Advanced Computer Networks

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Probability Basics (Intro to Queueing Systems)

DSSRG: Decentralized Smart Systems Research Group

https://sites.google.com/iitbbs.ac.in/dssrg
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Why Probability theory is important?

- Almost nothing in this universe is deterministic
- Sun rises in the East!......
- Truly? In the same place always ?
- Many things are apparently deterministic, but random in reality
- Human created things may be deterministic. But where nature is involved – things are mostly random.
- Examples…

Basics of probability theory – (Quick recap)

- **Probability (likelihood) of an event**

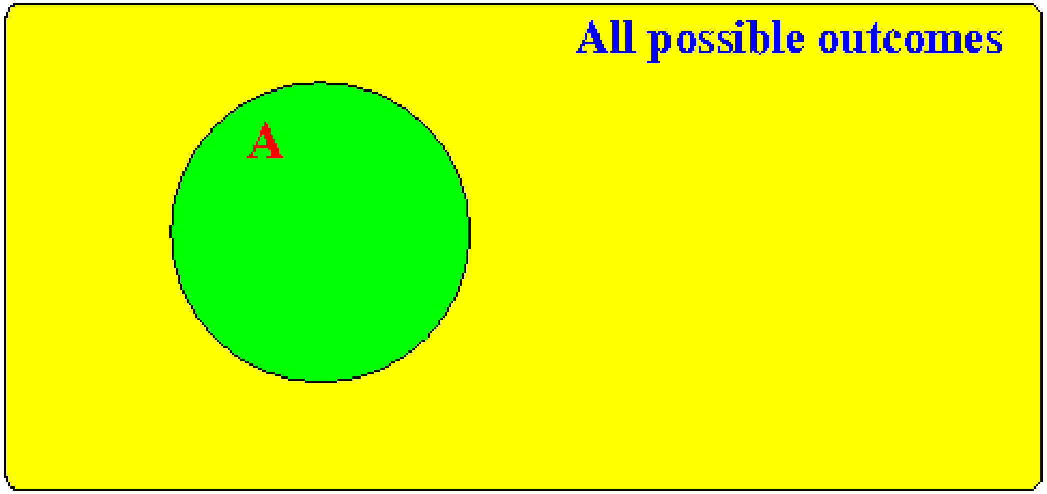
- The **probability** of an **event A** is the **likelihood** of the **occurring** of that event.

Basics of the Theory of Probability

- **Probability (likelihood) of an event**
- The **probability** of an **event A** is the **likelihood** of the **occurring** of that event.
- **Concretely: Consider an random experiment …**

Probability of an event A:

$$P[\text{ event A }] = \frac{\#\text{ outcomes A}}{\#\text{ posible outcomes}}$$



- The **probability distribution function** of a *discrete* **random variable** is known as a **probability mass function**.

- The **probability distribution function** a *continuous* **random variable** is known as a **probability density function**.

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- **Consider x is called a random variable [holding the outcomes of the random experiment…]**
- The **probability mass/density function** is:
- A **mathematical function** used to model the **frequencies (probabilities) of occurrences** of each event
- **Specifically**
- **p(k) = Probability[x = k]**
- **p(k)** is the **probability** that the **outcome** is equal to **k**

- **Example:** the **probability mass/density function** for **Binomial(p, n)** is:
-

$$p(k) = C(n, k)\, p^{k} (1 - p)^{n-k}$$

$$= \frac{n!}{k!(n-k)!}\, p^{k} (1 - p)^{n-k}$$

CDF

- The **(cumulative) probability *distribution* function** is the **cumulative sum** of the values of the **probability *density* function**:

