Prob V: When X is of the form x^m

Then

$$PI = \frac{1}{F(D)} X$$

$$= \frac{1}{F(D)} x^m$$

$$= \frac{1}{[1 \pm F(D)]} x^m$$

$$= [1 \pm F(D)]^{-1} x^m$$

(Expand binomially & simplify)

Note:

(i) $(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$

(ii) $(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$

(iii) $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$

(iv) $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$

(v) $(1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \cdots$

(vi) $(1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots$

✓ (vii) $(1+D)^m = 1 + mD + \frac{m(m-1)}{2!}D^2 + \frac{m(m-1)(m-2)}{3!}D^3 + \cdots$

Ex: $(D^2 - 2D + 1)\,y = x^2$

Soln: The AE is $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, 1$

$\therefore CF = (c_1 + c_2 x)\,e^x$

$$PI = \frac{1}{D^2 - 2D + 1} x^2$$

$$= \frac{1}{1 - 2D + D^2} x^2$$

$$= \frac{1}{(1-D)^2} x^2$$

$$= (1-D)^{-2} x^2$$

$$= (1 + 2D + 3D^2 + \cdots) x^2$$

$$= x^2 + 2(2x) + 3(2)$$

$$= x^2 + 4x + 6$$

$$\therefore \quad y = (c_1 + c_2 x)e^x + \underline{x^2 + 4x + 6}$$

NAc:

$D^n(x^n) = n!$

$D^{n+1}(x^n) = 0$

Ex: $(D^2+2D+2)y = x^2 \quad \left(D = \frac{d}{dx}\right)$

Soln: The A E is

$$m^2 + 2m + 2 = 0.$$

$$\Rightarrow \quad m^2 + 2m + 1 = -1$$

$$\Rightarrow \quad (m+1)^2 = i^2$$

$$\Rightarrow \quad m+1 = \pm i$$

$$\Rightarrow \quad m = -1 \pm i$$

$$\therefore \quad CF = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$$PI = \frac{1}{(D^2+2D+2)} x^2$$

$$= \frac{1}{2\left[1+\left(\frac{D^2+2D}{2}\right)\right]} x^2$$

$$= \frac{1}{2}\left(1+\left(\frac{D^2+2D}{2}\right)\right)^{-1} x^2$$

$$= \frac{1}{2}\left\{1-\left(\frac{D^2+2D}{2}\right) + \left(\frac{D^2+2D}{2}\right)^2 - \cdots\right\} x^2$$

$$= \frac{1}{2}\left\{1 - \frac{D^2}{2} - D + D^2 \cdots\right\} x^2$$

$$= \frac{1}{2}\left\{1 - D + \frac{D^2}{2} \cdots\right\} x^2$$

$$= \frac{1}{2}(x^2 - 2x + 1)$$

Prop VI :- when X is of the form $e^{ax} \cdot V$ where V is a fun^n of only x

Then

$$PI = \frac{1}{F(D)} X$$

$$= \frac{1}{F(D)} e^{ax} \cdot V$$

$$= e^{ax} \frac{1}{F(D+a)} \cdot V$$

$e^{ax} \cdot V$:

- $e^{ax} \cdot e^{bx} = e^{(a+b)x}$ ✗
- $e^{ax}(\sin ax \text{ or } \cos ax)$ ✓
- $e^{ax} \cdot x^m$ ✓

Ex: Solve $(D^2-2)y = e^x \sin x \quad (D = d/dx)$

$$PI = \frac{1}{D^2-2} e^x \sin x$$

$$= e^x \frac{1}{(D+1)^2-2} \sin x$$

$$= e^x \frac{1}{D^2+2D+1-2} \sin x$$

$$= e^x \frac{1}{D^2+2D-1} \sin x$$

$$= e^x \frac{1}{-1+2D-1} \sin x$$

$$= e^x \frac{1}{2D-2} \sin x$$

$$= \frac{e^x}{2} \frac{1}{(D-1)} \sin x$$

$$= \frac{e^x}{2} \frac{(D+1)}{(D^2-1)} \sin x$$

$$= \frac{e^x}{2} \frac{(D+1) \sin x}{(-1-1)}$$

$$= -\frac{1}{4} e^x (\cos x + \sin x)$$

Ex: $(D^2+D+1)y = e^x x^2$

Then $PI = \dfrac{1}{(D^2+D+1)} e^x x^2$

$$= e^x \frac{1}{(D+1)^2+(D+1)+1} x^2$$

$$= e^x \frac{1}{D^2+2D+1+D+2} x^2$$

$$= e^x \frac{1}{D^2+3D+3} x^2$$

$$= e^x \frac{1}{3} \frac{1}{\left[1+\left(\frac{D^2+3D}{3}\right)\right]} x^2$$

$$= \frac{e^x}{3}\left(1+\left(\frac{D^2+3D}{3}\right)\right)^{-1} x^2$$

$$= \frac{e^x}{3}\left(1-\left(\frac{D^2+3D}{3}\right) + \left(\frac{D^2+3D}{3}\right)^2 - \cdots\right) x^2$$

