Differential Equations

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1^{st} September 2021

1 Ordinary Differential Equation

Definition 1.1 *A differential Equation which contains one dependent variable and one independent variable is called ODE.*

$$1.\ \frac{dy}{dx} + y = x \qquad 2.\ \frac{d^2y}{dx^2} + y = x^3$$

$$3.\ \frac{dz}{dx} + z = x$$

Order of a Differential Equations:- The order of a differential equation is the highest order derivative appearing in the given differential equations.

Ex: 1. $\dfrac{d^2y}{dx^2} + y = x,$ order $= 2$

2. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 + y = x^2,$ order $= 2$

3. $\dfrac{dy}{dx} + y = \dfrac{d^3y}{dx^3}$ order $= 3$

Note: Order of a differential equation always exists and is a unique positive integer

Degree of a differential equation:- Highest power of the highest order derivative is called the degree of the differential equation, provided it is free from radicals and fractions. Ex:

Ex: 1. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 + y = x + \sin x$ order $= 2$, degree $= 1$.

2. $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{1/3} = a\dfrac{d^2y}{dx^2}$

i.e $1 + \left(\dfrac{dy}{dx}\right)^2 = a^3\left(\dfrac{d^2y}{dx^2}\right)^3$ order $= 2$, degree $= 3$.

3. $\left(\dfrac{d^2y}{dx^2}\right)^5 = \left(\dfrac{d^2y}{dx^2}\right)^7$ order $= 2$, degree $= 7$

Note:

i For degree of a differential equation, the D.E mut be a polynomial in its derivative.

ii The degree of a differential equation may or may not exist.

Ex: 1. $\dfrac{dy}{dx} + y = \sin\left(\dfrac{dy}{dx}\right)$
order $= 1$
but the degree is not defined ($\because$ it is not a polynomial in its detivative)

2. $\dfrac{d^2y}{dx^2} + y = e^{d^2y/dx^2}$
order $= 2$
but the degree is not defined ($\because$ it is not a polynomial in its derivative)

3. $\dfrac{d^2y}{dx^2} + y = e^{dy/dx}$

Linear Differential Equation:-

For Linear Differential Equation
1. The dependent varible and its derivative should not be multiplied to each other.
2. The degree of the dependent variable and all its derivative be 1.

Ex: i. $y\dfrac{dy}{dx} + y = x \longrightarrow$ non-linear.

ii. $x\dfrac{dy}{dx} + y = x^2 \longrightarrow$ linear

iii. $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x^2 \longrightarrow$ linear

iv. $\dfrac{dy}{dx} + y^2 = x^2 \longrightarrow$ non-linear

2 Linear Differential Equation with constant coefficents

Definition 2.1 *A differential equation in of the form*

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_ny = X$$

i.e

$$\left(a_0D^n + a_1D^{n-1} + a_2D_{n-2} + \cdots + a_n\right)y = X \tag{1}$$

where $D = \dfrac{d}{dx}$ and $a_0, a_1, a_2, \cdots, a_n$ are all constants and X is a function of only x or a constant is called L.D.E with constant coefficents.

The required solution is

$$y = \text{C.F} + \text{P.I}\ (= y_c + y_p)$$

C.F $\rightarrow$ Complementary function, P.I $\rightarrow$ Particular Integral. **If $X = 0$** Then equation (1) becomes

$$\left(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n\right)y = 0 \tag{2}$$

which is called the homogenous L.D.E with constant coefficients.

The required solution is.

$$y = \text{C.F}\ (= y_c)$$

let y $= e^{mx}$ be the solution of equation (2)
$\therefore$ Equation (2) becomes
$(a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n)e^{mx} = 0\ (\because e^{mx} \neq 0)$
$\implies a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n = 0$ — (3)
which is called the auxillary equation.

$(D - m)y = 0$

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{y} = mdx$$

$$\log y = mx + \log c$$

$$\boxed{y = ce^{mx}}$$

Case I: Roots are real and distinct: Let
$m = m_1, m_2$ be the roots (say) Then $y =$ C.F
$= c_1e^{m_1x} + c_2e^{m_2x}$

$$D(e^{mx}) = me^{mx}$$
$$D^2(e^{mx}) = m^2e^{mx}$$
$$\vdots$$
$$D^n(e^{mx}) = m^ne^{mx}$$

Principle of Superposition:

1 If y_1, y_2 be two solutions of a homogenous differential equation with constant coefficents, then their linear combination $c_1y_1 + c_2y_2$ is also a solution of that differential equation

2 Let $y_1, y_2, \cdots, y_n$ be n solutions of a homogenous L.D.E with constant coefficents, then their linear combination $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is also the solution of that L.D.E.

