Discrete Structures

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Chapter 1

Predicate Logic or Predicate Calculus

The statement "x is greater than 3" has two parts. The first part, the variable x, is the **subject** of the statement. The second part - the **predicate**, "is greater than 3" - refers to a property that the subject of the statement can have. We can denote the statement "x is greater than 3" by P(x), where P denotes the predicate "is greater than 3" and x is the variable. The statement P(x) is also said to be the value of the **propositional function** P at x. Once a value has been assigned to the variable x, the statement P(x) becomes a **proposition** and has a **truth value**.

Example 1 *Let $P(x)$ denote the statement "$x > 3$". What are the truth values of $P(4)$ and $P(2)$?*

Solution: *We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$". Hence, $P(4)$, which is the statement "$4 > 3$", is true. However, $P(2)$, which is the statement "$2 > 3$", is false.*

Example 2 *Let $Q(x, y)$ denote the statement "$x = y + 3$". What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?*

Solution: *To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$", which is false. The statement*

Q(3, 0) is the proposition "3 = 0 + 3", which is true.

In general, a statement of the form P$(x_1, x_2, \ldots, x_n)$ is the value of the **propositional function** P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called an **n-place predicate** or an **n-ary predicate**.

Propositional functions may occur in computer programs as follows.

Example 3 *Consider the statement*

if $x > 0$ then $x = x + 1$.

When this statement is encountered in a program, the value of the variable x at that point in the execution of the program is inserted into P(x), which is "$x > 0$". If P(x) is true for this value of x, the assignment statement $x = x + 1$ is executed, so the value of x is increased by 1. If P(x) is false for this value of x, the assignment statement is not executed, so the value of x is not changed.

Predicates are also used to establish the correctness of computer programs, that is, to show that computer programs always produce the desired output when given valid input. (Note that unless the correctness of a computer program is established, no amount of testing can show that it produces the desired output for all input values, unless every input value is tested). The statements that describe valid input are known as **preconditions** and the conditions that the output should satisfy when the program has run are known as **postconditions**. As the following illustrates, we use predicates to describe both preconditions and postconditions.

Example 4 *Consider the following program for swapping the values of two variables x and y.*

temp $= x$

$x = y$

$y =$ *temp*

Find predicates that we can use as the precondition and the postcondition to verify the correctness of this program. Then explain how to use them to verify that for all valid input the program does what is intended.

Solution: *For the precondition, we need to express that x and y have particular values before we run the program. So, for this precondition we can use the predicate $P(x, y)$, where $P(x, y)$ is the statement "$x = a$ and $y = b$", where a and b are the values of x and y before we run the program.*

Because we want to verify that the program swaps the values of x and y for all input values, for the postcondition we can use $Q(x, y)$, where $Q(x, y)$ is the statement "$x = b$ and $y = a$".

To verify that the program always does what it is supposed to do, suppose that the precondition $P(x, y)$ holds. That is, we suppose that the statement "$x = a$ and $y = b$" is true. This means that $x = a$ and $y = b$.

The first step of the program, $temp = x$, assigns the value of x to the variable temp, so after this step we know that $x = a$, $temp = a$, and $y = b$.

After the second step of the program, $x = y$, we know that $x = b$, $temp = a$, and $y = b$.

Finally, after the third step, we know that $x = b$, $temp = a$, and $y = a$.

Consequently, after this program is run, the postcondition $Q(x, y)$ holds, that is, the statement "$x = b$ and $y = a$" is true.

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. However, there is another important way, called **quantification**, to create a proposition from a propositional function. Quantification expresses the extent to which a predicate is true over a range of elements. We will focus on two types of quantification here.

Definition 5 (Universal Quantification) *A predicate is true for all values of a variable in a particular domain, called the* **domain of discourse** *or the* **universe of discourse***, often just referred to as the* **domain***. The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not defined.*

The universal quantification of P(x) is the statement

"P(x) for all values of x in the domain".

The notation $\forall x P(x)$ denotes the universal quantification of P(x). Here $\forall$ is called the universal quantifier. We read $\forall x P(x)$ as "for all $xP(x)$" or "for every $xP(x)$". An element for which P(x) is false is called a **counter example** *to $\forall x P(x)$.*

Example 6 *What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all positive integers?*

Solution: *The universal quantification $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false for all real numbers x with $0 < x < 1$.*

Note that $x^2 \geq x$ if and only if $x^2 \geq x = x(x-1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x(x^2 \geq x)$ is false if the domain consists of all real numbers.

