Applied Graph Theory - Jan-May 2025.
Notes on Hall's Theorem and its generalisations
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Definition 0.1 *Given a graph $G = (V, E)$, a subset $I \subseteq V$ is* **independent** *in G if there exists no edge $uv \in E$ with both u and v from I. A subset $S \subseteq V$ is a* **vertex cover** *if, for every edge $uv \in E$, either $u \in S$ or $v \in S$. S is a vertex cover* **if and only if** *$I = V \setminus S$ is an independent set. We use $\alpha(G)$ and $\nu(G)$ to denote respectively the sizes $|I|$ and $|S|$ of a maximum-sized independent set I and minimum-sized vertex cover S in G.*

Definition 0.2 *Let $G = (V, E)$ be a graph. A subset $M \subseteq E$ is a* **matching** *if no two edges in M share any vertex in common. A matching M in G* **saturates** *a vertex u if u is one endpoint of some edge in M. We use $\mu(G)$ to denote the maximum size $|M|$ of a matching M in G. Given a matching M, an M*-**alternating path** *P is a finite sequence $u_0, e_1, u_1, e_2, u_2, e_3 \ldots$ which forms a path starting from an un-saturated vertex u_0 in G and with edges alternately from $E \setminus M$ (like $e_1, e_3, \ldots$) and from M (like $e_2, e_4, \ldots$). An M*-**augmenting path** *P is an alternating path which starts and ends at un-saturated vertcies. An M*-**alternating cycle** *is a cycle in G in which the edges are alternately from M and $E \setminus M$. Every M-alternating cycle is of even length.*

Theorem 0.1 *For a graph $G = (V, E)$, a matching M is a maximum-sized matching* **if and only if** *M admits no augmenting path.*

Proof: $\Rightarrow$ **:** Suppose M admits an augmenting path $P = (u_0, e_1, u_1, \ldots, e_{2k+1}, u_{2k+1})$ where u_0 and u_{2k+1} are unsaturated and $u_1, \ldots, u_{2k}$ are all saturated. Also, $e_1, e_3, \ldots, e_{2k+1}$ are from $E \setminus M$ and $e_2, \ldots, e_{2k}$ are from M. Treating P and M as sets of edges, consider M' defined as $M' = M \Delta P := (M \setminus P) \bigcup (P \setminus M)$. It is seen that M' is a matching and is of size $|M| + 1$ contradicting that M is a maximum-sized matching.

$\Leftarrow$: Suppose M admits no M-augmenting path and M is not maximum-sized. Then, there exists a matching M' of size larger than M. Consider $G' = (V, M \cup M')$. G' is a spanning subgraph of G whose maximum degree is at most 2. It follows that G' is the vertex disjoint union of an independent set I, a collection $\mathcal{P} = \{P_1, \ldots, P_k\}$ of M-alternating paths, each P_i starting and ending at degree-1 vertices and a collection $\mathcal{C} = \{C_1, \ldots, C_r\}$ of M-alternating even cycles. Since $|M'| > |M|$, there exists a path, say,

P_1 which has exactly one M'-edge more than M-edges and hence starts and ends at an un-saturated (by M in G') vertices. Thus P_1 is an M-augmenting path in G' and hence in G also, contradicting our assumption. Hence M is of maximum size. ∎

Theorem 0.2 *For any bipartite $G = (X \cup Y, E)$, we have $\mu(G) = \nu(G)$.*

Proof: Since any vertex cover S has to include at least one vertex from each edge in M, we have $\nu(G) \geq \mu(G)$. It remains to prove $\nu(G) \leq \mu(G)$. Let M be a maximum sized (that is, $|M| = \mu(G)$) matching in G. Let Y' denote the set of those vertices $y \in Y$ saturated by M such that there is some alternating path in G which starts at an unsaturated vertex $u \in X$ and ends at y. Let X' denote the set of those vertices $x \in X$ saturated by some edge $e = xy \in M$ for which $y \notin Y'$. Let S denote the set $X' \cup Y'$. Clearly, for every $e = xy \in M$, either $x \in X'$ and $y \notin Y'$ or $x \notin X'$ and $y \in Y'$. Hence, $\mu(G) = |X'| + |Y'| = |M| = |S|$. It suffices to prove that S is a vertex cover since that implies $\nu(G) \leq \mu(G)$.

