Case II :- Roots are real and equal:

say $m = m_1, m_1$

Then $CF = (c_1 + c_2 x) e^{m_1 x}$

Expn :- Then the DE becomes

$$(D - m_1)^2 y = 0$$

$$\Rightarrow (D - m_1)\big((D - m_1) y\big) = 0 \quad \text{——— (1)}$$

Let $(D - m_1) y = u$ ——— (2)

Then eqn (1) becomes

$$(D - m_1) u = 0$$

$$\Rightarrow \frac{du}{dx} - m_1 u = 0$$

$$\Rightarrow \frac{du}{u} = m_1 dx$$

Integrate

$$\log u = m_1 x + \log c_2$$

$$\Rightarrow \quad \frac{u}{c_2} = e^{m_1 x}$$

$$\Rightarrow \quad u = c_2 e^{m_1 x}$$

Put the value of u in eqⁿ ②

$$\therefore (D - m_1) y = c_2 e^{m_1 x}$$

$$\Rightarrow \quad \frac{dy}{dx} - m_1 y = c_2 e^{m_1 x}$$

which is of the form $\frac{dy}{dx} + Py = Q$

$$IF = e^{\int P dx} = e^{\int -m_1 dx} = e^{-m_1 x}$$

$\therefore$ The required soln is

$$y\,(IF) = \int Q\,(IF)\,dx + \text{constant}$$

$$\therefore \quad y\,e^{-m_1 x} = \int \left(c_2\, e^{m_1 x}\right) e^{-m_1 x}\, dx + c_1$$

$$\therefore \quad y\,e^{-m_1 x} = c_2 x + c_1$$

$$\therefore \quad y = (c_1 + c_2 x)\, e^{m_1 x}$$

If three roots are equal

say $m = m_1, m_1, m_1$

Then $$CF = (c_1 + c_2 x + c_3 x^2)\, e^{m_1 x}$$

$\therefore$ In general

If $m = m_1, m_1, m_1 \cdots m_1$ (n times)

Then $CF = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{m_1 x}$.

Ex: Solve $(D^2 - 2D + 1) y = 0$ $\left(D = \frac{d}{dx}\right)$

Soln: The AE is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow (m-1)^2 = 0$$

$$\Rightarrow m = 1, 1$$

$$\therefore y = CF = (c_1 + c_2 x) e^x$$

$$\therefore y = (c_1 + c_2 x) e^x$$

is the reqd soln.

Ex: Solve $((D^2-5D+6)(D^2-4D+4))y=0$ $\left(D=\frac{d}{dx}\right)$

Soln: The AE is

$$(m^2-5m+6)(m^2-4m+4)=0$$

$$\Rightarrow (m-2)(m-3)(m-2)^2=0$$

$$\Rightarrow m=2,2,2,3$$

$$\therefore y=CF=(c_1+c_2x+c_3x^2)e^{2x}+c_4e^{3x}$$

Case III:- Roots are imaginary.

say $m=\alpha\pm i\beta$

$$CF=e^{(RP)x}\{c_1\cos(IP)x+c_2\sin(IP)x\}$$

$$CF = e^{\alpha x}\{c_1 \cos\beta x + c_2 \sin\beta x\}$$

Ex: $(D^2 + 2D + 2)y = 0 \quad \left(D = \frac{d}{dx}\right)$

Soln: The AE is

$$m^2 + 2m + 2 = 0$$

$$\Rightarrow m^2 + 2m + 1 = -1$$

$$\Rightarrow (m+1)^2 = i^2$$

$$\Rightarrow m + 1 = \pm i$$

$$\Rightarrow m = -1 \pm i$$

$$y = CF = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$$= c_1(e^{-x}\cos x) + c_2(e^{-x}\sin x)$$

Ex: $(D^2+2D+2)^2 y = 0$ $\left(D = \frac{d}{dx}\right)$

Soln:- The AE is

$$(m^2+2m+2)^2 = 0$$

$\Rightarrow$ $m^2+2m+2 = 0$ (Twice)

$\Rightarrow$ $m^2+2m+1 = -1$ (Twice)

$\Rightarrow$ $(m+1)^2 = i^2$ (Twice)

$\Rightarrow$ $m+1 = \pm i$ (Twice)

$\Rightarrow$ $m = -1 \pm i$ (Twice)

$$\therefore y = CF = e^{-x}\left((c_1+c_2x)\cos x + (c_3+c_4x)\sin x\right)$$

Ex: Find a diff'tal eq^n whose one root is $x^2 \sin x$ & also find order of D.E.

Sol^n :-

$$CF = \{ (c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x \}$$

$\therefore$ order of diff'tal eq^n = no. of arbitrary constant $= 6$.