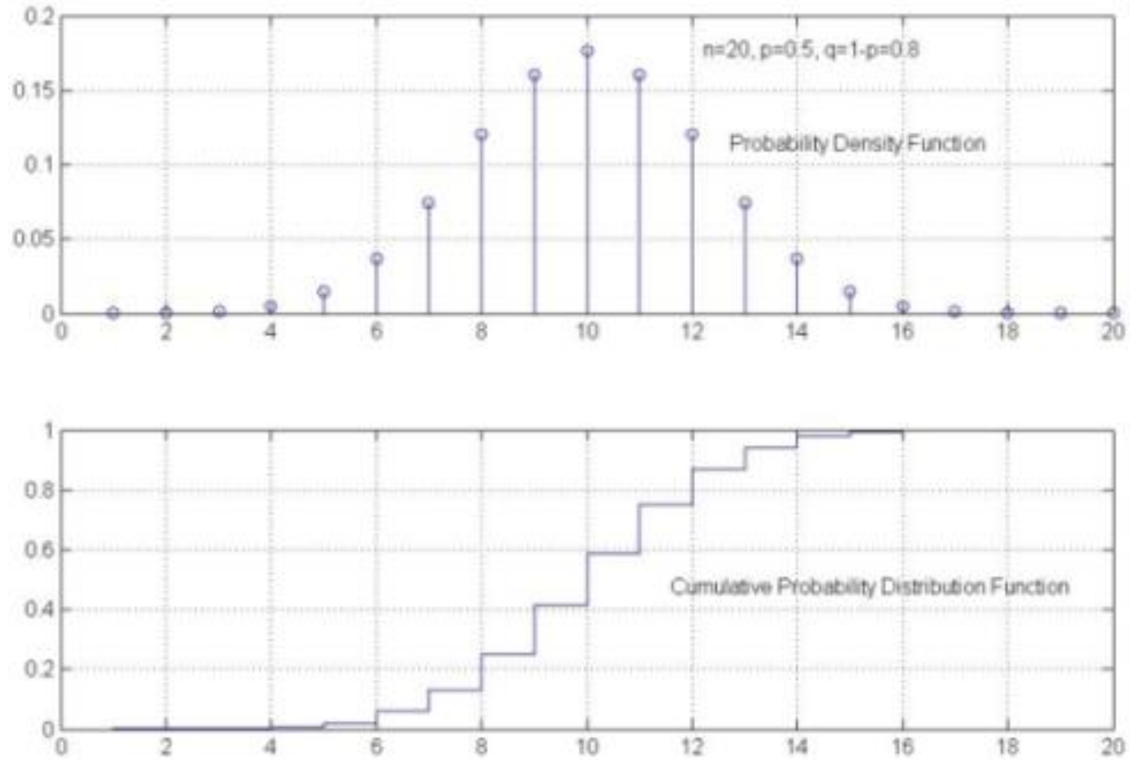
- **Discrete p(x):**
  - $Q(k) = P[\underline{x} \leq k] = \text{Sum}_{x \leq k}\, p(x)$

- **Continuous p(x):**
  - $Q(k) = P[\underline{x} \leq k] = \text{Integration}_{x \leq k}\, p(x)$
  - Note that: $p(x) = d\,Q(x) / dx$

- **Property of every probability *distribution* function:**

- Lim $_{(x \to \infty)}$ Q(x) = 1

- Example
- **Binomial(0.5, 20)**



Expectation

- **Expected value or the mean (average) number of a *random variable***
- The **expected value** of a **random variable** is the **mean/average value** of the **random variable**
- **Mathematical definition of *expected value***:

- x is a random variable with a density function P[x] (P[x] is a short hand for P[x = x]) The expected value E[x] of x is:

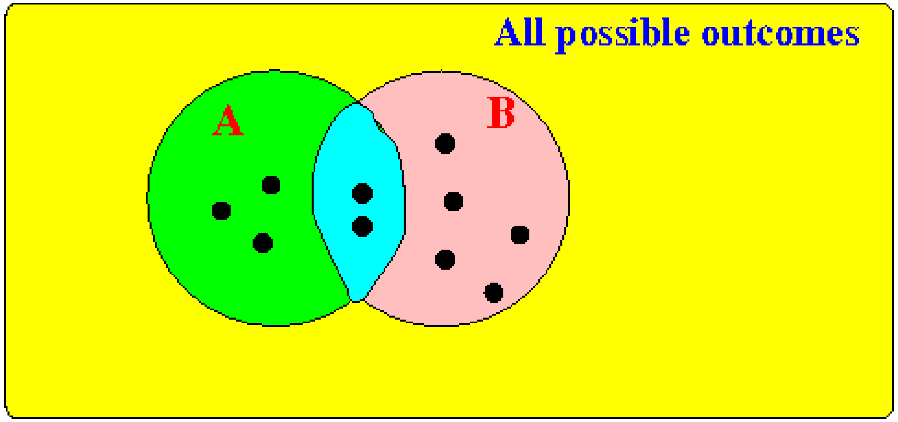
- $E[\underline{x}] = \sum_{(\text{all values } k)} k\, P[k]$

Conditional Probability

The **conditional probability of A for given B** – i.e.

