Linear Algebra Assignment - Part II

Advanced Course on Mathematics for IIT JAM'22 - Part I

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RANKER'S BATCH

PRACTICE SET 1

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REAL ANALYSIS

Q.1 – Which of the following is/are True for a real sequence $\{X_n\}$ with the general term x_n?

(a) $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is a Cauchy Sequence

(b) $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is not a Cauchy Sequence

(c) $x_n = \int_1^n \frac{\cos(t)}{t^2} dt$ is a Cauchy Sequence

(d) $x_n = \int_1^n \frac{\cos(t)}{t^2} dt$ is not a Cauchy Sequence

Q.2-Which of the following is/are True for a real sequence $\{X_n\}$ with the general term x_n?

(a) $x_n = \frac{n^2}{\sqrt{n^6+1}} + \frac{n^2}{\sqrt{n^6+2}} + \cdots + \frac{n^2}{\sqrt{n^6+n}}$ is a Convergent Sequence

(b) $x_n = \frac{[\alpha]+[2\alpha]+\cdots+[n\alpha]}{n^2}$ is a Convergent sequence, [.] being the greatest integer function, α being an arbitrary real number

(c) $x_n = \frac{n^2}{\sqrt{n^6+1}} + \frac{n^2}{\sqrt{n^6+2}} + \cdots + \frac{n^2}{\sqrt{n^6+n}}$ is a Divergen t Sequence

(d) $x_n = \frac{[\alpha]+[2\alpha]+\cdots+[n\alpha]}{n^2}$ is a Divergent sequence, [.] being the greatest integer function, α being an arbitrary real number

Q.3- Consider a real sequence whose general term is given by $x_n = \frac{\alpha^n - \beta^n}{\alpha^n + \beta^n}$, where
where α and β are real numbers such that $|\alpha| \neq |\beta|$. Then which of the following is/are True ?

(a) Limit is -1 if $|\alpha| > |\beta|$

(b) Limit is 1 if $|\alpha| > |\beta|$

(c) Limit is 0 if $|\alpha| < |\beta|$

(d) Limit is 0 if $|\alpha| < |\beta|$

Q.4- Consider the following statements about x_n,

(I) If $\{x_n\}$ converges then, $y_n = \frac{x_1+x_2+\cdots+x_n}{n}$ also converges to the same limit

(II) If $\lim_{n\to\infty} x_{n+1} - x_n = l$, then $\lim_{n\to\infty} \frac{x_n}{n} = l$

Which of the following statements is False?

(a) I but not II
(b) II but not I
(c) Both I and II
(d) Neither I nor II

Q.5- Let $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ for a sequence $\{a_n\}$ and $b_1 = 1$, and $b_{n+1} = \sqrt{b_n^2 + \frac{1}{2^n}}$ for a sequence $\{b_n\}$. Which of the following is/are True ?

(a) $\{a_n\}$ and $\{b_n\}$ both converge

(b) $\{a_n\}$ converges but $\{b_n\}$ diverges

(c) $\{a_n\}$ diverges but $\{b_n\}$ converges

(d) $\{a_n\}$ and $\{b_n\}$ both diverge

Q.6- Let $\{x_n\}$ be a bounded sequence. Then, which of the following must be True?

(a) There must exist a subsequence which converges to $\liminf_{n\to\infty} x_n$

(b) There must exist a subsequence which converges to $\limsup_{n\to\infty} x_n$

(c) There need not exist a subsequence which converges to $\liminf_{n\to\infty} x_n$

(d) There need not exist a subsequence which converges to $\liminf_{n\to\infty} x_n$

Q.7 – $\{\cos n : n \in \mathbb{N}\}$ is dense in $[-1,1]$. (T/F)

Q.8 - Let $a_1, a_2, \dots, a_p$ be fixed positive numbers. Consider the sequences $s_n = \frac{a_1^n+a_2^n+\cdots+a_p^n}{p}$ and $x_n = \sqrt[n]{s_n}$, $n \in \mathbb{N}$. Then the sequence is monotonically increasing.

Differential Calculus

Q.1 - $\lim_{x\to 0}\frac{\sin(x)}{\sqrt{1-\cos(x)}}$

(a) is $\sqrt{2}$

(b) is $-\sqrt{2}$

(c) Does not exist

(d) None of these

Q.2- $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \\ \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$ is

(a) Continuous at each rational and discontinuous at each irrational in (0, 1)

(b) Discontinuous at each rational and continuous at each irrational in (0, 1)

(c) Continuous at each rational and continuous at each irrational in (0, 1)

(d) Discontinuous at each rational and discontinuous at each irrational in (0, 1)

Q.3- Which of the following is True for the function $f(x) = \begin{cases} x^2 e^{-x^2} & \text{if } |x| \le 1 \\ \frac{1}{e} & \text{if } |x| > 1 \end{cases}$?

(a) Continuous but not differentiable at x=1

(b) Not differentiable at x = 1

(c) Not continuous at x = 2

(d) None of these

Q.4- If f is differentiable at a, then $\lim_{n\to\infty}\frac{a^n f(x)-x^n f(a)}{x-a}$ has the value,

(a) $a^n f'(a) - f(a)na^{n-1}$

(b) $a^n f'(a) - f(a)na^n$

(c) $a^n f'(a) + f(a)na^{n-1}$

(d) $a^n f'(a) + f(a)na^n$

Q.5 - Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Consider the sequence $x_0 \in \mathbb{R}$ and $x_{n+1} = f(x_n)$.

