Null Space and Range Space

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 19 • Sept 23, 2021

$$N(T) = \{ x \in V \mid T(x) = 0 \}$$

$N(T) \neq \phi$ $\qquad$ $0 \in N(T)$ always.

(eg) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T(x, y, z) = (x+z, y, 0)$$

(a) $N(T)$

(b) Basis of $N(T)$

(c) $\eta(T)$

$$N(T) = \{ (x, y, z) \in \mathbb{R}^3 \mid T(x, y, z) = (0,0,0) \}$$

$$T(x, y, z) = (0,0,0)$$

$\downarrow$

$$(x+z, y, 0) = (0,0,0)$$

$$y = 0$$

$$x + z = 0$$

free variable.

$$N(T) = \{ (-z, 0, z) \mid z \in \mathbb{R} \}$$

$N(T) = \{ (-z, 0, z) \mid z \in \mathbb{R} \}$

$\dim N(T) = \eta(T) = 1$

$\text{Basis} = \{ (-1, 0, 1) \}$

$y = 0$

$x + z = 0$

$z = -x$

$N(T) = \{ (x, 0, -x) \mid x \in \mathbb{R} \}$

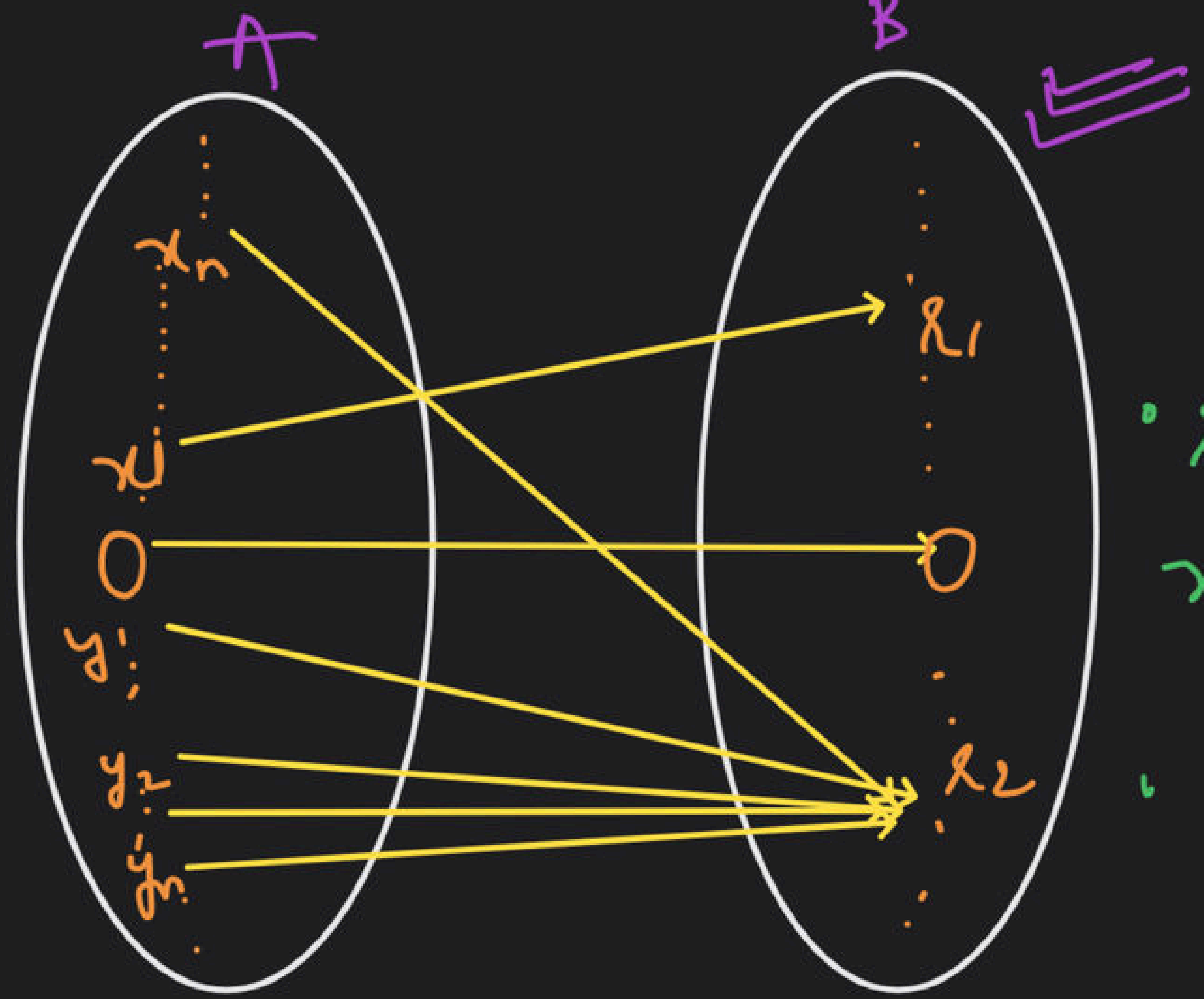
$\text{Basis} = \{ (1, 0, -1) \}$

dim 1

$$f : A \to B$$

$$\text{Range}(f) = \{ y \in B \mid \exists\, x \in A : T(x) = y \}$$

$$R(f) = \{ r_1, r_2, 0 \}$$



$$f(x_1) = r_1$$

- r_1 is image of x_1 under f
- x_1 is preimage of r_1 under f

Range space

let V and W are F.D.V.S. over the same field F. let $T: V \to W$ be a LT. then

$$R(T) = \{ y \in W \mid \exists x \in V ,\ T(x) = y \}$$

$\downarrow$

range space of T.

$$R(T) \subseteq W$$

