Examples of Subspace

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 6 • Sept 10, 2021

INTERSECTION OF SUBSPACES

let V be a vector space over the field F. let W_1 and W_2 be two sub-spaces of vector space V.

$W = W_1 \cap W_2$

$\Rightarrow$ $\because$ W_1 and W_2 are sub-spaces of V

$$0 \in W_1 \text{ and } 0 \in W_2$$

$$0 \in W_1 \cap W_2$$

$$W = (W_1 \cap W_2) \neq \phi$$

$\Rightarrow \forall \alpha, \beta \in F$ and $\forall u, v \in W = (W_1 \cap W_2)$

Claim : $\alpha u + \beta v \in W = (W_1 \cap W_2)$

$\because u, v \in W$

$u, v \in W_1 \cap W_2$

$u, v \in W_1$ and $u, v \in W_2$

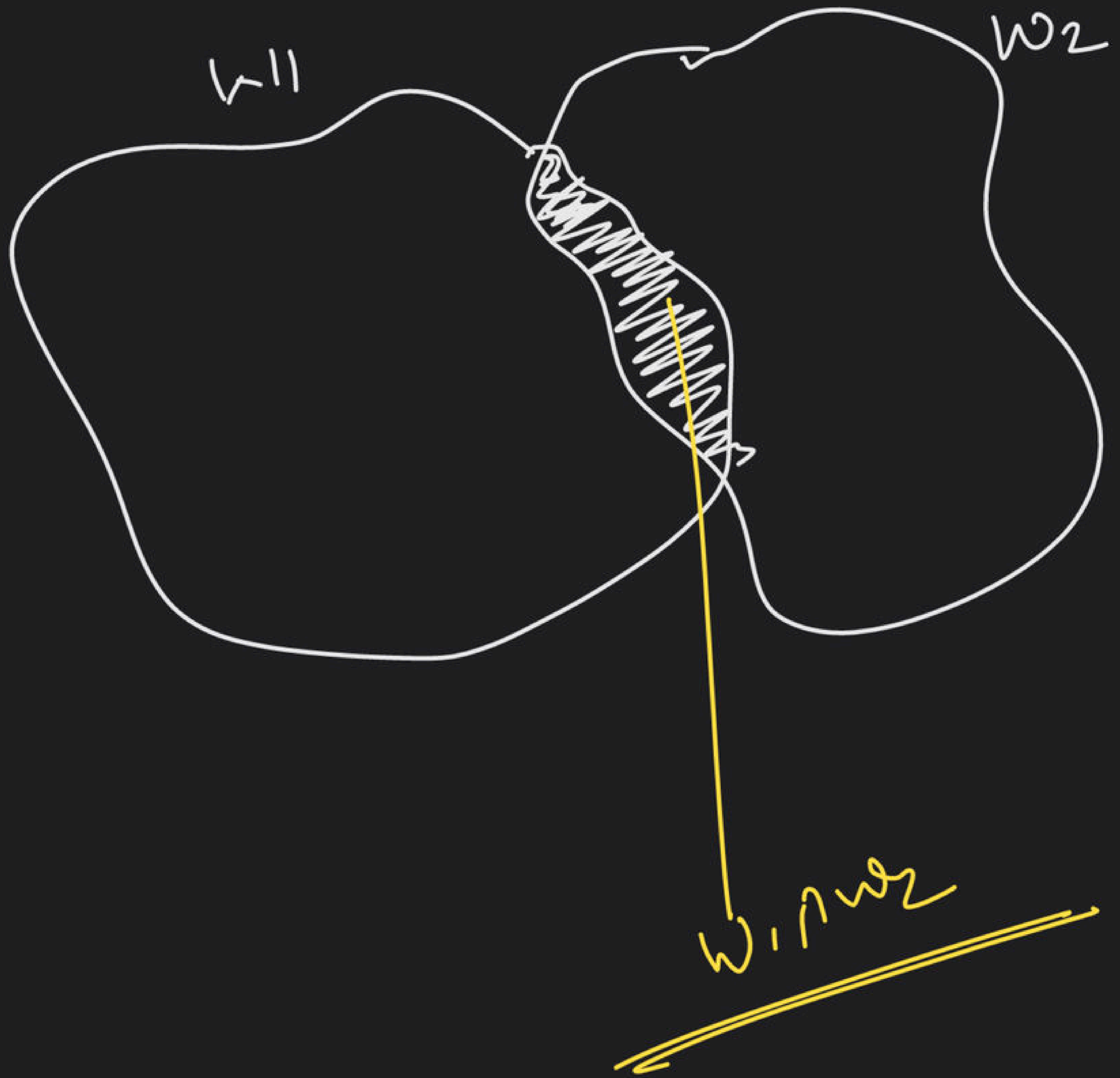
$\because W_1$ and W_2 are sub-spaces of V

$\forall \alpha, \beta \in F$

$\alpha u + \beta v \in W_1$ $\qquad$ $\alpha u + \beta v \in W_2$

$\Rightarrow \alpha u + \beta v \in W = W_1 \cap W_2$

$\Rightarrow W_1 \cap W_2$ is a sub-space.



w1
w2
w1∩w2

Let V be a vector space over the field F. Let W_i be the arbitrary subspaces where $i \in \mathbb{N}$.

