Wronskian.

① General theory of linear diffel eqⁿ of higher order:

The general linear diffel eq^n of the nth order is of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = Q(x)$$

$$\forall\, x \in I = [a, b] \qquad \text{—— ①}$$

where $a_0(x), a_1(x), a_2(x) \ldots\ldots a_n(x)$ & $Q(x)$ are all continuous fu^n of x and $a_0(x) \neq 0 \ \forall x$

The above eq^n can be written as

$$(a_0(x) D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = Q(x) \qquad \text{—— ②}$$

Classification of L.D.E:

Homogeneous and non-homo diffal eqⁿ:-

The L.D.E. ① or ② is said to be homogeneous if $Q(x)=0$ & is said to be non-homogenous linear diffal eqⁿ if $Q(x)\neq 0$

Variable coefficient / Constant coefficient:-

If all $a_0(x), a_1(x), a_2(x) \dots a_n(x)$ are constant. Then it called L.D.E with constant coefficient otherwise it is called variable coefficient.

Ex: ① $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ is called homo L.D.E. with variable coefficient

(ii) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + y = 0$ is called LDE with constant coefficient.

(iii) $\frac{d^2y}{dx^2} - y = e^x$ is called non-homo L.D.E with constant coefficient

(iv) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2x$ is called non-homo. L.D.E with variable coefficient.

Linear Combination:- Let $f_1, f_2, f_3 \dots f_n$ are n fn^n defined on a domain D, Then the expression $c_1f_1 + c_2f_2 + \dots + c_nf_n$ where $c_1, c_2 \dots c_n$ are constant is called linear combination of $f_1, f_2, f_3 - - f_n$.

Convex Combination :- A linear combination is called convex combination if

$$\sum_{i=1}^{n} c_i = 1 \quad \& \quad c_i \geq 0 \ \forall i$$

Linearly Independent fn :- The n fn $f_1, f_2, f_3 - - f_n$ are called linearly indep. on a common domain D if $\exists$ scalar $c_1, c_2 - - c_n$ s.t.

$$c_1 f_1 + c_2 f_2 + c_3 f_3 + - + c_n f_n = 0$$

$$\Rightarrow c_1 = 0, c_2 = 0 \cdot - c_n = 0 \quad \forall x \in D.$$

Linearly Dependent fn : The n-fns $f_1, f_2, f_3 - - f_n$ are

called linearly dependent on a common domain D
if $\exists$ scalar $c_1, c_2 - - c_n$ (not all zero) s.t.

$$c_1 f_1 + c_2 f_2 + \cdot - + c_n f_n = 0 \quad \forall x \in D.$$

<u>Ex:-</u> ① $f_1(x) = |x|$, $f_2(x) = x$

$$\frac{\text{fn}}{\text{fn}} = \text{fn} \text{ ——— linearly independent}$$

$$\frac{\text{fn}}{\text{fn}} = \text{constant on whole real line} \longrightarrow \text{Linearly dependent.}$$

$$\frac{f_1}{f_2} = \frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & \text{if } x \geq 0 \\ \frac{x}{-x} = -1 & \text{if } x < 0 \end{cases}$$

$\therefore$ f_1 & f_2 are l. independent

But f_1 & f_2 are L.D on $\mathbb{R}^+$ or $\mathbb{R} \cup \{0\}$

Also f_1 and f_2 are L.D on $\mathbb{R}^-$

② $f_1(x) = e^{4x}$, $f_2(x) = e^{6x}$

$$\frac{f_1(x)}{f_2(x)} = \frac{e^{4x}}{e^{6x}} = e^{-2x}$$

$\therefore$ $f_1(x)$ and $f_2(x)$ are L.I.

③ $f_1(x) = |x|^3$, $f_2(x) = x^3$

Then f_1 & f_2 are L. Independent

Principle of superposition :- Consider the nth order diff. eqn

① (1st slide)

If $y_1, y_2, y_3 - \cdot - y_n$ be any n solns of the diffal eqn (1)

then $y(x) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

is also a soln of the diffal eqn (1) iff

either $Q(x) = 0$

or $\sum_{i=1}^{n} c_i = 1$

Note :: (1) If y_1, y_2 be the soln of homo. linear diffal eqn. Then their linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a soln of above homo. L.D.E.

(2) If y_1, y_2 be the soln of non-homo. L.DE

Then their linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a soln of non-homo. L.D.E if $c_1 + c_2 = 1$

** (3) If y_1 & y_2 be the soln of the non-homo LDE

Then $y = y_1 - y_2$ is a soln of homo. L.DE

Ex:- The diff'al eqn $(D^3 - 6D^2 + 11D + 5)y = e^{2x}$ —— (1)

then

× (1) $2y_1 - y_2 - y_3$ is a soln of eqn (1)

$c_1 + c_2 + c_3$
$= 2 - 1 - 1 = 0$

(2) ✓ $\frac{1}{3}y_1 + \frac{1}{3}y_2 + \frac{1}{3}y_3$ " "

(3) × $2y_1 + y_2 - y_3$ is a soln of eqn (1)

If y_1, y_2 be the soln of non homo. diff tal eqn

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^x \quad \text{——— (1)}$$

Then their difference $y_1 - y_2$ is a soln of homo. diff tal eqn $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \cdot$ — (2)

