Ex:: $y(y\,dx - x\,dy) + x^2(2y\,dx + 2x\,dy) = 0$

which is of the form

$$x^{\alpha} y^{\beta}(m y\,dx + n x\,dy) + x^{\alpha_1} y^{\beta_1}(m_1 y\,dx + n_1 x\,dy) = 0$$

$\alpha = 0,\ \beta = 1,\ m = 1,\ n = -1$

$\alpha_1 = 2,\ \beta_1 = 0,\ m_1 = 2,\ n_1 = 2$

$$IF = x^{km-1-\alpha} y^{kn-1-\beta} = x^{k-1-0} y^{-k-1-1} = x^{k-1} y^{-k-2}$$

$$IF = x^{k_1 m_1 - 1 - \alpha_1} y^{k_1 n_1 - 1 - \beta_1} = x^{2k_1 - 1 - 2} y^{2k_1 - 1 - 0} = x^{2k_1 - 3} y^{2k_1 - 1}$$

$$\begin{aligned} k - 1 &= 2k_1 - 3 \\ -k - 2 &= 2k_1 - 1 \end{aligned} \Rightarrow \begin{aligned} k - 2k_1 &= -2 \\ -k - 2k_1 &= 1 \end{aligned}$$

$$-4k_1 = -1 \Rightarrow k_1 = \tfrac{1}{4}$$

$$\therefore IF = x^{\frac{1}{2} - 3} y^{\frac{1}{2} - 1} = \underline{x^{-5/2} y^{-1/2}}$$

Note: I — If the given diff^l eq^n contains $(x\,dy - y\,dx)$ as a term, then its multiplication by

(i) $\frac{1}{x^2}$ gives $\frac{x\,dy - y\,dx}{x^2} = d\left(y/x\right)$

(ii) $\frac{1}{y^2}$ gives $\frac{x\,dy - y\,dx}{y^2} = -\left(\frac{y\,dx - x\,dy}{y^2}\right) = -d\left(x/y\right)$

(iii) $\frac{1}{xy}$ gives $\frac{x\,dy - y\,dx}{xy} = \frac{dy}{y} - \frac{dx}{x} = d\left(\log y/x\right)$

(iv) $\frac{1}{x^2+y^2}$ gives $\frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(v) $\frac{1}{x\sqrt{x^2-y^2}}$ gives $\frac{x\,dy - y\,dx}{x\sqrt{x^2-y^2}} = d\left(\sin^{-1}\frac{y}{x}\right)$

Ex: $x\,dy - y\,dx = (x^2+y^2)\,dy$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2+y^2} = dy$$

$$\Rightarrow d\left(\tan^{-1}\frac{y}{x}\right) = dy$$

Integrate $\tan^{-1}\frac{y}{x} = y + C$

(II) If the diffal eqn contains $(x\,dy + y\,dx)$ as a term, then its multiplication by

(i) $\frac{1}{xy}$ gives $\frac{x\,dy + y\,dx}{xy} = \frac{dy}{y} + \frac{dx}{x} = d(\log(xy))$

(ii) $\frac{1}{(xy)^n}$ gives $\frac{x\,dy + y\,dx}{(xy)^n} = \frac{d(xy)}{(xy)^n} = d\left(\frac{-1}{(n-1)(xy)^{n-1}}\right)$ $n \neq 1$

Linear diffal eqⁿ:- (1) A diffal eqⁿ of the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q both are fnⁿ of only x or constant is called <u>linear diffal eqⁿ</u>.

In this case $IF = e^{\int P\,dx}$

$\therefore$ The required soln is

$$y(IF) = \int Q(IF)\,dx + C$$

(2) A diffal eqⁿ of the form

$$\frac{dx}{dy} + Px = Q$$

where P and Q both are fnⁿ of only y or constant is called linear diff^tal eq^n.

In this case $IF = e^{\int P\,dy}$

& the required sol^n is

$$x(IF) = \int Q(IF)\,dy + c$$

Bernoulli's eq^n :- A diff^tal eq^n is of the form

$$\frac{dy}{dx} + Py = Qy^n \quad (n \neq 0,\ n \neq 1) \quad \text{--- } (1)$$

where P and Q both are fn^n of only x or constant

is called Bernoulli's eq^n

If $n = 0$. Then eq^n (1) reduce to linear diff^tal eq^n

If $n=1$:- Then eqⁿ ① directly convert to separable variable.

Divided by y^n in eqⁿ ①

$$\frac{1}{y^n}\frac{dy}{dx} + P\cdot\frac{1}{y^{n-1}} = Q$$

Let $\boxed{\frac{1}{y^{n-1}} = v}$

$$\Rightarrow y^{1-n} = v$$

$$\Rightarrow (1-n)\, y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}$$

$$\therefore \frac{1}{(1-n)}\frac{dv}{dx} + Pv = Q$$

$$\Rightarrow \frac{dv}{dx} + (1-n) P v = Q (1-n)$$

which is called reducible linear diff[l] eq[n].

