Advanced Computer Networks

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Markov Process

DSSRG: Decentralized Smart Systems Research Group

https://sites.google.com/iitbbs.ac.in/dssrg
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Back to … Queuing Theory ….

- Stochastic Modeling
- Application of probability theory to understand real-world phenomena
- Many places we have seen Queues - or in a general terms – Waiting line
- Queueing theory is the mathematical study of the queues
- **Analysis** of systems that provides **services to some random demand**

Examples -

- Grocery store
- Airport
- Traffic Signal

- Common questions –
- **What is the average time to be spent in the queue**
- [Customer – Cars, Calls, packets]
- **How long the lines are on average**
- How many customers are waiting for **more than two minutes**
- **How many servers are needed?**

Networks and Queues …

- Dealing with quantitative models –
- *Probabilistic and Stochastic*

- When ARPANET was being considered – There are many queues it was supposed to deal with.

- Its feasibility was mathematically shown by researchers through Queueing theory – Some people were **Erlang, Kleinrock**

- Similar places we can use the theories -

Source to read…

- Web2.uwindor.ca/math/hlynka/queue.html
- Shortle, Thomson, Gross and Harris: Fundamentals of Queueing Theory: Wiley
- **J Medhi: Stochastic Modeling, Academic Press**
- Kleinrock, Queuing Systems, Vol 1, Wiley
- Cooper, Introduction to Queueing Theory, North Holland
- Nelson: Probability, Stochastic Process and Queuing Theory: The Mathematics of Performance Modeling: Springer
- Gelenbe and Pujolle, Introduction to Queuing Networks, Wiley

Markov processes and Queueing theory:

- **Markov processes**
- A **Markov process (MP) { X(t), t ∈ T}** is a process such that:

$$\mathbb{P}[\,X(t+1) = x_{t+1} \mid X(1) = x_1,\ X(2) = x_2,\ \ldots,\ X(t) = x_t\,] = \mathbb{P}[\,X(t+1) = x_{t+1} \mid X(t) = x_t\,]$$

- X(t+1) = State at t+1
- X(1) = state at x1, X(2), state at x2, …

- In other words: to determine the **next state X(t+1)**, we only have to look at the **present state X(t)**.

Categories of Markov processes

- **Markov processes** are **categorized** by the **nature of the *time* and *space* domains**:

Space →

Time ↓

	Discrete	Continuous
Discrete	Discrete time Markov chain	Discrete time, continuous space Markov process
Continuous	Continuous time Markov chain	Diffusion processes

Link with Queuing System -

- **X(t) = number** of **customers** in the system
-

```
t ∈ [0, ∞)                  (all positive time values)

X(t) ∈ [0, 1, 2, .... )     (# of customers)

ℙ[ X(t+1) = k-1 | X(t) = k ] = probab that a customer departs in the next time unit

ℙ[ X(t+1) = k+1 | X(t) = k ] = probab that a customer arrives in the next time unit
```

Discrete time Markov chains

- **State Transition –**

- **Recall** that in a **Markov process**, **only the *last* state** determines the **next state** that the **Markov process** will visit:

- The **state** at **time t** is **X(t)**
- The **state** at **time t+1** is **X(t+1)**
- **Moving** from state **X(t)** to the state **X(t+1)** is called - **State transition**

One-step transition probability

- **Given** that the **Markov chain** is in **state i** at **time t**
- The **probability** that the **Markov chain** will be in **state j** at **time t+1** is defined as

$$P_{ij} = P[\, X(t+1) = j \mid X(t) = i \,]$$

One-step transition probability

- Notice that this **relationship** is **valid** for **any value of t**.
- i.e.: This is **time-independent**

$$
\begin{aligned}
P_{ij} &= \mathbb{P}[\ X(t+1) = j \mid X(t) = i\ ] \\
&= \mathbb{P}[\ X(t+2) = j \mid X(t+1) = i\ ] \\
&= \mathbb{P}[\ X(t+3) = j \mid X(t+2) = i\ ]
\end{aligned}
$$

- The **probability** P_{ij} is called the **one-step (state) transition probability**

One-step transition probability matrix

- Let **N** denote the **number of states** in the **Markov chain**
- The **collection** of ***all*** **one-step transition probabilities** forms a **matrix**: (TPM)
-

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \ldots & P_{1N} \\ P_{21} & P_{22} & P_{23} & \ldots & P_{2N} \\ .. & .. & .. & & .. \\ .. & .. & .. & & .. \\ P_{N1} & P_{N2} & P_{N3} & \ldots & P_{NN} \end{bmatrix}$$

Properties of one-step transition probability matrix

Every row-sum is 1

$$
\begin{aligned}
&P_{11} + P_{12} + P_{13} + \;\ldots\; + P_{1N} = 1 \\
&P_{21} + P_{22} + P_{23} + \;\ldots\; + P_{2N} = 1 \\
&.. \\
&P_{N1} + P_{N2} + P_{N3} + \;\ldots\; + P_{NN} = 1
\end{aligned}
$$

Why?

When a MP is at state **i**, at **t**, in the next time state, it must need to go to another state which may be **i** itself or something different but within the set of states only…

Stochastic Matrix

- An **N×N matrix P** is a **stochastic matrix** if
- $\forall$ **i** $= 1, 2, \ldots, N$: $\sum_{\{j = 1, 2, \ldots, N\}} P_{ij} = 1$
- (i.e., every **row sum** is equal to **1**)

Double Stochastic Matrix

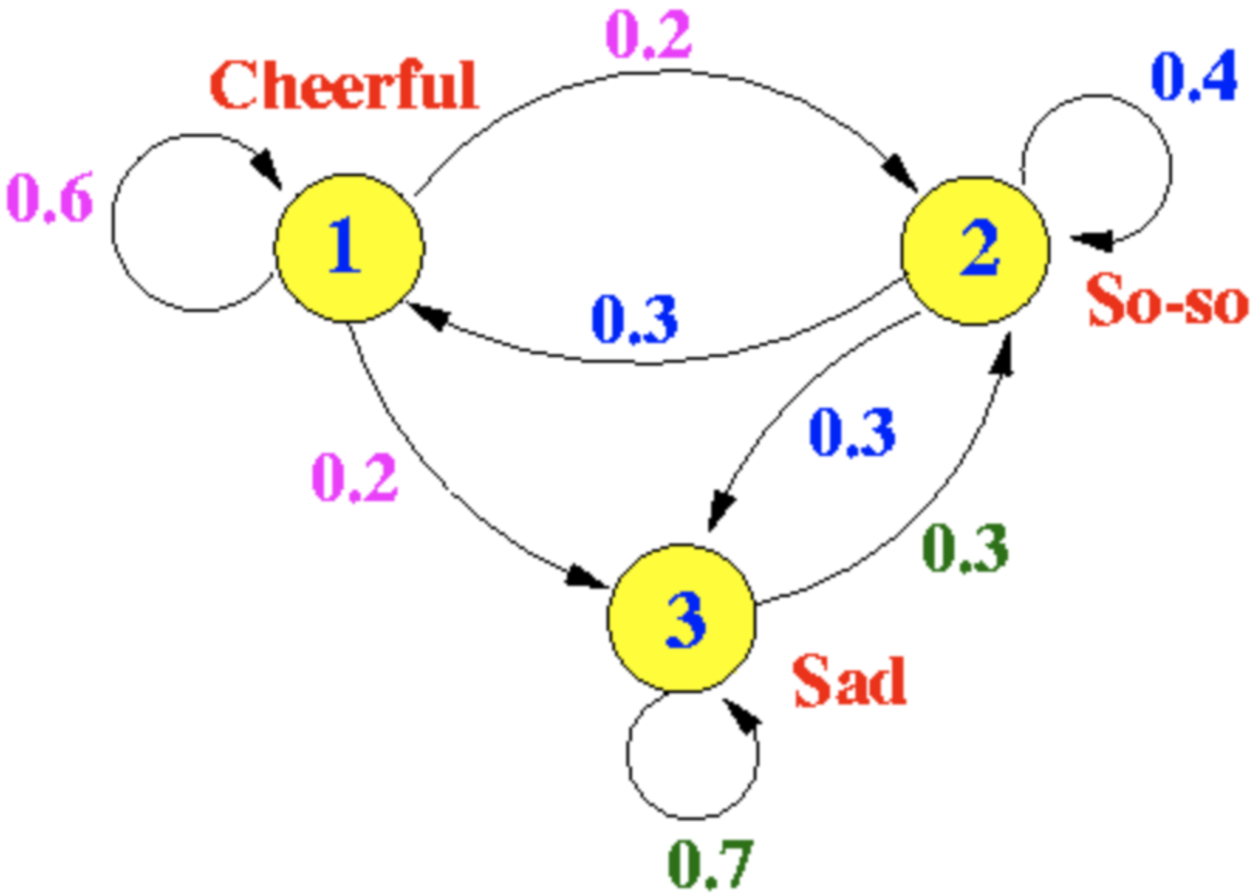
- An **N×N matrix P** is a ***double*** **stochastic matrix** if

