differ by a renaming of labels. What *append* does, then, is find

$\frac{(C, M[V/x]) \Downarrow (D, W)}{(C, (\lambda x_A.M) V) \Downarrow (D, W)}$	$\frac{(C, M[V/x][W/y]) \Downarrow (D, X)}{(C, \text{let } \langle x, y \rangle = \langle V, W \rangle \text{ in } M) \Downarrow (D, X)}$
$\frac{(C, M) \Downarrow (D, V)}{(C, \text{ifz } 0 \text{ then } M \text{ else } N) \Downarrow (D, V)}$	$\frac{(C, M) \Downarrow (D, V)}{(C, \text{force}(\text{lift } M)) \Downarrow (D, V)}$
$\frac{(E, \bar{q}) = \text{append}(C, \bar{t}, (\bar{\ell}, D, \bar{k}))}{(C, \text{apply}((\bar{\ell}, D, \bar{k}), \bar{t})) \Downarrow (E, \bar{q})}$	$\frac{(Q, \bar{\ell}) = \text{freshlabels}(T) \quad (id_Q, V \bar{\ell}) \Downarrow (D, \bar{k})}{(C, \text{box}_T V) \Downarrow (C, (\bar{\ell}, D, \bar{k}))}$
$\frac{}{(C, \text{return } V) \Downarrow (C, V)}$	$\frac{(C, M) \Downarrow (E, V) \quad (E, N[V/x]) \Downarrow (D, W)}{(C, \text{let } x = M \text{ in } N) \Downarrow (D, W)}$

Fig. 2. Proto-Quipper-C big-step operational semantics.

$(\bar{t}, D', \bar{q}) \cong (\bar{\ell}, D, \bar{k})$ and return \bar{q} , along with a circuit E defined as follows:



Overall, this semantics is the same as the one given in [Colledan and Dal Lago 2024], except for the *box* rule, which is modified to align with the modifications made in the typing rules.

3.3 Type Preservation

We write $Q \vdash (C, M) : A; Q'$ and say that the configuration (C, M) is *well-typed under Q and Q'* if $C : Q \rightarrow L, Q' \text{ and } L \vdash_c M : A$ for some label context L disjoint from Q' . Similarly, we write $Q \vdash (C, V) : A; Q'$ if $Q \vdash (C, \text{return } V) : A; Q'$. We have the following type preservation theorem.

Theorem 2 (Type Preservation). *If $Q \vdash (C, M) : A; Q'$ and $(C, M) \Downarrow (D, V)$, then $Q \vdash (D, V) : A; Q'$.*

PROOF. By induction of the derivation of $(C, M) \Downarrow (D, V)$. □

4 A Monadic Semantics for Proto-Quipper-C

Here we define a new monadic denotational semantics for Proto-Quipper-C, introduced in Section 3. We first explain the categorical structure we use such as the indexed monad for circuits. We then define the interpretation, and prove its soundness and adequacy.

4.1 The Circuit Monad

We now review the notion of indexed monads [Atkey 2009] (aka parameterized monads), which plays a key role in our model. Intuitively, an indexed monad is a “multi-sorted generalization” of a monad. Its formal definition is given as follows.

Definition 3 (Indexed Monad [Atkey 2009]). Let C be any cartesian category and S be a category. A (strong) S -indexed monad on C is a quadruple $(\mathcal{T}, \eta, \mu, \tau)$ where

- $\mathcal{T} : S^{\text{op}} \times S \times C \rightarrow C$ is a functor
- the *unit* η is a family of morphisms $\eta_{S,X} : X \rightarrow \mathcal{T}(S, S, X)$ natural in X and dinatural in S

- the *multiplication* μ is a family of morphisms $\mu_{S_1, S_2, S_3, X}: \mathcal{T}(S_1, S_2, \mathcal{T}(S_2, S_3, X)) \rightarrow \mathcal{T}(S_1, S_3, X)$ natural in S_1, S_3 and X and dinatural in S_2 .
- the *strength* τ is a family of morphisms $\tau_{X, S_1, S_2, Y}: X \times \mathcal{T}(S_1, S_2, Y) \rightarrow \mathcal{T}(S_1, S_2, X \times Y)$ natural in X, S_1, S_2 and Y .

The unit and multiplication must obey the evident monad laws and the axiom for strength (cf. Definition 9). \triangleleft

The indexed monad we are interested in is the circuit monad $\mathcal{T}_M: M^{\text{op}} \times M \times \text{Set} \rightarrow \text{Set}$. The circuit monad is defined as follows⁶:

$$\begin{aligned}\mathcal{T}_M(T, U, X) &\stackrel{\text{def}}{=} X \times M(T, U) \\ \eta_{T, X}(x) &\stackrel{\text{def}}{=} (x, \text{id}_T) \\ \mu_{T_1, T_2, T_3, X}((x, f), g) &\stackrel{\text{def}}{=} (x, f; g) \\ \tau_{X, T, U, Y}(x, (y, f)) &\stackrel{\text{def}}{=} ((x, y), f).\end{aligned}$$

So, the unit augments a value with the identity circuit and multiplication is just a sequential composition of circuits. Since we assumed that the category of circuits is premonoidal, this premonoidal structure lifts to \mathcal{T}_M (in the sense of [Atkey 2009, Def. 5]). That is, there is a natural transformation $(T \rtimes -)_{U_1, U_2, X}^\dagger: \mathcal{T}_M(U_1, U_2, X) \rightarrow \mathcal{T}_M(T \otimes U_1, T \otimes U_2, X)$ satisfying certain desired properties. In elementary terms, this is merely the map that associates (x, C) to $(x, T \rtimes C)$.

As one might expect, we will interpret terms in the Kleisli category of \mathcal{T}_M , which we now briefly explain. Given an S -indexed monad \mathcal{T} on C , its *Kleisli category* $C_{\mathcal{T}}$ is a category whose objects are pairs of C and S objects and homsets $C_{\mathcal{T}}((X, T), (Y, U)) \stackrel{\text{def}}{=} C(X, \mathcal{T}(T, U, Y))$. The identity morphisms and composition of morphisms are defined using units and multiplication as the obvious generalization of those in the Kleisli category of an ordinary monad. It is known that the Kleisli category induces a *parameterized Freyd category* [Atkey 2009] $J: C \times S \rightarrow C_{\mathcal{T}}$ as is the case for the ordinary monad and Freyd category; here J is an identity on objects functor that strictly preserves the premonoidal structure of C . Since Freyd categories are known to have a better match with the fine-grained call-by-value syntax, our semantical model will be based on the parametrized Freyd category induced by \mathcal{T}_M . The circuit monad \mathcal{T}_M also has Kleisli exponentials: there is a functor $X \Rightarrow_T -: \text{Set}_{\mathcal{T}_M} \rightarrow \text{Set}$ for every objects X, T , and there is a natural isomorphism $\Lambda_{Y, (X, T), (Z, U)}: \text{Set}_{\mathcal{T}_M}((Y \times X, T), (Z, U)) \cong \text{Set}(Y, X \Rightarrow_T (Z, U))$. This means that the parameterized Freyd category induced by \mathcal{T}_M is a closed parameterized Freyd category. We note that the object $X \Rightarrow_T (Z, U)$ is just $X \Rightarrow_{\text{Set}} Z \times M(U, T)$, the set of functions from X to $Z \times M(U, T)$. The counit of this adjunction is written as $\text{ev}: ((X \Rightarrow_T (Y, U)) \times X, T) \rightarrow (Y, U)$. The Kleisli category $\text{Set}_{\mathcal{T}_M}$ is a premonoidal category, where $(X, T) \otimes (Y, U) = (X \times Y, T \otimes U)$ and the binoidal functors are defined in a way analogous to how the premonoidal structure was lifted to \mathcal{T}_M .⁷

4.2 Interpreting Types, Programs, and Configurations

We now give the denotational semantics of Proto-Quipper-C using the Freyd category induced by \mathcal{T}_M . In Figure 3, we summarize the overall structure of the interpretation. Terms will be interpreted in the Kleisli category $\text{Set}_{\mathcal{T}_M}$, and values will be interpreted in $\text{Set} \times \text{disc}(\text{Obj}(M))$, where $\text{disc}(\text{Obj}(M))$ is the discrete category whose objects are those of M .

⁶This is just an indexed monad for category actions [Atkey 2009] and we do not claim any novelty in this definition.

⁷This should not be confused with the premonoidal structure of a parameterized Freyd category with respect to the cartesian category, which exist even if S is not premonoidal. See also Appendix C for a review on parameterized Freyd categories.

$$\begin{array}{ccc}
 \mathbf{Set} \times \mathbf{disc}(\mathbf{Obj}(\mathcal{M})) = \mathcal{V} & \xleftarrow{\iota_T} & \mathbf{Set} \\
 \downarrow & & \uparrow X \Rightarrow_T - \\
 \mathbf{Set} \times \mathcal{M} & \xrightarrow{J} & \mathbf{Set}_{\mathcal{T}_M}
 \end{array}$$

Fig. 3. Overview of the model for Proto-Quipper- \mathcal{M} .

$$\begin{aligned}
 & \boxed{[\![A]\!] \in \mathbf{Set} \times \mathcal{M}} \\
 & [\![P]\!] \stackrel{\text{def}}{=} ([\![P]\!]_P, I) \quad [\![T]\!] \stackrel{\text{def}}{=} (1, [\![T]\!]_{\mathcal{M}}) \quad [\![A \multimap T]\!] \stackrel{\text{def}}{=} (\flat[\![A]\!] \Rightarrow_{[\![T]\!]_{\mathcal{M}} \otimes \#[\![A]\!]_{\mathcal{M}}} [\![B]\!], [\![T]\!]_{\mathcal{M}}) \\
 & [\![A \otimes B]\!] \stackrel{\text{def}}{=} (X \times Y, T \otimes U) \text{ where } [\![A]\!] = (X, T) \text{ and } [\![B]\!] = (Y, U) \\
 & \boxed{[\![P]\!]_P \in \mathbf{Set}} \\
 & [\![1]\!]_P \stackrel{\text{def}}{=} 1 \quad [\![\mathbf{Nat}]\!]_P \stackrel{\text{def}}{=} \mathbb{N} \\
 & [\![P \otimes R]\!]_P \stackrel{\text{def}}{=} [\![P]\!]_P \times [\![R]\!]_P \quad [\![!A]\!]_P \stackrel{\text{def}}{=} 1 \Rightarrow_I [\![A]\!] \quad [\![\mathbf{Circ}(T, U)]\!]_P \stackrel{\text{def}}{=} \mathcal{M}([\![T]\!]_{\mathcal{M}}, [\![U]\!]_{\mathcal{M}}) \\
 & \boxed{[\![T]\!]_{\mathcal{M}} \in \mathcal{M}} \\
 & [\![\mathbf{Qubit}]\!]_{\mathcal{M}} \stackrel{\text{def}}{=} Q \quad [\![\mathbf{Bit}]\!]_{\mathcal{M}} \stackrel{\text{def}}{=} B \quad [\![I]\!]_{\mathcal{M}} \stackrel{\text{def}}{=} I \quad [\![T \otimes U]\!]_{\mathcal{M}} \stackrel{\text{def}}{=} [\![T]\!]_{\mathcal{M}} \otimes [\![U]\!]_{\mathcal{M}}
 \end{aligned}$$

Fig. 4. Interpretation of Simple Types.

The interpretation of types is given in Figure 4. Types are interpreted as objects in $\mathbf{Set} \times \mathcal{M}$, i.e. as pairs of a set and an object of \mathcal{M} . We write $\# : \mathbf{Set} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\flat : \mathbf{Set} \times \mathcal{M} \rightarrow \mathbf{Set}$ for the obvious forgetful functors. The object $X \Rightarrow_T Y$ used in the interpretation of the arrow type is the parameterized Kleisli arrow of the closed parameterized Freyd category. It is used to model the type of a “code” under the idea that a closure is a pair of a code and an environment. The subscript T represents the type of the additional bundled typed arguments corresponding to the free variables. Note that $!A$ is interpreted as $(1 \Rightarrow_I [\![A]\!], I)$ representing closures that do not capture any free variables having a bundle type. The objects Q and B are the interpretation of qubits and bits, respectively, that we assumed to exist in \mathcal{M} .

Lemma 1. *Given a type A , we have $[\![\#A]\!]_{\mathcal{M}} = \#\![\![A]\!]$.*

In our interpretation, circuit types and function types for bundle types are isomorphic, and this supports our design choice for the typing rule *box*.

Proposition 1. *We have an isomorphism $\mathbf{box} : \flat[\![T \multimap U]\!] \cong [\![\mathbf{Circ}(T, U)]\!]_P$ in \mathbf{Set} . Therefore, $[\![T \multimap U]\!] \cong [\![\mathbf{Circ}(T, U)]\!]$ in $\mathbf{Set}_{\mathcal{T}_M}$.*

PROOF. By unrolling the definition, we have the following obvious isomorphisms for sets

$$\begin{aligned}
 \flat[\![T \multimap U]\!] &= 1 \Rightarrow_{[\![T]\!]_{\mathcal{M}}} (1, [\![U]\!]_{\mathcal{M}}) \\
 &\cong 1 \Rightarrow_{\mathbf{Set}} 1 \times \mathcal{M}([\![T]\!]_{\mathcal{M}}, [\![U]\!]_{\mathcal{M}}) \\
 &\cong \mathcal{M}([\![T]\!]_{\mathcal{M}}, [\![U]\!]_{\mathcal{M}}) \\
 &= [\![\mathbf{Circ}(T, U)]\!]_P.
 \end{aligned}$$

The isomorphism between $[\![T \multimap U]\!] \cong [\![\mathbf{Circ}(T, U)]\!]$ is given by $J(\mathbf{box}, \text{id}_I)$. \square

$$\begin{aligned}
\llbracket \Gamma \vdash_v V : A \rrbracket &= \boxed{\llbracket \Gamma \vdash_v V : A \rrbracket} \\
\llbracket \Phi \vdash_v * : \mathbb{1} \rrbracket &\stackrel{\text{def}}{=} (!_{[\Phi]}, \text{id}_I) \\
\llbracket \Phi, \ell : w \vdash_v \ell : w \rrbracket &\stackrel{\text{def}}{=} (!_{[\Phi]_P}, \text{id}_{[\ell]_M}) \\
\llbracket \Phi, x : A \vdash_v x : A \rrbracket &\stackrel{\text{def}}{=} (\pi_x, \text{id}_I) \\
\llbracket \Gamma \vdash_v \lambda x. M : A \multimap_{\#(\Gamma)} B \rrbracket &\stackrel{\text{def}}{=} (\Lambda([\llbracket M \rrbracket]), \text{id}_{[\#(\Gamma)]_M}) \\
\llbracket \Phi \vdash_v \text{lift } M : !A \rrbracket &\stackrel{\text{def}}{=} (\Lambda([\llbracket \Phi \vdash_c M : A \rrbracket]), \text{id}_I) \\
\llbracket \Phi \vdash_v (\bar{\ell}, C, \bar{k}) : \text{Circ}(T, U) \rrbracket &\stackrel{\text{def}}{=} (!_{[\Phi]}; \widehat{[\ell, C, k]}_M, \text{id}_I) \\
\text{where } \widehat{[\ell, C, k]}_M &\stackrel{\text{def}}{=} \llbracket T \rrbracket_M \xrightarrow{\cong} \llbracket Q \rrbracket_M \xrightarrow{C} \llbracket L \rrbracket_M \xrightarrow{\cong} \llbracket U \rrbracket_M \\
\hat{C} &\stackrel{\text{def}}{=} 1 \xrightarrow{\Delta(J(\text{id}_1, C))} 1 \Rightarrow_{[\Gamma]_M} (1, \llbracket U \rrbracket_M) \xrightarrow[\cong]{\text{box}} \mathcal{M}([\llbracket T \rrbracket_M, \llbracket U \rrbracket_M]) \\
\llbracket \Phi, \Gamma_1, \Gamma_2 \vdash_v \langle V, W \rangle : A \otimes B \rrbracket &\stackrel{\text{def}}{=} [\Phi] \otimes [\Gamma_1] \otimes [\Gamma_2] \xrightarrow{(\Delta_{[\Phi]}, \text{id}_I) \otimes \text{id}} [\Phi] \otimes [\Phi] \otimes [\Gamma_1] \otimes [\Gamma_2] \\
&\xrightarrow{\cong} ([\Phi] \otimes [\Gamma_1]) \otimes ([\Phi] \otimes [\Gamma_2]) \\
&\xrightarrow{[\Gamma] \otimes [\Gamma]} \llbracket A \otimes B \rrbracket
\end{aligned}$$

Fig. 5. Interpretation of Values.

