Homogeneous L.D.E. with constant coefficients:-

A diff^{tal} eq^n of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = X$$

ie $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} + \cdots + a_n) y = X$

is called homo linear diff^{tal} eq^n with variable coefficient (which is also called Cauchy-Euler's eq^n) where $a_0, a_1, a_2 - - a_n$ are all constants and X is a fun^n of only x or constant. $\left(D = \frac{d}{dx}\right)$ — (1)

Let $x = e^z \Rightarrow z = \log x$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} \cdot \frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\Rightarrow x\frac{d}{dx} = \frac{d}{dz}$$

$$\left(D = \frac{d}{dx}\right) \Rightarrow \underline{xD = D} \quad \left(D = \frac{d}{dz}\right)$$

How

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow x^2\frac{d^2}{dx^2} = \frac{d^2}{dz^2} - \frac{d}{dz}$$

$$\Rightarrow x^2D^2 = D^2 - D$$

y

$\uparrow$

$\Rightarrow \quad x^2 D^2 = D(D-1)$

Similarly $\quad x^3 D^3 = D(D-1)(D-2)$

$\therefore$ In general

$$x^n D^n = D(D-1)(D-2)\cdots(D-\overline{n-1})$$

Ex: Solve $(x^2D^2 + xD + 1)y = x \quad \left(D = \frac{d}{dx}\right)$

Soln:- Let $x = e^z \Rightarrow z = \log x$

$\therefore$ The eqn becomes

$$\left(D(D-1) + D + 1\right) y = e^z \quad \left(D = \frac{d}{dz}\right)$$

i.e. $(D^2 + 1)\, y = e^z$

The AE is

$$m^2+1=0$$

$$\Rightarrow \quad m = \pm i = 0 \pm i$$

$$\therefore \quad CF = c_1 \cos Z + c_2 \sin Z$$

$$\& \quad PI = \frac{1}{D^2+1} e^Z$$

$$= \frac{1}{1+1} e^Z = \frac{1}{2} e^Z$$

$$\therefore \quad y = c_1 \cos Z + c_2 \sin Z + \frac{1}{2} e^Z$$

$$\therefore \quad y = c_1 \cos(\log x) + c_2 \sin(\log x) + \frac{1}{2} x$$

Ex ② $\left((x+1)^2 D^2 + (x+1) D + 1\right) y = x \quad \left(D = \frac{d}{dx}\right)$

Let $x+1 = e^z \Rightarrow x = e^z - 1$

Also $z = \log(x+1)$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} \cdot \frac{1}{(x+1)}$$

$$\Rightarrow (x+1) \frac{dy}{dx} = \frac{dy}{dz}$$

$$\Rightarrow (x+1) \frac{d}{dx} = \frac{d}{dz}$$

$$\Rightarrow (x+1) D = D \quad \left(D = \frac{d}{dz}\right)$$

Similarly

$(x+1)^2 D^2 = D(D-1)$

$\therefore$ The given diff^{tial} eq^n becomes

$$(D(D-1)+D+1)y = e^{z}-1 \qquad \left(D=\frac{d}{dz}\right)$$

ie $(D^2+1)\,y = e^{z}-1$

$\therefore$ The AE is

$$m^2+1=0$$

$$\Rightarrow m = \pm i$$

$\therefore$ CF $= c_1 \cos z + c_2 \sin z$

$$PI = \frac{1}{(D^2+1)}(e^{z}-1)$$

$$= \frac{1}{D^2+1}e^{z} - \frac{1}{D^2+1}e^{0.z}$$

$$= \frac{1}{2}e^{z}-1$$

$\therefore y = c_1 \cos z + c_2 \sin z + \frac{1}{2}e^{z}-1$

$\therefore y = c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) + \frac{1}{2}(x+1)-1$

$\therefore y = c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) + \frac{1}{2}(x-1)$

Ex: $\left((2x+1)^2 D^2 + (2x+1) D + 1\right) y = x \quad \left(D = \frac{d}{dx}\right)$

Let $2x+1 = e^z \Rightarrow z = \log(2x+1)$

$\therefore \quad x = \frac{1}{2}(e^z - 1)$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} \cdot \frac{1}{(2x+1)} \cdot 2$$

$$\therefore (2x+1) \frac{dy}{dx} = 2 \cdot \frac{dy}{dz}$$

$$\therefore (2x+1) \frac{d}{dx} = 2 \cdot \frac{d}{dz}$$

$$\therefore (2x+1) D = 2D \quad \left(D = \frac{d}{dz}\right)$$

$$(2x+1)^2 D^2 = 2^2 D(D-1)$$

$$(2x+1)^3 D^3 = 2^3 D(D-1)(D-2)$$

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$\therefore$ The given diff. eq^n becomes

$$\left(z^2 D(D-1) + 2D + 1\right) y = \frac{1}{2} e^z - \frac{1}{2} \quad \left(D = \frac{d}{dz}\right)$$