Then $W = \bigcap_{i=1}^{\infty} W_i$ is also a sub-space of V

UNION OF SUB-SPACES

let V be a vector space over the field F. let W_1 and W_2 be two sub-spaces of V. then

$W = W_1 \cup W_2$

$\Rightarrow$ $W_1 \cup W_2$ need not to be a sub-space.

$V = \mathbb{R}^2 \qquad F = \mathbb{R}$

$W_1 = \{ (x, 0) \mid x \in \mathbb{R} \}$

$W_2 = \{ (0, y) \mid y \in \mathbb{R} \}$

W_1 and W_2 are sub-spaces of V

$W = W_1 \cup W_2$ is not a sub-spaces of $\mathbb{R}^2$.

$W = W_1 \cup W_2 = \{ (x, 0), (0, y) \mid x, y \in \mathbb{R} \}$

$\forall\, u, v \in W$, $u + v$ need not to belong to W.

$(1, 0) \in W \qquad (0, 1) \in W$

$(1, 0) + (0, 1) = (1, 1) \notin W$

$\Rightarrow$ $\boxed{\text{union of two sub-spaces is a sub-space iff one is contained in the other.}}$

let V be a V.S. over the field F.

let W_1 and W_2 be sub-spaces of V over F.

Then.

$W = W_1 \cup W_2$ is a sub-space of V iff. either $W_1 \subseteq W_2$

or $W_2 \subseteq W_1$.

$\Rightarrow$ $W_1 \cup W_2$ is a sub-space of V.

<u>Claim</u> : either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Suppose neither $W_1 \subseteq W_2$ nor $W_2 \subseteq W_1$

$W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$

let $x \in W_1$ s.t. $x \notin W_2$ let $y \in W_2$ s.t $y \notin W_1$

$\Rightarrow x \in W_1 \cup W_2$ $y \in W_1 \cup W_2$

$\because W_1 \cup W_2$ is a sub-space.

$x + y \in W_1 \cup W_2$

$\Rightarrow x+y \in W_1$ or $x+y \in W_2$

and $x \in W_1$ $y \in W_2$

$\because W_1$ is a sub-space $\because W_2$ is a sub-space

$(x+y)-x \in W_1$ or $x+y-y \in W_2$

$y \in W_1$ $x \in W_2$

contradiction to our assumption $y \notin W_1$

contradiction to our assumption $x \notin W_2$

Hence,

$W_1 \cup W_2$ is a sub-space of $V \implies$

$W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Assume either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ say

claim : $W = W_1 \cup W_2$ is a subspace.

If $W_1 \subseteq W_2 \implies W_1 \cup W_2 = W_2$

$\because W_2$ is a sub-space.

$\implies W_1 \cup W_2$ is a sub.

If $W_2 \subseteq W_1 \implies W_1 \cup W_2 = W_1$

$\because W_1$ is a sub-space.

$\implies W_1 \cup W_2$ is a sub-space.

SUM OF SUB-SPACES

let V be a vector space over ~~the~~ field F. let W_1 and W_2 be two sub-spaces of V. Then $W = W_1 + W_2$

$W = W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2 \}$

$\rightarrow$ $\because$ W_1 and W_2 are sub-spaces of V

$0 \in W_1$ and $0 \in W_2$

$\underbrace{0}_{\in W_1} + \underbrace{0}_{\in W_2} = 0 \in W_1 + W_2$ $\qquad W_1 + W_2 \neq \phi$

$\rightarrow \quad \forall \alpha, \beta \in F \quad , \quad \forall u, v \in W_1 + W_2$

<u>Claim</u> : $\alpha u + \beta v \in W_1 + W_2$

$\because\ u, v \in W_1 + W_2$

$\exists\ u_1, v_1 \in W_1$ and $u_2, v_2 \in W_2$ st

$u = u_1 + u_2$ and $v = v_1 + v_2$

$$\Rightarrow \alpha u + \beta v = \alpha(u_1 + u_2) + \beta(v_1 + v_2)$$

$$= (\alpha u_1 + \alpha u_2) + (\beta v_1 + \beta v_2)$$

$$= (\alpha u_1 + \beta v_1) + (\alpha u_2 + \beta v_2)$$

$$= w_1 + w_2$$

$w_1 = \alpha u_1 + \beta v_1 \in W_1$

$w_2 = \alpha u_2 + \beta v_2 \in W_2$

$\alpha u + \beta v = w_1 + w_2$; $w_1 \in W_1$ and $w_2 \in W_2$

$\Rightarrow \alpha u + \beta v \in (W_1 + W_2) = W$

$W_1 + W_2$ is a sub-space of V.

X, Y

$$X + Y = \{ u \mid \exists\, x \in X, \text{ and } y \in Y \text{ s.t. } u = x + y \}$$

$x + y$

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