Remark 1. As briefly mentioned, in Proto-Quipper calculi, the box operator usually coerces a function of type $!(T \multimap U)$ to a circuit type. This is because, in presheaf models of Proto-Quipper, there is an isomorphism $![T \multimap U] \cong \llbracket \text{Circ}(T, U) \rrbracket$. In our model, adding a bang to the type $T \multimap U$ means to additionally thunk a function that has no free variable having a bundle type. Invoking this thunk may cause some effects, i.e. produce a circuit while returning a function that corresponds to a boxed circuit. Hence, we do not have an isomorphism between $![T \multimap U]$ and $\llbracket \text{Circ}(T, U) \rrbracket$. While our calculus draws inspiration from those by Colledan and Dal Lago [2024, 2025], they also coerce $!(T \multimap U)$ to $\text{Circ}(T, U)$. In their works, the issue of the effect is circumvented using the effect system; either by requiring that $!(T \multimap U)$ has zero effect or by adding an effect annotation to the circuit type. \triangleleft

Now we define the interpretation of typing judgments. The interpretation is defined in Figure 5 and 6. A valid typing judgment for values $\Gamma \vdash_v V : A$ is interpreted as a morphism $[\Gamma \vdash_v V : A] : [\Gamma] \rightarrow [A]$ in $\mathcal{V} (= \text{Set} \times \text{disc}(\mathcal{M}))$ capturing the fact that values only produces trivial circuits. In contrast, a computational judgment $\Gamma \vdash_c M : A$ is interpreted as $[\Gamma \vdash_c M : A] : [\Gamma] \rightarrow [A]$ in $\text{Set}_{\mathcal{T}_M}$. We sometimes denote these morphisms by $[\Gamma]$ and $[M]$. The morphisms, Δ , π_x and $!$ in Figure 5 are the diagonal map, projection, and the unique map to the terminal object 1 , respectively, which exist in Set ; the morphism dup_X in Figure 6 is defined as $J(\Delta_X, \text{id}_I)$ and it is used for duplicating a values with a parameter type. Some obvious coherence isomorphisms of the cartesian product in Set are omitted for simplicity.

The interpretation of values is standard, except for λ -abstractions and boxed circuits. A λ -abstraction is interpreted as a closure. The first element of $[\lambda x. M]$ is the semantic counterpart of the function that takes variables of types $\sharp(\Gamma)$ as additional parameters, and the second element $id_{[\# \Gamma]}$ is the “environment”. The interpretation of lift M is just the interpretation of a thunk $\lambda().M$.

$$\begin{aligned}
\llbracket \Gamma \vdash_c M : A \rrbracket &= \\
\llbracket \Phi, \Gamma_1, \Gamma_2 \vdash_c V W : B \rrbracket &\stackrel{\text{def}}{=} \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \xrightarrow{\text{dup}_{\llbracket \Phi \rrbracket} \otimes \llbracket \Gamma_2 \rrbracket \otimes \llbracket \Gamma_1 \rrbracket} \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \\
&\xrightarrow{\cong} (\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_1 \rrbracket) \otimes (\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket) \\
&\xrightarrow{J(\llbracket V \rrbracket) \otimes J(\llbracket W \rrbracket)} \llbracket A \multimap_T B \rrbracket \otimes \llbracket A \rrbracket \xrightarrow{\text{ev}} \llbracket B \rrbracket \\
\llbracket \Phi \vdash_c \text{force } V : A \rrbracket &\stackrel{\text{def}}{=} J(\llbracket \Phi \vdash_v V : !A \rrbracket); \text{ev} \\
\llbracket \Phi, \Gamma_1, \Gamma_2 \vdash_c \text{apply}(V, W) : U \rrbracket &\stackrel{\text{def}}{=} \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \xrightarrow{\text{dup}_{\llbracket \Phi \rrbracket} \otimes \llbracket \Gamma_2 \rrbracket \otimes \llbracket \Gamma_1 \rrbracket} \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \\
&\xrightarrow{\cong} (\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_1 \rrbracket) \otimes (\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket) \\
&\xrightarrow{J(\llbracket V \rrbracket) \otimes J(\llbracket W \rrbracket)} \llbracket \text{Circ}(T, U) \rrbracket \otimes \llbracket U \rrbracket \\
&\xrightarrow{\text{apply}} \llbracket U \rrbracket \\
\llbracket \Phi \vdash_c \text{box}_T V : \text{Circ}(T, U) \rrbracket &\stackrel{\text{def}}{=} \llbracket \Phi \rrbracket \xrightarrow{J(\llbracket V \rrbracket)} \llbracket T \multimap U \rrbracket \xrightarrow[\cong]{J(\text{box}, \text{id}_I)} \llbracket \text{Circ}(T, U) \rrbracket \\
\llbracket \Gamma \vdash_c \text{return } V : A \rrbracket &\stackrel{\text{def}}{=} J(\llbracket \Gamma \vdash_v V : A \rrbracket) \\
\llbracket \Phi, \Gamma_2, \Gamma_1 \vdash_c \text{let } x = \text{Min } N : B \rrbracket &\stackrel{\text{def}}{=} \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \xrightarrow{(\text{dup}_{\llbracket \Phi \rrbracket} \otimes \llbracket \Gamma_1 \rrbracket \otimes \llbracket \Gamma_2 \rrbracket); \cong} (\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket) \otimes (\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_1 \rrbracket) \\
&\xrightarrow{(\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket) \rtimes \llbracket M \rrbracket} \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \otimes \llbracket A \rrbracket \xrightarrow{\llbracket N \rrbracket} \llbracket B \rrbracket \\
\llbracket \Gamma_1, a : A, b : B, \Gamma_2 \vdash_c M : C \rrbracket &= \llbracket \Gamma_1 \rrbracket \otimes \llbracket A \rrbracket \otimes \llbracket B \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \xrightarrow{\cong} \llbracket \Gamma_1 \rrbracket \otimes \llbracket B \rrbracket \otimes \llbracket A \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket C \rrbracket
\end{aligned}$$

Fig. 6. Interpretation of computational judgments of the simple type system (excerpt).

In the interpretation of boxed circuits, we use isomorphisms $\llbracket T \rrbracket_{\mathcal{M}} \cong \llbracket Q \rrbracket_{\mathcal{M}}$ and $\llbracket L \rrbracket_{\mathcal{M}} \cong \llbracket U \rrbracket_{\mathcal{M}}$ that exist thanks to the premises of the typing rule *circ* such as $Q \cong_{\sigma} Q'$ and $Q' \vdash_v \ell : T$. The important part of the interpretation of a boxed circuit C is the map \hat{C} , which is just the global element $\hat{C}(\ast) \stackrel{\text{def}}{=} C$. The definition using Λ and box emphasizes the idea that boxed circuits can be seen as special functions.

As for the interpretation of terms, the interpretation of $\text{apply}(V, W)$ and $\text{box}_T V$ are the most interesting cases. The morphism $\text{apply}_{T, U} : (\mathcal{M}(T, U), I) \otimes (1, T) \rightarrow (1, U)$ is defined by

$$(\mathcal{M}(T, U), I) \otimes (1, T) \xrightarrow{J(\text{box}^{-1}, \text{id}_I) \otimes \text{id}_{(1, T)}} (1 \Rightarrow_T (1, U), I) \otimes (1, T) \xrightarrow{\text{ev}} (1, U).$$

The box operator is interpreted by the post-composition of the isomorphism between function types and circuit types given in Proposition 1. The interpretation of the let operator is also worth explaining. The premonoidal product $(\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket) \rtimes \llbracket M \rrbracket$ adds wires of type $\sharp \llbracket \Gamma_2 \rrbracket$ (and $\sharp \llbracket \Phi \rrbracket = I$ which can be ignored) to the circuit produced by M so that N can use these wires.

Remark 2. The interpretation is somewhat unorthodox in that some syntactic constructs do not have a corresponding semantic operator. This is because we are doing two things at once. The interpretation can be factorized into (1) a syntactic translation from Proto-Quipper-C to the internal language of parametrized Freyd categories (called the command calculus) [Atkey 2009] and (2)

interpreting the translated term. The syntactic translation resembles a variant of closure conversion, but we are not sure if a category theoretic explanation can be given to this translation. \triangleleft

The categorical semantics is correct with respect to the big-step operational semantics. To further elaborate on this result we first extend the interpretation to configurations. Intuitively, $\llbracket (C, M) \rrbracket$ is the morphism obtained by post-composing $\llbracket M \rrbracket$, together with some parallel wires, to C .

Definition 4 (Interpretation of Configurations). Suppose that $Q \vdash (C, V) : A; Q'$ and suppose that L and L' are the label contexts that satisfy $C : Q \rightarrow L', Q'; L \cong_{\sigma} L'$; and $L \vdash_v V : A$. Then we define $\llbracket (C, V) \rrbracket$ as $(\text{id}_1, C; \text{perm}); (\llbracket L \vdash_v V : A \rrbracket \otimes \text{id}_{\llbracket Q' \rrbracket})$, which is a morphism in $\text{Set} \times \mathcal{M}$. Here, perm is the isomorphism $\llbracket L' \rrbracket_{\mathcal{M}} \otimes \llbracket Q' \rrbracket_{\mathcal{M}} \xrightarrow{\cong} \llbracket L \rrbracket_{\mathcal{M}} \otimes \llbracket Q' \rrbracket_{\mathcal{M}}$. Similarly, for $Q \vdash (C, M) : A; Q'$, we define $\llbracket (C, M) \rrbracket$ as $J(\text{id}_1, C); (\llbracket L \vdash_c M : A \rrbracket \ltimes \llbracket Q' \rrbracket)$, which is a morphism in $\text{Set}_{\mathcal{T}_{\mathcal{M}}}$. Here, L is the label context that types M as in the case of V . \triangleleft

4.3 Main Results

We are now ready to state the soundness and computational adequacy properties.

Theorem 5 (Soundness). *Suppose that $Q \vdash (C, M) : A; Q'$ and $(C, M) \Downarrow (C', V)$. Then $\llbracket (C, M) \rrbracket = J(\llbracket (C', V) \rrbracket)$.*

PROOF. By induction on the derivation of the big-step evaluation relation. See Appendix A for the details. \square

Theorem 6 (Computational Adequacy). *Suppose that $\emptyset \vdash (C, M) : \mathbb{1}; \emptyset$ and $\llbracket (C, M) \rrbracket = J(\llbracket (D, V) \rrbracket)$. Then $(C, M) \Downarrow (D, V)$ (possibly up to renamings of labels).*

PROOF. By an argument utilizing logical relations similar to those defined in [Colledan and Dal Lago 2024], which are given in Appendix A. \square

5 Effect System

We extend the type system defined in Section 3 with effect annotations that estimate the properties (e.g. size) of the circuit generated by a program. To put it another way, we introduce the type system underlying (a non-dependent version of) Proto-Quipper-R. Then we give the interpretation of programs in Proto-Quipper-R by using the category graded monad [Orchard et al. 2020]. This exemplifies that the type-and-effect system of Proto-Quipper-R, although rooted in an operational perspective, also has a natural denotational reading.

5.1 Effects for Circuits

When designing an effect system, the key question to ask is “What kind of structure should we assume on effects?”. A common choice is to use a preordered monoid [Katsumata 2014], where the monoid multiplication is used to compute the effect of sequential execution and the preorder is used for subtyping. We make the same choice, but use *categories* instead of *monoids* because circuits are *many-sorted* in the sense that circuits have various input and output interfaces. Moreover, since we have postulated that circuits form a premonoidal category, it is natural to require that the algebraic structure representing the effect—dubbed *circuit algebra*—also be premonoidal.

Definition 7 (Circuit Algebra). A *circuit algebra* \mathcal{E} is a strict symmetric premonoidal category that is preorder enriched. The preorder enrichment means that:

- each homset $\mathcal{E}(t, u)$ is a preordered set;
- composition of morphisms preserves the order: if $e_1 \lesssim d_1$ and $e_2 \lesssim d_2$, then $e_1; e_2 \lesssim d_1; d_2$;

- $t \times -$ (resp. $- \times t$) preserves the order for any t : if $e \lesssim d$, then $t \times e \lesssim t \times d$.

We call morphisms of \mathcal{E} , ranged over by e, d, \dots , *effect annotations*. The identity over the monoidal unit i of \mathcal{E} is denoted by ε , and we call it the *null effect*. Objects of \mathcal{E} will often be written in lowercase script letters so that they are distinguishable from objects of \mathcal{M} . \triangleleft

Effect annotations are meant to abstract the actual effect. We propose to consider this abstraction as a functor.

Definition 8 (Abstraction). An abstraction α from the category of circuits \mathcal{M} to a circuit algebra \mathcal{E} is a strict symmetric premonoidal functor $\alpha: \mathcal{M} \rightarrow \mathcal{E}$. That is, α is a functor satisfying $\alpha(T \otimes U) = \alpha(T) \otimes \alpha(U)$ (equality on the nose), $\alpha(T \times f) = \alpha(T) \times \alpha(f)$, $\alpha(g \times U) = \alpha(g) \times \alpha(U)$ and preserves the symmetry. \triangleleft

In case \mathcal{M} is the syntactic category of circuits, defining an abstraction is no different from giving a functorial semantics to circuits. We believe that allowing arbitrary interpretations of circuits as effect annotations is not only conceptually clean but also helps us conceive of a wide variety of examples—though in practice, we should seek efficiently implementable effects.

Remark 3. We defined abstractions as *strict* premonoidal functors because it is known that a non-strict premonoidal functor is tricky to define [Román and Sobociński 2025; Staton and Levy 2013]. A way to circumvent this issue is to use *effectful categories* [Román and Sobociński 2025], which are premonoidal categories endowed with a chosen family of central morphisms, as the definition of circuits. However, using premonoidal categories and strict premonoidal functors are enough to deal with examples of circuit algebras we show below, which contains examples for resource estimations that have been considered in the literature. \triangleleft

Examples of Circuit Algebras. Here we give some examples of circuit algebras that capture some notions of circuit metrics.

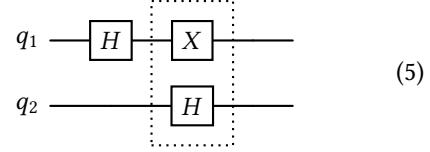
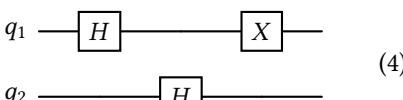
Since the concept of circuit metrics is intrinsically intensional, namely the way gates are placed is important, we consider the syntactic category of circuits for the category \mathcal{M} in the following examples. A *signature* is a tuple $\Sigma = (\Sigma_0, \Sigma_1)$ where Σ_0 is the set of *object variables*, Σ_1 is the set of *generators*, which are typed constants of the form $f: \sigma \rightarrow \tau$ with $\sigma, \tau \in \Sigma_0^*$. We write \mathcal{M}_Σ for the free strict symmetric premonoidal category (with trivial center) generated by the signature Σ , which can be defined by designing an appropriate term calculus (as in [Joyal and Street 1991]) or by considering string diagrams. To facilitate understanding, we shall informally deal with string diagrams by depicting the graph and considering them as “monoidal string diagrams without interchange law” (see e.g. [Román and Sobociński 2025] for a more mathematically formal definition). Note that a functor $\alpha: \mathcal{M}_\Sigma \rightarrow \mathcal{E}$ is determined if we define how object variables and generators are mapped to objects and morphisms of \mathcal{E} , respectively.

As a prototypical example, we shall consider string diagrams over the signature Σ^{QC} for quantum circuits. The set Σ_0^{QC} is defined as $\{\text{Qubit}, \text{Bit}\}$ and the generators are finite sequences over the set of gates Σ_G . We assume that the set Σ_G , which is also a set of generators, contains usual quantum gates such as the Hadamard gate $H: \text{Qubit} \rightarrow \text{Qubit}$, CNOT gate $\text{CNOT}: (\text{Qubit}, \text{Qubit}) \rightarrow (\text{Qubit}, \text{Qubit})$ and measurement $\text{meas}: \text{Qubit} \rightarrow \text{Bit}$. A sequence $\tilde{g} = g_1 \cdots, g_n \in \Sigma_1^{\text{QC}}$ has the type $\sigma_1 \cdots \sigma_n \rightarrow \tau_1 \cdots \tau_n$ provided that $g_i: \sigma_i \rightarrow \tau_i$, and intuitively corresponds to applying the gates g_i in parallel. In a sense, this means that it is users responsibility to explicitly state which gates to be placed in parallel, and users cannot expect the gates to automatically slide and be parallelized.

Example 1 (Gate count and naive depth). The simplest notion of circuit metric we consider is *gate count*. The number of gates can be captured by the monoid $(\mathbb{N}, +)$, which can be seen as a single object circuit algebra (where we denote the only object as \star) whose morphisms are natural

numbers $n: \star \rightarrow \star$. Sequential composition is defined as addition and the identity morphism is given by 0. The functor $\star \times -$ is simply given as the identity functor: $\star \times \star \stackrel{\text{def}}{=} \star$ and $\star \times n \stackrel{\text{def}}{=} n$. Obviously, this category is order enriched by considering the standard ordering for the natural numbers. The abstraction functor $\alpha_G: \mathcal{M}_{\Sigma^{\text{QC}}} \rightarrow \mathbb{N}$ is defined by mapping $\tilde{g} \in \Sigma_1^{\text{QC}}$ to the number of gates in \tilde{g} .

