Vertex Coloring

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Basic Terminology

- K-coloring: Assignment of colors to the vertices of a graph
- K-colorable graph: If a graph can be colored with at most K colors such that adjacent vertices are assigned different colors (proper coloring), then the graph is called K-colorable graph.
- Chromatic Number (χ) (pronounced as"chi"): Minimum number of colors needed to do a proper K-coloring.
- K-critical graph: A graph is called a K-critical graph if chromatic number of every proper subgraph is strictly less than K.
- Clique Number(ω): The number of vertices in the largest clique that appears as a subgraph.

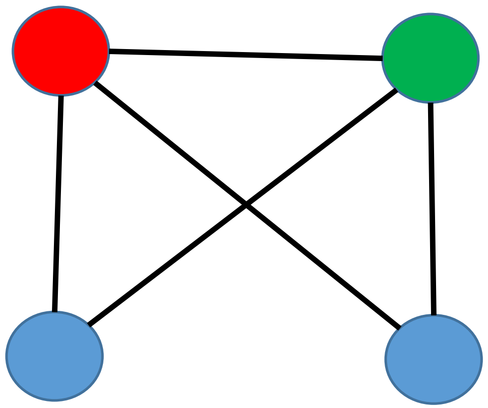
Proof of Lower Bounds

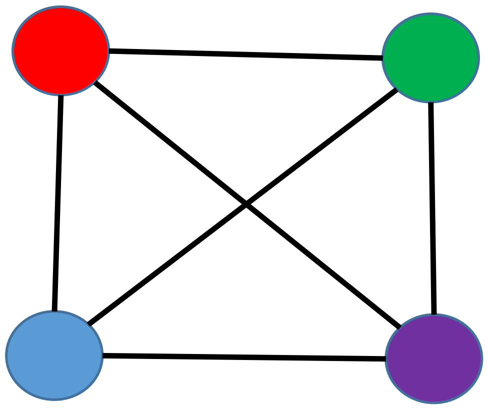
- $\chi(G) \geq \omega(G)$

Proof: All vertices that gives $\omega(G)$ are adjacent to each other. It implies that minimum $\omega(G)$ colors are required to properly color the graph.

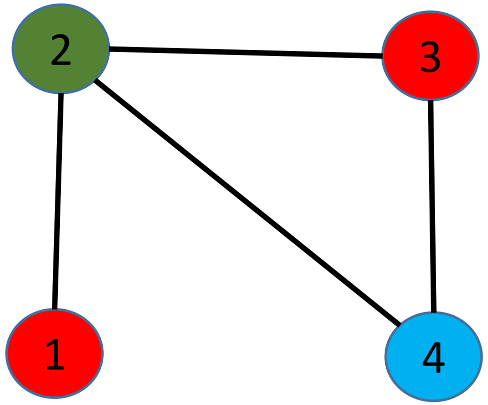
- $\chi(G) \geq V(G)/\alpha(G)$

Proof: All vertices that gives $\alpha(G)$ are non-adjacent to each other, and cardinality of the set of these vertices is maximum. All these vertices can be given same color.





Greedy Coloring



Vertex Ordering: 1 2 3 4

Infinite Colors: N (No of Vertices)

Strategy: While coloring vertex I, find the smallest-index color that is not used to color lowest-indexed neighbors of I.

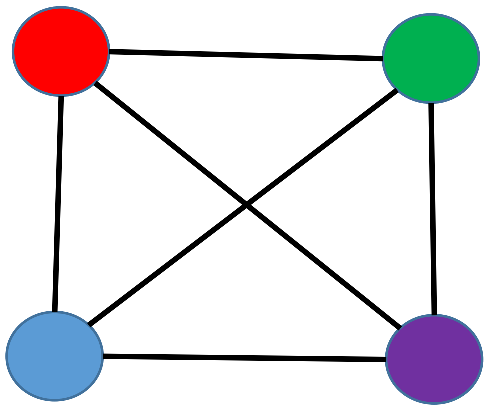
Proof of Upper Bounds

- $\chi(G) \leq \Delta(G) + 1$ (Maximum degree of a vertex in G)

Proof: We follow Greedy Coloring Strategy. Suppose P is a vertex of maximum degree. It may so happen that all neighbors of P appear before P in the vertex ordering. In the worst case, all Δ neighbors of P are adjacent to each other. In that case, P will receive a color that is different from colors assigned to all Δ neighbors of P.

Why I need a better upper bound?

