Direct Sum

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 13 • Sept 17, 2021

$\text{Span } B = V \implies$ every vector v in V.S. V can be written as L.C. of vectors in B.

BASIS

Let V be a vector space over the field F. Let B be a non-empty subset of V.

Then B is called as basis of V if

(a) B is LI.

(b) Span B = V

$\boxed{\text{DIMENSION}}$: The no. of elements present in the basis is called dimension of V.

or

cardinality of B is called dimension of V.

$C(B) < \infty \implies$ V is a finite dimen. V.S.

$C(B) \not< \infty \implies$ V is inf. dim vector space.

No of Basis of any v.s. can be infinite

but

dimension of v.s. is unique.

$\mathbb{R}^2(\mathbb{R})$

$\Rightarrow \quad B_1 = \{(1,0), (0,1)\}$

(a) LI (checked) (b) span $B_1 = \mathbb{R}^2$ yes

$(x,y) = x(1,0) + y(0,1)$

B_1 is a basis of $\mathbb{R}^2$.

$\dim(\mathbb{R}^2) = 2$

$\Rightarrow \quad B_2 = \{(2,0), (0,2)\}$

(a) LI ✓

(b) $(x,y) = \frac{x}{2}(2,0) + \frac{y}{2}(0,2)$ ✓

B_2 is also a basis of $\mathbb{R}^2$

$\dim(\mathbb{R}^2) = 2$

(3) $B_3 = \{ (1,2), (-1,-1) \}$

(a) $c_1(1,2) + c_2(-1,-1) = 0$

$(c_1 - c_2, 2c_1 - c_2) = 0$

$c_1 - c_2 = 0$

$2c_1 - c_2 = 0$

$c_1 = 0 = c_2$

LI

$\text{Span } B_3 = \mathbb{R}^2$

$(x,y) = c_1(1,2) + c_2(-1,-1)$

$(x,y) = (c_1 - c_2, 2c_1 - c_2)$

$c_1 - c_2 = x$

$2c_1 - c_2 = y$

$c_1 = y - x$

$c_2 = y - 2x$

(eg) $B_4 = \{ (1,2), (-1,3) \; (0,1) \}$ not a basis.

(a) $c_1(1,2) + c_2(-1,3) + c_3(0,1) = (0,0)$

$(c_1 - c_2, \; 2c_1 + 3c_2 + c_3) = (0,0)$

$c_1 - c_2 = 0$ $c_1 = c_2$

$2c_1 + 3c_2 + c_3 = 0$ non-zero soln.

(LD)

(b) span $B_4 = \mathbb{R}^2$

(b) $(x, y) = c_1(1, 2) + c_2(-1, 3) + c_3(0, 1)$

$(x, y) = (c_1 - c_2,\ 2c_1 + 3c_2 + c_3)$

Span $B_4 = \mathbb{R}^2$

$c_1 - c_2 = x$

$2c_1 + 3c_2 + c_3 = y$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & x \\ 2 & 3 & 1 & y \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & x \\ 0 & 5 & 1 & y-2x \end{array}\right]$$

$\rho(A) = 2 = \rho(A|B)$ consistent

$\Rightarrow$ Let S be a subset of V.

Span $S = V$ $\qquad$ $S = \{u_1, u_2, u_3, \ldots, u_n\}$

(a) any $w \in V$ then

$S \cup \{w\} = \{u_1, u_2, u_3, \ldots, u_n, w\}$

Span $(S \cup \{w\}) = V$

(b) If any u_i is l.c. of $u_1, u_2, \ldots, u_{i-1}$

then $S \setminus \{u_i\}$ also spans V.

