Linear Algebra

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1 Field

Let F be non-empty set. Let '+' and '.' be two binary operations. Then,
$\implies (F, +)$ must be an Abelian group

a **Closure:** For all $x, y \in$ F, $x + y = z \in$ F

b **Associative:** For all $x, y, z \in$ F, $(x + y) + z = x + (y + z)$

c **Identity:** For all $x \in$ F, there exists an element $e \in$ F such that $x + e = e + x = x$

d **Inverse:** For all $x \in$ F, there exists an element $y \in$ F such that $x + y = e$

e **Commutative:** For all $x, y \in$ F, $x + y = y + x$

First Four properties forms a group and all the five properties forms an Abelian group on '+'

$\implies (F, *)$ must be an Abelian group

a **Closure:** For all $x, y \in$ F, $x \cdot y = z \in$ F

b **Associative:** For all $x, y, z \in$ F, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

c **Identity:** For all $x \in$ F, there exists an element $e \in$ F such that $x \cdot e = e \cdot x = x$

d **Inverse:** For every **non-zero** element $x \in$ F, there exists an element $y \in$ F such that $x \cdot y = e$

e **Commutative:** For all $x, y \in$ F, $x \cdot y = y \cdot x$

First Four properties forms a group and all the five properties forms an Abelian group on '$\cdot$'

Note: Identity is the property of the set F and Inverse is the property of the elements of the set F

$\implies$ **Distributve Laws:**

1. $a \cdot (b + c) = a \cdot b + a \cdot c$
2. $(b + c) \cdot a = b \cdot a + c \cdot a$

6^{th} September 2021

2 Vector Space

Let V be a non empty set and F be a field. Let '+' and '$\cdot$' be two binary operations.

Vector Addition:

- For all $\boldsymbol{u}, \boldsymbol{v} \in$ V, $\boldsymbol{u} + \boldsymbol{v} \in$ V.
- For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in$ V, $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) \in$ V.
- For all $\boldsymbol{u} \in$ V, there exists an $\boldsymbol{e} \in$ V such that $\boldsymbol{u} + \boldsymbol{e} = \boldsymbol{u} = \boldsymbol{e} + \boldsymbol{u}$.
- For all $\boldsymbol{u} \in$ V, there exists a $\boldsymbol{v} \in V$ such that $u + v = v + u = e$.
- For all $\boldsymbol{u}, \boldsymbol{v} \in$ V, $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.

Scalar Multiplication:

- Let $\boldsymbol{u} \in$ V and $k \in$ F, then $ku \in$V.
- Let $\boldsymbol{u} \in V$ and $k_1, k_2 \in$ F, then $(k_1 + k_2)\boldsymbol{u} = k_1\boldsymbol{u} + k_2\boldsymbol{u}$
- Let $\boldsymbol{u}, \boldsymbol{v} \in V$ and $k \in$ F, then $k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$
- Let $k_1, k_2 \in$ F and $\boldsymbol{u} \in$ V, then $(k_1k_2)\boldsymbol{u} = k_1(k_2\boldsymbol{u})$
- There exists $1 \in F$ such that for all $\boldsymbol{u} \in$ V, $1 \cdot \boldsymbol{u} = \boldsymbol{u}$

If a non empty set V satisfies all these properties over the Field F under the binary operations '+' and '$\cdot$', then V is said to be a Vector space over field F.

Notations:

- $\mathbb{R} = (-\infty, \infty)$
- $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$
- $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
- $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) \mid x_1, x_2, \cdots, x_n \in \mathbb{R}\}$

3 Vector Subspaces

7th September 2021

Definition 3.1. *Let V be a vector space over the field F and W be its subset. W is called the subspace of V if W is itself a vector space over the same field F with the same binary operations as of V.*

OR

Definition 3.2. *Let V be a vector space over the field F and W be its subset. W is called the subspace of V over the same field F with the same binary operations if for all $\alpha, \beta \in F$ and u, v in W.*

$$\alpha \cdot \boldsymbol{u} + \beta \cdot \boldsymbol{v} \in W$$

OR

Definition 3.3. *Let V be a vector space over the field F and W be its subset. W is called the subspace of V over the same field F with the same binary operations if for all $u, v \in W$ and $\alpha \in F$.*

$$\boldsymbol{u} + \boldsymbol{v} \in W \text{ and } \alpha \boldsymbol{u} \in W$$

3.1 Examples

1. Let V be a set of all functions from S to F. Is V a vector space under the binary operations $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = c \cdot f(x)$ where S is a non-empty set.

$V = \{f \mid f : S \to V\}$, $S \neq \phi$
Vector Addition:

a Closure: Let $f, g \in V \implies f : S \to F,\ g : S \to F$

$$(f + g)(x) = f(x) + g(x)\ \forall x \in S$$

and since

$$f(x),\ g(x) \in F,\ f(x) + g(x) \in F$$

therefore

$$(f + g) : S \to F \implies (f + g) \in V$$

b Association: For all $x \in S$

$$\begin{aligned} ((f + g) + h)(x) &= (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)), \text{ since } f(x),\ g(x),\ h(x) \in F \\ &= f + (g + h)(x) \end{aligned}$$

c Identity: For all f we have to find an e (if it exists) such that

$$f + e = f$$

or

$$f(x) + e(x) = f(x),\ \forall x \in S$$

which implies

$$e(x) = 0\ \forall x \in S$$

$e(x) = 0$ is the Identity element.

d Inverse: For every f there must exist g such that

$$\begin{aligned} f + g &= e \\ \implies f(x) + g(x) &= e(x),\ \forall x \in S \\ \implies g(x) &= 0 - f(x) \\ \implies g(x) &= -f(x) \end{aligned}$$

There exist $g = -f$ for every f in V.

e Commutative: For all $x \in S$

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ &= g(x) + f(x), \text{ since } f(x),\ g(x) \in F \\ &= (g+f)(x) \end{aligned}$$

Scalar Multiplication:

a Let $f \in V$ and $k \in F$, then as per definition, for $x \in S$

$$(k \cdot f)(x) = k \cdot f(x)$$

since $k, f(x) \in F$ therefore $k \cdot f(x) \in F$. This implies $c \cdot f \in V$

b Let $f \in V$ and $k_1, k_2 \in F$, then for all x

$$\begin{aligned} ((k_1 + k_2) \cdot f)(x) &= (k_1 + k_2) \cdot f(x) \\ &= k_1 \cdot f(x) + k_2 \cdot f(x) \\ &= (k_1 \cdot f)(x) + (k_2 \cdot f)(x) \end{aligned}$$

Therefore $(k_1 + k_2) \cdot f = k_1 \cdot f + k_2 \cdot f$

c Let $f \in V$ and $k_1, k_2 \in F$, then for all x

$$\begin{aligned} ((k_1 k_2) \cdot f)(x) &= (k_1 k_2) \cdot f(x) \\ &= k_1(k_2 \cdot f(x)), \text{ since } k_1, k_2, f(x) \in F \\ &= k_1(k_2 \cdot f)(x) \end{aligned}$$

which implies $(k_1 k_2) \cdot f = k_1(k_2 \cdot f)$.

d Let $f, g \in V$ and $k \in F$, then for all x

$$\begin{aligned} (k \cdot (f+g))(x) &= k \cdot (f(x) + g(x)) \\ &= k \cdot f(x) + k \cdot g(x), \text{ since } k, f(x), g(x) \in F \\ &= (k \cdot f)(x) + (k \cdot g)(x) \end{aligned}$$

e Let $f \in V$, $1 \in F$ then for all x

$$1 \cdot f(x) = f(x)$$

which implies $1 \cdot f = f$

Hence V is a vector space.

