I Testing problems relating to a univariate normal distribution.

10th March '23

i) Parameter of interest is μ (we are to ~~test~~ for $H_0: \mu = \mu_0$)

(a) σ is known, say $= \sigma_0$

(b) σ is unknown.

(1) Problems related to a single univariate normal distribution. | 24th Feb '23

$$X \sim N(\mu, \sigma^2)$$

1. Testing problems on μ

(a) σ is known say $= \sigma_0$ ($\sigma_0 > 0$)

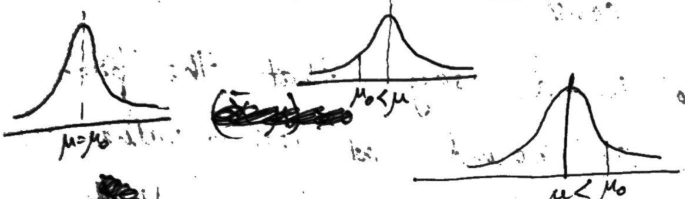
(i) To test $H_0 : \mu = \mu_0$ ag $H_1 : \mu > \mu_0$

(μ_0 : pre specified value $\in \mathbb{R}$)

Let $X_1, \dots, X_n$ r.s of size n from the dist of

$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ = sample mean

$(\bar{x} - \mu_0) \approx 0$ If H_0 is true, then $(\bar{x} - \mu_0)$ would be close to 0.



$(\bar{x} - \mu_0) \approx 0$ If H_0 is true, then $(\bar{x} - \mu_0)$ would be close to 0

[We only reject H_0 iff the situation sufficiently favours H_1 else it will favour H_0]

$(\bar{x} - \mu_0) > 0$ $\exists$ reason to favour H_1 over H_0.

$(\bar{X} - \mu_0) < 0$ $\exists$ reason to ~~favour~~ stick to H_0.

Thus the difference $(\bar{x} - \mu_0)$ provides us with an idea of favouring or disfavouring H_0. In particular, a large +ve value of $(\bar{x} - \mu_0)$ indicates a departure from H_0 in favour of H_1.

Thus the test statistic to test (H_0) should be based on the quantity $(\bar{x} - \mu_0)$

We know from theory ~~that~~ $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Since σ is known $= \sigma_0$ we have $\bar{X} \sim N\left(\mu, \frac{\sigma_0^2}{n}\right)$

Under H_0 (assuming H_0 as true) : $\bar{X} \sim N\left(\mu_0, \frac{\sigma_0^2}{n}\right)$

or $\frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma_0} \sim N(0, 1)$

An appropriate test statistic to test H_0 ag H_1 is given by $Z = \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma_0}$ which under H_0 follows a $N(0,1)$ $dist^n$

A high ~~the~~ value of Z indicates that there is strong evidence to doubt the validity of H_0

Thus the critical region of the test should be of the form $Z = \frac{\sqrt{n}}{\sigma_0}(\bar{x}-\mu_0) > K$ choose K is an appropriately chosen const.

K is chosen in a way that the prob of type 1 error is bound at desired value α $(0 < \alpha < 1)$ i.e, K is so chosen that

$P_{\mu_0}\left(Z = \frac{\sqrt{n}}{\sigma_0}(\bar{x}-\mu_0) \geq K\right) = \alpha$ i.e, k = upper α- point of a $N(0,1)$ $dist^n$ $= \tau_\alpha$

