Linear Transformation

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 18 • Sept 22, 2021

(eg) $T: \mathbb{R}^2 \to \mathbb{R}^3$ s.t.

$T(x,y) = (x, y, x+y) \rightarrow$ is a LT.

$\forall\, u = (x_1, y_1) \in \mathbb{R}^2$

$v = (x_2, y_2) \in \mathbb{R}^2$

$\forall\, \alpha, \beta \in F$

$$T(\alpha u + \beta v) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2)$$

$$= (\alpha x_1, \alpha y_1, \alpha x_1 + \alpha y_1) + (\beta x_2, \beta y_2, \beta x_2 + \beta y_2)$$

$$= \alpha(x_1, y_1, x_1 + y_1) + \beta(x_2, y_2, x_2 + y_2)$$

$$= \alpha\, T(x_1, y_1) + \beta\, T(x_2, y_2)$$

$$= \alpha\, T(u) + \beta\, T(v)$$

eg) $T : \mathbb{R}^n \to \mathbb{R}^m$ and A is a fixed matrix of order $m \times n$. Then define

$$\boxed{T(x) = Ax}$$

$$\left(A_{m\times n}(x)_{n\times 1}\right)_{m\times 1}$$

$\forall\, u, v \in \mathbb{R}^n, \ \forall\, \alpha, \beta \in F$

$$T(\alpha u + \beta v) = A\cdot(\alpha u + \beta v)$$

$$= \alpha A u + \beta A v$$

$$= \alpha\cdot T(u) + \beta T(v)$$

(eg) $T : F^{m\times n} \to F^{m\times n}$

$T(A) = PAQ$, P and Q are fixed matrices $P = (p_{ij})_{m\times m}$ and $Q = (q_{ij})_{n\times n}$

also a L.T.

$$\Big(\underbrace{\left(P_{m\times m} \; A_{m\times n} \right)}_{m\times n} Q_{n\times n} \Big)_{m\times n}$$

$\forall\, A, B \in F^{m\times n}, \; \forall\, \alpha, \beta \in F$

$$T(\alpha A + \beta B) = P(\alpha A + \beta B)Q$$
$$= \alpha PAQ + \beta PBQ$$
$$= \alpha T(A) + \beta T(B)$$

KERNEL of T or Null space of T

Let T be linear trans. from V to W.