A very rough estimation of the circuit *depth* can be given by the same circuit algebra. We define $\alpha_D: \mathcal{M}_{\Sigma^{\text{QC}}} \rightarrow \mathbb{N}$ by $\alpha_D(\tilde{g}) = 1$ for every $g \in \Sigma_1^{\text{QC}}$. For example, let us consider the following circuits.



Here, in (5), the X and Hadamard gate are placed parallelly. The number of gates is estimated as 3 both in (4) and (5). On the other hand, the depth is estimated as 3 in (4) since the three gates are sequentially composed, but 2 in (5). While this way of counting depth has its own benefit of being easy to compute, it is not satisfactory because typically the depth of (4) is also defined as 2; below we shall see a better way to count depth. \triangleleft

Example 2 (Width [Colledan and Dal Lago 2025]). We explain a circuit algebra \mathcal{W} that is used to estimate (upper bounds on) the width of a circuit and an abstraction $\alpha_{\mathcal{W}}: \mathcal{M}_{\Sigma^{\text{QC}}} \rightarrow \mathcal{W}$. Recall that the *width* of a circuit is just a natural number that is defined as the maximum number of wires active at any point in the circuit. Therefore, *morphisms* in \mathcal{W} should be natural numbers. For example, a quantum circuit C depicted as



has with width 3, i.e. $\alpha_{\mathcal{W}}(C) = 3$. Note that a wire of type Qubit is counted as a circuit of width 1. This leads us to define $\alpha(\text{id}_{\text{Qubit}}): \alpha(\text{Qubit}) \rightarrow \alpha(\text{Qubit})$ as the natural number 1. Now how should we define $\alpha(\text{Qubit})$? Since this object should contain enough information to define the width of wires of type Qubit, a natural choice is to define this as the natural number 1. Hence, we also define *objects* of \mathcal{W} as natural numbers. The sequential composition of morphisms $k_1 \xrightarrow{m} k_2$ and $k_2 \xrightarrow{n} k_3$ in \mathcal{W} is defined as $k_1 \xrightarrow{\max(m,n)} k_3$ reflecting the definition of the width of a circuit. The functor $k \times -$ (resp. $- \times k$) represents parallelly adding k wires. Hence, we define $k \times m \stackrel{\text{def}}{=} k + m$. To summarize, \mathcal{W} is given by the following data:

- $\text{Obj}(\mathcal{W}) \stackrel{\text{def}}{=} \mathbb{N}$,
- $\mathcal{W}(k_1, k_2) \stackrel{\text{def}}{=} (\mathbb{N}, \leq_{\mathbb{N}})$, and the composition is defined by \max , with the identity morphism over an object k being the morphism k itself.
- functors $k \times -$ (resp. $- \times k$) such that the action on objects and morphisms are both defined as $k + (-)$. \triangleleft

Example 3 (Depth). We give a better circuit algebra for depth, which may be seen as a way to count the naive depth of a circuit after optimizing it by sliding gates. The idea is to track the depth using *matrices over max-plus tropical semiring* $(\mathbb{N} \cup \{-\infty\}, \max, +)$, where the (i, j) component

of the matrix describes the cost for traversing from the i -th input to the j -th output. We define a circuit algebra \mathcal{D} as follows:

- $\text{Obj}(\mathcal{D}) \stackrel{\text{def}}{=} \mathbb{N}$,
- $\mathcal{D}(k_1, k_2) \stackrel{\text{def}}{=} M_{k_1, k_2}(\mathbb{N}) \times M_{1, k_1}(\mathbb{N}) \times M_{k_2, 1}(\mathbb{N})$, where $M_{k_1, k_2}(\mathbb{N})$ is the set of $k_1 \times k_2$ matrices over the tropical semiring. The composition $(A_1, v_1, w_1) \circ (A_2, v_2, w_2)$ is given as $(A_1 A_2, \max(v_1 A_2, v_2), \max(w_1, A_1 v_2))$; here \max acts on vectors component-wise. The identity morphism over k is given as $(I_k, 0_{1 \times k}, 0_{k \times 1})$, where I_k is the $k \times k$ identity matrices and $0_{n \times m}$ is the $n \times m$ zero matrix over the tropical semiring. These should not be confused with the standard identity and zero matrices, say over \mathbb{Q} . For example, $0_{1 \times k}$ is $(-\infty, \dots, -\infty)$. The ordering on morphisms is given by ordering over matrices where $A \leq B$ if $a_{i,j} \leq b_{i,j}$ for every (i, j) .
- functors $k \rtimes -$ (resp. $- \ltimes k$) are defined by the “direct sum” of matrices. Concretely, for $(A, v, w) \in \mathcal{D}(k_1, k_2)$, we define $(A, v, w) \rtimes k$ as $(\begin{pmatrix} A & 0_{k_1 \times k} \\ 0_{k \times k_2} & I_k \end{pmatrix}, (v, 0_{1 \times k}), (\begin{smallmatrix} w \\ 0_{k \times 1} \end{smallmatrix}))$. Here again, the zero and identity matrices are those over the tropical semiring.

As mentioned, components of (A, v, w) represents the cost of the paths in a circuit as illustrated in Figure 7. The role of the vectors v and w is to track the depth of wires that have “dead ends”, for instance, created by qubit creation or annihilation. The i -th component of v describes the maximum cost for traversing the circuit from the i -th input until it reaches an end. Conversely, the i -th component of the vector w tracks the maximum depth of a path starting from an “end” and ending at the i -th output.

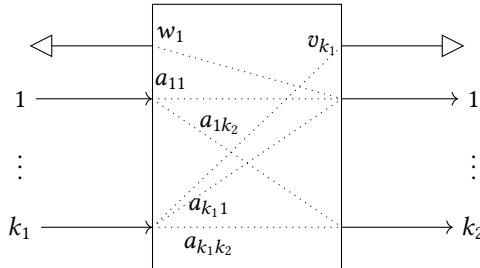


Fig. 7. Components of (A, v, w) .

The depth of the circuit (4) of Example 1, is given as $((\begin{smallmatrix} 1 & -\infty \\ -\infty & 2 \end{smallmatrix}), (-\infty, -\infty), (\begin{smallmatrix} -\infty \\ -\infty \end{smallmatrix}))$. Now we can correctly conclude that the depth is 2, not 3, as the maximum number in the tuple is 2. As an example containing a “dead end”, we show the depth of the circuit (6) of Example 2. It is given as $((\begin{smallmatrix} 2 & 2 & -\infty \\ -\infty & -\infty & 0 \end{smallmatrix}), (-\infty, -\infty), (\begin{smallmatrix} 1 \\ -\infty \end{smallmatrix}))$. \triangleleft

It is natural to ask whether any matrix circuit algebra can capture optimized circuit width taking into account the so-called qubit recycling. Given well-known results on the NP-hardness of qubit recycling [Jiang 2024], this seems difficult. On the other hand, it is possible to capture the so-called qubit dependency graph, which serves as input to a (heuristic) solver that outputs a qubit recycling strategy [Jiang 2024]. The graph can be calculated as an adjacent matrix (over the boolean semiring) by employing an approach similar to that used for capturing circuit depth.

Example 4 (Assertion-Based Optimization). We consider the size of quantum circuits modulo an optimization that removes operations that act trivially on states that are known to satisfy certain conditions. For example, if we know that the input of a CNOT gate is in a state $|\psi\rangle =$

$\alpha_{00} |00\rangle + \alpha_{01} |01\rangle$, then we know that this *CNOT* operation is *trivial* in the sense that it behaves as the identity operation because $|0b\rangle \xrightarrow{\text{CNOT}} |0b\rangle$ for $b \in \{0, 1\}$. This is actually the case for the circuit (6) of Example 2, meaning that we may remove the *CNOT* gate since it is redundant. In (6), the fact that the first qubit is zero is evident because it was created right before the *CNOT* gate, but we may also *assert* such properties against inputs of the circuit and remove gates based on these assertions. This is the core idea of an automated optimization methodology proposed by Häner et al. [2020].⁸

Our aim here is not to give an effect annotation that works as an optimizer that transforms a given circuit but to capture the size of the circuit after the optimization as an effect annotation. For the size, we consider gate counts for simplicity, but the other circuit metrics can be used as well. Since the size of the circuit depends on the precondition, we consider a function of the type $\mathcal{P}X \rightarrow \mathcal{P}Y \times \mathbb{N}$, essentially a forward predicate transformer combined with a “cost monad” (i.e. a writer monad) $(-) \times \mathbb{N}$. The reason for returning not just the size but also the postcondition in $\mathcal{P}Y$, is simply to make the effect annotations compose. The choice of the set X and Y is the key to obtaining a tractable notion of effect annotation. Here, we follow Häner et al. [2020] and take $X \stackrel{\text{def}}{=} \{0, 1\}^m$ and $Y \stackrel{\text{def}}{=} \{1, 0\}^n$, where m and n are the number of input and output qubits, respectively. A bitstring $b \in \{0, 1\}^n$ represents the b -th (written in the binary format) computational base state, and $L \subseteq \{0, 1\}^n$ can be considered as the set of possible outcomes of the quantum state. Formally, we say that a (pure) state $|\psi\rangle \in \mathbb{C}^{2^n}$ satisfies the predicate L if $|\psi\rangle = \sum_{b \in \{0, 1\}^n} \alpha_b |b\rangle$ and $|\alpha_b| > 0$ implies $b \in L$.⁹ For *CNOT*, we define a function $e : \{0, 1\}^2 \rightarrow \{0, 1\}^2 \times \mathbb{N}$, which, for instance, associates $\{00, 01\}$ to $(\{00, 01\}, 0)$ and $\{00, 10\}$ to $(\{00, 11\}, 1)$. The reason why we assign the cost 0 to $\{00, 01\}$ is because *CNOT* acts as an identity for $|\psi\rangle$ satisfying $\{00, 01\}$, meaning that this *CNOT* gate can be removed by optimization. To capture the linearity of the operation, we require that the effect $\mathcal{P}(\{0, 1\}^m) \rightarrow \mathcal{P}(\{0, 1\}^n) \times \mathbb{N}$ to be join preserving.¹⁰

The circuit algebra we consider, denoted as $\mathcal{A}\text{sr}t$ is defined as follows:

- Objects are natural numbers
- A morphism $e \in \mathcal{A}\text{sr}t(m, n)$ is a function from $\mathcal{P}(\{0, 1\}^m)$ to $\mathcal{P}(\{0, 1\}^n) \times \mathbb{N}$ that is join preserving. That is, $e(L_1 \cup L_2) = (L'_1 \cup L'_2, \max(c_1, c_2))$ where $e(L_i) = (L'_i, c_i)$. Sequential composition is defined as that of the writer monad, and the identity morphism is simply $L \mapsto (L, 0)$. The ordering between morphisms is defined as $e \lesssim d$ if, for every L , $e(L) \lesssim_{\mathcal{P}(\{0, 1\}^k) \times \mathbb{N}} d(L)$, where $\lesssim_{\mathcal{P}(\{0, 1\}^k) \times \mathbb{N}}$ is the product order of $(\mathcal{P}(\{0, 1\}^k), \subseteq)$ and $(\mathbb{N}, \leq_{\mathbb{N}})$.
- the functor $k \rtimes -$ that acts on $e : m \rightarrow n$ as $(k \rtimes e)(L) \stackrel{\text{def}}{=} \bigcup_{(b_1, b_2) \in L} e(\{b_1\}) \times \{b_2\}$. On objects, it just acts as $k + (-)$ as the previous examples.

If we consider a subsignature of Σ^{QC} that only has Qubit as object variable and unitary gates as generators and the free premonoidal category generated by it, then we can define the abstraction function to $\mathcal{A}\text{sr}t$ by giving the interpretation to these unitary operators as we did for *CNOT*. We can also handle Bit and non unitary operation such as $\text{meas} : \text{Qubit} \rightarrow \text{Bit}$ by considering the set of predicates $\mathcal{P}(\{0, 1\}^m \times \{0, 1\}^n)$ for a state with m qubits and n classical bits. \triangleleft

The circuit algebra examples we have discussed encompass all those previously considered by Colledan and Dal Lago [2025]. In contrast, Example 4 represents a novel contribution: to the best of the authors’ knowledge, no existing resource analysis techniques for quantum programs in the literature account for optimizations, i.e., no such techniques is capable of deriving bounds that reflect the improvement in size induced by optimizations. It is also worth noting that the notions of

⁸This optimization method is implemented as a transpiler pass in Qiskit [Qiskit API reference 2025].

⁹It is easy to extend this satisfaction relation to mixed states.

¹⁰It is well-known that there is a bijection between join preserving functions from $\mathcal{P}X$ to $\mathcal{P}Y$ and Kleisli morphisms $f : X \rightarrow \mathcal{P}Y$. Similarly, we may think that we are working with morphisms $X \rightarrow \mathcal{P}Y \times \mathbb{N}$.

width and depth used by Colledan and Dal Lago differ in nature: the former is global, assigning a single numerical value to each circuit, while the latter is local, assigning a value to each individual qubit. These two types of metrics are captured in different ways in *op. cit.* Circuit algebras provide a unified formalism that can accommodate both global and local metrics within the same framework. As we will see, the proof of soundness of the resulting type system will be done just once.

5.2 Type-and-Effect System

Now we add effect annotations to the types. The type system is parameterized by an abstraction to a circuit algebra $\alpha: \mathcal{M} \rightarrow \mathcal{E}$. As usual, arrow types have annotations that estimate the scope of effect caused by invoking the function. We also annotate $!$ and $\text{Circ}(T, U)$ because thunks and boxed circuits can be thought of as special functions. The grammar of types and parameter types now become as follows:

$$\begin{aligned} \text{Types} \quad A, B ::= P &| T &| A \otimes B &| A \xrightarrow{e: t \rightarrow u} B \\ \text{Parameter types} \quad P, R ::= \mathbb{1} &| \text{Nat} &| P \otimes R &| !^{e: t \rightarrow u} A &| \text{Circ}^{e: u \rightarrow s}(T, U) \end{aligned}$$

The definition of bundle types remains the same.

The typing judgment for computations now takes the form $\Gamma \vdash_c M : A; e: t \rightarrow u$. The annotation $e: t \rightarrow u$ gives the information about the circuit generated by M ; the type of the effect annotation $t \rightarrow u$ are sometimes omitted for readability. The shape of typing judgment for values is the same as before.

Typing rules are given in Figure 8. Rules for lambda abstraction, application, thunking and forcing are the standard rules. The rule for boxed circuit adds the effect e calculated by abstracting the circuit C as the annotation. In practice, the boxed circuits are the primitive constant circuits for which the abstraction is predefined. The rules of greatest interest may be the rules for return and let. Usually, the rule for return in a type-and-effect system adds the null effect since values are effectless. In our calculus, we cannot give such a uniform treatment to values. Instead, we add the annotation id_t , which represents the effect of the identity circuit of type $\#\llbracket A \rrbracket$. This is different from the null effect; for example, if we are interested in width, the width of identity circuits cannot be treated as zero. For the same reason that we cannot ignore identity circuits, the rule for let is annotated by $(\text{id} \times e_1); e_2$ rather than $e_1; e_2$. We also have a new subsumption rule, which allows us to relax the effect annotation.¹¹ We note that the domain (resp. codomain) of the effect is an abstraction of the bundle type of the type environment (resp. return type).

Lemma 2. Suppose that $\Gamma \vdash_c M : A; e: t \rightarrow u$. Then we have $\alpha(\#\llbracket \Gamma \rrbracket) = t$ and $\alpha(\#\llbracket A \rrbracket) = u$. □

We also adjust the typing for configurations. We write $Q \vdash (C, M) : A; Q'; e : t \rightarrow u$ if $C: Q \rightarrow L$, $Q' \vdash_c M : A; e : t \rightarrow u$ and $L \vdash_c M : A; e : t \rightarrow u$ for some label context L disjoint from Q . Type preservation (Theorem 2) holds even after this modification.

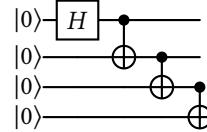
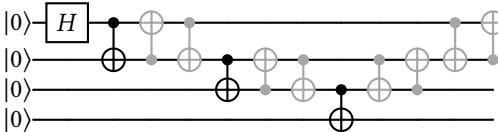
Example 5 (A program with information about assertion-based optimization). We provide a simple example of how Proto-Quipper-R can be used to verify the resource usage of a circuit at the language level. We use a program that generates an *inefficient* circuit for a linear nearest-neighbor (LNN) architecture, taken from Figure 3 of [Häner et al. 2020]. In a LNN architecture gates can be applied only to adjacent qubits, and because of this restriction, a programmer (or a compiler) might write an inefficient code following a certain idiom. The program¹² in Figure 9 is an example of such a program, which generates the left circuit given below. This circuit can be optimized into the one in the right by optimizing the trivial CNOT gates, which are gray in the left circuit.

