Invertible Maps

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 21 • Sept 25, 2021

THEOREM $T: V \to W$ be a linear transformation. If $\{v_1, v_2, v_3, \dots v_n\}$ be a spanning set of V then $\{T(v_1), T(v_2), \dots, T(v_n)\}$ spans range of T.

Proof: $\text{span}\{v_1, v_2, \dots, v_n\} = V$

$\forall v \in V$, $\exists c_1, c_2, \dots, c_n \in F$ st

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

for $v \in V$, $T(v) \in \text{Range}(T)$

$$T(v) = T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$$

$$= c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n)$$

$\because T$ is a LT

$$T(v) \in \operatorname{span}\{T(v_1), T(v_2), \ldots, T(v_n)\}$$

$$\forall v \in V$$

$$\operatorname{span}\{T(v_1), T(v_2), \ldots, T(v_n)\} = \operatorname{Range}(T).$$

THEOREM : Let $T: V \to W$ be a linear transformation. Let $\{v_1, v_2, \ldots v_n\}$ be a subset of V. If $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is LI set in W then $\{v_1, v_2, \ldots, v_n\}$ is also a LI set in V.

Proof :

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = T(0)$$

$$c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) = 0$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

$\{T(v_i) \mid i = 1, 2 \ldots n\}$ is LI.

$\Rightarrow \{v_1, v_2, \ldots v_n\}$ is LI.

(eg) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(x, y) = (0, 0)$

$T(0, 1) = (0, 0)$

$T(1, 0) = (0, 0)$

THEOREM : let V and W be two two F.D.V.S. over the same field F. let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. and $u_1, u_2, \ldots, u_n$ be any vectors in W. then, $\exists$ a unique linear transforma s.t

$$T(v_i) = u_i \quad \forall \; 1 \le i \le n.$$

(eg) $T: \mathbb{R}^3 \to \mathbb{R}^2$ find a LT which following.

$T(1,0,0) = (0,1)$

$T(1,1,0) = (1,-1)$

$T(0,0,1) = (2,0)$

$\{(1,0,0), (1,1,0), (0,0,1)\}$ basis of $\mathbb{R}^3$

$(x,y,z) = c_1(1,0,0) + c_2(1,1,0) + c_3(0,0,1)$

$T(x,y,z) = c_1 T(1,0,0) + c_2 T(1,1,0) + c_3 T(0,0,1)$

$(x,y,z) = (x-y)(1,0,0) + y(1,1,0) + z(0,0,1)$

$T(x,y,z) = (x-y)(0,1) + y(1,-1) + z(2,0)$

$$T(x, y, z) = (y + 2z, \; x - 2y)$$

(eg) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(-1,0) = (0,0)$$

$$T(1,3) = (1,2)$$

$$(x,y) = c_1(-1,0) + c_2(1,3)$$

$$(x,y) = \left(x + \frac{y}{3}\right)(-1,0) + \frac{y}{3}(1,3)$$

$$T(x,y) = \left(-x + \frac{y}{3}\right)T(-1,0) + \frac{y}{3}T(1,3)$$

$$T(x,y) = \left(-x + \frac{y}{3}\right)(0,0) + \frac{y}{3}(1,2)$$

$$\boxed{T(x,y) = (y/3,\ 2y/3)}$$

$T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(\underline{1,1}) = (0,1)$$

$$T(\underline{2,2}) = (0,-1)$$

We can't find y

RANK NULLITY THEOREM

Let $T: V \to W$ be a linear transformation. Let V be a F.D.V.S. Then,

$$\boxed{\eta(T) + \rho(T) = \dim V}$$

or

$$\boxed{\dim N(T) + \dim R(T) = \dim V}$$

$\Rightarrow \quad T : P_3[x] \longrightarrow P_4[x] \to B = \{1, x, x^2, x^3, x^4\}$

dim $= 5$

$$T(p(x)) = p'(x) + \int_0^x p(t)\,dt$$

$N(T) = \{ 0 \}$

$\eta(T) = 0$

By R.N.T.

$$\rho(T) + \eta(T) = \dim V$$

$$\rho(T) + 0 = 4$$

$$\rho(T) = 4$$

$$f(x^3) = \frac{d}{dx}(x^3) + \int_0^x t^3\,dt$$

$$= 3x^2 + \frac{x^4}{4}.$$

ALGEBRA OF LT.

Let $T, S : V \to W$ be two linear transformation.

(a) $T+S$ is also a linear transformation.

$\forall\, u, v \in V$ and $\forall\, \alpha, \beta \in F$

$T'(\alpha u+\beta v) = (T+S)(\alpha u+\beta v) = T(\alpha u+\beta v) + S(\alpha u+\beta v)$

$= \alpha T(u) + \beta T(v) + \alpha S(u) + \beta S(v)$

$= \alpha (T(u) + S(u)) + \beta (T(v) + S(v))$

$= \alpha (T+S)(u) + \beta (T+S) v$

$= \alpha T'(u) + \beta T'(v)$

(2) If α be a non-zero scalar from field F then αT is also an LT.

$\alpha T : V \to W$

$\forall\ u, v \in V, \quad \forall\ c_1, c_2 \in F$

$$\alpha T(c_1 u + c_2 v) = \alpha\left(c_1 T(u) + c_2 T(v)\right)$$

$(\because T \text{ is a LT})$

$$= c_1\, \alpha T(u) + c_2\, \alpha T(v)$$

$$= c_1 (\alpha T)(u) + c_2 (\alpha T)(v)$$

αT is a LT for $\alpha (\neq 0) \in F$

(3) $T : \underline{\underline{V}} \to W$ and $S : W \to \underline{\underline{V}}$ be two LT.

$T \circ S$ (or $S \circ T$) is also a linear trans.

from W to W

$T \circ S : W \to W$ $\qquad S \circ T : V \to V$

$\forall\ u, v \in W,\ \forall\ \alpha, \beta \in F$ $\qquad S(u), S(v) \in V$

$$(T \circ S)(\alpha u + \beta v) = T\big(S(\alpha u + \beta v)\big)$$
$$= T\big(\alpha \underbrace{S(u)} + \beta \underbrace{S(v)}\big) \quad (\because S \text{ is a LT})$$
$$= \alpha\, T(S(u)) + \beta\, T(S(v))$$
$$= \alpha\, (T \circ S)(u) + \beta (T \circ S)(v)$$

Every linear transformation is a function

Every func. need not to be a Linear transformation.

$T : \mathbb{R} \to \mathbb{R}$

func. + LT. — $T(x) = x$

$T(x) = |x|$ — func but not LT.

$/ \ x^2 \ / \ x+1 \ / \ \sin x \ /$
$\cos x$

$T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x, y, z) = (x^2, y, z)$

$\times$