Imp **Linear diff^tal eq^n of IInd order :**

The general form of eq^n of IInd order is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R. \quad \text{——— (1)}$$

where P, Q, R be the fn^n of x only.

Let $y = u$ be one integral part of complementary fn^n (CF)

Let $y = uv$ ——— (2)

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\cdot\frac{du}{dx} \quad \text{——— (3)}$$

$$\therefore \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{dv}{dx} + \frac{dv}{dx}\cdot\frac{du}{dx} + v\cdot\frac{d^2u}{dx^2}$$

$$\therefore \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\cdot\frac{du}{dx}\frac{dv}{dx} + v\cdot\frac{d^2u}{dx^2} \quad \text{--- (4)}$$

Put the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in eqn (1) and simplify, we get

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u} \quad \text{--- (5)}$$

Let $\frac{dv}{dx} = p$

$\therefore$ Eqn (5) becomes

$$\frac{dp}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right)p = \frac{R}{u} \quad \text{--- (6)}$$

Which is linear in p.

$$\therefore \quad IF = e^{\int \left(\frac{2}{u}\frac{du}{dx} + P\right) dx}$$

$$= e^{\int \frac{2}{u} du + \int P dx}$$

$$= e^{2 \log u + \int P dx}$$

$$= e^{\log u^2 + \int P dx}$$

$$= e^{\log u^2} \cdot e^{\int P dx}$$

$$= u^2 e^{\int P dx}$$

$\therefore$ The required soln is

$$p \cdot u^2 e^{\int P dx} = \int \frac{R}{u} u^2 e^{\int P dx} dx + c_1$$

$$\therefore\ p = u^{-2} e^{-\int P\,dx} \int R u e^{\int P\,dx}\,dx + c_1 u^{-2} e^{-\int P\,dx}$$

$$\therefore\ \frac{dv}{dx} = u^{-2} e^{-\int P\,dx} \int R u e^{\int P\,dx}\,dx + c_1 u^{-2} e^{-\int P\,dx}$$

Integrate

$$v = \int \left(u^{-2} e^{-\int P\,dx} \int R u e^{\int P\,dx} \right) dx + \boxed{c_1 \int u^{-2} e^{-\int P\,dx}\,dx + c_2}$$

$\therefore$ The required soln is

$$y = u \int \left(u^{-2} e^{-\int P\,dx} \int R u e^{\int P\,dx} \right) dx + c_1 u \int u^{-2} e^{-\int P\,dx}\,dx + c_2 u$$

$$\therefore\ y = \underbrace{c_2 u + c_1\left(u \int u^{-2} e^{-\int P\,dx}\,dx\right)}_{\text{Complementary function}}$$

$$\underbrace{+\, u \int \left(u^{-2} e^{-\int P\,dx} \int R\, u\, e^{\int P\,dx}\,dx \right) dx}_{\text{Particular Integral.}}$$

Note: (1) The Ind integral part of CF

$$= u \int u^{-2} e^{-\int P\,dx}\,dx$$

$$= u \int \frac{e^{-\int P\,dx}}{u^2}\,dx$$

(2) $$PI = u\int\left(\frac{e^{-\int P\,dx}}{u^2}\int R u e^{\int P\,dx}\,dx\right)dx$$

Ex:- Let $y = x$ be one integral part of C.F of the DE

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3.$$

The other integral part is - - - - - -

& also PI is - - - - - -

Soln: The given diff'tal eqn can be written as

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x$$

which is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$\therefore P = -2\left(\frac{1}{x}+1\right), \quad Q = 2\left(\frac{1}{x^2}+\frac{1}{x}\right), \quad R = x$$

given $y = x$ be one soln of CF

The other integral part of CF

$$= x\int \frac{e^{2\int\left(\frac{1}{x}+1\right)dx}}{x^2}\,dx$$

If

$P + Qx = 0$.

Then $y = x$

is one of

integral

part of

CF.

$$= x \int \frac{e^{2(\log x + x)}}{x^2}\, dx$$

$$= x \int \frac{e^{\log x^2 + 2x}}{x^2}\, dx$$

$$= x \int \frac{x^2 e^{2x}}{x^2}\, dx$$

$$= x \int e^{2x}\, dx$$

$$= \frac{x e^{2x}}{2}$$

Note :- To find the one integral part of C.F. of the gen. eqⁿ

$$\frac{d^2y}{dx^2} + P\cdot\frac{dy}{dx} + Qy = R.$$

(i) If $P + Qx = 0$ then $y = x$ is one integral part of C.F.

(ii) If $2 + 2Px + Qx^2 = 0$ " $y = x^2$ " " " "

(iii) If $m(m-1) + Pmx + Qx^2 = 0$ then $y = x^m$ " " " "

(iv) If $1 + P + Q = 0$ then $y = e^x$ " " " "

(v) If $1 - P + Q = 0$ then $y = e^{-x}$ " " "

(vi) If $m^2 + Pm + Q = 0$ then $y = e^{mx}$ " " "

2019:. Let $y = xv$ be a soln of the diff'tal eqn

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 0$$

If $v(0) = 0$ & $v(1) = 1$ then $v(-2)$ is equal to ....