¹¹It is also possible to define a subtyping relation based on the $e \lesssim e'$ in a standard way.

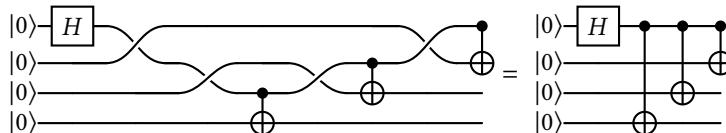
¹²For readability, we did not write the program using the syntax of Section 3, but a variant that uses a ML-like syntax.

$\frac{\Gamma, x : A \vdash_c M : B; e : t \rightarrow u}{\Gamma \vdash_v \lambda x_A.M : A \xrightarrow{e: t \rightarrow u}_{\#(\Gamma)} B}$	$\frac{\Phi, \Gamma_1 \vdash_v V : A \xrightarrow{e: s \rightarrow u}_T B \quad \Phi, \Gamma_2 \vdash_v W : A}{\Phi, \Gamma_1, \Gamma_2 \vdash_c V W : B; e : s \rightarrow u}$
$\frac{\Phi \vdash_c M : A; e : t \rightarrow u}{\Phi \vdash_v \text{lift } M : !^e A}$	$\frac{\Phi \vdash_v V : !^e A}{\Phi \vdash_c \text{force } V : A; e : t \rightarrow u}$
$\frac{C : Q \rightarrow L \quad Q \cong_{\sigma} Q' \quad L \cong_{\sigma} L' \quad Q' \vdash_v \bar{\ell} : T \quad L' \vdash_v \bar{k} : U \quad \alpha(\llbracket (\bar{\ell}, C, \bar{k}) \rrbracket_M) = e}{\Phi \vdash_v (\bar{\ell}, C, \bar{k}) : \text{Circ}^e(T, U)}$	$\frac{\Phi \vdash_v V : T \xrightarrow{e: t \rightarrow u}_I U}{\Phi \vdash_c \text{box}_T V : \text{Circ}^{e: t \rightarrow u}(T, U); e}$
$\frac{\Phi, \Gamma_1 \vdash_v V : \text{Circ}^{e: t \rightarrow u}(T, U) \quad \Phi, \Gamma_2 \vdash_v W : T}{\Phi, \Gamma_1, \Gamma_2 \vdash_c \text{apply}(V, W) : U; e : t \rightarrow u}$	
$\frac{\Phi, \Gamma_1 \vdash_v V : A \otimes B \quad \Phi, \Gamma_2, x : A, y : B \vdash_c M : C; e}{\Phi, \Gamma_2, \Gamma_1 \vdash_c \text{let } \langle x, y \rangle = V \text{ in } M : C; e}$	$\frac{\Phi \vdash_v V : \text{Nat} \quad \Phi, \Gamma \vdash_c M : A; e \quad \Phi, \Gamma \vdash_c N : A; e}{\Phi, \Gamma \vdash_c \text{ifz } V \text{ then } M \text{ else } N : A; e}$
$\frac{\Gamma \vdash_v V : A \quad \alpha(\#\llbracket A \rrbracket) = t}{\Gamma \vdash_c \text{return } V : A; \text{id}_t : t \rightarrow t}$	$\frac{\Phi, \Gamma_1 \vdash_c M : A; e_1 : t_1 \rightarrow t'_1 \quad \Phi, \Gamma_2, x : A \vdash_c N : B; e_2 : t_2 \rightarrow t'_2 \quad \alpha(\#\llbracket \Gamma_i \rrbracket) = u_i \quad e = (\text{id}_{u_2} \bowtie e_1); e_2}{\Phi, \Gamma_2, \Gamma_1 \vdash_c \text{let } x = M \text{ in } N : B; e}$
	$\frac{\Gamma \vdash_c M : A; e_1 : t \rightarrow u \quad e_1 \lesssim e_2}{\Gamma \vdash_c M : A; e_2 : t \rightarrow u}$

Fig. 8. Typing Rules for the Effect System of Proto-Quipper-R (excerpt).



These circuits are circuits that entangle four qubits as is clear from the right circuit; the left circuit is also a somewhat natural implementation of such a circuit that a compiler may emit. The three consecutive *CNOT* gates acting on the top two qubits of the left circuit, as well as those acting on the second and third qubits, are implementations of swap gates using *CNOT* gates. These swaps are often inserted to naively implement an operation that acts at a distance. The left circuit above is just an implementation of the following circuit (followed by a simple optimization that removes two consecutive applications of the same *CNOT* gate) that entangles the qubits by repeatedly applying *CNOT* to the first qubit and the other qubits.



```

1 -- init4 : Circ(1, qubit ⊗ qubit ⊗ qubit ⊗ qubit) [e1]
2 let (q1, q2, q3, q4) = apply(init4, *) in
3 -- hadamard : Circ(qubit, qubit) [e2]
4 let q1 = apply(hadamard, q1) in
5 -- cnot12 : Circ(qubit ⊗ qubit, qubit ⊗ qubit) [e3]
6 let (q1, q2) = apply(cnot12, (q1, q2)) in
7 -- cnot21cnot12 : Circ(qubit ⊗ qubit, qubit ⊗ qubit) [e4]
8 -- The sequential composition of cnot21 and cnot12
9 let (q1, q2) = apply(cnot21cnot12, (q1, q2)) in
10 let (q2, q3) = apply(cnot12, (q2, q3)) in
11 let (q2, q3) = apply(cnot21cnot21) in
12 ...
13 return (q1, q2, q3, q4)

```

Fig. 9. A program that generates a chain of redundant CNOTs for a LNN architecture.

We show how, by using the circuit algebra given in Example 4, the effect system can capture that the number of gates of the produced circuit *after being optimized*. In Figure 9, the types of circuits that are being applied are given as comments, where $\text{Circ}(T, S)^e$ is written as $\text{Circ}(T, S)$ [e]. Here we explain how each e_i is defined. The effect annotation e_1 for the four qubit initialization is defined by $e_1(S) = (\{0000\}, 0)$. (We are not counting the initializations as gates.) For the Hadamard gate, the effect annotation e_2 is defined by $e_2(S) = (\{1, 0\}, 1)$; the first element is $\{1, 0\}$ because the result after applying the Hadamard gate is a superposition of base states, and the second element 1 is the count. The effect annotation e_3 for the CNOT gate is the one that we explained in Example 4, which, in particular, satisfies $e_3(\{00, 10\}) = (\{00, 11\}, 1)$. Note that this means we have $q_1 = q_2$ after line 6. We can define the effect annotation e_4 for cnot21cnot12 to satisfy $e_4(\{00, 11\}) = (\{00, 11\}, 0)$ because $(\text{CNOT}_{12} \circ \text{CNOT}_{21})(|00\rangle) = |00\rangle$ and $(\text{CNOT}_{12} \circ \text{CNOT}_{21})(|11\rangle) = |11\rangle$. That is, cnot21cnot12 acts as identity and can be removed if $q_1 = q_2$. Hence, the effect annotation e (restricted to the first two qubits) for the program from line 1 to line 9, satisfies $e(\{\varepsilon\}) = (\{00, 11\}, 3)$ meaning that $q_1 = q_2$ and we only need three gates, as opposed to five, after the optimization. \triangleleft

5.3 Categorical Semantics

5.3.1 Circuit Monad with Effect Annotation. We refine the circuit monad from Section 3 by annotating it with effects. An established approach to giving a semantics of a type-and-effect system is to use *graded monads* [Katsumata 2014; Mellies 2012]. We follow this approach and refine the circuit monad as a *category-graded monad* [Orchard et al. 2020], which can be thought of as a many-sorted generalization of graded monads.

Definition 9 (Cat-graded Monads [Orchard et al. 2020]). A (preorder enriched) category-graded monad (or an \mathcal{A} -graded monad) on \mathbf{Set} consists of a family of endofunctors $\mathcal{T}^f : \mathbf{Set} \rightarrow \mathbf{Set}$ indexed by morphisms f in \mathcal{A} and families of natural transformations

- $\eta_a : \text{Id}_{\mathbf{Set}} \rightarrow \mathcal{T}^{\text{id}_a}$ for $a \in \text{Obj}(\mathcal{A})$,
- $\mu_{f,g} : \mathcal{T}^f \mathcal{T}^g \rightarrow \mathcal{T}^{f \circ g}$ for $f : a \rightarrow b$ and $g : b \rightarrow c$,
- $\mathcal{T}^{f \leq f'} : \mathcal{T}^f \rightarrow \mathcal{T}^{f'}$ for $f, f' : a \rightarrow b$ such that $f \leq f'$,

satisfying the following unital and associativity laws.

$$\begin{array}{ccc} \begin{array}{c} \mathcal{T}^f \xrightarrow{\eta_a \mathcal{T}^f} \mathcal{T}^{\text{id}_a} \mathcal{T}^f \\ \downarrow \mathcal{T}^f \eta_b \quad \downarrow \mu_{\text{id}_a, f} \\ \mathcal{T}^f \mathcal{T}^{\text{id}_b} \xrightarrow{\mu_{f, \text{id}_b}} \mathcal{T}^f \end{array} & \quad & \begin{array}{c} \mathcal{T}^f \mathcal{T}^g \mathcal{T}^h \xrightarrow{\mathcal{T}^f \mu_{g, h}} \mathcal{T}^f \mathcal{T}^{g; h} \\ \downarrow \mu_{f, g} \mathcal{T}^h \quad \downarrow \mu_{f, g; h} \\ \mathcal{T}^f; g \mathcal{T}^h \xrightarrow{\mu_{f; g, h}} \mathcal{T}^{f; g; h} \end{array} \end{array}$$

Moreover, we have the following commutativity concerning the preordering

$$\begin{array}{ccc} \begin{array}{c} \mathcal{T}^f \\ \downarrow \mathcal{T}^{f \leq f} \quad \searrow \text{id}_{\mathcal{T}^f} \\ \mathcal{T}^f = \mathcal{T}^f \end{array} & \quad & \begin{array}{c} \mathcal{T}^f \\ \downarrow \mathcal{T}^{f \leq g} \quad \searrow \mathcal{T}^{f \leq h} \\ \mathcal{T}^g \xrightarrow{\mathcal{T}^{g \leq h}} \mathcal{T}^h \end{array} & \quad & \begin{array}{c} \mathcal{T}^f \mathcal{T}^g \xrightarrow{\mathcal{T}^{f \leq f'} \mathcal{T}^{g \leq g'}} \mathcal{T}^{f'} \mathcal{T}^{g'} \\ \downarrow \mu_{f, g} \quad \downarrow \mu_{f', g'} \\ \mathcal{T}^{f; g} \xrightarrow{\mathcal{T}^{f; g \leq f' \cdot g'}} \mathcal{T}^{f' \cdot g'} \end{array} \end{array}$$

△

We are interested in category-graded monads of a specific kind, namely those constructed from abstractions to circuit algebras. Given $\alpha: \mathcal{M} \rightarrow \mathcal{E}$, we define an endofunctor on Set , parameterized by e , by $\mathcal{T}_{\mathcal{M}}^e(X) \stackrel{\text{def}}{=} X \times \mathcal{M}^{\leq e}(T, U)$ where the set $\mathcal{M}^{\leq e}(T, U) \subseteq \mathcal{M}(T, U)$ is defined as $\{C \in \mathcal{M}(T, U) \mid \alpha(C) \leq e\}$. This construction is reminiscent of graded monads arising from effect observations [Katsumata 2014].¹³ There is a caveat to this definition: this is not exactly an \mathcal{E} -graded monad, as the grading is defined with respect to a slight modification of \mathcal{E} . The annotation $e: t \rightarrow u$ does not tell us the type of circuits, but only the type of circuits *after the abstraction*. To remedy the problem, for the category of grades, we use $\tilde{\mathcal{E}}$ whose objects are those of \mathcal{M} and whose homset $\tilde{\mathcal{E}}(T, U)$ is defined as $\mathcal{E}(\alpha(T), \alpha(U))$.

We spell out the definition of the $\tilde{\mathcal{E}}$ -graded monad constructed from abstraction to circuit algebra. As mentioned, the endofunctor $\mathcal{T}_{\mathcal{M}}^e: T \rightarrow U$ acts on an object X as $X \times \mathcal{M}^{\leq e}(T, U)$. The unit and multiplication are defined exactly the same way as in the (ordinary) circuit monad. That is, we have

$$\eta_{T, X}(x) \stackrel{\text{def}}{=} (x, \text{id}_T), \quad \mu_{e_1, e_2, X}((x, C), D) \stackrel{\text{def}}{=} (x, C; D).$$

Note that the multiplication is well-defined because if $\alpha(C) \leq e_1$ and $\alpha(D) \leq e_2$, we have $\alpha(C; D) = \alpha(C); \alpha(D) \leq e_1; e_2$. Each component of the natural transformation $\mathcal{T}_{\mathcal{M}}^{e_1 \leq e_2}$ is the inclusion from $X \times \mathcal{M}^{\leq e_1}(T, U)$ to $X \times \mathcal{M}^{\leq e_2}(T, U)$. Moreover, the category-graded monad constructed this way, has premonoidal lifting as in the case for the ordinary circuit monad. For example, there is a natural transformation $(T \rtimes -)^{\dagger}_{e: T \rightarrow U, X}: \mathcal{T}_{\mathcal{M}}^e X \rightarrow \mathcal{T}_{\mathcal{M}}^{T \rtimes e} X$ that respects unit and the multiplication of the monad, which is defined as $(x, C) \mapsto (x, T \rtimes C)$

5.3.2 Interpretation. The interpretation of types can be obtained by replacing $\mathcal{M}(T, U)$ with $\mathcal{M}^{\leq e}(T, U)$. For example, we define

$$\begin{aligned} \llbracket \text{Circ}^{e: t \rightarrow u}(T, U) \rrbracket_P &\stackrel{\text{def}}{=} \mathcal{M}^{\leq e}(\llbracket T \rrbracket_{\mathcal{M}}, \llbracket U \rrbracket_{\mathcal{M}}) \\ \llbracket A \xrightarrow{e: u \rightarrow s} B \rrbracket &\stackrel{\text{def}}{=} (\flat \llbracket A \rrbracket \Rightarrow_{\text{Set}} \flat \llbracket B \rrbracket \times \mathcal{M}^{\leq e}(\sharp \llbracket A \rrbracket \otimes \llbracket T \rrbracket_{\mathcal{M}}, \sharp \llbracket B \rrbracket), \llbracket T \rrbracket_{\mathcal{M}}). \end{aligned}$$

The first element of $\llbracket A \xrightarrow{e: u \rightarrow s} B \rrbracket$ can also be written as $\flat \llbracket A \rrbracket \Rightarrow_{\text{Set}} \mathcal{T}_{\mathcal{M}}^{e: \sharp \llbracket A \rrbracket \otimes \llbracket T \rrbracket_{\mathcal{M}} \rightarrow \sharp \llbracket B \rrbracket}(\flat \llbracket B \rrbracket)$. The interpretation of type $!^e A$ is similar to that of the function type. Note that we have an isomorphism

¹³Our recipe is different from Katsumata's in that (1) we deal with indexed monads rather than ordinary monads, and (2) it is tailored for a specific monad, i.e. category-action monads.

$\mathbb{b}[\![T \xrightarrow{e} U]\!] \cong \mathbb{b}[\![\text{Circ}^e(T, U)]]\!$ as before.¹⁴ The rest of the types are interpreted as in the case of the simple type system.

We now discuss how the judgments are interpreted. Computational judgments of the shape $\Gamma \vdash_c M : A; e$ are interpreted as morphisms $\llbracket M \rrbracket : \mathbb{b}[\![\Gamma]\!] \rightarrow \mathcal{T}_M^{e: \#[\![\Gamma]\!] \rightarrow \#[\![A]\!]} \mathbb{b}[\![A]\!]$ in **Set**. Note that this is almost identical to the interpretation in the simple type system when the interpreted term is regarded as a morphism in **Set**. Although the interpretation of the simply typed terms were given as morphisms in the Kleisli category, we define the interpretation of terms typed in the type-and-effect system in **Set**. This is because the notion of Kleisli category for category-graded monads is tricky to define (see Remark 4 for a further discussion). The value judgments remain to be interpreted as morphisms in $\mathbf{Set} \times \mathbf{disc}(\mathbf{Obj}(\mathcal{M}))$.