$\therefore$ In general let $m = m_1, m_2, \cdots, m_n$ be the roots of the auxillary equation of a homogenous L.D.E with constant coefficents, then

$$y = \text{C.F} = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$$

Ex: Solve $(D^2 - 5D + 6)y = 0 \left(D = \dfrac{d}{dx}\right)$

Soln: The Auxilary Equation is

$$\begin{aligned} m^2 - 5m + 6 &= 0 \\ \implies (m-2)(m-3) &= 0 \\ \implies m = 2, 3 & \end{aligned}$$

$\therefore y = \text{C.F} = c_1e^{2x} + c_2e^{3x}$is the required solution

Ex: If $y = ae^{2x} + be^{3x}$, form its D.E

Soln: $(D-2)(D-3)y = 0$

$$(D^2 - 5D + 6)y = 0$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

<u>**Note:**</u> All constants are present in the C.F, the P.I dosen't contain any constant.

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Case 1:- Roots are Real and distinct.

Case 2:- Roots are Real and repeated.
Say m, m are the real repeated roots, then C.F $= (c_1 + c_2x)e^{mx}$

Expln:- Then

$$\begin{aligned} &(D-m)^2y = 0 \\ \implies &(D-m)(D-m)y = 0 \qquad (1) \\ &\text{Let}(D-m)y = u \qquad (2) \end{aligned}$$

Then eqn (5) becomes

$$\begin{aligned} &(D-m)u = 0 \\ \implies &\frac{du}{dx} - mu = 0 \\ \implies &\frac{du}{u} = mdx \\ \implies &\log u = mx + \log c \\ \implies &u = c_2e^{mx} \end{aligned}$$

Put the value of u in equation (6)

$$\begin{aligned} &\therefore (D-m)y = c_2e^{mx} \\ \implies &\frac{dy}{dx} - my = c_2e^{mx} \end{aligned}$$

which is of the form $\dfrac{dy}{dx} + Py = Q$, therefore the I.F $= e^{\int P dx} = e^{\int -m dx} = e^{-mx}$
Therefore the required solution is

$$
\begin{aligned}
y(\text{I.F}) &= \int Q(I.F)dx + \text{constant} \\
ye^{-mx} &= \int (c_2 e^{mx}) e^{-mx} dx + c_1 \\
ye^{-mx} &= c_2 x + c_1
\end{aligned}
$$

$$\boxed{y = (c_1 + c_2 x) e^{mx}}$$

In general if n roots are repeated, then

$$\boxed{\text{C.F} = (c_1 + c_2 x + \cdots + c_n x^{n-1}) e^{mx}}$$

Ex. Solve $(D^2 - 2D + 1)y = 0$
Soln: The A.E is

$$
\begin{aligned}
& m^2 - 2m + 1 = 0 \\
\implies & (m-1)^2 = 0 \\
\implies & m = 1, 1 \\
\therefore \quad & y = \text{C.F} = (c_1 + c_2 x) e^{x}
\end{aligned}
$$

is the required solution.
Ex. Solve $(D^2 - 5D + 6)(D^2 - 4D + 4)y = 0$
Soln: The A.E is

$$
\begin{aligned}
\implies & (m^2 - 5m + 6)(m^2 - 4m + 4)y = 0 \\
\implies & m = 2, 2, 2, 3 \\
\therefore \; & y = \text{C.F} = (c_1 + c_2 x + c_3 x^2) e^{2x} + e^{3x}
\end{aligned}
$$

Case 3:- Roots are imaginary.

$$\text{say } m = \alpha \pm \iota\beta$$

$$\text{C.F} = e^{\Re(m)x}(\cos(|\Im(m)|x) + \sin(|\Im(m)|x))$$

$$\boxed{\text{C.F} = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)}$$

Ex #1 $(D^2 + 2D + 2)y = 0$
Soln:. The A.E is

$$
\begin{aligned}
\implies & m^2 + 2m + 2 = 0 \\
\implies & m^2 + 2m + 1 + 1 = 0 \\
\implies & (m+1)^2 + 1 = 0 \\
\implies & (m+1)^2 = -1 \\
\implies & m + 1 = \pm\iota \\
\implies & m = -1 \pm \iota
\end{aligned}
$$

