Applied Graph Theory (Jan'25-May'25)
Problem Sheet-2 on Colorings

Notations :

- $\chi(G)$ denotes the chromatic number of G.
- $\omega(G)$ and $\alpha(G)$ denote respectively the maximum sizes of an induced complete subgraph and an independent set in G.
- $\overline{G}$ denotes the complement of G. $G \times H$ denotes the cartesian product of G and H, defined as $(V(G) \times V(H), \{(uv, u'v') : (u = u' \wedge vv' \in E(H)) \vee (v = v' \wedge uu' \in E(G))\})$.
- Join of two vertex disjoint graphs G and H, denoted by $G \vee H$, is the graph $(V(G) \cup V(H), E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\})$.
- $\chi(G; k)$ denotes the number of k-colorings of G.
- Given G and H, their union is the graph $G \cup H = (V_G \cup V_H, E_G \cup E_H)$.
- Given G and H, their intersection is the graph $G \cap H = (V_G \cap V_H, E_G \cap E_H)$.

Problems :

1. Suppose that $\chi(G) = \omega(G) + 1$ as in an odd cycle C_{2k+1}. Let $H_1 = G$ and define $H_k = H_{k-1} \vee G$ where $H \vee K$ denotes the join of H and K. for $k > 1$. Prove that $\chi(H_k) = \omega(H_k) + k$ for each k.
2. Determine the chromatic number of $\chi((K_4 \setminus \{e\}) \times C_5 \times P_{1000})$ where $G \times H$ denotes the cartesian product of G and H.
3. Prove or refute : Every k-chromatic graph G ($\chi(G) = k$) has a proper k-coloring in which some color class is a maximum independent set.
4. Let G be a graph whose odd cycles are pairwise intersecting, that is, every two odd cycles in G share a common vertex. Prove that $\chi(G) \leq 5$.
5. Suppose every edge of a graph G appear in at most one cycle. Prove that $\chi(G) \leq 3$.

6. Given a set of lines in the plane with no three of them meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$.

7. Given non-empty finite sets $S_1, \ldots, S_m$, let $U = S_1 \times \ldots \times S_m$. Define a graph $G = (U, E)$ where $u, v \in U$ are adjacent *if and only if* u and v differ in *every* coordinate. Determine $\chi(G)$. Also, determine $\chi(H)$ where H denotes the complement of G.

8. Prove that $\chi(G)\chi(\overline{G}) \geq n(G)$. Use this to prove that $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n(G)}$. Here, $\overline{G}$ denotes the complement of G and $n(G)$ denotes the number of vertices in G.

9. Prove that every k-chromatic graph has at least $\binom{k}{2}$ edges. Use this to prove that if G is the union $(V_1 \cup \ldots \cup V_m, E_1 \cup \ldots \cup E_m)$ of m complete graphs $G_i = (V_i, E_i)$ of order $(|V_i|)$ m each, then $\chi(G) \leq 1 + m\sqrt{m-1}$.

10. Prove that a graph G is m-colorable *if and only if* $\alpha(G \times K_m) \geq n(G)$.

11. Prove that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$ for any G. Use induction on $n(G)$.

12. Prove that $\chi(G) = \omega(G)$ when $\overline{G}$ is bipartite.

13. A flat circular city of radius four miles will get 18 cellular-phone transmitters. Each station can transmit to others within a radius of three miles. Prove that no matter where the transmitters are placed in the city, at least two of them will each be able to transmit to at least five others.

14. Let G and H be graphs possibly sharing vertices and edges.

    (a) Prove that $\chi(G \cup H; k) = \frac{\chi(G;k) \cdot \chi(H;k)}{\chi(G \cap H;k)}$, provided $G \cap H$ is complete.

    (b) Consider two internally vertex disjoint paths $P(u, v)$ and $Q(u, v)$ to show that formula above will fail when $G \cap H$ is not complete.