Applied Graph Theory (Jan'25-May'25)

Problem Sheet-3 on connectivity

Notations :

- V_G and E_G denote respectively the sets of vertices and edges of G.
- $\kappa(G), \kappa'(G)$ denote respectively the vertex connectivity and edge connectivity of G.
- $\kappa_G(u,v), \kappa'_G(u,v)$ denote respectively the vertex and edge connectivities between u and v.
- $\lambda_G(u,v)$, $\lambda'_G(u,v)$ denote respectively the maximum number of internally vertex disjoint and edge disjoint paths between u and v.
- $\overline{G}$ denotes the complement of G.
- Join of two vertex disjoint graphs G and H, denoted by $G \vee H$, is the graph $(V_G \cup V_H, E_G \cup E_H \cup \{uv : u \in V_G, v \in V_H\})$.
- Given G and H, their union is the graph $G \cup H = (V_G \cup V_H, E_G \cup E_H)$.
- Given G and H, their intersection is $G \cap H = (V_G \cap V_H, E_G \cap E_H)$.
- The block-cutpoint graph $BC(G)$ of G is the bipartite graph $(\mathcal{B} \cup \mathcal{C}, E)$ where $\mathcal{B}$ is the set of all blocks of G and $\mathcal{C}$ is the set of all cut vertices of G, E is the collection of all pairs Bc where $B \in \mathcal{B}$, $c \in \mathcal{C}$ and $c \in V_B$.
- An edge-cut of a graph $G = (V, E)$ is a set of the form $E(S, V \setminus S) := \{uv \in E : u \in S, v \notin S\}$ for some $S \subseteq V, S \neq \emptyset, V$.

Problems :

1. Prove that a graph G is k-connected *if and only if* $G \vee K_r$ is $(k+r)$-connected.
2. Let $G = (V, E)$ be a connected graph with at least three vertices. Form G' from G by adding an edge $uv \notin E$ whenever $dist_G(u,v) = 2$. Prove that G' is 2-connected.

3. For each choice of integers k, l, m with $0 < k \leq l \leq m$, construct a simple graph G with $\kappa(G) = k$, $\kappa'(G) = l$, and $\delta(G) = m$.

4. Let G be a connected graph in which for every edge e, there are cycles C_1 and C_2 whose only common edge is e. Prove that G is 3-edge-connected.

5. Let F be a set of edges in G. Prove that F is an edge-cut *if and only if* F contains an even number of edges from every cycle in G.

6. Prove that the symmetric difference of two different edge cuts is an edge cut.

7. Let G be a simple graph on $V_G = \{1, 2, \ldots, 11\}$ defined by $ij \in E_G$ if and only if i and j have a common factor bigger than 1. Determine the blocks of G.

8. A **cactus** is a connected graph in which every block is an edge or a cycle. Prove that the maximum number of edges in a simple n-vertex cactus is $\lfloor 3(n-1)/2 \rfloor$.

9. Let n, k be positive integers with n even, k odd, and $n > k > 1$. Let G be the k-regular simple graph formed by placing uniformly n vertices on a circle and making each vertex adjacent to the diametrically opposite vertex and to the $(k-1)/2$ nearest vertices in each direction. Prove that $\kappa(G) = k$.

10. Prove that hypercube Q_k is k-connected by constructing k pairwise internally vertex disjoint u, v-paths for every pair $u, v \in Q_k$.

11. Let G be a graph with no isolated vertices. Prove that if G has no even cycles, then every block of G is either an edge or an odd length cycle.

12. Let G be a k-connected graph, and let S and T be disjoint subsets of V_G with size at least k each. Prove that G has k pairwise disjoint S, T-paths.

13. Let X and Y be disjoint sets of vertices in a k-connected graph G. Let $u(x)$ for $x \in X$ and $w(y)$ for $y \in Y$ be nonnegative integers such that $\sum_{x \in X} u(x) = \sum_{y \in Y} w(y) = k$. Prove that G has k internally vertex

disjoint X, Y-paths so that $u(x)$ of them start at x and $w(y)$ end at y, for each $x \in X$ and $y \in Y$.

14. Let G be a simple k-connected graph. Suppose C and D are two maximum length cycles in G. Prove that C and D share at least k vertices for the case $k = 2$. Can you obtain a similar conclusion for the case $k = 3$? Justify.

15. Let G_1 and G_2 be two vertex disjoint k-connected graphs with $k \geq 2$. For $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$, introduce a bipartite graph B beween $N_{G_1}(u_1)$ and $N_{G_2}(u_2)$ such that B has no isolated vertex and B admits a matching of size at least k. Prove that $(G_1 - u_1) \cup (G_2 - u_2) \cup B$ is k-connected.

16. Prove that if G is 2-connected, then $G - uv$ is 2-connected if and only if u and v lie on a cycle in $G - uv$. Conclude that a 2-connected graph is minimally 2-connected if and only if every cycle is an induced subgraph.