- $\forall\, i = 1, 2, \ldots, N$: $\sum_{\{j = 1, 2, \ldots, N\}} P_{ij} = 1$
- and:
- $\forall\, j = 1, 2, \ldots, N$: $\sum_{\{i = 1, 2, \ldots, N\}} P_{ij} = 1$

- (i.e., every **row sum** and every **column sum** are equal to **1**)

Can you find some examples ?
(Stochastic but not double, both double and single etc??)

The following **Markov chain** models the **mode** of a person:



State transition probabilities

```
P[ X(t+1) = Cheerful | X(t) = Cheerful ] =
P[ X(t+1) = So-so | X(t) = Cheerful ]    =
P[ X(t+1) = Sad | X(t) = Cheerful ]      =
```

P11 = 0.6
P12 = 0.2
P13 = 0.2

- **One-step transition matrix:**

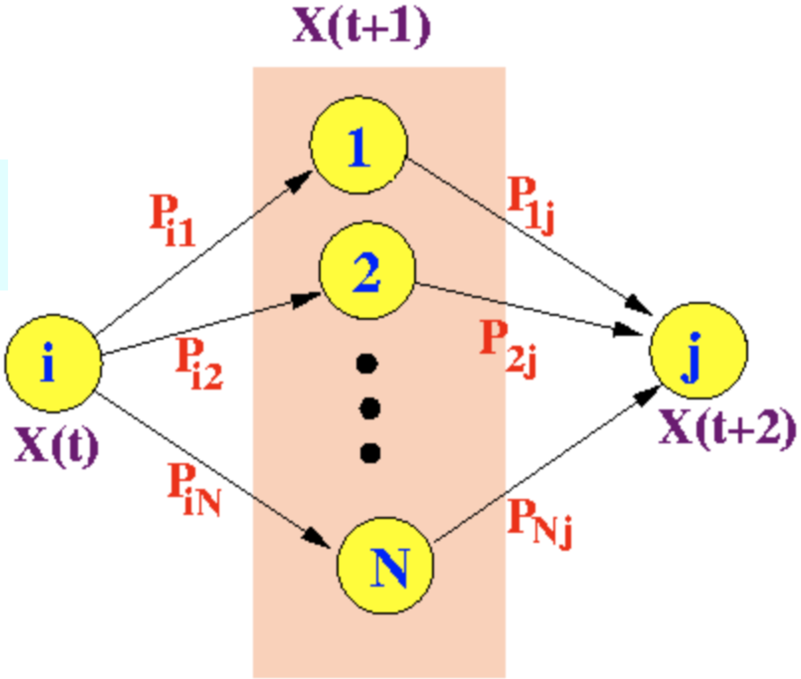
$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}$$

Two-steps transition probability matrix

- **Two steps probability:**
- $P^2_{ij} = P[\, X(t+2) = j \mid X(t) = i \,]$

- **Possible ways to arrive in state j from state i in 2 steps:**
-

$$P^2_{ij} = P_{i1}\times P_{1j} + P_{i2}\times P_{2j} + \ldots + P_{iN}\times P_{Nj}$$



Note: this **formula** is used to **multiply 2 matrices** !!!

Thus …

- **Relationship between the *2-steps* and *one-step* transition probability matrices:**

- $P^2 = P \times P$

- Example –

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}$$

-

$$P^2 = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.42 & 0.26 & 0.32 \\ 0.30 & 0.31 & 0.39 \\ 0.09 & 0.33 & 0.58 \end{bmatrix}$$

N-step …

- **N-steps transition probability matrix**
- The **2-steps transition matrix** can be generalized to an **N-steps process**
- It is easy to show that the **N-steps transition matrix** $\mathbf{P}^N$ is equal to:

- $P^N = P \times P \times \dots \times P$

Example

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0. & 0.3 & 0.7 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.42 & 0.26 & 0.32 \\ 0.30 & 0.31 & 0.39 \\ 0.09 & 0.33 & 0.58 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.330 & 0.284 & 0.386 \\ 0.273 & 0.301 & 0.426 \\ 0.153 & 0.324 & 0.523 \end{bmatrix}$$

$$P^6 = \begin{bmatrix} 0.245490 & 0.304268 & 0.450242 \\ 0.237441 & 0.306157 & 0.456402 \\ 0.218961 & 0.310428 & 0.470611 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.2313863532 & 0.3075488830 & 0.4610647637 \\ 0.2310495033 & 0.3076271722 & 0.4613233244 \\ 0.2302738212 & 0.3078074437 & 0.4619187352 \end{bmatrix}$$

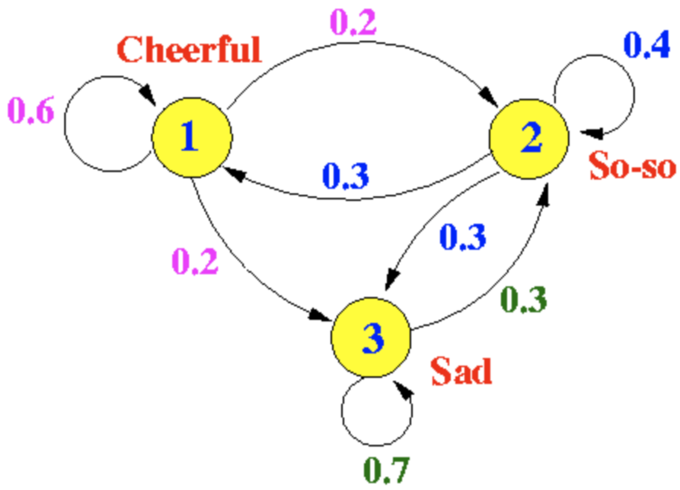
It converges !!!

$$P^{20} = \begin{bmatrix} 0.2307712768 & 0.3076918322 & 0.4615368910 \\ 0.2307701600 & 0.3076920917 & 0.4615377482 \\ 0.2307675883 & 0.3076926893 & 0.4615397223 \end{bmatrix}$$

Two issues -

- Markov Process models some real-life process –
- So there are two issues -
- **Initial State of a Markov Process**
  - How the system starts
- **Progress of a Markov process**
  - How it evolves at each time step

Initial state



- **Define:**
- $\pi^{(0)} = (\pi_1^{(0)}, \pi_2^{(0)}, \ldots, \pi_N^{(0)})$

- The **initial state $\pi^{(0)}$** is the **vector of probability values** that the **Markov chain** is in **state i**