$T : V \to W$

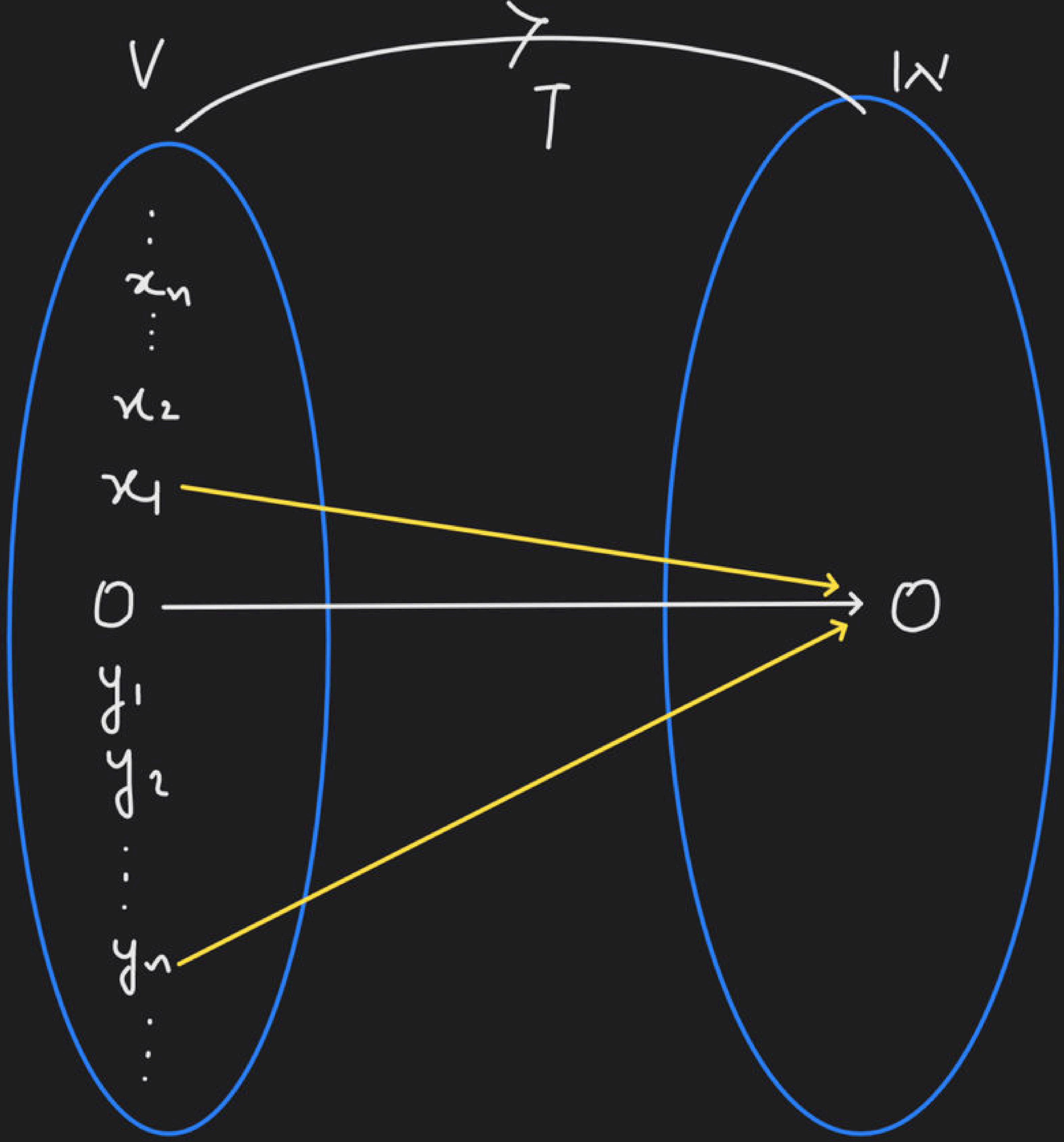
$$\boxed{N(T) = Ker(T) = \{ x \in V \mid T(x) = 0 \}}$$

(Note): Null space of any linear transformation is always non-empty.

$\because T$ is a LT. So,

$T(0) = 0$, $\Rightarrow$ $0 \in Null(T)$ (always)

