10th March '23

I Testing problems relating to a univariate normal distribution.

i) Parameter of interest is μ [we are to test for $H_0: \mu = \mu_0$]

(a) σ is known, say $= \sigma_0$

(b) σ is unknown.

ii) Our parameter of interest is σ, that is the std deviation of normal dist^n

Let X be the r.v of interest which follows a $N(\mu, \sigma^2)$ dist^n $\mu \in \mathbb{R}$, $\sigma > 0$. where we are interested to test $H_0: \sigma = \sigma_0$

σ_0 is a pre-specified value of σ, $\sigma_0 > 0$.

(a) Suppose μ is known, say $\mu = \mu_0$; $\mu_0 \in \mathbb{R}$

Let us consider a random sample of size n from the dist^n of X. Let the sample denoted by $X_1, \ldots, X_n$.

Since μ is known to be μ_0, an apt estimator of σ^2

then $s_0^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2$ } unbiased

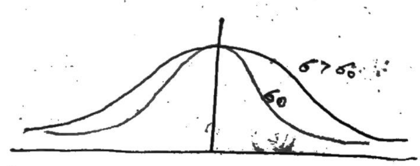
[when μ not known, $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$]

4 Consider the ratio $\frac{s_0^2}{\sigma_0^2}$

- If H_0 is true, that is, the true variance of the dist^n is equal to the value claimed under H_0, the quantity s_0^2 will be close to σ_0^2 and hence the ratio $\frac{s_0^2}{\sigma_0^2}$ will be close to 1

$H_1 : \sigma > \sigma_0$ | $H_1 : \sigma < \sigma_0$ | $H_1 : \sigma \neq \sigma_0$

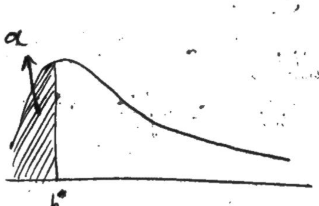
Case 1 : The alternative is $H_1 : \sigma > \sigma_0$. The random sample then comes from the true dist^n and s_0^2 will be close to the true variance of the dist^n which is higher than σ_0^2. Then the ratio s_0^2/σ_0^2 will be much higher than one. Thus a high value of the ratio s_0^2/σ_0^2 will indicate departure.



Case 2 : The alt $H_1 : \sigma < \sigma_0$ in line with the similar logic as case I, the ratio $\frac{s_0^2}{\sigma_0^2} < 1$.
Thus a low value of $\frac{s_0^2}{\sigma_0^2}$ will indicate departure from H_0.

Now $P(\text{Type I error}) = \alpha$

$\Rightarrow P(\text{Reject } H_0 \mid H_0 \text{ is true}) = \alpha$



$\Rightarrow P(T < k^* \mid H_0 \text{ is true}) = \alpha$

χ_n^2

Thus, $k^* =$ lower α-point of a χ_n^2 dist.

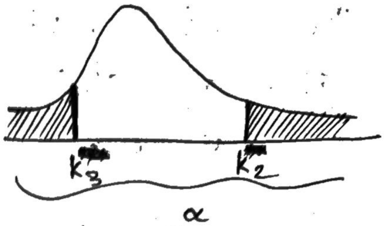
= upper $(1-\alpha)$ pt of χ_n^2 dist.

$= \chi^2_{n;1-\alpha}$ or $\chi^2_{1-\alpha;n}$

∴ Test Rule Reject H_0 at α l.o.s iff $T_{obs} < \chi^2_{n;1-\alpha}$

Case 3 : $H_1: \sigma \neq \sigma_0$

The critical region is of the form $T > K_2$ and $T < k_3$ where k_2 & K_3 are so chosen that $P(\text{Type I error}) = \alpha$.



$\Rightarrow P(\text{Reject } H_0 \mid H_0 \text{ is true}) = \alpha \Rightarrow P(T > k_2 \cup T < k_3 \mid \sigma = \sigma_0) = \alpha$

$$\Rightarrow P(T > K_2 \mid \sigma = \sigma_0) + P(T < K_3 \mid \sigma = \sigma_0) = \alpha$$

$K_2 =$ upper $\alpha/2$ point of $\chi^2_n = \chi^2_{n;\alpha/2}$

$K_3 =$ " $(1-\alpha/2)$ pt of $\chi^2_n = \chi^2_{n;1-\alpha/2}$

Test rule: Reject H_0 at α l.o.s iff $T_{obs} > \chi^2_{n;\alpha/2}$ or $T_{obs} < \chi^2_{n;1-\alpha/2}$

(b) Here μ is unknown. [μ_0 is not given]

Here to test H_0 against any of the three alt we can't use $T = \frac{nS_0^2}{\sigma_0^2}$ → [∵ μ_0 is not known]

In this case, we need to replace S_0^2 with something similar [as well as it should be unbiased]; with an estimator that does not involve μ. Such an estimator of σ^2 is $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Note that s^2 is unbiased for σ^2. An apt test statistic for testing H_0 is $T = \frac{(n-1)s^2}{\sigma_0^2}$

From sampling distⁿ theory, we know that under H_0, $T \sim \chi^2_{n-1}$

The test rules are formulated in line with a similar logic as is the case where μ was known.

Test rules:

Case I: $H_1 : \sigma > \sigma_0$; Reject H_0 at α l.o.s iff $T_{obs} > \chi^2_{n-1;\alpha}$

Case II: $H_1 : \sigma < \sigma_0$; Reject H_0 at α l.o.s iff $T_{obs} < \chi^2_{n-1;\alpha}$

Case III: Reject H_0 at α l.o.s iff $T_{obs} > \chi^2_{n-1;\alpha/2}$ or $T_{obs} < \chi^2_{n-1;\alpha/2}$