Change of Basis - I

Comprehensive Course on Linear Algebra

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CHANGE OF BASIS

$T: V \to V$. Let B_1 and B_2 be two bases of V. Then

$B_1 = \{ v_1, v_2, \dots, v_n \}$

$B_2 = \{ u_1, u_2, \dots, u_n \}$

$$v_1 = a_{11} u_1 + a_{12} u_2 + \dots + a_{1n} u_n$$
$$v_2 = a_{21} u_1 + a_{22} u_2 + \dots + a_{2n} u_n$$
$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$$
$$v_n = a_{n1} u_1 + a_{n2} u_2 + \dots + a_{nn} u_n$$

$$[T]_{B_2}^{B_1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}^T$$

from B_2 to B_1

$$u_1 = b_{11} v_1 + b_{12} v_2 + \cdots + b_{1n} v_n$$

$$u_2 = b_{21} v_1 + b_{22} v_2 + \cdots + b_{2n} v_n$$

$$\vdots$$

$$u_n = b_{n1} v_1 + b_{n2} v_2 + \cdots + b_{nn} v_n$$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

from B_1 to B_2

(1) find a change of basis matrix from B_1 to B_2 where $B_1 = \{(1,1)\ (0,2)\}$, $B_2 = \{(1,2),(2,0)\}$

(i) $[T]_{B_1}^{B_2} = \begin{bmatrix} 1 & 2 \\ 1/2 & -1 \end{bmatrix}$ (ii) $[T]_{B_2}^{B_1} = \begin{bmatrix} 1/2 & 1 \\ 1/4 & -1/2 \end{bmatrix}$

$$[T]_{B_2}^{B_1}$$

$$(x,y) = c_1(1,2) + c_2(2,0)$$

$$c_1 = y/2$$

$$c_1 + 2c_2 = x$$

$$c_2 = \frac{2x - y}{4}$$

$$(x,y) = \frac{y}{2}(1,2) + \frac{2x-y}{4}(2,0)$$

$$(1,1) = \frac{1}{2}(1,2) + 1/4\,(2,0)$$

$$(0,2) = 1\cdot(1,2) + (-1/2)(2,0)$$

$$[T]_{B_2}^{B_1} = \begin{bmatrix} 1/2 & 1 \\ 1/4 & -1/2 \end{bmatrix}_{2\times 2}$$

$B_1 = \{(1,1), (0,2)\}$ $\qquad$ $B_2 = \{\underline{(1,2)}, \underline{(2,0)}\}$

$[T]_{B_1}^{B_2}$

$(x,y) = c_1(1,1) + c_2(0,2)$

$c_1 = x$

$c_1 + 2c_2 = y$

$c_2 = \frac{y-x}{2}$

$(x,y) = x(1,1) + \frac{y-x}{2}(0,2)$

$(1,2) = 1\cdot(1,1) + \frac{1}{2}(0,2)$

$(2,0) = 2(1,1) + (-1)(0,2)$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 1 & 2 \\ 1/2 & -1 \end{bmatrix}_{2\times 2}$$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 1 & 2 \\ 1/2 & -1 \end{bmatrix}_{2\times 2} = P$$

$$P^{-1} = \frac{1}{-1-1} \begin{bmatrix} -1 & -2 \\ -1/2 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 & -2 \\ -1/2 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1/2 & 1 \\ 1/4 & -1/2 \end{bmatrix} = [T]_{B_2}^{B_1}$$

Note : If $P = [T]_{B_1}^{B_2}$ and $Q = [T]_{B_2}^{B_1}$

then

(a) P and Q are invertible matrices

(b) P and Q are inverse of each other

$$P^{-1} = Q , \quad Q^{-1} = P$$

(eg) find the change of basis from B_1 to B_2 st

$B_1 = \{(1,0)\ (0,1)\}$ $B_2 = \{(1,2), (2,0)\}$

\# $[T]_{B_1}^{B_2}$

$(1,2) = 1\cdot(1,0) + 2\cdot(0,1)$

$(2,0) = 2(1,0) + 0\cdot(0,1)$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$[T]_{B_1}^{B_2} = [u_1 \quad u_2 \quad u_3 \; . \; . \; . \; , u_n]$$

B_1 — SB

where B_1 is standard basis and B_2 be any other basis

$$B_2 = \{u_1, u_2, \ldots, u_n\}$$

(eg) $B_1 = \{(-1,0), (0,2)\}$ $B_2 = \{(1,4)(2,7)\}$

$$\Rightarrow [T]_{B_1}^{B_2} = \begin{bmatrix} -1 & -2 \\ 2 & 7/2 \end{bmatrix}$$

$$\Rightarrow [T]_{B_2}^{B_1} = \begin{bmatrix} 7/4 & 1 \\ -1 & -1/2 \end{bmatrix}$$

$$(x,y) = c_1(-1,1,0) + c_2(0,1,2)$$

$$(x,y) = -x(-1,1,0) + \frac{y}{2}(0,1,2)$$

$$(1,4) = -1(-1,1,0) + 2(0,1,2)$$

$$(2,7) = -2(-1,1,0) + 7/2\,(0,1,2)$$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} -1 & -2 \\ 2 & 7/2 \end{bmatrix}$$

$$[T]_{B_2}^{B_1} = \frac{1}{-\frac{7}{2}+4} \begin{bmatrix} 7/2 & 2 \\ -2 & -1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 7/2 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ -4 & -2 \end{bmatrix}$$

(eg) $V = \mathbb{R}^3$ $B_1 = \{(1,1,0), (0,1,0), (0,0,1)\}$

$B_2 = \{\underline{(-1,2,0)}, (0,3,0), (0,0,1)\}$

$[T]_{B_1}^{B_2} =$

$(x,y,z) = c_1(1,1,0) + c_2(0,1,0) + c_3(0,0,1)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 1 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right]$$

$c_3 = z$ $\quad c_1 = x$

$c_2 = y - x$

$$(x, y, z) = x(1, 1, 0) + (y - x)(0, 1, 0) + z(0, 0, 1)$$

$$(-1, 2, 0) = -1 \cdot (1, 1, 0) + 3(0, 1, 0) + 0 \cdot (0, 0, 1)$$

$$(0, 3, 0) = 0 \cdot (1, 1, 0) + 3 \cdot (0, 1, 0) + 0 \cdot (0, 0, 1)$$

$$(0, 0, 1) = 0 \cdot (1, 1, 0) + 0 \cdot (0, 1, 0) + 1 \cdot (0, 0, 1)$$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]_{B_2}^{B_1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(eg) $V = \mathbb{R}^2$

$B_1 = \{ (2,3), (5,6) \}$

$B_2 = \{ (0,1) (-1,2) \}$

$P = [T]_{B_1}^{B_2}$

1st oct

8 am $\xrightarrow{\text{live}}$ Function of one variable.