# Dimension

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 15 • Sept 19, 2021

$AX = B$ $\qquad A_{m\times n} \quad X_{n\times 1} \quad B_{m\times 1}$

System of linear eq$^n$

Soln. of this system.

$\Rightarrow W = \{ X \mid \underline{AX = 0} \}$

W is a sub-space.

$\Rightarrow A \cdot 0 = 0 \qquad 0_{n \times 1} \in W \qquad W \neq \phi$

$\alpha X + \beta Y \in W!$

$\Rightarrow \forall X, Y \in W, \forall \alpha, \beta \in F$

Claim : $\alpha X + \beta Y \in W$

$\because X, Y \in W \Rightarrow AX = 0$ and $AY = 0$

$$\begin{aligned} \text{LHS} = A(\alpha X + \beta Y) &= \alpha \cdot 0 + \beta \cdot 0 \\ = A(\alpha X) + A(\beta Y) &= 0 + 0 \\ = \alpha AX + \beta AY &= 0 \end{aligned}$$

$$W = \{ x \mid AX = 0 \}$$

$$\begin{aligned}
\dim W &= \text{no. of LI solns.} \\
&= \text{no. of free variables} \\
&= \text{nullity of } A \; \Big( \underbrace{n}_{\text{no. of variables involved}} - \overset{\text{rank}(A)}{r} \Big)
\end{aligned}$$

$$W = \{ x \mid AX = B,\ B \neq 0 \}$$

$$A \cdot 0 = 0 \neq B$$

$$0 \notin W$$

Set of soln. of a non-homo. sys. can never form a sub-space.

(1) $x + 2y + 2z - s + 3t = 0$

$x + 2y + 3z + s + t = 0$

$3x + 6y + 8z + s + 5t = 0$

$AX = 0 \quad X = \begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix}$

$W = \{ X \mid AX = 0 \}$

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim W = n - r$

$= 5 - 2$

$= 3$

$$x + 2y + 2z - s + 3t = 0$$

$$z + 2s - 2t = 0$$

| $y$ | $s$ | $t$ | $\longrightarrow$ | $x$ | $z$ |
|---|---|---|---|---|---|
| 1 | 0 | 0 | | -2 | 0 |
| 0 | 1 | 0 | | 5 | -2 |
| 0 | 0 | 1 | | -7 | 2 |

$$B = \{ (-2, 1, 0, 0, 0) \; (5, 0, -2, 1, 0),$$
$$(-7, 0, 2, 0, 1) \}$$

$x + 2y + 3z = 0$

LV

F.

$(x, y, z) \rightarrow$

LV.

$0 \cdot x + y + 2z = 0$

FV

(eg)

$$x + 2y + z - 2t = 0$$

$$2x + 4y + 4z - 3t = 0$$

$$3x + 6y + 7z - 4t = 0$$

$W = \{ X \mid AX = 0 \}$

$\downarrow$

Basis and dim. of $W$.

$$\begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 4 & 4 & -3 \\ 3 & 6 & 7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= n - r$$

$$= 4 - 2 = 2$$

$$x + 2y + z - 2t = 0$$

$$(y) \qquad 2z + t = 0$$

$$(t)$$

$$x + 0 - \frac{1}{2} - 2 = 0$$

$$x = 5/2$$

| $y$ | $t$ | $z$ | $x$ |
|---|---|---|---|
| | | $0$ | $-2$ |
| $1$ | $0$ | | |
| $0$ | $1$ | $-1/2$ | $5/2$ |

$$B = \left\{ (-2, 1, 0, 0), \left(5/2, 0, -1/2, 1\right) \right\}$$

Theorem : Let $V$ be the FDVS over the field $F$. Let $B$ be a subset of $V$. Then,

$B$ is a basis of $V$ iff every $v \in V$ can be expressed as a unique linear combination of members of $B$.

$1 \Rightarrow 2$ Let $B = \{v_1, v_2, \dots, v_n\}$ is a basis of $V$.

