Numerical Analysis Assignment 1
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In this assignment, we examined three different root-finding methods to find the solutions to problems. For each method, we compare the speed of convergence (we use the number of iterations to represent the speed), ability to converge and difficulty in finding initial values. For all problems, we first pre-processing the problem, plot the function, and choose appropriate initial values to satisfy the assumption of each method.

Problem ($\tan(x) = 1000$). We let $f(x) = \tan(x) - 1000$ and solve the equation. As a result, we obtain $x = 1.56980$. For $f(x)$, when $x \to \frac{\pi}{2}$, $f(x) \to \infty$. So we pick initial value within the interval $[0, \frac{\pi}{2})$. In solving for $f(x) = 0$, we need to be careful so that during the iteration, no point would exceed $\frac{\pi}{2}$, or secant and newton's method will not be able to find the solution. Therefore, it is difficult when choosing the initial value for these two methods. We also did an experiment on the possible interval of initial values for both methods. After several attempts, we find that the initial value for Secant method should be between $\frac{3.137589}{2}$ and $\frac{\pi}{2}$, and the initial value for Newton's method should be between $\frac{3.137593}{2}$ and $\frac{\pi}{2}$.

From the table (see appendix), all three methods are able to converge. Also notice that for $f(x)$, secant method converges the fastest and bisection method, as expected, converges the slowest. Secant method converges faster is probably because fast increase of $\tan(x)$ when it approaches $\frac{\pi}{2}$ and the selection of initial values.

Problem 2.3.19. The distance function between $y = x^2$ and point $(1, 0)$ is $d(x) = \sqrt{(x^2 - 0)^2 + (x - 1)^2} = \sqrt{x^4 + x^2 - 2x + 1}$. We want to minimize $[d(x)]^2 = x^4 + x^2 - 2x + 1$. That being said, we are going to compute the derivative of $d(x)^2$ and set it equal to 0. The resulting x value corresponds to the desired solution. In other words, we obtain the minimum distance. As can be seen from the table in the appendix, the bisection method converges the slowest and Newton's method converges the fastest. In terms of convergence ability, all three methods can converge. For $f(x) = d(x)^2$, when $x \in [0, 1]$, $f'(x) \neq 0$. Therefore, any initial value between 0 and 1 works for Newton's and Secant Method. As a result, when $x = 0.58976$, i.e. the point $(0.58976, 0.34781)$ is closest to $(1, 0)$.

Problem 2.3.21. Set $a + b = 20$, and $a = x$, then $b = 20 - a = 20 - x$. In the problem, $(a + \sqrt{a})(b + \sqrt{b}) = 155.5$, substituting a and b with expression

of x, we have $(x+\sqrt{x})((20-x)+\sqrt{20-x})=155.5$.
Let $f(x)=(x+\sqrt{x})((20-x)+\sqrt{20-x})-155.5$, and we want to find the root of $f(x)$. For the bisection method, it is easy to find a initial value. As we can see from the graph in the appendix, we just need to be careful if we want a certain solution. In the graph, however, we notice a local maximum, i.e., derivative equals to 0 at the point. We need to avoid this point when choosing the initial values of Newton's and Secant Method. Similarly, bisection method goes through the most iterations. Newton's method is the quickest and then the secant method. As a result, we obtained two solutions for, where $x_1=6.51285$ and $x_2=13.48715$. These two solutions add up to 20, which makes sense as the equation is symmetric.

Problem 2.3.28a. To begin this problem, we first need to solve for the local maximum of the equation, $f(t)=te^{-\frac{t}{3}}$. By calculating its derivative and set it to 0, we find when $t=3$, $f(t)$ reaches its maximum. We plug in $t=3$ to solve for the value of A.
We also know the maximum concentration is $1mg/mL$ and we want to get as close as possible. We then obtain, $g(A)=A*\frac{3}{e}-1$, and we want to find the root of $g(A)$. Using the three root-finding methods, we obtain $A=0.90610$. Thus, the amount should be injected is 0.90610 units and it reaches its maximum 3 hours after injection.
As we can see from the table (see appendix), all three methods converge nicely and there is no difficulty in choosing initial values as the derivative of $g(A)$ is always nonzero. Similarly, bisection method goes through the most iterations. Newton's method is the quickest and then the secant method.

Problem (x and $\tan(x)$). The relative error between x and $\tan(x)$ is denoted by the equation *relative error* $=\frac{|\tan(x)-x|}{|\tan(x)|}$. Since $\tan(x)-x>0$ when $0<x<\pi$, we can ignore the absolute value and obtain the following function to solve the problem, $f(x)=\frac{\tan(x)-x}{\tan(x)}-0.01=0$. The resulting x value is 0.17303. All of methods converges, and Newton's method takes the least number of iterations. According to graph of $f(x)$, see Appendix 2, there is no such point a, that $f'(a)=0$, and there is no asymptote. Thus, for all three methods, there is no difficulty in determining the initial value, as long as the initial value is in the interval $(0,\pi)$.