2. Let V be the set of all polynomials from $F \to F$. Is V a vector space over the field F?

$V = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \mid a_i \in F\ \forall i\}$

Vector Addition:

- Let $p, q \in V$, then $p + q \in V$
- Let $p, q, r \in V$, then $(p + q) + r = p + (q + r)$
- Let $p \in V$, then for $q = \mathbf{0} \in V$, $p + q = p$
- Let $p \in V$, then for $q = -p \in V$, $p + q = 0$

- Let $p, q \in V$, then $p + q = q + p$

Scalar Multiplication:

- Let $p \in V, k \in F$, then $k \cdot f \in V$
- Let $p \in V$ and $k_1, k_2 \in F$, then $(k_1 + k_2) \cdot p = k_1 \cdot p + k_2 \cdot p$
- Let $p \in V$ and $k_1, k_2 \in F$, then $(k_1 k_2) \cdot p = k_1(k_2 \cdot p)$
- Let $p, q \in V$ and $k \in F$, then $k \cdot (p + q) = k \cdot p + k \cdot q$
- Let $p \in V$ and $1 \in F$, then $1 \cdot p = p$

3. Let V be the set of all polynomials of degree n. Is V a vector space?

V is not a vector space because the following properties fails:

- **Closure:** Consider the polynomials $p = x^n + x + 1$ and $q = -x^n + x + 1$, then

$$p + q = 2x + 1$$

which doesn't belong to V because it degree is not n.

- **Identity:** The degree of the polynomial $p = 0$ is not defined, but it is certainly doesn't belong to V

4. V is the set of all polynomials whose degree is atmost n.

$V = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \mid a_i \in F \; \forall i \text{ and } k \leq n\}$
It is a vector space over F because it satisfies all the properties of vector space and for that of the identity element $\mathbf{0}$, it belongs to V (if we put $a_i = 0$ for all i's)

5. V is the set of all polynomials whose degree is atleast n.

Same as **Ex #3**

6. V is the set of all mxn matrices.

Yes V is a vector space

9^{th} September

7. $W_1 = \{f \in V | \; f \text{ is a continous function}\}$
$W_1 \subseteq V$, where V is the vector space of all functions $f : S \to F$. We know that $\mathbf{0}(x) = 0$ for all $x \in S$, hence $\mathbf{0}$ in W_1. Now for f and $g \in W_1$ consider

$$(\alpha f + \beta g)(x), \text{ for all } x \in S$$

This function is continous because it is the sum of two constinous functions. Hence $(\alpha f + \beta g)(x)$ is also in W_1. Therefore W_1 is a subspace

8. $W_3 = \{f \in V \mid f(-x) = f(x) \quad \forall x \in S\}$
$=$ set of all even functions
$\mathbf{0}(x) = 0$, for all $x \in S$, hence $\mathbf{0}(x) = \mathbf{0}(-x)$ and therefore $\mathbf{0} \in W_3$. Now Let

$$h(x) = (\alpha \cdot f + \beta \cdot g)(x) \quad \text{for all } x \in S$$

Then

$$\begin{aligned} h(x) &= (\alpha \cdot f)(x) + (\beta \cdot g)(x) \\ &= \alpha f(x) + \beta g(x) \\ &= \alpha f(-x) + \beta g(-x) \\ &= (\alpha \cdot f)(-x) + (\beta \cdot g)(-x) \\ &= (\alpha \cdot f + \beta \cdot g)(-x) \\ &= h(-x) \end{aligned}$$

which implies $(\alpha \cdot f + \beta \cdot g)$ is an even function. Therefore $(\alpha \cdot f + \beta \cdot g) \in W_3$ and W_3 is a subspace.

9. $W_4 = \{f \in V \mid f(x) = -f(-x) \quad \forall x \in S\}$
$=$ set of all odd functions
$\mathbf{0}(x) = 0$, for all $x \in S$, hence $\mathbf{0}(x) = -\mathbf{0}(-x)$ and therefore $\mathbf{0} \in W_4$. Now Let

$$h(x) = (\alpha \cdot f + \beta \cdot g)(x) \quad \text{for all } x \in S$$

Then

$$\begin{aligned} h(x) &= (\alpha \cdot f)(x) + (\beta \cdot g)(x) \\ &= \alpha f(x) + \beta g(x) \\ &= \alpha f(-x) + \beta g(-x) \\ &= -(\alpha \cdot f)(-x) - (\beta \cdot g)(-x) \\ &= -(\alpha \cdot f + \beta \cdot g)(-x) \\ &= -h(-x) \end{aligned}$$

which implies $(\alpha \cdot f + \beta \cdot g)$ is an odd function. Therefore $(\alpha \cdot f + \beta \cdot g) \in W_4$ and W_4 is a subspace

10. $V = \mathbb{R}^2, F = \mathbb{R}$ and

$$W = \{(x, y) | x, y \in \mathbb{R} \text{ and } x \geq 0\}$$

$(0, 0) \in W$. Let u $= (1, 1)$ and $\alpha = -1$, therefore $\alpha \cdot u \notin W$ because $(-1, -1)$ has its x component is negative and W is not a subspace.

11. $W_5 = \{A = M_{n \times n}(F) \mid a_{ij} = a_{ji} \text{ for all } i, j\}$
W_5 is the set of all $n \times n$ symmetric matrices, hence for all $A \in W_5, A = A^\intercal$.

Null matrix $= \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. It is clear that $\mathbf{0} = \mathbf{0}^\intercal$. Hence $\mathbf{0} \in W_5$.

Let $C = \alpha A + \beta B$, where $A, B \in W_5$ and $\alpha, \beta \in F$, then

$$\begin{aligned} C^\intercal &= (\alpha A + \beta B)^\intercal \\ &= (\alpha A)^\intercal + (\beta B)^\intercal \\ &= \alpha A^\intercal + \beta B^\intercal \\ &= \alpha A + \beta B \\ &= C \end{aligned}$$

which implies $C = C^\intercal$ for all $A, B \in W_6$ and $\alpha, \beta \in F$, and hence $C \in W_6$. W_6 is a subspace

12. $W_6 = \{A = M_{n \times n}(F) \mid a_{ij} = -a_{ji} \text{ for all } i, j\}$
W_6 is the set of all $n \times n$ symmetric matrices, hence for all $A \in W_6, A = -A^\intercal$.

Null matrix $= \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. It is clear that $\mathbf{0} = \mathbf{0}^\intercal$. Hence $\mathbf{0} \in W_6$.

Let $C = \alpha A + \beta B$, where $A, B \in W_6$ and $\alpha, \beta \in F$, then

$$\begin{aligned} C^\intercal &= (\alpha A + \beta B)^\intercal \\ &= (\alpha A)^\intercal + (\beta B)^\intercal \\ &= -\alpha A^\intercal - \beta B^\intercal \\ &= -(\alpha A + \beta B) \\ &= -C \end{aligned}$$

which implies $C = -C^\intercal$ for all $A, B \in W_6$ and $\alpha, \beta \in F$, and hence $C \in W_6$. W_6 is a subspace

13. $W_7 = \{A = M_{n \times n}(F) \mid A = A^\theta\}$

Null matrix $= \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. It is clear that $\mathbf{0} = \mathbf{0}^\theta$. Hence $\mathbf{0} \in W_7$.