$\because$ $B$ is a basis $\Rightarrow$ $\operatorname{span} B = V$ $\Rightarrow$ $\forall\, v \in V$, $\exists\, c_1, c_2, \dots, c_n \in F$ s.t.

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \quad \text{---} \ (1)$$

<u>Claim</u>: $c_i \in F$ $\forall i$ must be unique.

Suppose $c_i$'s are not unique

$\exists\, k_1, k_2, \dots, k_n \in F$

$$v = k_1 v_1 + k_2 v_2 + \dots + k_n v_n \quad \text{---} \ (2)$$

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = k_1 v_1 + k_2 v_2 + \dots + k_n v_n$$

$$(c_1 - k_1) v_1 + (c_2 - k_2) v_2 + \dots + (c_n - k_n) v_n = 0$$

$\because$ B is a basis $\Rightarrow$ B is LI set vector.

$$c_i - k_i = 0 \quad \forall\, i = 1, 2 \dots n$$

$$c_i = k_i \quad \forall\, i = 1, 2 \dots n$$

contradiction

$2 \Rightarrow 1$ Every $v \in V$ can be exp. as unique L.C. of B

Claim: B is a basis of V.

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

$\Rightarrow$ span B = V (by given inf.).

$$0 = 0 v_1 + 0 \cdot v_2 + \dots + 0 v_n$$

$\Rightarrow \{v_1, v_2, \dots v_n\}$ is LI.

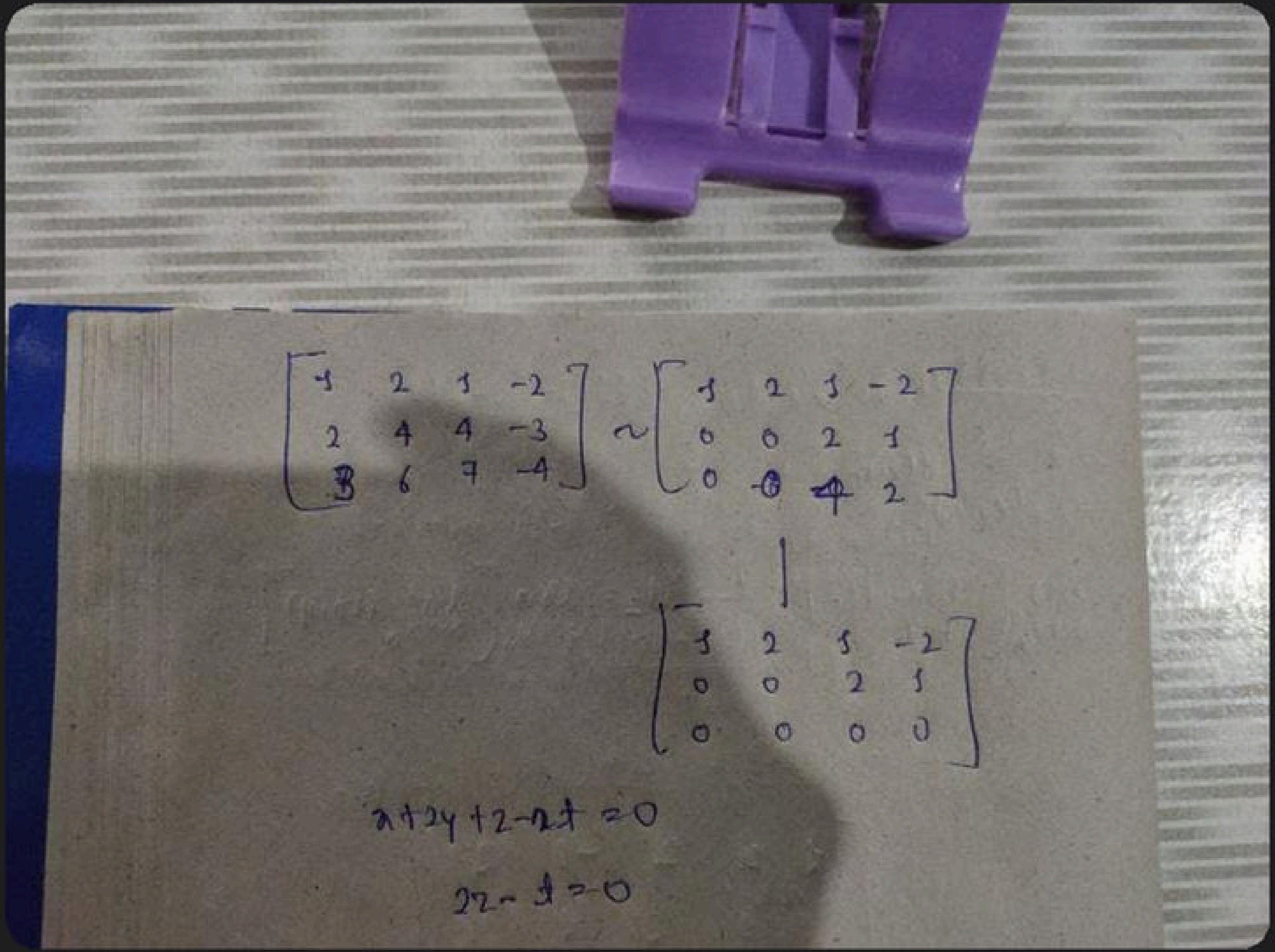
$\Rightarrow$ B is a basis of V.

