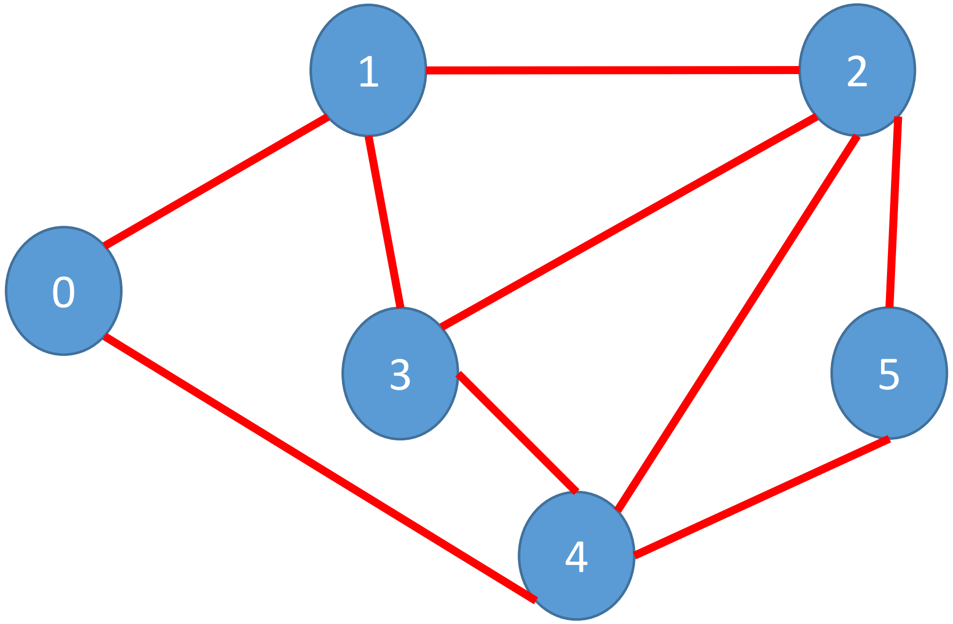
Euler Graph

Joy Mukherjee

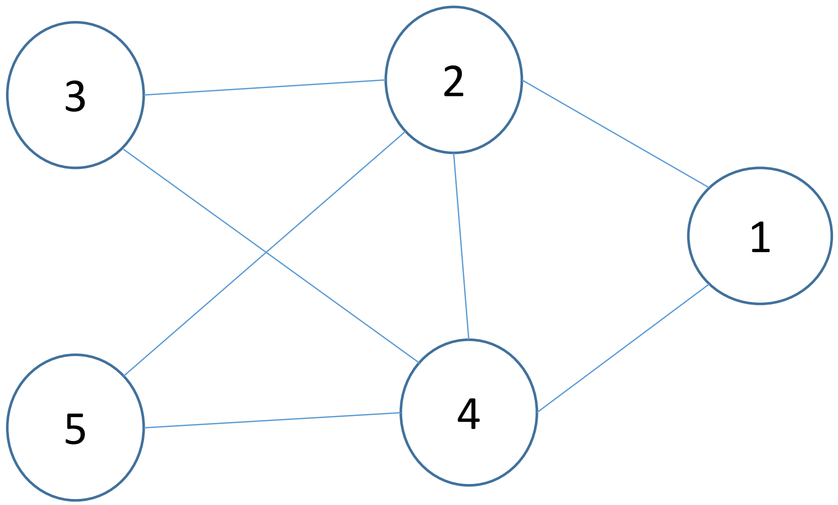
Walk, Trail, Path

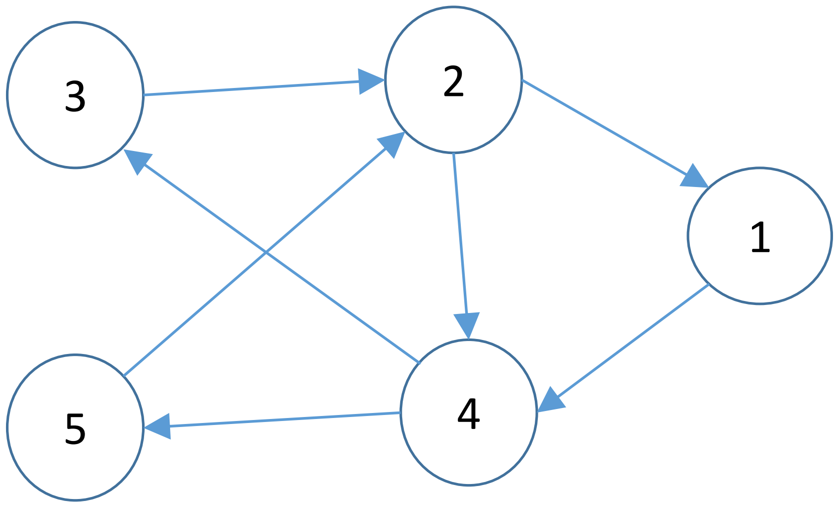
- Walk: Sequence of vertices in the graph where adjacent vertices must have an edge between them. Example: 1 2 5 4 3 2 5 is a walk
- Trail: A walk with no repeated edge. Example: 1 2 5 4 3 2 4 is a trail. In trail, vertex can be repeated.
- Path: A trail/walk with no repeated vertex. Example: 1 2 5 4 3 is a path. If no vertex is repeated, then edges can't be repeated.



Euler Graph

- A trail is a walk with no repeated edges.
- A graph is Eulerian if it has a closed trail containing all edges.
- An even graph is a graph with vertex degrees all even.
- A vertex is odd (or even) when its degree is odd (or even).