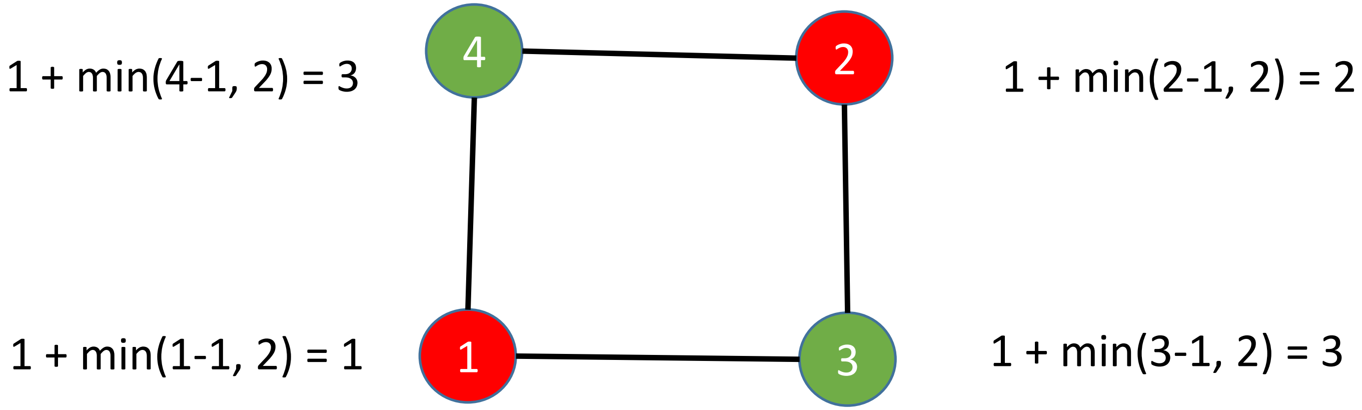
This result only depends on the vertex P of maximum degree, but it does not consider the index of P in the vertex ordering. [Solution: Welsh-Powell Theorem]



Proof of Upper Bounds: Welsh-Powell Theorem

- If a graph G has degree sequence d1 ≥ d2 ≥ … ≥ dn, then $\chi(G) \leq \max_i \min\{di, i\text{-}1\} + 1$.

Proof: Apply greedy coloring to the vertices in non-increasing order of degree. When we color the i-th vertex vi , it has **at most min{di , i-1}** earlier neighbors.

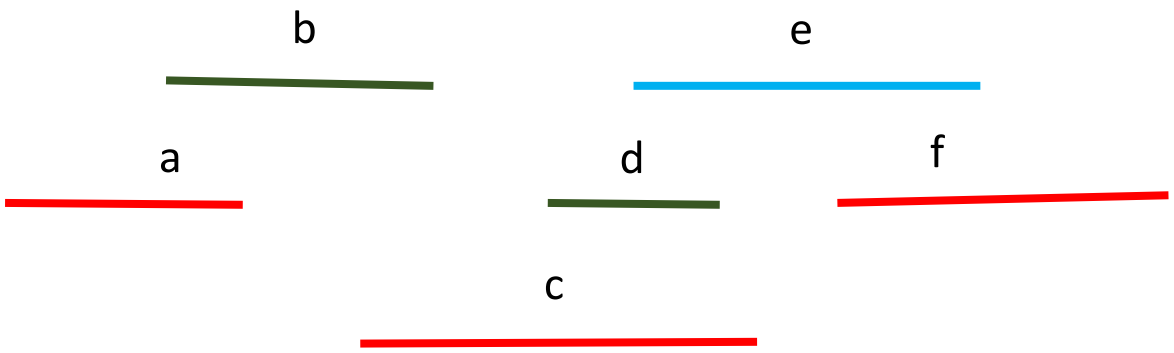


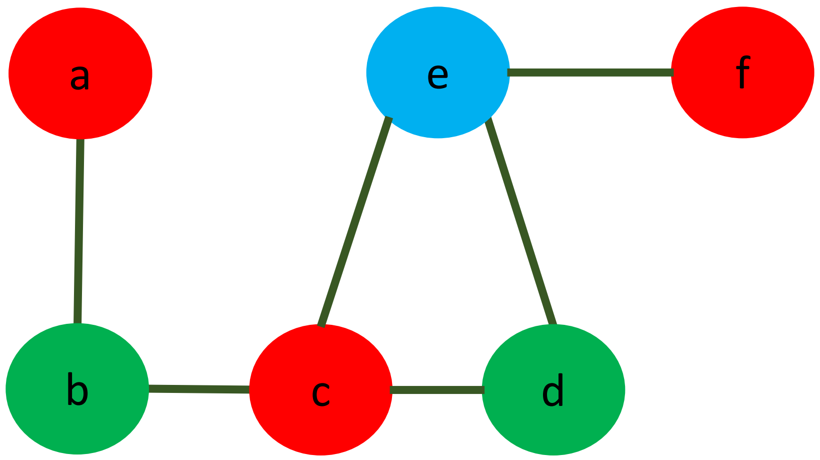
Integers in vertices indicate vertex ordering

Register Allocation Problem

- An assembly program is a set of instructions.
- Each instruction use some variables.
- To store a variable, a CPU register is needed.
- With each variable, we can associate an interval [start time, end time] that defines the time for which it is stored in a register.
- A register can be used to store two variables if their intervals do not overlap.
- Question: Minimize the number of CPU register needed to execute the program.

Modeling as Interval Graph Coloring





Vertices: Intervals
Edges: Intersection of Intervals

Intervals

Interval Graph

Algorithm

- Vertex Ordering: We sort the intervals(vertices) in non-decreasing order of start time.
- Greedy coloring algorithm: While coloring vertex I, find the smallest-index color that is not used to color lowest-indexed neighbors of I.

Interval Graph Coloring is Optimal

- For any graph G, $\chi(G) \geq \omega(G)$.
- For interval graph H, $\chi(H) \geq \omega(H)$. To prove $\chi(H) \leq \omega(H)$.
- Suppose x be the vertex in H, that received the maximum color k.
- $k \geq \chi(H)$ (From definition of chromatic number)
- Why I color x with k?
- x is overlapped with exactly (k-1) vertices having different colors.
- It implies I discovered a clique of size k.
- $\omega(H) \geq k$. (From definition of clique number)
- $\omega(H) \geq k \geq \chi(H)$. We are done.