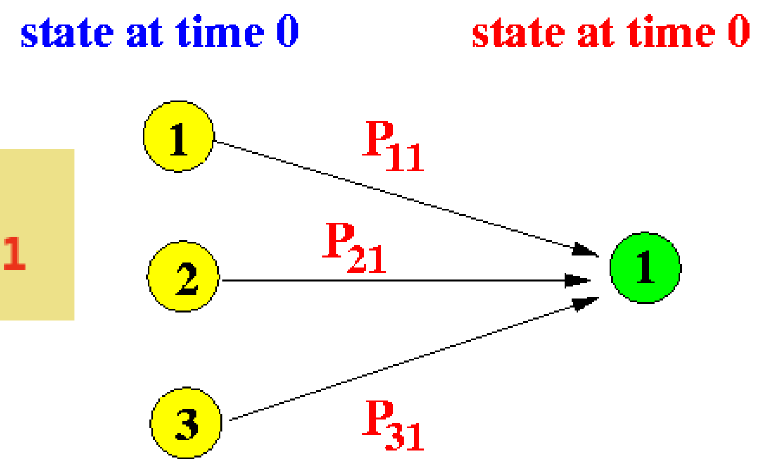
  with **probability $\pi_i^{(0)}$**

- Suppose that **initially** a person **always** starts off in the **"Cheerful" state**
- $\pi^{(0)} = (1, 0, 0)$

Progress of a Markov Process

- **Starting** in the **initial state**, a **Markov process (chain)** will make a **state transition** at each time unit.
- The following shows the **possible ways** to **reach the state 1** after **one step**:
- Therefore, the **probability** that the **Markov chain** is in **state 1** is equal to:

$$\pi_1^{(1)} = \pi_1^{(0)} P_{11} + \pi_2^{(0)} P_{21} + \pi_3^{(0)} P_{31}$$



State Probability Vector

$\boldsymbol{\pi}^{(1)} = (\pi_1^{(1)}, \pi_2^{(1)}, \ldots, \pi_N^{(1)})$ after **one step transition** can be computed **using $\boldsymbol{\pi}^{(0)}$** as follows:

$$\pi_1^{(1)} = \pi_1^{(0)} P_{11} + \pi_2^{(0)} P_{21} + \pi_3^{(0)} P_{31}$$
$$\pi_2^{(1)} = \pi_1^{(0)} P_{12} + \pi_2^{(0)} P_{22} + \pi_3^{(0)} P_{32}$$
$$\pi_3^{(1)} = \pi_1^{(0)} P_{13} + \pi_2^{(0)} P_{23} + \pi_3^{(0)} P_{33}$$

Notice: this is a **vector-matrix multiplication** (rather than a **matrix-vector multiplication**)

in matrix form:

$$\pi^{(1)} = \pi^{(0)} \times P$$

State Probability Vector after k steps

2 steps:

$$\pi_1^{(2)} = \pi_1^{(1)} P_{11} + \pi_2^{(1)} P_{21} + \pi_3^{(1)} P_{31}$$
$$\pi_2^{(2)} = \pi_1^{(1)} P_{12} + \pi_2^{(1)} P_{22} + \pi_3^{(1)} P_{32}$$
$$\pi_3^{(2)} = \pi_1^{(1)} P_{13} + \pi_2^{(1)} P_{23} + \pi_3^{(1)} P_{33}$$

Or:

$$\begin{aligned}\pi^{(2)} &= \pi^{(1)} \times P \\ &= (\pi^{(0)} \times P) \times P \\ &= \pi^{(0)} \times (P \times P) \\ &= \pi^{(0)} \times P^2\end{aligned}$$

In general:

$$\pi^{(k)} = \pi^{(0)} \times P^k$$

Example

- Initial State Probability Vector (SPV) = (1,0,0)
- State Transition Probability Matrix (TPM)
- P = [0.6, 0.2, 0.2]
  [0.3, 0.4, 0.3]
  [0.0, 0.3, 0.7]

So, SPV after 1st time step, after 2 time steps and so on …

$x^{(1)} =$	[0.6,	0.2,	0.2]
$x^{(2)} =$	[0.42,	0.26,	0.32]
$x^{(4)} =$	[0.2832,	0.2954,	0.4214]
$x^{(8)} =$	[0.23490798,	0.30673034,	0.45836168]
$x^{(16)} =$	[0.2307693926,	0.3076922701,	0.4615383374]

It Converges!... -

Stationary State & Transient State

Stationary state

- The **stationary state** is the following **limiting probability**:

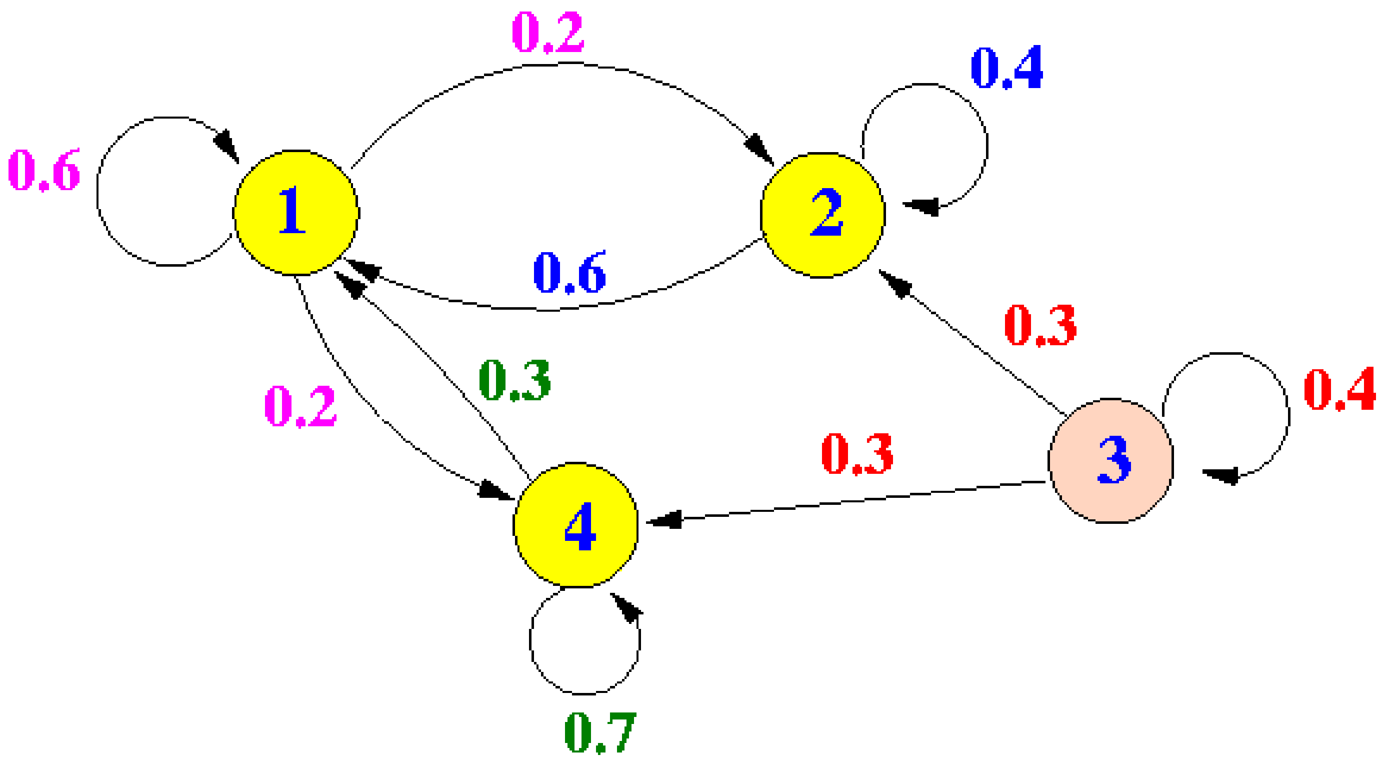
$$\pi^{(\infty)} = \text{Lim}_{(k \to \infty)}\, \pi^{(k)}$$