<u>Test Rule</u>: Reject H_0 at level of significance (los)

iff $Z_{obs} = \frac{\sqrt{n}}{\sigma_0}(\bar{x}-\mu_0) > \tau_\alpha$

accept o.w

Suppose we are to test

(ii) $H_0: \mu = \mu_0$ ag $H_1: \mu < \mu_0$

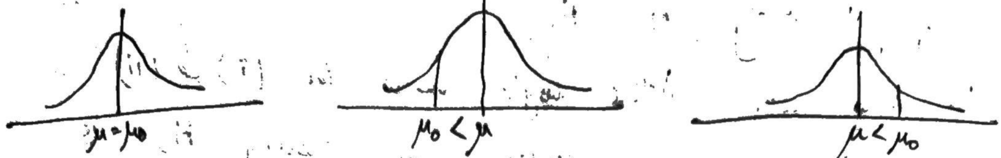
let $x_1, \ldots, x_n$ be r.s of size n drawn from the $dist^n$ of X. $\bar{x}$ = sample mean $= \frac{1}{n}\sum_{i=1}^{n} X_i$

In line with logic of (i)

$(\bar{X}-\mu_0) \approx 0$ (H_0) seems to be true

$(\bar{X}-\mu_0) > 0$ Better to stick to H_0

$(\bar{X}-\mu_0) < 0$ Disfavour H_0 in favour of H_1



Thus a large negative value of $(\bar{x} - \mu_0)$ would indicate that $\exists$ sufficient evidence to doubt the validity of H_0. In line with the logic used in (i) is an apt test statistic for testing H_0 is

$$Z = \frac{\sqrt{n}}{\sigma_0}(\bar{x} - \mu_0) \underset{\sim}{H_0} N(0,1)$$

<u>Critical region</u> : $Z = \frac{\sqrt{n}}{\sigma_0}(\bar{x} - \mu_0) < k'$ where k' is so chosen that the prob of type 1 error is bound at a desired level α $(0<\alpha<1)$

ie. $P_{\mu_0}\left\{ Z = \frac{\sqrt{n}}{\sigma_0}(\bar{x} - \mu_0) < k' \right\} = \alpha$.

i.e k' = lower α-pt of $N(0,1)$

$$\Rightarrow \tau_{(1-\alpha)} = -\tau_\alpha$$

$\boxed{\text{Test rule : Reject } H_0 \text{ at } \alpha \text{ los iff } Z_{obs} = \frac{\sqrt{n}}{\sigma_0}(\bar{x} - \mu_0) < -\tau_\alpha}$

(iii) Suppose we are to test $H_0 : \mu = \mu_0$ ag $H_1 : \mu \neq \mu_0$
let $x_1, \ldots, x_n$ be a r.s of size n drawn from the dist^n of X.

$\bar{x}$ = sample mean = $\frac{1}{n}\sum_{i=1}^{n} x_i$ In line with logic of (i) & (ii).

$(\bar{x} - \mu_0) \approx 0$: H_0 seems to be true.

$(\bar{x} - \mu_0) <, > 0$: Disfavour H_0 in favour of H_1.

Thus a high +ve value of $|\bar{x} - \mu_0|$ would indicate that $\exists$ sufficient evidence to doubt the validity of H_0

In line with the logic used in (i) & (ii) an apt test statistic for testing H_0 is

$$Z = \frac{\sqrt{n}}{\sigma_0}(\bar{x} - \mu_0) \overset{H_0}{\sim} N(0,1)$$

Critical region, $Z = \frac{\sqrt{n}}{\sigma_0}(\bar{x} - \mu_0) > k''$ and

$$\frac{\sqrt{n}}{\sigma_0}(\bar{x} - \mu_0) < k'''$$

To maintain symmetry, we choose k'' & k''' to be equal to k^*, in magnitude.

i.e $Z = \frac{\sqrt{n}}{\sigma_0}(\bar{x} - \mu_0) > k^*$ & $\frac{\sqrt{n}}{\sigma_0}(\bar{x} - \mu_0) < -k^*$

i.e, $|Z| = \frac{\sqrt{n}\,|\bar{x} - \mu_0|}{\sigma_0} > k^*$

where k^* is so chosen that the prob of Type 1 error is bound at α, $0 < \alpha < 1$.

