Matrix Representation - I

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 23 • Sept 27, 2021

$Hom(V, W)$ = set of all LT from V to W

$= \{ T : V \to W \mid T \text{ is a LT} \}$

$Hom(V, V) = \{ T : V \to V \mid T \text{ is a LT} \}$

If $T : V \to V$ is a LT. from V to V then T is a linear operator.

$$\text{Hom}(V, W) = \{ T : V \to W \mid T \text{ is a LT} \}$$

$0 : V \to W \quad 0(x) = 0 \quad \forall x \in V$

$0 \in \text{Hom}(V, W) \implies \text{Hom}(V, W) \neq \phi$

$\forall\ T, S \in \text{Hom}(V, W)$ and $\forall \alpha, \beta \in F$,

<u>claim</u> : $\alpha T + \beta S \in \text{Hom}(V, W)$

$T, S \in \text{Hom}(V, W)$

$T, S : V \to W$ are LT.

By algebra, $\alpha T, \beta S : V \to W$ are also LT.

$\Rightarrow \quad \alpha T + \beta S : V \to W$ is also a LT.

$\Rightarrow \quad \alpha T + \beta S \in \mathrm{Hom}(V, W)$.

Hom(V, W) forms a sub-space.

$\dim V = m \qquad \dim W = n$

$$\boxed{\dim \operatorname{Hom}(V, W) = mn.}$$

$$\boxed{\dim \operatorname{Hom}(V, V) = n^2.}$$

INVERTIBLE MAPS

Let $T: V \to V$ be a L.T. If V is a $\boxed{\text{F.D.V.S.}}$ then

$$\text{Ker}(T) = \{0\} \iff T \text{ is one-one}$$

$$\iff T \text{ is onto}$$

$$\iff T \text{ is bijective}$$

$$\iff T \text{ is invertible}$$

$$\iff T \text{ is non-singular.}$$

Let $T: V \to V$ s.t.

$\ker T = \{0\} \iff T$ is one-one

$\iff T$ is non-singular.

Let $T: V \to W$ be a LT. Then

* T cannot be one-one if $\dim(V) > \dim W$

* T cannot be onto if $\dim V < \dim W$

$f : A \to B$

Car(A) $\leq$ car(B) one-one

car(A) $\geq$ car(B) onto

$\ker T = \{0\}$ iff T is one-one

- Suppose $\ker T = \{0\}$

<u>Claim</u>: T is one-one

$T : V \to W$

$T(x_1) = T(x_2)$

$T(x_1 - x_2) = 0$

$x_1 - x_2 \in \ker T.$

$x_1 - x_2 = 0$

$x_1 = x_2$

Suppose T is one-one

<u>Claim</u>: $\ker T = \{0\}$

$x \in \ker T$

$T(x) = 0$

$\because T$ is a LT.

$T(0) = 0$

$\because T$ is one-one

$T(x) = T(0)$

$\boxed{x = 0}$

$T : V \to W$

$\dim V = \dim W = \text{finite}$

$\ker T = \{0\} \iff T$ is 1-1

$\iff$ onto

$\iff$ invertible

$\iff$ non-sing.

$\iff$ bijec.

$T : V \to W$

$\dim V = \text{finite}$

$\ker T = \{0\} \iff T$ is 1-1

$\iff T$ is non-sing.

(eg) $f : \mathbb{R}^2 \to \mathbb{R}^2$

$f(x, y) = (xy, x)$

not a LT

$f(\alpha u) = f(\alpha x, \alpha y) = (\alpha^2 xy, \alpha x)$

$\alpha f(u) = \alpha f(x, y) = \alpha(xy, x)$

$= (\alpha xy, \alpha x)$

$\checkmark$

(eg) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (x+y, x)$

1-1

onto

inv.

$N(T) = \{ (x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0) \}$

$(x+y, x) = (0,0)$

$x+y=0 \qquad x=0$

$y=0$

$\ker(T) = \{(0,0)\}$

(eg) $\quad T: P_3(x) \xrightarrow{} P_2(x)$, where $V \to W$

$$T(p(x)) = p'(x)$$

onto

dim $V = 4$ $\quad$ dim $W = 3$

dim $V >$ dim W

~~one-one~~

invertible.

not one one $\Rightarrow$ $\eta(T) > 0$

$N(T)$

$n(T)$?

R.N.T. $\rightarrow$ $l(T) = 4 - n(T)$

$= 3$ onto

$\dim V \leq \dim W$ one-one

$\dim V \geq \dim W$ onto

$\dim V \geq \dim W$. not one.

$\dim V \leq \dim W$ not onto

$g: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ st $g(x,y) = (x+y, x-2y, 3x+y)$

onto ×

invertible ×

bijective ×

$N(T) = \{ u \in \mathbb{R}^2 \mid T(u) = (0,0,0) \}$

$N(T) = \{ (0,0) \}$

$\downarrow$

one - one

$f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x,y) = (x-y, x-2y)$

$\downarrow$

L.O.