Let us look at the interpretation of return and let since they highlight the use of the monad. As usual, return is interpreted as the interpretation of a value post-composed with the unit.

$$\left[\frac{\Gamma \vdash_v V : A \quad \alpha(\#[\![A]\!]) = t}{\Gamma \vdash_c \text{return } V : A; \text{id}_t : t \rightarrow t} \right] \stackrel{\text{def}}{=} \mathbb{b}[\![\Gamma \vdash_v V : A]\!]; \eta_{\#[\![A]\!]}$$

The interpretation of let is essentially the same as how composition is defined in the Kleisli category of a (standard) monad:

$$\left[\frac{\Phi, \Gamma_1 \vdash_c M : A; e_1 : t_1 \rightarrow t'_1 \quad \Phi, \Gamma_2, x : A \vdash_c N : B; e_2 : t_2 \rightarrow t'_2 \quad \llbracket \#[\![\Gamma_i]\!] \triangleright u_i \quad e = (\text{id}_{u_2} \rtimes e_1); e_2}{\Phi, \Gamma_2, \Gamma_1 \vdash_c \text{let } x = M \text{ in } N : B; e} \right]$$

is given as

$$\begin{aligned} \llbracket \Phi \rrbracket \times \mathbb{b}[\![\Gamma_2]\!] \times \mathbb{b}[\![\Gamma_1]\!] &\xrightarrow{\Delta \times \text{id}; \cong} \llbracket \Phi \rrbracket \times \mathbb{b}[\![\Gamma_2]\!] \times \llbracket \Phi \rrbracket \times \mathbb{b}[\![\Gamma_1]\!] \\ &\xrightarrow{\text{id} \times \eta_{u_2} \times \llbracket M \rrbracket} \llbracket \Phi \rrbracket \times \mathcal{T}_M^{\text{id}_{u_2}} \mathbb{b}[\![\Gamma_2]\!] \times \mathcal{T}_M^{e_1} \mathbb{b}[\![A]\!] \\ &\xrightarrow{\text{id} \times \otimes; \tau} \mathcal{T}_M^{\text{id} \rtimes e_1} (\llbracket \Phi \rrbracket \times \mathbb{b}[\![\Gamma_2]\!] \times \mathbb{b}[\![A]\!]) \\ &\xrightarrow{\mathcal{T}_M^{\text{id} \rtimes e_1} \llbracket N \rrbracket} \mathcal{T}_M^{\text{id} \rtimes e_1} (\mathcal{T}_M^{e_2} (\mathbb{b}[\![B]\!])) \\ &\xrightarrow{\mu_{\text{id} \rtimes e_1, e_2}} \mathcal{T}_M^{(\text{id} \rtimes e_1); e_2} (\mathbb{b}[\![B]\!]). \end{aligned}$$

The morphism τ is the strength (which can be defined in a straightforward way), and $\otimes_{T_1, U_1, T_2, U_2}$ is the morphism¹⁵

$$\otimes : \mathcal{T}_M^{e_1: T_1 \rightarrow U_1} X \times \mathcal{T}_M^{e_2: T_2 \rightarrow U_2} Y \rightarrow \mathcal{T}_M^{(\text{id} \rtimes e_1); (e_2 \ltimes \text{id})} (X \times Y)$$

$$((x, C), (y, D)) \mapsto ((x, y), (T_2 \rtimes C); (D \ltimes U_1)).$$

Intuitively, \otimes pairs x, y and, at the same time, “parallelly composes” C and D in the order C first followed by D . In the interpretation of let, it is simply used to parallelly augment wires of the type $\#[\![\Gamma_2]\!]$ to the circuit generated by M .

The interpretation for the other constructs is also essentially unchanged from the interpretation in Figure 6. A minor difference is that instead of using the functor J to map value morphisms into

¹⁴By abuse of notation, we also denote this isomorphism as **box**.

¹⁵Aside from this elementary definition, \otimes can be defined using liftings of $\text{id} \rtimes -$ and $- \ltimes \text{id}$, and the strength and multiplication of the cat-graded monad.

the category of computations, we use η as we did for the interpretation of return V . For example, we have

$$\llbracket \Phi \vdash_c \text{box}_T V : \text{Circ}^{e: t \rightarrow u}(T, U); \varepsilon \rrbracket \stackrel{\text{def}}{=} \flat \llbracket \Phi \vdash_v V : T \xrightarrow{e: t \rightarrow u} U \rrbracket; \text{box}; \eta_{I, \text{Circ}^e(T, U)}.$$

The subsumption rule is interpreted by postcomposing the component of the natural transformation $\mathcal{T}_M^{e_1 \leq e_2}$ at $\llbracket A \rrbracket$. (See Appendix B for the interpretation of the other constructs.)

Remark 4. While we used (parameterized) Freyd category in Section 3, in this section, we had to define the interpretation explicitly using the structures of monads. Freyd categories have a better match with the fine-grained call-by-value calculus that Proto-Quipper-R is based on. The difficulty lies in defining a suitable notion of “locally graded category” [Levy 2019; Wood 1978] for category graded monads. A locally graded category is a category-like structure whose homsets are indexed by grades (i.e. elements of a preordered monoid), and this is central to the definition of graded Freyd categories [Gaboardi et al. 2021]. Naively, we may define a category-like structure in which homsets are indexed by morphisms, but determining the laws such a structure should satisfy seems non-trivial and is left for future work. \triangleleft

5.4 Correctness

Analogous to the case of simple types, the semantics is sound and adequate. Interpretation of configurations are defined as in Section 3 by considering J as a map that associates $(f, C) \in (\text{Set} \times M)((X, T), (Y, U))$ to a morphism from X to $Y \times M^{\leq \alpha(C)}(T, U)$ in Set .

Theorem 10 (Soundness). *Suppose that $Q \vdash (C, M) : A; Q'; e: t \rightarrow u$ and $(C, M) \Downarrow (C', V)$. Then $\llbracket (C, M) \rrbracket = J(\llbracket (C', V) \rrbracket)$.* \square

Theorem 11 (Computational adequacy). *Suppose that $\emptyset \vdash (C, M) : \mathbb{1}; \emptyset; e: t \rightarrow i$ and $\llbracket (C, M) \rrbracket = J(\llbracket (D, V) \rrbracket)$. Then $(C, M) \Downarrow (D, V)$ (possibly up to renaming of labels). Moreover, there must exist a circuit E such that $\alpha(E) \lesssim e$ and $C; E = D$.* \square

6 Discussion: Dependent Types

One feature that some of the languages of the Proto-Quipper family have is a form of dependent types [Colledan and Dal Lago 2024, 2025; Fu et al. 2020, 2022a]. Dependent types are useful in the context of quantum circuit programming because one can express, at the level of types, the number of qubits a function takes as input. For example, a function implementing the so-called Quantum Fourier Transform has the type $qft: \prod(n : \text{Nat}), \text{Vec Qubit } n \multimap \text{Vec Qubit } n$. We briefly discuss the possibility of adding dependent types to the languages and models from the previous sections.

The denotational model given in Section 3 can be straightforwardly extended to support some dependent types. We can apply the families construction as shown in Figure 10. Since ordinary adjunctions lift to fibered adjunctions over Set by a pointwise definition, this construction gives an instance of a parameterized version of the fibered adjunction models [Ahman et al. 2016]. Fibered adjunction models are models for a language with dependent types and computational effects. On the syntax side, this means allowing dependent types with “value restriction”: types can only depend on values with parameter type, which correspond to morphisms in Set . Although such a type system is less expressive compared to the type system of Proto-Quipper-D [Fu et al. 2022a] that allows types to depend on (the shape of) terms with quantum data types, it is expressive enough to express the type for qft we described above.

What is more challenging is to model effect annotations that are dependent on terms. Colledan and Dal Lago [2024, 2025] considered a type system in which the function qft has the type $qft: \prod(n : \text{Nat}), \text{Vec Qubit } n \xrightarrow{n: n \rightarrow n} \text{Vec Qubit } n$, where the n over the arrow is the effect annotation

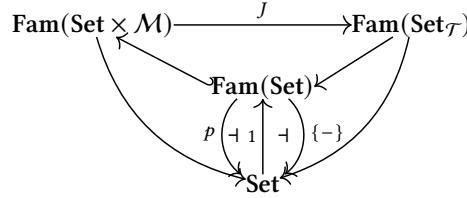


Fig. 10. Model for the dependently typed Proto-Quipper-R. The functors $\mathbf{1}$ and $\{-\}$ are the terminal object functor and the comprehension functor, respectively.

expressing the estimated width of the circuit qft generates.¹⁶ Their syntax forces annotations to be arithmetic terms depending on arithmetic variables so that type inference based on SMT-solving can semi-automatically infer the effect information. If we are only interested in annotations that are arithmetic expressions, then it seems possible to write out the interpretation just by indexing the interpretation we described in Section 5 with (tuples of) natural numbers. However, it is not clear (a) to what extent the effect annotations can be generalized and (b) whether there is a categorical explanation (e.g. an explanation in the form of some generalization of fibered monads) that captures the nature of the interpretation. Further investigation is left for future work.

7 Related Work

Giving a denotational semantics to quantum programming languages has generally been more challenging than the corresponding problem for classical languages. Even in the case of imperative QRAM languages, this task is not so simple, given that quantum data can be interpreted by semantic domains which are different from the usual ones, see [Ying 2016] for an overview.

When, in addition to a quantum store, the underlying language is also endowed with higher-order functions, the task becomes even more complicated. A few years after the quantum λ -calculus [Selinger and Valiron 2005] was introduced, a fully abstract model for the *linear fragment* of the quantum λ -calculus was given by using the category \mathbf{CPM} of completely positive maps [Selinger and Valiron 2008]. However, designing a model of (variants of) the full quantum λ -calculus remained a challenge. One difficulty is the tension between the finite and the infinite. The category \mathbf{CPM} or its subcategory \mathbf{Q} of trace-preserving completely positive maps are inherently finite since their definition relies on finite Hilbert spaces, whereas quantum λ -calculus comes with infinite features such as the $!$ modality or term level recursion. To overcome this difficulty, various approaches have been studied. These include (but are not limited to) studies based on presheaves [Malherbe et al. 2013], $(\Sigma\text{-monoid})$ enriched presheaves [Tsukada and Asada 2024], quantitative semantics of linear logic [Pagani et al. 2014], operator algebra [Cho and Westerbaan 2016], geometry of interaction [Hasuo and Hoshino 2017; Yoshimizu et al. 2014] and game semantics [Clairambault and de Visme 2020; Clairambault et al. 2019]. The first four approaches can be considered as taking a certain “completion” of \mathbf{CPM} or \mathbf{Q} to support higher-order and infinite types. The latter two approaches are more operational, since they are based on interactive semantics. We note that some of these [Clairambault and de Visme 2020; Pagani et al. 2014; Tsukada and Asada 2024] are fully abstract models of the full quantum λ -calculus.

In CDLs, the types of problems encountered in giving a denotational semantics are different. On the one hand, a semantics of CDL needs to cope with the distinction between circuit *generation* time and circuit *execution* time that does not exist in languages such as the quantum λ -calculus. Moreover,

¹⁶The syntax we are using here is not exactly the syntax used in [Colledan and Dal Lago 2024, 2025]

circuits can, like in Quipper, support specific operators which do not exist for other data structures and which allow programs to be interpreted as circuits e.g., Quipper’s *box* operator. On the other hand, giving denotational semantics to CDLs is somehow easier because the characteristics of quantum circuits, say, compared to Boolean circuits, are abstracted away. This simplicity, however, no longer holds if the CDL allows an interplay between the host and the circuit level language, known as *dynamic lifting*.

As already mentioned, since the introduction of the Proto-Quipper family with Proto-Quipper-S and Proto-Quipper-M, a presheaf-based denotational semantics has been known to be adequate [Lindenholz et al. 2018; Rios and Selinger 2017]. Later, extensions of the language with dependent types [Fu et al. 2022a] and dynamic lifting [Fu et al. 2022b, 2023] have been considered. The model for dependent types uses the families construction as discussed in Section 6. At a superficial level, the model for dynamic lifting also shares an idea with the semantics we introduced in Section 5. Both models hold a morphism representing a quantum circuit and a morphism representing its interpretation, either as a quantum operation or effect annotation. However, the foundations of these models remain fundamentally unaltered in that they use presheaves, and further comparison with our models is left for future work. It should be noted that the members of the Proto-Quipper family intended for the analysis of circuit size, like Proto-Quipper-R [Colledan and Dal Lago 2024], lack a denotational account, although being solidly grounded from an operational point of view. We believe this to be a result of the intrinsic difficulty of reflecting intensional properties of the underlying circuit in the aforementioned presheaf-based semantics.

Another commonly used type of CDLs, aside from the Proto-Quipper family, consists of those with a clear stratification between classical and quantum layers. QWire [Paykin et al. 2017] and EWire [Rennella and Staton 2020] are languages that have a dedicated language for circuits with a linear type system that can be embedded into a non-linear host language. On the semantic side, this embedding has been nicely captured using enriched category theory [Rennella and Staton 2020]. VQPL [Jia et al. 2022] is another quantum programming language with two subcalculi, one for classical programs and the other for quantum programs, where the quantum programs are more high-level than mere circuits. They support rich features such as classical recursive types, inductive quantum types and dynamic lifting. Furthermore, VQPL has an adequate denotational model that unites domain-theoretic models of classical programming and von Neumann algebras for quantum interpretation. Overall, these languages cannot have data structures with *both* quantum data and classical data, unlike languages in the Proto-Quipper family. While our semantics also has a clear separation between *Set* and the category of circuits, our languages allow *mixing* classical types and quantum types. Our key observation here is that the introduction of the closure types allows us to decompose these mixed types into classical and quantum parts.

Since the pioneering works of Moggi [1989, 1991] it can certainly be said that the concept of a monad is one of the most powerful mathematical constructions in giving meaning to computational effects. Various generalizations on plain monads, such as indexed (aka parameterized) monads [Atkey 2009], graded monads [Katsumata 2014; Mellies 2012] or category-graded monads [Orchard et al. 2020] have been proposed in the literature. The circuit monads in Section 4 and Section 5 can be seen as an instance of an indexed monad and category-graded monads, respectively. Defining effect annotations as abstractions of computational effects is a general idea that could be applied to other things besides circuits. Identifying the essences of the construction in Section 5 and deriving a recipe to construct category-graded monad might be of independent interest. Additionally, as discussed in Remark 4, a “Freyd category” for category-graded monads appears to be absent from the literature and is worth investigating further. The literature, by the way, offers some examples of circuit monads [Elliott 2013; Valiron 2016], none of which has been applied to languages in the Proto-Quipper family.

8 Conclusion

In this work, we introduce a monadic denotational semantic model for circuit description languages in which the role of the monad is played by a circuit construction. This way, the structure of the produced circuit can be observed as an effect, making the model potentially capable of reflecting intensional features of circuits produced by terms of any type. This allows us to give semantics to members of the Proto-Quipper family for which a denotational account is not, to the authors' knowledge, known. Remarkably, by considering an abstract notion of circuit algebra, different notions of circuit metrics are treated uniformly, including simple forms of circuit optimization; we believe that this abstraction may help reveal further concrete circuit metrics.

Among the future developments of this work, we must certainly mention the study of denotational semantic models for Proto-Quipper-RA, a recently introduced member of the Proto-Quipper family able to support the analysis of a wide range of circuit metrics via types. Some of its features, such as effect and dependent typing, have already been treated separately (see Sections 5 and 6), but the study of their combination in the same semantic framework is left to future work. Although our work primarily focuses on denotational semantics, it is natural to consider implementing the type system described in Section 5, e.g., by adapting or extending the QuRA tool [Colledan and Dal Lago 2025]. While supporting the circuit algebra examples discussed in this paper should not pose major technical challenges, aspects such as efficient automatic type inference merit further investigation. We plan to explore these directions in future work. We would also like to extend the language and the model with features of Quipper that we did not cover in this paper such as term and type level recursion and dynamic lifting.

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A Soundness & Adequacy

We prove the soundness and the computational adequacy of the interpretation given in Section 3. The soundness and the adequacy for the interpretation given in Section 5 can be shown by almost the same argument (and thus we do not repeat the argument).

A.1 Soundness

As usual, the soundness is proved by induction on the derivation of the evaluation relation. The proof also uses the following standard substitution lemma.

