Practice Session

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 17 • Sept 21, 2021

LINEAR TRANSFORMATION

Let V and W be two vectors spaces over the same field F. Then $T: V \to W$ is said to be a linear transformation if

(a) $\forall\ u, v \in V$, $T(u+v) = T(u) + T(v)$ --- (1)

(b) $\forall\ \alpha \in F$ and $\forall\ u \in V$, $T(\alpha u) = \alpha T(u)$ --- (2)

OR

$\forall\ \alpha, \beta \in F$ and $\forall\ u, v \in V$.

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

(Note)

(i) from (2), $\forall\ \alpha \in F$ and $\forall\ u \in V$,

$$T(\alpha u) = \alpha T(u)$$

choose $\alpha = 0$, $u = 0$

$$\boxed{T(0) = 0.}$$

(ii) from (2). $\alpha = -1$, $\forall\ u \in V$

$$T(\alpha u) = \alpha T(u)$$

$$\boxed{T(-u) = -T(u)}$$

(3) $T : \mathbb{R} \to \mathbb{R}$

$T(x) = x + 1$

(Note)

$T(0) = 0 + 1 = 1 \neq 0$

Not a LT.

$\forall\, u, v \in \mathbb{R}$ and $\forall\, \alpha, \beta \in F$,

$LHS = T(\alpha u + \beta v)$

$= \alpha u + \beta v + 1$

$RHS = \alpha T(u) + \beta T(v)$

$= \alpha \cdot (u + 1) + \beta (v + 1)$

$= \alpha u + \beta v + (\alpha + \beta)$

This is not a LT.

(2) $T : \mathbb{R} \longrightarrow \mathbb{R}$

$T(x) = |x|$

$\forall\ u, v \in \mathbb{R}$ and $\forall\ \alpha, \beta \in F$

$$LHS = T(\alpha u + \beta v)$$
$$= |\alpha u + \beta v|$$

$$RHS = \alpha T(u) + \beta T(v)$$
$$= \alpha \cdot |u| + \beta |v|$$

$$LHS \neq RHS$$

This is not a linear trans.

$$T(-x) = |-x| = |x| \neq -|x| = -T(x)$$

not a LT.

(1) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(x, y) = (y, x)$ is a linear trans.

$\forall \alpha, \beta \in F$ and $\forall u, v \in \mathbb{R}^2$

$u = (x_1, y_1)$ and $v = (x_2, y_2)$

$$\text{LHS} = T(\alpha u + \beta v) = T(\alpha(x_1, y_1) + \beta(x_2, y_2))$$

$$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$= (\alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2)$$

$$\text{RHS} = \alpha T(u) + \beta T(v) = \alpha T(x_1, y_1) + \beta T(x_2, y_1)$$

$$= \alpha(y_1, x_1) + \beta(y_2, x_2)$$

$$= (\alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2) = \text{LHS}.$$

__(eg)__ $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x_1, y_1) = (1 - x_1, y_1)$$

$$0 \xrightarrow[T]{\times} 0$$

$\to$ $$T(0,0) = (1-0, 0)$$
$$= (1, 0)$$
$$\neq (0, 0)$$

not a LT.

$\to$ $$T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2)$$
$$= (1 - x_1 - x_2, y_1 + y_2)$$

$$T(x_1, y_1) + T(x_2, y_2) = (1 - x_1, y_1) + (1 - x_2, y_2)$$
$$= (2 - x_1 - x_2, y_1 + y_2)$$

(eg) $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x,y) = (\sin x, y)$$

$$T(0,0) = (\sin 0, 0) = (0,0)$$

$$T(-(x,y)) = T(-x,-y) = (\sin(-x), -y)$$

$$= (-\sin x, -y)$$

$$= -(\sin x, y)$$

$$= -T(x,y)$$

can't say

$$T(u+v) = T(x_1+x_2, y_1+y_2)$$
$$= \left(\underbrace{\sin(x_1+x_2)}, \underbrace{y_1+y_2}\right)$$

Not a LT.

