

Linear Algebra

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1 Field

Let F be non-empty set. Let '+' and '·' be two binary operations. Then,
 $\implies (F, +)$ must be an Abelian group

- a **Closure:** For all $x, y \in F, x + y = z \in F$
- b **Associative:** For all $x, y, z \in F, (x + y) + z = x + (y + z)$
- c **Identity:** For all $x \in F$, there exists an element $e \in F$ such that $x + e = e + x = x$
- d **Inverse:** For all $x \in F$, there exists an element $y \in F$ such that $x + y = e$
- e **Commutative:** For all $x, y \in F, x + y = y + x$

First Four properties forms a group and all the five properties forms an Abelian group on '+'

$\implies (F, *)$ must be an Abelian group

- a **Closure:** For all $x, y \in F, x \cdot y = z \in F$
- b **Associative:** For all $x, y, z \in F, (x \cdot y) \cdot z = x \cdot (y \cdot z)$
- c **Identity:** For all $x \in F$, there exists an element $e \in F$ such that $x \cdot e = e \cdot x = x$
- d **Inverse:** For every **non-zero** element $x \in F$, there exists an element $y \in F$ such that $x \cdot y = e$
- e **Commutative:** For all $x, y \in F, x \cdot y = y \cdot x$

First Four properties forms a group and all the five properties forms an Abelian group on '·'

Note: Identity is the property of the set F and Inverse is the property of the elements of the set F

\implies **Distributive Laws:**

1. $a \cdot (b + c) = a \cdot b + a \cdot c$
2. $(b + c) \cdot a = b \cdot a + c \cdot a$

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2 Vector Space

Let V be a non empty set and F be a field. Let '+' and '·' be two binary operations.

Vector Addition:

- For all $\vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$.
- For all $\vec{u}, \vec{v}, \vec{w} \in V, (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \in V$.
- For all $\vec{u} \in V$, there exists an $\vec{e} \in V$ such that $\vec{u} + \vec{e} = \vec{u} = \vec{e} + \vec{u}$.
- For all $\vec{u} \in V$, there exists a $\vec{v} \in V$ such that $\vec{u} + \vec{v} = \vec{v} + \vec{u} = \vec{e}$.
- For all $\vec{u}, \vec{v} \in V, \vec{u} + \vec{v} = \vec{v} + \vec{u}$.

Scalar Multiplication:

- Let $\vec{u} \in V$ and $k \in F$, then $k\vec{u} \in V$.
- Let $\vec{u} \in V$ and $k_1, k_2 \in F$, then $(k_1 + k_2)\vec{u} = k_1\vec{u} + k_2\vec{u}$
- Let $\vec{u}, \vec{v} \in V$ and $k \in F$, then $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$
- Let $k_1, k_2 \in F$ and $\vec{u} \in V$, then $(k_1 k_2)\vec{u} = k_1(k_2\vec{u})$
- There exists $1 \in F$ such that for all $\vec{u} \in V, 1 \cdot \vec{u} = \vec{u}$

If a non empty set V satisfies all these properties over the Field F under the binary operations '+' and '·', then V is said to be a Vector space over field F .

Notations:

- $\mathbb{R} = (-\infty, \infty)$
- $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$
- $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
- $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{R}\}$

3 Vector Subspaces

7th September 2021

Definition 3.1. Let V be a vector space over the field F and W be its subset. W is called the subspace of V if W is itself a vector space over the same field F with the same binary operations as of V .

OR

Definition 3.2. Let V be a vector space over the field F and W be its subset. W is called the subspace of V over the same field F with the same binary operations if for all $\alpha, \beta \in F$ and u, v in W .

$$\alpha \cdot \mathbf{u} + \beta \cdot \mathbf{v} \in W$$

OR

Definition 3.3. Let V be a vector space over the field F and W be its subset. W is called the subspace of V over the same field F with the same binary operations if for all $u, v \in W$ and $\alpha \in F$.

$$\mathbf{u} + \mathbf{v} \in W \text{ and } \alpha \mathbf{u} \in W$$

3.1 Examples

1. Let V be a set of all functions from S to F . Is V a vector space under the binary operations $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = c \cdot f(x)$ where S is a non-empty set.

$$V = \{f \mid f : S \rightarrow F\}, S \neq \emptyset$$

Vector Addition:

a Closure: Let $f, g \in V \implies f : S \rightarrow F, g : S \rightarrow F$

$$(f + g)(x) = f(x) + g(x) \quad \forall x \in S$$

and since

$$f(x), g(x) \in F, f(x) + g(x) \in F$$

therefore

$$(f + g) : S \rightarrow F \implies (f + g) \in V$$

b Association: For all $x \in S$

$$\begin{aligned} ((f + g) + h)(x) &= (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)), \text{ since } f(x), g(x), h(x) \in F \\ &= f + (g + h)(x) \end{aligned}$$

c Identity: For all f we have to find an e (if it exists) such that

$$f + e = f$$

or

$$f(x) + e(x) = f(x), \quad \forall x \in S$$

which implies

$$e(x) = 0 \quad \forall x \in S$$

$e(x) = 0$ is the Identity element.

d Inverse: For every f there must exist g such that

$$\begin{aligned} f + g &= e \\ \implies f(x) + g(x) &= e(x), \forall x \in S \\ \implies g(x) &= 0 - f(x) \\ \implies g(x) &= -f(x) \end{aligned}$$

There exist $g = -f$ for every f in V .

e Commutative: For all $x \in S$

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ &= g(x) + f(x), \text{ since } f(x), g(x) \in F \\ &= (g + f)(x) \end{aligned}$$

Scalar Multiplication:

a Let $f \in V$ and $k \in F$, then as per definition, for $x \in S$

$$(k \cdot f)(x) = k \cdot f(x)$$

since $k, f(x) \in F$ therefore $k \cdot f(x) \in F$. This implies $c \cdot f \in V$

b Let $f \in V$ and $k_1, k_2 \in F$, then for all x

$$\begin{aligned} ((k_1 + k_2) \cdot f)(x) &= (k_1 + k_2) \cdot f(x) \\ &= k_1 \cdot f(x) + k_2 \cdot f(x) \\ &= (k_1 \cdot f)(x) + (k_2 \cdot f)(x) \end{aligned}$$

Therefore $(k_1 + k_2) \cdot f = k_1 \cdot f + k_2 \cdot f$

c Let $f \in V$ and $k_1, k_2 \in F$, then for all x

$$\begin{aligned} ((k_1 k_2) \cdot f)(x) &= (k_1 k_2) \cdot f(x) \\ &= k_1(k_2 \cdot f(x)), \text{ since } k_1, k_2, f(x) \in F \\ &= k_1(k_2 \cdot f)(x) \end{aligned}$$

which implies $(k_1 k_2) \cdot f = k_1(k_2 \cdot f)$.

d Let $f, g \in V$ and $k \in F$, then for all x

$$\begin{aligned} (k \cdot (f + g))(x) &= k \cdot (f(x) + g(x)) \\ &= k \cdot f(x) + k \cdot g(x), \text{ since } k, f(x), g(x) \in F \\ &= (k \cdot f)(x) + (k \cdot g)(x) \end{aligned}$$

e Let $f \in V$, $1 \in F$ then for all x

$$1 \cdot f(x) = f(x)$$

which implies $1 \cdot f = f$

Hence V is a vector space.