$$y = \text{C.F} = e^{-1}x(c_1 \cos x + c_2 \sin ax)$$

Ex #2 $(D^2 + 2D + 2)^2 y = 0$

Soln:. The A.E is

$$\begin{aligned}
&\implies (m^2+2m+2)^2 = 0 \\
&\implies m^2+2m+1+1 = 0 && \text{(Twice)} \\
&\implies (m+1)^2+1 = 0 && \text{(Twice)} \\
&\implies (m+1)^2 = -1 && \text{(Twice)} \\
&\implies m+1 = \pm\iota \\
&\implies m = -1\pm\iota, -1\pm\iota
\end{aligned}$$

$$y = \text{C.F} = e^{-x}\left((c_1+c_2x)\cos x + (c_3+c_4x)\sin x\right)$$

Ex #3 Find the order of the differential equation whose one root is $x^2\sin x$.

Soln:. Let C.F of the differential equation be

$$e^{\alpha x}\left[(c_1+c_2x+c_3x^2)\cos x + (c_4+c_5x+c_6x^2)\sin x\right]$$

since the order of the differential equation is equal to the number of the arbitrary constants, therefore order $= 6$.

Note: If $m = \alpha\pm\beta$, then

$$\begin{aligned}
\text{C.F} &= c_1e^{(\alpha+\beta)x} + c_2e^{(\alpha-\beta)x} \\
\text{or C.F} &= c_1\cosh\beta x + c_2\sinh\beta x
\end{aligned}$$

Ex #4 $(D^2+2D-2)^2y = 0$

Soln:. The A.E is

$$\begin{aligned}
&\implies (m^2+2m-2)^2 = 0 \\
&\implies m^2+2m+1-3 = 0 \\
&\implies (m+1)^2-3 = 0 \\
&\implies (m+1)^2 = 3 \\
&\implies m+1 = \pm\sqrt{3} \\
&\implies m = -1\pm\sqrt{3}
\end{aligned}$$

$$y = \text{C.F} = e^{-x}\left(c_1\cosh(\sqrt{3}x) + c_2\sinh(\sqrt{3}x)\right)$$

2.1 Particular Integral

The Equation (1) can be written as

$$F(D)y = X$$

2.1.1 Properties:

I When X is of the form e^{ax} provided $F(a)\neq 0$

$$\begin{aligned}
\text{Then P.I} &= \frac{1}{F(D)}X \\
&= \frac{1}{F(D)}e^{ax} \\
&= \frac{1}{F(a)}e^{ax} \quad (F(a)\neq 0)
\end{aligned}$$

Expln: $D(e^{ax}) = D(e^{ax})$

$$D^2(e^{ax}) = a^2(e^{ax})$$

$$\vdots$$

$$D^n(e^{ax}) = a^n(e^{ax})$$

$$\therefore F(D)e^{ax} = F(a)e^{ax}$$

$$\implies \frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}, \text{ provided F(a)} \neq 0$$

Ex:- $(D^2 - 5D + 6)y = e^{5x}, \left(D = \frac{d}{dx}\right)$

Soln:- The A.E is

$$\begin{aligned} m^2 - 5m + 6 &= 0 \\ (m-2)(m-3) &= 0 \\ m &= 2, 3 \\ \therefore \text{C.F} &= c_1e^{2x} + c_2e^{3x} \end{aligned}$$

Now the P.I

$$\begin{aligned} \text{P.I} &= \frac{1}{F(D)}e^{5x} \\ &= \frac{1}{D-2)(D-3)}e^{5x} \\ &= \frac{1}{(5-2)(5-3)}e^{5x} \\ &= \frac{e^{5x}}{6} \end{aligned}$$

Hence the solution is

$$y = c_1e^{2x} + c_2e^{3x} + \frac{e^{5x}}{6}$$

II When X is of the form e^{ax} provided $F(a) = 0$

$$\begin{aligned} \text{Then P.I} &= \frac{1}{F(D)}X \\ &= \frac{1}{F(D)}e^{ax} \\ &= \frac{1}{(D-a)^r}e^{ax} \\ &= \frac{x^r}{r!}e^{ax} \end{aligned}$$