$$T(u)+T(v) = T(x_1, y_1) + T(x_2, y_2)$$
$$= (\sin x_1, y_1) + (\sin x_2, y_2)$$
$$= \left(\underbrace{\sin x_1 + \sin x_2}, \underbrace{y_1+y_2}\right)$$

$$\sin(x_1+x_2) \neq \sin(x_1) + \sin(x_2)$$
$$\downarrow$$
$$\sin(x_1)\cos x_2 + \cos x_1 \sin x_2$$

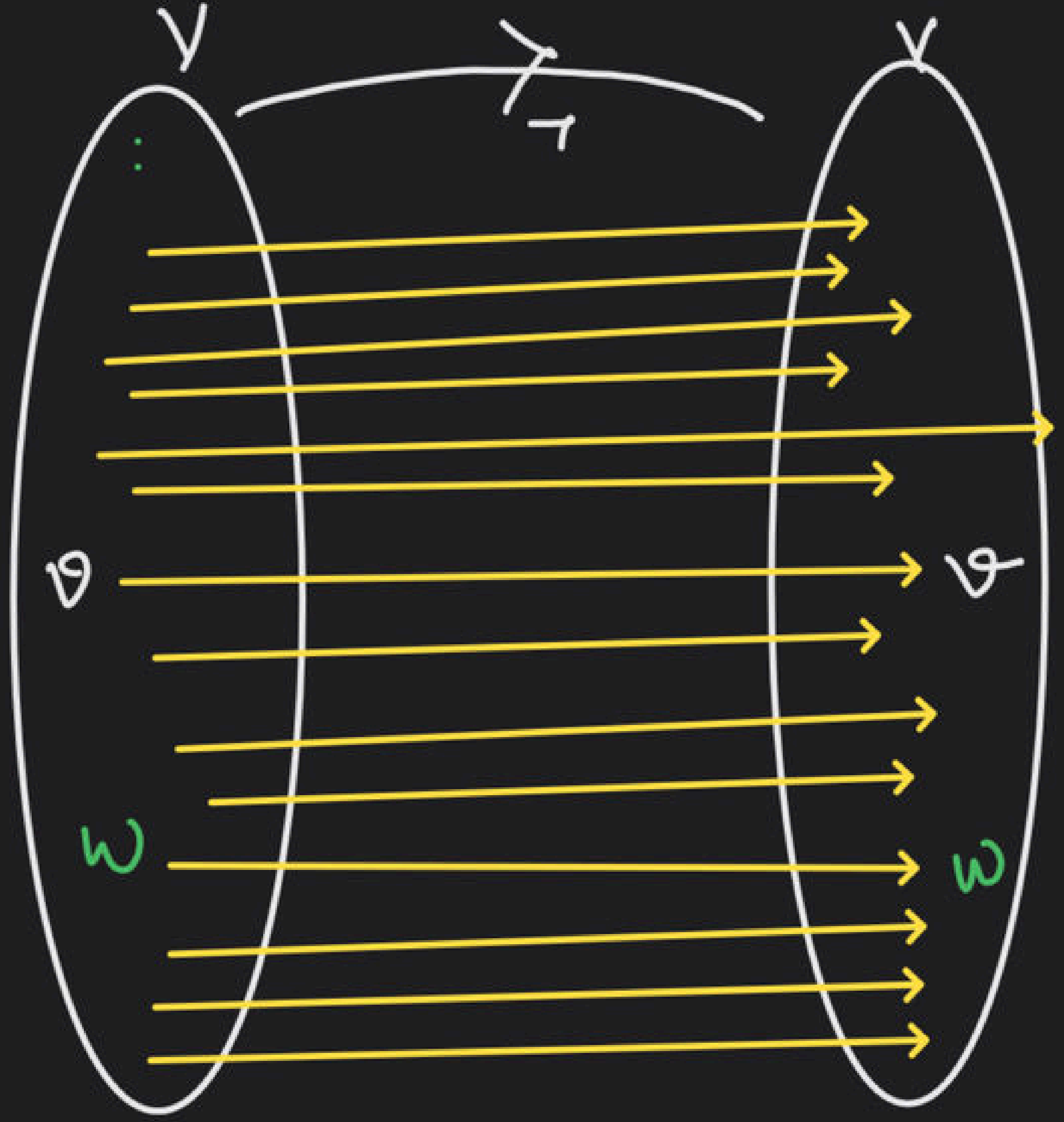
$$f : A \longrightarrow A$$

$$f(x) = x \quad \forall \, x \in A$$

IDENTITY TRANSFORMATION

Let V be a v.s. over the field F, $T: V \to V$ s.t.