ie $\left(4D^2 - 2D + 1\right) y = \frac{1}{2} e^z - \frac{1}{2}$

$\therefore$ The AE is

$$4m^2 - 2m + 1 = 0$$

$$m = \frac{2 \pm \sqrt{4-16}}{8}$$

$$m = \frac{2 \pm 2\sqrt{3}\, i}{8}$$

$$m = \frac{1}{4} \pm \frac{\sqrt{3}}{4} i$$

$(e^z)^{1/4}$

$$\therefore CF = e^{\frac{1}{4}z}\left(c_1 \cos \frac{\sqrt{3}}{4} z + c_2 \sin \frac{\sqrt{3}}{4} z\right)$$

$$PI = \frac{1}{4D^2 - 2D + 1}\left(\frac{1}{2} e^z - \frac{1}{2}\right)$$

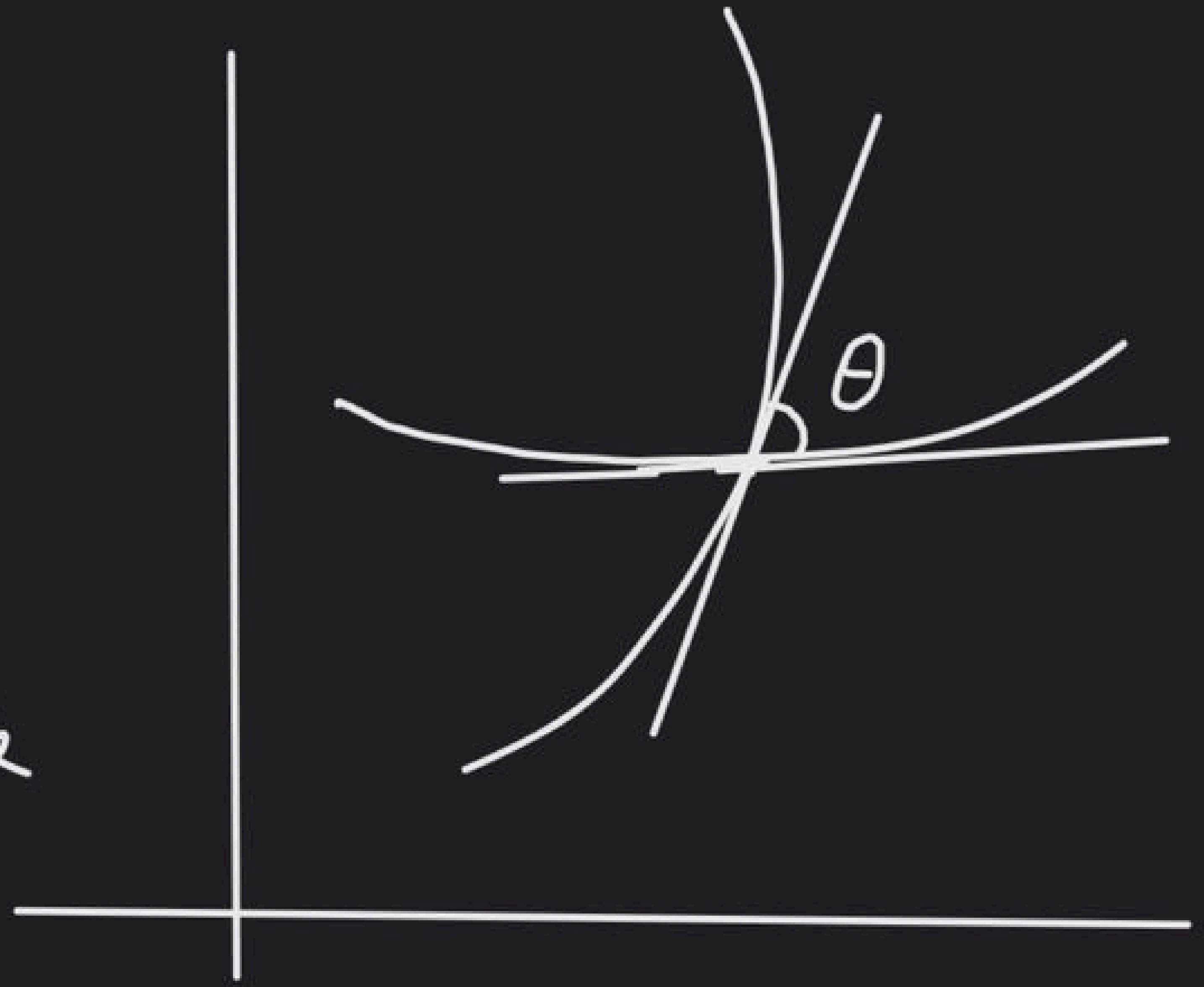
$$= \frac{1}{2} \frac{1}{(4-2+1)} e^z - \frac{1}{2}$$

$$= \frac{1}{6} e^z - \frac{1}{2}$$

Orthogonal Trajectory:

Angle between two curves:

Angle between two curves is the angle between their tangents at the common point of intersection. If θ be the angle between two curves, then



$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

m_1 = slope of Ist curve

m_2 = slope of IInd curve

If $\theta = 90^\circ$

Then

$$\tan 90^\circ = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\frac{1}{0} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$\Rightarrow \quad 1 + m_1 m_2 = 0$

$\Rightarrow \quad m_1 m_2 = -1$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{I} \left(\frac{dy}{dx}\right)_{II} = -1$$

i.e. Two curves intersect orthogonally iff Product of their slope $= -1$

Cartesian form: $y = f(x)$ or $f(x, y) = c$

Step I :- find $\frac{dy}{dx}$

Step II :- eliminate the constant

Step III : Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

Then integrate and find the sol^n.

