Note:

① Consider the LDE of Ind order

$$a_0(x) y'' + a_1(x) y' + a_2(x) y = 0 \quad \text{—— ①}$$

where $a_0(x) \neq 0$ and $a_0(x)$, $a_1(x)$, $a_2(x)$ are all continuous fn of x $\forall x \in [a, b)$

ⓐ Then two solns y_1 and y_2 of eqn ① are L.I. iff $W(y_1, y_2) \neq 0$ $\forall x \in [a, b]$

ⓑ The two solns y_1 and y_2 of eqn ① are LD iff $W(y_1, y_2) = 0$ $\forall x \in [a, b)$

② If y_1 and y_2 are two solns of a diffal eqn and diffal eqn is not given Then

ⓐ If $W(y_1, y_2) \neq 0 \Rightarrow y_1$ & y_2 are L.I.

(b) If $W(y_1, y_2) = 0 \Rightarrow$ Then we can't say anything.

Ex: The fn $y_1 = x^3$ and $y_2 = |x^3|$ are linearly indep soln on the real line of the eqn $x^2 y'' - 3xy' + 3y = 0$, varify that $W(y_1, y_2)$ is identically zero.

Expn:- If $x > 0$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0$$

Case II: **If $x < 0$**

Then $y_1 = x^3$, $y_2 = -x^3$

$$\therefore W(y_1, y_2) = \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix}$$

$$= -3x^5 + 3x^5 = 0$$

$$W(y_1, y_2) = 0$$

Case III: If $x = 0$

$$W(y_1, y_2) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

$$W(y_1, y_2) = 0$$

$$x^2 y'' - 3xy' + 3y = 0$$

$$y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 0 \quad \text{—— (1)}$$

At $x = 0$, The eqn (1) is not defined

OR

$$\frac{y_1}{y_2} = \frac{x^3}{|x^3|} = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\left.\begin{array}{l} LHL = -1 \\ RHL = 1 \end{array}\right\} \text{ at } x = 0$$

The fun $\frac{y_1}{y_2}$ is not cts at $x = 0$

$\therefore$ y_1 & y_2 are L.I.

③ If $y_1(x) = e^{m_1 x}$, $y_2(x) = e^{m_2 x}$, $y_3(x) = e^{m_3 x}$ be three L.I. soln of a DE

$$a_0 y''' + a_1 y'' + a_2 y' + a_3 = 0 \quad \text{where } a_0 \neq 0$$

Then

$$W(y_1, y_2, y_3) = e^{(m_1+m_2+m_3)x} \begin{vmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \\ m_1^2 & m_2^2 & m_3^2 \end{vmatrix}$$

If $y_1 = e^{m_1 x}$ . . . . . . $y = e^{m_n x}$ be n soln s. Then

$$W(y_1, y_2 \cdots y_n) = e^{(m_1+m_2+\cdots+m_n)x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ m_1 & m_2 & \cdots & m_n \\ \vdots & \vdots & & \vdots \\ m_1^{n-1} & m_2^{n-2} & \cdots & m_n^{n-1} \end{vmatrix}$$

Vendermonde determinant

(4) Let $y = y(x)$, Then consider the DE $\frac{dy}{dx} = y^a$, $y(b) = 0$, $b \in \mathbb{R}$. & $a \in (0, 1)$, Then the number of real valued soln of diff^{tal} eq^n is infinite and it has infinite number of L.I. soln.

But if $y(b) = 1$ $b \in \mathbb{R}$. $a \in (0, 1)$

Then no. of soln is unique.

Ex:- The diff^{tal} eq^n $\frac{dy}{dx} = y^{1/3}$, $y(0) = 0$ has infinite soln.

