# Sequence and Series

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**Terms of Sequence:** Let $< a_n >$ be a sequence then $a_1, a_2, \cdots, a_n$ are called terms of a sequence

**Range of a Sequence:** The set of all distinct terms of a sequence is called its range

**Bounded and Unbounded sequence:**

1. **Bounded Above Sequences:** A sequence $< a_n >$ is said to be bounded above if there exists a real number $k$ such that $a_n \leq k$ for all $n \in \mathbb{N}$

2. **Bounded Below Sequences:** A sequence $< a_n >$ is said to be bounded below if there exists a real number $k$ such that $a_n \geq k$ for all $n \in \mathbb{N}$

   A sequence which is bounded above and bounded below is called a bounded sequence.

3. **Unbounded Sequence:** A sequence is said to be unbounded if it is not bounded.

**Ex** Bound Above and Bound Below sequence: $a_n =< \frac{1}{n} >$

**Ex** Neither Bound Above nor Bound Below sequence: $a_n =< (-1)^n n >$

**Supremum(least-upper bound):** The supremum of the range set the sequence is called Supremum of that sequence.

**Infimum(greatest-lower bound):** The infimum of the range set the sequence is called Infimum of that sequence.

**Note:**

1. If a sequence is unbounded above then its Supremum is $\infty$

2. If a sequence is unbounded below then its Infimum is $-\infty$

3. If for any sequence its Supremum and Infimum are finite (or it exists) then it is a bounded sequence

**Monotonocity of a Sequence:**

1. **Monotonic Increasing Sequence:** Let $< a_n >$ be a sequence, the this sequence is called monotonically increasing sequence if $a_{n+1} \geq a_n$ for all $n > N, N \in \mathbb{N}$.

2. **Strictly Monotonic Increasing Sequence:** Let $< a_n >$ be a sequence, the this sequence is called strictly monotonically increasing sequence if $a_{n+1} \geq a_n$ for all $n \geq N, N \in \mathbb{N}$.

3. **Monotonic Decreasing Sequence:** Let $< a_n >$ be a sequence, the this sequence is called monotonically decreasing sequence if $a_{n+1} \leq a_n$ for all $n \geq N, N \in \mathbb{N}$.

4. **Strictly Monotonic Decreasing Sequence:** Let $< a_n >$ be a sequence, the this sequence is called strictly monotonically decreasing sequence if $a_{n+1} < a_n$ for all $n \geq N, N \in \mathbb{N}$.

| **Sequence** | **Bounded Above** | **Bounded Below** | **Range Set** |
|---|---|---|---|
| **Ex.1** $a_n =< \frac{1}{n} >, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ | yes, 1 | yes, 0 | $\text{R} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ |
| **Ex.2** $a_n =< (-1)^n >, -1, 1, -1, 1, \cdots$ | yes, 1 | yes, $-1$ | $\text{R} = \{-1, 1\}$ |
| **Ex.3** $a_n =< \frac{(-1)^n}{n} >, -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \cdots$ | yes, $\frac{1}{2}$ | yes, $-1$ | $\text{R} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \cdots\right\}$ |
| **Ex.4** $a_n =< 1 + (-1)^n >, 0, 2, 0, 2, \cdots$ | yes, 0 | yes, 2 | $\text{R} = \{0, 2\}$ |
| **Ex.5** $a_n =< \frac{n}{n+1} >, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$ | yes, 1 | yes, $\frac{1}{2}$ | $\text{R} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right\}$ |
| **Ex.6** $a_n =< n >, 1, 2, 3, 4, \cdots$ | no | yes, 1 | $\text{R} = \{1, 2, 3, 4, \cdots\}$ |
| **Ex.7** $a_n =< -n >, -1, -2, -3, -4, \cdots$ | yes, -1 | no | $\text{R} = \{-1, -2, -3, -4, \cdots\}$ |
| **Ex.8** $a_n = \begin{cases} 2, \text{if } n \text{ is prime} \\ n, \text{if } n \text{ is not prime} \end{cases}$ | no | yes, 1 | $\text{R} = \{1, 2, 2, 4, 2, 6, \cdots\}$ |
| **Ex.9** $a_n =< (-1)^n n >, -1, 2, -3, 4, \cdots$ | no | no | $\text{R} = \{-1, 2, -3, 4, \cdots\}$ |

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**Limit point / Cluster point**

- If a sequence is bounded and has only one limit point, then that sequence converges to that point.
- If a sequence has more than one limit points then its limit does not exist.
- **Examples:**
  1. $a_n = \begin{cases} 2, \text{if n is prime} \\ n, \text{if n is not prime} \end{cases}$

     It has a limit point at 2, because every neighbourhood of 2 has infinite number of terms if the sequence.
- **Results:**
  1. **Bolzano - Weierstrass Theorem:** Every bounded sequence has a limit point.
  2. Unbounded sequence may have a limit point.

**Limit of a Sequence:** Let $< a_n >$ be a sequence, limit of the sequence is denoted by $\lim\limits_{n \to \infty} a_n$.

- **Result:**
  1. A sequence can have atmost one limit.
  2. Unbounded sequence cannot have limit.
  3. A non-monotonic sequence can have limit. Ex: $< \dfrac{(-1)^n}{n} >$.
  4. A bounded sequence may not have a limit. Ex: $< (-1)^n >$.
  5. Limit of a sequence is also a limit point, but the converse is not true.