- The **stationary state** is also called the **steady state**

- (State Probability Vector after long run)

Transient state:

- **Introduced through an example of Markov chain**
- Consider the following **Markov chain**:



The **multi-step probability matrices**:

P

```
[0.6, 0.2, 0.0, 0.2,
 0.6, 0.4, 0.0, 0.0,
 0.0, 0.3, 0.4, 0.3,
 0.3, 0.0, 0.0, 0.7]
```

P2

```
[0.54      0.20      0.        0.26]
[                                  ]
[0.60      0.28      0.        0.12]
[                                  ]
[0.27      0.24      0.16      0.33]
[                                  ]
[0.39      0.06      0.        0.55]
```

P3

```
[0.5130     0.1796       0.         0.3074]
[                                        ]
[0.5388     0.2056       0.         0.2556]
[                                        ]
[0.4617     0.1794       0.0256     0.3333]
[                                        ]
[0.4611     0.1278       0.         0.4111]
```

P5

```
[0.5000282111   0.1666948777         0.          0.3332769112]
[                                                           ]
[0.5000846332   0.1667513000         0           0.3331640668]
[                                                           ]
[                                      -6                   ]
[0.4999993559   0.1666668815   0.4294967296 10   0.3333333333]
[                                                           ]
[0.4999153666   0.1665820333         0.          0.3335025999]
```

P4

```
[0.50167962     0.16834628        0.           0.32997410]
[                                                         ]
[0.50503884     0.17170552        0.           0.32325564]
[                                                         ]
[0.49901697     0.16699434     0.00065536      0.33333333]
[                                                         ]
[0.49496115     0.16162782        0.           0.34341103]
```

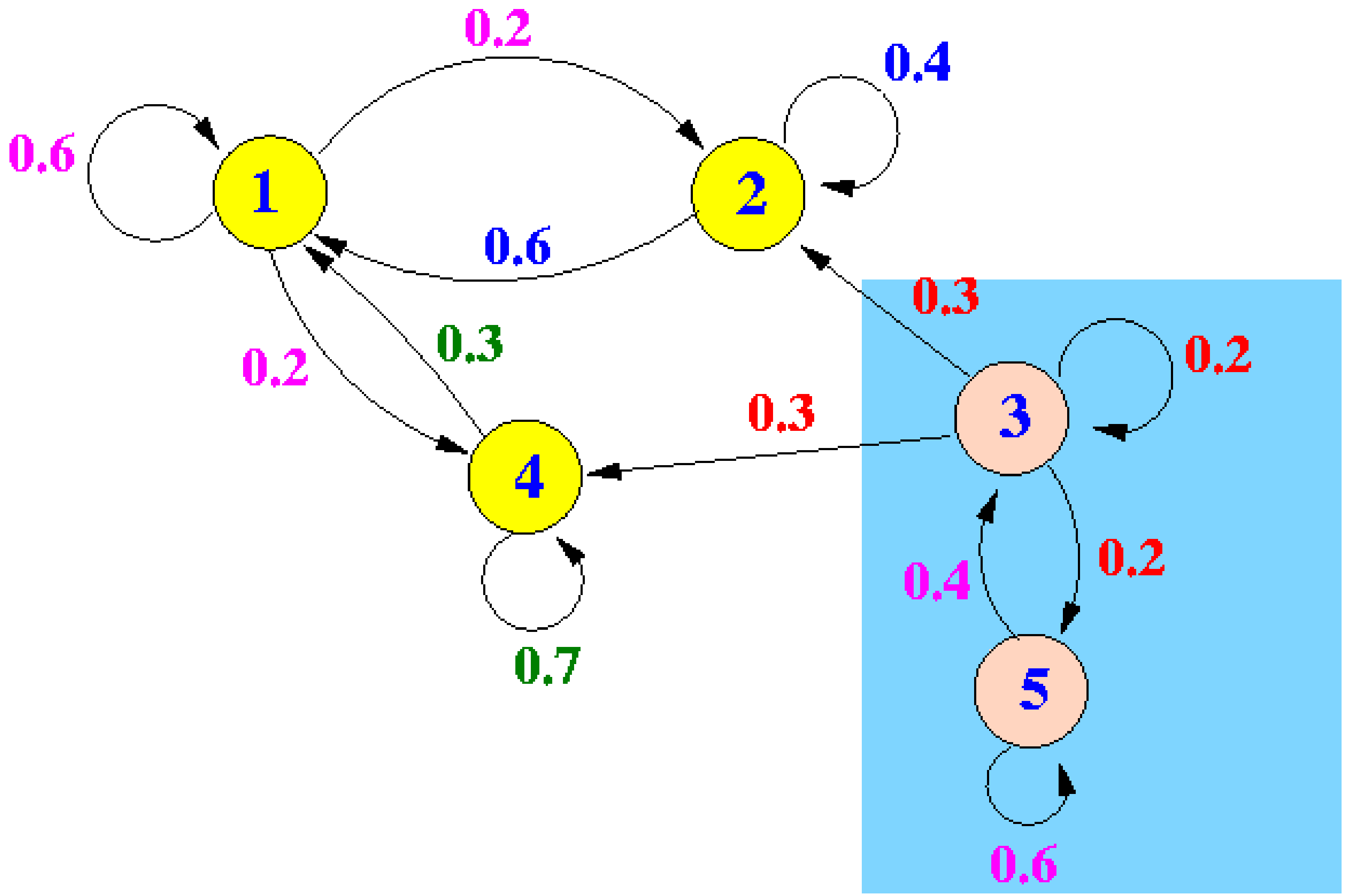
The **probability** that the **Markov chain** is found
in **state 3** becomes **smaller and smaller** with time !!!

Transient and Steady State

- **Transient state:**
- A **transient state** of a **Markov chain** is a state where the **stationary probability** is equal to **zero (0)**

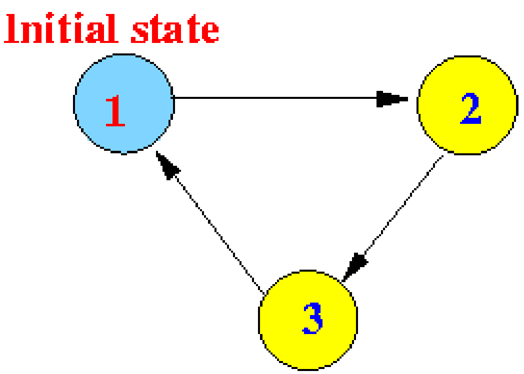
- **Recurrent state:**
- A **recurrent state** of a **Markov chain** is a ***non*-transient state (stationary probability nonzero)**

States **3 and 5** are **transient states** because once the **Markov chain leaves theses states**, it will **never return back** to them.



Periodic states:

- A **periodic state** is a **special subclass** of **recurrent states**
- A periodic state is a **recurrent state say** with a **period d** if it **re-occurs after exactly d steps**
- $p^{(1)}_{ii} = 0$
- $p^{(2)}_{ii} = 0$ ...
- $p^{(d-1)}_{ii} = 0$
- $p^{(d)}_{ii} = 1$
- $p^{(d+1)}_{ii} = 0$ ....