Assume that $\lim_{n\to\infty} x_n = l$ and $f'(l)$ exists. If $|f'(l)| \le$ k, then k is ___.

Q.6 - Let $f:[a,b] \to \mathbb{R}$ be continuous and differentiable everywhere in (a,b) except maybe at $c \in (a,b)$. Assume that $\lim_{x \to c} f'(x) = l$. Then, which of these must be True ?

(a) f must be differentiable at c

(b) f need not be differentiable at c

(c) f '(c) =l

(d) None of these

Q.7- Let $f(x)$ be a continuous function on $[a,b]$, differentiable on (a,b), and $f'(x) \neq 0$ for any x in (a,b). Which of the following is/are True ?

(a) $f(x)$ is one − to − one

(b) $f'(x) > 0$ for every $x \in (a,b)$ or $f'(x) < 0$ for every $x \in (a,b)$

(c) f'(x) satisfies the intermediate value theorem on (a,b)

(d) None of these

Q.8- Let $f:\mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) > f(x)$ for all $x \in \mathbb{R}$, and $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then,

(a) $f(x) > 0$ for all $x > x_0$

(b) $f(x) < 0$ for all $x > x_0$

(c) $ae^x = 1 + x + \frac{x^2}{2}$, where a > 0 has exactly one root

(d) None of these

Q.9- Let $f:\mathbb{R} \to \mathbb{R}$. Assume that for any $x, t \in \mathbb{R}$ we have $|f(x) - f(t)| \leq |x - t|^{1+\alpha}$ where $\alpha > 0$.

Which of the following must be True for f(x)?

(a) f is continuous over $\mathbb{R}$

(b) f is differentiable over $\mathbb{R}$

(c) f is constant over $\mathbb{R}$

(d) None of these

Q.10 - Let $f:[0,\infty) \to \mathbb{R}$ differentiable everywhere. Assume that $\lim_{x \to \infty} f(x) + f'(x) = 0$. Then, $\lim_{x \to \infty} f(x)$= ___.

Q.11- Let $f:[0,1]\to\mathbb{R}$ be continuous and differentiable inside $(0,1)$ such that

(i) f(0)=0

(ii) there exists $M>0$ such that $|f'(x)|\le M|f(x)|$, for $x\in(0,1)$

then

(a) f is continuous over R
(b) f is differentiable over R
(c) f is identically 0 over R
(d) None of these

Q.12- $f(x)=\begin{cases} e^{-1/x^2} & \text{if } x\neq 0, \\ 0 & \text{if } x=0. \end{cases}$

For natural n, $f^{[n]}(0)$ = ____.

Q.13 -

Consider a function $f(x)$ whose second derivative $f''(x)$ exists and is continuous on (a,b) . Let $c\in(a,b)$. Then,

$$\lim_{h\to 0}\frac{f(c+h)-2f(c)+f(c-h)}{h^2}=f''(c)$$

[T/F]

Is the existence of the second derivative necessary to prove the existence of the above limit?

Q.14 -

Consider a function $f(x)$ whose second derivative $f''(x)$ exists and is continuous on $[0,1]$. Assume that $f(0) = f(1) = 0$ and suppose that there exists $A > 0$ such that $|f''(x)| \leq A$ for $x \in [0,1]$.

Which of the following must be True ?

(a) $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{A}{4}$

(b) $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{A}{2}$

(c) $|f'(x)| \leq \frac{A}{2}$

(d) $|f'(x)| \leq \frac{A}{2}$

Q.15 – Suppose a function $f: (-a, a) \setminus \{0\} \to (0, +\infty)$ satisfies $\lim_{x \to 0} \left(f(x) + \frac{1}{f(x)}\right) = 2$. Then,

$\lim_{x \to 0} f(x) =$ ___.

Q.16– Suppose a function $f: (-a, a) \setminus \{0\} \to (0, +\infty)$ satisfies $\lim_{x \to 0} \left(f(x) + \frac{1}{|f(x)|}\right) = 0$. Then,

$\lim_{x \to a} f(x) =$ ___.

Q.17- If f is a bounded function on $[0,1]$ satisfying $f(ax) = bf(x)$ for $0 \leq x \leq \frac{1}{a}$ and $a, b > 1$, then $\lim_{x \to 0^+} f(x) = f(0)$.

[T/F]

Q.18- Evaluate

(i) $\lim_{x \to 0} \left(x^2 \left(1 + 2 + 3 + \cdots + \left[\frac{1}{|x|}\right]\right)\right)$

(ii) $\lim_{x \to 0^+} \left(x \left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \cdots + \left[\frac{k}{x}\right]\right)\right), k \in \mathbb{N}$

Q.19 - If in a deleted neighborhood of zero the inequalities $f(x) \geq |x|^{\alpha}, \frac{1}{2} < \alpha < 1$, and $f(x)f(2x) \leq |x|$ hold, then $\lim_{x \to 0} f(x) =$ ___.

Q.20 - Given a real α, assume that $\lim_{x \to \infty} \frac{\mu(ax)}{x^{\alpha}} = g(a)$ for each positive a. Then,

$\to \infty$

there exists some c such that $g(a) = ca^{\alpha}$