Take any $e = xy \notin M$. If $x \in X'$, it takes care of e. Hence, assume that $x \notin X'$. Since M is maximum-sized, either x or y or both are saturated by M. Let $e_1 = xy'$ denote the matching edge incident at x if x is saturated by M. If x is saturated, let P be an alternating path starting at an unsaturated $u \in X$ and ending at y'. P is of odd length. Let $e_2 = x'y$ denote the matching edge incident at x if y is saturated by M.

Claim 0.1 *Either x is not saturated or y is saturated.*

Proof: Suppose, on the contrary, that x is saturated and y is not saturated. Then, the path P' defined by $P' = uPy'e_1xey$ is an alternating path starting at an unsaturated $u \in X$ and ending at an unsaturated $y \in Y$ and hence is an augmenting path, contradicting that M is maximum-sized. ∎

Hence, either both x and y are saturated or only y is saturated.

Case 1: Suppose both x and y are saturated. Clearly, $e_1 \neq e_2$. We have $y' \in Y'$. The path P' defined by $P' = uPy'e_1xey$ is an alternating path ending at y and hence we have $y \in Y'$. **Case 2 :** y is saturated and x is not saturated. The path $P' = xey$ establishes that $y \in Y'$. ∎

Theorem 0.3 (Hall's Marriage Theorem :) *Let $G = (X \cup Y, E)$ be any bipartite graph. Then, G admits a matching M saturating each vertex of X* **if and only if** *$|N_G(S)| \geq |S|$ for each $S \subseteq V$.*

Proof: $\Rightarrow$: Let M saturate all vertices of X. For each x, let y_x be the other endpoint of the unique M-edge incident at x. Then, for any $S \subseteq X$, $\{y_x : x \in S\} \subseteq N_G(S)$ and hence $|N_G(S)| \geq |S|$.

$\Leftarrow$: Let M be a maximum size matching in G and suppose M does not saturate all vertices of X. Let S be a minimum-sized vertex cover of S. Define $X' = X \cap S$ and $Y' = Y \cap S$. Then $|X'| + |Y'| = \nu(G) = \mu(G) < |X|$. Hence, $|Y'| < |X| - |X'| = |X \setminus X'|$. It follows from the definition of X' and Y' that there is no edge from any vertex in $X \setminus X'$ to $Y \setminus Y'$. Hence, $N_G(X \setminus X') \subseteq Y'$. Hence, for $S = X \setminus X'$, $|N_G(S)| \leq |Y'| < |X \setminus X'|$, contradicting our assumption. ∎

The above theorem has the following corollaries.

Definition 0.3 *For a (not necessarily disjoint) collection* $\mathcal{A} = \{A_1, \ldots, A_n\}$ *of sets, a* System of Distinct Representatives *(SDR) is a collection* $\{x_i \in A_i : 1 \leq i \leq n\}$ *satisfying* $x_i \neq x_j$ *for* $i \neq j$.

Corollary 0.1 *Let* $\mathcal{A} = \{A_1, \ldots, A_n\}$ *be a collection of (not necessarily disjoint) sets. Then,* $\mathcal{A}$ *admits a SDR* if and only if $| \cup_{j \in J} A_j| \geq |J|$ *for all* $J \subseteq [n]$.

Proof: Consider the bipartite graph $G = (X \cup Y, E)$ where $X = [n] := \{1, \ldots, n\}$ and $Y := \cup_{j=1}^{n} A_j$. Define $E := \{(i, y) : i \in X, y \in Y, y \in A_i\}$. It follows that $\mathcal{A}$ admits a SDR *if and only if* G admits a matching M saturating all vertices of X. By Theorem 0.3, G admits such a matching M *if and only if* $| \cup_{j \in J} A_j| = |N_G(J)| \geq |J|$ for all $J \subseteq [n]$. ∎

Corollary 0.2 *Suppose* $\mathcal{C} = \{C_1, \ldots, C_n\}$ *be a list of courses offered in an institution and let* $\mathcal{F} = \{f_1, \ldots, f_m\}$ *be the set of faculty members. Each course is to be assigned to one faculty member from among those who are willing to teach. Suppose each member is ready to teach at most one course. Then, an assignment matching each course to one faculty member willing to teach it can be found* if and only if *for any subset* $\mathcal{D}$ *of courses, there are at least* $|\mathcal{D}|$ *members each willing to teach at least one of the courses in* $\mathcal{D}$.