$N(T) = \{0, x_1, y_n\}$



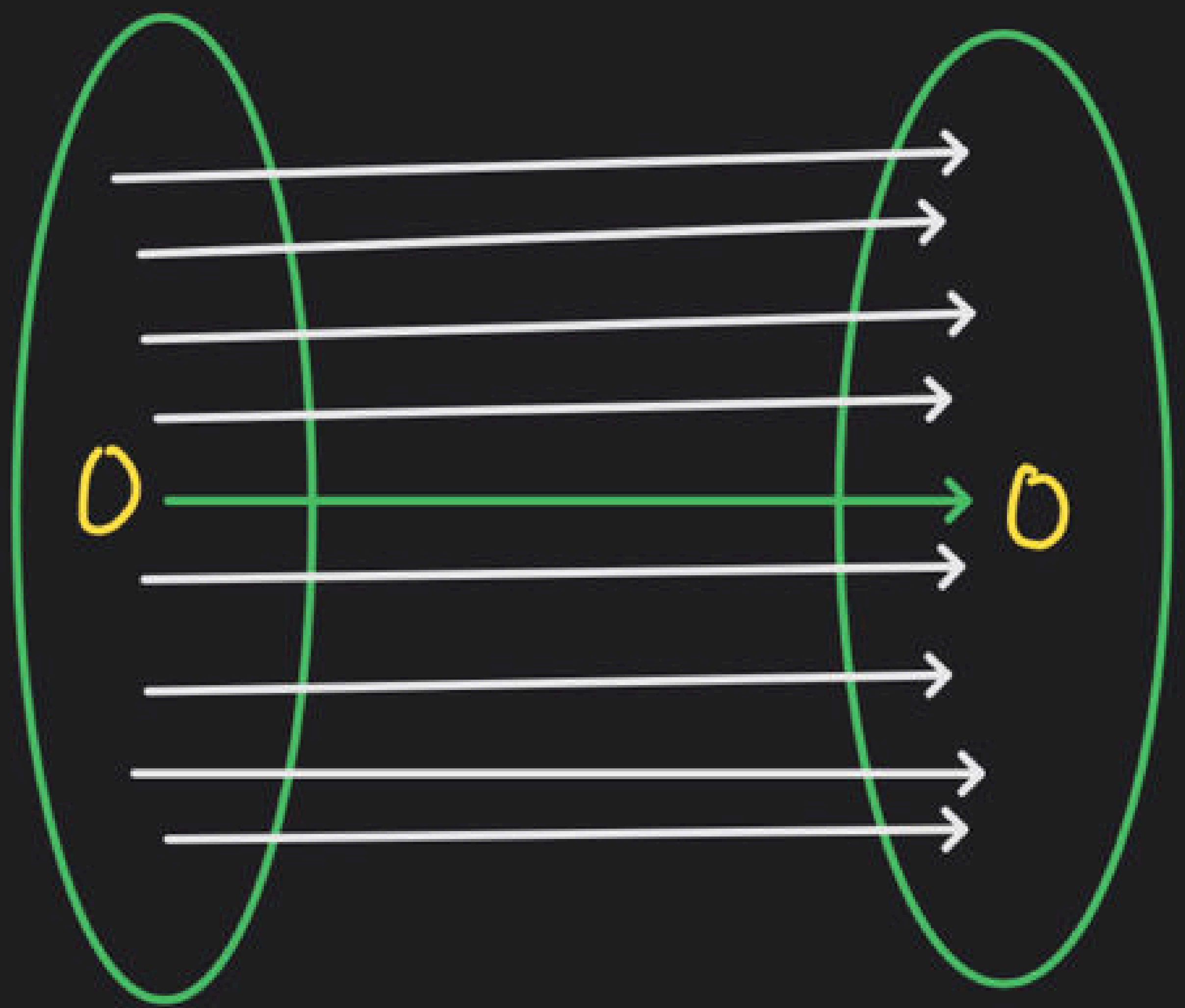
(eg)

$I : V \to V$

$I(x) = x \quad \forall\, x \in V$

$N(I) = \{ x \in V \mid I(x) = 0 \}$

$= \{0\}$



$0 \longrightarrow 0$

$v \longrightarrow v \quad (v \neq 0)$

$$N(I) = \{ x \in V \mid I(x) = 0 \}$$

$$= \{ x \in V \mid x = 0 \} \quad \because I(x) = x$$

$$= \{ 0 \}$$

(eg)

$T : V \to W$

$T(x) = 0 \quad \forall x \in V$

$$N(T) = \{ x \in V \mid T(x) = 0 \}$$

$$= \{ x \in V \}$$

$$= V$$

(eg) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x,y) = (x,0)$$

$$N(T) = \{ u \in V \mid T(u) = 0 \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0) \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid (x,0) = (0,0) \}$$

$$= \{ (\underline{0,y}) \mid y \in \mathbb{R} \}$$

$$T(0,y) = (0,0)$$

$$T(\underline{0, y}) = (0, 0)$$

(eg) $T : \mathbb{R}^2 \to \mathbb{R}^3$

$T(x,y) = (x, y, x+y)$

$N(T) = \{ u \in V \mid T(u) = 0 \}$ ---------- $\circledast$

$= \{ (x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0,0) \}$

$\downarrow$

$(x, y, x+y) = (0,0,0)$

$x = 0, \; y = 0, \; x+y = 0$

$= \{ (0,0) \}$

$T: V \to W$

$$N(T) = \{ \underline{\underline{u \in V}} \mid T(u) = 0 \}$$

$\Downarrow$

$$N(T) \subseteq V$$

THEOREM: Let V and W be vector spaces over the same field F. Let $T: V \to W$ be a linear transformation. Then, Null space of T is a sub-space of V.