2. Let V be the set of all polynomials from $F \rightarrow F$. Is V a vector space over the field F ?

$$V = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \mid a_i \in F \forall i\}$$

Vector Addition:

- Let $p, q \in V$, then $p + q \in V$
- Let $p, q, r \in V$, then $(p + q) + r = p + (q + r)$
- Let $p \in V$, then for $q = \mathbf{0} \in V$, $p + q = p$
- Let $p \in V$, then for $q = -p \in V$, $p + q = 0$

- Let $p, q \in V$, then $p + q = q + p$

Scalar Multiplication:

- Let $p \in V, k \in F$, then $k \cdot p \in V$
- Let $p \in V$ and $k_1, k_2 \in F$, then $(k_1 + k_2) \cdot p = k_1 \cdot p + k_2 \cdot p$
- Let $p \in V$ and $k_1, k_2 \in F$, then $(k_1 k_2) \cdot p = k_1(k_2 \cdot p)$
- Let $p, q \in V$ and $k \in F$, then $k \cdot (p + q) = k \cdot p + k \cdot q$
- Let $p \in V$ and $1 \in F$, then $1 \cdot p = p$

3. Let V be the set of all polynomials of degree n . Is V a vector space?

V is not a vector space because the following properties fails:

- **Closure:** Consider the polynomials $p = x^n + x + 1$ and $q = -x^n + x + 1$, then

$$p + q = 2x + 1$$

which doesn't belong to V because its degree is not n .

- **Identity:** The degree of the polynomial $p = 0$ is not defined, but it is certainly doesn't belong to V

4. V is the set of all polynomials whose degree is at most n .

$$V = \{a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \mid a_i \in F \forall i \text{ and } k \leq n\}$$

It is a vector space over F because it satisfies all the properties of vector space and for that of the identity element $\mathbf{0}$, it belongs to V (if we put $a_i = 0$ for all i 's)

5. V is the set of all polynomials whose degree is at least n .

Same as **Ex #3**

6. V is the set of all $m \times n$ matrices.

Yes V is a vector space

9th September

7. $W_1 = \{f \in V \mid f \text{ is a continuous function}\}$

$W_1 \subseteq V$, where V is the vector space of all functions $f : S \rightarrow F$. We know that $\mathbf{0}(x) = 0$ for all $x \in S$, hence $\mathbf{0} \in W_1$. Now for f and $g \in W_1$ consider

$$(\alpha f + \beta g)(x), \text{ for all } x \in S$$

This function is continuous because it is the sum of two continuous functions. Hence $(\alpha f + \beta g)(x)$ is also in W_1 . Therefore W_1 is a subspace

8. $W_3 = \{f \in V \mid f(-x) = f(x) \quad \forall x \in S\}$

= set of all even functions

$\mathbf{0}(x) = 0$, for all $x \in S$, hence $\mathbf{0}(x) = \mathbf{0}(-x)$ and therefore $\mathbf{0} \in W_3$. Now Let

$$h(x) = (\alpha \cdot f + \beta \cdot g)(x) \quad \text{for all } x \in S$$

Then

$$\begin{aligned} h(x) &= (\alpha \cdot f)(x) + (\beta \cdot g)(x) \\ &= \alpha f(x) + \beta g(x) \\ &= \alpha f(-x) + \beta g(-x) \\ &= (\alpha \cdot f)(-x) + (\beta \cdot g)(-x) \\ &= (\alpha \cdot f + \beta \cdot g)(-x) \\ &= h(-x) \end{aligned}$$

which implies $(\alpha \cdot f + \beta \cdot g)$ is an even function. Therefore $(\alpha \cdot f + \beta \cdot g) \in W_3$ and W_3 is a subspace.

9. $W_4 = \{f \in V \mid f(x) = -f(-x) \quad \forall x \in S\}$
 = set of all odd functions
 $\mathbf{0}(x) = 0$, for all $x \in S$, hence $\mathbf{0}(x) = -\mathbf{0}(-x)$ and therefore $\mathbf{0} \in W_4$. Now Let

$$h(x) = (\alpha \cdot f + \beta \cdot g)(x) \quad \text{for all } x \in S$$

Then

$$\begin{aligned} h(x) &= (\alpha \cdot f)(x) + (\beta \cdot g)(x) \\ &= \alpha f(x) + \beta g(x) \\ &= \alpha f(-x) + \beta g(-x) \\ &= -(\alpha \cdot f)(-x) - (\beta \cdot g)(-x) \\ &= -(\alpha \cdot f + \beta \cdot g)(-x) \\ &= -h(-x) \end{aligned}$$

which implies $(\alpha \cdot f + \beta \cdot g)$ is an odd function. Therefore $(\alpha \cdot f + \beta \cdot g) \in W_4$ and W_4 is a subspace

10. $V = \mathbb{R}^2, F = \mathbb{R}$ and

$$W = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x \geq 0\}$$

$(0, 0) \in W$. Let $u = (1, 1)$ and $\alpha = -1$, therefore $\alpha \cdot u \notin W$ because $(-1, -1)$ has its x component is negative and W is not a subspace.