Corollary 0.3 *Suppose* $\mathcal{C} = \{C_1, \ldots, C_n\}$ *be a list of courses offered in an institution and let* $\mathcal{F} = \{r_1, \ldots, r_m\}$ *be the set of lecture halls available. Each course is to be assigned to one lecture hall from among those which are capable of accommodating all students of that course Then, an assignment* *matching each course to one lecture hall large enough for that course can be found* if and only if *for any subset $\mathcal{D}$ of courses, there are at least $|\mathcal{D}|$ lecture halls each large enough to accommodate all students of at least one of the courses in $\mathcal{D}$.*

Theorem 0.4 Generalized Hall's Theorem : *Let $G = (X \cup Y, E)$ be a simple bipartite graph. Suppose $r_x \geq 0$ ($x \in X$) is a set of integers assigned for each $x \in X$. Then, G has a spanning subgraph $H = (X \cup Y, F)$ satisfying (i) $deg_H(x) = r_x$ for each $x \in X$, (ii) $N_H(x) \cap N_H(x') = \emptyset$ for each $x \neq x'$* **if and only if** *$|N_G(S)| \geq \sum_{x \in S} r_x$ for each $S \subseteq X$.*

Note : By setting $r_x = 1$ for each $x \in X$, the above theorem specialises to the well-known Hall's Theorem on matchings.

Proof:

Only if : Suppose H exists. Then, for any $S \subseteq V$, $N_H(S)$ is the disjoint union of $N_H(x)$'s over all $x \in S$ and hence $|N_G(S)| \geq |N_H(S)| = \sum_{x \in S} r_x$.

If : Define a bipartite graph $G' = (X' \cup Y', E')$ as follows. $Y' = Y$. For each $x \in X$, introduce r_x new vertices $x_1, \ldots, x_{r_x}$. Let R_x denote the set of these r_x vertices. If $r_x = 0$, then $R_x = \emptyset$. Also, $R_x \cap R_{x'} = \emptyset$ for $x \neq x'$. $X' = \cup_{x \in X} R_x$. E' consists of exactly the edges (x_j, y) (for each $1 \leq j \leq r_x$), for each $(x, y) \in E$. We have the following claims.

Claim 0.2 (A) : *$|N_{G'}(T)| \geq |T|$ for each $T \subseteq X'$* $\iff$ **(B) :** *$|N_G(S)| \geq \sum_{x \in S} r_x$ for each $S \subseteq X$.*

Claim 0.3 (C) : *G has a spanning subgraph H meeting the neighborhood requirements for each $x \in X$* $\iff$ **(D) :** *G' has a matching saturating all vertices of X'.*

The **If** part follows from **(B)** $\Rightarrow$ **(A)** $\Rightarrow$ **(D)** $\Rightarrow$ **(C)**. The second implication follows from applying Hall's Theorem to G'. Hence, it suffices to only prove the claims.

Proof: (**of Claim** 0.2)

(B) $\Rightarrow$ **(A)** : For $T \subseteq X'$, define $S = S(T) := \{x \in X : R_x \cap T \neq \emptyset\}$. Then, $|N_{G'}(T)| = |N_G(S(T))| \geq \sum_{x \in S(T)} r_x \geq |T|$ since, by construction, we have $|T| \leq \sum_{x \in S(T)} r_x$ for any $T \subseteq X'$.