Ex: Solve $\frac{dy}{dx} + xy = xy^2$

$$\therefore \frac{1}{y^2} \frac{dy}{dx} + x \frac{1}{y} = x$$

Let $\frac{1}{y} = v \Rightarrow -\frac{1}{y^2} \frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} = -\frac{dv}{dx}$$

$$-\frac{dv}{dx} + x v = x$$

$$\frac{dv}{dx} - x v = -x$$

$$IF = e^{-\int x \, dx} = e^{-x^2/2}$$

$$\therefore \; v \cdot e^{-x^2/2} = \int -x e^{-x^2/2} \, dx + c$$

$$v e^{-x^2/2} = e^{-x^2/2} + C$$

$$\therefore \; \frac{1}{y} = 1 + c e^{x^2/2}$$

$$\Rightarrow \; \underline{y\left(1 + c e^{x^2/2}\right) = 1}$$

Seprable - Variable :

I A diffetal eqⁿ of the form

$$\frac{dy}{dx} = \frac{f_1(x)}{f_2(y)}$$

i.e. $f_2(y) \, dy = f_1(x) \, dx$

Integrate and find the soln.

Ex! : $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

Soln : : $\frac{dy}{dx} = e^x \cdot e^y + x^2 e^y$

$= e^y(e^x + x^2)$

$\Rightarrow e^{-y} dy = (e^x + x^2) dx$

Integrate

$-e^{-y} = e^x + \frac{x^3}{3} + c.$

II A diff^al eq^n is of the form

$\frac{dy}{dx} = f(ax+by+c)$ — (1)

In this case, let $\boxed{ax+by+c = v}$ — (2)

$\Rightarrow a + b\frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow b\frac{dy}{dx} = \frac{dv}{dx} - a$

$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$ — (3)

$\therefore$ from eq^n (1) (2) & (3)

$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = f(v)$

$\Rightarrow \frac{dv}{dx} = b f(v) + a$

$\Rightarrow \frac{dv}{b f(v) + a} = dx$

Integrate & find the soln

Ex: $\dfrac{dy}{dx} = (x+y+1)^2$ —— (1)

Let $x+y+1 = v$

$\Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$

$\therefore$ from eqⁿ (1)

$\dfrac{dv}{dx} - 1 = v^2$

$\therefore \dfrac{dv}{dx} = v^2 + 1$

$\therefore \dfrac{dv}{v^2+1} = dx$

Integrate

$\tan^{-1} v = x + c$

$\therefore v = \tan(x+c)$

$x+y+1 = \tan(x+c)$

Note :- A diff'tial eqⁿ is of the form

$\dfrac{dy}{dx} = f(ax+by)$

Let $ax+by = v$

Ex: $\dfrac{dy}{dx} = \sin(x+y) + \cos(x+y) \longleftarrow (1)$

Let $x + y = v$

$\Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$

$\therefore$ from eqn (1)

$\dfrac{dv}{dx} - 1 = \sin v + \cos v$

$\therefore \dfrac{dv}{dx} = \sin v + (1 + \cos v)$

$\therefore \dfrac{dv}{\sin v + (1 + \cos v)} = dx$

$\therefore \dfrac{dv}{2\sin\frac{v}{2}\cos\frac{v}{2} + 2\cos^2\frac{v}{2}} = dx$

$\dfrac{dv}{2\cos^2\frac{v}{2}\left(1 + \tan\frac{v}{2}\right)} = dx$

$\therefore \dfrac{\frac{1}{2}\sec^2\frac{v}{2}\,dv}{1 + \tan\frac{v}{2}} = dx$

integrating

$\log\left(1 + \tan\frac{v}{2}\right) = x + C$

$\therefore \log\left(1 + \tan\frac{x+y}{2}\right) = x + C$

Homogeneous eqⁿ - An eqⁿ is of the form

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n \quad \text{--- (1)}$$

is called homo. eqⁿ in x and y of degree n.

