Discrete Structures

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Chapter 1

The Principle of Inclusion-Exclusion

Example 1 *How many elements are in the union of two finite sets A and B?*

Solution: *The number of elements in the union of the two sets A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection. That is, $|A \cup B| = |A| + |B| - |A \cap B|$.*

Example 2 *How many positive integers not exceeding 1000 are divisible by 7 or 11? How many positive integers not exceeding 1000 are not divisible by either 7 or 11?*

Solution: *Let A be the set of positive integers not exceeding 1000 that are divisible by 7, then $|A| = \lfloor 1000/7 \rfloor = 142$. Let B be the set of positive integers not exceeding 1000 that are divisible by 11, then $|B| = \lfloor 1000/11 \rfloor = 90$. Then, $A \cap B$ is the set of integers not exceeding 1000 that are divisible by both 7 and 11. Because 7 and 11 are relatively prime, the integers divisible by both 7 and 11 are those divisible by $7 \times 11 = 77$.*

Consequently, $|A \cap B| = \lfloor 1000/77 \rfloor = 12$. Then $A \cup B$ is the set of integers not exceeding 1000 that are divisible by either 7 or 11 or both is $|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$.

As a result, $\overline{A \cup B} = U \setminus (A \cup B)$ *is the set of integers not exceeding 1000 that are not divisible by either 7 or 11 is* $1000 - 220 = 780$.

Example 3 *How many elements are in the union of three finite sets A, B and C?*

Solution: *Note that* $|A| + |B| + |C|$ *counts each element that is in exactly one of the three sets once, elements that are in exactly two of the sets twice, and elements in all three sets three times.*

To remove the overcount of elements in more than one of the sets, we subtract the number of elements in the intersections of all pairs of the three sets. We obtain $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$.

This expression still counts elements that occur in exactly one of the sets once. An element that occurs in exactly two of the sets is also counted exactly once, because this element will occur in one of the three intersections of sets taken two at a time. However, those elements that occur in all three sets will be counted zero times by this expression, because they occur in all three intersections of sets taken two at a time.

To remedy this undercount, we add the number of elements in the intersection of all three sets. This final expression counts each element once, whether it is in one, two, or three of the sets. Thus,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example 4 *A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?*

Solution:

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|.$$

$$|S \cap F \cap R| = |S \cup F \cup R| - |S| - |F| - |R| + |S \cap F| + |S \cap R| + |F \cap R|.$$

$|S \cap F \cap R| = 2092 - (1232 + 879 + 114) + (103 + 23 + 14) = 2092 - 2225 + 140 = 7.$

Example 5 *How many elements are in the union of three finite sets A, B, C and D?*

Solution:

$$\begin{aligned} |A \cup B \cup C \cup D| \;=\; & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D|. \end{aligned}$$

Theorem 6 *Let A_1, A_2, ..., A_n be finite sets. Then*

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots \cup A_n| \;=\; & \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| \\ & - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|. \end{aligned}$$

Proof: We will prove the formula by showing that an element in the union is counted exactly once by the right-hand side of the equation. Suppose that a is a member of exactly r of the sets A_1, A_2, ..., A_n where $1 \le r \le n$. This element is counted $C(r, 1)$ times by $\sum |A_i|$. It is counted $C(r, 2)$ times by $\sum |Ai \cap Aj|$. In general, it is counted $C(r, m)$ times by the summation involving m of the sets A_i. Thus, this element is counted exactly

$$C(r, 1) - C(r, 2) + C(r, 3) - \ldots + (-1)^{r+1} C(r, r)$$

times by the expression on the right-hand side of this equation. Our goal is to evaluate this quantity. Applying binomial theorem, we can prove that

$$C(r, 0) - C(r, 1) + C(r, 2) - \ldots + (-1)^r C(r, r) = 0.$$

Hence,

$$1 = C(r, 0) = C(r, 1) - C(r, 2) + \ldots + (-1)^{r+1} C(r, r).$$

Therefore, each element in the union is counted exactly once by the expression on the right-hand side of the equation. This proves the principle of inclusion-exclusion. ■

1.1 Applications

There is an alternative form of the principle of inclusion–exclusion that is useful in counting problems. In particular, this form can be used to solve problems that ask for the number of elements in a set that have none of n properties $P_1, P_2, \ldots, P_n$.