Let $C = \alpha A + \beta B$, where $A, B \in W_7$ and $\alpha, \beta \in F$, then

$$\begin{aligned} C^\theta &= (\alpha A + \beta B)^\theta \\ &= (\overline{\alpha A + \beta B})^\top \\ &= (\overline{\alpha A})^\intercal + (\overline{\beta B})^\intercal \\ &= \overline{\alpha}\overline{A}^\intercal + \overline{\beta}\overline{B}^\intercal \\ &= \overline{\alpha} A^\theta + \overline{\beta} B^\theta \\ &= \overline{\alpha} A + \overline{\beta} B \end{aligned}$$

which is not equal to C for all $\alpha, \beta \in F$. For $F = \mathbb{R}$, W_7 is vector subpace but for $F = \mathbb{C}$, W_7 is not a subspace

14. $W_7 = \{A = M_{n\times n}(F) \mid A = -A^\theta\}$

Null matrix $= \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. It is clear that $\mathbf{0} = -\mathbf{0}^\theta$. Hence $\mathbf{0} \in W_7$.

Let $C = \alpha A + \beta B$, where $A, B \in W_7$ and $\alpha, \beta \in F$, then

$$\begin{aligned} C^\theta &= (\alpha A + \beta B)^\theta \\ &= (\overline{\alpha A + \beta B})^\top \\ &= (\overline{\alpha A})^\intercal + (\overline{\beta B})^\intercal \\ &= \overline{\alpha}\overline{A}^\intercal + \overline{\beta}\overline{B}^\intercal \\ &= \overline{\alpha} A^\theta + \overline{\beta} B^\theta \\ &= -(\overline{\alpha} A + \overline{\beta} B) \end{aligned}$$

which is not equal to C for all $\alpha, \beta \in F$. For $F = \mathbb{R}$, W_7 is vector subpace but for $F = \mathbb{C}$, W_7 is not a subspace

10th September 2021

3.2 Intersection of two subspaces

Theorem 3.1. *Let V be a vector space over the field F and W_1 and W_2 be two subspaces of the vector space V. Then $W = W_1 \cap W_2$ is also a subspace of V.*

Proof. W_1 and W_2 are subspaces of W. Hence

$$\begin{aligned} &\mathbf{0} \in W_1 \text{ and } \mathbf{0} \in W_2 \\ \Longrightarrow\ &\mathbf{0} \in W_1 \cap W_2 \\ \Longrightarrow\ &W = (W_1 \cap W_2) \neq \phi \end{aligned}$$

Let $\alpha, \beta \in F$ and $u, v \in W$. Since $u, v \in W$, then $u, v \in W_1$ and $u, v \in W_2$ which implies $\alpha u + \beta v \in W_1$ and $\alpha u + \beta v \in W_2$, because W_1 and W_2 are subspaces. Since $\alpha u + \beta v \in W_1$ and $\alpha u + \beta v \in W_2$, $\alpha u + \beta v \in W$ for all $\alpha, \beta \in F$ and hence $W = W_1 \cap W_2$ is a subspace. ☐

Corollary 3.1.1. *Let V be a vector space over the field F and W_i the be arbitrary subspaces of the vector space V where $i \in \mathbb{N}$. Then $W = \bigcap_{i=1}^{\infty} W_i$ is also a subspace of V.*

3.3 Union of two subspaces

Theorem 3.2. *Union of two subspaces of the same vector space is a subspace if and only if one of them is contained in another.*

Proof. Let V be a vector space over the field F and W_1 and W_2 be two subspaces of the vector space V.

- **Forward:** $W = W_1 \cup W_2$ is a subspace (Hypothesis)
  (Proof by contradiction) Let us assume that one doesn't contain other ($W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$).

$W_1 \not\subseteq W_2$	and	$W_2 \not\subseteq W_1$
There exists a $x \in W_1$ such that $x \notin W_2$, which implies $x \in W = W_1 \cup W_2$		There exists a $y \in W_2$ such that $y \notin W_1$, which implies $y \in W = W_1 \cup W_2$

$$\because W \text{ is a subspace } \therefore x + y \in W$$

$$\because x + y \in W, \text{ then either}$$

$x + y \in W_1$ and $x \in W_1$	or	$x + y \in W_2$ and $y \in W_2$
$\because W_1$ is a subspace $(x + y) - x \in W_1$, that is $y \in W_1$, which is a contradiction because $y \in W_2$ and $y \notin W_1$.		$\because W_2$ is a subspace $(x + y) - y \in W_2$, that is $x \in W_2$, which is a contradiction because $x \in W_1$ and $x \notin W_2$.

  Hence our assumption that one subspace doesnt contain other is false.

- **Backward:** $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ (Hypothesis)

$W_1 \subseteq W_2$	or	$W_2 \subseteq W_1$
$W = W_1 \cup W_2 = W_2$ ($\because W_1 \subseteq W_2$), hence W is subspace		$W = W_1 \cup W_2 = W_1$ ($\because W_2 \subseteq W_1$), hence W is subspace

□

3.4 Sum of two subspaces

Theorem 3.3. *Let V be a vector space over the field F and W_1 and W_2 be two subspaces of the vector space V. Then $W = W_1 + W_2$ is also a subspace of V.*

Proof. $W = W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}$. $\mathbf{0} \in W_1$ and W_2, hence $\mathbf{0} + \mathbf{0} = \mathbf{0}$ which also belongs to $W \neq \boldsymbol{\phi}$. For all $\alpha, \beta \in F$ and $x, y \in W$ consider

$$\alpha \cdot x + \beta \cdot y = \alpha \cdot (u + v) + \beta \cdot (p + q)$$

where $x = u + v$ for some $u \in W_1$ and $v \in W_2$ and $y = p + q$ for some $p \in W_1$ and $q \in W_2$. Therefore

$$\alpha \cdot x + \beta \cdot y = (\alpha u + \beta p) + (\alpha v + \alpha q)$$

since $u, p \in W_1$ and $v, q \in W_2$

$$\alpha x + \beta y = s + t$$

where $s \in W_1$ and $t \in W_2$ and therefore $\alpha x + \beta y \in W$ for all $\alpha, \beta \in F$ and $x, y \in W$. Hence W is subspace of V □

4 Linear Dependency and Independency

4.1 Linear Combination

Definition 4.1. *Let v be a vector of the vector space V. Then v is a linear combination of $u_1, u_2 \in V$ if there exist $c_1, c_2 \in F$ such that*

$$\boxed{v = c_1 u_1 + c_2 u_2}$$

Generalized form: *For $u_1, u_2, \cdots, u_n \in V$ $v \in V$ is Linear Combination of $u_1, u_2, \cdots, u_n$ if there exist $c_1, c_2 \cdots, c_n \in F$ such that*

$$\boxed{v = c_1 u_1 + c_2 u_2 + \cdots c_n u_n}$$

Theorem 4.1. *Set of all possible linear combinations of $u_1, u_2, \cdots, u_n \in V$, forms a subspace of V over the field F*

Proof. The set $S = \{c_1 u_1 + c_2 u_2 + \cdots + c_n u_n \in V \mid c_i \in F, \forall i's\} = \left\{\sum_i^n c_i u_i \mid c_i \in F, \forall i's\right\}$.