However, if the domain consists of the positive integers, $\forall x(x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

Definition 7 (Existential quantification) *A predicate is true for one or more values of a variable in a particular domain, called the* **domain of discourse** *or the* **universe of discourse***, often just referred to as the* **domain***.*

The existential quantification of P(x) is the proposition

"There exists an element x in the domain such that P(x)",

"There is an x such that P(x)",

"There is at least one x such that P(x)", or

"For some xP(x)".

We use the notation $\exists x P(x)$ for the existential quantification of P(x). Here $\exists$ is called the existential quantifier. To prove $\exists x P(x)$ is false, one has to show that either the domain is empty or P(x) is false for every x.

Example 8 *Let P(x) denote the statement "$x > 3$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?*

Solution: *Because "$x > 3$" is sometimes true - for instance, when x = 4 - the existential quantification of P(x), which is $\exists x P(x)$, is true.*

Example 9 *Let Q(x) denote the statement "x = x + 1". What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?*

Solution: *Because Q(x) is false for every real number x, the existential quantification of Q(x), which is $\exists x Q(x)$, is false.*

The area of logic that deals with predicates and quantifiers is called the **predicate logic** or **predicate calculus**.

Precedence of Quantifiers: The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus. For example, $\forall x P(x) \lor Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)) \lor Q(x)$ rather than $\forall x (P(x) \lor Q(x))$.

1.1 Quantifiers over Finite Domains

When the domain of a quantifier is finite, that is, when all its elements can be listed, quantified statements can be expressed using propositional logic.

In particular, when the elements of the domain are x_1, x_2, ..., x_n, where n is a positive integer, the universal quantification $\forall x P(x)$ = P(x_1) $\land$ P(x_2) $\land$ ... $\land$ P(x_n), because this conjunction is true if and only if P(x_1), P(x_2), ..., P(x_n) are all true.

Similarly, when the elements of the domain are x_1, x_2, ..., x_n, where n is a positive integer, the existential quantification $\exists x P(x)$ = P(x_1) $\lor$ P(x_2) $\lor$ ... $\lor$ P(x_n), because this disjunction is true if and only if at least one of P(x_1), P(x_2), ..., P(x_n) is true.

1.2 Quantifiers with Restricted Domains

To restrict a domain, a condition a variable must satisfy is included after the quantifier.

The restriction of a universal quantification is the same as the universal quantification of a conditional statement. For instance, $\forall x < 0(x^2 > 0)$ is another way of expressing $\forall x(x < 0 \rightarrow x^2 > 0)$, where the domain consists of the real numbers.

The restriction of an existential quantification is the same as the existential quantification of a conjunction. For instance, $\exists z > 0(z^2 = 2)$ is another way of expressing $\exists z(z > 0 \wedge z^2 = 2)$, where the domain consists of the real numbers.

1.3 Free and Bound Variables

When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**. All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

The part of a logical expression to which a quantifier is applied is called the **scope of the quantifier**. Consequently, a variable is free if it is outside the scope of all quantifiers in the formula that specify the variable.

Example 10 *In the statement $\exists\ x(x + y = 1)$, the variable x is bound by the existential quantification $\exists\ x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to y. This illustrates that in the statement $\exists\ x(x + y = 1)$, x is bound, but y is free.*

In the statement $\exists\ x(P(x) \wedge Q(x)) \vee \forall\ xR(x)$, all variables are bound. The scope of the first quantifier, $\exists\ x$, is the expression $P(x) \wedge Q(x)$, because

$\exists$ x is applied only to P(x) $\wedge$ Q(x) and not to the rest of the statement.

Similarly, the scope of the second quantifier, $\forall$ x, is the expression R(x). That is, the existential quantifier binds the variable x in P(x) $\wedge$ Q(x) and the universal quantifier $\forall$ x binds the variable x in R(x).

Observe that we could have written our statement using two different variables x and y, as $\exists$ x(P(x) $\wedge$ Q(x)) $\vee$ $\forall$ yR(y), because the scopes of the two quantifiers do not overlap.

The reader should be aware that in common usage, the same letter is often used to represent variables bound by different quantifiers with scopes that do not overlap.