Soln:. Here $u = x$, $P = -\frac{3}{x}$, $Q = \frac{3}{x^2}$,

$$\boxed{v = c_1 \int \frac{e^{-\int P\,dx}}{u^2}\,dx + c_2}$$

$$= c_1 \int \frac{e^{-\int -\frac{3}{x}\,dx}}{x^2}\,dx + c_2$$

$$\therefore v = c_1 \int \frac{e^{3\log x}}{x^2}\,dx + c_2$$

$$v = c_1 \int \frac{x^3}{x^2}\,dx + c_2$$

$$\boxed{v = c_1 \frac{x^2}{2} + c_2}$$

$$v(x) = c_1 \cdot \frac{x^2}{2} + c_2$$

$$\therefore v(0) = 0 + c_2$$

$$0 = c_2$$

$$\therefore v(1) = \frac{c_1}{2} + 0$$

$$1 = \frac{c_1}{2}$$

$$\Rightarrow c_1 = 2$$

$$\therefore \boxed{v = x^2}$$

$$\therefore v(-2) = (-2)^2$$

$$\boxed{v(-2) = 4}$$

Ex: Let $y = xv$ be a soln of the DE

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 0$$

If $y(1) = 2$ $y(2) = 3$ find $y(-2)$ ...

Soln:

$$y = xv$$

Here $v = c_1 \frac{x^2}{2} + c_2$

$$\therefore y = \left(c_1 \frac{x^2}{2} + c_2\right) x$$

$$\therefore y(x) = \frac{c_1}{2} x^3 + c_2 x \quad — (1)$$

$$\therefore y(1) = \frac{c_1}{2} + c_2$$

$$2 = \frac{c_1}{2} + c_2 \quad — (2)$$

$$\& \; y(2) = \frac{8c_1}{2} + 2c_2$$

$$3 = 4c_1 + 2c_2 \quad — (3)$$

$$c_1 + 2c_2 = 4$$

$$\underline{4c_1 + 2c_2 = 3}$$

$$-3c_1 = 1$$

$$c_1 = -\tfrac{1}{3}$$

$$-\frac{1}{3} + 2c_2 = 4$$

$$2c_2 = 4 + \tfrac{1}{3} = \tfrac{13}{3}$$

$$\therefore\ y(x) = -\frac{1}{6}x^3 + \frac{13}{6}x$$

$$\therefore\ y(-2) = \frac{8}{6} - \frac{13}{6} = \frac{-5}{6}$$

Removal of Ist Derivative:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{--- (1)}$$

Let $y = uv$ be the soln of eqn (1)

$\therefore$ Eqn (1) reduce to

$$\frac{d^2v}{dx^2} + Xv = Y$$

where $X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$

$$y = R e^{\frac{1}{2}\int P\,dx}$$

$$\Delta \quad u = e^{-\frac{1}{2}\int P\,dx}$$

2018 :- If $y = v \sec x$ is a soln of $y'' - 2\tan x\, y' + 5y = 0$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ satisfying $y(0) = 0$, $y'(0) = \sqrt{6}$

Then $v\left(\frac{\pi}{6\sqrt{6}}\right)$ is ....

soln :-

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

Here $P = -2\tan x$

$$\therefore \frac{dP}{dx} = -2\sec^2 x$$

$\therefore\ x = 5 - \frac{1}{2}(-2\sec^2 x) - \frac{1}{4}(-2\tan x)^2$

$= 5 + \sec^2 x - \tan^2 x$

$= 5 + 1$

$= 6$

$\therefore\ \frac{d^2v}{dx^2} + 6v = 0$

$(D^2 + 6)v = 0$

The AE is

$m = \pm\sqrt{6}\, i$

$\therefore\ v(x) = c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x$ ——— (i)

$\therefore\ y = v \sec x$

$y(x) = (c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x) \sec x$ ——— (ii)

$$\therefore\ y(0) = (c_1 \cos 0 + c_2 \sin 0) \sec 0$$

$$0 = c_1$$

$$\therefore\ c_1 = 0$$

$$y'(x) = \left(c_1 \cos\sqrt{6}\,x + c_2 \sin\sqrt{6}\,x\right)\sec x \tan x$$

$$+ \left(-c_1 \sin\sqrt{6}\,x \cdot \sqrt{6} + c_2 \cos\sqrt{6}\,x \cdot \sqrt{6}\right)\sec x$$

$$\therefore\ y'(0) = 0 + (0 + c_2\sqrt{6}) \cdot 1$$

$$\sqrt{6} = c_2\sqrt{6}$$

$$\therefore\ c_2 = 1$$

$$\therefore\ v(x) = \sin\sqrt{6}\,x$$

$$v\left(\frac{\pi}{6\sqrt{6}}\right) = \sin\sqrt{6}\cdot\frac{\pi}{6\sqrt{6}} = \sin\frac{\pi}{6} = \frac{1}{2} = \underline{\underline{0.5}}$$

Slide no. 17:

Particular integral

$$PI = u\int \frac{e^{-\int P\,dx}}{u^2}\left(\int R u\, e^{\int P\,dx}\,dx\right)dx$$

$$= x\int \frac{e^{2\int\left(\frac{1}{x}+1\right)dx}}{x^2}\left(\int x\cdot x\, e^{-2\int\left(\frac{1}{x}+1\right)dx}\right)dx$$

$$= x\int \frac{1}{x^2}\, e^{2\log x + 2x}\left(\int x^2 e^{-2\log x - 2x}\,dx\right)dx$$

$$= x\int \frac{1}{x^2}\cdot x^2\cdot e^{2x}\left(\int x^2\cdot\frac{1}{x^2}\, e^{-2x}\,dx\right)dx$$

$$= x\int e^{2x}\left(\int e^{-2x}\,dx\right)dx$$

$$= x\int e^{2x}\cdot\frac{e^{-2x}}{-2}\,dx$$

$$= x\left(-\frac{1}{2}\int dx\right) = -\frac{x^2}{2}.$$