Putting $c_i = 0$ for all i, we get the $\mathbf{0}$ vector, implying $\mathbf{0} \in S$. Consider $u, v \in S$, then there exists $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\beta_1, \beta_2, \cdots \beta_n \in F$ such that

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$
$$v = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$$

Then

$$\begin{aligned} u + v &= \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n \\ &= (\alpha_1 + \beta_1)u_1 + (\alpha_2 + \beta_2)u_2 + \cdots + (\alpha_n + \beta_n)u_n \end{aligned}$$

since $\alpha_i, \beta_i \in F$ for all i's $\implies$ $\alpha_i + \beta_i \in F$ for all i's and hence $u + v \in S$.
Let $\alpha \in F$, then

$$\begin{aligned} \alpha u &= \alpha(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) \\ &= (\alpha\alpha_1)u_1 + (\alpha\alpha_2)u_2 + \cdots + (\alpha\alpha_n)u_n \end{aligned}$$

also belongs in S because for all $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\alpha \in F$, $\alpha\alpha_i \in F$ for all i's.
Therefore S is subspace by Definition (3.3). □

4.2 Span

Definition 4.2. *Let V be a vector space over the field $F$and $S \subseteq V$. Then the set of all possible linear combinations of the set S is called the span of that set or the spanning set of S and is written as* span(S).

$$\begin{aligned} S &= \{v_1, v_2, \cdots, v_n\} \\ \text{span}(S) &= \{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n | c_i \in F, v_i \in S\} \\ &= \left\{ \sum_{i=1}^{n} c_i v_i \;\middle|\; c_i \in F, v_i \in S \right\} \end{aligned}$$

OR

Definition 4.3. *Span S is the intersection of all subspaces of V containing S.*

Corollary 4.1.1. *Let V be a vector space over the field F and $S \subseteq V$, then* $\text{Span}(S)$ *is a subspace of V.*

Proof. For $S = \phi$, from definition **(4.3)** S is subspace of V and for $S \neq \phi$, from Definition **(4.2)** and Theorem **(4.1)** S is a subspace of V. □

4.3 Linearly Independent (L.I) and Linearly Dependent (L.D)

Definition 4.4. *Let V be a vector space and $S = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. Then the set S is **linearly independent** if*

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \implies c_i = 0 \text{ for all } i\text{'s}$$

*where $v_1, v_2, \cdots, v_n \in S$ and $c_1, c_2 \cdots, c_n \in F$. If it is not linearly independent then its **linearly dependent**, i.e there exists atleast one $c_i \neq 0$ when $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$.*

Theorem 4.2. *Let V be a vector space over the field F and S be a non empty subset of V. S is linearly dependent if and only if atleast one vector of S can be written as a linear combination of others.*

Proof. Let $S = \{v_1, v_2, \cdots v_n\}$ be non-empty subset of V.
Forward: S is linearly dependent. There exists a $c_i \neq 0$ such that

$$c_1 v_1 + \cdots + c_{j-1} v_{j-1} + c_j v_j + c_{j+1} v_{j+1} + \cdots + c_n v_n = 0$$

Without loss of generality we assume $c_j \neq 0$.

$$\begin{aligned} \therefore c_j v_j &= -c_1 v_1 - \cdots - c_{j-1} v_{j-1} - c_{j+1} v_{j+1} - \cdots - c_n v_n \\ v_j &= -\frac{c_1}{c_j} v_1 - \cdots - \frac{c_{j-1}}{c_j} v_{j-1} - \frac{c_{j+1}}{c_j} v_{j+1} - \cdots - \frac{c_n}{c_j} v_n \\ v_j &= k_1 v_1 + \cdots + k_{j-1} v_{j-1} + k_{j+1} v_{j+1} + \cdots + k_n v_n \end{aligned}$$

which says that v_j is the linear combination of the remaining vectors of S.
Backward: Atleast one vector of S can be written as remaining others. This implies that

$$\begin{aligned} v_i &= c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \cdots + c_n v_n \\ \implies& c_1 v_1 + \cdots + c_{i-1} v_{i-1} + (-1) v_i + c_{i+1} v_{i+1} + \cdots + c_n v_n = 0 \end{aligned}$$

Let $c_i = -1$, then the linear combination of the vectors of S is 0 for atleast one $c_j \neq 0, (c_i = -1)$ and hence S is linearly dependent. □

4.3.1 Properties

1. $S = \phi$ is a Linearly Independent set.
2. $S = \{v\}$ is a singleton set, then
   (a) If $v \neq 0$, then $c \cdot v \implies c = 0$.
   (b) If $v = 0$, then $c \cdot v \implies c = 0$ for all values of c.

   Hence S is linearly independent if $v \neq 0$ else linearly dependent.
3. If S is linearly dependent set in V, then subset of S may or may not be linearly dependent.
   Ex. $S = \{(0,1),(1,0),(1,1),(2,2)\}$
   Consider $W_1 = \{(0,1),(1,0)\} \subseteq S$ and $W_2 = \{(1,1),(2,2)\} \subseteq S$. W_1 is linearly independent and W_2 is linearly dependent
4. If S is linearly dependent set in V, then superset of S is also linearly dependent.

   Proof. Suppose $S = \{v_1, v_2, \cdots, v_m\}$, and $W = \{v_1, \cdots, v_m, v_{m+1} \cdots, v_n\}$
   Clearly $W \supseteq S$. Now consider the linear combination of the vectors in W.

   $$\underbrace{c_1v_1 + c_2v_2 + \cdots + c_mv_m}_{= 0 \text{ for atleast one } c_j \neq 0} + c_{m+1}v_{m+1} + c_{m+2}v_{m+2} + \cdots + c_nv_n$$

   Therefore
   $$c_1v_1 + c_2v_2 + \cdots + c_mv_m + c_{m+1}v_{m+1} + c_{m+2}v_{m+2} + \cdots + c_nv_n = 0$$
   implies there exists atleast one $c_i = c_j \neq 0$. Hence $W \supseteq S$ is linearly dependent. □
5. If S is a linearly independent set in V then any subset of S is also linearly independent.