Polar form: $r = f(\theta)$

Step I : find $\frac{dr}{d\theta}$

Step II : eliminate the constant

Step III : - Replace

$\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ and integrate

find the soln.

Ex: ① The orthogonal trajectory of

(i) $r = a(1+\cos\theta) \rightleftarrows r = b(1-\cos\theta)$

(ii) $r^n = a^n \cos n\theta \rightleftarrows r^n = b^n \sin n\theta$

(iii) $r^n \cos n\theta = a^n \rightleftarrows r^n \sin n\theta = b^n$

(iv) $r = a\theta \longrightarrow r = b e^{-\theta^2/2}$

② A parabola $y^2 = 4a(x+a)$ is self orthogonal.

Expn:- $r = a(1+\cos\theta)$

Taking log both the sides

$$\log r = \log a + \log(1+\cos\theta)$$

$$\therefore \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{(1+\cos\theta)}(-\sin\theta)$$

$$\therefore \frac{1}{r}\frac{dr}{d\theta} = -\frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\therefore \frac{1}{r}\frac{dr}{d\theta} = -\tan\theta/2$$

Replace $dr/d\theta$ by $-r^2\, d\theta/dr$

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan\theta/2$$

$$\therefore r\frac{d\theta}{dr} = \tan\theta/2$$

$$\Rightarrow \frac{dr}{r} = \cot\theta/2\ d\theta$$

Integrate

$$\log r = \frac{\log\sin\frac{\theta}{2}}{1/2} + \log c$$

$$\log r = 2\log\sin\theta/2 + \log c$$

$$\log r = \log\sin^2\theta/2 + \log c$$

$$\log r = \log(c\sin^2\theta/2)$$

$$r = c\sin^2\theta/2$$

$$r = \frac{c}{2}(2\sin^2\theta/2)$$

$$\underline{\underline{r = b(1-\cos\theta)}}$$

(iv) $r = a\theta$

$$\log r = \log a + \log \theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta}$$

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{1}{\theta}$$

$$-r\frac{d\theta}{dr} = \frac{1}{\theta}$$

$$\Rightarrow \frac{dr}{r} = -\theta\, d\theta$$

Integrate

$$\log r = -\frac{\theta^2}{2} + \log c$$

$$\underline{r = c\, e^{-\theta^2/2}}$$

(2) $y^2 = 4a(x+c)$ ——— (1)

$$\therefore 2y\frac{dy}{dx} = 4a \quad \text{——— (2)}$$

$$\Rightarrow a = \frac{1}{2}y\frac{dy}{dx} \quad \text{——— (3)}$$

$\therefore$ from (1), (2) & (3)

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{1}{2}y\frac{dy}{dx}\right)$$

$$\therefore y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 \quad \text{——— (4)}$$

Replace dy/dx by $-(dx/dy)$

$$\therefore \quad y^2 = 2xy\left(-\frac{dx}{dy}\right) + y^2\left(-\frac{dx}{dy}\right)^2$$

$$\therefore \quad y^2 = -2xy\,\frac{1}{\left(\frac{dy}{dx}\right)} + y^2 \cdot \frac{1}{\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \quad y^2 \cdot \left(\frac{dy}{dx}\right)^2 = -2xy\left(\frac{dy}{dx}\right) + y^2$$

$$\Rightarrow \quad y^2 = 2xy\left(\frac{dy}{dx}\right) + y^2\left(\frac{dy}{dx}\right)^2 \quad \text{——— } (5)$$

$\therefore$ eqⁿ (4) & (5) are same

$\therefore$ eqⁿ (1) i.e. $y^2 = 4a(x+a)$ is

self orthogonal.

Ex: Find the orthogonal trajectory of $y = ax^2$

$$y = ax^2$$

$$\log y = \log a + 2\log x$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$$

Replace dy/dx by $-dx/dy$

$$\therefore \frac{1}{y}\left(-\frac{dx}{dy}\right) = \frac{2}{x}$$

$$x\,dx + 2\cdot y\,dy = 0$$

Integrate $\dfrac{x^2}{2} + y^2 = C$

Ex: 1. Find the orthogonal trajectory of $x^{2/3} + y^{2/3} = a^{2/3}$

2. Find the orthogonal trajectory of $y = \dfrac{x^3 - a^3}{3x}$.

Note: ① The orthogonal trajectory of

$$r = c(\cos\theta + \sin\theta) \text{ is } r = b(\sin\theta - \cos\theta)$$

② $$r = \frac{2a}{1+\cos\theta} \text{ is } r = \frac{b}{1-\cos\theta}$$