The **period** of the above **Markov process** is **3**:

$p^{(1)}11 = 0$ (Probability that from state 1, in one step, we reach state 1)
$p^{(2)}11 = 0$ (Probability that from state 1, in two steps, we reach state 1)
$p^{(3)}11 = 1$ (Probability that from state 1, in three steps, we reach state 1)

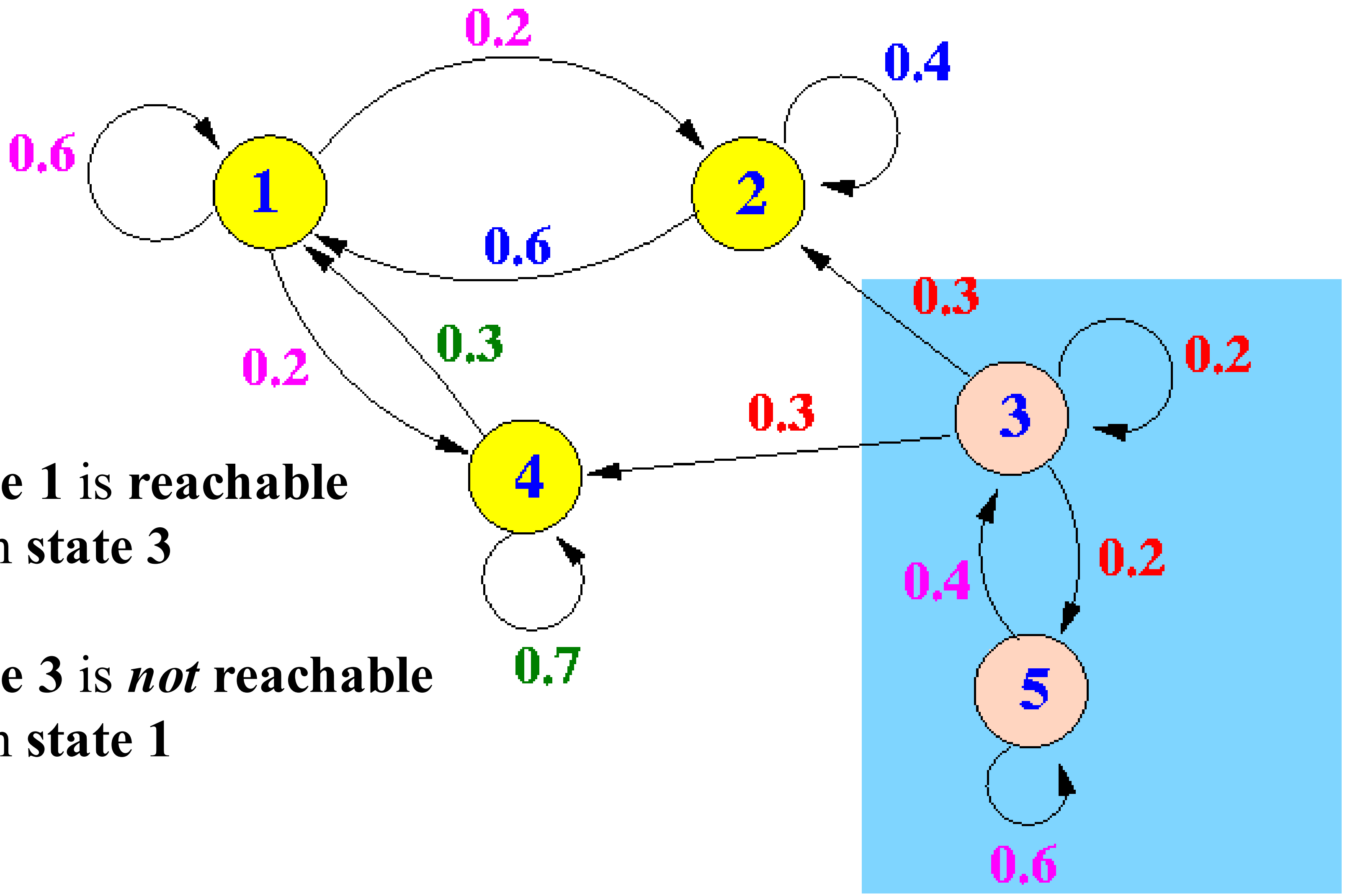
Reachable state:

- A **state j** is **reachable** from a **state i** iff:

- $p^{(n)}_{ij} > 0$ for some $n > 0$

i.e., there is a **non-zero probability** that we **end up in state j starting from state i** after a **finite number of steps n**

Example

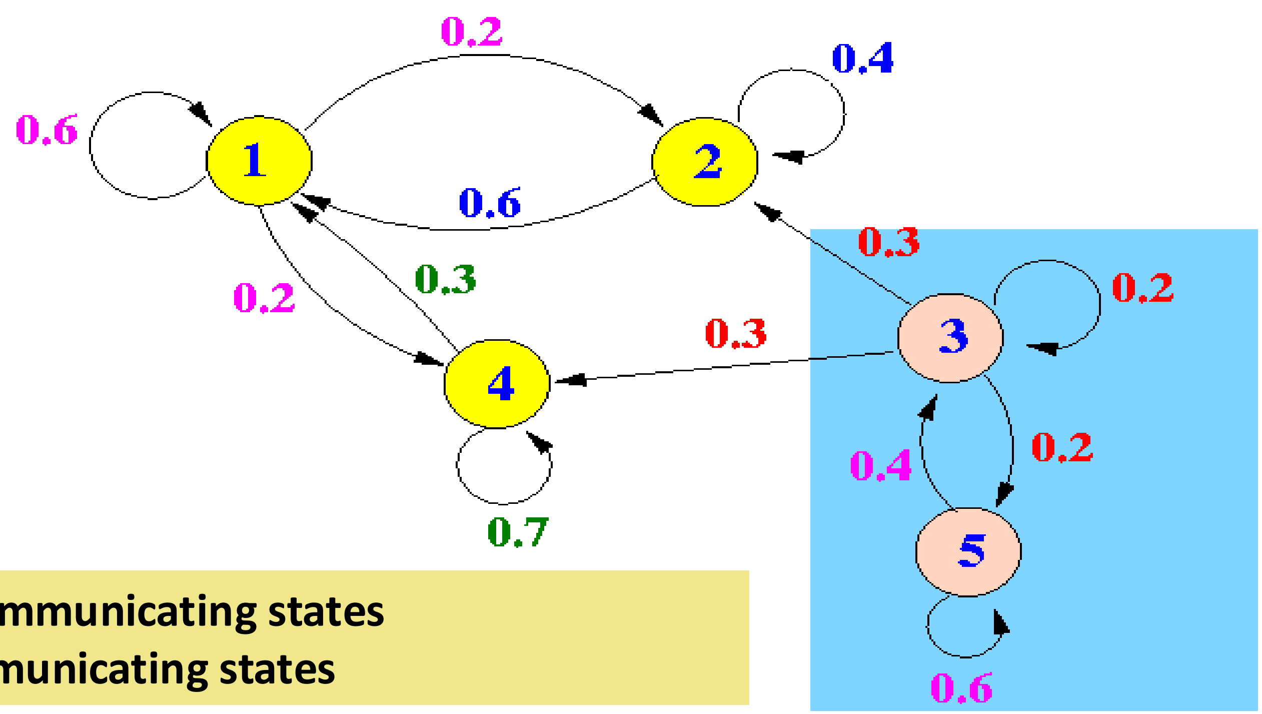


State 1 is **reachable** from **state 3**

State 3 is ***not*** **reachable** from **state 1**

Communicating states:

- The **states i and j** are **communicating states** iff:
- **State j** is **reachable** from **state i, and State i** is **reachable** from **state j**



•**States 1, 2 and 4** are **communicating states**
•**States 3 and 5** are **communicating states**