Let A_i be the subset containing the elements that have property P_i. The number of elements with all the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ will be denoted by N$(P_{i_1}, P_{i_2}, \ldots, P_{i_k})$. Writing these quantities in terms of sets, we have

$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = N(P_{i_1}, P_{i_2}, \ldots, P_{i_k}).$$

If the number of elements with none of the properties $P_1, P_2, \ldots, P_n$ is denoted by $N(P_1'P_2' \ldots P_n')$ and the number of elements in the set is denoted by N, it follows that

$$N(P_1'P_2' \ldots P_n') = N - |A1 \cup A2 \cup \ldots \cup An|.$$

From the inclusion–exclusion principle, we see that

$$\begin{aligned} N(P_1'P_2' \ldots P_n') &= N - \sum_i N(P_i) + \sum_{i<j} N(P_iP_j) \\ &- \sum_{i<j<k} N(P_iP_jP_k) + \ldots + (-1)^n N(P_1P_2 \ldots P_n). \end{aligned}$$

Example 7 *How many solutions does $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, and x_3 are non-negative integers with $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$?*

Solution: *To apply the principle of inclusion-exclusion, let a solution have property P_1 if $x_1 > 3$, property P_2 if $x_2 > 4$, and property P_3 if $x_3 > 6$. The number of solutions satisfying the inequalities $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$ is $N(P_1'P_2'P_3')$ = N - $N(P_1)$ - $N(P_2)$ - $N(P_3)$ + $N(P_1P_2)$ + $N(P_1P_3)$ + $N(P_2P_3)$ - $N(P_1P_2P_3)$. Now*

- *N = total number of solutions = C(3 + 11 - 1, 11) = 78,*
- *$N(P_1)$ = (number of solutions with $x_1 \geq 4$) = C(3 + 7 - 1, 7) = C(9, 7) = 36,*

- $N(P_2)$ = (number of solutions with $x_2 \geq 5$) = C(3 + 6 - 1, 6) = C(8, 6) = 28,

- $N(P_3)$ = (number of solutions with $x_3 \geq 7$) = C(3 + 4 - 1, 4) = C(6, 4) = 15,

- $N(P_1P_2)$ = (number of solutions with $x_1 \geq 4$ and $x_2 \geq 5$) = C(3 + 2 - 1, 2) = C(4, 2) = 6,

- $N(P_1P_3)$ = (number of solutions with $x_1 \geq 4$ and $x_3 \geq 7$) = C(3 + 0 - 1, 0) = 1,

- $N(P_2P_3)$ = (number of solutions with $x_2 \geq 5$ and $x_3 \geq 7$) = 0,

- $N(P_1P_2P_3)$ = (number of solutions with $x_1 \geq 4$, $x_2 \geq 5$, and $x_3 \geq 7$) = 0.

Inserting these quantities into the formula for $N(P_1'P_2'P_3')$ *shows that the number of solutions with* $x_1 \leq 3$, $x_2 \leq 4$, *and* $x_3 \leq 6$ *equals* $N(P_1'P_2'P_3')$ *= 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6.*

Example 8 *A composite integer is divisible by a prime not exceeding its square root. Using the fact, find the number of primes not exceeding 100?*

Solution: *Note that composite integers not exceeding 100 must have a prime factor not exceeding 10. Because the only primes not exceeding 10 are 2, 3, 5, and 7, the primes not exceeding 100 are these four primes and those positive integers greater than 1 and not exceeding 100 that are divisible by none of 2, 3, 5, or 7.*

To apply the principle of inclusion-exclusion, let P_1 *be the property that an integer is divisible by 2, let* P_2 *be the property that an integer is divisible by 3, let* P_3 *be the property that an integer is divisible by 5, and let* P_4 *be the property that an integer is divisible by 7. Thus, the number of primes not exceeding 100 is 4 +* $N(P_1'P_2'P_3'P_4')$.

$$\begin{aligned}
N(P_1'P_2'P_3'P_4') &= N - N(P_1) - N(P_2) - N(P_3) - N(P_4) \\
&+ N(P_1P_2) + N(P_1P_3) + N(P_1P_4) + N(P_2P_3) \\
&+ N(P_2P_4) + N(P_3P_4) - N(P_1P_2P_3) - N(P_1P_2P_4) \\
&- N(P_1P_3P_4) - N(P_2P_3P_4) + N(P_1P_2P_3P_4) \\
&= 99 - \lfloor \tfrac{100}{2} \rfloor - \lfloor \tfrac{100}{3} \rfloor - \lfloor \tfrac{100}{5} \rfloor - \lfloor \tfrac{100}{7} \rfloor \\
&+ \lfloor \tfrac{100}{2.3} \rfloor + \lfloor \tfrac{100}{2.5} \rfloor + \lfloor \tfrac{100}{2.7} \rfloor + \lfloor \tfrac{100}{3.5} \rfloor + \lfloor \tfrac{100}{3.7} \rfloor + \lfloor \tfrac{100}{5.7} \rfloor \\
&- \lfloor \tfrac{100}{2.3.5} \rfloor - \lfloor \tfrac{100}{2.3.7} \rfloor - \lfloor \tfrac{100}{2.5.7} \rfloor - \lfloor \tfrac{100}{3.5.7} \rfloor + \lfloor \tfrac{100}{2.3.5.7} \rfloor \\
&= 99 - (50+33+20+14) + (16+10+7+6+4+2) - (3+2+1+0) + 0 \\
&= 99 - 117 + 45 - 6 = 21
\end{aligned}$$

Hence, the number of primes not exceeding 100 is 4 + 21 = 25.