Discrete Structures

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Chapter 1

Pigeon Hole Principle

Theorem 1 (Pigeonhole Principle: Simple Form) *If $n + 1$ objects are distributed into n boxes, then at least one box contains two or more of the objects.*

Proof: The proof is by contradiction. If each of the n boxes contains at most one of the objects, then the total number of objects is at most n. Since we distribute $n + 1$ objects, some box contains at least two of the objects. ■

Theorem 2 (Pigeonhole Principle: General Form) *If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.*

Proof: We will use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil$ - 1 objects. Then, the total number of objects is at most

$$k(\lceil N/k \rceil - 1) < k((N/k + 1) - 1) = N$$

Thus, the total number of objects is less than N. This completes the proof by contraposition. ■

Example 3 *A function f from a set with k + 1 or more elements to a set with k elements is not one-to-one.*

Proof: *Suppose that for each element y in the codomain of f, we have a box that contains all elements x of the domain of f such that $f(x) = y$. Because the domain contains k + 1 or more elements and the codomain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means that f cannot be one-to-one.*

Example 4 *How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?*

Solution: *There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.*

Example 5 *Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.*

Proof: *Let n be a positive integer. Consider the n + 1 integers 1, 11, 111, ..., 11...1, where the last integer in this list is the integer with n + 1 1s in its decimal expansion. Note that there are n possible remainders when an integer is divided by n. Because there are n + 1 integers in this list, by the pigeonhole principle there must be two with the same remainder when divided by n. The larger of these integers less the smaller one is a multiple of n, which has a decimal expansion consisting entirely of 0s and 1s.*

Example 6 *Among 100 people there are at least $\lceil 100/12 \rceil$ = 9 who were born in the same month.*

Example 7 *What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?*

Solution: *The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil$*

= 6. The smallest such integer is $N = 5 \times 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade, so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

Example 8 *A standard deck of 52 cards has 13 kinds of cards, with four cards of each of kind, one in each of the four suits, hearts, diamonds, spades, and clubs. a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are selected?*

b) How many must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?

Solution: *a) Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for cards of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least $\lceil N/4 \rceil$ cards. Consequently, we know that at least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \times 4 + 1 = 9$, so nine cards suffice. Note that if eight cards are selected, it is possible to have two cards of each suit, so more than eight cards are needed. Consequently, nine cards must be selected to guarantee that at least three cards of one suit are chosen. One good way to think about this is to note that after the eighth card is chosen, there is no way to avoid having a third card of some suit.*

b)We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts.

Example 9 *What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)*

Solution: *There are eight million different phone numbers of the form NXX-XXXX. Hence, by the generalized pigeonhole principle, among 25 million telephones, at least* $\lceil 25,000,000/8,000,000 \rceil = 4$ *of them must have identical phone numbers. Hence, at least four area codes are required to ensure that all 10-digit numbers are different.*

Example 10 *During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.*

Proof: *Let* a_j *be the number of games played on or before the* j*-th day of the month. Then* $a_1, a_2, \ldots, a_{30}$ *is an increasing sequence of distinct positive integers, with* $1 \leq a_j \leq 45$*. Moreover,* $a_1 + 14$, $a_2 + 14$, $\ldots$, $a_{30} + 14$ *is also an increasing sequence of distinct positive integers, with* $15 \leq a_j + 14 \leq 59$*.*

The 60 positive integers $a_1, a_2, \ldots, a_{30}$, $a_1 + 14$, $a_2 + 14$, $\ldots$, $a_{30} + 14$ *are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Because the integers* a_j, $j = 1, 2, \ldots, 30$ *are all distinct and the integers* $a_j + 14$, $j = 1, 2, \ldots, 30$ *are all distinct, there must be indices* i *and* j *with* $a_i = a_j + 14$*. This means that exactly 14 games were played from day* $j + 1$ *to day* i*.*

Example 11 *Show that among any* $n + 1$ *positive integers not exceeding* $2n$ *there must be an integer that divides one of the other integers.*

Proof: *Express each of the* $n + 1$ *integers* $a_1, a_2, \ldots, a_{n+1}$ *as a power of 2 times an odd integer. In other words, let* $a_j = 2^{k_j} \times q_j$ *for* $j = 1, 2, \ldots, n+1$*, where* k_j *is a non-negative integer and* q_j *is an odd integer. The integers* $q_1, q_2, \ldots, q_{n+1}$ *are all odd positive integers less than* $2n$*. Because there are only* n *odd positive integers less than* $2n$*, it follows from the pigeonhole principle that two of the integers* $q_1, q_2, \ldots, q_{n+1}$ *must be equal. Therefore, there are distinct integers* i *and* j *such that* $q_i = q_j = q$*. Then,* $a_i = 2^{k_i} \times q$ *and* $a_j = 2^{k_j} \times q$*. It follows that if* $k_i < k_j$*, then* a_i *divides* a_j*; while if* $k_i > kj$*, then* a_j *divides* a_i*.*

Example 12 *Every sequence of* $n^2 + 1$ *distinct real numbers contains a* *subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.*

Proof: *Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. Associate an ordered pair with each term of the sequence, namely, associate (i_k, d_k) to the term a_k, where i_k is the length of the longest increasing subsequence starting at a_k, and d_k is the length of the longest decreasing subsequence starting at a_k.*

Suppose that there are no increasing or decreasing subsequences of length $n + 1$. Then i_k and d_k are both positive integers less than or equal to n, for $k = 1, 2, \ldots, n^2 + 1$. Hence, by the product rule there are n^2 possible ordered pairs for (i_k, d_k). By the pigeonhole principle, two of these $n^2 + 1$ ordered pairs are equal. In other words, there exist terms a_s and a_t, with $s < t$ such that $i_s = i_t$ and $d_s = d_t$. We will show that this is impossible. Because the terms of the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$.

If $a_s < a_t$, then, because $i_s = i_t$, an increasing subsequence of length $i_t + 1$ can be built starting at a_s, by taking a_s followed by an increasing subsequence of length i_t beginning at a_t. This is a contradiction.

If $a_s > a_t$, then, because $d_s = d_t$, a decreasing subsequence of length $d_t + 1$ can be built starting at a_s, by taking a_s followed by an decreasing subsequence of length d_t beginning at a_t. This is a contradiction.

Example 13 *Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.*

Proof: *Let G be the graph of the friendship relation, and let x be one of the people. Since x has 5 potential friends, x has either at least 3 friends or at least 3 strangers.*

Case 1: *x has at least 3 friends. If any two of the three friends of x are friends, then with x we have 3 mutual friends. If all the three friends of x are strangers, then there are three mutual strangers.*

Case 2: *x has at least 3 strangers. If any two of the three strangers are strangers, then with x we have 3 mutual strangers. If all the three strangers of x are friends, then there are three mutual friends.*