# Differential eqⁿ of Ist order & Ist Degree

**Note:-** Every diff^tal eq^n of Ist order and Ist degree can be solved by either exact or integrating factor.

**Exact Diff^tal eq^n:-** A diff^tal eq^n of the form

$$M\,dx + N\,dy = 0$$

'is s.t.b. exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

**Expn:-** $f(x, y) = c$

$\Rightarrow df = 0$

$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

Comparing with $M\,dx + N\,dy = 0$

$\therefore \; M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}$

$\therefore \; \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$

$\therefore$ f is IInd order cts partial derivative

$$\therefore \; \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\therefore \; \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

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The required soln is

$$\underset{\text{regarding 'y' as constant}}{\int M\,dx} + \underset{\text{only those terms which are indep of 'x'}}{\int N\,dy} = C$$

Ex: Solve $(x^2+y^2)\,dx + 2xy\,dy = 0$ —— ①

Sol<sup></sup>:- Comparing with $M\,dx + N\,dy = 0$

$$M = x^2+y^2, \quad N = 2xy$$

$$\therefore \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Eqn ① is exact.

$\therefore$ The required soln is

$$\int (x^2+y^2)\,dx + 0 = \text{Constant}$$

$$\frac{x^3}{3} + y^2x = C$$

OR $\int (x^2 + y^2)\, dx + \int 2xy\, dy = c$

$\Rightarrow \frac{x^3}{3} + y^2 x + 2x \frac{y^2}{2} = c$

$\Rightarrow \frac{x^3}{3} + y^2 x + xy^2 = c$

$\Rightarrow \frac{x^3}{3} + y^2 x = \text{Constant}$

Ex: Solve $(x+2y+3)\,dx + (2x+y+4)\,dy = 0 \quad \text{--- (1)}$

Soln: Comparing with $M\,dx + N\,dy = 0$

$M = x+2y+3, \quad N = 2x+y+4$

$\frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 2$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\therefore$ Eqⁿ (1) is exact.

$\therefore$ The required soln is

$$\int (x+2y+3)\,dx + \int (y+4)\,dy = \text{Constant}$$

$$\Rightarrow \frac{x^2}{2} + 2yx + 3x + \frac{y^2}{2} + 4y = \text{Constant}$$

$$\Rightarrow x^2 + 4xy + y^2 + 6x + 8y = C \quad \text{———} \quad (2)$$

$h = 2,\ a = 1,\ b = 1 \quad \therefore h^2 - ab = (2)^2 - 1 \cdot 1 = 4 - 1 = 3 > 0$

$\therefore$ The eqⁿ (2) represents hyperbola

**Note**: The general eqⁿ of 2nd degree eqⁿ is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0$$

represents:

$\Delta = 0$

(Pair of st. lines)

where

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(i) coeff of $x^2$ = coeff of $y^2$

(ii) & coeff of $xy = 0$

(Circle)

(i.e. $a = b$, $h = 0$)

$\Delta \neq 0$

(Conic)

$h^2 - ab > 0$

Hyperbola

$h^2 - ab = 0$

Parabola

$h^2 - ab < 0$

Ellipse

**Integrating factor !** Sometime the given DE is not an exact Then we multiplying some fun of $x$ & $y$ to make it exact is called integrating factor (I F).

**Rule I :-** If the given DE $M\,dx + N\,dy = 0$ is homogeneous

Then $IF = \dfrac{1}{Mx+Ny}$ provided $Mx + Ny \neq 0$

Ex: Solve $(x^2+y^2)\,dx - xy\,dy = 0$ ——— (1)

Soln :- $M = x^2+y^2$, $N = -xy$

$\dfrac{\partial M}{\partial y} = 2y$, $\dfrac{\partial N}{\partial x} = -y$

$\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ $\therefore$ eqn (1) is not an exact.

$\because$ eqⁿ ① is homo.

$\therefore IF = \dfrac{1}{Mx+Ny}$

$= \dfrac{1}{(x^2+\cancel{y^2})x + (-\cancel{xy})y}$

$IF = \dfrac{1}{x^3}$

multiplying in eqⁿ ① by $\dfrac{1}{x^3}$

$$\frac{1}{x^3}(x^2+y^2)\,dx - \frac{y}{x^2}\,dy = 0$$

$\therefore$ The required soln is

$$\int\left(\frac{1}{x} + \frac{y^2}{x^3}\right)dx + 0 = \text{Constant}$$

$$\log x - \frac{y^2}{2x^2} = c$$

**Prop II** :- If the diff^tial eq^n $M\,dx + N\,dy$ is of the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$$

Then $$I.F = \frac{1}{Mx - Ny}$$ provided $Mx - Ny \neq 0$

Ex: Solve. $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$ — (1)

OR

$(x^2y^3 + xy^2 + y)\,dx + (x^3y^2 - x^2y + x)\,dy = 0$

Sol^n :

$M = (x^2y^2 + xy + 1)y$

$N = (x^2y^2 - xy + 1)x$

clearly $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (verify)

$\therefore$ eq^n (1) is not an exact.