1.4 Logical Equivalences Involving Quantifiers

Example 11 *Show that $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$ (where the same domain is used throughout). This logical equivalence shows that we can distribute a universal quantifier over a conjunction.*

Solution: *Suppose that $\forall x(P(x) \wedge Q(x))$ is true. This means that if a is in the domain, then $P(a) \wedge Q(a)$ is true. Hence, P(a) is true and Q(a) is true. Because P(a) is true and Q(a) is true for every element a in the domain, we can conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both true. This means that $\forall x P(x) \wedge \forall x Q(x)$ is true.*

Suppose that $\forall x P(x) \wedge \forall x Q(x)$ is true. It follows that $\forall x P(x)$ is true and $\forall x Q(x)$ is true. Hence, if a is in the domain, then P(a) is true and Q(a) is true [because P(x) and Q(x) are both true for all elements in the domain, there is no conflict using the same value of a here]. It follows that for all a, $P(a) \wedge Q(a)$ is true. It follows that $\forall x(P(x) \wedge Q(x))$ is true.

Example 12 *Show that $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$ (where the same domain is used throughout). This logical equivalence shows that we can distribute an existential quantifier over a disjunction.*

Solution: *Suppose that $\exists x(P(x) \vee Q(x))$ is true. This means that if a is* *in the domain, then* $P(a) \vee Q(a)$ *is true. Hence, P(a) is true or Q(a) is true. Because P(a) is true or Q(a) is true for at least one element a in the domain, we can conclude that* $\exists x P(x)$ *or* $\exists x Q(x)$ *are true. This means that* $\exists x P(x) \vee \exists x Q(x)$ *is true.*

Suppose that $\exists x P(x) \vee \exists x Q(x)$ *is true. It follows that* $\exists x P(x)$ *is true or* $\exists x Q(x)$ *is true. Hence, for some a in the domain, P(a) is true or Q(a) is true [because P(x) or Q(x) is true for at least one element in the domain, there is no conflict using the same value of a here]. It follows that for some a,* $P(a) \vee Q(a)$ *is true. It follows that* $\exists x(P(x) \vee Q(x))$ *is true.*

Example 13 *Show that* $\neg \forall x P(x) \equiv \exists x \neg P(x)$.

Solution: *To show that* $\neg \forall x P(x)$ *and* $\exists x \neg P(x)$ *are logically equivalent no matter what the propositional function P(x) is and what the domain is, first note that* $\neg \forall x P(x)$ *is true if and only if* $\forall x P(x)$ *is false. Next, note that* $\forall x P(x)$ *is false if and only if there is an element x in the domain for which P(x) is false. This holds if and only if there is an element x in the domain for which* $\neg P(x)$ *is true. Finally, note that there is an element x in the domain for which* $\neg P(x)$ *is true if and only if* $\exists x \neg P(x)$ *is true. Putting these steps together, we can conclude that* $\neg \forall x P(x)$ *is true if and only if* $\exists x \neg P(x)$ *is true. It follows that* $\neg \forall x P(x)$ *and* $\exists x \neg P(x)$ *are logically equivalent.*

When the domain has n elements $x_1, x_2, \ldots, x_n$, it follows that $\neg \forall x P(x)$ is the same as $\neg(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n))$, which is equivalent to $\neg P(x_1) \vee \neg P(x_2) \vee \ldots \vee \neg P(x_n)$ by De Morgan's laws, and this is the same as $\exists x \neg P(x)$.

Example 14 *Show that* $\neg \exists x P(x) \equiv \forall x \neg P(x)$.

Solution: *To show that* $\neg \exists x P(x)$ *and* $\forall x \neg P(x)$ *are logically equivalent no matter what the propositional function P(x) is and what the domain is, first note that* $\neg \exists x P(x)$ *is true if and only if* $\exists x P(x)$ *is false. Next, note that* $\exists x P(x)$ *is false if and only if for all element x in the domain, P(x) is false. This holds if and only if for all element x in the domain for which* $\neg P(x)$ *is true. Finally, note that for all element x in the domain for which* $\neg P(x)$ *is true if and only if* $\forall x \neg P(x)$ *is true. Putting these steps together, we can conclude that* $\neg \exists x P(x)$ *is true if and only if* $\forall x \neg P(x)$ *is true. It follows that* $\neg \exists x P(x)$ *and* $\forall x \neg P(x)$ *are logically equivalent.*

When the domain has n elements $x_1, x_2, \ldots, x_n$, it follows that $\neg\exists x P(x)$ is the same as $\neg(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n))$, which is equivalent to $\neg P(x_1) \wedge \neg P(x_2) \wedge \ldots \wedge \neg P(x_n)$ by De Morgan's laws, and this is the same as $\forall x \neg P(x)$.