Expln:- $\because$ y_1, y_2 be the soln of eqn (1)

$$\therefore \frac{d^2y_1}{dx^2} + \frac{dy_1}{dx} + y_1 = e^x$$

$$\& \quad \frac{d^2y_2}{dx^2} + \frac{dy_2}{dx} + y_2 = e^x$$

$$\frac{d^2}{dx^2}(y_1 - y_2) + \frac{d}{dx}(y_1 - y_2) + (y_1 - y_2) = 0$$

$\Rightarrow$ $y_1 - y_2$ is soln of homo. L.D.E (2)

Wronskian! Consider the IInd order L.D.E

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0 \quad \text{——} \quad (1)$$

where $a_0(x),\ a_1(x),\ a_2(x)$ are continuous fn of x

& $a_0(x) \neq 0 \quad \forall x \in I = [a, b]$

Then $$W(y_1, y_2) = W(x) = A\, e^{-\int \frac{a_1(x)}{a_0(x)} dx}$$ (Abel's formula)

Expl^n:- Let y_1 & y_2 be two sol^{ns} of eq^n (1)

$\therefore$ $$a_0 y_1'' + a_1 y_1' + a_2 y_1 = 0 \quad \text{——} \quad (2)$$

& $$a_0 y_2'' + a_1 y_2' + a_2 y_2 = 0 \quad \text{——} \quad (3)$$

Now $$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W = y_1 y_2' - y_2 y_1'$$

$$\therefore W' = (y_1 y_2'' + y_1' y_2') - (y_2 y_1'' + y_2' y_1')$$

$$\therefore W' = y_1 y_2'' - y_2 y_1''$$

$$a_0(x) W' = y_1 (a_0 y_2'') - y_2 (a_0 y_1'')$$

$$= y_1 (-a_1 y_2' - a_2 y_2) - y_2 (-a_1 y_1' - a_2 y_1).$$

$$= -a_1 y_1 y_2' + a_1 y_2 y_1'$$

$$= -a_1 (y_1 y_2' - y_2 y_1')$$

$$= -a_1 W$$

$$\therefore \frac{W'(x)}{W(x)} = \frac{-a_1(x)}{a_0(x)}$$

Integrating

$$\log W(x) = \int \frac{-a_1(x)}{a_0(x)}\,dx + \log A$$

$$\therefore \log\left(\frac{W(x)}{A}\right) = -\int \frac{a_1(x)}{a_0(x)}\,dx$$

$$\therefore \frac{W(x)}{A} = e^{-\int \frac{a_1(x)}{a_0(x)}\,dx}$$

$$\therefore W(x) = A\,e^{-\int \frac{a_1(x)}{a_0(x)}\,dx}$$

Ex: Let $y_1(x)$ & $y_2(x)$ be two L.I. soln of the D.E. $x y'' - 2x^2 y' + e^x y = 0$ satisfying $y_1(0)=1$, $y_2(0)=-1$, $y_1'(0)=1$, $y_2'(0)=1$

Then the wronskian of y_1 & y_2 at $x = 2$ is $W(y_1, y_2)\big|_{x=0}$ is equal to . . . . —

Soln :-

$$W(x) = W(y_1, y_2) = A e^{-\int \frac{a_1(x)}{a_0(x)} dx}$$

$$= A e^{-\int \frac{-2x^2}{x} dx}$$

$$\therefore W(x) = A e^{\int 2x \, dx}$$

$$W(x) = A e^{x^2} \quad \text{—— (1)}$$

$$\therefore W(0) = A e^{0}$$

$$W(0) = A \quad \text{—— (2)}$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1 = 2$$

$W(0) = 2$ —— (3)

$\therefore$ from eqⁿ (2) & (3)

$A = 2$

$\therefore$ from eqⁿ (1) $W(x) = 2e^{x^2}$

$\therefore$ $\underline{W(2) = 2e^4}$

Ex:- Let $y_1(x)$ and $y_2(x)$ be two solns of diff'al eqn.

$$(1-x^2)y'' - 2xy' + (\sec x)y = 0$$ with wronskian

$W(x)$. If $y_1(0)=1$, $y_1'(0)=0$, $W\left(\frac{1}{2}\right)=\frac{1}{3}$

Then $y_2'(0) = ?$

Soln :-

$$W(x) = A\, e^{-\int \frac{a_1(x)}{a_0(x)}dx}$$

$$= A\, e^{-\int \frac{-2x}{(1-x^2)}dx}$$

$$= A\, e^{\int \frac{2x}{1-x^2}dx}$$

$$= A\, e^{-\log(1-x^2)}$$

$$= A\, e^{\log(1-x^2)^{-1}} = A\, e^{\log(1-x^2)^{-1}} = A(1-x^2)^{-1}$$

$$\therefore\ W(x) = \frac{A}{(1-x^2)} \quad \text{--- } 1$$

$$\Rightarrow W\left(\frac{1}{2}\right) = A \cdot \frac{1}{\left(1-\frac{1}{4}\right)}$$

$$\Rightarrow \frac{1}{3} = A \cdot \frac{4}{3}$$

$$\Rightarrow A = \frac{1}{4}$$

$$\therefore\ W(x) = \frac{1}{4(1-x^2)} \quad \text{--- } 2$$

Again $W(0) = \begin{vmatrix} y_1(0) & y_1'(0) \\ y_2(0) & y_2'(0) \end{vmatrix}$

$$\therefore\ W(0) = \begin{vmatrix} 1 & 0 \\ y_2(0) & y_2'(0) \end{vmatrix}$$

$$\therefore\ W(0) = y_2'(0)$$

$$\frac{1}{4} = y_2'(0)$$

$$\therefore\ \boxed{y_2'(0) = \frac{1}{4}}$$

Ex! :