Span$(S \setminus \{u_i\}) = V$

$$\therefore \quad \operatorname{Span} S = V$$

$$\exists\, c_i \in \mathbb{R}$$

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_{i-1} u_{i-1} + c_i u_i + \cdots + c_n u_n$$

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_{i-1} u_{i-1} + c_i (\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_{i-1} u_{i-1}) + c_{i+1} u_{i+1} + \cdots + c_n u_n$$

$$v = (c_1 + \alpha_1 c_i) \underline{u_1} + (c_2 + \alpha_2 c_i) \underline{u_2} + \cdots + (c_{i-1} + \alpha_{i-1} c_i) \underline{u_{i-1}} + c_{i+1} \underline{u_{i+1}} + \cdots + c_n \underline{u_n}$$

$\Rightarrow \mathbb{R}^{2}$

$B = \{(1,0), (0,1)\}$ — Standard Basis of $\mathbb{R}^2$.

$\Rightarrow \mathbb{R}^{3}$

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$ — Standard basis of $\mathbb{R}^3$

dim of $\mathbb{R}^3 = 3$

$\Rightarrow \mathbb{R}^n$

$$B = \{(1,0,0,\ldots 0), (0,1,0,\ldots 0), (0,0,1,\ldots 0) \ldots (0,0,0\ldots 1)\}$$

$$\dim(\mathbb{R}^n) = n$$

Standard Basis of $\mathbb{R}^n$.

$$P[x] = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \}$$

$$a_i \in F \quad \forall\, i$$

$$B = \{ 1, x, x^2, x^3, x^4, \ldots\ldots \}$$

$$\left.\begin{array}{l} \text{span } B = P[x] \\ B \text{ is LI} \end{array}\right\} \Longrightarrow B \text{ is basis of } P[x]$$

$P[x]$ is a infinite dimensional V.S.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{---} \quad (1)$$

Let W be the set of all solutions of a second order homogeneous linear D.E.

Then W forms a sub-space

$$W = \left\{ y(x) \;\middle|\; \frac{d^2 y(x)}{dx^2} + P(x)\frac{d}{dx}y(x) + Q(x)y(x) = 0 \right\}$$

$\Rightarrow$ zero soln. $y(x) = 0$ satisfies DE.

$y(x) = 0 \in W$ $\quad W \neq \phi$

$\Rightarrow$ $\forall\, y_1$ and $y_2 \in W$ and $\forall\, \alpha, \beta \in F$

claim : $\alpha y_1 + \beta y_2 \in W$

$\because$ y_1 and $y_2 \in W$

y_1 and y_2 are soln of DE (satis W)

$y_1'' + P(x)y_1' + Q(x)y_1 = 0$ and $y_2'' + P(x)y_2' + Q(x)y_2 = 0$

$$LHS = \frac{d^2}{dx^2}(\alpha y_1 + \beta y_2) + P(x)\frac{d}{dx}(\alpha y_1 + \beta y_2) + Q(x)(\alpha y_1 + \beta y_2)$$

$$= \alpha\left(\frac{d^2 y_1}{dx^2} + P(x)\frac{dy_1}{dx} + Q(x)y_1\right) + \beta\left(y_2'' + P(x)y_2' + Q(x)y_2\right)$$

$$= \alpha \cdot 0 + \beta \cdot 0$$

$$= 0 = RHS.$$

$\alpha y_1 + \beta y_2$ is a solution (1)

$$\alpha y_1 + \beta y_2 \in W$$

$\Rightarrow$ W is a sub-space.

general soln.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \text{--- } (*)$$

In gene. soln. $\{y_1, y_2\}$ is LI

$$B = \{y_1, y_2\}$$ is a basis of

LI

Car(B) = 2 = order of DE.