Now, $P_{\mu_0}\left\{ \frac{\sqrt{n}}{\sigma_0}|\bar{x} - \mu_0| \geq k^* \right\} = \alpha$

i.e, $k^* =$ upper $\alpha/2$ th point of $N(0,1)$ dist^n

$= \tau_{\alpha/2}$

Test Rule: Reject H_0 at α l.o.s iff $|\tau_{obs}| = \frac{\sqrt{n}|\bar{x} - \mu_0|}{\sigma_0} > \tau_{\alpha/2}$

Accept o.w

3rd March '23

$X \sim N(\mu, \sigma_0^2)$ | $H_0 : \mu = \mu_0$ ag. $H_1 : \mu > \mu_1$

or

$H_1 : \mu < \mu_1$

or

$H_1 : \mu \neq \mu_1$

(b) σ is unknown

We can not use the test statistic $Z = \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma}$ as it involves the unknown ~~sample~~ constant σ.

Thus we replace σ by an estimate obtained from the sample.

$$\hat{\sigma} = s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

An appropriate test statistic for testing H_0 is given by $T = \frac{\sqrt{n}(\bar{X} - \mu_0)}{s}$ which under H_0 follows a t distribution with $(n-1)$ dof.

$$\left[t_n = \frac{N(0,1)}{\sqrt{\chi^2_n / n}} \qquad T = \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} = \frac{\frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma}}{\sqrt{\frac{(n-1)s^2}{\sigma^2} / (n-1)}} \right.$$

$$\left. T = \frac{N(0,1)}{\sqrt{\chi^2_{n-1}/(n-1)}} \right]$$

In line with the logic in case when σ is known,

→ for $H_1 : \mu > \mu_0$ high +ve values of T indicate departure from H_0

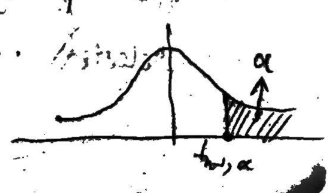
→ for $H_1 : \mu < \mu_0$ high -ve values of T indicate departure from H_0

→ For $H_1 : \mu \neq \mu_0$, high ±ve values of T indicate departure from H_0

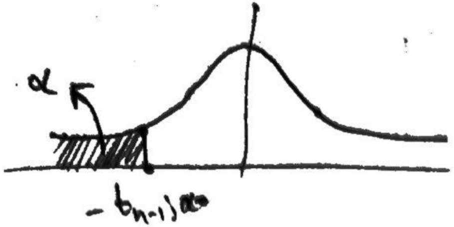
The test rules are given by

1) when $H_1 : \mu > \mu_0$: Reject H_0 iff $T_{obs} > t_{n-1;\alpha}$ (same $t_{\alpha;n-1}$)

where $t_{n-1;\alpha}$ is the upper α-point of a t-dist^n with $(n-1)$ dof and α is known.



ii) when $H_1 : \mu < \mu_0$: Reject H_0 iff $T_{obs} < t_{n-1;1-\alpha}$



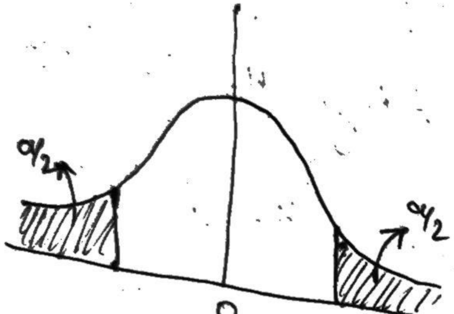
$\equiv T_{obs} < -t_{n-1;\alpha}$

(because t-distribution is symm about 0)

where α is desired l.o.s.

iii) when $H_1 : \mu \neq \mu_0$: Reject H_0 iff $T_{obs} > t_{n-1;\alpha/2}$ or

$T_{obs} < -t_{n-1;\alpha/2} \Rightarrow |T_{obs}| > t_{n-1;\alpha/2}$



where α is the desired l.o.s.