The above eqⁿ can be written as

$$f(x,y) = x^n \left(a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(y/x\right)^2 + \dots + a_n (y/x)^n \right)$$

Degree

$$\therefore f(x,y) = x^n F(y/x) \quad \text{--- (2)}$$

Euler's th. on homo fnⁿ: (ADC) (Partial differentiation)

If $f(x,y)$ is homo in x & y of degree n

Then $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$

Ex!: If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$

Then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \cdots$

Soln: $z = \tan u = \dfrac{x^3+y^3}{x+y}$

Here u is not homo. fn in x & y

Here $\tan u$ is homo fn in x & y of degree

$3-1=2$

∴ By Euler's th. on homo-fn

$$x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2 \tan u$$

$$\therefore \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u}$$

$$\therefore \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \sin u \cos u$$

$$\therefore \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

Trick :-

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \text{degree} \left(\frac{\text{fu}^n}{\text{Derivative of fu}^n} \right)$$

$$\therefore \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\tan u}{\sec^2 u} = \sin 2u$$

Ex: If $u = \log\left(\frac{x^5+y^5}{x^2+y^2}\right)$

Then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$

$$= (5-2)\cdot\frac{e^u}{e^u}$$

$$= 3.$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

Ex: $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

Then find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \text{- - -}$

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$$

Then $$\tan u = \left(\frac{x^3+y^3}{x+y}\right)$$

Then $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (3-1)\frac{\tan u}{\sec^2 u}$$

$$= \sin 2u = g(u)\ (\text{say})$$

$$\therefore\quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left(g'(u)-1\right)$$

$$= \sin 2u\left(2\cos 2u - 1\right)$$

Homogeneous diff'al eqⁿ :- A diff'al eqⁿ is of the form

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)} \quad \text{--- (1)}$$

where $f_1(x,y)$ and $f_2(x,y)$ both are homogeneous in x and y of degree n

$$\therefore \frac{dy}{dx} = \frac{x^n F_1(y/x)}{x^n F_2(y/x)}$$

$$\therefore \frac{dy}{dx} = \frac{F_1(y/x)}{F_2(y/x)}$$

$$\therefore \frac{dy}{dx} = F(y/x) \quad \text{--- (2)}$$

Let

$$\frac{y}{x} = v$$

$$\therefore y = vx$$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = F(v)$$

$$\therefore x\frac{dv}{dx} = F(v) - v$$

Ex: solve $\dfrac{dy}{dx} = \left(\dfrac{x^2+y^2}{xy}\right)$

Let $y = vx$

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore v + x\dfrac{dv}{dx} = \dfrac{x^2+v^2x^2}{x(vx)}$

$\therefore v + x\dfrac{dv}{dx} = \dfrac{1+v^2}{v}$

$\therefore x\dfrac{dv}{dx} = \dfrac{1+v^2}{v} - v$

$\therefore x\dfrac{dv}{dx} = \dfrac{1}{v}$

$\therefore v\,dv = \dfrac{dx}{x}$

Integrate

$\dfrac{v^2}{2} = \log x + \log c$

$\dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = \log(cx)$

$cx = e^{y^2/2x^2}$

$\therefore x e^{-y^2/2x^2} = c$

Non-homogeneous diff^tal eq^n:

A diff^tal eq^n is of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad \text{—— (1)}$$

is called non-homogeneous diff^tal eq^n.

Case I:- If $\frac{a}{a'} \neq \frac{b}{b'}$

In this case, let

$$x = X+h \quad \Rightarrow \quad dx = dX$$
$$y = Y+k \qquad\quad dy = dY$$

$\therefore$ Eq^n (1) becomes

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$$

$$\Rightarrow \frac{dy}{dx} = \frac{ax+by+(ah+bk+c)}{a'x+b'y+(a'h+b'k+c')}$$

Let $ah+bk+c=0$

$a'h+b'k+c'=0$

solve $h=?$, $k=?$

$\therefore$ The eqⁿ becomes

$$\frac{dy}{dx} = \frac{ax+by}{a'x+b'y}$$

Let $y = vx$

Oblique Trajectory:

Ex: Find the family of oblique trajectories whose tangent form the angle $\pi/4$ with the hyp' $xy = c$

Soln:

$$xy = c$$

$$\Rightarrow x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow p = \frac{-y}{x} \quad \left(p = \frac{dy}{dx}\right)$$

Replace p by $\dfrac{p+\tan\alpha}{1-p\tan\alpha}$

i.e. $\dfrac{p+\tan\pi/4}{1-p\tan\pi/4}$

i.e. $\dfrac{p+1}{1-p}$

$$\therefore \frac{p+1}{1-p} = -\frac{y}{x}$$

$$x(p+1) + y(1-p) = 0$$

$$x\left(\frac{dy}{dx}+1\right) + y\left(1-\frac{dy}{dx}\right) = 0$$

$$x(dy+dx) + y(dx-dy) = 0$$

$$\Rightarrow \quad x\,dy + x\,dx + y\,dx - y\,dy = 0$$

$$\Rightarrow \quad (x\,dy + y\,dx) + (x\,dx - y\,dy) = 0$$

$$\Rightarrow \quad d(xy) + x\,dx - y\,dy = 0$$

Integrating

$$xy + \frac{x^2}{2} - \frac{y^2}{2} = C.$$