Lemma 3 (Substitution). *Suppose that $\Phi, \Gamma_1 \vdash_v V : A$.*

(1) *If $\Phi, \Gamma_2, x : A, \Gamma'_2 \vdash_v W : B$, then*

$$\begin{aligned} \llbracket \Phi, \Gamma_2, \Gamma_1, \Gamma'_2 \vdash_v W[V/x] : B \rrbracket &= \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \otimes \llbracket \Gamma'_2 \rrbracket \xrightarrow{(\Delta, \text{id}) \otimes \text{id}} \\ &\llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \otimes \llbracket \Gamma'_2 \rrbracket \xrightarrow{\cong} \\ &\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \otimes \llbracket \Gamma'_2 \rrbracket \xrightarrow{\text{id} \otimes \llbracket V \rrbracket \otimes \text{id}} \\ &\llbracket \Phi \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \otimes \llbracket A \rrbracket \otimes \llbracket \Gamma'_2 \rrbracket \xrightarrow{\llbracket W \rrbracket} \llbracket B \rrbracket. \end{aligned}$$

(2) *If $\Phi, \Gamma, x : A \vdash_c M : B$, then*

$$\begin{aligned} \llbracket \Phi, \Gamma, \Gamma_1 \vdash_v M[V/x] : B \rrbracket &= \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \xrightarrow{\text{dup} \otimes \text{id}} \\ &\llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \xrightarrow{\cong} \\ &\llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma_1 \rrbracket \xrightarrow{\text{id} \otimes J(\llbracket V \rrbracket)} \\ &\llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket A \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket B \rrbracket. \end{aligned}$$

PROOF. By a simultaneous induction on the structure of the type derivation. \square

Theorem 5 (Soundness). *Suppose that $Q \vdash (C, M) : A; Q'$ and $(C, M) \Downarrow (C', V)$. Then $\llbracket (C, M) \rrbracket = J(\llbracket (C', V) \rrbracket)$.*

PROOF. By induction on the derivation of $(C, M) \Downarrow (C', V)$. Throughout the proof we write Q_M (resp. Q_V) for the label context that types the term M (resp. V); this means that we have $Q = Q_M, Q'$ and similarly for V .

Case app: We have $M = (\lambda x.N) W$ with $(C, N[W/x]) \Downarrow (D, V)$. It suffices to prove that $\llbracket (\lambda x.N) W \rrbracket = \llbracket N[W/x] \rrbracket$ because then we can close this case by the induction hypothesis. By inversion on the typing rule, we must have

$$Q_N, x : B \vdash_c N : A \quad \text{and} \quad Q_W \vdash_v W : B$$

with $Q = Q_N, Q_W$ and $\# \llbracket Q_W \rrbracket = \# \llbracket B \rrbracket$.

$$\begin{aligned} &\llbracket (\lambda x.N) W \rrbracket \\ &= (J((\Lambda(\llbracket Q_N, x : B \vdash_c N : A \rrbracket), \text{id}_{\#(Q_N, x : B)})) \otimes J(\llbracket Q_W \vdash_v W : B \rrbracket)); \mathbf{ev} \quad (\text{by def.}) \\ &= (J(\Lambda(\llbracket Q_N, x : B \vdash_c N : A \rrbracket), \#(Q_N, x : B)) \otimes J(\llbracket Q_W \vdash_v W : B \rrbracket)); \mathbf{ev} \\ &\quad (\text{by def. of the functor } J(-, T)) \\ &= (J(\text{id}_{Q_N} \otimes \llbracket Q_W \vdash_v W : B \rrbracket)); \llbracket Q_N, x : B \vdash_c N : A \rrbracket \quad (\text{by the universality of ev}) \\ &= \llbracket N[W/x] \rrbracket \quad (\text{by substitution lemma (Lemma 3)}) \end{aligned}$$

Case dest: Similar to *let*.

Case force: Similar to the case *app*, it suffices to show that $\llbracket \text{force}(\text{lift } N) \rrbracket = \llbracket N \rrbracket$, for some N such that $M = \text{force}(\text{lift } N)$. Since *lift* is a special case of abstraction (i.e. thunking) and *force* is a special case for application, this can be proved as in the case of *app*.

Case apply: In this case, we have $M = \text{apply}((\bar{\ell}, E, \bar{k}), \bar{t})$ for some circuit E and wire bundles $\bar{\ell}$, \bar{k} and \bar{t} . Moreover, we have $C; (E \ltimes \llbracket Q' \rrbracket_M) = C'$. By inversion on the typing rules, we must have

$$\emptyset \vdash_v (\bar{\ell}, E, \bar{k}) : \text{Circ}(T, U) \quad \text{and} \quad \bar{t} : T \vdash_v \bar{t} : T$$

with $Q_M \vdash_v \bar{t} : T$ and $A = U$ for some T and U . We first show that $\llbracket \text{apply}((\bar{\ell}, E, \bar{k}), \bar{t}) \rrbracket = J(\text{id}_1, E)$. By definition, we have

$$\begin{aligned} \llbracket \text{apply}((\bar{\ell}, E, \bar{k}), \bar{t}) \rrbracket &= (\llbracket (\bar{\ell}, E, \bar{k}) \rrbracket \otimes \text{id}_{Q_M}); \text{apply} \\ &= ((J(\Lambda(\text{id}_1, \llbracket (\bar{\ell}, E, \bar{k}) \rrbracket_M)); J(\text{box}, \text{id}_I)) \otimes \text{id}_{Q_M}); (J(\text{box}^{-1}, \text{id}_I) \otimes \text{id}_{Q_M}); \text{ev} \\ &= ((J(\Lambda(\text{id}_1, \llbracket (\bar{\ell}, E, \bar{k}) \rrbracket_M)) \otimes \text{id}_{Q_M}); \text{ev} \\ &= J(\text{id}_1, \llbracket (\bar{\ell}, E, \bar{k}) \rrbracket_M). \end{aligned}$$

Therefore,

$$\begin{aligned} \llbracket (C, M) \rrbracket &= J(\text{id}_1, C); (\llbracket \text{apply}((\bar{\ell}, E, \bar{k}), \bar{t}) \rrbracket \otimes \text{id}_{Q'}) \\ &= J(\text{id}_1, C); (J(\text{id}_1, \llbracket (\bar{\ell}, E, \bar{k}) \rrbracket_M) \otimes J(\text{id}_1, \text{id}_{Q'})) \\ &= J(\text{id}_1, C); J(\text{id}_1, \llbracket (\bar{\ell}, E, \bar{k}) \rrbracket_M \ltimes \llbracket Q' \rrbracket) \\ &= J(\text{id}_1, C'; \text{perm}) \\ &= \llbracket (C', \bar{k}) \rrbracket \end{aligned}$$

as desired. Here, **perm** is the isomorphism induced by the codomain of C' and type of \bar{k} .

Case box: It must be the case that $M = \text{box}_T W$ and $V = (\bar{\ell}, D, \bar{k})$ for some value W , circuit D , and wire bundles $\bar{\ell}$ and \bar{k} , and moreover, $(\text{id}_Q, W \bar{t}) \Downarrow (D, \bar{k})$. By inversion on the typing rules, we have

$$\emptyset \vdash_v W : T \multimap U \quad \text{and} \quad \emptyset \vdash_v (\bar{\ell}, D, \bar{k}) : \text{Circ}(T, U)$$

where $A = \text{Circ}(T, U)$. Since the underlying circuit does not change during the reduction, it suffices to show that $\llbracket \text{box}_T W \rrbracket = J(\llbracket (\bar{\ell}, D, \bar{k}) \rrbracket)$. Moreover, since $\llbracket \text{box}_T W \rrbracket = J(\llbracket W \rrbracket); J(\text{box}, \text{id}_I)$ and $J(\llbracket (\bar{\ell}, D, \bar{k}) \rrbracket) = J(\widehat{\llbracket (\bar{\ell}, D, \bar{k}) \rrbracket_M, \text{id}_I}); J(\text{box}, \text{id}_I)$ by the definitions, we only need to show that $\llbracket W \rrbracket = \widehat{\llbracket (\bar{\ell}, D, \bar{k}) \rrbracket_M, \text{id}_I}$. By the induction hypothesis, we have

$$\llbracket (\text{id}_Q, W \bar{t}) \rrbracket = \llbracket W \bar{t} \rrbracket = J(\text{id}_1, \llbracket (\bar{\ell}, D, \bar{k}) \rrbracket_M) = \llbracket (D, \bar{k}) \rrbracket.$$

Thus, we have

$$\llbracket W \rrbracket = (\Lambda(\llbracket W \bar{t} \rrbracket), \text{id}_I) = (\Lambda(J(\text{id}_1, \llbracket (\bar{\ell}, D, \bar{k}) \rrbracket_M)), \text{id}_I) = (\widehat{\llbracket (\bar{\ell}, D, \bar{k}) \rrbracket_M, \text{id}_I})$$

as desired.

Case return: In this case, we have $M = \text{return } V$ and $D = C$. Since $D = C$, it suffices to show that $\llbracket M \rrbracket = J((\text{id}_1, \text{perm}); \llbracket V \rrbracket)$ for some suitable permutation isomorphism **perm**, and this is obvious by the definition of $\llbracket \text{return } V \rrbracket$.

Case let: In this case, we must have

$$\frac{(C, N_1) \Downarrow (D, W) \quad (E, N_2[W/x]) \Downarrow (D, V)}{(C, \text{let } x = N_1 \text{ in } N_2) \Downarrow (C', V)}$$

with $M = \text{let } x = N_1 \text{ in } N_2$ for some terms N_1, N_2 , value W and circuit E . By the inversion on the typing rule, we must also have

$$Q_{N_1} \vdash_c N_1 : B \quad \text{and} \quad Q_{N_2}, x : B \vdash_c N_2 : A$$

for some Q_{N_1} and Q_{N_2} such that $Q_M = Q_{N_2}, Q_{N_1}$.

$$\begin{aligned} \llbracket (C, M) \rrbracket &= J(\text{id}_1, C); (\llbracket \text{let } x = N_1 \text{ in } N_2 \rrbracket \times \llbracket Q' \rrbracket) \\ &= J(\text{id}_1, C); ((\llbracket Q_{N_2} \rrbracket \times \llbracket N_1 \rrbracket); \llbracket N_2 \rrbracket) \times \llbracket Q' \rrbracket \\ &= J(\text{id}_1, C); (\llbracket Q_{N_2} \rrbracket \times \llbracket N_1 \rrbracket \times \llbracket Q' \rrbracket); (\llbracket N_2 \rrbracket \times \llbracket Q' \rrbracket) \\ &= J(\text{id}_1, D); (\text{id}_{\llbracket Q_{N_2} \rrbracket} \otimes J(\llbracket W \rrbracket)) \times \llbracket Q' \rrbracket; (\llbracket N_2 \rrbracket \times \llbracket Q' \rrbracket) \quad (\text{by I.H.}) \\ &= J(\text{id}_1, D); \left((\text{id}_{\llbracket Q_{N_2} \rrbracket} \otimes J(\llbracket W \rrbracket)); \llbracket Q_{N_2}, x : B \vdash_c N_2 : A \rrbracket \right) \times \llbracket Q' \rrbracket \\ &= J(\text{id}_1, D); (\llbracket N_2[W/x] \rrbracket \times \llbracket Q' \rrbracket) \quad (\text{by substitution lemma (Lemma 3)}) \\ &= J(\text{id}_1, C'); (J(\llbracket V \rrbracket) \times \llbracket Q' \rrbracket) \quad (\text{by I.H.}) \\ &= \llbracket (C', V) \rrbracket \end{aligned}$$

□

A.2 Computational Adequacy

The proof of computational adequacy also follows a standard strategy: we define a logical relation between semantics and syntax of Proto-Quipper-M. Similar logical relations have been considered by Colledan and Dal Lago [Colledan and Dal Lago 2024, 2025] to prove the correctness of the type system of Proto-Quipper-R. (These are defined by purely operational means.)

We say that a value (resp. term) is *closed* if the value (resp. term) does not have any free variables (but it may have some labels). The set of closed values of type (resp. term) A that is well-typed under the labeled context Q is denoted by $\text{CVal}_{Q \vdash A}$ (resp. $\text{CTerm}_{Q \vdash A}$). We define relations (indexed by judgment of the form $Q \vdash A$) $\mathcal{R}_{Q \vdash A} \subseteq (\mathbb{b}\llbracket A \rrbracket \times \mathcal{M}(\llbracket Q \rrbracket_M, \# \llbracket A \rrbracket)) \times \text{CTerm}_{Q \vdash A}$ and $\mathcal{V}_{Q \vdash A} \subseteq \mathbb{b}\llbracket A \rrbracket \times \text{CVal}_{Q \vdash A}$ as the smallest relations satisfying the following conditions.

- $(v, C; \text{perm}) \mathcal{R}_{Q \vdash A} M$ if and only if $(\text{id}_{\llbracket Q \rrbracket_M}, M) \Downarrow (C, V)$, $C: Q \rightarrow L'$ and $v \mathcal{V}_{L \vdash A} V$ for some label contexts L' and L such that $\llbracket L' \rrbracket_M \xrightarrow[\cong]{\text{perm}} \llbracket L \rrbracket_M$ where perm is a permutation isomorphism.
- $* \mathcal{V}_{\emptyset \vdash \perp} *$
- $n \mathcal{V}_{\emptyset \vdash \text{Nat}} n$
- $* \mathcal{V}_{\ell: w \vdash w} \ell$
- $f \mathcal{V}_{Q \vdash A \multimap_T B} V$ if and only if for all W such that $w \mathcal{V}_{L \vdash A} W$, we have $f w \mathcal{R}_{Q, L \vdash B} V W$.
- $f \mathcal{V}_{\emptyset \vdash !A} V$ if and only if $f * \mathcal{R}_{\emptyset \vdash B}$ force V
- $\llbracket (\bar{\ell}, C, \bar{k}) \rrbracket \mathcal{V}_{\emptyset \vdash \text{Circ}(T, U)} (\bar{\ell}, C, \bar{k})$
- $(v, w) \mathcal{V}_{Q_1, Q_2 \vdash A \otimes B} \langle V, W \rangle$ if and only if $v \mathcal{V}_{Q_1 \vdash V}$ and $w \mathcal{V}_{Q_2 \vdash W}$.

We extend the relation to a relation between pairs of a value and a substitution and typing contexts by

$$(v, \gamma) \mathcal{V}_Q (a_1 : A_1, \dots, a_n : A_n) \iff v \mathcal{V}_{Q \vdash A_1 \otimes \dots \otimes A_n} \langle \gamma(a_1), \dots, \gamma(a_n) \rangle$$

where

- v is an element of $\mathbb{b}\llbracket A_1 \rrbracket \times \dots \times \mathbb{b}\llbracket A_n \rrbracket$ and
- γ is a map from $\text{dom}(\Gamma)$ to closed values such that, for each a_i , the only labels appearing in $\gamma(a_i)$ are those in Q .

The definition of $\mathcal{R}_{Q \vdash A}$ evaluates M with the identity circuit, but this does not loose generality because if the initial configuration has a circuit that is not the identity we can just concatenate the initial circuit to the circuit obtained by evaluating M with the identity circuit.

Lemma 4. Suppose that $C: Q_0 \rightarrow Q_1, L, Q_2$ and $L \vdash M: A$. If $(\text{id}_{\llbracket L \rrbracket_M}, M) \Downarrow (D, V)$ and $(C, M) \Downarrow (E, W)$, then $V = W$ (up to renaming of labels) and $E = C; (\llbracket Q_1 \rrbracket_M \rtimes (D \ltimes \llbracket Q_2 \rrbracket_M))$.

PROOF. By induction on the derivation of $(C, M) \Downarrow (E, W)$. \square

The fundamental property of the logical relations hold. And, as usual, the theorem is a direct consequence of the fundamental property.

Lemma 5 (Fundamental Property). Let Γ be a type environment, Q be a label context such that $\llbracket Q \rrbracket_M = \#\llbracket \Gamma \rrbracket$. Suppose that $(v, \gamma) \mathcal{V}_Q \Gamma$. Then the following holds.

- (1) $\Gamma \vdash_v V : A$ implies $b\llbracket V \rrbracket(v) \mathcal{V}_{Q \vdash A} \gamma(V)$.
- (2) $\Gamma \vdash_c M : A$ implies, $\llbracket M \rrbracket(v) \mathcal{R}_{Q \vdash A} \gamma(M)$

PROOF. By simultaneous induction on the type derivation. We show two of the most interesting cases: the case for *box* and *let*. The other cases can be proved in a similar manner.

Case box: In this case, we must have $M = \text{box } V$, $\Gamma = \Phi$ and $A = \text{Circ}(T, U)$ for some value V , parameter context Φ and bundle types T and U . Moreover, we have $\Phi \vdash_v V : T \multimap U$.