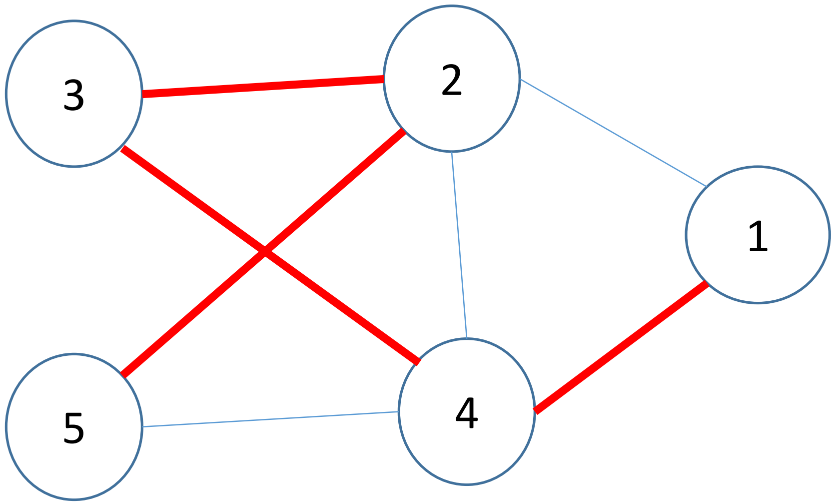


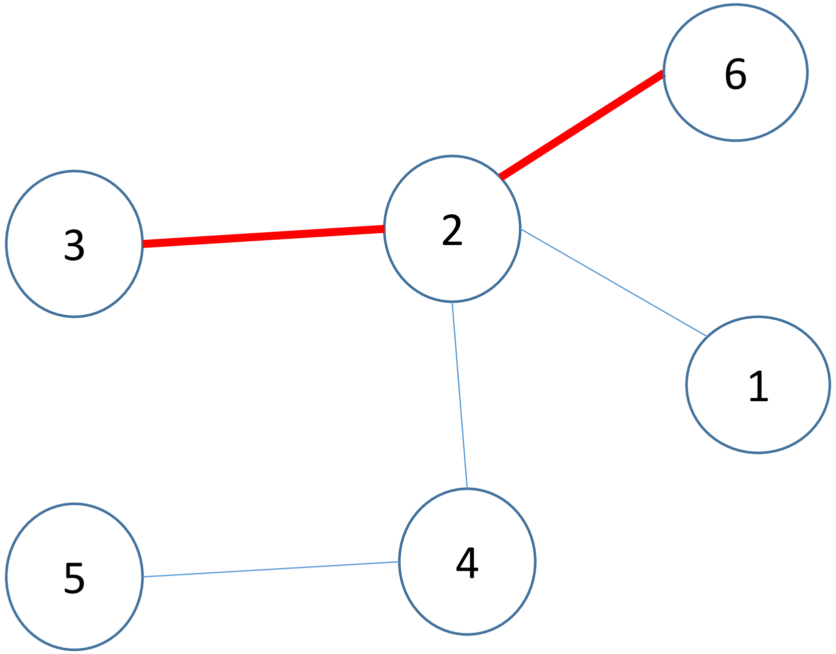


1 4 5 2 4 3 2 1

Maximal Path

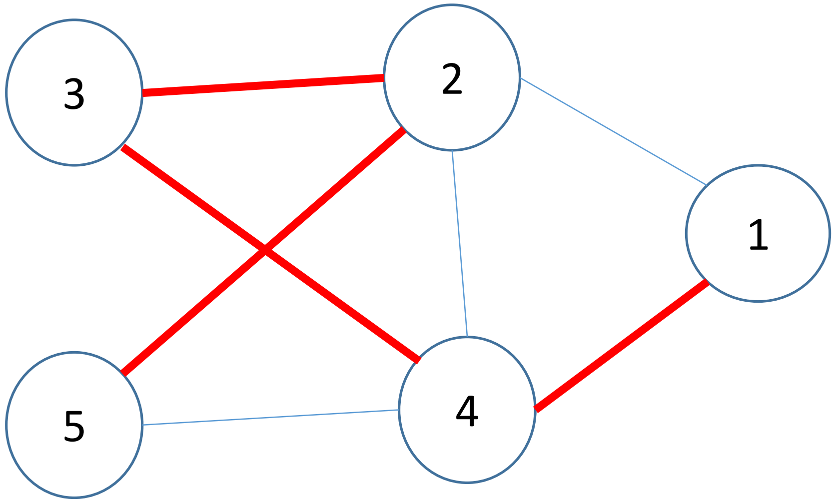
- A maximal path in a graph G is a path P in G that is not contained in a longer path.
  - When a graph is finite, no path can extend forever, so maximal (non-extendible) paths exist.





Proof by Extremality

- **Theorem**: If every vertex of a graph G has degree at least 2, then G contains a cycle.
- **Proof**: Let P be a maximal path in G, and let u be an endpoint of P. Since P cannot be extended, every neighbor of u must already be a vertex of P. Since u has degree at least 2, it has a neighbor v in V(P) via an edge not in P. The edge uv completes a cycle with the portion of P from v to u.

