ii) Test for equality of variances. We are to test $H_0: \sigma_X^2 = \sigma_Y^2$

(a) Means μ_X and μ_Y are known say $\mu_X = \mu_{X_0}$ and $\mu_Y = \mu_{Y_0}$

Consider a random sample of size n_1 $(X_1, \dots, X_{n_1}) \overset{iid}{\sim} X$ and
" " " " " n_2 $(Y_1, \dots, Y_{n_2}) \overset{iid}{\sim} Y$

Based on these we need to obtain σ_X^2 & σ_Y^2

Apt estimator of σ_X^2 is $S_{X_0}^2 = \frac{1}{n_1}\sum_{i=1}^{n_1}(X_i - \mu_X)^2$

$$S_{Y_0}^2 = \frac{1}{n_2}\sum_{i=1}^{n_2}(\cancel{X_i} - \mu_Y)^2$$

Note that H_0 can also be expressed as $H_0 : \frac{\sigma_X^2}{\sigma_Y^2} = 1$

We consider the ratio $\frac{S_{X_0}^2}{S_{Y_0}^2}$. If H_0 is true, then the ratio $\frac{S_{X_0}^2}{S_{Y_0}^2}$ will be close to 1

Case I : $H_1 : \frac{\sigma_X^2}{\sigma_Y^2} > 1$. In this case $\frac{S_{X_0}^2}{S_{Y_0}^2}$ will be much higher than 1. Thus such high values of $\frac{S_{X_0}^2}{S_{Y_0}^2}$ indicate departure of H_0 towards H_1.

Case II : $H_1 : \frac{\sigma_X^2}{\sigma_Y^2} < 1$, following a similar logic low values of $\frac{S_{X_0}^2}{S_{Y_0}^2}$ indicate departure of H_0 towards H_1.

Case III : $H_1 : \frac{\sigma_X^2}{\sigma_Y^2} \neq 1$. Following same logic, $\pm$ high values of $\frac{S_{X_0}^2}{S_{Y_0}^2}$ indicate departure of H_0 towards H_1

Thus, in all cases, the ratio $\frac{S_{X_0}^2}{S_{Y_0}^2}$ indicate how close H_0 is towards the truth, the test statistic for testing H_0 will be based on this ratio.

We have $\frac{n_1 S_{X_0}^2}{\sigma_X^2} \sim \chi^2_{n_1}$ and $\frac{n_2 S_{Y_0}^2}{\sigma_Y^2} \sim \chi^2_{n_2}$

Also, $S_{X_0}^2$ & $S_{Y_0}^2$ are independently distributed as X and Y are independent.

Thus by knowledge of sampling distributions

$$\frac{\left(\frac{n_1 S_{X_0}^2}{\sigma_X^2}\right)\cdot\left(\frac{1}{n_1}\right)}{\left(\frac{n_2 S_{Y_0}^2}{\sigma_Y^2}\right)\left(\frac{1}{n_2}\right)} \sim F_{n_1, n_2}$$

i.e $F = \frac{S_{X_0}^2}{S_{Y_0}^2} \cdot \frac{\sigma_Y^2}{\sigma_X^2} \sim F_{n_1, n_2}$. Under $H_0 : \frac{\sigma_X^2}{\sigma_Y^2} = 1$,

$F = \frac{S_{X_0}^2}{S_{Y_0}^2} \sim F_{n_1, n_2}$. The apt test statistic for testing H_0 is $F = \frac{S_{Y_0}^2}{S_{X_0}^2}$. Critical Region:

Case I : $H_1 : \frac{\sigma_X^2}{\sigma_Y^2} > 1$. Test rule: Reject H_0 at α los iff $F_{obs} > F_{n_1, n_2, \alpha}$

Critical ~~reg~~ Region : $F > K$ where K is so chosen that $P(\text{Type I error}) = \alpha$

so $K =$ upper α-point

Case II : $H_1 : \frac{\sigma_X^2}{\sigma_Y^2} < 1$. Critical region : $F < K^*$ where K^* is so chosen that prob. of Type I error $= \alpha$ $= F_{n_1, n_2, 1-\alpha}$.

Test rule : Reject H_0 at α l.o.s iff $F_{obs} < F_{n_1, n_2, 1-\alpha}$

Case III $H_1 : \frac{\sigma_X^2}{\sigma_Y^2} \neq 1$. Critical region : $F > K^{**}$ and $F < K^{***}$ where K^{**} and K^{***} are chosen s.t $P(\text{Type I error}) = \alpha$.

From similar calculations, we see $K^{**} = F_{n_1, n_2, \alpha/2}$ and $K^{***} = F_{n_1, n_2, 1-\alpha/2}$.

Test rule : Reject H_0 at α l.o.s iff $F_{obs} > F_{n_1, n_2, \alpha/2}$ ~~and~~ or $F_{obs} < F_{n_1, n_2, 1-\alpha/2}$

(b) Means are unknown. Here we cannot use $S_{X_0}^2$ and $S_{Y_0}^2$ since they involve the unknown parameters μ_X & μ_Y. So in this case we need to estimate σ_X^2 & σ_Y^2 entirely from the samples.

Apt estimator of σ_X^2 is given by

$$\hat{\sigma}_X^2 = s_X^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (X_i - \bar{X})^2 \text{ and}$$

$$\hat{\sigma}_Y^2 = s_Y^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2$$

Following a similar logic as in (a) an apt test statistic for testing H_0 is given by $F = \frac{s_X^2}{s_Y^2}$

Under H_0 : $F \sim F_{n_1-1, n_2-1}$. Test rules are given by ($\alpha \equiv$ l.o.s):

Case I : $H_1 : \frac{\sigma_X^2}{\sigma_Y^2} > 1$: Reject H_0 iff $F_{obs} > F_{n_1-1, n_2-1, \alpha}$

Case II : $H_1 : \frac{\sigma_X^2}{\sigma_Y^2} < 1$: Reject H_0 iff $F_{obs} < F_{n_1-1, n_2-1, 1-\alpha}$