Homo and linear D.E. of $\boxed{n^{th}\text{ order.}}$

$$G.S = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

LI

$$B = \{y_1, y_2, \ldots, y_n\}.$$

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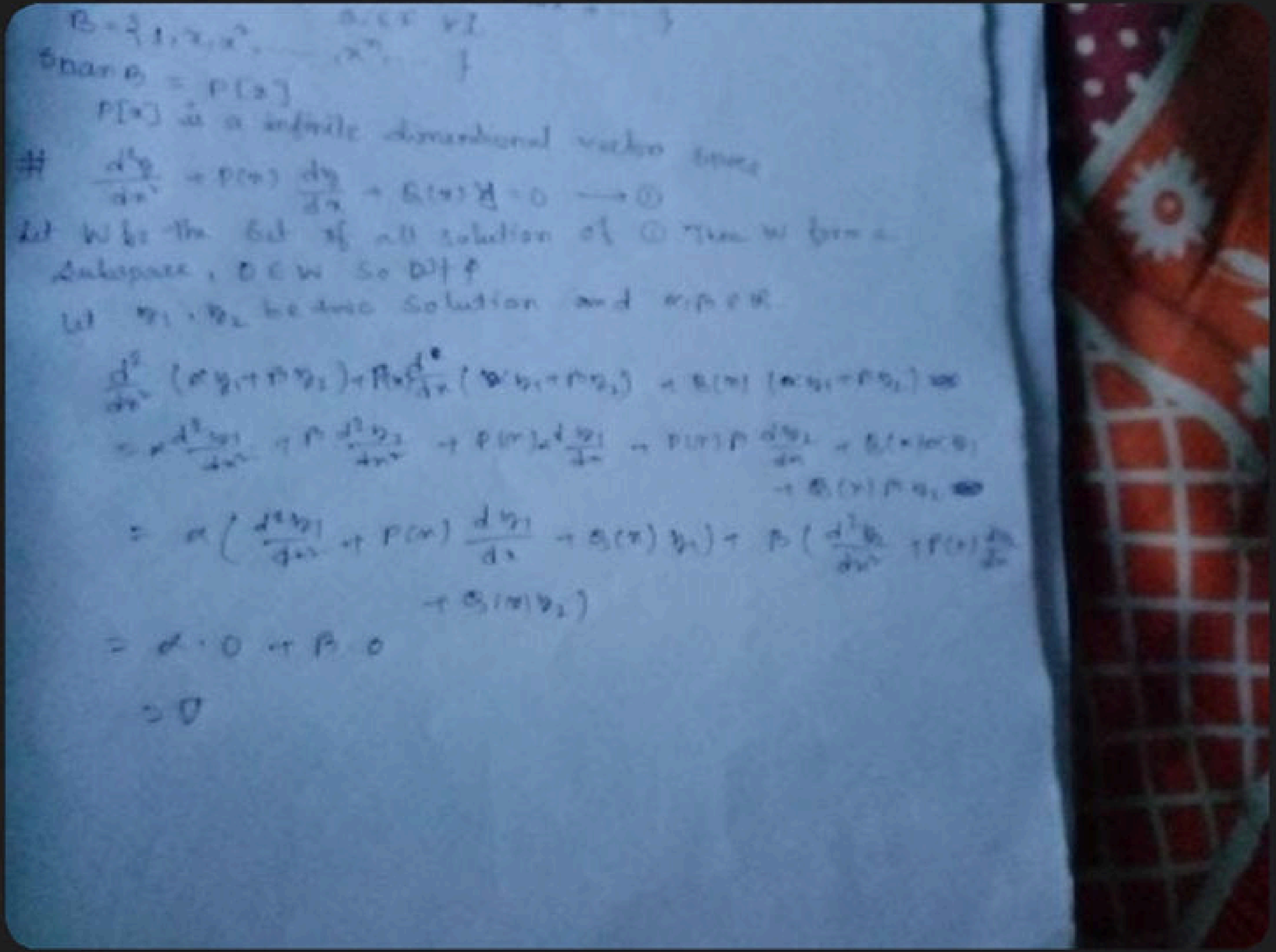
▲ 1 • Asked by Anamika

Please help me with this doubt



(eg) $S = \{ (1,2,3), (-1,0,2)\ (0,0,1) \}$

$\rightarrow$ LI

$\rightarrow$ span $S = \mathbb{R}^3$ ✓

$\rightarrow$ [illegible]

[illegible] $= 4$ [illegible]