   Proof. **(Proof by contradiction)** Let us assume that a subset T of S is linearly dependent, then the linear combination

   $$c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0 \implies \text{that there exist atleast one } c_i \neq 0, 1 \leq i \leq m \tag{1}$$

   Now consider the original set S which is linearly independent. The equation

   $$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

   implies that $c_i = 0$ for all i's. Lets inspect the linear combination more carefully.

   $$\underbrace{c_1v_1 + c_2v_2 + \cdots + c_mv_m}_{= 0 \implies \text{ there exist a } c_j \neq 0} + \underbrace{c_{m+1}v_{m+1} + c_{m+2}v_{m+2} + \cdots + c_nv_n}_{= 0 \text{ if we put } c_i = 0 \text{ for all } i > n}$$
   $$= 0$$

   Hence
   $$c_1v_1 + c_2v_2 + \cdots + c_mv_m + c_{m+1}v_{m+1} + c_{m+2}v_{m+2} + \cdots + c_nv_n = 0$$
   implies there exist a $c_j \neq 0$. This shows that S is a linearly dependent set which contradicts with our assumption and proves T is a linearly independent set. □
6. If S is a linearly independent set in V, then the superset of S may or may not be linearly independent.
   Ex. $S = \{(-1,0)\}$ and $T_1 = \{(-1,0),(0,1)\}$ and $T_1 = \{(-1,0),(-2,0)\}$ then, T_1 is linearly independent and T_2 is linearly dependent

5 Basis

$\implies$ span S is the subspace of V

$\implies$ span $\phi = 0$

$\implies$ span S is the intersection of all subspace of V containing S

$\implies$ If S is a subspace of V, then span $S = S$

$\implies$ span(span S) = span S, span(span($\cdots$ span(S)$\cdots$)) = span S

$\implies$ If S is a subspace, then span(span($\cdots$ span(S)$\cdots$)) = span S

Definition 5.1. *Let V be a vector space over the field F and B be non empty subset of V. Then B is called the basis of V if*

(a) *B is linearly independent*

(b) span(B) $= V$

Dimension of a Basis: The number of elements in the basis is called the dimension of that basis.

OR

The cardinality of the basis is called its dimension.
$C(B) < \infty \implies V$ is a finite dimensional vector space
$C(B) \not< \infty \implies V$ is an infinite dimensional vector space
Number of Basis of any vector space can be infinite but dimension of a vector space is unique.

5.1 Properties

Let S be a subset of V and span(S) = V

1. For any $w \in V$, $S = \{u_1, u_2, \cdots, u_n\}$ then $S \cup \{w\}$ also spans V, i.e span($S \cup \{w\}$) $= V$
2. If any u_i is a linear combination of $u_1, u_2, \cdots, u_{i-1}$ then $S \setminus \{u_i\}$ also spans V, i.e
$$\text{span}\,(S \setminus \{u_i\}) = V$$
3. Standard basis of the Euclidean space $\mathbb{R}^n$ is
$$\{(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1)\}$$
4. $P[x] = \{a_0, a_1x + \cdots + a_nx_n + \cdots \mid a_i \in F \text{ for all } i\}$. Then
$$B = \{1, x, x^2, \cdots, x^n, \cdots\}$$
is the basis for $P[x]$
5. Let W be the set of all solution of the second order differential equation
$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{2}$$
Then W is forms a subspace.

Proof.
$$W = \left\{ y(x) \;\middle|\; \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \right\}$$

$y(x) = 0$ satisfies equation (2). Hence $y(x) = 0$ is a solution of the differential equation. Let $y_1(x)$ and $y_2(x)$ be two solutions of (2), then consider $\alpha y_1 + \beta y_2$.

$$\begin{aligned}
&\frac{d^2(\alpha y_1 + \beta y_2)}{dx^2} + P(x)\frac{d(\alpha y_1 + \beta y_2)}{dx} + Q(x)(\alpha y_1 + \beta y_2) \\
&= \left(\alpha\frac{d^2 y_1}{dx^2} + \alpha P(x)\frac{dy_1}{dx} + \alpha Q(x)y_1\right) + \left(\beta\frac{d^2 y_2}{dx^2} + \beta P(x)\frac{dy_2}{dx} + \beta Q(x)y_2\right) \\
&= \alpha\left(\frac{d^2 y_1}{dx^2} + P(x)\frac{dy_1}{dx} + Q(x)y_1\right) + \beta\left(\frac{d^2 y_2}{dx^2} + P(x)\frac{dy_2}{dx} + Q(x)y_2\right) \\
&= 0 + 0 = 0
\end{aligned}$$

Hence W is a subspace The general solution y of the above differential equation is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{3}$$

where $y_1(x)$ and $y_2(x)$ are two solutions of (2). From (3) we can say that any solution can be written as the linear combination of y_1 and y_2. Hence y_1 and y_2 spans W and since (3) is the general solution y_1 and y_2 are linearly independent. Therefore $B = \{y_1, y_2\}$ is the basis for W whose dimension is 2.

Generalization: Let W be the set of all solutions of the differential equation

$$a_n\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{dy}{dx} + \cdots + a_0(x)y = 0 \tag{4}$$

Then W forms a subspace(vector space?) and its basis is given by $B = \{y_1, y_2, \cdots, y_n\}$ with dimension equal to the order of the D.E $= n$ and where $y_1, y_2, \cdots, y_n$ are the solutions of (4) and they form the general solution of (4), that is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

. □

Theorem 5.1. *Let V be finite dimensional vector space over the field F and let L be the set of linearly independent vectors in V and S be spanning set of V, then cardinality of L is less than or equal to cardinality of* span(S).

Proof. span$(S) = V$ and S can be either linearly independent or linearly dependent. If S is linearly independent then by definition S becomes the basis for V. If S is not linearly independent then its linearly dependent and atleast one vector of S can be written as linear combination of other vectors of S say v_1. Since S spans V, any vector $v \in V$ can be written as linear combination of vectors in S.

$$\begin{aligned}
v &= c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \\
&= c_1(k_1 v_2 + k_2 v_3 + \cdots + k_{n-1} v_n) + c_2 v_2 + \cdots + c_n v_n \\
&= (c_1 k_1 + c_2)v_2 + (c_1 k_2 + c_3)v_3 + \cdots + (c_1 k_{n-1} + c_n)v_n \\
&= \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n
\end{aligned}$$

which implies there exists α_i for every $v \in V$ such that v can be written as a linear combination of vectors in S. Hence span$(S) = V$ □

Theorem 5.2. *Let V be a **finite dimensional vector space**, with dimension of V equal to $\boldsymbol{n}$, then any subset of V with $\boldsymbol{(n+1)}$ or more vectors is a linearly dependent set (it cannot form a basis for V).*

Notes:
Let V be a **finite dimensional vector space**, with dim $V = n$ and B be a subset of V with $\boldsymbol{n}$ number of vectors, then
If span$(B) = V$, then B is a basis of V.

OR

If B is linearly independent, then B is a basis of V
That is for checking whether a subset B of V with n vectors is a basis of V either check if it is Linearly Independent or if it spans V
Also the vector space $V = \{0\}$ has the set $B = \boldsymbol{\phi}$ as its basis

Theorem 5.3. *Let V be a* ***finite dimensional vector space*** *over the field F and B be a subset of V. The set B is a basis for V if and only if every vector of V can be written as a unique linear combination of the vectors in B.*

Proof. **Forward direction:** $B = \{u_1, u_2, \cdots, u_n\}$ is a basis for V.
Let's assume that there exist two distinct set of scalars $\{c_1, c_2, \cdots, c_n\}$ and $\{k_1, k_2, \cdots, k_n\}$ for any vector $v \in V$ such that

$$v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$$

$$v = k_1u_1 + k_2u_2 + \cdots + k_nu_n.$$

Substracting one from another, we get

$$(c_1 - k_1)u_1 + (c_2 - k_2)u_2 + \cdots + (c_n - k_n)u_n = 0$$

since the vectors from the basis are all linearly independent we have $c_i = k_i$ for all i's, which is a contradiction because they are suppose to be different from our assumption.
Backward direction: every vector of V can be written as a unique linear combination of the vectors in B.
There exists a unique set of scalars $c_1, c_2, \cdots, c_n$ for every vector $v \in V$ such that

$$v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$$

That is B spans V. Now we have to show that the vectors in B are linearly independent, for that consider the 0 vector.