$\because$ T is a LT $\quad T: V \to W$

$$T(0) = 0$$

$\downarrow \qquad \downarrow$

$\in V \qquad \in W$

R(T) will always be a non-empty set.

for $0 \in W$ $\exists$ always $0 \in V$ s.t.

$0 \in R(T)$.

THEOREM: Let V and W are F.D.V.S. over the same field F. Let $T: V \to W$ be a LT. Then range space of T is a sub-space of W.

Proof: $R(T) = \{ y \in W \mid \exists x \in V : T(x) = y \}$

$\because T(0) = 0 \Rightarrow 0 \in R(T) \Rightarrow R(T) \neq \phi$

$\Rightarrow \forall u, v \in R(T), \forall \alpha, \beta \in F$

Claim: $\alpha u + \beta v \in R(T)$.

$\because u, v \in R(T)$, $\exists\ x_1, x_2 \in V$ st

$T(x_1) = u$ and $T(x_2) = v$

$\because u, v \in W$ and W is a V.S. $\Rightarrow \alpha u + \beta v \in W$

$\forall\ \alpha, \beta \in F.$

$$\alpha u + \beta v = \alpha \cdot T(x_1) + \beta T(x_2)$$

$$= T(\alpha x_1) + T(\beta x_2) \quad (\because T \text{ is LT})$$

$$= T(\alpha x_1 + \beta x_2)$$

$\alpha x_1 + \beta x_2 \in V$ $\because$ V is a V.S.

for $\alpha u+\beta v$, $\exists\ \alpha x_1+\beta x_2 \in V$ s.t.

$$T(\alpha x_1+\beta x_2)=\alpha u+\beta v$$

$\Rightarrow \alpha u+\beta v \in R(T)$

$\Rightarrow R(T)$ is a sub-space of W.

$T: V \to W$

$\because R(T) \leq W \Rightarrow R(T)$ is a sub-space of W.