The rules for negations for quantifiers are called **De Morgan's laws for quantifiers**.

1.5 Applications of Predicates, Single/Nested Quantifiers

1.5.1 Translating Sentences

Example 15 *Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.*

Solution: *we introduce a variable x that represents a person.*

"For every person x, if person x is a student in this class, then x has studied calculus". If S(x) represents the statement that person x is in this class, and C(x) represents the statement "x has studied calculus", we see that our statement can be expressed as $\forall x(S(x) \rightarrow C(x))$.

Caution: *Our statement cannot be expressed as* $\forall x(S(x) \wedge C(x))$ *because this statement says that all people are students in this class and have studied calculus.*

Finally, when we are interested in the background of people in subjects besides calculus, we may prefer to use the two-variable quantifier Q(x, y) for the statement "Student x has studied subject y". Then we would replace C(x) by Q(x, calculus) in both approaches to obtain $\forall xQ(x, \text{calculus})$ *or* $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$.

Example 16 *Express the statement "Some student in this class has visited Mexico" using predicates and quantifiers.*

Solution: *We introduce M(x), which is the statement "x has visited Mexico".*

If the domain for x consists of the students in this class, we can translate this first statement as $\exists\, xM(x)$.

If the domain for the variable x consists of all people. We introduce $S(x)$ *to represent "x is a student in this class". Our solution becomes* $\exists\, x(S(x) \wedge M(x))$ *because the statement is that there is a person x who is a student in this class and who has visited Mexico.*

Caution: *Our statement cannot be expressed as* $\exists\, x(S(x) \rightarrow M(x))$, *which is true when there is someone not in the class because, in that case, for such a person x,* $S(x) \rightarrow M(x)$ *becomes either* $F \rightarrow T$ *or* $F \rightarrow F$, *both of which are true.*

Example 17 *Express the statement "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.*

Solution: *"For every person x, if x is a student in this class, then x has visited Mexico or x has visited Canada". In this case, the statement can be expressed as* $\forall\, x(S(x) \rightarrow (C(x) \vee M(x)))$.

Example 18 *Express the statement "Every student in your school has a computer or has a friend who has a computer" using predicates and quantifiers.*

Solution: $C(x)$ *is "x has a computer",* $F(x, y)$ *is "x and y are friends", and the domain for both x and y consists of all students in your school. In this case, the statement can be expressed as* $\forall\, x(C(x) \vee \exists\, y(C(y) \wedge F(x, y)))$

Example 19 *Translate the statement* $\exists\, x \forall\, y \forall\, z((F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y, z))$ *into English, where* $F(a, b)$ *means a and b are friends and the domain for x, y, and z consists of all students in your school.*

Solution: *This expression says that if students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends. It follows that the original statement, which is triply quantified, says that there is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends. In other words, there is a student none of whose friends are also friends with each other.*

Example 20 *Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.*

Solution: *The statement "If a person is female and is a parent, then this person is someone's mother" can be expressed as "For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y". We introduce the propositional functions F(x) to represent "x is female", P(x) to represent "x is a parent", and M(x, y) to represent "x is the mother of y". The original statement can be represented as* $\forall\, x((F(x) \wedge P(x)) \rightarrow \exists\, yM(x, y))$, *or* $\forall\, x\exists\, y((F(x) \wedge P(x)) \rightarrow M(x, y))$.

Example 21 *Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.*

Solution: *The statement "Everyone has exactly one best friend" can be expressed as "For every person x, person x has exactly one best friend". Introducing the universal quantifier, we see that this statement is the same as "∀ x(person x has exactly one best friend)", where the domain consists of all people. To say that x has exactly one best friend means that there is a person y who is the best friend of x, and furthermore, that for every person z, if person z is not person y, then z is not the best friend of x. When we introduce the predicate B(x, y) to be the statement "y is the best friend of x". Consequently, our original statement can be expressed as* $\forall\, x\exists\, y(B(x, y) \wedge \forall\, z((z \neq y) \rightarrow \neg B(x, z)))$.