$\text{Ker}\,T = \{0\}$

$N(T) = \{(x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0)\}$

$N(T) = \{(0,0)\}$

one-one, onto, bij, inver,

non-sing.

f^{-1} exists

$$f(x, y) = (x - y,\ x - 2y) = (a, b)$$

$$(x, y) = f^{-1}(a, b)$$

$$(2a - b,\ a - b) = f^{-1}(a, b)$$

$$a \longrightarrow x$$

$$b \longrightarrow y$$

$$x - y = a$$

$$x - 2y = b$$

$$\boxed{y = a - b}$$

$$\boxed{x = 2a - b}$$

$$\boxed{f^{-1}(x, y) = (2x - y,\ x - y)}$$

(eg) $f : \mathbb{R}^3 \to \mathbb{R}^2$ $\qquad$ $g : \mathbb{R}^2 \to \mathbb{R}^2$

$f(x, y, z) = (2x, y+z)$ $\qquad$ $g(x, y) = (y, x)$

$$\boxed{g \circ f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2}$$

$$(g \circ f)(x, y, z) = g(f(x, y, z))$$

$$= g(2x, y+z)$$

$$\boxed{(g \circ f)(x, y, z) = (y+z, 2x)}$$

Onto $f: \mathbb{R}^3 \to \mathbb{R}^2$

$f(x, y, z) = (2x, y+z)$

never be 1-1. singular

$N(T) = \{(0, y, -y) \mid y \in \mathbb{R}\}$

$2x = 0$

$y+z = 0$

$n(T) = 1$

$r(T) = 3-1$

$= 2 = r$

$g: \mathbb{R}^2 \to \mathbb{R}^2$

$g(x, y) = (y, x)$

$N(T) = \{(0, 0)\}$

One-one ✓

Onto ✓

bij ✓

inv. ✓

non-singular ✓

$g(x, y) = (y, x) = (a, b)$

$(x, y) = g^{-1}(a, b)$

$a = y$

$b = x$

$(b, a) = g^{-1}(a, b)$

$g^{-1}(x, y) = (y, x)$

$$(g \circ g^{-1})(\underline{\underline{x}}) = I(x) = \underline{\underline{x}}$$

$$(g \circ g^{-1})(\underline{\underline{x, y}}) = g(g^{-1}(x, y))$$

$$= g(y, x)$$

$$= \underline{\underline{(x, y)}}$$

(eg) $T : \mathbb{R}^3 \to \mathbb{R}^3$

one-one, onto, non-s.

bij, inv.

$T(x,y,z) = (\underbrace{x-2y}_{=0}, \underbrace{2x+y+z}_{=0}, \underbrace{x+y-3z}_{=0})$

$T(x,y,z) = (0,0,0)$

(x,y,z) ??

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & -18 \end{bmatrix}$$

$n - r$

$3 - 3 = 0$

$\eta(T) = 0$

unacademy

1 • Asked by Prasant

Please help me with this doubt

backward shift

Recall that $\mathbf{F}^\infty$ denotes the vector space of all sequences of elements of $\mathbf{F}$. Define $T \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

from $\mathbf{R}^3$ to $\mathbf{R}^2$

Define $T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R}^2)$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z).$$

from $\mathbf{F}^n$ to $\mathbf{F}^m$

Generalizing the previous example, let m and n be positive integers, let $A_{j,k} \in \mathbf{F}$ for $j = 1, \dots, m$ and $k = 1, \dots, n$, and define $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ by

