# Basis

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 14 • Sept 18, 2021

Let $V$ be the F.D. vector-space over the field $F$. Let $L$ be the set of linearly independent vectors in $V$ and let $S$ be the spanning set of $V$.

$$\operatorname{Car}(L) \leq \operatorname{car}(S)$$

Span $S = V$

→ If $S$ is LI $\Rightarrow$ $S$ is a basis

→ If $S$ is not LI $\Rightarrow$ $S$ is LD. (atleast one vector can be written as L.C. of others) (say $v_1$)

→ If $S \setminus \{v_1\}$ is LI $\Rightarrow$ $S \setminus \{v_1\}$ is basis of $V$.

→ If $S \setminus \{v_1\}$ is not LI $\Rightarrow$ $S \setminus \{v_1\}$ is LD.

( L.C. of oth) (say $v_2$ )

→ $S \setminus \{v_1, v_2\}$ is LI → Basis

$\mathbb{R}^2$

$$S = \{ \overset{v_1}{(1,2)} \quad \overset{v_2}{(-1,0)} \quad \overset{v_3}{(0,1)} \}$$ not a basis of $\mathbb{R}^2$

→ $S$ is a spanning set of $\mathbb{R}^2$ but $S$ is not LI.

$$(1,2) = -1(-1,0) + 2(0,1)$$

→ $S \setminus \{(1,2)\} = \{(-1,0), (0,1)\}$ → Basis

$$\text{span}(S \setminus \{v_1\}) = \mathbb{R}^2$$

$S \setminus \{v_1\}$ is LI.

(eq) $\mathbb{R}^2$, $S = \{ \overset{v_1}{(1,2)}, \overset{v_2}{(3,-1)}, \overset{v_3}{(0,2)}, \overset{v_4}{(-1/2,0)} \}$

dim V = 2

→ span S = $\mathbb{R}^2$

→ $(1,2) = 1 \cdot (0,2) + (-2)(-1/2,0)$

→ $S \setminus \{v_1\} = \{ (3,-1), (0,2), (-1/2,0) \}$

$(3,-1) = -\frac{1}{2}(0,2) - 6(-1/2,0)$

→ $\underbrace{S \setminus \{v_1, v_2\}}_{\text{Basis of } \mathbb{R}^2} = \{ (0,2), (-1/2,0) \}$

LI

# ** Let V be a F.D.V.S. with $\dim V = n$

Let B be a subset of V. of n no. of vectors

If $\operatorname{span} B = V \Rightarrow$ B is a basis of V.

OR

If B is LI set of vectors $\Rightarrow$ B is a basis of V

$S = \{ u_1, u_2 \ldots u_n \}$

$v = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$

$\text{span } S = V$

$c_1, c_2, c_n$

$!!?$

$\Rightarrow \mathbb{R}^3 = V \qquad \dim V = 3$

Basis

$S = \{ (1,-1,0) \quad (0,-1,1) \quad (1,0,3) \} \rightarrow 1$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 = c_2 = 0 = c_3$

$\Rightarrow$ Let V be a F.D.V.S. with $\dim V = n$.

Any set with $(n+1)$ no. of elements (or more) is always linearly dependent. (Hence can't form basis)

$\dim V = n$

$S \subseteq V$

Step (1) no. of elements in S.

If $C(S) = n$

Step (2)

Check either LI or span

$C(S) > n$

S is LD

S is not a basis

$C(S) < n$

span $S \neq V$

S is not a basis

(eg) $V = \mathbb{R}^2$ $S = \{(-1,0)\}$ → not a basis

$l(S) = 1 < \dim V = 2$

S is L.I ✓

spans $\mathbb{R}^2$

$(x,y) = c_1(-1,0)$

$(x,y) = (-c_1, 0)$

$x = -c$

$\underline{c = -x}$ $y = 0$

$(1,2) = c_1 \; (-1,0)$

$(x,0) = c_1(-1,0)$

$= -x(-1,0)$

# Let $V$ be a F.D.V.S. with $\dim V = n$.

Let $W$ be a sub-space of $V$.

$$\boxed{\dim W \leq \dim V}$$

$V = \mathbb{R}^2$

$W_1 = \{ (x,0) \in \mathbb{R}^2 \mid x \in \mathbb{R} \}$

$S = \{ (1,0) \}$ LI ✓

$(x,0) = x \cdot (1,0)$ span $S = W_1$

Basis of $W_1$

$\dim W_1 = 1 < 2 = \dim V$

(Notes)

(1) $V = \{0\}$

$\text{span}\ \phi = \{0\} = V$

$\phi$ is LI.

(2) $V = \mathbb{R}^3$ $\quad W < V$ $\quad \dim W = 3, 2, 1, 0$

$\dim V = 3$ $\quad\quad W = V$

$W_1 = \{ (x, y, 0) \mid x, y \in \mathbb{R} \}$ $\quad \dim W_1 = 2$

representing xy plane ( $z = 0$ )

$W_2 = \{ (x, 0, 0) \mid x \in \mathbb{R} \}$ represents X- axis

$\dim W_2 = 1$

$\checkmark\ W_3 = \{(0,0,0)\}$

$\dim W_3 = 0 \implies$ Basis contains 0 no. of vectors.

$B = \phi$

$$|w|_3 = \{ (1,2,3) \} \longrightarrow \quad \mathrm{Car}(W_3) = 1$$

$$\in \mathbb{R}^3$$

$$|x|_3 = \{ 1, 2, 3 \} \Rightarrow \quad \mathrm{Car}(W_3) = 3$$