Ex: $(D^2 - 4D + 4)y = e^{2x} \left(D = \frac{d}{dx}\right)$

Soln: A.E is $m^2 - 4m + 4 = 0 \implies m = 2, 2$

$$\therefore \text{C.F} = (c_1 + c_2x)e^{2x}$$

For P.I

$$\begin{aligned} \text{P.I} &= \frac{1}{F(D)}e^{2x} \\ &= \frac{1}{(D-2)^2}e^{2x} \end{aligned}$$

Since $F(a) = F(2) = 0$, there are two ways to do this

Formula based

$$\begin{aligned} \text{P.I} &= \frac{x^r}{r!}e^{ax} \\ &= \frac{x^2}{2!}e^{2x} \\ &= \frac{x^2}{2}e^{2x} \end{aligned}$$

Differentiate and multiply by x till $F(a) \neq 0$

$$\begin{aligned} \text{P.I} &= \frac{1}{F(D)}e^{2x} \\ &= \frac{1}{(D^2 - 4D + 4)}e^{2x} \\ &= x\left(\frac{1}{2D - 4}\right)e^{2x} \\ &= x^2\left(\frac{1}{2}\right)e^{2x} \\ &= \frac{x^2}{2}e^{2x} \end{aligned}$$

$\therefore y = (c_1 + c_2x)e^{2x} + \dfrac{x^2}{2}e^{2x}$

Ex: $(D^2 - 5D + 6)y = e^{3x} \quad \left(D = \dfrac{d}{dx}\right)$

Soln: A.E is $m^2 - 5m + 6 = 0 \implies m = 2, 2$

$$\therefore \text{C.F} = c_1e^{2x} + c_2e^{3x}$$

$$\text{For P.I} = \frac{1}{F(D)}e^{2x} = \frac{1}{(D-2)(D-3)}e^{2x}$$

Since $F(a) = F(3) = 0$, there are two ways to do this

Formula based

$$\begin{aligned} \text{P.I} &= \frac{1}{F(D-2)(D-3)}e^{3x} \\ &= \frac{1}{(D-3)}\left(\frac{1}{(D-2)}e^{3x}\right) \\ &= \frac{1}{(D-3)}e^{3x} \\ &= \frac{x}{1!}e^{3x} \\ &= xe^{3x} \end{aligned}$$

Differentiate and multiply by x till $F(a) \neq 0$

$$\begin{aligned} \text{P.I} &= \frac{1}{F(D)}e^{3x} \\ &= \frac{1}{(D^2 - 5D + 6)}e^{3x} \\ &= x\left(\frac{1}{2D - 5}\right)e^{3x} \\ &= xe^{3x} \end{aligned}$$

$\therefore y = c_1e^{2x} + c_2e^{2x} + xe^{3x}$

III When X is of the form $\sin ax$ or $\cos ax$ provided $F(-a^2) \neq 0$

$$\begin{aligned} \text{Then P.I } &= \frac{1}{F(D)}X \\ &= \frac{1}{F(D^2)}(\sin ax \text{ or } \cos ax) \\ &= \frac{1}{F(-a^2)}(\sin ax \text{ or } \cos ax) \end{aligned}$$

(Replace D^2 by $-a^2$, D^4 by a^4, D^6 by $-a^6$, $\cdots$)

$$\begin{aligned} D(\sin ax) &= a(\cos ax) \\ D^2(\sin ax) &= -a^2(\sin ax) \\ &\vdots \end{aligned}$$

Ex:- $D^2 - 2D + 3 = \sin x \left(D = \dfrac{d}{dx}\right)$

$$\begin{aligned}
\text{P.I} &= \frac{1}{(D^2-2D+3)}\sin x \\
&= \frac{1}{(-1-2D+3)}\sin x \\
&= \frac{1}{(2-2D)}\sin x \\
&= \frac{1}{2}\frac{1}{(1-D)}\sin x \\
&= \frac{1}{2}\frac{(1+D)}{1-D^2}\sin x \\
&= \frac{1}{2}\frac{(1+D)}{(1-(-1))}\sin x \\
&= \frac{1}{4}(\sin x + \cos x)
\end{aligned}$$