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \cdots + A_{m,n}x_n).$$

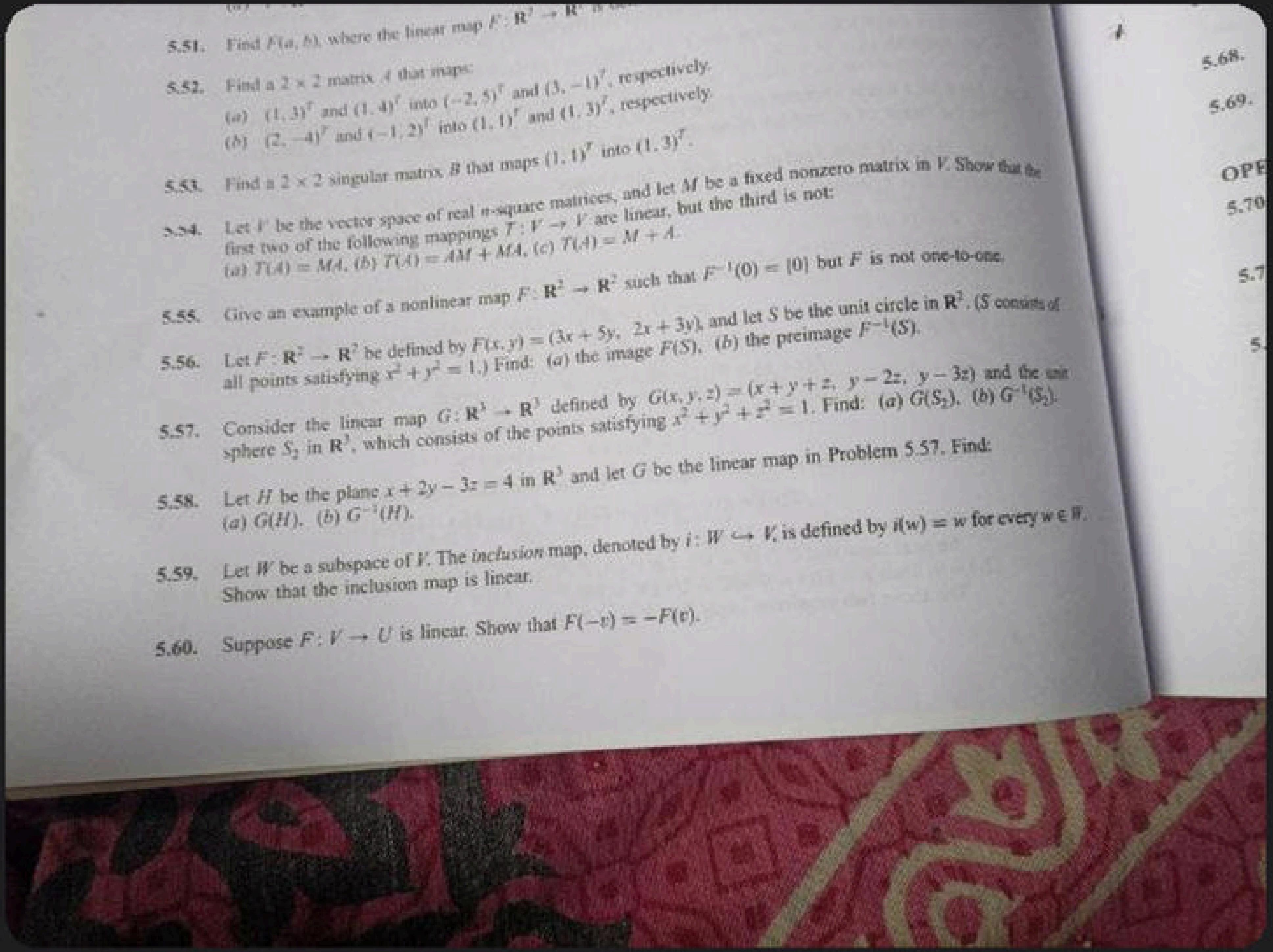
Actually every linear map from $\mathbf{F}^n$ to $\mathbf{F}^m$ is of this form.

Mw

δ^{-1}

▲ 2 • Asked by Papon

5.59 what is inclusion map??



5.51. Find $F(a, b)$, where the linear map $F: \mathbf{R}^2 \to \mathbf{R}^2$ ...

5.52. Find a 2×2 matrix A that maps:
(a) $(1, 3)^T$ and $(1, 4)^T$ into $(-2, 5)^T$ and $(3, -1)^T$, respectively.
(b) $(2, -4)^T$ and $(-1, 2)^T$ into $(1, 1)^T$ and $(1, 3)^T$, respectively.

5.53. Find a 2×2 singular matrix B that maps $(1, 1)^T$ into $(1, 3)^T$.

5.54. Let V be the vector space of real n-square matrices, and let M be a fixed nonzero matrix in V. Show that the first two of the following mappings $T: V \to V$ are linear, but the third is not:
(a) $T(A) = MA$, (b) $T(A) = AM + MA$, (c) $T(A) = M + A$.

5.55. Give an example of a nonlinear map $F: \mathbf{R}^2 \to \mathbf{R}^2$ such that $F^{-1}(0) = \{0\}$ but F is not one-to-one.

5.56. Let $F: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $F(x, y) = (3x + 5y,\ 2x + 3y)$, and let S be the unit circle in $\mathbf{R}^2$. (S consists of all points satisfying $x^2 + y^2 = 1$.) Find: (a) the image $F(S)$, (b) the preimage $F^{-1}(S)$.

5.57. Consider the linear map $G: \mathbf{R}^3 \to \mathbf{R}^3$ defined by $G(x, y, z) = (x + y + z,\ y - 2z,\ y - 3z)$ and the unit sphere S_2 in $\mathbf{R}^3$, which consists of the points satisfying $x^2 + y^2 + z^2 = 1$. Find: (a) $G(S_2)$, (b) $G^{-1}(S_2)$.

5.58. Let H be the plane $x + 2y - 3z = 4$ in $\mathbf{R}^3$ and let G be the linear map in Problem 5.57. Find:
(a) $G(H)$, (b) $G^{-1}(H)$.

5.59. Let W be a subspace of V. The *inclusion* map, denoted by $i: W \hookrightarrow V$, is defined by $i(w) = w$ for every $w \in W$. Show that the inclusion map is linear.

5.60. Suppose $F: V \to U$ is linear. Show that $F(-v) = -F(v)$.

5.68.

5.69.

OPE

5.70

5pm → 5pal

6pm → IC