Suppose that $(v, \gamma) \mathcal{V}_\emptyset \Phi$. Our goal is to show $\llbracket M \rrbracket(v) \mathcal{R}_{\emptyset \vdash \text{Circ}(T, U)} \gamma(M)$. By induction hypothesis, we have $b\llbracket V \rrbracket(v) \mathcal{V}_{\emptyset \vdash T \multimap U} \gamma(V)$. Hence, we have $b\llbracket V \rrbracket(v)(*) \mathcal{R}_{\ell: T \vdash U} \gamma(V) \ell$. By the definition of $\mathcal{R}_{\ell: T \vdash U}$, we have

$$(\text{id}_{\llbracket T \rrbracket_M}, \gamma(V) \ell) \Downarrow (C, \bar{k}) \quad (7)$$

$$b\llbracket V \rrbracket(v)(*) = (*, C; \mathbf{perm}) \quad (8)$$

$$* \mathcal{V}_{L \vdash U} \bar{k} \quad (9)$$

$$\mathbf{perm}: \llbracket L' \rrbracket_M \xrightarrow{\cong} \llbracket L \rrbracket_M \quad (10)$$

where $C: \llbracket T \rrbracket_M \rightarrow \llbracket L' \rrbracket_M$. By applying the *box* rule to (7), we have $(\text{id}_I, \gamma(\text{box } V)) \Downarrow (\text{id}_I, (\ell, C, \bar{k}))$. It remains to show that $\llbracket M \rrbracket(v) = (\llbracket (\ell, C, \bar{k}) \rrbracket_M, \text{id}_I)$, and this is an immediate consequence of (8) and the interpretation of *box*.

Case let: It must be the case that

$$M = (\text{let } x = N \text{ in } P) \quad \Gamma = \Phi, \Gamma_2, \Gamma_1$$

$$\Phi, \Gamma_1 \vdash_c N : B$$

$$\Phi, \Gamma_2, x : B \vdash_c P : A$$

for some terms N and M , some type B and contexts Φ , Γ_1 and Γ_2 . Let Q_1 and Q_2 be label contexts such that $\llbracket Q_1 \rrbracket_M = \#\llbracket \Gamma_1 \rrbracket$ and $\llbracket Q_2 \rrbracket_M = \#\llbracket \Gamma_2 \rrbracket$.

Suppose that $(v, \gamma) \mathcal{V}_{Q_2, Q_1} \Gamma$. We need to show that $\llbracket M \rrbracket(v) \mathcal{R}_{\Gamma_2, Q_1} \gamma(M)$. Let us write v_0, v_1 and v_2 for the Φ , Γ_1 and Γ_2 part of v , respectively, and let $\gamma_1 \stackrel{\text{def}}{=} \gamma \upharpoonright_{\text{dom}(\Phi, \Gamma_1)}$ and $\gamma_2 \stackrel{\text{def}}{=} \gamma \upharpoonright_{\text{dom}(\Phi, \Gamma_2)}$. By the induction hypothesis, we have

$$\llbracket N \rrbracket(v_0, v_1) \mathcal{R}_{Q_1 \vdash B} \gamma_1(N) \quad (11)$$

$$\llbracket P \rrbracket(v_0, v_2, w) \mathcal{R}_{Q_2, Q_1 \vdash A} [W/x]\gamma_2(P) \quad (12)$$

where

$$\llbracket N \rrbracket(v_0, v_1) = (w, C) \quad (13)$$

$$(\text{id}_{\llbracket Q_1 \rrbracket_M}, \gamma_1(N)) \Downarrow (C', W) \quad (14)$$

By (11), we must have $w \mathcal{V}_{Q_1 \vdash B} W$ and $C = C'$ (up to permutation which we shall ignore for simplicity). By applying Lemma 4 to (14), we also have

$$(\text{id}_{[\![Q_2, Q_1]\!]_{\mathcal{M}}}, \gamma_1(N)) \Downarrow ([\![Q_2]\!]_{\mathcal{M}} \rtimes C', W). \quad (15)$$

Now suppose that

$$[\![P]\!](v_0, v_2, w) = (v_{\text{ret}}, D) \quad (16)$$

$$(\text{id}_{[\![Q_1]\!]_{\mathcal{M}}}, \gamma_1(N)) \Downarrow (D', V) \quad (17)$$

By the definition of (12), it must be the case that $v_{\text{ret}} \mathcal{V}_{Q_2, Q_1 \vdash B} V$ and $D = D'$ (again, up to permutation which we ignore). Then, once again, by Lemma 4, we have

$$([\![Q_2]\!]_{\mathcal{M}} \rtimes C', [W/x]\gamma_2(P)) \Downarrow (([\![Q_2]\!]_{\mathcal{M}} \rtimes C'); D', V) \quad (18)$$

It follows that

$$(\text{id}_{[\![Q_2, Q_1]\!]_{\mathcal{M}}}, \gamma(M)) \Downarrow (([\![Q_2]\!]_{\mathcal{M}} \rtimes C'); D, V)$$

by applying the rule *let* to (15) and (18). From the definition of $[\![M]\!]$, (13), and (16) together with $v_{\text{ret}} \mathcal{V}_{Q_2, Q_1 \vdash B} V$ we have $b([\![M]\!](v_0, v_2, v_1)) \mathcal{V}_{Q_2, Q_1 \vdash A} V$. The equality (up to permutation) on the circuit part also holds. We, therefore, have $[\![M]\!](v) \mathcal{R}_{Q_2, Q_1 \vdash A} \gamma(M)$. \square

Theorem 6 (Computational Adequacy). *Suppose that $\emptyset \vdash (C, M) : \mathbb{1}; \emptyset$ and $[\!(C, M)\!] = J([\!(D, V)\!])$. Then $(C, M) \Downarrow (D, V)$ (possibly up to renamings of labels).*

PROOF. The assumption $\emptyset \vdash (C, M) : \mathbb{1}; \emptyset$ means that there is a label context Q such that $Q \vdash_c M : \mathbb{1}$. Since M is closed, using the previous lemma, we obtain $[\![M]\!](*) \mathcal{R}_{Q \vdash \mathbb{1}} M$. By the definition of $\mathcal{R}_{Q \vdash \mathbb{1}}$, we have $(\text{id}_{[\![Q]\!]_{\mathcal{M}}}, M) \Downarrow (E, *)$ where $[\![M]\!](*) = (*, E)$. By $[\!(C, M)\!] = J([\!(D, V)\!])$, it follows that $D = C; E$. From this and Lemma 4, we have $(C, M) \Downarrow (D, V)$ as desired. \square

Remark 5. To prove the adequacy for terms typed in the type-and-effect system, we just need to add effect annotations to the logical relations. That is, the logical relation for computations, now becomes $\mathcal{R}_{Q \vdash A}^e \subseteq (b[\![A]\!] \times \mathcal{M}^{\leq e}([\![Q]\!]_{\mathcal{M}}, \#[\![A]\!])) \times \text{CTerm}_{Q \vdash A; e}$ where $\text{CTerm}_{Q \vdash A; e}$ is the set of terms M such that $Q \vdash_c M : A; e$. Note that the set of circuits are now restricted to $\mathcal{M}^{\leq e}([\![Q]\!]_{\mathcal{M}}, \#[\![A]\!])$. Relations $\mathcal{V}_{Q \vdash A}$ for arrow type, thunk type, and circuit type are changed accordingly. \triangleleft

B Omitted Definitions from Section 5

This section shows the typing rules and the interpretation that was omitted from Section 5.

B.1 Full Definition of the Typing Rules

Here we list the full list of typing rules for reference. The rules that were omitted from Section 5 are mostly identical to the corresponding rules of Proto-Quipper-C.

$\frac{\text{unit}}{\Phi \vdash_v * : \mathbb{1}}$	$\frac{\text{nat}}{\Phi \vdash_v n : \text{Nat}}$	$\frac{\text{lab}}{\Phi, \ell : w \vdash_v \ell : w}$	$\frac{\text{var}}{\Phi, x : A \vdash_v x : A}$
$\frac{\Gamma, x : A \vdash_c M : B; e : t \rightarrow u}{\Gamma \vdash_v \lambda x_A.M : A \xrightarrow[e:t \rightarrow u]{\#(\Gamma)} B}$	$\frac{\Phi, \Gamma_1 \vdash_v V : A \xrightarrow[e:s \rightarrow u]{\circ_T} B \quad \Phi, \Gamma_2 \vdash_v W : A}{\Phi, \Gamma_1, \Gamma_2 \vdash_c V W : B; e : s \rightarrow u}$		
$\frac{\Phi \vdash_c M : A; e : t \rightarrow u}{\Phi \vdash_v \text{lift } M : !^e A}$	$\frac{\Phi \vdash_v V : !^e A}{\Phi \vdash_c \text{force } V : A; e : t \rightarrow u}$		
$\frac{C : Q \rightarrow L \quad Q \cong_{\sigma} Q' \quad L \cong_{\sigma} L' \quad \mathcal{Q}' \vdash_v \bar{\ell} : T \quad L' \vdash_v \bar{k} : U \quad \alpha(\llbracket(\bar{\ell}, C, \bar{k})\rrbracket) = e}{\Phi \vdash_v (\bar{\ell}, C, \bar{k}) : \text{Circ}^e(T, U)}$	$\frac{\Phi \vdash_v V : T \xrightarrow[e:t \rightarrow u]{\circ_I} U}{\Phi \vdash_c \text{box}_T V : \text{Circ}^{e:t \rightarrow u}(T, U); e}$		
	$\frac{\Phi, \Gamma_1 \vdash_v V : \text{Circ}^{e:t \rightarrow u}(T, U) \quad \Phi, \Gamma_2 \vdash_v W : T}{\Phi, \Gamma_1, \Gamma_2 \vdash_c \text{apply}(V, W) : U; e : t \rightarrow u}$		
$\frac{\Phi, \Gamma_1 \vdash_v V : A \otimes B \quad \Phi, \Gamma_2, x : A, y : B \vdash_c M : C; e}{\Phi, \Gamma_2, \Gamma_1 \vdash_c \text{let } \langle x, y \rangle = V \text{ in } M : C; e}$	$\frac{\Phi \vdash_v V : \text{Nat} \quad \Phi, \Gamma \vdash_c M : A; e \quad \Phi, \Gamma \vdash_c N : A; e}{\Phi, \Gamma \vdash_c \text{ifz } V \text{ then } M \text{ else } N : A; e}$		
$\frac{\Phi, \Gamma_1 \vdash_v V : A \Phi, \Gamma_2 \vdash_v W : B}{\Phi, \Gamma_1, \Gamma_2 \vdash_v \langle V, W \rangle : A \otimes B}$	$\frac{\Gamma \vdash_v V : A \quad \alpha(\#\llbracket A \rrbracket) = t}{\Gamma \vdash_c \text{return } V : A; \text{id}_t : t \rightarrow t}$		
$\frac{\Phi, \Gamma_1 \vdash_c M : A; e_1 : t_1 \rightarrow t'_1 \quad \Phi, \Gamma_2, x : A \vdash_c N : B; e_2 : t_2 \rightarrow t'_2 \quad \alpha(\#\llbracket \Gamma_i \rrbracket) = u_i \quad e = (\text{id}_{u_2} \rtimes e_1) ; e_2}{\Phi, \Gamma_2, \Gamma_1 \vdash_c \text{let } x = M \text{ in } N : B; e}$	$\frac{\Gamma \vdash_c M : A; e_1 : t \rightarrow u \quad e_1 \lesssim e_2}{\Gamma \vdash_c M : A; e_2 : t \rightarrow u}$		
$\frac{\text{perm} : \llbracket \Gamma_1 \rrbracket \otimes \llbracket A \rrbracket \otimes \llbracket B \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \xrightarrow{\cong} \llbracket \Gamma_1 \rrbracket \otimes \llbracket B \rrbracket \otimes \llbracket A \rrbracket \otimes \llbracket \Gamma_2 \rrbracket \quad \Gamma_1, a : A, b : B, \Gamma_2 \vdash_c M : C; e : t \rightarrow u}{\Gamma_1, b : B, a : A, \Gamma_2 \vdash_c M : C; (\alpha(\text{perm}); e)}$			

Fig. 11. Typing rules for the effect system of Proto-Quipper-R (full definition).

B.2 Interpretation of Value Judgments

We first show how value judgments are interpreted. As explained, value judgments are interpreted in $\mathcal{V} (= \text{Set} \times \text{disc}(\mathcal{M}))$ and the interpretation is essentially the same as that of the simply-typed

system. Here we only show the interpretation of values with types that have effect annotations.

$$\begin{aligned}\llbracket \Gamma \vdash_v \lambda x_A.M : A \xrightarrow{e: t \rightarrow u} B \rrbracket &\stackrel{\text{def}}{=} (\Lambda_e(\llbracket \Gamma, x : A \vdash_c M : B; e \rrbracket), \text{id}_{\llbracket \#(\Gamma) \rrbracket_M}) \\ \llbracket \Phi \vdash_v \text{lift } M : !^e A \rrbracket &\stackrel{\text{def}}{=} (\Lambda_e(\llbracket \Phi \vdash_c M : A \rrbracket), \text{id}_I) \\ \llbracket \Phi \vdash_v (\bar{\ell}, C, \bar{k}) : \text{Circ}^e(T, U) \rrbracket &\stackrel{\text{def}}{=} (!_{\llbracket \Phi \rrbracket}; \widehat{\llbracket (\bar{\ell}, C, \bar{k}) \rrbracket}, \text{id}_I)\end{aligned}$$

The above interpretations are almost identical to those of Figure 5. The only difference is the natural bijection Λ_e , which is now indexed with e :

$$\text{Set}(X \times Y, Z \times \mathcal{M}^{\leq e}(T, U)) \xrightarrow[\cong]{\Lambda_{e,X,Y,Z}} \text{Set}(X, Y \Rightarrow_{\text{Set}} Z \times \mathcal{M}^{\leq e}(T, U)).$$

The morphism $\hat{C} : 1 \rightarrow \mathcal{M}^{\leq e}(\llbracket T \rrbracket_M, \llbracket U \rrbracket_M)$ is the global element $\hat{C}(*) = C$ (which in turn can be defined using Λ_e).

B.3 Interpretation of Computational Judgments

Recall that $\Gamma \vdash_c M : A; e$ is interpreted as a morphism in Set from $\flat[\Gamma]$ to $\mathcal{T}^{e: \sharp[\Gamma] \rightarrow \sharp[A]}(\flat[A])$.

Application and forcing.

$$\begin{aligned}&\left[\frac{\Phi, \Gamma_1 \vdash_v V : A \xrightarrow{e: s \rightarrow u} B \quad \Phi, \Gamma_2 \vdash_v W : A}{\Phi, \Gamma_1, \Gamma_2 \vdash_c V W : B; e: s \rightarrow u} \right] \\ &\stackrel{\text{def}}{=} \llbracket \Phi \rrbracket_P \times \flat[\Gamma_1] \times \flat[\Gamma_2] \xrightarrow{\Delta_{[\Phi]_P} \times \text{id}} \llbracket \Phi \rrbracket_P \times \llbracket \Phi \rrbracket \times \flat[\Gamma_1]_P \times \flat[\Gamma_2] \\ &\xrightarrow{\cong} (\llbracket \Phi \rrbracket_P \times \flat[\Gamma_1]) \times (\llbracket \Phi \rrbracket_P \times \flat[\Gamma_2]) \\ &\xrightarrow{\flat[V] \times \flat[W]} \flat[A \multimap_T B] \times \flat[A] \\ &\xrightarrow{\text{ev}} \mathcal{T}^e(\flat[B]) \\ \left[\frac{\Phi \vdash_v V : !^e A}{\Phi \vdash_c \text{force } V : A; e: t \rightarrow u} \right] &\stackrel{\text{def}}{=} \llbracket \Phi \rrbracket_P \xrightarrow{\flat[V]} \flat[!^e A] \times 1 \xrightarrow{\text{ev}} \mathcal{T}^e(\flat[A])\end{aligned}$$

Here, unlike in Section 4, ev is the evaluation morphism for the exponential objects in Set .

Circuit operations.

$$\begin{aligned}
& \left[\frac{\Phi \vdash_v V : T \xrightarrow{e: t \rightarrow u} U}{\Phi \vdash_c \text{box}_T V : \text{Circ}^{e: t \rightarrow u}(T, U); e} \right] \\
& \stackrel{\text{def}}{=} [\Phi]_P \xrightarrow{[\![V]\!]} b[\![T \xrightarrow{e} U]\!] \\
& \xrightarrow{\text{box}} \mathcal{M}^{\leq e}([\![T]\!]_{\mathcal{M}}, [\![U]\!]_{\mathcal{M}}) \\
& \xrightarrow{\eta_I} \mathcal{T}^e(\mathcal{M}^{\leq e}([\![T]\!]_{\mathcal{M}}, [\![U]\!]_{\mathcal{M}})) \\
\\
& \left[\frac{\Phi, \Gamma_1 \vdash_v V : \text{Circ}^{e: t \rightarrow u}(T, U) \quad \Phi, \Gamma_2 \vdash_v W : T}{\Phi, \Gamma_1, \Gamma_2 \vdash_c \text{apply}(V, W) : U; e: t \rightarrow u} \right] \\
& \stackrel{\text{def}}{=} [\Phi]_P \times b[\![\Gamma_1]\!] \times b[\![\Gamma_2]\!] \xrightarrow{\Delta[\![\Phi]\!]_P \times \text{id}} [\Phi]_P \times [\Phi]_P \times b[\![\Gamma_1]\!] \times b[\![\Gamma_2]\!] \\
& \xrightarrow{\cong} ([\Phi]_P \times b[\![\Gamma_1]\!]) \times ([\Phi]_P \times b[\![\Gamma_2]\!]) \\
& \xrightarrow{b[\![V]\!] \times b[\![W]\!]} \mathcal{M}^{\leq e}([\![T]\!]_{\mathcal{M}}, [\![U]\!]_{\mathcal{M}}) \\
& \xrightarrow{\eta_{[\![T]\!]_{\mathcal{M}}}} \mathcal{T}^{\text{id}_{[\![T]\!]}}(\mathcal{M}^{\leq e}([\![T]\!]_{\mathcal{M}}, [\![U]\!]_{\mathcal{M}}))) \\
& \xrightarrow{\text{comp}} \mathcal{T}^e(1).
\end{aligned}$$

Here $\text{comp}_{T_1, T_2, T_3} : \mathcal{M}(T_1, T_2) \times \mathcal{M}(T_2, T_3) \rightarrow \mathcal{M}(T_1, T_3)$ is the composition morphism.