▲ 1 • Asked by Reena

ma'am ye phli class h dimension ki mene abhi subscription liya h unacademy

1 • Asked by Sumit

Please help me with this doubt



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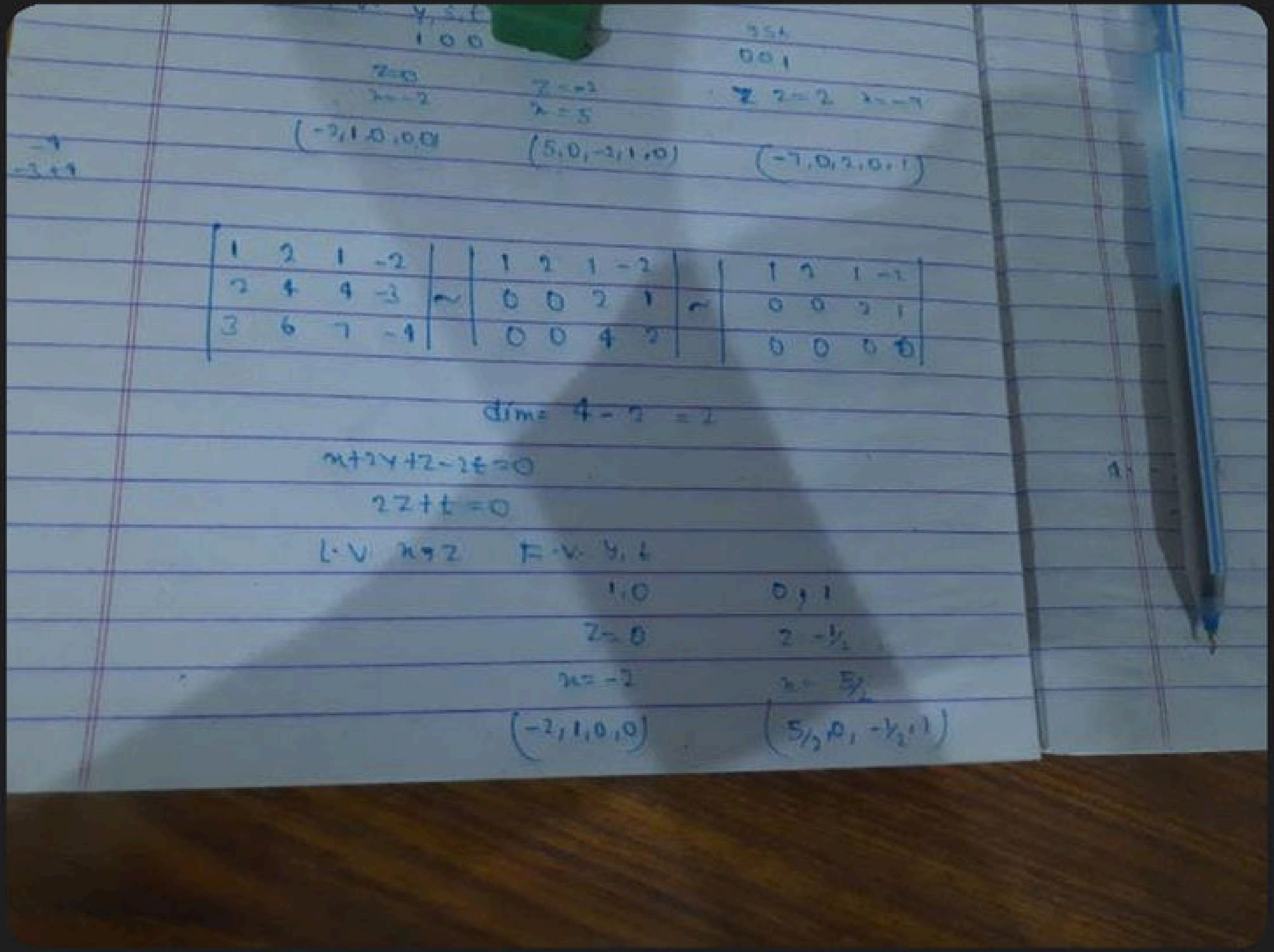
1 • Asked by Khushi