Ex:- $(D^3 + 5D)y = \sin 2x \left(D = \dfrac{d}{dx}\right)$

$$\begin{aligned}
\text{P.I} &= \frac{1}{(D^3+5D)}\sin x \\
&= \frac{1}{(D^2\cdot D+5D)}\sin 2x \\
&= \frac{1}{(-4D+5D)}\sin 2x \\
&= \frac{1}{D}\sin 2x \qquad \frac{D}{D^2}\sin 2x \\
&= \int \sin 2x\,dx \qquad \frac{2\cos 2x}{-4}
\end{aligned}$$

$$-\frac{1}{2}\cos x$$

IV When X is of the form $\sin ax$ **or** $\cos ax$ provided $F(-a^2) = 0$

Ex: $(D^2 + a^2)y = \sin ax$

$$y = \frac{1}{(D^2+a^2)}\sin ax$$

Differentiate till $F(-a^2) \neq 0$

$$\begin{aligned}
&= x\frac{1}{(2D)}\sin ax \\
&= \frac{x}{2}\int \sin ax\,dx \\
&= \frac{x}{2}\left(-\frac{\cos ax}{a}\right) \\
&= -\frac{x}{2a}\cos ax
\end{aligned}$$

Ex: $(D^2 + a^2)y = \cos ax$

$$y = \frac{1}{(D^2+a^2)}\cos ax$$

Differentiate till $F(-a^2) \neq 0$

$$\begin{aligned}
&= x\frac{1}{(2D)}\cos ax \\
&= \frac{x}{2}\int \cos ax\,dx \\
&= \frac{x}{2}\left(\frac{\sin ax}{a}\right) \\
&= \frac{x}{2a}\sin ax
\end{aligned}$$

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V When X is of the form x^m

$$\begin{aligned}
\text{Then P.I} &= \frac{1}{F(D)}X \\
&= \frac{1}{F(D)}x^m \\
&= \frac{1}{1 \pm G(D)} \\
&= [1 \pm G(D)]^{-1}\,x^m
\end{aligned}$$

Expand binomially and multiply

Note:

1 $(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$

2 $(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$

3 $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$

4 $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$

5 $(1+D)^n = 1 + \dfrac{n}{1!}D + \dfrac{n(n-1)}{2!}D^2 + \dfrac{n(n-1)(n-3)}{3!}D^3 + \cdots$

Ex: $(D^2 - 2D + 1)y = x^2$
Soln: The A.E is $m^2 - 2m + 2 = 0 \implies (m-1)^2 + 1 = 0 \implies (m-1)^2 = -1 \implies m = 1 \pm \iota$

C.F $= e^x(c_1 \sin x + c_2 \cos x)$, and P.I is given by:

$$
\begin{aligned}
&\frac{1}{F(D)}x^2\\
=&\frac{1}{(D^2-2D+2)}x^2\\
=&\frac{1}{2\left(1+\dfrac{D^2-2D}{2}\right)}x^2\\
=&\frac{1}{2}\left(1+\left(\frac{D^2-2D}{2}\right)\right)^{-1}x^2\\
=&\frac{1}{2}\left(1-\left(\frac{D^2-2D}{2}\right)+\left(\frac{D^2-2D}{2}\right)^2+\cdots\right)x^2\\
=&\frac{1}{2}\left(1+D+\frac{D^2}{2}+\cdots\right)x^2\\
=&\frac{1}{2}\left(x^2+2x+1\right)\\
=&\frac{1}{2}\left(x-1\right)^2
\end{aligned}
$$

Therefore $y = e^x(c_1 \sin x + c_2 \cos x) + \frac{1}{2}\left(x-1\right)^2$

VI When X is of the form $e^{ax}V$ where V is a function of only x. Then

$$
\begin{aligned}
\text{P.I} &= \frac{1}{F(D)}X\\
&= \frac{1}{F(D)}e^{ax}\cdot V\\
&= e^{ax}\frac{1}{F(D+a)}V
\end{aligned}
$$

Ex: $(D^2+D+1)y = e^x x^2$

$$
\begin{aligned}
\text{Then P.I} &= \frac{1}{D^2+D+1}e^x x^2\\
&= e^x\frac{1}{(D+1)^2+(D+1)+1}x^2\\
&= e^x\frac{1}{D^2+3D+3}x^2\\
&= \frac{e^x}{3}\left(\frac{1}{1+\left(\dfrac{D^2}{3}+D\right)}\right)x^2\\
&= \frac{e^x}{3}\left(1+\left(\frac{D^2}{3}+D\right)\right)^{-1}x^2\\
&= \frac{e^x}{3}\left(1-\left(\frac{D^2}{3}+D\right)+\left(\frac{D^2}{3}+D\right)^2+\cdots\right)^{-1}x^2\\
&= \frac{e^x}{3}\left(1-D+\frac{2D^2}{3}\right)x^2\\
&= \frac{e^x}{3}\left(x^2-2x+\frac{4}{3}\right)
\end{aligned}
$$