which is of the form

$$f_1(xy)\, y\,dx + f_2(xy)\, x\,dy.$$

$$\therefore IF = \frac{1}{Mx - Ny}$$

$$= \frac{1}{xy(2xy)}$$

$$= \frac{1}{2x^2y^2}$$

$$If = \frac{1}{x^2y^2}$$

Multiplying in eq$^n$ ① by $\frac{1}{x^2y^2}$

$$\left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx$$

$$+ \left(x - \frac{1}{y} + \frac{1}{xy^2}\right)dy = 0$$

$\therefore$ The required soln is

$$\int \left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx$$

$$+ \int -\frac{1}{y}\,dy = Const$$

$$yx + \log x - \frac{1}{xy} - \log y = C$$

**Rule III :-** $M\,dx + N\,dy = 0$

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a fn of $x$ say $f(x)$

Then $IF = e^{\int f(x)\,dx}$

**Rule IV :-** $M\,dx + N\,dy = 0$

If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a fn of $y$ say $f(y)$

Then $IF = e^{-\int f(y)\,dy}$

OR

If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a fn of $y$ say $f(y)$, Then

$IF = e^{\int f(y)\,dy}$

Ex: $(x^2+y^2+x)\,dx + xy\,dy = 0$ —— ①

Soln :- $M = x^2+y^2+x$, $N = xy$

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Eqn ① is not an exact.

Now $$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(y) = \frac{1}{x}$$

$$If = e^{\int 1/x\,dx} = e^{\log x} = x$$

$\therefore$ multiplying in eqn ① by $x$

$\therefore (x^3+y^2x+x^2)\,dx + x^2y\,dy = 0$

$\therefore$ The required soln is

$$\int (x^3+y^2x+x^2)\,dx = \text{constant}$$

$$\frac{x^4}{4} + \frac{y^2x^2}{2} + \frac{x^3}{3} = c$$

OR $(x^2+y^2+x)\,dx + xy\,dy = 0$

$$\Rightarrow xy\frac{dy}{dx} + y^2 = -(x^2+x)$$

$$\Rightarrow y\frac{dy}{dx} + \frac{1}{x}y^2 = -(x+1)$$

$$\Rightarrow 2y\frac{dy}{dx} + \frac{2}{x}y^2 = -2(x+1)$$

Let $y^2 = v$

$$\Rightarrow 2y\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} + \frac{2}{x}v = -(2x+2)$$

$$IF = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

The required soln is

$$v\,x^2 = -\int (2x+2)x^2\,dx + c$$

$$y^2x^2 = -2\frac{x^4}{4} - \frac{2x^3}{3} + c$$

$$\text{i.e. } \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2y^2}{2} = c'$$

OR. Let $IF = x^h y^k$

$\therefore\quad x^h y^k (x^2+y^2+x)\,dx + x^h y^k xy\,dy = 0 \quad \text{—— (1)}$

$M = x^{h+2} y^k + x^h y^{k+2} + x^{h+1} y^k$

$N = x^{h+1} y^{k+1}$

$\therefore$ eqⁿ (1) is exact

$\therefore\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$k x^{h+2} y^{k-1} + (k+2)\,\underline{x^h y^{k+1}} + k x^{h+1} y^{k-1}$
$= (h+1)\,\underline{x^h y^{k+1}}$

$\therefore\ k+2 = h+1$

$k = 0$

$\therefore\ h = 1$

$\therefore\ IF = x^1 y^0$

$\boxed{IF = x}$

Prop V :- If the diff$^l$ eq$^n$ $M\,dx + N\,dy = 0$ is of the form

$$x^{\alpha} y^{\beta} (my\,dx + nx\,dy) + x^{\alpha_1} y^{\beta_1} (m_1 y\,dx + n_1 x\,dy) = 0 \quad \text{--- (1)}$$

$$\text{If} = x^{km-1-\alpha}\, y^{kn-1-\beta}$$

$$\text{If} = x^{k_1 m_1 - 1 - \alpha_1}\, y^{k_1 n_1 - 1 - \beta_1}$$

$$\Rightarrow \quad km - 1 - \alpha = k_1 m_1 - 1 - \alpha_1 \quad \text{--- (2)}$$

$$kn - 1 - \beta = k_1 n_1 - 1 - \beta_1 \quad \text{--- (3)}$$

Solving eq$^n$ (2) & (3)

$$k = ?\,,\quad k_1 = ?$$

## Prop VI. (Method by Inspection)

Ex: $x\,dy - y\,dx = x^2 y\,dy$

OR $(x - x^2 y)\,dy - y\,dx = 0$

Soln: $x\,dy - y\,dx = x^2 y\,dy$

$\Rightarrow \dfrac{x\,dy - y\,dx}{x^2} = y\,dy$

$\Rightarrow d\left(y/x\right) = y\,dy$

Integrate $\dfrac{y}{x} = \dfrac{y^2}{2} + C$

$\Rightarrow \dfrac{y}{x} - \dfrac{y^2}{2} = C$

Note:

(i) $y\,dx + x\,dy = d(xy)$

(ii) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(x/y\right)$

(iii) $\dfrac{y\,dx - x\,dy}{x^2 + y^2}$

$= d\left(\tan^{-1}\left(y/x\right)\right)$

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