Extra Credit

Problem 1.

$$f(x) = -3x^2 + x^3 + -3x^4 + 4x^5 - x^6 - 3x^7 + x^8$$

This equation has a solution $x = 3$ and is not around $x = 3$. Thus, if the initial values for bisection is 2.5 and 3.5, it gives us the result in one iteration and stops with condition $f(p) = 0$. However, if we use the same set of initial values with secant method. It takes 33 iterations if the tolerance is 0.0001.

Problem 2.

$$f(x) = 117x^2 - 17x^3 - 2x^4 + 4x^5 - x^6 - 3x^7 + x^8 - 348x + 369$$

The function has a solution at $x = 3$ and we picked initial values to be 2.5 and 3.5. It takes the secant method 55 iterations to find a solution with tolerance 0.000001.

A Tables

1. The table for the Problem $\tan(x) = 1000$.

Methods	Ability to Convergence	Number of Iterations
Bisection Method	can converge	31
Newton's Method	can converge	14
Secant Method	can converge	7

2. The table for Problem 2.3.19

Methods	Ability to Convergence	Number of Iterations
Bisection Method	can converge	24
Newton's Method	can converge	4
Secant Method	can converge	5

3. The table for Problem 2.3.21

Methods	Ability to Convergence	Number of Iterations
Bisection Method	can converge	28
Newton's Method	can converge	3
Secant Method	can converge	5

4. The table for Problem 2.3.28(a)

Methods	Ability to Convergence	Number of Iterations
Bisection Method	can converge	18
Newton's Method	can converge	1
Secant Method	can converge	2

5. The table for $\tan(x)$ and x problem.

Methods	Ability to Convergence	Number of Iterations
Bisection Method	can converge	17
Newton's Method	can converge	4
Secant Method	can converge	5

