Applied Graph Theory (Jan'25-May'25)

Problem Sheet-3 on Network Flows

Notations :

- $\kappa(G), \kappa'(G)$ denote respectively the vertex connectivity and edge connectivity of G.
- $\kappa_G(u, v), \kappa'_G(u, v)$ denote respectively the vertex and edge connectivities between u and v.
- $\lambda_G(u, v)$, $\lambda'_G(u, v)$ denote respectively the maximum number of internally vertex disjoint and edge disjoint paths between u and v.
- $\overline{G}$ denotes the complement of G.

Problems :

1. Use network flows, prove the edge version of Menger's theorem for undirected graphs $G = (V, E)$: For $u, v \in V$, $\kappa'_G(u, v) = \lambda'_G(u, v)$.

2. Let G be a directed graph with two distinct vertices s and t. Suppose we introduce capacities c_u on vertices (other than s and t) and not on the edges. For each $u \neq s, t$, the netflow out of u equals zero (as before) but total flow out of u (=total flow into u) is at most c_u. There is no upper bound on flow through edges of G. Use standard edge-capacitated network flows to formulate the maximum vertex-capacitated flow problem from s to t.

3. Use network flows to prove Konig-Egervary Theorem : $\alpha'(G) = \beta(G)$ for bipartite graphs. Here, $\alpha'(G)$ and $\beta(G)$ denote respectively the maximum size of a matching and minimum size of a vertex cover in G.

4. In a huge bag, there are objects of p different types and for each type i, there are m_i objects. There are q bins $B_1, \ldots, B_q$ with respective capacities (number of objects it can contain) $b_1, \ldots, b_q$. Assume (after renumbering the object types and the bin indices) that $m_1 \geq \ldots \geq m_p$ and $b_1 \leq \ldots \leq b_q$. Each bin B_i can have at most one object of a given type but it can have objects of different types. The goal is to assign each object in the bag to one of the bins while obeying the constraints

above. Formulate this problem as a network flow problem and prove that there is such an assignment if and only if, for all $0 \leq k \leq p$ and $0 \leq l \leq q$, it holds that $k(q-l) + \sum_{j=1}^{l} b_j \geq \sum_{i=1}^{k} m_i$.

5. In a large university with k (divisible by 3) academic departments, a committee of k members must be formed with each of the k departments represented by exactly one faculty member (Assistant/Associate/Full Professor) of that department. A person can be possibly be a member of more than one department but can represent at most one department. Also, the committee should have exactly $k/3$ members for each of the three academic designations. Model this assignment problem as a network flow problem to determine if there exists such an assignment. Characterise the solutions of the given problem in terms of the network flows.

6. Given lists $\mathbf{r} = (r_1, \ldots, r_n)$ and $\mathbf{s} = (s_1, \ldots, s_n)$ of nonnegative integers, obtain necessary and sufficient conditions for the existence of a digraph D on vertices $\{v_1, \ldots, v_n\}$ such that each ordered pair (v_i, v_j) occurs at most once as an edge and $d^+(v_i) = r_i$ and $d^-(v_i) = s_i$ for all i.