Expln:-

$$\frac{dy}{dx} = y^{1/3} \Rightarrow \frac{dy}{y^{1/3}} = dx$$

$\Rightarrow y^{-1/3}\,dy = dx$

Integrate

$$\frac{y^{-1/3+1}}{-1/3+1} = x + c$$

$$\Rightarrow \frac{3}{2} y^{2/3} = x + c \quad \text{—— (1)}$$

given $y(0)=0 \Rightarrow$ when $x=0,\ y=0$

$$\Rightarrow \frac{3}{2}\cdot 0 = 0 + c \Rightarrow c = 0$$

$\therefore$ from (1) $\frac{3}{2} y^{2/3} = x$

$$\Rightarrow y^{2/3} = \frac{2}{3} x$$

$$\Rightarrow y^2 = \left(\frac{2}{3}x\right)^3 \Rightarrow y^2 = \frac{8}{27} x^3$$

$$\Rightarrow 27 y^2 = 8 x^3 .$$

Ex:.. Let $y_1(x), y_2(x), y_3(x)$ be the soln of the diff eqn

$$\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

If the Wronskian $W(y_1, y_2, y_3)$ is of the form ke^{bx} f.s. constant k. Then the value of b is ------

Soln:- $(D^3 + 6D^2 + 11D - 6)y = 0 \quad D \equiv d/dx$

The AE is

$m^3 + 6m^2 + 11m \cdot 6 = 0$

$\Rightarrow (m-1)(m-2)(m-3) = 0$

$\Rightarrow m = 1, 2, 3 \quad y = CF = c_1e^{x} + c_2e^{2x} + c_3e^{3x}$.

$\therefore y_1 = e^{x}, y_2 = e^{2x}, y_3 = e^{3x}$ be 3 L.I. soln

$$\therefore \quad W(y_1, y_2, y_3) = e^{(1+2+3)x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= (6 - 6 + 2) e^{6x}$$

$$= 2e^{6x}$$

Given $W(y_1, y_2, y_3) = k e^{bx}$

$\therefore \boxed{b = 6}$ f.s. constant $k = 2$.

PYQ.

2005:- ① An IF of $x\frac{dy}{dx} + (3x+1)y = xe^{-2x}$ is ....

Soln. The given diff'tal eqn can be written as

$$\frac{dy}{dx} + \left(3 + \frac{1}{x}\right) y = e^{-2x}$$

$$IF = e^{\int\left(3+\frac{1}{x}\right)dx} = e^{3x+\log x} = \underline{xe^{3x}}$$

② The general soln of $x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 0$ is ...

Soln: The DE is

$$(x^2D^2 - 5xD + 9)y = 0$$

Let $x = e^Z \Rightarrow Z = \log x$

$$(D(D-1) - 5D + 9)y = 0 \qquad D = d/dz$$

$$(D^2 - 6D + 9)y = 0$$

The AE is $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3, 3.$

$\therefore y = CF = (c_1 + c_2 z)e^{3z}$

$= (c_1 + c_2 z)(e^z)^3$

$= (c_1 + c_2 \log x)x^3$

2006:. ③ If $\left(\dfrac{c_1 + c_2 \log x}{x}\right)$ is general soln of the DE $(x^2D^2 + kxD + 1)y = 0$, $x > 0$, Then $k = \ldots$

$$y = \left(\frac{c_1 + c_2 \log x}{x}\right)$$

The DE is

$$(x^2D^2 + kxD + 1)\,y = 0$$

Let $x = e^z \Rightarrow z = \log x$

$$\therefore (D(D-1) + kD + 1)\,y = 0$$

$$(D^2 + \underline{(k-1)}D + 1)\,y = 0$$

$$\therefore y = (c_1 + c_2 \log x)\,x^{-1}$$

$$\therefore y = (c_1 + c_2 z)\,e^{-z}$$

$\therefore$ The roots are $m = -1, -1$

$$(D+1)^2 y = 0$$

$$(D^2 + \underline{2}D + 1)\,y = 0$$

$$\Rightarrow k - 1 = 2$$

$$\Rightarrow \boxed{k = 3}$$

2007 :- 4 If k is constant s.t.

$$xy + k = e^{\frac{(x-1)^2}{2}}$$ satisfies the DE.

