# Practice Session

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 7 • Sept 11, 2021

## Linear Combination

Let $v$ be a vector of vector space $V$. Then $v$ is a linear combination of $u_1$ and $u_2 \in V$. if $\exists$ $c_1, c_2 \in F$ s.t.

$$\boxed{v = c_1 u_1 + c_2 u_2}$$

$u_1, u_2, u_3, \ldots, u_n.$

$v \in V$ is a L.C. of $\{u_1, u_2, \ldots, u_n\}$ if $\exists$

$c_1, c_2, c_3, \ldots, c_n \in F$ st

$$v = c_1 u_1 + c_2 u_2 + c_3 u_3 + \cdots + c_n u_n$$

(eg) $v = (1,2)$ Is $v$ a L.C. of

$u_1 = (1,0)$ and $u_2 = (0,1)$ ? Yes

$v = c_1 u_1 + c_2 u_2$ ; $\boxed{c_1, c_2 \in F}$

$(1,2) = c_1 (1,0) + c_2 (0,1)$

$(1,2) = (c_1, c_2)$

$c_1 = 1 \quad c_2 = 2$

$$\boxed{(1,2) = 1 \cdot (1,0) + 2 \cdot (0,1)}$$

(eg) $v = (1, 2, 3)$. Is $v$ a L.C. of

$u_1 = (1, 0, -1)$ & $u_2 = (2, -3, 0)$ ?

$$(1, 2, 3) = c_1 (1, 0, -1) + c_2 (2, -3, 0)$$

$$(1, 2, 3) = (c_1, 0, -c_1) + (2c_2, -3c_2, 0)$$

$$(1, 2, 3) = (c_1 + 2c_2, -3c_2, -c_1)$$

$\nexists$ any $c_1, c_2 \in F$ st

$$v = c_1 u_1 + c_2 u_2$$

$c_1 + 2c_2 = 1$ --- (1)

$-3c_2 = 2 \Rightarrow c_2 = -2/3$

$-c_1 = 3 \Rightarrow c_1 = -3$

given system has no solution.

$c_1 = -3$ and $c_2 = -2/3$

$LHS = c_1 + 2c_2$

$= -3 + 2 \times (-2/3) = -3 - \frac{4}{3} \neq 1$

$= -13/3$

$\neq RHS$

$\Rightarrow$ Set of all possible linear combinations of $u_1, u_2, u_3, \ldots, u_n \in V$ forms a sub-space of $V$ over the field $F$.

$$S = \left\{ c_1 u_1 + c_2 u_2 + \cdots + c_n u_n \;\middle|\; c_i \in F \;\; \forall i \right\}$$

or

$$S = \left\{ \sum_{i=1}^{n} c_i u_i \;\middle|\; c_i \in F \;\; \forall i \right\}$$

$$S = \left\{ \sum_{i=1}^{n} c_i u_i \;\middle|\; c_i \in F \right\}$$

$\Rightarrow$ We can choose $c_1 = c_2 = \cdots = c_n = 0$

$$0 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n = 0$$

$$0 \in S \Rightarrow S \neq \phi$$

$\Rightarrow$ $\forall\, u, v \in S$ <u>claim</u>: $u + v \in S$

$\because u, v \in S$ $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \in F$ s.t.

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

$\exists\, \beta_1, \beta_2, \ldots, \beta_n \in F$ s.t.

$$v = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$$

$$u+v = (\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) + (\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n)$$

$$= (\alpha_1 + \beta_1) u_1 + (\alpha_2 + \beta_2) u_2 + \cdots + (\alpha_n + \beta_n) u_n$$

$\because \alpha_i, \beta_i \in F \quad \forall i$

$\alpha_i + \beta_i \in F \quad \forall i$

for $u+v$, $\exists (\alpha_1 + \beta_1), (\alpha_2 + \beta_2), \ldots, (\alpha_n + \beta_n)$

st

$$u+v = (\alpha_1 + \beta_1) u_1 + (\alpha_2 + \beta_2) u_2 + \cdots + (\alpha_n + \beta_n) u_n$$

$\Rightarrow u+v \in S$

$\Rightarrow \quad \forall u \in S, \quad \forall \alpha \in F \qquad \underline{\text{claim}} : \quad \alpha u \in S$

$\because \; u \in S \Rightarrow \exists \; c_1, c_2, \ldots c_n$ st

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

$$\underline{\alpha u} = \alpha ( c_1 u_1 + c_2 u_2 + \cdots + c_n u_n )$$

$$= (\underline{\alpha c_1}) u_1 + \underline{(\alpha c_2})u_2 + \cdots + (\underline{\alpha c_n}) u_n$$

$\underline{c_i \in \mathbb{R}} \quad \forall i$

$\underline{\alpha \in F} \qquad (\alpha . c_i) \in F \quad \forall i$

for $\alpha u$, $\exists \quad \alpha c_i \in F \quad i = 1, 2 \ldots n$

$$\alpha u = \sum_{i=1}^{n} (\alpha c_i) u_i . \quad \Rightarrow \alpha u \in S$$

Let $V$ be a vector space over the field $F$.