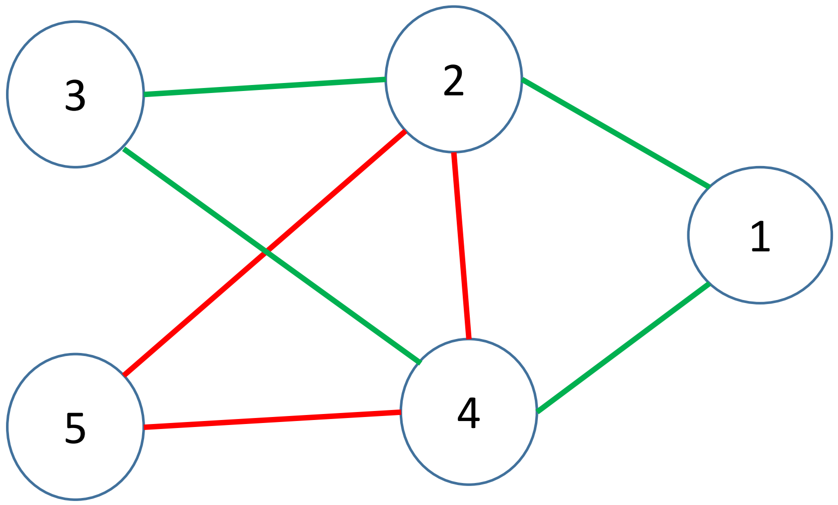


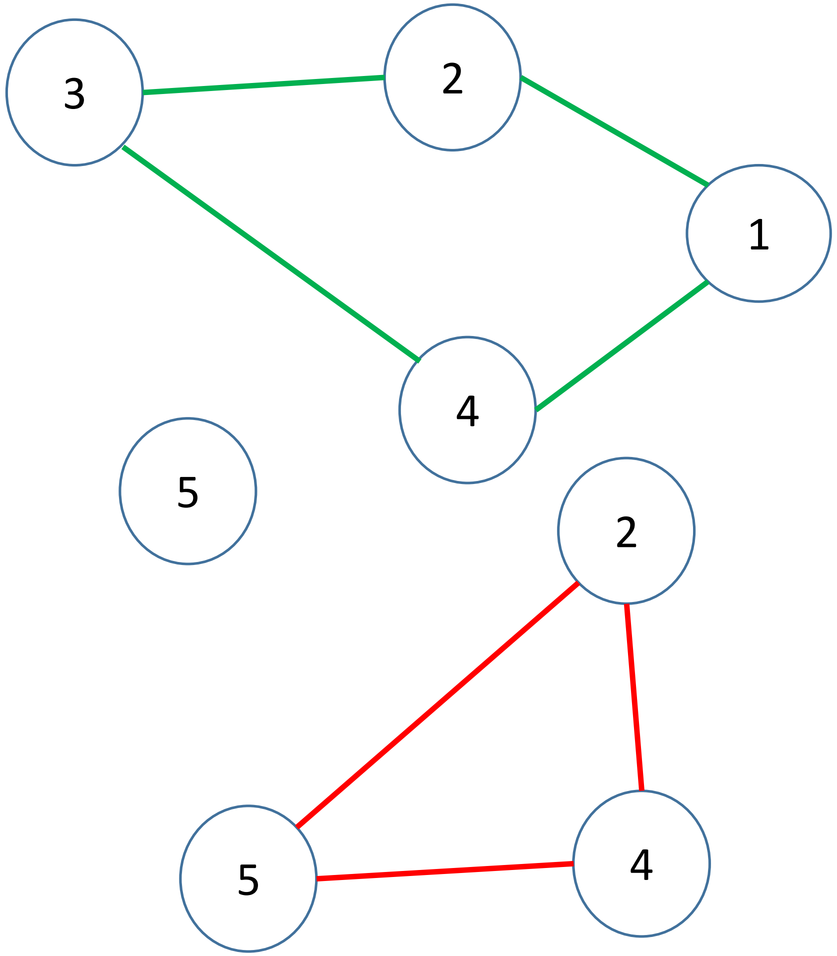
Property of Euler Graph

- **Theorem**: A graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.
- **Proof [Only if]**:Suppose that G is Eulerian. G has a closed trail C containing all edges.
- Each passage of C through a vertex uses two incident edges, and the first edge is paired with the last edge at the first vertex. Hence every vertex has even degree.
- Also, two edges can be in the same trail only when they lie in the same component, so there is at most one nontrivial component.

Property of Euler Graph

- **Proof [If]**:Proof by induction on the number of edges, m.
- **Basis Step**: m = 0. A closed trail consisting of one vertex suffices.
- **Induction Step**:
- **Induction Hypothesis**: Claim holds for < m.
- For m > 0, each vertex in the nontrivial component of G has degree at least 2. It implies that the nontrivial component has a cycle C.
- Let G' be the graph obtained from G by deleting E(C). Since C has 2 edges at each vertex, each component of G' is also an even graph.
- Since each component also is connected and has fewer than m edges, we can apply the induction hypothesis to conclude that each component of G' has an Eulerian circuit.