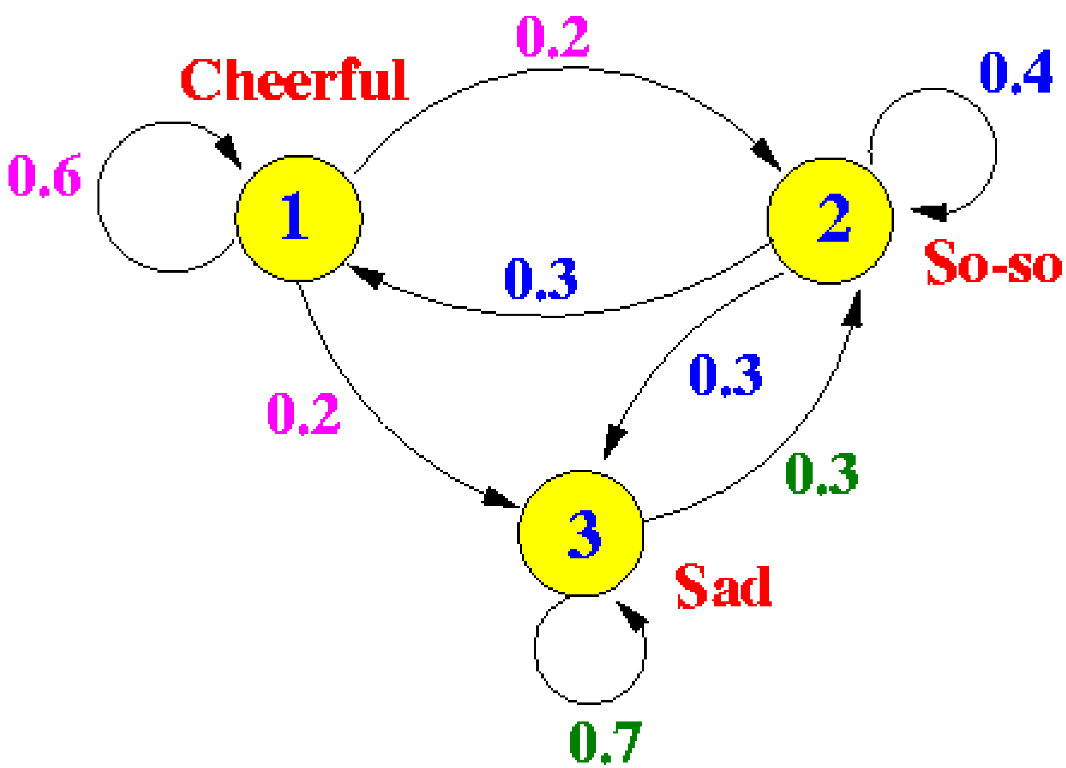
Chain

- A **chain C** is a **set of states** where **all members are mutually communicating**

- $\forall$ i, j $\in$ C:
- i is reachable from j and j is reachable from i

Single chain Markov process:

- A **single chain Markov process** is a **Markov process** that does **not have any *transient* states**



Steady-state property of Single Chain Markov processes

- The **steady state probability** (limiting state probability) of a state is the **likelihood** that the **Markov chain** is in that state **after a long period of time**

- **Mathematically speaking:** we must find this **limit**

$$\lim_{\{n \to \infty\}} \pi^{(n)}_j$$

Lemma 1:

- If a **Markov Chain** has a **single chain** and **no *periodic* states**, then: the **limiting state probabilities exists** and **independent** from the **initial state**

$$\lim_{\{n \to \infty\}} \pi^{(n)}_j = \pi$$

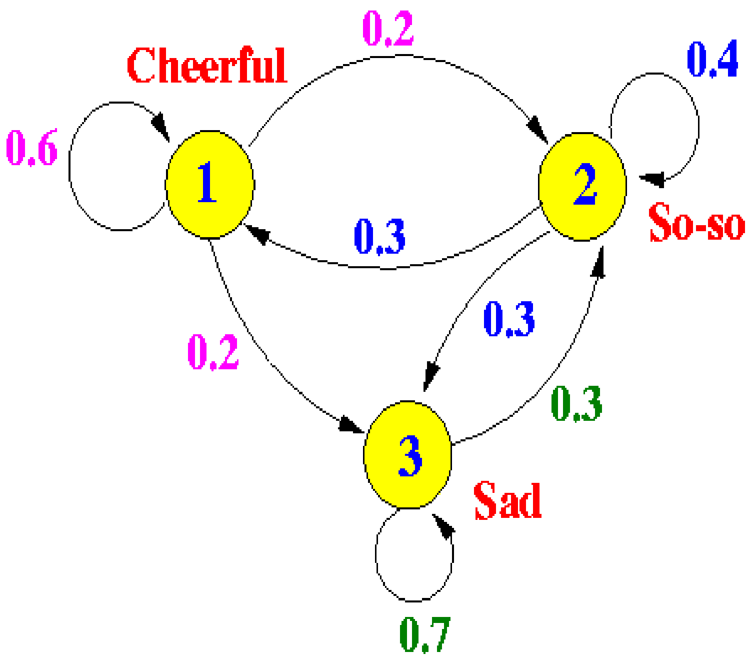
(π is some constant)

independent from the value of $\pi^{(0)}_j$

Computing steady-state probabilities (Limiting state probabilities)

One-step probability matrix:

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}$$



The **example Markov chain** has: one single chain no recurrent states

According to **Lemma 1**, the **Markov chain** has a **steady state** and the **steady state** is reached from **any initial state**

Finding the steady-state probability π

- When **Markov chain** starts in **initial state $\pi^{(0)}$** and make **1 step**:

$$\pi_1^{(1)} = \pi_1^{(0)} P_{11} + \pi_2^{(0)} P_{21} + \pi_3^{(0)} P_{31}$$
$$\pi_2^{(1)} = \pi_1^{(0)} P_{12} + \pi_2^{(0)} P_{22} + \pi_3^{(0)} P_{32}$$
$$\pi_3^{(1)} = \pi_1^{(0)} P_{13} + \pi_2^{(0)} P_{23} + \pi_3^{(0)} P_{33}$$

Or:

$$\pi^{(1)} = \pi^{(0)} \times P$$

Forming the equations …

- If the **Markov chain** is in the **steady state**, and **makes one step**, the **next state** equal to the **steady state**.
- Therefore, in the **steady state** $\boldsymbol{\pi = (\pi_1, \pi_2, \pi_3)}$, we have:

$$\pi_1 = \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31}$$
$$\pi_2 = \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32}$$
$$\pi_3 = \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33}$$

Or:

$$\pi = \pi^{Tr} \times P \qquad (Tr = \text{transpose})$$

Caveat in solving for π

- The **system of equations** obtained from the **one step transition probability matrix** is a ***dependent*** **system of equations** because **one of the equation** is a **linear combination of the *other* two equations**:

$$
\begin{array}{rl}
 & \pi_1 = \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31} \\
 & \pi_2 = \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32} \\
+ & \pi_3 = \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33}
\end{array}
$$

Solving equations…

$$
\begin{array}{lrcl}
 & \pi_1 & = & \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31} \\
 & \pi_2 & = & \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32} \\
+ & \pi_3 & = & \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33} \\
\hline
\end{array}
$$

$$\pi_1 + \pi_2 + \pi_3 =$$

$$\pi_1(P_{11} + P_{12} + P_{13}) + \pi_2(P_{21} + P_{22} + P_{23}) + \pi_3(P_{31} + P_{32} + P_{33})$$