Proof: $T: V \to W$

$$N(T) = \{ u \in V \mid T(u) = 0 \}$$

clearly, $N(T) \subseteq V$

$$\Rightarrow \because T \text{ is a LT} \Rightarrow T(\underset{\in V}{0}) = \underset{\in W}{0}$$

$$0 \in N(T) \Rightarrow N(T) \neq \phi$$

$\Rightarrow \quad \forall\, u, v \in N(T), \quad \forall\, \alpha, \beta \in F$

Claim : $\alpha u + \beta v \in N(T)$

$\because \; u, v \in N(T) \Rightarrow T(u) = T(v) = 0$

$N(T) = \{\, u \in V \mid T(u) = 0 \,\}$

$\because \; u, v \in N(T) \Rightarrow u, v \in V \Rightarrow \alpha u + \beta v \in V$
($\because$ V is a V.S.)

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \quad (\because T \text{ is a LT})$$
$$= \alpha \cdot 0 + \beta \cdot 0 = 0$$

$$\alpha u + \beta v \xrightarrow{T} 0$$

$$\Rightarrow \alpha u + \beta v \in N(T)$$

$\Rightarrow$ $N(T)$ is a sub-space of V.

V is a V.S. of finite dimension.

$N(T)$ is a sub-space of V.

$$\dim(N(T)) \leq \dim(V)$$

$\dim(N(T))$ = no. of vectors in the basis B of $N(T)$

$\text{span}\, B = N(T)$

B is LI.

Nullity of T : The dimension of the null-space of any LT is called nullity of that LT.

It is represented by $\eta(T)$.

eg)

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T(x, y, z) = (x+y+z,\ x-y-z,\ y+2z)$$

$$N(T) = ?$$

$$N(T) = \{ (x, y, z) \in \mathbb{R}^3 \mid T(x, y, z) = (0, 0, 0) \}$$

$$T(x, y, z) = (0, 0, 0)$$

$$(x+y+z,\ x-y-z,\ y+2z) = (0, 0, 0)$$

$$x+y+z = 0$$
$$x-y-z = 0$$
$$y+2z = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dim N(T) = 0$$
$$\eta(T) = 0$$

$$= n - r \quad = \dim$$
$$= 3 - 3 = 0$$

$$x+y+z = 0 \quad x = 0$$
$$z = 0 \quad y+z = 0 \quad y = 0$$

$N(T) = \{ (0,0,0) \}$

$B = \phi$ $\quad \operatorname{span} \phi = \{ (0,0,0) \} = N(T)$

LI

$\dim N(T) = 0 = \eta(T)$

(eg) $T : P[x] \to P[x]$ $N(T) = \{0\}$

$$T(p(x)) = p''(x) + p(x)$$

$$N(T) = \{ p(x) \in P[x] \mid T(p(x)) = 0 \}$$

?

$$T(p(x)) = 0$$

$$p''(x) + p(x) = 0$$

$y'' + y = 0$ → L.D.E with C.C.

$y = p(x)$

$p''(x) = y''$

$$(D^2+1)\, y = 0 \qquad \frac{d^2y}{dx^2} + y = 0$$

$$\frac{d}{dx} = D$$

$$\frac{d^2}{dx^2} = D^2$$

$$(D^2+1)y = 0$$

$$m^2+1=0$$

$$m = \pm i$$

$p(x) \in P[x]$ s.t.

$T(p(x)) = 0$

$T(p(x)) = 0$

$$y(x) = \boxed{p(x) = c_1 \cos x + c_2 \sin x}$$

No. $p(x) \notin P[x]$.

$$T(0) = 0'' + 0 = 0$$

(eg) $T : P(x) \to P[x]$

$T(p(x)) = p'(x)$

$N(T) = \{ c \mid c \in \mathbb{R} \}$

$N(T) = \{ p(x) \in P[x] \mid T(p(x)) = 0 \}$

$p'(x) = 0$

$p(x) = c$

$N(T) = \{ c \}$ $\to \dim N(T) = 1$

Basis $= \{ 1 \}$

$c1$

Range Space