Case 1 Testing for mean of Normal distribution when σ is known (given)

$\therefore$ The test we will opt is Z-test

The concerned test statistic is $Z = \frac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} \sim N(0,1)$ under $H_0 : (\mu = \mu_0)$

Given significance level is α.

Depending on H_1

$\hookrightarrow H_1 : \mu > \mu_0$, then we reject H_0 if $Z_{obs} > Z_\alpha$

$\hookrightarrow H_1 : \mu < \mu_0$, then we reject H_0 if $Z_{obs} < -Z_\alpha$

$\hookrightarrow H_1 : \mu \neq \mu_0$, then we reject H_0 if $Z_{obs} < Z_{1-\alpha/2} = -Z_{\alpha/2}$ [$\because Z$ is symm]

or

$Z_{obs} > Z_{\alpha/2}$

i.e. $|Z_{obs}| > Z_{\alpha/2}$

Case 2 Testing for mean for Normal distribution when σ is unknown

Let X follow a distribution $N(\mu, \sigma^2)$ where σ^2 is not known.

let $X_1, X_2, \ldots, X_n$ be n iid random samples drawn from the distribution of X.

Let $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ where $\bar{X} = \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)$

let $H_0 : \mu = \mu_0$ & given significance level be α.

Consider the test-statistic $t = \frac{\sqrt{n}(\bar{X}-\mu_0)}{s}$ under H_0.

Now we see $\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right) \sim Z(0,1)$

and $\frac{\cancel{(n-1)} s^2}{\sigma^2} = \frac{1}{n-1}\sum(X_i - \bar{X})^2 \sim \chi^2_{n-1}$

$$\therefore t = \frac{\sqrt{n}\left(\frac{\bar{X}-\mu_0}{\sigma}\right)}{\frac{s}{\sigma}} \sim \frac{Z(0,1)}{\sqrt{\chi^2_{n-1}}} = t_{n-1}$$

$$\Rightarrow t = \frac{\sqrt{n}(\bar{X}-\mu_0)}{s} \sim t_{n-1}$$

Now depending H_1:

↳ $H_1 : \mu < \mu_0$, we reject H_0 if

$$t_{obs} < t_{n-1; 1-\alpha}$$

↳ $H_1 : \mu > \mu_0$, we reject H_0 if

$$t_{obs} > t_{n-1;\alpha}$$

↳ $H_1 : \mu \neq \mu_0$, we reject H_0 if

$$|t_{obs}| > t_{n-1;\alpha/2}$$

Case 3 : Testing for variance of a normal population whose mean is known

Let X follow normal distribution with mean ~~σ_0~~

$X \sim N(\mu_0, \sigma^2)$ where ~~σ^2~~ is being tested

Consider n iid random samples $X_1, X_2, \dots, X_n$ drawn from the distribution of X.

Let H_0 : $\sigma > \sigma_0$

Consider $S_0^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2$

~~Then~~

Then under H_0, consider test-statistic,

$$t = \frac{n S_0^2}{\sigma_0^2} \sim \chi_n^2$$

Depending on H_A, at α-level of significance

↳ If $H_1 : \sigma > \sigma_0$, we reject H_0 if

$$t_{obs} > \chi^2_{n;\alpha}$$

↳ If $H_1 : \sigma < \sigma_0$ we reject H_0 if

$$t_{obs} < \chi^2_{n;1-\alpha}$$

↳ If $H_1 : \sigma \neq \sigma_0$, we reject H_0 if

$$t_{obs} < \chi^2_{n;1-\alpha/2}$$

or

$$t_{obs} > \chi^2_{n;\alpha/2}.$$

case 4: Testing for variance of a normal population when mean is unknown.

let X follow a normal distribution $N(\mu, \sigma^2)$ where μ is unknown & σ^2 is being tested for.

let n iid random variables be $X_1, X_2, \ldots, X_n$ drawn from the distribution of X.

$\therefore$ let $H_0 : \sigma = \sigma_0$

Now consider the test statistic $t = \frac{(n-1)S^2}{\sigma_0^2}$ under H_0

where $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})$ &

$\bar{X} = \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)$

under H_0, $t \sim \chi^2_{n-1}$

Thus dependending on H_1, at α l.o.s,

↳ If $H_1 : \sigma < \sigma_0$, we will reject H_0 if

$$t_{obs} < \chi^2_{n-1;\,1-\alpha}$$

↳ If $H_1 : \sigma > \sigma_0$, we will reject H_0 if

$$t_{obs} > \chi^2_{n-1;\,\alpha}$$

↳ If $H_1 : \sigma \neq \sigma_0$, we will reject H_0 if

$$t_{obs} > \chi^2_{n-1;\,\alpha/2}$$

or

$$t_{obs} < \chi^2_{n-1;\,1-\alpha/2}.$$