Facts:

$$P_{11} + P_{12} + P_{13} = 1$$
$$P_{21} + P_{22} + P_{23} = 1$$
$$P_{31} + P_{32} + P_{33} = 1$$

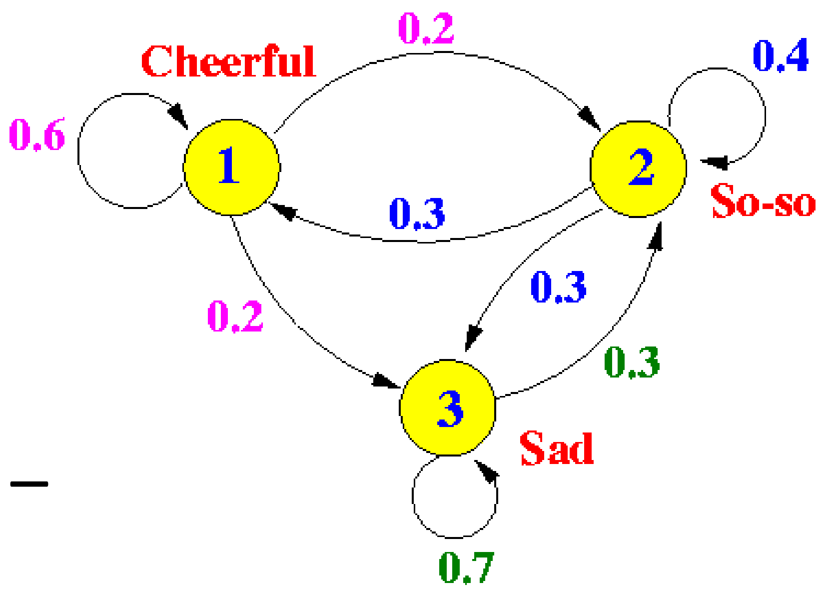
Solving…

- In order to **obtain a *non-dependent* system of equations**, you ***must* replace *any* one equation** in the **system of equations** with this equation:

- So we need to apply $\pi_1 + \pi_2 + \pi_3 = 1$

- And then solve it…

Example -



Steady state probability satisfies –

$$\pi_1 = 0.6\ \pi_1 + 0.3\ \pi_2 + 0.0\ \pi_3 \quad .... \ (1)$$
$$\pi_2 = 0.2\ \pi_1 + 0.4\ \pi_2 + 0.3\ \pi_3 \quad .... \ (2)$$
$$\pi_3 = 0.2\ \pi_1 + 0.3\ \pi_2 + 0.7\ \pi_3 \quad .... \ (3)$$

And:

$$\pi_1 + \pi_2 + \pi_2 = 1 \quad .... \ (4)$$

Solution…

Drop equation (3) and re-write into canonical form:

$$
\begin{aligned}
-0.4\,\pi_1 + 0.3\,\pi_2 + 0.0\,\pi_3 &= 0 \quad \text{.... (1)}\\
0.2\,\pi_1 - 0.6\,\pi_2 + 0.3\,\pi_3 &= 0 \quad \text{.... (2)}\\
\pi_1 + \pi_2 + \pi_3 &= 1 \quad \text{.... (4)}
\end{aligned}
$$

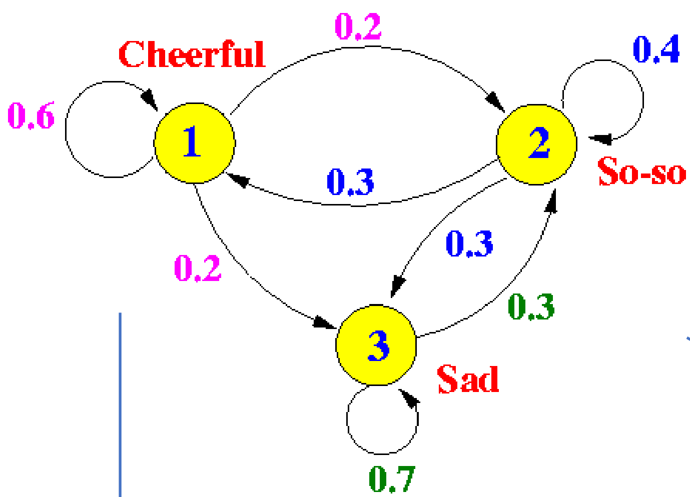
Solving the equations—

$\pi 1 = 0.2307692308$

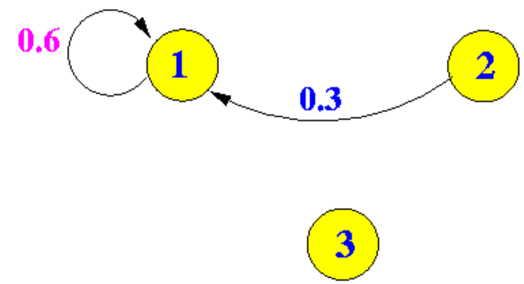
$\pi\ 2 = 0.3076923077$

$\pi\ 3 = 0.4615384615$

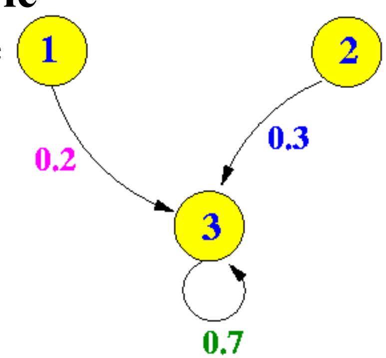
A common way to find *equilibrium equations* for Markov chains



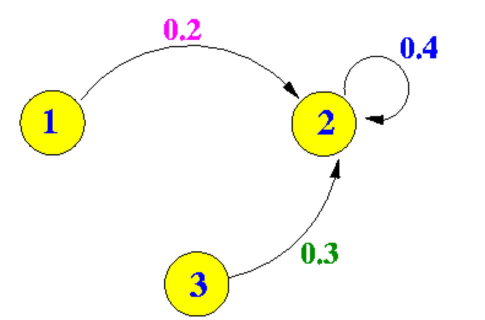
Focussing **only** on **the possible ways to** ***get to*** the **state 1**, we see:



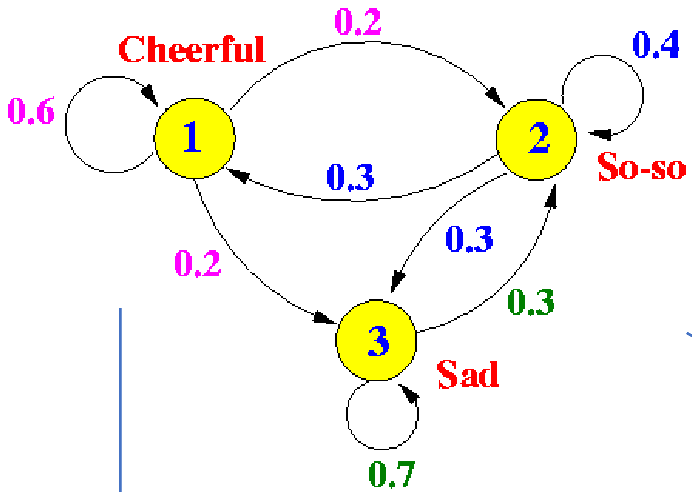
Focussing **only** on **the possible ways to** ***get to*** the **state 3**, we see:



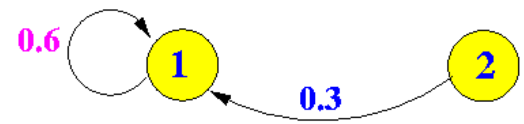
Focussing **only** on **the possible ways to** ***get to*** the **state 2**, we see:



A common way to find *equilibrium equations* for Markov chains

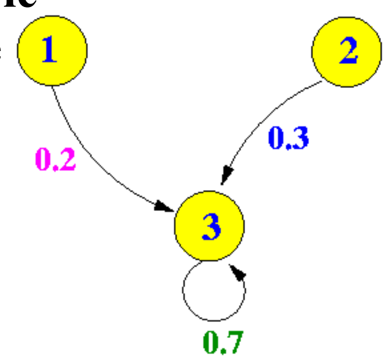


Focussing **only** on **the possible ways to *get to*** the **state 1**, we see:

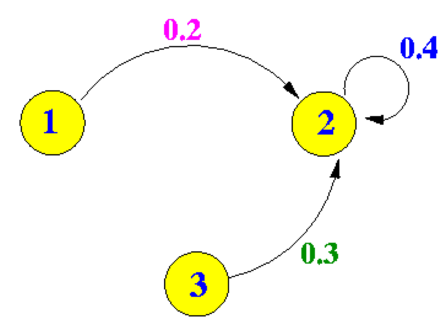


$$\pi_1 = 0.6 \times \pi_1 + 0.3 \times \pi_2 + 0.0 \times \pi_3$$

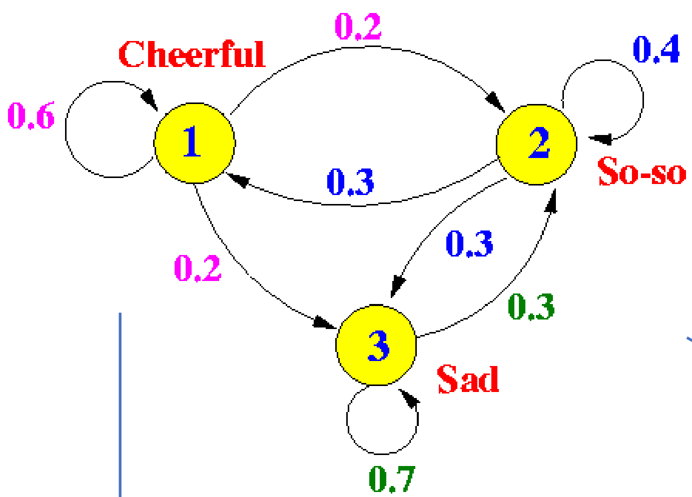
Focussing **only** on **the possible ways to *get to*** the **state 3**, we see:



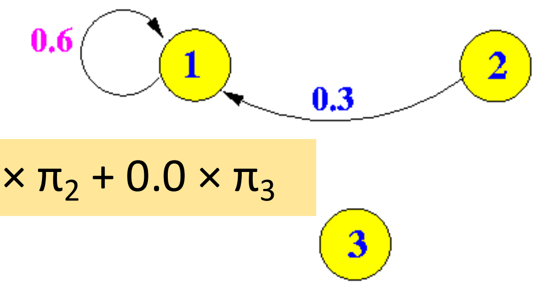
Focussing **only** on **the possible ways to *get to*** the **state 2**, we see:



A common way to find *equilibrium equations* for Markov chains

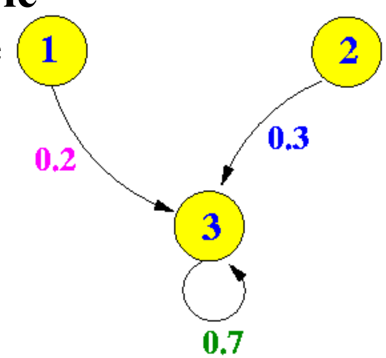


Focussing **only** on **the possible ways to** ***get to*** the **state 1**, we see:

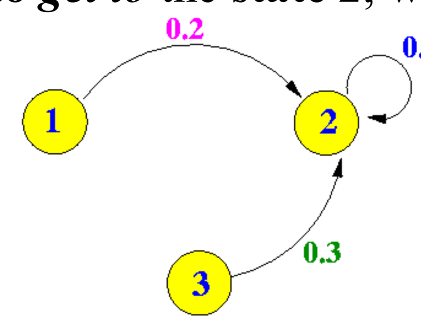


$$\pi_1 = 0.6 \times \pi_1 + 0.3 \times \pi_2 + 0.0 \times \pi_3$$

Focussing **only** on **the possible ways to** ***get to*** the **state 3**, we see:

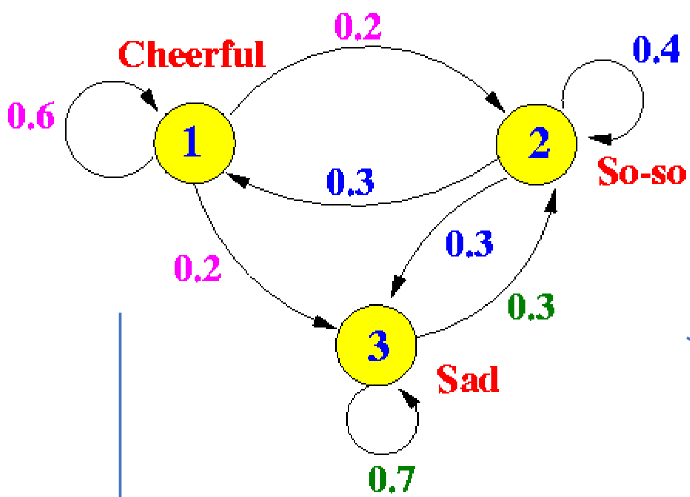


Focussing **only** on **the possible ways to** ***get to*** the **state 2**, we see:

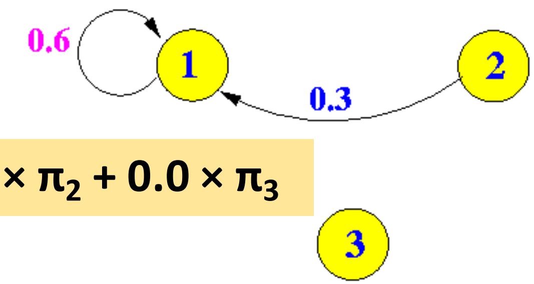


$$\pi_2 = 0.2 \times \pi_1 + 0.4 \times \pi_2 + 0.3 \times \pi_3$$

A common way to find *equilibrium equations* for Markov chains

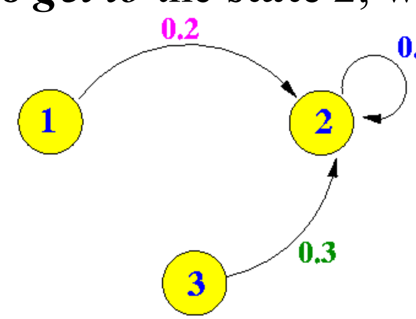


Focussing **only** on **the possible ways to** ***get to*** the **state 1**, we see:



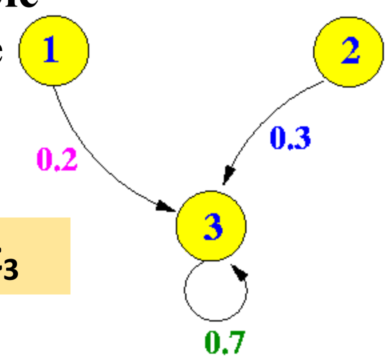
$$\pi_1 = 0.6 \times \pi_1 + 0.3 \times \pi_2 + 0.0 \times \pi_3$$

Focussing **only** on **the possible ways to** ***get to*** the **state 2**, we see:



$$\pi_2 = 0.2 \times \pi_1 + 0.4 \times \pi_2 + 0.3 \times \pi_3$$

Focussing **only** on **the possible ways to** ***get to*** the **state 3**, we see:



$$\pi_3 = 0.2 \times \pi_1 + 0.3 \times \pi_2 + 0.7 \times \pi_3$$

Complicated situation

- In **more complicated Markov chains**, we may need to use **multiple transitions** to establish **equilibrium equations**