Return and let.

$$\begin{aligned}
& \left[\frac{\Gamma \vdash_v V : A \quad \alpha(\#[\![A]\!]) = t}{\Gamma \vdash_c \text{return } V : A; \text{id}_t : t \rightarrow t} \right] \stackrel{\text{def}}{=} b[\![\Gamma]\!] \xrightarrow{b[\![V]\!]} b[\![A]\!] \xrightarrow{\eta_{\#[\![A]\!]}} \mathcal{T}^{\text{id}_{\#[\![A]\!]}}(b[\![A]\!]) \\
\\
& \left[\frac{\Phi, \Gamma_1 \vdash_c M : A; e_1 : t_1 \rightarrow t'_1 \quad \Phi, \Gamma_2, x : A \vdash_c N : B; e_2 : t_2 \rightarrow t'_2 \quad [\![\#\Gamma_i]\!] \triangleright u_i \quad e = (\text{id}_{u_2} \rtimes e_1); e_2}{\Phi, \Gamma_2, \Gamma_1 \vdash_c \text{let } x = M \text{ in } N : B; e} \right] \\
& \stackrel{\text{def}}{=} [\Phi] \times b[\![\Gamma_2]\!] \times b[\![\Gamma_1]\!] \xrightarrow{(\Delta \times \text{id}); \cong} [\Phi] \times b[\![\Gamma_2]\!] \times [\Phi] \times b[\![\Gamma_1]\!] \\
& \xrightarrow{\text{id} \times \eta_{u_2} \times [\![M]\!]} [\Phi] \times \mathcal{T}_M^{\text{id}_{u_2}} b[\![\Gamma_2]\!] \times \mathcal{T}_M^{e_1} b[\![A]\!] \\
& \xrightarrow{\text{id} \times \otimes; \tau} \mathcal{T}_M^{\text{id} \rtimes e_1}([\Phi] \times b[\![\Gamma_2]\!] \times b[\![A]\!]) \\
& \xrightarrow{\mathcal{T}_M^{\text{id} \rtimes e_1} [\![N]\!]} \mathcal{T}_M^{\text{id} \rtimes e_1}(\mathcal{T}_M^{e_2}(b[\![B]\!])) \\
& \xrightarrow{\mu_{\text{id} \rtimes e_1, e_2}} \mathcal{T}_M^{(\text{id} \rtimes e_1); e_2}(b[\![B]\!]).
\end{aligned}$$

Subsumption.

$$\left[\frac{\Gamma \vdash_c M : A; e_1 : t \rightarrow u \quad e_1 \lesssim e_2}{\Gamma \vdash_c M : A; e_2 : t \rightarrow u} \right] \stackrel{\text{def}}{=} b[\![\Gamma]\!] \xrightarrow{[\![M]\!]} \mathcal{T}^{e_1: \#[\![\Gamma]\!] \rightarrow \#[\![A]\!]}(b[\![A]\!]) \xrightarrow{\mathcal{T}_{b[\![A]\!]}^{e_1 \lesssim e_2}} \mathcal{T}^{e_2}(b[\![A]\!])$$

C Supplementary Materials on Indexed Monads and Parameterized Freyd Categories

As supplementary material, we include a brief review of indexed monads and the parameterized Freyd category to make the paper self-contained. Most of the definitions are drawn from [[Atkey 2009](#)]. We present only the minimal definitions required to understand this paper, with explanations tailored to our purposes.

C.1 Parameterized Freyd category

We give the precise definition of parameterized Freyd category that was omitted from the body of the paper.

Definition 12 ([[Atkey 2009](#)]). A parameterized Freyd category consists of three functors $J : C \times S \rightarrow \mathcal{K}$, $\otimes : C \times \mathcal{K} \rightarrow \mathcal{K}$ and $\boxtimes : \mathcal{K} \times C \rightarrow \mathcal{K}$ such that

- J is identity on objects,
- the cartesian product of C is respected: $X \otimes J(Y, S) = J(X, S) \otimes Y = J(X \times Y, S)$,
- for each $S \in S$, the transformations given by the associativity $J(\alpha, S)$, the left unit $J(\lambda, S)$, the right unit $J(\rho, S)$ and the symmetry $J(s, S)$ of the symmetric monoidal structure arising from the finite products of C must be natural in the variables in all combinations of \times , \otimes and \boxtimes that make up their domain and codomain.

As is clear from the above conditions, we are assuming that C has finite products. The pair of functors \otimes and \boxtimes is often called the premonoidal structure of the parameterized Freyd category (with respect to C). \triangleleft

The most important example (and the only example appearing in the paper) of a parameterized Freyd category is the one given by the Kleisli construction of an indexed monad. As already explained, the Kleisli category of $\mathcal{T} : S^{\text{op}} \times S \times C \rightarrow C$ has pairs of objects of C and S as its objects and the homset $C_{\mathcal{T}}((X, S), (Y, T))$ is defined as $C(X, \mathcal{T}(S, T, Y))$. Compositions of morphisms $f : (X, S) \rightarrow (Y, T)$ and $g : (Y, T) \rightarrow (Z, U)$ are defined by the Kleisli composition:

$$X \xrightarrow{f} \mathcal{T}(S, T, Y) \xrightarrow{\mathcal{T}(S, T, g)} \mathcal{T}(S, T, \mathcal{T}(T, U, Z)) \xrightarrow{\mu_{S, T, U, Z}} \mathcal{T}(S, U, Z).$$

The (morphism part of the) functor \boxtimes is defined by

$$f \boxtimes c \stackrel{\text{def}}{=} X \times W \xrightarrow{f \times c} Y \times \mathcal{T}(S, T, Z) \xrightarrow{\tau_{Y, S, T, Z}} \mathcal{T}(S, T, Y \times Z). \quad (19)$$

for $f : X \rightarrow Y$, a morphism in C , and $c : (W, S) \rightarrow (Z, T)$.

C.2 Lifting of the Premonoidal Structure

In the main part of the paper, we claimed that the premonoidal structure is lifted to the indexed monad, parameterized Freyd category, and the category-graded monad. Here we make precise what we mean by that. This is just a review of an existing, but perhaps not so known, notion [[Atkey 2009](#)], and we believe that having this supplementary section would help readers understand the technical details. However, *readers not interested in the technical definitions may safely ignore this section*. To understand the interpretation, it suffices to understand that there is an operation, called the lifting of $- \times T$, that acts on computations by modifying the underlying circuit C to $C \times T$. It may be worth mentioning that our interpretation, as well as the soundness result, does not use any specific property of the circuit monad except for circuit related operations such as box and lift. The interpretation of the other constructs only relies on the parameterized Freyd structure *and the existence of the premonoidal lifting*.

We start by reviewing the notion of lifting for indexed monads.

Definition 13 (Lifting for indexed monads [Atkey 2009]). Let \mathcal{T} be a \mathcal{S} -indexed monad over a cartesian category C and $F: \mathcal{S} \rightarrow \mathcal{S}$ be an endofunctor. A *lifting* of F to \mathcal{T} is a natural transformation $F_{S,T,X}^\dagger: \mathcal{T}(S, T, X) \rightarrow \mathcal{T}(FS, FT, X)$ that commutes with the unit, multiplication (and strength of the monad):

$$\begin{array}{ccc}
X & \xrightarrow{\eta_{S,X}} & \mathcal{T}(S, S, X) \\
& \searrow \eta_{FS,X} & \downarrow F_{S,S,X}^\dagger \\
& & \mathcal{T}(FS, FS, X) \\
\\
\mathcal{T}(S, T, \mathcal{T}(T, U, X)) & \xrightarrow{F_{S,T,\mathcal{T}(T,U,X)}^\dagger} & \mathcal{T}(FS, FT, \mathcal{T}(T, U, X)) \\
\downarrow \mu_{S,T,U,X} & & \downarrow \mathcal{T}(FS, FT, F^\dagger) \\
\mathcal{T}(S, U, X) & \xrightarrow{F_{S,U,X}^\dagger} & \mathcal{T}(FS, SU, X) \xleftarrow{\mu_{FS,FT,FU,X}} \mathcal{T}(FS, FT, \mathcal{T}(FT, FU, X))
\end{array}$$

A natural transformation $\theta: F \rightarrow G$ is *natural for liftings* F^\dagger and G^\dagger if the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{T}(S, T, X) & \xrightarrow{F_{S,T,X}^\dagger} & \mathcal{T}(FS, FT, X) \\
\downarrow G_{S,T,X}^\dagger & & \downarrow \mathcal{T}(FS, \theta_T, X) \\
\mathcal{T}(GS, GT, X) & \xrightarrow{\mathcal{T}(\theta_S, GT, X)} & \mathcal{T}(FS, GT, X)
\end{array}$$

△

Definition 14. We say that an \mathcal{S} -indexed monad \mathcal{T} has *premonoidal lifting* if there are liftings for the functors $- \ltimes S$ and $S \rtimes -$, written $(- \ltimes S)^\dagger$ and $(S \rtimes -)^\dagger$, for every $S \in \mathcal{S}$, such that all the associativity and left and right unitors are natural for them. Furthermore, the indexed monad is said to have a symmetric premonoidal lifting if the symmetry natural transformations are also natural. △

As we briefly explained, the circuit monad \mathcal{T}_M has premonoidal lifting.

Next we review the notion of liftings for the parameterized Freyd category. The liftings for an indexed monad and the liftings for the parameterized Freyd category obtained as the Kleisli category of that indexed monad are mutually related.

Definition 15 (Lifting for parameterized Freyd categories [Atkey 2009]). Let $F: \mathcal{S} \rightarrow \mathcal{S}$ be an endofunctor and (J, \otimes, \otimes) be a parametric Freyd category where $J: C \times \mathcal{S} \rightarrow \mathcal{K}$. The parameterized Freyd category has a lifting of F if it has a functor $F^*: \mathcal{K} \rightarrow \mathcal{K}$ such that

- $F^*(J(X, S)) = J(X, FS)$ and $F^*(J(f, s)) = J(f, Fs)$ for $f \in C(X, Y)$ and $s \in \mathcal{S}(S, T)$
- F^* respects the premonoidal structure: $F^*(X \otimes (Y, S)) = F^*((X, S) \otimes Y) = (X \times Y, FS)$ and $F^*(f \otimes c) = f \rtimes F^*c$ (and similarly for \otimes).

A natural transformation $\theta: F \rightarrow G$ is natural for liftings F^* and G^* if the following diagram commutes for all $c: (X, S) \rightarrow (Y, T)$.

$$\begin{array}{ccc}
J(X, FS) & \xrightarrow{J(X, \theta_S)} & J(X, GS) \\
\downarrow F^\dagger f & & \downarrow G^\dagger f \\
J(Y, FT) & \xrightarrow{J(Y, \theta_T)} & J(Z, GT)
\end{array}$$

△

Proposition 2 ([Atkey 2009, Theorem 3]). *Let \mathcal{T} be an indexed monad over a cartesian category C . If F^\dagger is a lifting of an endofunctor $F : \mathcal{S} \rightarrow \mathcal{S}$, then we can construct a lifting F^* on the parameterized Freyd category $\mathcal{C}_\mathcal{T}$ and vice versa. These operations are inverse. If a natural transformation from F to G is natural for liftings F^\dagger and G^\dagger , then it is also natural for liftings F^* and G^* , and vice versa.*

The premonoidal lifting for parameterized Freyd category is defined as in Definition 14, and it is easy to check that if \mathcal{T} has a premonoidal lifting so does its Kleisli category. We note that a parameterized Freyd category with a premonoidal lifting, in a sense, has two premonoidal structures one respect to C and the other respect to \mathcal{S} . This allows us to define a binoidal functor $(X, S) \bowtie - : \mathcal{K} \rightarrow \mathcal{K}$, for each $(X, S) \in \mathcal{K}$, defined as $(X \otimes -); (S \bowtie -)^*$. The functor $- \bowtie (X, S) : \mathcal{K} \rightarrow \mathcal{K}$ is defined analogously. It is not hard to see that this makes \mathcal{K} a premonoidal category. In particular, for $\text{Set}_{\mathcal{T}_M}$, we have

$$((X, S) \bowtie f)(x, y) = ((x, z), S \bowtie C)$$

provided that $f : Y \rightarrow \mathcal{T}_M(T, U, Z)$ maps y to (z, C) . In other words, $(X, S) \bowtie f$ just performs the computation against the second element and, at the same time, augments wires to the underlying circuit of the computation. It is this premonoidal structure that is used in the interpretation of the computational judgments given in Section 4. We note that $J(f, s)$ is a central morphism if s is with respect to the premonoidal structure of \mathcal{S} . Therefore, the values are interpreted as central morphisms in the interpretation.

The notion of lifting of functors can be naturally extended to category-graded monads.

Definition 16 (Lifting for category-graded monad). *Let \mathcal{T} be a \mathcal{A} -graded monad over a cartesian category C and $F : \mathcal{A} \rightarrow \mathcal{A}$ be an endofunctor. A lifting of F to \mathcal{T} is a family of natural transformation $F_{f,X}^\dagger : \mathcal{T}^f X \rightarrow \mathcal{T}^{Ff} X$ indexed by morphisms f in \mathcal{A} that commutes with the unit, multiplication (and strength of the monad):*

$$\begin{array}{ccc} X & \xrightarrow{\eta_{a,X}} & \mathcal{T}^{\text{id}_a} \\ & \searrow \eta_{Fa,X} & \downarrow F_{\text{id}_a,X}^\dagger \\ & & \mathcal{T}^{\text{id}_{Fa}} \end{array} \quad \begin{array}{ccccc} \mathcal{T}^f(\mathcal{T}^g X) & \xrightarrow{F_{f,\mathcal{T}^g X}^\dagger} & \mathcal{T}^{Ff}(\mathcal{T}^g X) & \xleftarrow{\mathcal{T}^{Ff}(F^\dagger)} & \mathcal{T}^{Ff}(\mathcal{T}^{Fg} X) \\ \mu_{f,g X} \downarrow & & & & \downarrow \mu_{Ff,Fg,X} \\ \mathcal{T}^{f \circ g} X & \xrightarrow{F_{f \circ g,X}^\dagger} & \mathcal{T}^{Ff;Fg} X & \xleftarrow{\mu_{Ff,Fg,X}} & \mathcal{T}^{Ff}(\mathcal{T}^{Fg} X) \end{array}$$

A natural transformation $\theta : F \rightarrow G$ is natural for liftings F^\dagger and G^\dagger if there is a generalized unit [Orchard et al. 2020] $\eta_{\theta_a,X} : X \rightarrow \mathcal{T}^{\theta_a} X$ for each component θ_a such that

$$\begin{array}{ccccc} \mathcal{T}^f X & \xrightarrow{F_{f,X}^\dagger} & \mathcal{T}^{Ff} & \xrightarrow{\mathcal{T}^{Ff}\eta_{\theta_b,X}} & \mathcal{T}^{Ff}\mathcal{T}^{\theta_b} X \\ \downarrow G_{f,X}^\dagger & & & & \downarrow \mu \\ \mathcal{T}^{Gf} X & \xrightarrow{\eta_{a,\mathcal{T}^{Gf} X}} & \mathcal{T}^{\theta_a} \mathcal{T}^{Gf} X & \xrightarrow{\mu_{\theta_a,Gf}} & \mathcal{T}^{\theta_a;Gf} X \end{array}$$

commutes for each $f : a \rightarrow b$. \triangleleft

For the \mathcal{T}_M^e the lift of the premonoidal product $T \bowtie -$ is once again just the operation that maps the circuit part of a computation C to $T \bowtie C$. The only natural transformation we are interested in is the symmetry (as we are working in a strict premonoidal category), and the generalized unit for the symmetry s is just the map $x \mapsto (x, s)$.