P[A | B] is the **probability** of the **event A given** that the **event B** is **true**

- The **conditional probability P[A | B]** can be **computed** as follows



- **Conditional Probability P[A | B]:**

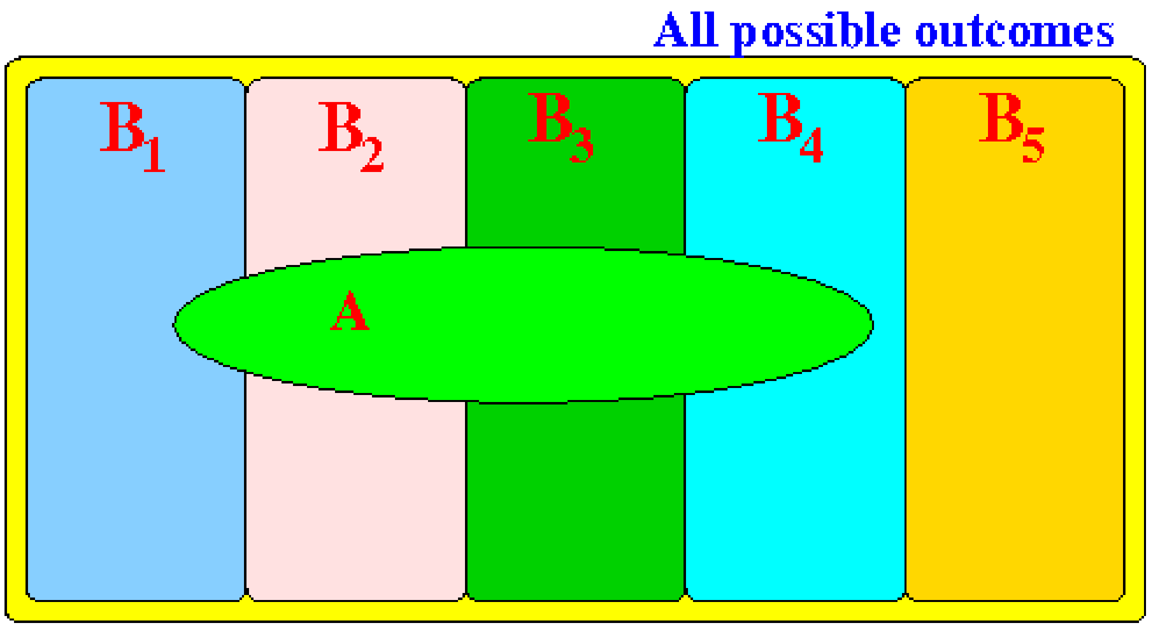
$$P[A \mid B] = \frac{P[A \wedge B]}{P[B]}$$