$$0 = 0 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n.$$

Since this combination is unique, there is no other way of writing the 0 vector. This states that the equation

$$c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0$$

has only one solution and that is $c_i = 0$ for all i's. Hence B is a basis for V

□

5.2 How to check whether a set is a basis?

Suppose dimension of V is n and S is a subset of V to be checked for basis.

$$\dim\ V = n \text{ and } S \subseteq V$$

If $|\boldsymbol{S}| < \boldsymbol{n}$, then S can never span V and hence S is not a basis of V

If $|\boldsymbol{S}| = \boldsymbol{n}$, then check either S is linearly independent or span$(S) = V$ for it to be a basis of V

If $|\boldsymbol{S}| > \boldsymbol{n}$, then S becomes L.D and hence S cannot form a basis of V

Theorem 5.4. *Let V be a finite dimensional vector space over the field F, and W be its subspace, then* **dim $\boldsymbol{W} \leq$ dim $\boldsymbol{V}$**

5.3 Ordered basis and Direct Sum

Coordinate vector: Let V be a finite dimensional vector space over the field F and B be a basis of V ($B = \{v_1, v_2, \cdots, v_n\}$). Then for any $v \in V$,

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

and

$$[\boldsymbol{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $[\boldsymbol{v}]_B$ is called the coordinate vector of v w.r.t the basis B.

Direct Sum: Let W_1 and W_2 be two sub-spaces of a vector space V, then $W_1 + W_2$ is called the direct sum of V if every member of V can be uniquely expressed as w_1+w_2, where $w_1 \in W_1$ and $w_2 \in W_2$. The direct sum of V is denoted by $W_1 \bigoplus W_2$

Examples of finding the basis:

Theorem 5.5. *Let W_1 and W_2 be two subspaces of V, then $W_1 + W_2$ is said to be the direct sum of V if and only if $W_1 \cap W_2 = \{0\}$*

Example: Let $V = \mathbb{R}^3$ and $W_1 = \{(0, y, z) : y, z \in \mathbb{R}\}$ and $W_2 = \{(x, y, 0) : x, y \in \mathbb{R}\}$. Is V the direct sum of W_1 and W_2.
Solution: $W_1 \cap W_2 = \{(0, y, 0) : y \in \mathbb{R}\}$. V is not a direct sum of W_1 and W_2. Also there is no unique sum for every element of V. That is

$$\begin{aligned}(1, -2, 1) &= (0, -2, 1) + (1, 0, 0)\\ &= (0, -1, 1) + (1, -1, 0)\\ &\vdots\end{aligned}$$

Note:

- $\dim(\mathbb{C}^n(\mathbb{R})) = 2n$
- $\dim(\mathbb{C}^n(\mathbb{C})) = n$

$$\dim(W_1 \cup W_2) = \dim\, W_1 + \dim\, W_2 - \dim(W_1 \cap W_2)$$

6 Linear Transformation

Definition 6.1. *Let V and W be two vector spaces over the same field F. Then $T : v \to W$ is said to be a linear transformation if*

(a) For all $u, v \in V$, $T(u + v) = T(u) + T(v)$

(b) For all $\alpha \in F$ and $u \in V$, $T(\alpha u) = \alpha T(u)$

OR

For all $\alpha, \beta \in F$ and $u, v \in V$, $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$

Note:

(i) If $T(0) \neq 0$, then the map T is not a linear transformation, where $0 \in V$ and W

6.1 Special Transformations

1 Identity Transformation: Let V be a vector space over the field F, define $I : V \to V$ such that

$$T(v) = v$$

for all $v \in V$, then I is a linear transformation

Proof. Consider $\alpha, \beta \in F$ and $v, u \in I$, then

$$\begin{aligned}I(\alpha u + \beta v) &= \alpha u + \beta v \quad \text{since} \quad \alpha u + \beta v \in V\\ &= \alpha I(u) + \beta I(v)\end{aligned}$$

Therefore I is a linear transformation. □

2 Zero Transformation: Let V, W be two subspaces of the vector space V over the field F, then the map $O : V \to W$ where

$$O(v) = 0$$

is called the linear transformation.

Proof. Consider $\alpha, \beta \in F$ and $v, u \in O$, then

$$\begin{aligned} O(\alpha u + \beta v) &= 0 \quad \text{since} \quad \alpha u + \beta v \in V \\ \alpha O(v) + \beta O(u) &= \alpha 0 + \beta 0 \\ &= 0 \\ \therefore O(\alpha u + \beta v) &= \alpha u + \beta v \end{aligned}$$

Hence O is a linear transformation. □

6.2 Null Space (Kernel) and Range Space of T

Definition 6.2. *Let T be a linear transformation from V to W, $T : V \to W$. Then*

$$N(T) = \ker(T) = \{x \in V \mid T(x) = 0\}$$

is called the ***Null Space or Kernel of*** *T*

Note: Null space of any linear transformation is always non empty because $T(0)$ is always equal to 0 for any linear transformation T.

Theorem 6.1. *Let V and U be vector spaces over the same field F. Let $T : V \to W$ be a linear transformation, then $N(T)$ is a subspace of V*

Proof. Since T is a linear transformation, $T(0) = 0$, hence $0 \in N(T)$ and $N(T) \neq \phi$ and also $N(T) \subseteq V$. Now consider $\alpha, \beta \in F$ and $u, v \in N(T)$, then $T(u) = T(v) = 0$.

$$\begin{aligned} T(\alpha u + \beta v) &= \alpha \cdot T(u) + \beta \cdot T(v) \\ &= \alpha \cdot 0 + \beta \cdot 0 \\ &= 0 \end{aligned}$$

Therefore $\alpha u + \beta v$ also belongs to $N(T)$, which implies $N(T)$ is a subspace of V □

Nullity of T: The dimension of the Null Space of any L.T T is called the nullity of T, It is represented as $\boldsymbol{\eta(T)}$
For a finite dimensional vector space V, $\dim(N(T)) \leq \dim(V)$

Definition 6.3. *Let V and W be finite dimensional vector spaces over the same field F and $T : V \to W$ be a linear transformation, then*

$$R(T) = \{w \in W \mid \exists\, v \in V, T(v) = w\}$$

is called the ***Range Space of*** *T*

Theorem 6.2. *Let $T : V \to W$ be a linear transformation where V and W are finite dimensional vector spaces over same field F, then the range space of T, $R(T)$ is a subpace of W. i.e $R(T) \leq W$*

Proof. Very simple, similar to null space being a subspace of V □

Note: The dimension of range space of T is denoted as $\rho(T)$

6.3 Linear Transformation Examples

1 $T : P_4(x) \to P_3(x)$ and $T\left(p(x)\right) = \int_0^x p(t)dt$

The linear transformation is not well defined, all polynomials of degree ≥ 3 lie outside of the codomain

2 $T : P(x) \to P(x)$ and $T\left(p(x)\right) = \int_0^x p(t)dt$

$$\begin{aligned} T(\alpha p + \beta q) &= \int_0^x \left(\alpha p(t) + \beta q(t)\right) dt \\ &= \alpha \int_0^x p(t)dt + \beta \int_0^x q(t)dt \\ &= \alpha T(p) + \beta T(q) \end{aligned}$$

3 $T : P(x) \to P(x)$ and $T(p(x)) = p''(x)$

It is a linear transformation.