$$= \frac{e^x}{3}\left\{1-\frac{D^2}{3} - D + D^2 - \cdots\right\} x^2$$

$$= \frac{e^x}{3}\left\{1 - D + \frac{2D^2}{3} - \cdots\right\} x^2$$

$$= \frac{e^x}{3}\left(x^2 - 2x + \frac{4}{3}\right)$$

Prop VII :- when X is of the form $x \cdot V$ where V is a fn of x.

$$x V \begin{cases} x\,x^m = x^{m+1} & \times \\ x(\sin ax \text{ or } \cos ax) & \checkmark \\ x\,e^{ax} & \times \end{cases}$$

Then

$$PI = \frac{1}{F(D)} \cdot X$$

$$= \frac{1}{F(D)}(x \cdot V)$$

$$= x \cdot \frac{1}{F(D)} V - \frac{F'(D)}{(F(D))^2} \cdot V$$

Ex: Solve $(D^2 + 2D + 1)\,y = x \sin x$

Then $PI = \dfrac{1}{(D^2+2D+1)}\, x \sin x$

$$= x \cdot \frac{1}{D^2+2D+1} \sin x - \frac{(2D+2)}{(D^2+2D+1)^2} \sin x$$

$$= x \cdot \frac{1}{-1+2D+1} \sin x - \frac{(2D+2)}{(-1+2D+1)^2} \sin x$$

$$= x \frac{1}{2D} \sin x - \frac{(2D+2)}{4D^2} \sin x$$

$$= \frac{x}{2} \int \sin x \, dx - \frac{1}{2} \cdot \frac{(D+1) \sin x}{-1}$$

$$= \frac{x}{2} (-\cos x) + \frac{1}{2} \left(\cos x + \sin x \right)$$

$$= -\frac{1}{2} x \cos x + \frac{1}{2} (\cos x + \sin x)$$

Prop VIII: - $\frac{1}{(D-a)}X = e^{ax}\int X e^{-ax}\,dx$.

Expn: Let $\frac{1}{(D-a)}X = u$

$\Rightarrow (D-a)u = X$

$\Rightarrow \frac{du}{dx} - au = X$

$IF = e^{-\int a\,dx} = e^{-ax}$

$\therefore u e^{-ax} = \int X e^{-ax}\,dx$

$u = e^{ax}\int X e^{-ax}\,dx \Rightarrow \frac{1}{(D-a)}X = e^{ax}\int X e^{-ax}\,dx$

Ex: $(D^2 - 3D + 2)y = e^{3x}$

Method I:

$$PI = \frac{1}{(D^2-3D+2)} e^{3x}$$

$$= \frac{1}{(D-1)(D-2)} e^{3x}$$

$$= \frac{1}{(3-1)(3-2)} e^{3x}$$

$$= \frac{1}{2} e^{3x}$$

Method II:

$$PI = \frac{1}{(D-1)(D-2)} e^{3x}$$

$$= \frac{1}{(D-1)} \left\{ \frac{1}{(D-2)} e^{3x} \right\}$$

$$= \frac{1}{(D-1)} \left\{ e^{2x} \int e^{3x} \cdot e^{-2x} dx \right\}$$

$$= \frac{1}{(D-1)} \left\{ e^{2x} \int e^{x} dx \right\}$$

$$= \frac{1}{(D-1)} \left\{ e^{2x} e^{x} \right\}$$

$$= \frac{1}{(D-1)} e^{3x}.$$

$$= e^{x}\int e^{3x}e^{-x}\,dx$$

$$= e^{x}\int e^{2x}\,dx$$

$$= e^{x}\cdot\frac{e^{2x}}{2}$$

$$= \frac{1}{2}e^{3x}.$$

OR

$$PI = \frac{1}{(D-1)(D-2)}e^{3x}$$

$$= \frac{(D-1)-(D-2)}{(D-1)(D-2)}e^{3x}$$

$$= \left(\frac{1}{(D-2)} - \frac{1}{D-1}\right)e^{3x}$$

$$= \frac{1}{(D-2)}e^{3x} - \frac{1}{(D-1)}e^{3x}$$

$$= e^{2x}\int e^{3x}\cdot e^{-2x}\,dx$$

$$- e^{x}\int e^{3x}\cdot e^{-x}\,dx$$

$$= e^{2x}e^{x} - e^{x}\cdot\frac{e^{2x}}{2}$$

$$= e^{3x}\left(1-\frac{1}{2}\right)$$

$$= \frac{1}{2}e^{3x}$$

Variation of Parameter:

Ex: $(D^2 - 3D + 2)y = e^{3x}$ $\left(D = \frac{d}{dx}\right)$

The AE is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-1)(m-2) = 0$$

$$\Rightarrow m = 1, 2$$

$$\therefore CF = c_1 e^{x} + c_2 e^{2x}$$

Let $y_1 = e^{x}$, $y_2 = e^{2x}$

be two L.I. soln.