$$x\frac{dy}{dx} = \frac{(x^2-x-1)y + (x-1)}{}$$, Then k is ---- ← ①

Sol^n :

$$xy + k = e^{\frac{(x-1)^2}{2}}$$

$$\Rightarrow x\frac{dy}{dx} + y = e^{\frac{(x-1)^2}{2}} \cdot 2\frac{(x-1)}{2}$$

$$\Rightarrow x\frac{dy}{dx} + y = (xy+k)(x-1)$$

$$\Rightarrow x\frac{dy}{dx} = (xy+k)(x-1) - y$$

$$= (xy(x-1) - y) + k(x-1)$$

$$\Rightarrow x\frac{dy}{dx} = (x^2-x-1)y + k(x-1) \quad \text{— ②}$$

from ① & ②

$\underline{k = 1}$

(5) One of the IF of the DE $(y^2-3xy)\,dx + (x^2-xy)\,dy = 0$ is - - -

Expln: The given D.E is homo.

$$\therefore \text{IF} = \frac{1}{Mx+Ny}$$

$$= \frac{1}{(y^2-3xy)x + (x^2-xy)y}$$

$$= \frac{1}{\cancel{xy^2} - 3x^2y + x^2y - \cancel{xy^2}}$$

$$= \frac{1}{-2x^2y}$$

2008:- (6) Let $y_1(x)$ & $y_2(x)$ be twice diff'able fun^n on an interval I satisfying the DE.

$$\frac{dy_1}{dx} - y_1 - y_2 = e^x \quad \& \quad 2\frac{dy_1}{dx} + \frac{dy_2}{dx} - 6y_1 = 0$$

Then $y_1(x)$ is - - - - -

Soln :- **Simultaneous D.E:**

$$D\left[\begin{array}{l}(D-1)y_1 - y_2 = e^x \\ (2D-6)y_1 + Dy_2 = 0\end{array}\right.$$

$$(D^2 - D + 2D - 6)y_1 = D(e^x)$$

$$(D^2 + D - 6)y_1 = e^x$$

The AE is

$$m^2 + m - 6 = 0$$

$$(m+3)(m-2) = 0$$

$$\therefore m = -3, 2$$

$$\therefore CF = c_1 e^{-3x} + c_2 e^{2x}$$

$$\& \; PI = \frac{1}{D^2 + D - 6} e^x$$

$$= \frac{1}{1+1-6} e^x$$

$$= \frac{-1}{4} e^x$$

$$\therefore y_1 = c_1 e^{-3x} + c_2 e^{2x} - \frac{1}{4} e^x$$

Method II:

$$\frac{d^2y_1}{dx^2} - \frac{dy_1}{dx} - \frac{dy_2}{dx} = e^x$$

$$\Rightarrow \quad \frac{dy_2}{dx} = \frac{d^2y_1}{dx^2} - \frac{dy_1}{dx} - e^x$$

$\therefore$ from the DE.

$$2\frac{dy_1}{dx} + \frac{d^2y_1}{dx^2} - \frac{dy_1}{dx} - 6y_1 = e^x$$

$$(D^2 + D - 6)y_1 = e^x$$

2009: 7 Consider the DE $2\cos(y^2)\,dx - xy\sin(y^2)\,dy = 0$

(a) e^x is an IF (b) e^{-x} is an IF

(c) $3x$ is an IF (d) x^3 is an IF ✓

Soln :-

$$2\cos(y^2)\,dx - xy\sin(y^2)\,dy = 0$$

$$M = 2\cos(y^2), \quad N = -xy\sin(y^2)$$

$$\frac{\partial M}{\partial y} = -4y\sin y^2, \quad \frac{\partial N}{\partial x} = -y\sin(y^2)$$

