Subspace

Comprehensive Course on Linear Algebra

Alka Singh • Lesson 5 • Sept 9, 2021

V is a V.S. over ~~the~~ field F

$W \subseteq V$ $\quad$ W is called as sub-sp of V if

→ $\underline{W}$ is itself a V.S. over ~~the~~ F

→ $0 \in W$

$\forall \alpha, \beta \in F$, $\forall u, v \in W$, $\alpha u + \beta v \in W$.

→ $0 \in W$

$\forall u, v \in W$, $u + v \in W$

$\forall \alpha \in F$, $\forall u \in W$, $\alpha u \in W$.

$$M : (i,j) \to a_{ij} \qquad a_{ij} \in F$$

$$M : S \to F$$

$$1 \le i \le m$$

$$1 \le j \le n$$

$$S = \{ (i,j) \mid 1 \le i \le m \, ; \, 1 \le j \le n \}$$

$$S \neq \phi$$

$$V = \{ f \mid f : S \to F \}$$ over F

$$(f+g)(x) = f(x) + g(x)$$

$$(cf)(x) = c \cdot f(x)$$

$W \subseteq V$, $W_1 = \{ f \in V \mid f \text{ is a cts func} \}$

$W_1 = \{ f \in V \mid f$ is a continuous func$\} \subseteq V$ is a sub-space of V.

$\rightarrow$ $0 \in V$ $\quad 0 : S \rightarrow F$ is always a cts func.

$0(x) = 0$

$W_1 \neq \phi$

$\rightarrow$ $\forall \alpha, \beta \in F, \quad \forall f, g \in W_1$

<u>Claim</u> : $\alpha f + \beta g \in W_1$

$\because f, g \in W_1 \Rightarrow f, g \in V$ and f and g are cts. func. By algebra of cont:

$\alpha f + \beta g \in V$ $\quad (\because V$ is a V.S.)

$\alpha f + \beta g$ is a cts func $\Rightarrow \alpha f + \beta g \in W_1$

$\Rightarrow W_2 = \{ f \in V \mid f$ is a differentiable func$\} \subseteq V$

HW

$\Rightarrow W_3 = \{ f \in V \mid f(-x) = f(x) \ \forall x \in S \}$

= set of all even func from S to F. subspace of V

$\rightarrow \quad 0(-x) = 0 = 0(x)$

$0 : S \to F \quad$ 0 is an even func. $\quad 0 \in W_3$

$W_3 \neq \phi$

$\rightarrow \quad \forall \alpha, \beta \in F, \ \forall f, g \in W_3 \quad$ Claim: $\alpha f + \beta g \in W_3$

$\because f, g \in W_3$

$\Rightarrow f$ and g are even func.

$f(-x) = f(x), \ g(-x) = g(x) \ \forall x \in S$

$\Rightarrow W_4 = \{ f \in V \mid f(-x) = -f(x) \ \forall x \in S \}$

forms a sub-space of V.

$\longrightarrow$ $0(x) = 0 \quad \forall x \in S$ (zero func) $0 : S \to F$

$$0(-x) = 0 = -0 = -0(x)$$

$0 \in W_4$

$\longrightarrow$

$$(\alpha f+\beta g)(-x) = (\alpha f)(-x) + (\beta g)(-x)$$

$$= \alpha \cdot f(-x) + \beta g(-x)$$

$$= \alpha \cdot f(x) + \beta g(x)$$

$$= (\alpha f)(x) + (\beta g)(x)$$

$$= (\alpha f+\beta g)(x)$$

$\Rightarrow$ $\alpha f+\beta g$ is an even func

$\alpha f+\beta g \in W_3$

(eg) $V = \mathbb{R}^2 \quad F = \mathbb{R}$

$W = \{ (x, y) \in \mathbb{R}^2 \mid x \geq 0 \quad y \in \mathbb{R} \}$ → not a sub-space

$u = (1, 0) \in W$

$\alpha = -1 \in \mathbb{R}$

$\alpha u = -1(1, 0)$

$= \underline{(-1, 0) \in W}$

$\underline{x}$ no

$W = \{ A \in M_{n\times n}(F) \mid a_{ij} = a_{ji} \;\; \forall\, 1 \le i,j \le n \}$

= set of all symme. matrices over the field F

forms a sub-space of $M_{n\times n}(F)$.

$M_{n\times n}$ — set of $n\times n$ ordered matrices.