B Graphs

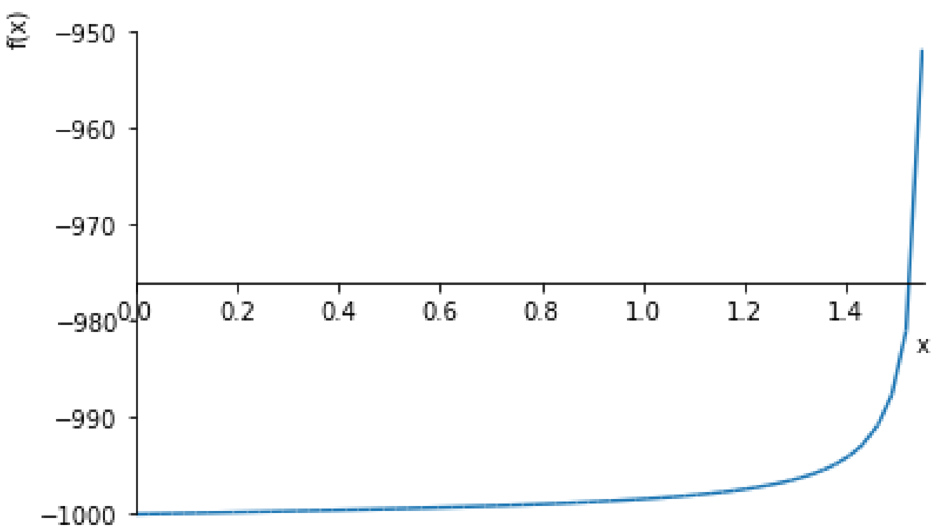

Figure 1: Graph of Function in Problem $\tan(x) = 1000$

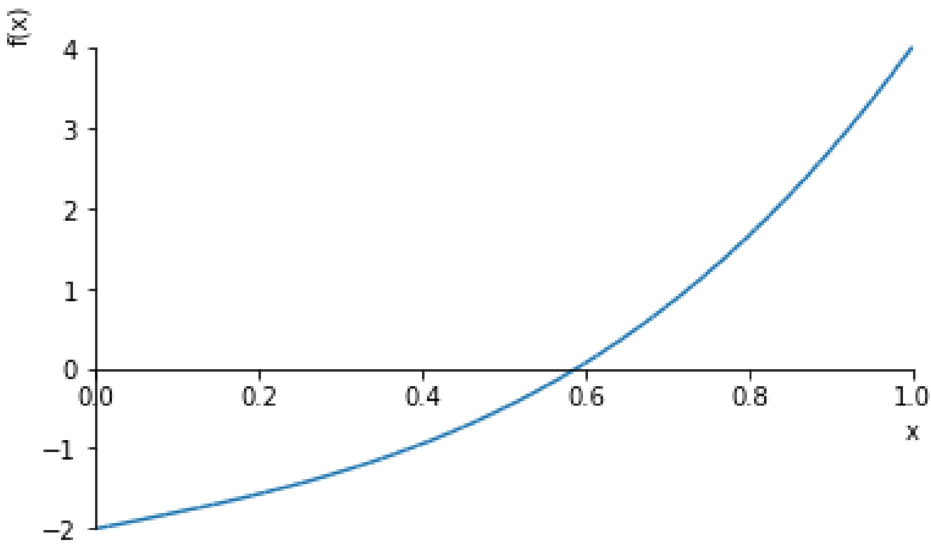

Figure 2: Graph of Function in Problem 2.3.19

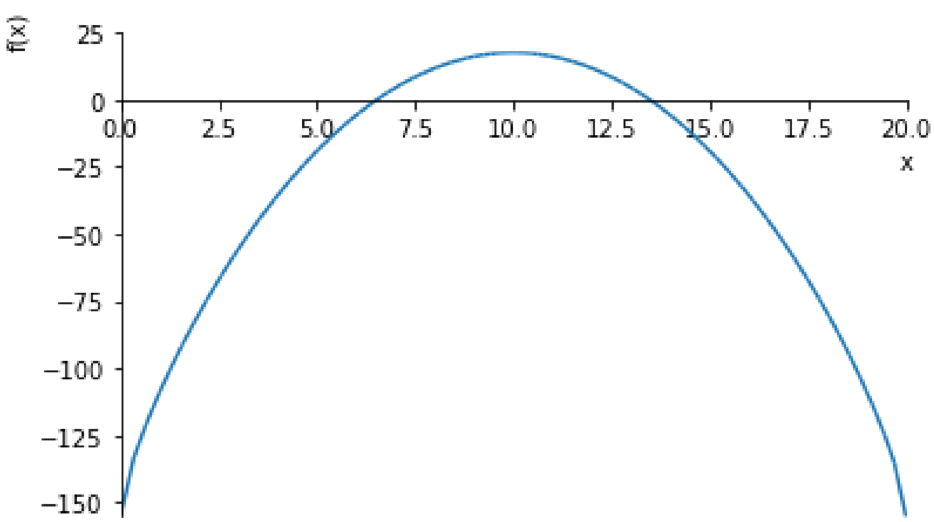

Figure 3: Graph of Function in Problem 2.3.21

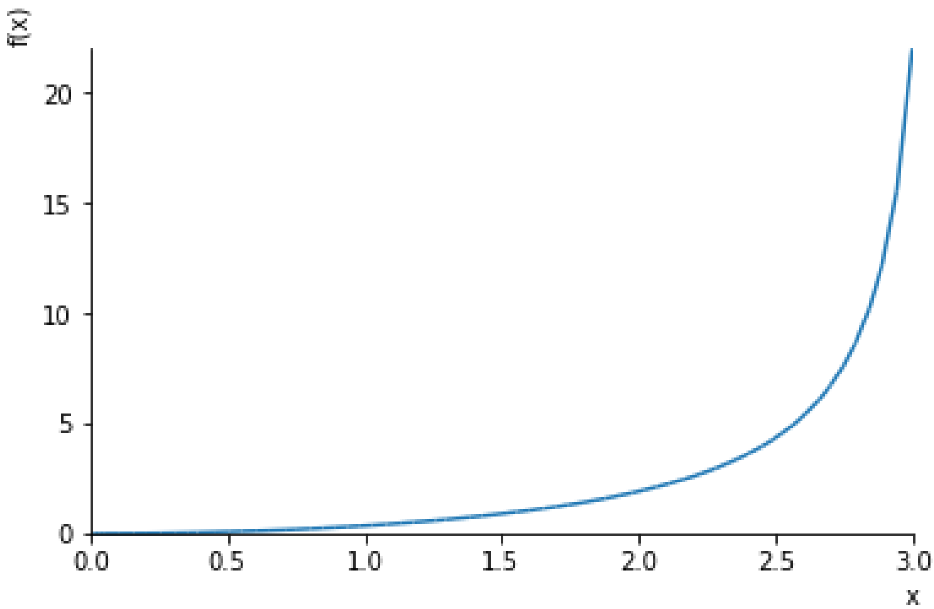

Figure 4: Graph of Function in $tan(x)$ and x problem