11. $W_5 = \{A = M_{n \times n}(F) \mid a_{ij} = a_{ji} \text{ for all } i, j\}$
 W_5 is the set of all $n \times n$ symmetric matrices, hence for all $A \in W_5, A = A^\top$.

$$\text{Null matrix} = \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \text{ It is clear that } \mathbf{0} = \mathbf{0}^\top. \text{ Hence } \mathbf{0} \in W_5.$$

Let $C = \alpha A + \beta B$, where $A, B \in W_5$ and $\alpha, \beta \in F$, then

$$\begin{aligned} C^\top &= (\alpha A + \beta B)^\top \\ &= (\alpha A)^\top + (\beta B)^\top \\ &= \alpha A^\top + \beta B^\top \\ &= \alpha A + \beta B \\ &= C \end{aligned}$$

which implies $C = C^\top$ for all $A, B \in W_5$ and $\alpha, \beta \in F$, and hence $C \in W_5$. W_5 is a subspace

12. $W_6 = \{A = M_{n \times n}(F) \mid a_{ij} = -a_{ji} \text{ for all } i, j\}$
 W_6 is the set of all $n \times n$ symmetric matrices, hence for all $A \in W_6, A = -A^\top$.

$$\text{Null matrix} = \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \text{ It is clear that } \mathbf{0} = \mathbf{0}^\top. \text{ Hence } \mathbf{0} \in W_6.$$

Let $C = \alpha A + \beta B$, where $A, B \in W_6$ and $\alpha, \beta \in F$, then

$$\begin{aligned} C^\top &= (\alpha A + \beta B)^\top \\ &= (\alpha A)^\top + (\beta B)^\top \\ &= -\alpha A^\top - \beta B^\top \\ &= -(\alpha A + \beta B) \\ &= -C \end{aligned}$$

which implies $C = -C^\top$ for all $A, B \in W_6$ and $\alpha, \beta \in F$, and hence $C \in W_6$. W_6 is a subspace

13. $W_7 = \{A = M_{n \times n}(F) \mid A = A^\theta\}$
 W_7 is the set of all $n \times n$ symmetric matrices, hence for all $A \in W_7, A = A^\theta$.

$$\text{Null matrix} = \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \text{ It is clear that } \mathbf{0} = \mathbf{0}^\theta. \text{ Hence } \mathbf{0} \in W_7.$$

Let $C = \alpha A + \beta B$, where $A, B \in W_7$ and $\alpha, \beta \in F$, then

$$\begin{aligned}
C^\theta &= (\alpha A + \beta B)^\theta \\
&= (\overline{\alpha A + \beta B})^\top \\
&= (\overline{\alpha A})^\top + (\overline{\beta B})^\top \\
&= \overline{\alpha} \overline{A}^\top + \overline{\beta} \overline{B}^\top \\
&= \overline{\alpha} A^\theta + \overline{\beta} B^\theta \\
&= \overline{\alpha} A + \overline{\beta} B
\end{aligned}$$

which is not equal to C for all $\alpha, \beta \in F$. For $F = \mathbb{R}$, W_7 is vector subspace but for $F = \mathbb{C}$, W_7 is not a subspace

14. $W_7 = \{A = M_{n \times n}(F) \mid A = -A^\theta\}$

Null matrix = $\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. It is clear that $\mathbf{0} = -\mathbf{0}^\theta$. Hence $\mathbf{0} \in W_7$.

Let $C = \alpha A + \beta B$, where $A, B \in W_7$ and $\alpha, \beta \in F$, then

$$\begin{aligned}
C^\theta &= (\alpha A + \beta B)^\theta \\
&= (\overline{\alpha A + \beta B})^\top \\
&= (\overline{\alpha A})^\top + (\overline{\beta B})^\top \\
&= \overline{\alpha} \overline{A}^\top + \overline{\beta} \overline{B}^\top \\
&= \overline{\alpha} A^\theta + \overline{\beta} B^\theta \\
&= -(\overline{\alpha} A + \overline{\beta} B)
\end{aligned}$$

which is not equal to C for all $\alpha, \beta \in F$. For $F = \mathbb{R}$, W_7 is vector subspace but for $F = \mathbb{C}$, W_7 is not a subspace

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3.2 Intersection of two subspaces

Theorem 3.1. Let V be a vector space over the field F and W_1 and W_2 be two subspaces of the vector space V . Then $W = W_1 \cap W_2$ is also a subspace of V .

Proof. W_1 and W_2 are subspaces of W . Hence

$$\begin{aligned}
&\mathbf{0} \in W_1 \text{ and } \mathbf{0} \in W_2 \\
&\implies \mathbf{0} \in W_1 \cap W_2 \\
&\implies W = (W_1 \cap W_2) \neq \phi
\end{aligned}$$

Let $\alpha, \beta \in F$ and $u, v \in W$. Since $u, v \in W$, then $u, v \in W_1$ and $u, v \in W_2$ which implies $\alpha u + \beta v \in W_1$ and $\alpha u + \beta v \in W_2$, because W_1 and W_2 are subspaces. Since $\alpha u + \beta v \in W_1$ and $\alpha u + \beta v \in W_2$, $\alpha u + \beta v \in W$ for all $\alpha, \beta \in F$ and hence $W = W_1 \cap W_2$ is a subspace. \square

Corollary 3.1.1. Let V be a vector space over the field F and W_i be arbitrary subspaces of the vector space V where $i \in \mathbb{N}$. Then $W = \bigcap_{i=1}^{\infty} W_i$ is also a subspace of V .

3.3 Union of two subspaces

Theorem 3.2. Union of two subspaces of the same vector space is a subspace if and only if one of them is contained in another.

Proof. Let V be a vector space over the field F and W_1 and W_2 be two subspaces of the vector space V .

- **Forward:** $W = W_1 \cup W_2$ is a subspace (Hypothesis)

(Proof by contradiction) Let us assume that one doesn't contain other ($W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$).

$W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$

There exists a $x \in W_1$ such that $x \notin W_2$, There exists a $y \in W_2$ such that $y \notin W_1$,
which implies $x \in W = W_1 \cup W_2$ which implies $y \in W = W_1 \cup W_2$

$\therefore W$ is a subspace $\therefore x + y \in W$

$\therefore x + y \in W$, then either

$x + y \in W_1$ and $x \in W_1$ or $x + y \in W_2$ and $y \in W_2$

$\therefore W_1$ is a subspace $(x + y) - x \in W_1$, that $\therefore W_2$ is a subspace $(x + y) - y \in W_2$, that
is $y \in W_1$, which is a contradiction because is $x \in W_2$, which is a contradiction because
 $y \in W_2$ and $y \notin W_1$. $x \in W_1$ and $x \notin W_2$.

Hence our assumption that one subspace doesn't contain other is false.

- **Backward:** $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ (Hypothesis)

$W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

$W = W_1 \cup W_2 = W_2$ ($\therefore W_1 \subseteq W_2$), hence $W = W_1 \cup W_2 = W_1$ ($\therefore W_2 \subseteq W_1$), hence
 W is subspace W is subspace

□

3.4 Sum of two subspaces

Theorem 3.3. Let V be a vector space over the field F and W_1 and W_2 be two subspaces of the vector space V . Then $W = W_1 + W_2$ is also a subspace of V .