Please help me with this doubt



(19)

$$x + y + 2z = 0$$
$$2x + 3y + 3z = 0$$
$$x + 3y + 5z = 0$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\dim = n - r = 3 - 3$$
$$= \underline{\underline{0}}$$

$$B = \phi$$

$$x + y + 2z = 0$$

$$y - z = 0 \longrightarrow y = 0$$

$$5z = 0 \longrightarrow z = 0$$

$x = 0$

LV

LV

LV.

Soln.

$$W = \{ x \mid Ax = 0 \}$$

$$W = \{ (0, 0, 0) \} \rightarrow \text{zero sub-space}$$

$$\dim W = 0 \qquad B = \phi$$

$$x + 2y + 3z = \lambda x$$
$$-x + y + z = \lambda y$$
$$-2x + 3y + 5z = \lambda z$$

$W = \{ X \mid AX = 0 \}$

for what values of $\lambda$,
dim $W = 0$.

$\lambda = ?$

$$0 = \dim W = n - r$$
$$= 3 - r = 0$$

$\lambda = 3 \Rightarrow \det A \neq 0$

$$\begin{pmatrix} (1-\lambda)x + 2y + 3z = 0 \\ -x + (1-\lambda)y + z = 0 \\ -2x + 3y + (5-\lambda)z = 0 \end{pmatrix}$$

$\lambda = 0$

$\lambda = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ -2 & 3 & 5 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 7 & 11 \end{bmatrix} \end{array}\begin{array}{l} \\ \times 7 \\ \times 3 \end{array} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$33 - 28$

$$\begin{vmatrix} (1-\lambda) & 2 & 3 \\ -1 & 1-\lambda & 1 \\ -2 & 3 & 5-\lambda \end{vmatrix} \neq 0$$

$$(1-\lambda)\left(5-6\lambda+\lambda^2-3\right)-2\left(-5+\lambda+2\right)+$$

$$3\left(-3+2-2\lambda\right) \neq 0$$

$$(1-\lambda)\left(\lambda^2-6\lambda+2\right)-\underline{2(\lambda-3)}+3\left(-1-2\lambda\right)$$

$$(1-\lambda)\left(\lambda^2-6\lambda+2\right)-7\lambda+9=0 \neq 0$$

8:00 am