Let $S \subseteq V$. Then the spanning set of $S$ is written as $\text{Span}(S)$.

$\text{Span}(S)$ is the set of all possible linear combinations of vectors in $S$.

Let $S = \{ v_1, v_2, \ldots, v_n \}$

$$\text{Span}(S) = \left\{ \sum_{i=1}^{n} c_i v_i \;\middle|\; c_i \in F,\ v_i \in S\ \forall i \right\}$$

Span $S$ is the intersection of all the sub-spaces of $V$ containing $S$.

Let $V$ be a V.S. over the field $F$.

Let $S \subseteq V$

$\Rightarrow$ $\text{Span}(S)$ is a sub-space of $V$ over the field $F$.

$\Rightarrow$ $\text{Span}(\phi) = \{0\}$

# LINEARLY INDEPENDENT (LI) / DEPENDENT (LD)

Let $V$ be a vector space over the field $F$.
Let $S = \{ v_1, v_2, v_3, \dots, v_n \}$ be a subset of $V$. Then.

→ $S$ is linearly independent if $c_i = 0 \ \forall \ i = 1, 2, \dots n$ s.t.

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

→ A set $S$ is LD if it is not linearly indep.

$\exists\, i$ s.t. $c_i \neq 0$ and

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

(eg)

$S = \{ (1,0), (1,2) \}$

$c_1(1,0) + c_2(1,2) = 0 = (0,0)$

$(c_1 + c_2, 2c_2) = (0,0)$

$c_1 + c_2 = 0$ —— $c_1 = 0$

$2c_2 = 0$ —— $c_2 = 0$

LI

(eg) $S = \{ (1,2) \ (2,4) \ (0,1) \}$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 - = (0,0) \in \mathbb{R}^2$

$c_1(1,2) + c_2(2,4) + c_3(0,1) = (0,0)$ — *

$(c_1 + 2c_2 \ , \ 2c_1 + 4c_2 + c_3) = (0,0)$

$c_1 + 2c_2 = 0$ —— 1

$2c_1 + 4c_2 + c_3 = 0$ —— 2

$2(c_1 + 2c_2) + c_3 = 0$

$2 \cdot 0 + c_3 = 0$

$\boxed{c_3 = 0}$

LD

$$c_1 + 2c_2 = 0$$

$$c_3 = 0$$

$$c_2 = K.$$

$$K \in \mathbb{R}$$

$$\begin{pmatrix} c_1 = -2K \\ c_2 = K \\ c_3 = 0 \end{pmatrix}$$

Soln.

$$\begin{pmatrix} c_2 = 1 \\ c_1 = -2 \\ c_3 = 0 \end{pmatrix}$$

$$\begin{pmatrix} c_2 = -1 \\ c_1 = 2 \\ c_3 = 0 \end{pmatrix}$$

(eg) $S = \{ x^2, \ 2+x, \ 3x^2+x^3 \} \subseteq P_3[x]$

LI

$$c_1 x^2 + c_2(2+x) + c_3(3x^2+x^3) = 0$$

$$= 0 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3$$

$$2c_2 + c_2 x + \underbrace{(c_1 + 3c_3)}x^2 + \underbrace{c_3}x^3 = \uparrow$$

$2c_2 = 0$

$c_2 = 0$

$c_1 + 3c_3 = 0$ ——— $c_1 = 0$

$c_3 = 0$

(eg)

$S = \{ (1,2,3), (-1,0,2) \}$

$c_1(1,2,3) + c_2(-1,0,2) = (0,0,0)$

$c_1 - c_2 = 0$

$2c_1 = 0$

$3c_1 + 2c_2 = 0$

$c_1 = 0$

$c_2 = 0$

LI

(eg)

$$S = \{ (1,2,3), (0,1,-1), (-2,3,2) \}$$

$$(c_1 v_1 + c_2 v_2 + c_3 v_3 = 0)$$

Homog. sy. of L.E.

$$A = [v_1 \quad v_2 \quad v_3]$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix}$$

det $A \neq 0$

$\rho(A) = 3$

$c_1 = c_2 = c_3 = 0$

LI

det $A = 0$

$\rho(A) < 3$

non. tri. v.

LD