Example 22 *Express the statement "There is a woman who has taken a flight on every airline in the world". using predicates and quantifiers.*

Solution: *Let P(w, f) be "w has taken f " and Q(f, a) be "f is a flight on a". We can express the statement as*

$$\exists\, w\forall\, a\exists\, f(P(w, f) \wedge Q(f, a))$$

, where the domains of discourse for w, f, and a consist of all the women in the world, all airplane flights, and all airlines, respectively.

Example 23 *Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."*

Solution:

$$\forall\ w \exists\ a \forall\ f(\neg P(w,f) \vee \neg Q(f,a))$$

This last statement states "For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline."

1.5.2 Translating Mathematical Statements

Example 24 *Translate the statement "The sum of two positive integers is always positive" into a logical expression.*

Solution: $\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$, *where the domain for both variables consists of all integers.*

$\forall x \forall y(x + y > 0)$, *where the domain for both variables consists of all positive integers.*

Example 25 *A* **multiplicative inverse** *of a real number x is a real number y such that $xy = 1$. Translate the statement "Every real number except zero has a multiplicative inverse."*

Solution: *We first rewrite this as "For every real number x except zero, x has a multiplicative inverse." We can rewrite this as "For every real number x, if $x \neq 0$, then there exists a real number y such that $xy = 1$." This can be rewritten as* $\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$.

Example 26 *Recall that the definition of the statement*

$$\lim_{x \to a} f(x) = L$$

is: For every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a in its domain.

Solution: *This definition of a limit can be phrased in terms of quantifiers by*

$$\forall \epsilon \ \exists \delta \ \forall x \ (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

, where the domain for the variables ϵ and δ consists of all positive real numbers and for x consists of all real numbers.

This definition can also be expressed as

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

where the domain for the variables ϵ, δ and x consists of all real numbers, rather than just the positive real numbers. Here, restricted quantifiers have been used. Recall that $\forall x > 0$ P(x) means that for all x with x > 0, P(x) is true.

Example 27 *To say that $\lim_{x \to a} f(x)$ does not exist means that for all real numbers L,*

$$\lim_{x \to a} f(x) \neq L$$

Use quantifiers and predicates to express the fact that $\lim_{x \to a} f(x)$ does not exist, where f(x) is a real-valued function of a real variable x and a belongs to the domain of f.

Solution: *This definition of a limit can be expressed as*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

where the domain for the variables ϵ, δ and x consists of all real numbers, rather than just the positive real numbers. Here, restricted quantifiers have been used. Recall that $\forall x > 0$ P(x) means that for all x with x > 0, P(x) is true.

$$\forall L \exists \epsilon > 0 \ \forall \delta > 0 \ \exists x \ (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)$$

This last statement says that for every real number L there is a real number $\epsilon >$ 0 such that for every real number $\delta >$ 0, there exists a real number x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.

1.5.3 Expressing System Specifications

Example 28 *Use predicates and quantifiers to express the system specification "Every mail message larger than one megabyte will be compressed".*

Solution: *Let M(x, y) be "Mail message x is larger than y megabytes", where the variable x has the domain of all mail messages and the variable y is a positive real number, and let C(x) denote "Mail message x will be compressed". Then the specification can be represented as* $\forall\ x\ (M(x,\ 1) \rightarrow C(x))$.

Example 29 *Use predicates and quantifiers to express the system specification "If a user is active, at least one network link will be available."*

Solution: $\forall\ x\ (A(x) \rightarrow \exists\ y\ (N(y) \wedge\ V(y)))$

1.6 Rules of Inference for Predicate Logic

We have discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal instantiation is the rule of inference used to conclude that P(c) is true, where c is a particular member of the domain, given the premise $\forall$ xP(x). Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise", where Lisa is a member of the domain of all women.

Universal generalization is the rule of inference to conclude that $\forall$ xP(x) is true, given the premise that P(c) is true for all elements c in the domain. Universal generalization is used when we show that $\forall$ xP(x) is true by taking an arbitrary element c from the domain and showing that P(c) is true. We have no control over c and cannot make any other assumptions about c other than it comes from the domain.