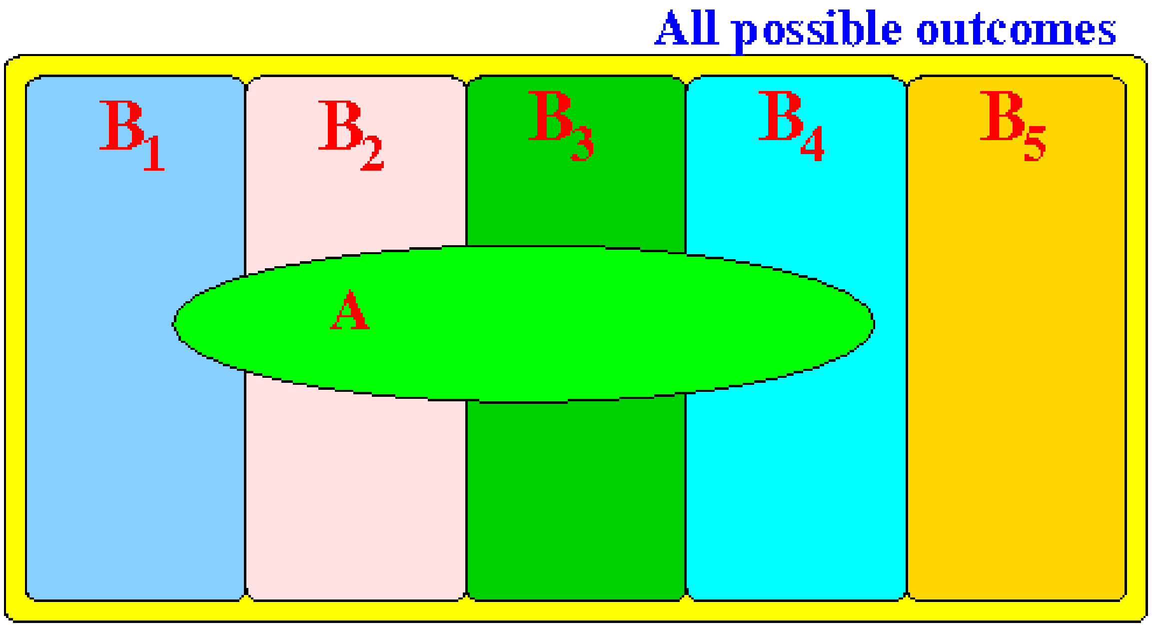
Law of Total Probability

- **(Law of Alternatives or Partitioning Theorem)**
- Let $\mathbf{B_1, B_2, \ldots B_N}$ **finite partition of a probability space**

Then -

- $P[A] = P[A \mid B_1] \times P[B_1] + P[A \mid B_2] \times P[B_2] + \ldots + P[A \mid B_N] \times P[B_N]$





$$P[\ A\ ] = P[\ A \cap B_1\ ] + P[\ A \cap B_2\ ] + \ldots + P[\ A \cap B_N\ ]$$

$$= P[\ A \mid B_1\ ] \times P[\ B_1\ ] + P[\ A \mid B_2\ ] \times P[\ B_2\ ] + \ldots + P[\ A \mid B_N\ ] \times P[\ B_N\ ]$$

Queuing Systems

A **Queueing System** consists of:

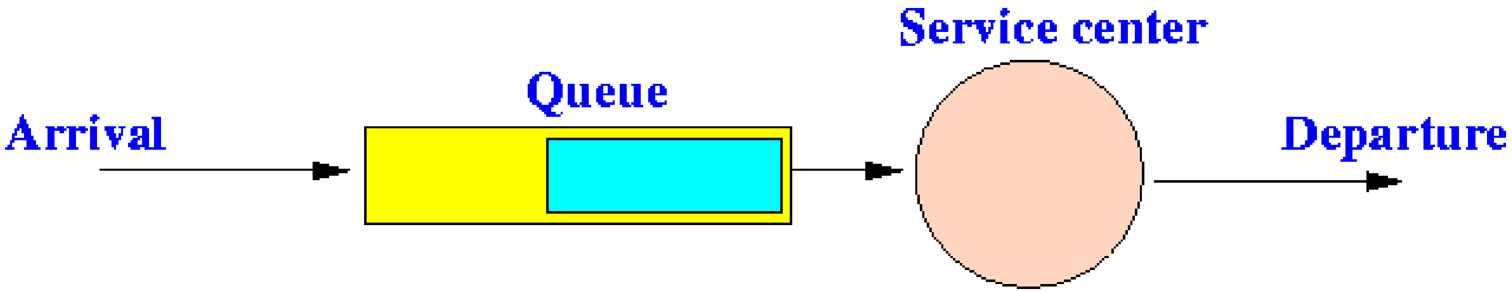
- An **arrival process** of **client** into a **holding area (queue)Clients** come (enter in) to the **queueing system** to obtain a **certain service**

- A **queue management process** that **organizes** the **clients in the queue.** The **most commonly used** queue management processes: **FIFO**

- A **service process** that **fulfils the service requests** of **clients.** After obtaining the **service** from the server, a **client** will **leave the queueing system**

- We call this process the **departure process**

Pictorial view …



Interesting measures of a queueing system

•**Average waiting time** inside the **queue.** i.e., what is the **average time** that **customers** must **wait** before they **starts** obtaining the **service**

•**Average time spent in system.** i.e., what is the **average time** needed for **customers** to **complete** the **service** (This is the duration from the **arrival** of the customer to its **departure**)

Stochastic and Deterministic process

- A **deterministic process** is a process with a **determined schedule of events.** We can tell what event will happen next.