4 $T : P(x) \to P(x)$ and $T(p(x)) = p'(x)$

It is a linear transformation.

5 $T : P(x) \to P(x)$ and $T(p(x)) = p''(x) + p(x)$

It is a linear transformation.

Theorem 6.3. *Let $T : V \to W$ be a linear transformation, where V and W are vector spaces over a same field F. If $S = \{v_1, v_2, \cdots, v_n\}$ be a spanning set of V, then the set $\{T(v_1), T(v_2), \cdots, T(v_n)\}$, say R is the spanning set of the range space of T.*

Proof. Range space of $T = \{w \in W \mid \exists v \in V, w = T(v)\}$ and since $\text{span}(S) = V$, for every $v \in V$ there exists scalars $c_1, c_2, \cdots, c_n$ from the field F such that

$$\begin{aligned} v &= c_1v_1 + c_2v_2 + \cdots + c_nv_n \\ T(v) &= T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) \\ w &= c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) \end{aligned}$$

Hence $R = \{T(v_1), T(v_2), \cdots, T(v_n)\}$ spans the range space of T. □

Theorem 6.4. *Let $T : V \to W$ be a linear transformation, where V and W are vector spaces over a same field F and $S = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. If the set $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is linearly independent set, then S is also a linearly independent set.*

Proof. Consider the equation

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

where c_i's are scalars from the field F. Then

$$\begin{aligned} T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) &= T(0) \\ c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) &= 0 \end{aligned}$$

since $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is an L.I set, $c_1 = c_2 = \cdots = c_n = 0$. We just showed that

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0$$

. Hence $\{v_1, v_2, \cdots, v_n\}$ is linearly independent. □

Theorem 6.5. *Let V and W be two finite dimensional vector spaces over F and $\{v_1, v_2, \cdots, v_n\}$ be a basis for V and $u_1, u_2, \cdots, u_n$ be any vectors in W. Then there exists a **unique** Linear Transformation from V to W such that $T(v_i) = u_i$ for $1 \leq i \leq n$*

Proof. Lenghty, hence skipped. □

Theorem 6.6. Rank Nullity Theorem *Let $T : V \to W$ be a linear transformation and V is a **finite dimensional vector space**. Then*

$$\eta(T) + \rho(T) = \dim V$$

OR

$$\dim N(T) + \dim R(T) = \dim V$$

Proof. skipped. □

Note:

- For homogeneous systems $\dim(R(T)) = n - r$
- For non-homogeneous systems $\dim(R(T)) = n - r + 1$

where r is the rank of the coefficient matrix and n is the number of variables (from the order $m \times n$)

6.4 Algebra of Linear Transformation

Theorem 6.7. *Let $T : V \to W and S : V \to W$ be two linear transformation, then their sum $T + S$ and the scalar multiplication $cT, c \in F$ are also linear transformations. V, W, F, c have their regular meanings.*

Corollary 6.7.1. *Let $\mathrm{Hom}(V, W)$ be the set of all linear transformations from vector spaces $V \to W$ over the same field F, then $\mathrm{Hom}(V, W)$ forms a subspace of the vector space of all functions or maps.*

If $\dim(V) = m$, $\dim(W) = n$, then $\dim(\mathrm{Hom}(V, W)) = mn$

Theorem 6.8. *Let $T : V \to W$ and $S : W \to V$ be two linear transformations, then $T \circ S$ and $S \circ T$ are also linear transformations from W to W and V to V respectively.*

Proof. For all $u, v \in V$ and $\alpha, \beta \in F$ the expression

$$\begin{aligned}(T \circ S)(\alpha u + \beta v) &= T(S(\alpha u + \beta v)) \\ &= T(S(\alpha u) + S(\beta v)) \\ &= T(\alpha S(u) + \beta S(v)) \\ &= T(S(\alpha u)) + T(S(\beta v)) \\ &= \alpha T(S(u)) + \beta T(S(v)) \\ &= \alpha(T \circ S)(u) + \beta(T \circ S)(v)\end{aligned}$$

□

Definition 6.4. ***Singular Map:*** *Let V and W be vector subspaces over the same field F and $T : V \to W$ be a linear transformation then T is called a singular map if there exists a non zero vector $x \in V$ such that $T(x) = 0$*

Definition 6.5. ***Non-Singular Map:*** *Let V and W be vector subspaces over the same field F and $T : V \to W$ be a linear transformation then T is called a singular map if there does not exist a non zero vector $x \in V$ such that $T(x) = 0$*

Note: if $\eta(T) > 0$, then T is a singular map and if $\eta(T) = 0$ then T is a non-singular map

6.5 Invertible Maps

Theorem 6.9. *Let $T : V \to V$ be a linear transformation where V is a **finite dimensional vector space**, then*

$$\begin{aligned}\ker(T) = 0 &\iff T \text{ is one-one} \\ &\iff T \text{ is onto} \\ &\iff T \text{ is bijective} \\ &\iff T \text{ is invertible} \\ &\iff T \text{ is non singular}\end{aligned}$$

Note: For the linear transformation $T : V \to W$

1. If $\dim(V)$ is finite, then

$$\begin{aligned}\ker(T) = 0 &\iff T \text{ is one-one} \\ &\iff T \text{ is non singular}\end{aligned}$$

2. If $\dim(V) = \dim(W)$, then

$$\begin{aligned}\ker(T) = 0 &\iff T \text{ is one-one} \\ &\iff T \text{ is onto} \\ &\iff T \text{ is bijective} \\ &\iff T \text{ is invertible} \\ &\iff T \text{ is non singular}\end{aligned}$$

6.6 Matrix Representation

Definition 6.6. *$T : V \to V$ be a Linear Transformation where V is a **finite dimensional vector space**. Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of V. Then matrix representation of T with respect to B is given as $[\boldsymbol{T}]_{\boldsymbol{B}}$*

$$\begin{aligned} T(v_1) &= a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ T(v_2) &= a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ &\cdots \\ T(v_n) &= a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n \end{aligned}$$

$$[\boldsymbol{T}]_{\boldsymbol{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^{\mathsf{T}}$$

$$[T]_{B_1} \sim [T]_{B_2} \sim \cdots [T]_{B_n} \sim \cdots$$

By similar we mean the matrix representations have **same eigen values, trace, detereminant** (for a linear operator, i.e a transformation from V to V)