Proof. $W = W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}$. $\mathbf{0} \in W_1$ and W_2 , hence $\mathbf{0} + \mathbf{0} = \mathbf{0}$ which also belongs to $W \neq \emptyset$. For all $\alpha, \beta \in F$ and $x, y \in W$ consider

$$\alpha \cdot x + \beta \cdot y = \alpha \cdot (u + v) + \beta \cdot (p + q)$$

where $x = u + v$ for some $u \in W_1$ and $v \in W_2$ and $y = p + q$ for some $p \in W_1$ and $q \in W_2$. Therefore

$$\alpha \cdot x + \beta \cdot y = (\alpha u + \beta p) + (\alpha v + \beta q)$$

since $u, p \in W_1$ and $v, q \in W_2$

$$\alpha x + \beta y = s + t$$

where $s \in W_1$ and $t \in W_2$ and therefore $\alpha x + \beta y \in W$ for all $\alpha, \beta \in F$ and $x, y \in W$. Hence W is subspace of V □

4 Linear Dependency and Independency

4.1 Linear Combination

Definition 4.1. Let v be a vector of the vector space V . Then v is a linear combination of $u_1, u_2 \in V$ if there exist $c_1, c_2 \in F$ such that

$$v = c_1 u_1 + c_2 u_2$$

Generalized form: For $u_1, u_2, \dots, u_n \in V$ $v \in V$ is Linear Combination of u_1, u_2, \dots, u_n if there exist $c_1, c_2, \dots, c_n \in F$ such that

$$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$$

Theorem 4.1. Set of all possible linear combinations of $u_1, u_2, \dots, u_n \in V$, forms a subspace of V over the field F

Proof. The set $S = \{c_1 u_1 + c_2 u_2 + \dots + c_n u_n \in V \mid c_i \in F, \forall i's\} = \left\{ \sum_{i=1}^n c_i u_i \mid c_i \in F, \forall i's \right\}$.

Putting $c_i = 0$ for all i , we get the $\mathbf{0}$ vector, implying $\mathbf{0} \in S$. Consider $u, v \in S$, then there exists $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n \in F$ such that

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$$

$$v = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_n u_n$$

Then

$$\begin{aligned} u + v &= \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n \\ &= (\alpha_1 + \beta_1)u_1 + (\alpha_2 + \beta_2)u_2 + \cdots + (\alpha_n + \beta_n)u_n \end{aligned}$$

since $\alpha_i, \beta_i \in F$ for all i 's $\implies \alpha_i + \beta_i \in F$ for all i 's and hence $u + v \in S$.

Let $\alpha \in F$, then

$$\begin{aligned} \alpha u &= \alpha(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) \\ &= (\alpha \alpha_1)u_1 + (\alpha \alpha_2)u_2 + \cdots + (\alpha \alpha_n)u_n \end{aligned}$$

also belongs in S because for all $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\alpha \in F$, $\alpha \alpha_i \in F$ for all i 's.

Therefore S is subspace by Definition (3.3). \square

4.2 Span

Definition 4.2. Let V be a vector space over the field F and $S \subseteq V$. Then the set of all possible linear combinations of the set S is called the span of that set or the spanning set of S and is written as $\text{span}(S)$.

$$\begin{aligned} S &= \{v_1, v_2, \dots, v_n\} \\ \text{span}(S) &= \{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \mid c_i \in F, v_i \in S\} \\ &= \left\{ \sum_{i=1}^n c_i v_i \mid c_i \in F, v_i \in S \right\} \\ &\text{OR} \end{aligned}$$

Definition 4.3. Span S is the intersection of all subspaces of V containing S .

Corollary 4.1.1. Let V be a vector space over the field F and $S \subseteq V$, then $\text{Span}(S)$ is a subspace of V .

Proof. For $S = \phi$, from definition (4.3) S is subspace of V and for $S \neq \phi$, from Definition (4.2) and Theorem (4.1) S is a subspace of V . \square

4.3 Linearly Independent (L.I) and Linearly Dependent (L.D)

Definition 4.4. Let V be a vector space and $S = \{v_1, v_2, \dots, v_n\}$ be a subset of V . Then the set S is **linearly independent** if

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \implies c_i = 0 \text{ for all } i\text{'s}$$

where $v_1, v_2, \dots, v_n \in S$ and $c_1, c_2, \dots, c_n \in F$. If it is not linearly independent then its **linearly dependent**, i.e there exists atleast one $c_i \neq 0$ when $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$.

Theorem 4.2. Let V be a vector space over the field F and S be a non empty subset of V . S is linearly dependent if and only if atleast one vector of S can be written as a linear combination of others.

Proof. Let $S = \{v_1, v_2, \dots, v_n\}$ be non-empty subset of V .

Forward: S is linearly dependent. There exists a $c_i \neq 0$ such that

$$c_1 v_1 + \cdots + c_{j-1} v_{j-1} + c_j v_j + c_{j+1} v_{j+1} + \cdots + c_n v_n = 0$$

Without loss of generality we assume $c_j \neq 0$.

$$\begin{aligned} \therefore c_j v_j &= -c_1 v_1 - \cdots - c_{j-1} v_{j-1} - c_{j+1} v_{j+1} - \cdots - c_n v_n \\ v_j &= -\frac{c_1}{c_j} v_1 - \cdots - \frac{c_{j-1}}{c_j} v_{j-1} - \frac{c_{j+1}}{c_j} v_{j+1} - \cdots - \frac{c_n}{c_j} v_n \\ v_j &= k_1 v_1 + \cdots + k_{j-1} v_{j-1} + k_{j+1} v_{j+1} + \cdots + k_n v_n \end{aligned}$$

which says that v_j is the linear combination of the remaining vectors of S .

Backward: Atleast one vector of S can be written as remaining others. This implies that

$$\begin{aligned} v_i &= c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \cdots + c_n v_n \\ \implies c_1 v_1 + \cdots + c_{i-1} v_{i-1} + (-1) v_i + c_{i+1} v_{i+1} + \cdots + c_n v_n &= 0 \end{aligned}$$

Let $c_i = -1$, then the linear combination of the vectors of S is 0 for atleast one $c_j \neq 0$, ($c_i = -1$) and hence S is linearly dependent. \square

4.3.1 Properties

1. $S = \phi$ is a Linearly Independent set.

2. $S = \{v\}$ is a singleton set, then

(a) If $v \neq 0$, then $c \cdot v \implies c = 0$.

(b) If $v = 0$, then $c \cdot v \implies c = 0$ for all values of c .

Hence S is linearly independent if $v \neq 0$ else linearly dependent.