- **Example: Sorting algorithm**

- A **stochastic process** is a process with a **probabilistic schedule of events.** The **next event** will occur with a **certain probability**

- **Example: Post office** (when the next customer arrive is a **probabilistic event**)

Poisson process

- The **Poisson process** is a **stochastic process** where

- (1) P[one customer arrives in the next time interval Δt] = $\lambda \times \Delta t + o(\Delta t)$
- (2) P[no customer arrives in the next time interval Δt] = $1 - \lambda \times \Delta t + o(\Delta t)$
- (3) P[≥ 2 customers arrive in the next time interval Δt] = $o(\Delta t)$

- The arrivals in non-overlapping time intervals are (probabilistically) independent

Base of Poisson Process …

- The **notation P[x]** means the **probability of the event x**

- The **parameter λ** is the **arrival rate** i.e., λ = **average number of arrivals per time unit**

- **Equation (1):** P[one customer arrives in the next time interval Δt] = $\lambda \times \Delta t + o(\Delta t)$ –

- It states that the **probability** of an **arrival** in the **Poisson process** is **linearly dependent** on the **arrival rate λ**

The **notation $o(\Delta t)$** means:

- $$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

i.e., terms **$o(\Delta t)$** is **negligible** compared to the term **Δt**

The probability *density* function

- The **probability *density* function** of the **Poisson arrival process** with **arrival rate λ** is defined as:

p(k) = P(k arrivals in an interval T)

```
     k arrival events
   |     |  |       |
   v     v  v       v
 |<--------------------->|
          T sec
```

Let us divide the interval

- Into n equal sub-intervals

```
  ΔT = T/n
 <-->
|<-->|<-->|<-->|................|<-->|

 <---------------------------------->
               T sec
```

Then …

- The **probability** that **one customer** arrives in the **interval ΔT** is

$$
\begin{aligned}
\mathbb{P}[\ 1 \text{ arrival in } \Delta T\ ] &= \lambda \times \Delta T + o(\Delta T) \\
&\sim= \lambda \times \Delta T \\
&= \lambda \times T/n
\end{aligned}
$$

Continuing …

- The **probability** that **k customers** arrives in the **interval T** is a **Binomial trial** with **probability of success** equal to $\lambda \times \Delta T + o(\Delta T)$, **Therefore:**

$$\mathbb{P}[\ k \text{ arrivals in } T\ ]$$

Substituting: **$\Delta T = T/n$**

$$= \frac{n!}{k!\,(n-k)!} \left(\mathbb{P}[\ 1 \text{ arrival in } \Delta T\ ]\right)^k \left(1 - \mathbb{P}[\ 1 \text{ arrival in } \Delta T\ ]\right)^{n-k}$$

$$= \lim_{(n \to \infty)} \frac{n!}{k!\,(n-k)!} (\lambda \times \Delta T)^k (1 - \lambda \times \Delta T)^{n-k}$$

Continuing …

$$= \lim_{(n \to \infty)} \frac{n!}{k!\,(n-k)!} (\lambda T/n)^k (1 - \lambda T/n)^{n-k}$$

$$= \lim_{(n \to \infty)} \frac{n!}{k!\,(n-k)!} (\lambda T)^k \times (1/n)^k \times (1 - \lambda T/n)^{n-k}$$

Move terms that are independent of **n** out of the limit...

$$= \frac{(\lambda T)^k}{k!} \times \lim_{n \to \infty} \frac{n!}{(n-k)!} (1/n)^n \times (1 - \lambda T/n)^{n-k}$$

$$= \frac{(\lambda T)^k}{k!} \times \lim_{n \to \infty} n\,(n-1)\,\ldots\,(n-k+1)\,(1/n)^k \times (1 - \lambda T/n)^{n-k}$$

$$= \frac{(\lambda T)^k}{k!} \times \lim_{n \to \infty} \frac{n\,(n-1)\,\ldots\,(n-k+1)}{n \quad n \quad \ldots \quad n} \times (1 - \lambda T/n)^{n-k}$$

Apply, for large n and constant x.

$$\frac{n - x}{n} \to 1$$

$$\lim_{(n \to \infty)} (1 - \lambda T/n)^{n-k}$$

$$= \lim_{(n \to \infty)} (1 - \lambda T/n)^{n} \times \lim_{(n \to \infty)} (1 - \lambda T/n)^{-k}$$

$$= \lim_{(n \to \infty)} (1 - \lambda T/n)^{n} \times (1 - 0)^{-k}$$

$$= \lim_{(n \to \infty)} (1 - \lambda T/n)^{n}$$

$= e^{-\lambda T}$ (a well-known Math limit)

Poisson Distribution…

$$P[\text{ k arrivals in T }] = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