Rule of Inference	Name
$\forall\, xP(x)$ ——— $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c ——— $\therefore \forall\, xP(x)$	Universal generalization
$\therefore \exists\, xP(x)$ ——— $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c ——— $\therefore \exists\, xP(x)$	Existential generalization
$\forall\, x(P(x) \rightarrow Q(x))$ $P(a)$, where a is a particular element in the domain ——— $\therefore Q(a)$	Universal modus ponens
$\forall\, x(P(x) \rightarrow Q(x))$ $\neg\, Q(a)$, where a is a particular element in the domain ——— $\therefore \neg\, P(a)$	Universal modus tollens

Table 1.1: Rules of Inference for Quantified Statements

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which P(c) is true if we know that $\exists\, xP(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which P(c) is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential generalization is the rule of inference that is used to conclude that $\exists\, xP(x)$ is true when a particular element c with P(c) true is known. That is, if we know one element c in the domain for which P(c) is true, then we know that $\exists\, xP(x)$ is true.

Example 30 *Show that the premises "Everyone in this discrete mathematics* *class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science".*

Solution: *Let D(x) denote "x is in this discrete mathematics class", and let C(x) denote "x has taken a course in computer science". Then the premises are* $\forall\, x(D(x) \rightarrow C(x))$ *and D(Marla). The conclusion is C(Marla). The following steps can be used to establish the conclusion from the premises.*

Step	Reason
1. $\forall\, x(D(x) \rightarrow C(x))$	Premise
2. $D(Marla) \rightarrow C(Marla)$	Universal instantiation from (1)
3. $D(Marla)$	Premise
4. $C(Marla)$	Modus ponens from (2) and (3)

Example 31 *Show that the premises "A student in this class has not read the book", and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book".*

Solution: *Let C(x) be "x is in this class", B(x) be "x has read the book", and P(x) be "x passed the first exam". The premises are* $\exists\, x(C(x) \wedge \neg B(x))$ *and* $\forall\, x(C(x) \rightarrow P(x))$*. The conclusion is* $\exists\, x(P(x) \wedge \neg B(x))$*. These steps can be used to establish the conclusion from the premises.*

Step	Reason
1. $\exists\, x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall\, x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists\, x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

Example 32 *Assume that "For all positive integers n, if n is greater than 4, then* n^2 *is less than* 2^n*" is true. Use universal modus ponens to show that* $100^2 < 2^{100}$.

Solution: *Let P(n) denote "*$n > 4$*" and Q(n) denote "*$n^2 < 2^n$*". The* *statement "For all positive integers n, if n is greater than 4, then* n^2 *is less than* 2^n*" can be represented by* $\forall\ n(P(n) \to Q(n))$*, where the domain consists of all positive integers. We are assuming that* $\forall\ n(P(n) \to Q(n))$ *is true. Note that P(100) is true because* $100 > 4$*. It follows by universal modus ponens that Q(100) is true. Therefore,* $100^2 < 2^{100}$.

Example 33 *Consider these statements. The first two are called premises and the third is called the conclusion. The entire set is called an argument.*

"All lions are fierce."

"Some lions do not drink coffee."

"Some fierce creatures do not drink coffee."

Determine whether the conclusion is a valid consequence of the premises.

Solution: *Let the domain consists of all creatures. Let P(x), Q(x), and R(x) be the statements "x is a lion", "x is fierce", and "x drinks coffee", respectively. We can express the statements in the argument as*

$\forall\ x(P(x) \to Q(x))$.

$\exists\ x(P(x) \land \neg R(x))$.

$\exists\ x(Q(x) \land \neg R(x))$.

Step	*Reason*
1. $\exists\ x(P(x) \land \neg R(x))$	*Premise*
2. $P(c) \land \neg R(c)$	*Existential instantiation from (1)*
3. $P(c)$	*Simplification from (2)*
4. $\neg R(c)$	*Simplification from (2)*
5. $\forall\ x(P(x) \to Q(x))$	*Premise*
6. $P(c) \to Q(c)$	*Universal instantiation from (5)*
7. $Q(c)$	*Modus ponens from (3) and (6)*
8. $Q(c) \land \neg R(c)$	*Conjunction from (7) and (8)*
9. $\exists\ x(Q(x) \land \neg R(x))$	*Existential generalization from (8)*