3. If S is linearly dependent set in V , then subset of S may or may not be linearly dependent.

Ex. $S = \{(0, 1), (1, 0), (1, 1), (2, 2)\}$

Consider $W_1 = \{(0, 1), (1, 0)\} \subseteq S$ and $W_2 = \{(1, 1), (2, 2)\} \subseteq S$. W_1 is linearly independent and W_2 is linearly dependent

4. If S is linearly dependent set in V , then superset of S is also linearly dependent.

Proof. Suppose $S = \{v_1, v_2, \dots, v_m\}$, and $W = \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$

Clearly $W \supseteq S$. Now consider the linear combination of the vectors in W .

$$\underbrace{c_1 v_1 + c_2 v_2 + \dots + c_m v_m}_{= 0 \text{ for atleast one } c_j \neq 0} + c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \dots + c_n v_n$$

Therefore

$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m + c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \dots + c_n v_n = 0$$

implies there exists atleast one $c_i = c_j \neq 0$. Hence $W \supseteq S$ is linearly dependent. \square

5. If S is a linearly independent set in V then any subset of S is also linearly independent.

Proof. (Proof by contradiction) Let us assume that a subset T of S is linearly dependent, then the linear combination

$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m = 0 \implies \text{that there exist atleast one } c_i \neq 0, 1 \leq i \leq m \quad (1)$$

Now consider the original set S which is linearly independent. The equation

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

implies that $c_i = 0$ for all i 's. Lets inspect the linear combination more carefully.

$$\underbrace{c_1 v_1 + c_2 v_2 + \dots + c_m v_m}_{= 0 \implies \text{there exist a } c_j \neq 0} + \underbrace{c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \dots + c_n v_n}_{= 0 \text{ if we put } c_i = 0 \text{ for all } i > n} = 0$$

Hence

$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m + c_{m+1} v_{m+1} + c_{m+2} v_{m+2} + \dots + c_n v_n = 0$$

implies there exist a $c_j \neq 0$. This shows that S is a linearly dependent set which contradicts with our assumption and proves T is a linearly independent set. \square

6. If S is a linearly independent set in V , then the superset of S may or may not be linearly independent.

Ex. $S = \{(-1, 0)\}$ and $T_1 = \{(-1, 0), (0, 1)\}$ and $T_2 = \{(-1, 0), (-2, 0)\}$ then, T_1 is linearly independent and T_2 is linearly dependent

5 Basis

\implies span S is the subspace of V

\implies span $\phi = 0$

\implies span S is the intersection of all subspace of V containing S

\implies If S is a subspace of V , then span $S = S$

\implies span(span S) = span S , span(span(\cdots span(S) \cdots)) = span S

\implies If S is a subspace, then span(span(\cdots span(S) \cdots)) = span S

Definition 5.1. Let V be a vector space over the field F and B be non empty subset of V . Then B is called the basis of V if

(a) B is linearly independent

(b) span(B) = V

Dimension of a Basis: The number of elements in the basis is called the dimension of that basis.

OR

The cardinality of the basis is called its dimension.

$C(B) < \infty \implies V$ is a finite dimensional vector space

$C(B) \not< \infty \implies V$ is an infinite dimensional vector space

Number of Basis of any vector space can be infinite but dimension of a vector space is unique.

5.1 Properties

Let S be a subset of V and span(S) = V

1. For any $w \in V$, $S = \{u_1, u_2, \dots, u_n\}$ then $S \cup \{w\}$ also spans V , i.e span($S \cup \{w\}$) = V

2. If any u_i is a linear combination of u_1, u_2, \dots, u_{i-1} then $S \setminus \{u_i\}$ also spans V , i.e

$$\text{span}(S \setminus \{u_i\}) = V$$

3. Standard basis of the Euclidean space \mathbb{R}^n is

$$\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$$

4. $P[x] = \{a_0, a_1x + \dots + a_nx_n + \dots \mid a_i \in F \text{ for all } i\}$. Then

$$B = \{1, x, x^2, \dots, x^n, \dots\}$$

is the basis for $P[x]$

5. Let W be the set of all solution of the second order differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad (2)$$

Then W is forms a subspace.

Proof.

$$W = \left\{ y(x) \mid \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \right\}$$

$y(x) = 0$ satisfies equation (2). Hence $y(x) = 0$ is a solution of the differential equation. Let $y_1(x)$ and $y_2(x)$ be two solutions of (2), then consider $\alpha y_1 + \beta y_2$.

$$\begin{aligned} & \frac{d^2(\alpha y_1 + \beta y_2)}{dx^2} + P(x) \frac{d(\alpha y_1 + \beta y_2)}{dx} + Q(x)(\alpha y_1 + \beta y_2) \\ &= \left(\alpha \frac{d^2 y_1}{dx^2} + \alpha P(x) \frac{dy_1}{dx} + \alpha Q(x) y_1 \right) + \left(\beta \frac{d^2 y_2}{dx^2} + \beta P(x) \frac{dy_2}{dx} + \beta Q(x) y_2 \right) \\ &= \alpha \left(\frac{d^2 y_1}{dx^2} + P(x) \frac{dy_1}{dx} + Q(x) y_1 \right) + \beta \left(\frac{d^2 y_2}{dx^2} + P(x) \frac{dy_2}{dx} + Q(x) y_2 \right) \\ &= 0 + 0 = 0 \end{aligned}$$

Hence W is a subspace. The general solution y of the above differential equation is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad (3)$$

where $y_1(x)$ and $y_2(x)$ are two solutions of (2). From (3) we can say that any solution can be written as the linear combination of y_1 and y_2 . Hence y_1 and y_2 spans W and since (3) is the general solution y_1 and y_2 are linearly independent. Therefore $B = \{y_1, y_2\}$ is the basis for W whose dimension is 2.

Generalization: Let W be the set of all solutions of the differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{dy}{dx} + \cdots + a_0(x)y = 0 \quad (4)$$

Then W forms a subspace (vector space?) and its basis is given by $B = \{y_1, y_2, \dots, y_n\}$ with dimension equal to the order of the D.E = n and where y_1, y_2, \dots, y_n are the solutions of (4) and they form the general solution of (4), that is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

□

Theorem 5.1. Let V be finite dimensional vector space over the field F and let L be the set of linearly independent vectors in V and S be spanning set of V , then cardinality of L is less than or equal to cardinality of $\text{span}(S)$.

Proof. $\text{span}(S) = V$ and S can be either linearly independent or linearly dependent. If S is linearly independent then by definition S becomes the basis for V . If S is not linearly independent then it is linearly dependent and at least one vector of S can be written as linear combination of other vectors of S say v_1 . Since S spans V , any vector $v \in V$ can be written as linear combination of vectors in S .

$$\begin{aligned} v &= c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \\ &= c_1 (k_1 v_2 + k_2 v_3 + \cdots + k_{n-1} v_n) + c_2 v_2 + \cdots + c_n v_n \\ &= (c_1 k_1 + c_2) v_2 + (c_1 k_2 + c_3) v_3 + \cdots + (c_1 k_{n-1} + c_n) v_n \\ &= \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n \end{aligned}$$

which implies there exists α_i for every $v \in V$ such that v can be written as a linear combination of vectors in S . Hence $\text{span}(S) = V$ □

Theorem 5.2. Let V be a **finite dimensional vector space**, with dimension of V equal to n , then any subset of V with $(n + 1)$ or more vectors is a linearly dependent set (it cannot form a basis for V).