Let $PI = u_1 y_1 + u_2 y_2$

W → Wronskian

where

$$u_1 = -\int \frac{y_2 R}{W}\,dx$$

$$u_2 = \int \frac{y_1 R}{W}\,dx$$

where $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

$$\therefore W = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

$$\therefore \; u_1 = -\int \frac{e^{2x} \cdot e^{3x}}{e^{3x}} \, dx = -\int e^{2x} \, dx$$

$$= -\frac{e^{2x}}{2}$$

$$\therefore \; u_1 = -\frac{e^{2x}}{2}$$

$$u_2 = \int \frac{y_1 R}{W} \, dx$$

$$= \int \frac{e^{x} \cdot e^{3x}}{e^{3x}} \, dx = \int e^{x} \, dx = e^{x}$$

$$\therefore \; u_2 = e^{x}$$

$$\therefore \; PI = -\frac{e^{2x}}{2} e^{x} + e^{x} \cdot e^{2x} = \left(-\frac{1}{2} + 1\right) e^{3x} = \frac{1}{2} e^{3x}$$

Ex: $(D^2 - 2D + 1)y = e^x$

Soln: The AE is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow (m-1)^2 = 0$$

$$\Rightarrow m = 1, 1$$

$$\therefore CF = (c_1 + c_2 x) e^x$$

$$\therefore CF = c_1 e^x + c_2 x e^x \longrightarrow (1)$$

& $PI = \dfrac{1}{D^2 - 2D + 1} e^x$

$$= x \frac{1}{2D - 2} e^x$$

$$= x^2 \cdot \tfrac{1}{2} e^x = \frac{1}{2} x^2 e^x .$$

By Variation of Parameter

Let $y_1 = e^x$, $y_2 = x e^x$

Let $PI = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix} e^x & x e^x \\ e^x & (x+1) e^x \end{vmatrix}$$

$$= (x+1) e^{2x} - x e^{2x}$$

$$= e^{2x}$$

$$\therefore u_1 = - \int \frac{x e^x \cdot e^x}{e^{2x}} dx$$

$$u_1 = - \frac{x^2}{2}$$

$$u_2 = \int \frac{y_1 R}{W} dx$$

$$u_2 = \int \frac{e^x e^x}{e^{2x}} dx$$

$$u_2 = x$$

$$\therefore PI = -\frac{x^2}{2} e^x + x \; x e^x$$

$$= -\frac{x^2}{2} e^x + x^2 e^x$$

$$= \frac{x^2}{2} e^x$$

$$y_1 = x e^x \; , \; y_2 = e^x$$

$$W = \begin{vmatrix} x e^x & e^x \\ (x+1) e^x & e^x \end{vmatrix}$$

$$= x e^{2x} - (x+1) e^{2x}$$

$$= - e^{2x}$$

$$\therefore u_1 = - \int \frac{e^x \cdot e^x}{-e^{2x}} dx = x$$

$$u_2 = \int \frac{x e^x \cdot e^x}{-e^{2x}} dx = -\frac{x^2}{2}$$

$$\therefore PI = x^2 e^x - \frac{x^2}{2} e^x$$

$$PI \cdot = \frac{x^2}{2} e^x$$

Ex: $(D^2 - 3D + 2)y = e^x$

Then $CF = C_1 e^x + C_2 e^{2x}$

$$PI = \frac{1}{(D^2 - 3D + 2)} e^x$$

$$PI = x \cdot \frac{1}{(2D - 3)} e^x$$

$$PI = -x e^x$$

OR

Let $y_1 = e^x$, $y_2 = e^{2x}$

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0$$

$$\therefore u_1 = -\int \frac{e^{2x} \cdot e^x}{e^{3x}} dx$$

$$= -x$$

$$u_2 = \int \frac{e^x \cdot e^x}{e^{3x}} dx$$

$$= \int e^{-x} dx$$

$$= -e^{-x}$$

$$\therefore PI = u_1 y_1 + u_2 y_2$$

$$= -x e^x + (-e^{-x}) e^{2x}$$

$$= -x e^x - e^x$$

$$= -(x+1) e^x$$

RHS $\longrightarrow \sec ax, \operatorname{cosec} ax, \cot ax, \tan ax, \log x, e^{e^x}$

$\frac{1}{x+1}$ . . . . . .

Ex: $(D^2+a^2)y = \sec ax$.