6.7 Change of Basis

Definition 6.7. *Let $T : V \to V$ be a linear transformation and B_1 and B_2 be two bases of V. $B_1 = \{v_1, v_2, \cdots, v_n\}$ and $B_2 = \{u_1, u_2, \cdots, u_n\}$. Then*

$$\begin{aligned} v_1 &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\ v_2 &= a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n \\ &\vdots \\ v_n &= a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n \end{aligned}$$

and

$$[\boldsymbol{T}]_{\boldsymbol{B_2}}^{\boldsymbol{B_1}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^{\mathsf{T}}$$

is called the change of basis matrix from B_1 to B_2

OR

Definition 6.8. *Let $T : V \to W$ be a linear transformation and B_1 and B_2 be the bases of V and W respectively. $B_1 = \{v_1, v_2, \cdots, v_m\}$ and $B_2 = \{u_1, u_2, \cdots, u_n\}$. Then*

$$\begin{aligned} T[v_1] &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\ T[v_2] &= a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n \\ &\vdots \\ T[v_m] &= a_{m1}u_1 + a_{m2}u_2 + \cdots + a_{mn}u_n \end{aligned}$$

and

$$[\boldsymbol{T}]_{\boldsymbol{B_2}}^{\boldsymbol{B_1}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^{\mathsf{T}}$$

is called the change of basis matrix from B_1 to B_2

Basically change of basis gives us a matrix which maps coordinate vectors from one basis to another, that is $[T]_{B_2}^{B_1}$ maps coordinate vector of vector in B_1 basis to coordinate vector of the same vector in B_2 basis

Note: If $P = [T]_{B_1}^{B_2}$ and $Q = [T]_{B_2}^{B_1}$, then

- P and Q are invertible matrices
- P and Q are inverses of each other

$$P^{-1} = Q \text{ and } Q^{-1} = P \text{ and } PQ = I = QP$$

- $[T]_{B_1}^{B_2} = [u_1 u_2 \cdots u_n]$ where B_1 is a standard basis and $B_2 = \{u_1, u_2, \cdots u_n\}$ is any other basis.

7 Eigen Values and Eigen Vectors

Definition 7.1. *Let $T : V \to V$. If for a non zero vector $v \in V$ there exists a scalar $\lambda \in F$, such that*

$$T(v) = \lambda v$$

, then λ is called the eigen value of T and v is called the eigen vector of T corresponding to the eigen value λ.

7.1 How to find Eigen Values ?

We know that $T(v) = \lambda v \implies (T - \lambda I)v = 0$. Find the null space of the transformation $(T - \lambda I)$.

OR

Since $T : V \to V$, take any matrix representation of T w.r.t to any basis B, i.e $[T]_B = A$ and then plug it in the equation $(T - \lambda I)v = 0$. We are looking for non zero v's, if any exists, hence for that we should make the matrix $A - \lambda I$ singular. Solve $\det(A - \lambda I)$ for λ
$\det(A - \lambda I)$ = characteristic polynomial $\det(A - \lambda I) = 0$ $\lambda \to$ eigen values / eigen roots / latent roots / characteristics

7.1.1 2×2 and 3×3 charateristics polynomial

$$A_{2\times 2} \implies \lambda^2 - (\text{trace})\lambda + \det(A) = 0$$
$$A_{3\times 3} \implies \lambda^3 - (\text{trace})\lambda^2 + (\text{sum of cofactors of diagonal elements})\lambda - \det(A) = 0$$

7.1.2 Special Matrices and their Eigen Values

Matrix	**Eigen Values**
diagonal matrix	diagonal entries
upper triangular matrix	diagonal entries
lower triangular matrix	diagonal entries
sum of each row or each column is s	s is an eigen value
$\det(A) = 0$	0 is an eigen value
scalar matrix $\begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{bmatrix}$	c
Back Diagonal Matrix of even order, example $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$	$\pm\sqrt{a_{n,n-i} \cdot a_{i,n-1}}$, where n is the order and i varies from 1 to $\frac{n}{2}$, that is $\pm\sqrt{4 \cdot 1}, \pm\sqrt{3 \cdot 2}$
Back Diagonal Matrix of odd order, example $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$	$\pm\sqrt{a_{n,n-i} \cdot a_{i,n-1}}$, where n is the order and i varies from 1 to $\left\lfloor \frac{n}{2} \right\rfloor$ and a_{kk}, where k is $\left\lceil \frac{n}{2} \right\rceil$, that is $\pm\sqrt{1 \cdot 3}, 2$

Definition 7.2. Eigen Space of a Transformation *Let $T : V \to V$ be a linear transformation, where V is a vector space over the field F and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be its eigen values, then consider the sets*

$$\begin{aligned} E_{\lambda_1} &= \{v \in V \mid T(v) = \lambda_1 v\} \\ E_{\lambda_2} &= \{v \in V \mid T(v) = \lambda_2 v\} \\ &\vdots \\ E_{\lambda_n} &= \{v \in V \mid T(v) = \lambda_n v\} \end{aligned}$$

These sets are called the Eigen Space of λ_n. In general

$$E_\lambda = \{v \in V \mid T(v) = \lambda v\}$$

is called the Eigen space of λ for linear transformation T.

Theorem 7.1. *The Eigen space E_λ of λ for linear transformation $T : V \to V$ is a subspace of V, where V is a vector space over the field F.*

Proof. Trivial ☐

Note: $\ker(T - \lambda I) = E_\lambda$. Kernal of $T - \lambda I$ can be written as $\{v \in V \mid (T - \lambda I)v = 0\}$

7.2 Algebraic and Geometric Multiplicity of an Eigen Value

Definition 7.3. Algebraic Multiplicity (AM) *The AM of an eigen value λ is the number of times it has occured.*

Definition 7.4. Geometric Multiplicity (GM) *The GM of an eigen value λ is equal to the number of linearly independent vectors corresponding to the eigen value λ.*

Theorem 7.2. *Let $T : V \to V$ be a vector space and $\dim(V) = n$. λ is an eigen value of T if and only if $T - \lambda I$ is singular i.e, non-invertible.*

7.2.1 Similar Matrices

Definition 7.5. *If there exists a non-singular matrix P such that*

$$A = P^{-1}BP \text{ or } B = PAP^{-1},$$

then A and B are called similar matrices, that is $A \sim B$.

For similar matrices their D.C.T.E.R.M's are same.

D $\to$ Determinant
C $\to$ Characteristic Polynimial
T $\to$ Trace
E $\to$ Eigen values
R $\to$ Rank
M $\to$ Minimal polynomial

7.2.2 Diagonalisibility

Let $T : V \to V$ be a linear transformation and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be its eigen values, then

- T is diagonalizable if all of its eigen values are distinct.
- A diagonal matrix is diagonalizable.
- T is diagonalizable if and onlt if $AM = GM$ for all eigen values of T
- T is diagonalizable if and only if $\sum_{i=1}^{n} E_{\lambda_i}$ =dim(V)

Note:

- AM $\leq$ GM
- $T(v) = \lambda v \implies (T(v) - \lambda v)(v) = 0$. Then number of linearly independent solutions of $Ev = 0$ is equal to the rank of E or nullity of the linear transformation $T - \lambda I$.

  GM of λ = nullity of $E = n - r$, where r is the rank of the matrix E (representation of $T - \lambda I$)

  Let A be the matrix representation w.r.t to any basis. For sake of simplicity take A as the matrix representation w.r.t the standard basis, i,e $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$, $[T]_{S.B} = A$. Eigen values of $A_{n\times n}$ are $\lambda_1, \lambda_2, \cdots, \lambda_n$. If

  $$\text{AM of } \lambda_1 = n$$

  then

  $$\text{GM of } \lambda_1 = 1/2/\cdots/n$$

- Range of T is spanned by all the eigen vectors corresponding to a non zero eigen value of T.
- Null Space of T is spanned by all eigen vectors corresponding to a 0 eigen value of T.
- c is an eigen value of T if and only if $T - cI$ is singular.
- If A and B are two similar matrices then A and B have same characteristic polynomial.