Notes:

Let V be a **finite dimensional vector space**, with $\dim V = n$ and B be a subset of V with n number of vectors, then

If $\text{span}(B) = V$, then B is a basis of V .

OR

If B is linearly independent, then B is a basis of V

That is for checking whether a subset B of V with n vectors is a basis of V either check if it is Linearly Independent or if it spans V

Also the vector space $V = \{0\}$ has the set $B = \emptyset$ as its basis

Theorem 5.3. Let V be a **finite dimensional vector space** over the field F and B be a subset of V . The set B is a basis for V if and only if every vector of V can be written as a unique linear combination of the vectors in B .

Proof. Forward direction: $B = \{u_1, u_2, \dots, u_n\}$ is a basis for V .

Let's assume that there exist two distinct set of scalars $\{c_1, c_2, \dots, c_n\}$ and $\{k_1, k_2, \dots, k_n\}$ for any vector $v \in V$ such that

$$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$$

$$v = k_1 u_1 + k_2 u_2 + \dots + k_n u_n.$$

Subtracting one from another, we get

$$(c_1 - k_1)u_1 + (c_2 - k_2)u_2 + \dots + (c_n - k_n)u_n = 0$$

since the vectors from the basis are all linearly independent we have $c_i = k_i$ for all i 's, which is a contradiction because they are suppose to be different from our assumption.

Backward direction: every vector of V can be written as a unique linear combination of the vectors in B .

There exists a unique set of scalars c_1, c_2, \dots, c_n for every vector $v \in V$ such that

$$v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$$

That is B spans V . Now we have to show that the vectors in B are linearly independent, for that consider the 0 vector.

$$0 = 0 \cdot u_1 + 0 \cdot u_2 + \dots + 0 \cdot u_n.$$

Since this combination is unique, there is no other way of writing the 0 vector. This states that the equation

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0$$

has only one solution and that is $c_i = 0$ for all i 's. Hence B is a basis for V

□

5.2 How to check whether a set is a basis?

Suppose dimension of V is n and S is a subset of V to be checked for basis.

$$\dim V = n \text{ and } S \subseteq V$$

If $ S < n$, then S can never span V and hence S is not a basis of V	If $ S = n$, then check either S is linearly independent or $\text{span}(S) = V$ for it to be a basis of V	If $ S > n$, then S becomes L.D and hence S cannot form a basis of V
--	--	--

Theorem 5.4. Let V be a finite dimensional vector space over the field F , and W be its subspace, then $\dim W \leq \dim V$

5.3 Ordered basis and Direct Sum

Coordinate vector: Let V be a finite dimensional vector space over the field F and B be a basis of V ($B = \{v_1, v_2, \dots, v_n\}$). Then for any $v \in V$,

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

and

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $[v]_B$ is called the coordinate vector of v w.r.t the basis B .

Direct Sum: Let W_1 and W_2 be two sub-spaces of a vector space V , then $W_1 + W_2$ is called the direct sum of V if every member of V can be uniquely expressed as $w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. The direct sum of V is denoted by $W_1 \oplus W_2$

Examples of finding the basis:

Theorem 5.5. Let W_1 and W_2 be two subspaces of V , then $W_1 + W_2$ is said to be the direct sum of V if and only if $W_1 \cap W_2 = \{0\}$

Example: Let $V = \mathbb{R}^3$ and $W_1 = \{(0, y, z) : y, z \in \mathbb{R}\}$ and $W_2 = \{(x, y, 0) : x, y \in \mathbb{R}\}$. Is V the direct sum of W_1 and W_2 .

Solution: $W_1 \cap W_2 = \{(0, y, 0) : y \in \mathbb{R}\}$. V is not a direct sum of W_1 and W_2 . Also there is no unique sum for every element of V . That is

$$\begin{aligned}(1, -2, 1) &= (0, -2, 1) + (1, 0, 0) \\ &= (0, -1, 1) + (1, -1, 0) \\ &\vdots\end{aligned}$$

Note:

- $\dim(\mathbb{C}^n(\mathbb{R})) = 2n$
- $\dim(\mathbb{C}^n(\mathbb{C})) = n$

$$\dim(W_1 \cup W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

6 Linear Transformation

Definition 6.1. Let V and W be two vector spaces over the same field F . Then $T : v \rightarrow W$ is said to be a linear transformation if

- (a) For all $u, v \in V$, $T(u + v) = T(u) + T(v)$
- (b) For all $\alpha \in F$ and $u \in V$, $T(\alpha u) = \alpha T(u)$

OR

For all $\alpha, \beta \in F$ and $u, v \in V$, $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$

Note:

- (i) If $T(0) \neq 0$, then the map T is not a linear transformation, where $0 \in V$ and W

6.1 Special Transformations

1 Identity Transformation: Let V be a vector space over the field F , define $I : V \rightarrow V$ such that

$$T(v) = v$$

for all $v \in V$, then I is a linear transformation

Proof. Consider $\alpha, \beta \in F$ and $v, u \in I$, then

$$\begin{aligned}I(\alpha u + \beta v) &= \alpha u + \beta v \quad \text{since } \alpha u + \beta v \in V \\ &= \alpha I(u) + \beta I(v)\end{aligned}$$

Therefore I is a linear transformation. □

2 Zero Transformation: Let V, W be two subspaces of the vector space V over the field F , then the map $O : V \rightarrow W$ where

$$O(v) = 0$$

is called the linear transformation.

Proof. Consider $\alpha, \beta \in F$ and $v, u \in O$, then

$$\begin{aligned} O(\alpha u + \beta v) &= 0 \quad \text{since} \quad \alpha u + \beta v \in V \\ \alpha O(u) + \beta O(v) &= \alpha 0 + \beta 0 \\ &= 0 \\ \therefore O(\alpha u + \beta v) &= \alpha u + \beta v \end{aligned}$$

Hence O is a linear transformation. □

6.2 Null Space (Kernel) and Range Space of T

Definition 6.2. Let T be a linear transformation from V to W , $T : V \rightarrow W$. Then

$$N(T) = \ker(T) = \{x \in V \mid T(x) = 0\}$$

is called the **Null Space or Kernel of T**

Note: Null space of any linear transformation is always non empty because $T(0)$ is always equal to 0 for any linear transformation T .