$$\boxed{T(v) = v \quad \forall \ v \in V}$$



$T(v) = v$

$\forall\ \alpha, \beta \in F\ ,\ \forall\ u, v \in V.$ $\Rightarrow T(u) = u\ ,\ T(v) = v$

claim : $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$

$$T(\alpha u + \beta v) = \alpha u + \beta v$$
$$= \alpha \cdot u + \beta \cdot v$$
$$= \alpha \cdot T(u) + \beta T(v)$$
$$= \alpha T(u) + \beta T(v)$$

$$LHS = T(\alpha u + \beta v)$$
$$= \alpha u + \beta v$$

$$RHS = \alpha T(u) + \beta T(v)$$
$$= \alpha \cdot u + \beta \cdot v$$
$$= \alpha u + \beta v$$

$$LHS = RHS$$

ZERO TRANSFORMATION

let V and W be two Vector spaces over the same field F. and $T : V \to W$. st.

$$\boxed{T(v) = 0 \quad \forall \ v \in V}$$

T is called as zero transformation.

$\forall u, v \in V, \quad \forall \alpha, \beta \in F$

<u>Claim</u> : $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$

$$T(\alpha u + \beta v) = 0$$

$$\alpha T(u) + \beta T(v) = \alpha \cdot 0 + \beta \cdot 0$$

$$= 0 + 0$$

$$= 0$$

(eg)

$T : \underbrace{P_4[x]} \to P_3[x]$

$T(p(x)) = \int_0^t p(t)\,dt$

not a LT

$T(\text{4 degree p.}) = \int_0^x (\text{4 dg. p.})\,dt$

$= 5$

eg

$T : P[x] \to P[x]$

$T(p(x)) = \int_0^x p(t)\,dt$ this is a L.T.

, $\forall \alpha, \beta \in F$

$\forall p(x), q(x) \in P(x)$

$LHS = T(\alpha p(x) + \beta q(x))$

$= \int_0^x (\alpha p(t) + \beta q(t))\,dt$

$= \alpha \boxed{\int_0^x p(t)\,dt} + \beta \boxed{\int_0^x q(t)\,dt}$

$= \alpha . T(p(x)) + \beta T(q(x))$

$= RHS$

(eg) $T : P[x] \to P[x]$

$T(p(x)) = p''(x) \;/\; p'(x) \;/\; p''(x) + p(x)$

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$\forall\, p(x), q(x) \in P[x]$ and $\forall\, \alpha, \beta \in F$

$$T(\alpha p(x) + \beta q(x)) = (\alpha p(x) + \beta q(x))''$$

$$= \alpha p''(x) + \beta q''(x)$$

$$= \alpha T(p(x)) + \beta T(q(x))$$

$$= RHS.$$

$$T(\alpha p(x) + \beta q(x)) = (\alpha p(x) + \beta q(x))'' + (\alpha p(x) + \beta q(x))$$

$$= \alpha p''(x) + \beta q''(x) + \alpha p(x) + \beta q(x)$$

$$= \alpha (p''(x) + p(x)) + \beta (q''(x) + q(x))$$

$$= \alpha T(p(x)) + \beta T(q(x))$$

$= \underline{\underline{\text{RHS}}}$

<u>This is a LT.</u>

$T : V \to W$

If $T(0) = 0$ $\longrightarrow$ T may or may not be LT (check by definition)

If $T(0) \neq 0$ $\longrightarrow$ T is not a linear Trans (eg1) $T: P_4[x] \longrightarrow P_2[x]$

$$T(p(x)) = p''(x)$$

4 degree poly. $\longrightarrow$ 2 deg. pol. $\in P_2$

not a linear Trans.

(eg2) $T: P_2[x] \longrightarrow P_4[x]$

$$T(p(x)) \longrightarrow p'(x)$$

8 am

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Seq. | Series