Note :- ① $m = \alpha \pm \sqrt{\beta}$

Then

$$CF = c_1 e^{(\alpha+\sqrt{\beta})x} + c_2 e^{(\alpha-\sqrt{\beta})x}$$

OR $$CF = e^{\alpha x} (c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x)$$

Expn:

$$\cosh x = \frac{e^{x}+e^{-x}}{2}, \quad \sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$CF = e^{\alpha x}\left(c_1 \cosh\sqrt{\beta}\,x + c_2 \sinh\sqrt{\beta}\,x\right)$$

$$= e^{\alpha x}\left\{c_1\left(\frac{e^{\sqrt{\beta}x}+e^{-\sqrt{\beta}x}}{2}\right)+c_2\left(\frac{e^{\sqrt{\beta}x}-e^{-\sqrt{\beta}x}}{2}\right)\right\}$$

$$= e^{\alpha x}\left\{e^{\sqrt{\beta}x}\left(\frac{c_1}{2}+\frac{c_2}{2}\right)+e^{-\sqrt{\beta}x}\left(\frac{c_1}{2}-\frac{c_2}{2}\right)\right\}$$

$$= A e^{(\alpha+\sqrt{\beta})x} + B e^{(\alpha-\sqrt{\beta})x}$$

Ex: $(D^2 + 2D - 2)y = 0$

Sol: The AE is

$$m^2 + 2m - 2 = 0$$

$\Rightarrow$ $m^2 + 2m + 1 = 3$

$\Rightarrow$ $(m+1)^2 = 3$

$\Rightarrow$ $m+1 = \pm\sqrt{3}$

$\Rightarrow$ $m = -1 \pm \sqrt{3}$

$\therefore$ $y = CF = e^{-x}\{c_1 \cosh\sqrt{3}x + c_2 \sinh\sqrt{3}x\}$

OR $y = CF = c_1 e^{(-1+\sqrt{3})x} + c_2 e^{(-1-\sqrt{3})x}$

Particular Integral:

The eqⁿ ① can be written as

$$F(D)\,y = X$$

Prop I:- when X is of the form e^{ax} provided $F(a) \neq 0$

Then
$$PI = \frac{1}{F(D)} X$$
$$= \frac{1}{F(D)} e^{ax}$$
$$= \frac{1}{F(a)} e^{ax} \quad (F(a) \neq 0)$$

(Replace D by a provided $F(a) \neq 0$)

Expn :-

$D(e^{ax}) = a(e^{ax})$

$D^2(e^{ax}) = a^2(e^{ax})$

- - . . . - -

- . - - - -

$D^n(e^{ax}) = a^n(e^{ax})$

$\therefore F(D)\, e^{ax} = F(a)\, e^{ax}$

$\frac{1}{F(D)} F(D)\, e^{ax} = \frac{1}{F(D)} F(a)\, e^{ax}$

$e^{ax} = F(a) \frac{1}{F(D)} e^{ax}$

$\Rightarrow \frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}$ provided $F(a) \neq 0$

$D = \frac{d}{dx}$

$\frac{1}{D} = \int dx$

$D e^{ax} = a e^{ax}$

$\frac{1}{D} D e^{ax} = \frac{1}{D} a e^{ax}$

$e^{ax} = a \frac{1}{D} e^{ax}$

$\frac{1}{D} e^{ax} = \frac{1}{a} e^{ax}$

Ex:- $(D^2-5D+6)y = e^{5x}$ $\left(D = \frac{d}{dx}\right)$

Soln:- The AE is

$$m^2-5m+6=0$$

$$\Rightarrow (m-2)(m-3)=0$$

$$\Rightarrow m = 2, 3$$

$$\therefore CF = c_1 e^{2x} + c_2 e^{3x}$$

& $PI = \frac{1}{(D^2-5D+6)} e^{5x}$

$$= \frac{1}{25-25+6} e^{5x} = \frac{1}{6} e^{5x}$$

$$\therefore y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6} e^{5x}.$$

Prop IV :- when X is of the form e^{ax} & $F(a) = 0$

Then $PI = \frac{1}{F(D)} X$

$$= \frac{1}{F(D)} e^{ax}$$

$$= \frac{1}{(D-a)^r} e^{ax}$$

$$= \frac{x^r}{r!} e^{ax}$$

Ex: $(D^2 - 4D + 4)y = e^{2x}$ $\left(D = \frac{d}{dx}\right)$

Soln: The AE is

$$m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

$$\therefore CF = (c_1 + c_2 x)\, e^{2x}$$

& $$PI = \frac{1}{(D^2 - 4D + 4)} e^{2x}$$

$$= \frac{1}{(D-2)^2} e^{2x}$$

$$= \frac{x^2}{2!} e^{2x}$$

$$= \frac{1}{2} x^2 e^{2x}$$

$$\therefore y = (c_1 + c_2 x)\, e^{2x} + \frac{1}{2} x^2 e^{2x}.$$

OR

$$PI = \frac{1}{D^2 - 4D + 4} e^{2x}$$

$$= x \cdot \frac{1}{2D - 4} e^{2x}$$

$$= x^2 \frac{1}{2} e^{2x}$$

$$= \frac{x^2}{2!} e^{2x}$$

Ex:- $(D^2 - 5D + 6)y = e^{3x}$ $\left(D = \frac{d}{dx}\right)$

Soln: $CF = c_1 e^{2x} + c_2 e^{3x}$.