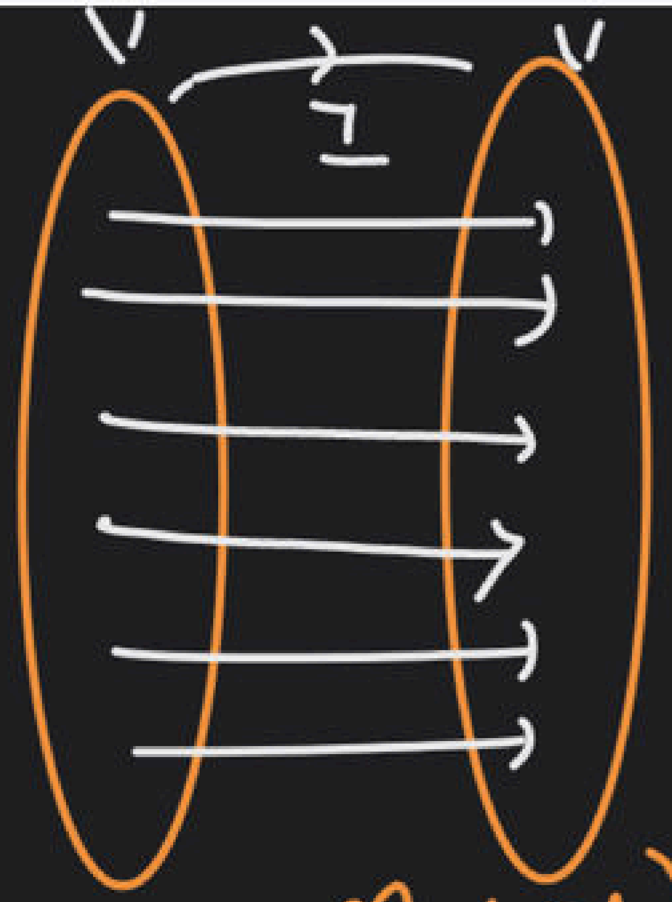
$\dim(R(T)) \leq \dim W$

dimension of range space of T is called as rank of T. and represented by $\rho(T)$.