$M_{n\times n}(F)$ — " over the field F

$\Rightarrow$ Null matrix is always a sy. mat.

$$0 = 0^T \qquad 0 \in W \qquad W \neq \phi$$

$\Rightarrow$

$\forall\, \alpha, \beta \in F, \quad \forall\, A, B \in W$

<u>claim</u>: $\alpha A + \beta B \in W \Rightarrow \alpha A + \beta B$ is a sym.

$\because\ A, B \in W \Rightarrow A$ and B sym.

$A = A^T$ and $B = B^T$

$$\underbrace{(\alpha A + \beta B)}^{T} = (\alpha A)^T + (\beta B)^T$$

$$= \alpha A^T + \beta B^T$$

$$= \underbrace{\alpha A + \beta B}$$

$\alpha A + \beta B$ is a symm. matrix.

$\alpha A + \beta B \in W$

$\Rightarrow$ W is a sub-space of $M_{n\times n}(F)$.

$\Rightarrow W = \{ A \in M_{n\times n}(F) \mid a_{ij} = -a_{ji} \ \forall\ 1 \le i, j \le n \}$

= set of all skew symmetric matrices of $M_{n\times n}(F)$.

forms a sub-space of $M_{n\times n}(F)$

Soln. $0 = -0^T \Rightarrow 0 \in W \Rightarrow W \neq \phi$

$\forall\ \alpha, \beta \in F, \ \forall\ A, B \in W$ claim: $\alpha A + \beta B \in W$