Theorem 6.1. Let V and U be vector spaces over the same field F . Let $T : V \rightarrow W$ be a linear transformation, then $N(T)$ is a subspace of V

Proof. Since T is a linear transformation, $T(0) = 0$, hence $0 \in N(T)$ and $N(T) \neq \phi$ and also $N(T) \subseteq V$. Now consider $\alpha, \beta \in F$ and $u, v \in N(T)$, then $T(u) = T(v) = 0$.

$$\begin{aligned} T(\alpha u + \beta v) &= \alpha \cdot T(u) + \beta \cdot T(v) \\ &= \alpha \cdot 0 + \beta \cdot 0 \\ &= 0 \end{aligned}$$

Therefore $\alpha u + \beta v$ also belongs to $N(T)$, which implies $N(T)$ is a subspace of V □

Nullity of T : The dimension of the Null Space of any L.T T is called the nullity of T , It is represented as $\eta(T)$

For a finite dimensional vector space V , $\dim(N(T)) \leq \dim(V)$

Definition 6.3. Let V and W be finite dimensional vector spaces over the same field F and $T : V \rightarrow W$ be a linear transformation, then

$$R(T) = \{w \in W \mid \exists v \in V, T(v) = w\}$$

is called the **Range Space of T**

Theorem 6.2. Let $T : V \rightarrow W$ be a linear transformation where V and W are finite dimensional vector spaces over same field F , then the range space of T , $R(T)$ is a subspace of W . i.e $R(T) \leq W$

Proof. Very simple, similar to null space being a subspace of V □

Note: The dimension of range space of T is denoted as $\rho(T)$

6.3 Linear Transformation Examples

$$1 \quad T : P_4(x) \rightarrow P_3(x) \text{ and } T(p(x)) = \int_0^x p(t) dt$$

The linear transformation is not well defined, all polynomials of degree ≥ 3 lie outside of the codomain

$$2 \quad T : P(x) \rightarrow P(x) \text{ and } T(p(x)) = \int_0^x p(t) dt$$

$$\begin{aligned} T(\alpha p + \beta q) &= \int_0^x (\alpha p(t) + \beta q(t)) dt \\ &= \alpha \int_0^x p(t) dt + \beta \int_0^x q(t) dt \\ &= \alpha T(p) + \beta T(q) \end{aligned}$$

3 $T : P(x) \rightarrow P(x)$ and $T(p(x)) = p''(x)$

It is a linear transformation.

4 $T : P(x) \rightarrow P(x)$ and $T(p(x)) = p'(x)$

It is a linear transformation.

5 $T : P(x) \rightarrow P(x)$ and $T(p(x)) = p''(x) + p(x)$

It is a linear transformation.

Theorem 6.3. Let $T : V \rightarrow W$ be a linear transformation, where V and W are vector spaces over a same field F . If $S = \{v_1, v_2, \dots, v_n\}$ be a spanning set of V , then the set $\{T(v_1), T(v_2), \dots, T(v_n)\}$, say R is the spanning set of the range space of T .

Proof. Range space of $T = \{w \in W \mid \exists v \in V, w = T(v)\}$ and since $\text{span}(S) = V$, for every $v \in V$ there exists scalars c_1, c_2, \dots, c_n from the field F such that

$$\begin{aligned} v &= c_1 v_1 + c_2 v_2 + \dots + c_n v_n \\ T(v) &= T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n) \\ w &= c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n) \end{aligned}$$

Hence $R = \{T(v_1), T(v_2), \dots, T(v_n)\}$ spans the range space of T . □

Theorem 6.4. Let $T : V \rightarrow W$ be a linear transformation, where V and W are vector spaces over a same field F and $S = \{v_1, v_2, \dots, v_n\}$ be a subset of V . If the set $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is linearly independent set, then S is also a linearly independent set.

Proof. Consider the equation

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

where c_i 's are scalars from the field F . Then

$$\begin{aligned} T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n) &= T(0) \\ c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n) &= 0 \end{aligned}$$

since $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is an L.I set, $c_1 = c_2 = \dots = c_n = 0$. We just showed that

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \implies c_1 = c_2 = \dots = c_n = 0$$

. Hence $\{v_1, v_2, \dots, v_n\}$ is linearly independent. □

Theorem 6.5. Let V and W be two finite dimensional vector spaces over F and $\{v_1, v_2, \dots, v_n\}$ be a basis for V and u_1, u_2, \dots, u_n be any vectors in W . Then there exists a **unique** Linear Transformation from V to W such that $T(v_i) = u_i$ for $1 \leq i \leq n$

Proof. Lengthy, hence skipped. □

Theorem 6.6. Rank Nullity Theorem Let $T : V \rightarrow W$ be a linear transformation and V is a **finite dimensional vector space**. Then

$$\eta(T) + \rho(T) = \dim V$$

OR

$$\dim N(T) + \dim R(T) = \dim V$$

Proof. skipped. □

Note:

- For homogeneous systems $\dim(R(T)) = n - r$
- For non-homogeneous systems $\dim(R(T)) = n - r + 1$

where r is the rank of the coefficient matrix and n is the number of variables (from the order $m \times n$)

6.4 Algebra of Linear Transformation

Theorem 6.7. Let $T : V \rightarrow W$ and $S : V \rightarrow W$ be two linear transformation, then their sum $T + S$ and the scalar multiplication cT , $c \in F$ are also linear transformations. V, W, F, c have their regular meanings.

Corollary 6.7.1. Let $\text{Hom}(V, W)$ be the set of all linear transformations from vector spaces $V \rightarrow W$ over the same field F , then $\text{Hom}(V, W)$ forms a subspace of the vector space of all functions or maps.

If $\dim(V) = m$, $\dim(W) = n$, then $\dim(\text{Hom}(V, W)) = mn$

Theorem 6.8. Let $T : V \rightarrow W$ and $S : W \rightarrow V$ be two linear transformations, then $T \circ S$ and $S \circ T$ are also linear transformations from W to W and V to V respectively.