(A) $\Rightarrow$ **(B)** : For $S \subseteq X$, define $T = T(S) := \cup_{x \in S} R_x$. We have $|T(S)| = \sum_{x \in S} r_x$ for every S. Hence $|N_G(S)| = |N_{G'}(T(S))| \geq |T(S)| = \sum_{x \in S} r_x$ for each S. ■

Proof: (**of Claim** 0.3)

(C) $\Rightarrow$ **(D)** : For each $x \in X$, x has r_x neighbors in H. Also, $N_H(x) \cap N_H(x') = \emptyset$ for all $x \neq x'$. Since each member of R_x is adjacent to each member of $N_H(x)$ by construction of G', we can find a perfect matching M_x in G' between R_x and $N_H(x)$. Also, $M_x \cap M_{x'} = \emptyset$ for $x \neq x'$. The union $\cup_{x \in X} M_x$ of these perfect matchings gets us a matching saturating all vertices of X' in G'.

(D) $\Rightarrow$ **(C)** : Given a matching M in G' saturating all vertices of X', partition M into $M = \cup_{x \in X} M_x$ where M_x is the set of those edges in M whose X'-endpoint belongs to R_x. Define Y_x be the collection of r_x Y-endpoints of the edges in M_x. We have $Y_x \subseteq N_G(x)$ by construction. Also, $Y_x \cap Y_{x'} = \emptyset$ for $x \neq x'$. Define H to be the subgraph formed by the union $\cup_{x \in X} \{(x, y) : y \in Y_x\}$. H is a spanning subgraph of G meeting the degree requirements of **(C)**. ■

This completes the proof of the theorem. ■

The above theorem has the following corollaries.

Definition 0.4 *For a (not necessarily disjoint) collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of sets and a collection $\{r_1, \ldots, r_n\}$ of nonnegative integers, a $(r_1, \ldots, r_n)$-type* System of Disjoint Representatives *(SDR) is a collection $\{D_i \subseteq A_i : 1 \leq i \leq n\}$ satisfying (i) $|D_i| = r_i$ for each i, and (ii) $D_i \cap D_j = \emptyset$ for $i \neq j$.*

Note : The SDR (System of Distinct Representatives) defined earlier is a $(1, 1, \ldots, 1)$-type SDR (System of Disjoint Representatives).

Corollary 0.4 *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a collection of (not necessarily disjoint) sets. Let $\{r_1, \ldots, r_n\}$ be a collection of nonnegative integers. Then, $\mathcal{A}$ admits a $(r_1, \ldots, r_n)$-type SDR* if and only if $|\cup_{j \in J} A_j| \geq \sum_{j \in J} r_j$ *for all $J \subseteq [n]$.*

Proof: Consider the bipartite graph $G = (X \cup Y, E)$ where $X = [n] := \{1, \ldots, n\}$ and $Y := \cup_{j=1}^{n} A_j$. Define $E := \{(i, y) : i \in X, y \in Y, y \in A_i\}$. It follows that $\mathcal{A}$ admits a $(r_1, \ldots r_n)$-type SDR *if and only if* G admits $(r_1, \ldots, r_n)$-type spanning subgraph H where vertices in X have mutually disjoint neighborhoods in Y satisfying the degree requirements $deg_H(i) = r_i$ for each $i \in X$. By Theorem 0.4, G admits such a spanning subgraph H *if and only if* $|\cup_{j \in J} A_j| = |N_G(J)| \geq \sum_{j \in J} r_j$ for all $J \subseteq [n]$. ■

Corollary 0.5 *Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a set of children and let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a set of fruits. Suppose each child C_j wants exactly r_j fruits from among those in $\mathcal{F}$ which it likes. Each fruit can be given to at most one child. Then, there is an assignment of fruits to children satisfying the requirements of children* if and only if *for each $J \subseteq [n]$, the number of fruits liked by at least one child from $\{C_j\}_{j \in J}$ is at least $\sum_{j \in J} r_j$.*

Corollary 0.6 *Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a set of committees to be formed from a set $\mathcal{E} = \{e_1, \ldots, e_m\}$ of employees. Suppose each committee C_j requires exactly r_j employees from among those in $\mathcal{E}$ who are qualified to be a member of C_j. Each employee can be a member of at most one committee. Then, there is an assignment of employees to committees satisfying the requirements of committees* if and only if *for each $J \subseteq [n]$, the number of employees who are qualified to be a member of at least one committee from $\{C_j\}_{j \in J}$ is at least $\sum_{j \in J} r_j$.*