$$PI = \frac{1}{D^2 - 5D + 6} e^{3x}$$

$$= x \cdot \frac{1}{2D - 5} e^{3x}$$

$$= x \frac{1}{2 \cdot 3 - 5} e^{3x}$$

$$= x e^{3x}$$

OR

$$PI = \frac{1}{(D-2)(D-3)} e^{3x}$$

$$= \frac{1}{(D-3)} \left\{ \frac{1}{(D-2)} e^{3x} \right\}$$

$$= \frac{1}{(D-3)} \left(\frac{1}{3-2} e^{3x} \right)$$

$$= \frac{1}{(D-3)} e^{3x}$$

$$= \frac{x^1}{1!} e^{3x} = x e^{3x}$$

Prop III :- when X is of the form $\sin ax$ or $\cos ax$ provided $F(-a^2) \neq 0$

$$PI = \frac{1}{F(D)} X$$

$$= \frac{1}{F(D^2)} (\sin ax \text{ or } \cos ax)$$

$$= \frac{1}{F(-a^2)} (\sin ax \text{ or } \cos ax)$$

(Replace D^2 by $-a^2$, D^4 by a^4

D^6 by $-a^6$ ....)

$D(\sin ax)$

$= a(\cos ax)$

$D^2(\sin ax)$

$= -a^2(\sin ax)$

Ex:- $(D^2 - 2D + 3)y = \sin x \quad \left(D = \frac{d}{dx}\right)$

$$PI = \frac{1}{(D^2 - 2D + 3)} \sin x$$

$$= \frac{1}{-1^2 - 2D + 3} \sin x$$

$$= \frac{1}{2 - 2D} \sin x$$

$$= \frac{1}{2} \frac{1}{(1 - D)} \sin x$$

$$= \frac{1}{2} \cdot \frac{(1 + D)}{(1 - D^2)} \sin x$$

$$= \frac{1}{2} \frac{(1 + D) \sin x}{(1 - (-1^2))}$$

$$= \frac{1}{4} (\sin x + \cos x)$$

Ex:- $(D^3 + 5D)y = \sin 2x$

$$PI = \frac{1}{D^3 + 5D} \sin 2x$$

$$= \frac{1}{D^2 \cdot D + 5D} \sin 2x$$

$$= \frac{1}{-4\,D + 5D} \sin 2x$$

$$= \frac{1}{D} \sin 2x$$

$$\int \sin 2x \, dx$$

$$= -\frac{\cos 2x}{2} + C$$

$$= \frac{D}{D^2} \sin 2x$$

$$= \frac{D(\sin 2x)}{-4}$$

$$= \frac{2\cos 2x}{-4} = -\frac{1}{2}\cos 2x$$

Prop IV :- when X is of the form of $\sin ax$ or $\cos ax$ provided $F(-a^2)=0$

Ex: $(D^2+a^2)y = \sin ax$

Soln:

$$PI = \frac{1}{D^2+a^2}\sin ax$$

$$= x\frac{1}{2D}\sin ax$$

$$= \frac{x}{2}\int \sin ax\,dx$$

$$= \frac{x}{2}\left(-\frac{\cos ax}{a}\right)$$

$$= -\frac{x}{2a}\cos ax.$$

Ex: $(D^2+a^2)y = \cos ax$

Then

$$PI = \frac{1}{(D^2+a^2)}\cos ax$$

$$= x\frac{1}{2D}\cos ax$$

$$= \frac{x}{2}\int \cos ax\,dx$$

$$= \frac{x}{2}\left(\frac{\sin ax}{a}\right)$$

$$= \frac{x}{2a}\sin ax$$

Ex: $(D^3 + 4D)y = \sin 2x$

Then

$$PI = \frac{1}{D^3+4D} \sin 2x$$

$$= x \frac{1}{3D^2+4} \sin 2x$$

$$= x \cdot \frac{1}{3(-4)+4} \sin 2x$$

$$= x \cdot \frac{1}{-8} \sin 2x$$

$$= -\frac{x}{8} \sin 2x$$