Expectation of Poisson Distribution

$$E[\underline{x}] = \sum_{\text{(all values } k)} k\, P[k]$$

$$= \sum_{(k = 0 \,..\, \infty)} k \times \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

$$= \sum_{(k = 1 \,..\, \infty)} \frac{(\lambda T)^k}{(k-1)!} e^{-\lambda T}$$

Move terms independent of k out of the sum....

$$E[\underline{x}] = e^{-\lambda T} \times \sum_{(k = 1\,..\,\infty)} \frac{(\lambda T)^k}{(k-1)!}$$

Adjust the running index (make k run from 0 ∞)

$$E[\underline{x}] = e^{-\lambda T} \times \sum_{(k = 0\,..\,\infty)} \frac{(\lambda T)^{k+1}}{k!}$$

Move one term λT out of the sum...

$$E[\underline{x}] = \lambda T \times e^{-\lambda T} \times \sum_{(k = 0\ ..\ \infty)} \frac{(\lambda T)^k}{k!}$$

Well-known Math serie: $\sum_{(k = 0\ ..\ \infty)} x^k/k! = e^x$

$$E[\underline{x}] = \lambda T \times e^{-\lambda T} \times e^{\lambda T}$$

$$= \lambda T$$

Arrival rate of a Poisson arrival process

- Previously, we found that the **expected value** of a **Poisson λ distributed** random variable $\underline{x}$ is:

- $E[\underline{x}] = \lambda T$

- The **random variable** $\underline{x}$ represents the ***number of arrivals*** **in a time interval of duration T**

- The **average (mean) number of arrivals** over a **time interval of duration T** is equal to **λ** × **T**

The **average number of arrivals** ***per time unit*** is:

- Avg # arrivals per second = $\lambda T/T = \lambda$

- **Arrival rate of a Poisson process**

- **λ** is the **arrival rate** of the **Poisson arrival process**
- **λ** = the **average number of arrivals per time unit (sec)**

Distribution of the interarrival times: time between 2 consecutive arrivals

- $\underline{y}$ = the **random variable** representing the **time** between **2 consecutive arrivals** in a **Poisson arrival process** (i.e., the **inter-arrival time**)
- **Probability density function of $\underline{y}$:**

$$P[\ \underline{y} > t\ ] = P[\ \text{no arrivals in interval } (0..t)\ ]$$

$$= \frac{(\lambda t)^0}{0!} e^{-\lambda t} \qquad \text{(i.e., k=0)}$$

$$= e^{-\lambda t}$$

Probability distribution function -

$$\mathbb{P}[\ \underline{y} \leq t\ ] = 1 - \mathbb{P}[\ \underline{y} > t\ ]$$

$$= 1 - e^{-\lambda t}$$

This is actually Q (i.e., the **Cumulative Density Function**)

(Probability distrubution function of $\underline{y}$)

Can be derived as follows (by taking derivative)

Probability density function

$$p(t) = \frac{d\,Q(t)}{dt}$$

Density function

$$Q_y(t) = 1 - e^{-\lambda t}$$

Therefore:

$$p_y(t) = \frac{d\,[1 - e^{-\lambda t}]}{dt}$$

$$= - e^{-\lambda t} \times (-\lambda)$$

$$= \lambda\, e^{-\lambda t}$$

Memory-less property of the Poisson arrival process

- A **process** is **memory-less** if it has the following **property**:

- P [number of events within next t sec
  | event has not happened for u sec]

= P[number of events within next t sec]

So, The **likelihood (probability)** of when the **next event** will happen

IS NOT AFFECTED

by the **given knowledge** that the event **has *not* happened for some time**

Example of *memory-full* processes:

- **Volcano eruptions:**

- The **probability** that a **volcano** will **not erupt** within the **next 100 yrs** is ***greatly decreased*** if we **knew** that the **volcano** has **not erupted for 1 million years**

- **Hunger:**

- The **probability** that a **person** does not become **hungry** within the **next hour** is ***greatly decreased*** if we **knew** that the **person** has **not eaten for 6 hours**

The Poisson process is *memory-less*

- P[number of arrivals occurs within next **t** sec
  | number of arrivals for u sec]
- = P[number of arrivals occurs within next **t** sec]

Proof ….[**Study in details**]

Tasks:

- **Make a detailed study of it.**
- **What is this memory less property.**
- **When the time intervals are overlapping –**
- **Does it still hold?**
- **What can we infer from this? What is the advantage and disadvantage?**