$\alpha A + \beta B$ is skew sy

$\because A, B \in W$

$A = -A^T$ and $B = -B^T$

$$(\alpha A + \beta B)^T = (\alpha A)^T + (\beta B)^T$$

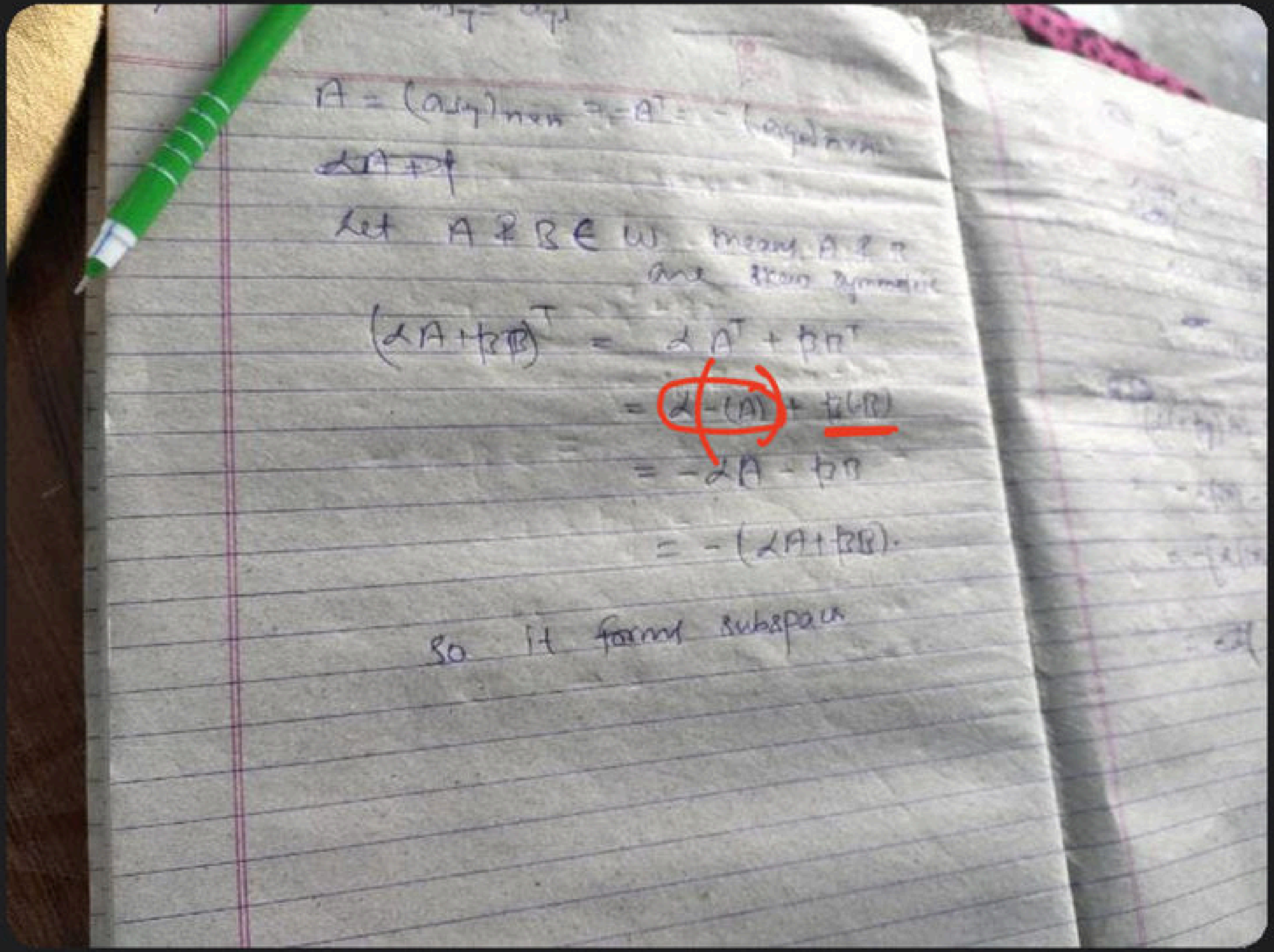
$$= \alpha A^T + \beta B^T$$

$$= \alpha(-A) + \beta(-B)$$

$$= -[\alpha A + \beta B]$$

1 • Asked by Amitesh

Please help me with this doubt



$A = (a_{ij})_{n\times n} \Rightarrow A^T = -(a_{ij})_{n\times n}$

~~$\alpha A+\beta$~~

Let A & $B \in W$ means A & B are skew symmetric

$(\alpha A+\beta B)^T = \alpha A^T + \beta B^T$

$= \alpha(-A) + \beta(-B)$

$= -\alpha A - \beta B$

$= -(\alpha A+\beta B)$.

So it forms subspace

$\Rightarrow \quad W = \{ A \in M_{n \times n}(F) \mid A = A^{\theta} \}$

$\rightarrow \quad 0^{\theta} = 0 \Rightarrow 0 \in W \Rightarrow W \neq \phi$

$\rightarrow \quad \forall A, B \in W$ Claim : $A + B \in W$ (i.e. $A + B$ is hermitian)

$\because A, B \in W \Rightarrow A$ and B are herm. matrices

$A = A^{\theta}$ and $B = B^{\theta}$

$$\underbrace{(A+B)^{\theta}} = \left(\overline{A+B}\right)^{T} = \left(\bar{A} + \bar{B}\right)^{T} = (\bar{A})^{T} + (\bar{B})^{T}$$

$$= A^{\theta} + B^{\theta}$$

$$= \underbrace{A + B}$$

$A + B \in W$

$\rightarrow \quad \forall \alpha \in F, \forall A \in W$ <u>Claim</u>: $\alpha A \in W$ (i.e. αA is a her. mat.)

$\because A \in W \Rightarrow A = A^\Theta$

$$\underbrace{(\alpha A)^\Theta} = (\overline{\alpha A})^T = (\bar{\alpha}\,\bar{A})^T$$

$$= (\bar{\alpha})^T (\bar{A})^T$$

$$= \bar{\alpha}\, A^\Theta$$

$$= \bar{\alpha}\, A \quad \left(\because \alpha \in \mathbb{R} \Rightarrow \bar{\alpha} = \alpha\right)$$

$\alpha A \in W$ $\quad = \alpha A$

W is a sub-space of $M_{n\times n}(\mathbb{F})$ iff $\mathbb{F} = \mathbb{R}$.

If $\mathbb{F} = \mathbb{C}$, $(\alpha A)^{\theta} = \bar{\alpha} A$

$\neq \alpha A$

$\Rightarrow \alpha A \notin W$

$\Rightarrow$ W is not a sub space.

$$W = \{ A \in M_{n\times n}(F) \mid A = -A^{\theta} \}$$

$\to$ $0 = -0^{\theta}$ $\quad 0 \in W \quad W \neq \phi$

$\to$ $\forall A, B \in W$, claim $A + B \in W$ $\quad \because A, B \in W$

$A = -A^{\theta}$

$B = -B^{\theta}$

$$(A+B)^{\theta} = A^{\theta} + B^{\theta} = -A - B$$

$$= -(A+B)$$

$A + B \in W$

$\to$ $\forall \alpha \in F$, $\forall A \in W$ claim: $\alpha A \in W$

$$(\alpha A)^{\theta} = \bar{\alpha} A^{\theta}$$

$(\because A \in W$

$A = -A^{\theta})$

$$= \bar{\alpha}(-A)$$

$$= -(\bar{\alpha} A) = -\alpha A$$

$(\alpha \in \mathbb{R})$

Is also forms a sub-space over the field of real numbers only.