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4y\sin y^3 + y\sin(y^2)$$

$$= -3y\sin(y^2)$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3}{x} \text{ (fn of } x\text{)}$$

$$\therefore IF = e^{\int \frac{3}{x}dx} = e^{3\log x} = e^{\log x^3}$$

$$\therefore IF = \underline{x^3}$$

(8) Let $f, g : [-1, 1] \longrightarrow \mathbb{R}$

$f(x) = x^3$, $g(x) = x^2 |x|$, Then

✓ (a) f & g are L.I. on $[-1, 1]$

✗ (b) f & g are L.D. on $[-1, 1]$

✗ (c) $f(x) g'(x) - f'(x) g(x)$ is not identically zero on $[-1, 1]$

✗ (d) There exist continuous fn $p(x)$ & $q(x)$ s.t. f & g satisfy $y'' + py' + qy = 0$ on $[-1, 1]$

Expn :-

$$\frac{f(x)}{g(x)} = \frac{x^3}{x^2 |x|} = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$\therefore$ f & g are L.I. on $[-1, 1]$

2010 :

(14) Consider the DE $\frac{dy}{dx} = ay - by^2$ where $a, b > 0$

& $y(0) = y_0$, if $x \to \infty$, then soln $y(x)$ tends to --

Soln :-

$$\frac{dy}{dx} = ay - by^2$$

$$\Rightarrow \frac{dy}{dx} - ay = -by^2$$

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + a \cdot \frac{1}{y} = b \qquad \text{(1)}$$

Let $\frac{1}{y} = v$

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$$

$\therefore$ from (1) $\quad \frac{dv}{dx} + a \cdot v = b$

$$IF = e^{\int a\,dx} = e^{ax}$$

$$\therefore v\,e^{ax} = \int b\,e^{ax}\,dx$$

$$\therefore v\,e^{ax} = \frac{b}{a}e^{ax} + c$$

$$\Rightarrow v = \frac{b}{a} + c\,e^{-ax}$$

$$\therefore \frac{1}{y} = \frac{b}{a} + c\,e^{-ax} \quad \text{— (1)}$$

given $y(0) = y_0$

when $x = 0$, $y = y_0$

$$\frac{1}{y_0} = \frac{b}{a} + c\,e^{-0}$$

$$\Rightarrow c = \left(\frac{1}{y_0} - \frac{b}{a}\right)$$

$\therefore$ from eqⁿ (1)

$$\frac{1}{y(x)} = \frac{b}{a} + \left(\frac{1}{y_0} - \frac{b}{a}\right)e^{-ax}$$

when $x \to \infty$

$$\frac{1}{y(x)} = \frac{b}{a} + \left(\frac{1}{y_0} - \frac{b}{a}\right)e^{-\infty}$$

$$\therefore \frac{1}{y(x)} = \frac{b}{a}$$

$$\Rightarrow \boxed{y(x) = \frac{a}{b}}$$

(11) Consider the DE $(x+y+1)\,dx + (2x+2y+1)\,dy = 0$
which of the following statements is true?

(a) The DE is linear.

(b) The DE is exact

(c) e^{x+y} is an I.F. of the DE

✓ (d) A suitable substitution transform the DE to the separable variable.

Expln:- MAE: A diffntl eqn is of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad \text{(Non-homo. DE)}$$

If $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{\lambda}$ (say)

$$\Rightarrow \quad a' = a\lambda$$
$$b' = b\lambda$$

$$\therefore \quad \frac{dy}{dx} = \frac{ax+by+c}{\lambda(ax+by)+c'} \quad \text{——— (1)}$$

Let $\boxed{ax+by = v}$

$$\Rightarrow \quad a + b\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

$\therefore$ Eqn (1) becomes

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = \frac{v+c}{\lambda v + c'} \quad \text{——— (2)}$$

which is in the form of separable - variable

Note: (i) If $\frac{a}{a'} = \frac{b}{b'}$ then non-homo. DE directly convert to seperable-variable.

(ii) The non-homo. DE. directly convert to seperable variable if $\frac{a}{a'} = \frac{b}{b'}$

2011 :-