Proof. For all $u, v \in V$ and $\alpha, \beta \in F$ the expression

$$\begin{aligned}(T \circ S)(\alpha u + \beta v) &= T(S(\alpha u + \beta v)) \\ &= T(S(\alpha u) + S(\beta v)) \\ &= T(\alpha S(u) + \beta S(v)) \\ &= T(S(\alpha u)) + T(S(\beta v)) \\ &= \alpha T(S(u)) + \beta T(S(v)) \\ &= \alpha(T \circ S)(u) + \beta(T \circ S)(v)\end{aligned}$$

□

Definition 6.4. Singular Map: Let V and W be vector subspaces over the same field F and $T : V \rightarrow W$ be a linear transformation then T is called a singular map if there exists a non zero vector $x \in V$ such that $T(x) = 0$

Definition 6.5. Non-Singular Map: Let V and W be vector subspaces over the same field F and $T : V \rightarrow W$ be a linear transformation then T is called a singular map if there does not exist a non zero vector $x \in V$ such that $T(x) = 0$

Note: if $\eta(T) > 0$, then T is a singular map and if $\eta(T) = 0$ then T is a non-singular map

6.5 Invertible Maps

Theorem 6.9. Let $T : V \rightarrow V$ be a linear transformation where V is a **finite dimensional vector space**, then

$$\begin{aligned}\ker(T) = 0 &\iff T \text{ is one-one} \\ &\iff T \text{ is onto} \\ &\iff T \text{ is bijective} \\ &\iff T \text{ is invertible} \\ &\iff T \text{ is non singular}\end{aligned}$$

Note: For the linear transformation $T : V \rightarrow W$

1. If $\dim(V)$ is finite, then

$$\begin{aligned}\ker(T) = 0 &\iff T \text{ is one-one} \\ &\iff T \text{ is non singular}\end{aligned}$$

2. If $\dim(V) = \dim(W)$, then

$$\begin{aligned}\ker(T) = 0 &\iff T \text{ is one-one} \\ &\iff T \text{ is onto} \\ &\iff T \text{ is bijective} \\ &\iff T \text{ is invertible} \\ &\iff T \text{ is non singular}\end{aligned}$$

6.6 Matrix Representation

Definition 6.6. $T : V \rightarrow V$ be a Linear Transformation where V is a **finite dimensional vector space**. Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V . Then matrix representation of T with respect to B is given as $[T]_B$

$$\begin{aligned} T(v_1) &= a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \\ T(v_2) &= a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n \\ &\vdots \\ T(v_n) &= a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nn}v_n \end{aligned}$$

$$[T]_B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}^T$$

$$[T]_{B_1} \sim [T]_{B_2} \sim \dots [T]_{B_n} \sim \dots$$

By similar we mean the matrix representations have **same eigen values, trace, detereminant** (for a linear operator, i.e a transformation from V to V)

6.7 Change of Basis

Definition 6.7. Let $T : V \rightarrow V$ be a linear transformation and B_1 and B_2 be two bases of V . $B_1 = \{v_1, v_2, \dots, v_n\}$ and $B_2 = \{u_1, u_2, \dots, u_n\}$. Then

$$\begin{aligned} v_1 &= a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n \\ v_2 &= a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n \\ &\vdots \\ v_n &= a_{n1}u_1 + a_{n2}u_2 + \dots + a_{nn}u_n \end{aligned}$$

and

$$[T]_{B_2}^{B_1} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}^T$$

is called the change of basis matrix from B_1 to B_2

OR

Definition 6.8. Let $T : V \rightarrow W$ be a linear transformation and B_1 and B_2 be the bases of V and W respectively. $B_1 = \{v_1, v_2, \dots, v_m\}$ and $B_2 = \{u_1, u_2, \dots, u_n\}$. Then

$$\begin{aligned} T[v_1] &= a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n \\ T[v_2] &= a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n \\ &\vdots \\ T[v_m] &= a_{m1}u_1 + a_{m2}u_2 + \dots + a_{mn}u_n \end{aligned}$$

and

$$[T]_{B_2}^{B_1} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}^T$$

is called the change of basis matrix from B_1 to B_2

Basically change of basis gives us a matrix which maps coordinate vectors from one basis to another, that is $[T]_{B_2}^{B_1}$ maps coordinate vector of vector in B_1 basis to coordinate vector of the same vector in B_2 basis

Note: If $P = [T]_{B_1}^{B_2}$ and $Q = [T]_{B_2}^{B_1}$, then

- P and Q are invertible matrices
- P and Q are inverses of each other

$$P^{-1} = Q \text{ and } Q^{-1} = P \text{ and } PQ = I = QP$$

- $[T]_{B_1}^{B_2} = [u_1 u_2 \cdots u_n]$ where B_1 is a standard basis and $B_2 = \{u_1, u_2, \cdots u_n\}$ is any other basis.

7 Eigen Values and Eigen Vectors

Definition 7.1. Let $T : V \rightarrow V$. If for a non zero vector $v \in V$ there exists a scalar $\lambda \in F$, such that

$$T(v) = \lambda v$$

, then λ is called the eigen value of T and v is called the eigen vector of T corresponding to the eigen value λ .

7.1 How to find Eigen Values ?

We know that $T(v) = \lambda v \implies (T - \lambda I)v = 0$. Find the null space of the transformation $(T - \lambda I)$.

OR

Since $T : V \rightarrow V$, take any matrix representation of T w.r.t to any basis B , i.e $[T]_B = A$ and then plug it in the equation $(T - \lambda I)v = 0$. We are looking for non zero v 's, if any exists, hence for that we should make the matrix $A - \lambda I$ singular. Solve $\det(A - \lambda I)$ for λ

$\det(A - \lambda I) = \text{characteristic polynomial}$ $\det(A - \lambda I) = 0 \implies \lambda \rightarrow \text{eigen values} / \text{eigen roots} / \text{latent roots} / \text{characteristics}$

7.1.1 2×2 and 3×3 characteristics polynomial

$$A_{2 \times 2} \implies \lambda^2 - (\text{trace})\lambda + \det(A) = 0$$

$$A_{3 \times 3} \implies \lambda^3 - (\text{trace})\lambda^2 + (\text{sum of cofactors of diagonal elements})\lambda - \det(A) = 0$$

7.1.2 Special Matrices and their Eigen Values

Matrix	Eigen Values
diagonal matrix	diagonal entries
upper triangular matrix	diagonal entries
lower triangular matrix	diagonal entries
sum of each row or each column is s	s is an eigen value
$\det(A) = 0$	0 is an eigen value
scalar matrix $\begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{bmatrix}$	c
Back Diagonal Matrix of even order, example $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$	$\pm \sqrt{a_{n,n-i} \cdot a_{i,n-1}}$, where n is the order and i varies from 1 to $\frac{n}{2}$, that is $\pm \sqrt{4 \cdot 1}, \pm \sqrt{3 \cdot 2}$
Back Diagonal Matrix of odd order, example $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$	$\pm \sqrt{a_{n,n-i} \cdot a_{i,n-1}}$, where n is the order and i varies from 1 to $\left\lfloor \frac{n}{2} \right\rfloor$ and a_{kk} , where k is $\left\lceil \frac{n}{2} \right\rceil$, that is $\pm \sqrt{1 \cdot 3}, 2$