(eg) 1

$I : V \to V$

$I(x) = x$



$N(T) = \{0\}$ B of $N(T) = \phi$ $\eta(T) = 0$

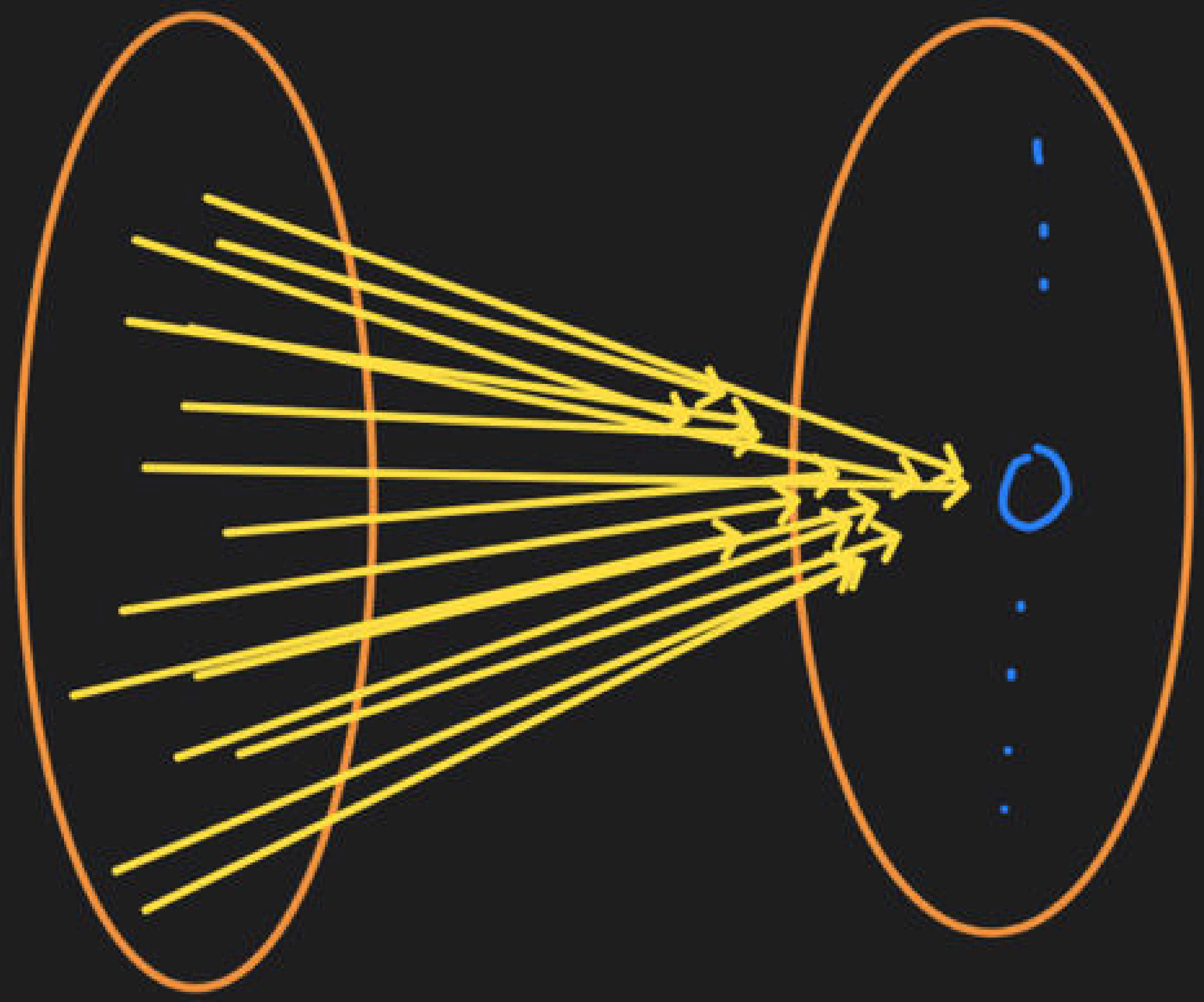
$R(T) = V$

B of $R(T)$ = Basis of V

dim $(R(T)) = \rho(T)$ = dim of V.

(eg) Zero Trans. $T : V \to W$

$T(x) = 0$



$N(T) = V$

$R(T) = \{ 0 \}$

$\eta(T) = \dim V$

$\ell(T) = 0$

(eg) $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x, y, z) = (x + y, y + z)$$

$$\Rightarrow N(T) = \{ (x, y, z) \in \mathbb{R}^3 \mid \underbrace{T(x, y, z) = (0, 0)}_{\Downarrow} \}$$

$$N(T) = \{ (z, -z, z) \mid z \in \mathbb{R} \}$$

$$(x + y, y + z) = (0, 0)$$

$$\dim N(T) = n(T) = 1$$

$$x + y = 0 \qquad x = -y \qquad x = z$$

$$y + z = 0 \qquad y = -z$$

$$B = \{ (1, -1, 1) \}$$

$T(x,y,z) = (x+y,\ y+z)$

$R(T) = \{ y \in \mathbb{R}^2 \mid \exists\, x \in \mathbb{R}^3 \ ;\ T(x) = y \}$

x_1, x_2, x_3 $\leftarrow$ $y = (y_1, y_2)$ $\quad \exists \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3$

?? ✓

s.t. $T(x_1, x_2, x_3) = (y_1, y_2)$

$(x_1 + x_2,\ x_2 + x_3) = (y_1, y_2)$

$x_1 + x_2 = y_1$

$x_2 + x_3 = y_2$

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$B = \{ (1,0), (0,1) \}$

$T(x, y, z) = (x+y, \; y+z)$

$T(e_1) = T(1,0,0) = (1,0)$

$T(e_2) = T(0,1,0) = (1,1)$

$T(e_3) = T(0,0,1) = (0,1)$

$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \sim$

$\begin{bmatrix} \boxed{1 \quad 0} \\ \boxed{0 \quad 1} \\ 0 \quad 0 \end{bmatrix}$

$\rho(T) = 2$

$x_1 + x_2 = y_1$

$x_2 + x_3 = y_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \end{array}\right]$$

$\rho(A) = 2 = \rho(A|B)$

soln. exists

(x_1, x_2, x_3) exists

$\dim R(T) = 2$

$= 3 - 2 + 1$

$= 2$

2

Ho sy.

$= n - r$

<u>Non mo.</u>

$= n - r + 1$