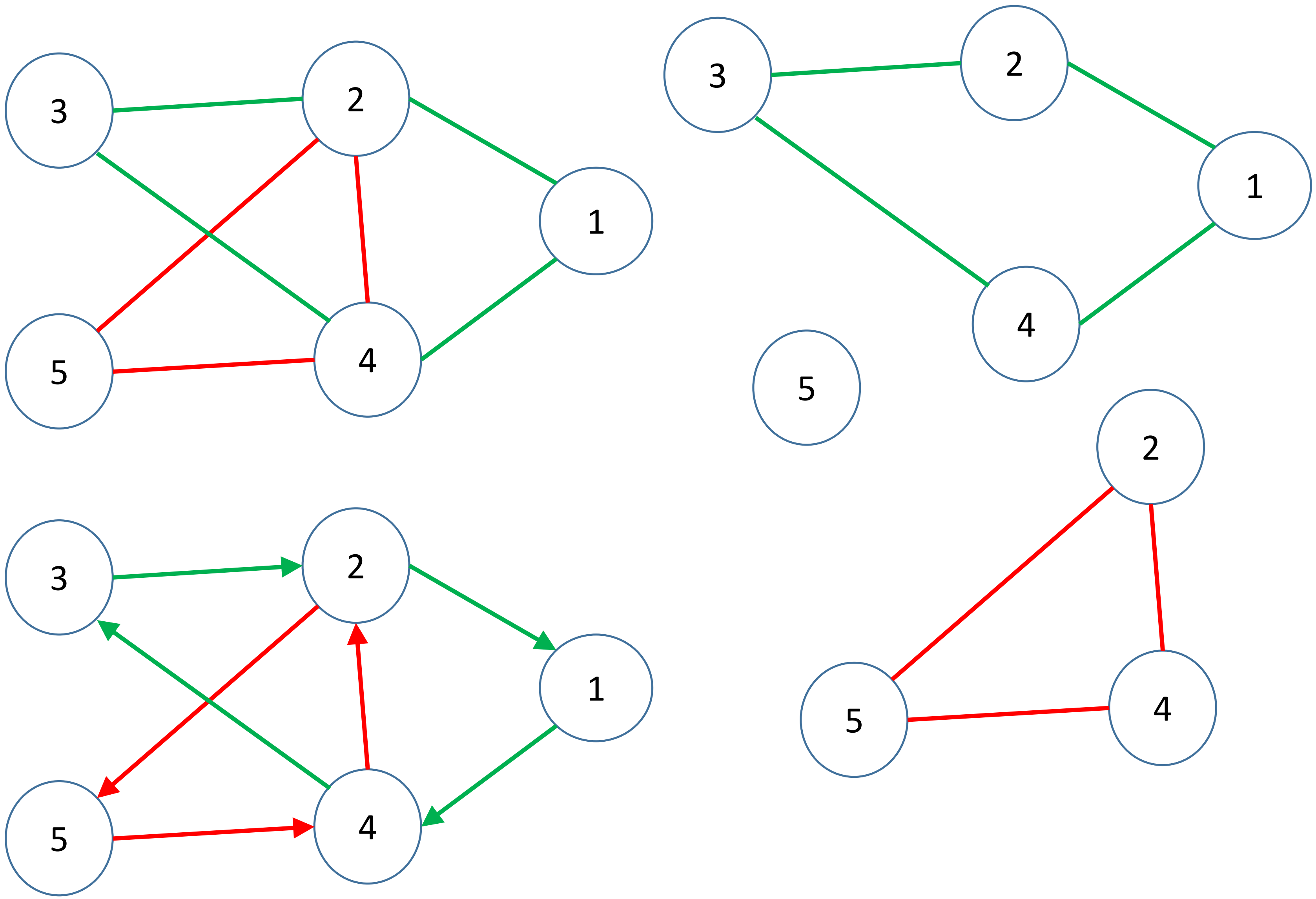




How to combine them to get an Euler trail?

Property of Euler Graph (Fleury’s Algorithm)

- To combine these into an Eulerian circuit of G, we traverse C, but when a component of G' is entered for the first time we detour along an Eulerian circuit of that component. This circuit ends at the vertex where we began the detour. When we complete the traversal of C, we have completed an Eulerian circuit of G.


3
2
1
5
4
3
2
1
4
5
2
3
2
1
5
4
5
4