Therefore $y = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + \frac{e^x}{3}\left(x^2-2x+\frac{4}{3}\right)$

VII When X is of the form $x \cdot V$, where V is a function of x
Then

$$\begin{aligned}
\text{P.I} &= \frac{1}{F(D)} \cdot X \\
&= \frac{1}{F(D)}(x \cdot V) \\
&= x\frac{1}{F(D)} \cdot V - \frac{F'(D)}{F(D)^2} \cdot V
\end{aligned}$$

Ex: $(D^2 + 2D + 1)y = x \sin x$

$$\begin{aligned}
\text{Then P.I} &= \frac{1}{F(D)} x \sin x \;=\; x\frac{1}{F(D)} \sin x - \frac{F'(D)}{F(D)^2} \sin x \\
&= x\left(\frac{1}{D^2 + 2D + 1}\right) \sin x - \left(\frac{2D + 2}{(D^2 + 2D + 1)^2}\right) \sin x \\
&= x\left(\frac{1}{2D}\right) \sin x - \left(\frac{2D + 2}{(-(1^2) + 2D + 1)^2}\right) \sin x \\
&= \frac{x}{2}\int \sin x dx - \frac{2}{4D^2}(\cos x + \sin x) \\
&= -\frac{x}{2} \cos x - \frac{1}{-2(1)^2}(\cos x + \sin x) \\
&= \frac{1}{2}(\cos x + \sin x) - \frac{x}{2} \cos x
\end{aligned}$$

VIII $\dfrac{1}{(D - a)} X = e^{ax} \displaystyle\int X e^{-ax} dx$

Expln: Let

$$\begin{aligned}
\frac{1}{(D - a)} X &= u \\
\implies (D - a)u &= X \\
\implies \frac{du}{dx} - au &= X
\end{aligned}$$

therefore the I.F $= e^{\int -a dx} = e^{-ax} \implies u e^{-ax} = \int X e^{-ax} dx \implies \boxed{\dfrac{1}{(D - a)} X = e^{ax} \displaystyle\int X e^{-ax} dx}$

2.2 Variation of parameters:

Ex: $(D^2 - 3D + 2)y = e^{3x}$ $\left(D = \dfrac{d}{dx}\right)$

Soln: The A.E is given by $m^2 - 3m + 2 = 0 \implies m = 2, 3$. Therefore the C.F is given by

$$\text{C.F} = c_1 e^x + c_2 e^{2x}$$

Let $y = u_1 y_1 + u_2 y_2$ be a solutionof the given differential equation. Then

$$u_1 = \int \frac{-y_2 R}{W} dx \qquad u_2 = \int \frac{y_1 R}{W} dx$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ and in this case $R = e^{3x}$. Therefore $W = \begin{vmatrix} e^x & e^{2x} \\ 2e^x & 3e^{2x} \end{vmatrix} = e^{3x}$ and

$$\begin{aligned} u_1 &= \int \frac{-y_2 R}{W} dx \\ &= \int \frac{-e^{2x} e^{3x}}{e^{3x}} dx \\ &= -\frac{e^{2x}}{2} \end{aligned} \qquad \begin{aligned} u_2 &= \int \frac{y_1 R}{W} dx \\ &= \int \frac{e^x e^{3x}}{e^{3x}} dx \\ &= e^x \end{aligned}$$

Therefore $y = -\left(\frac{e^{2x}}{2}\right) e^x + e^x e^{2x} = \frac{e^{3x}}{2}$

3 Homogeneous L.D.E with variable coefficients (Cauchy-Euler Equation)

Definition 3.1 *A differential equation of the form*

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X$$

that is

$$a_n x^n D^n + a_{n-1} x_{n-1} D^{n-1} + \cdots + a_n y = X$$

is called homogenous linear differential equation with variable coefficients or Cauchy-Euler's Equation, where $a_0, a_1, \cdots, a_n$ are all constants and X is a function of only x or a constant.

Put $x = e^z$, then (The D on the left is $\frac{d}{dx}$ and D on the right is $\frac{d}{dz}$)

$$\begin{aligned} x\frac{dy}{dx} &= \frac{dy}{dz} \\ xD &= D \end{aligned} \qquad \begin{aligned} x^2 \frac{d^2 y}{dx^2} &= \frac{d^2 y}{dz^2} - \frac{dy}{dz} \\ xD^2 &= D(D-1) \end{aligned}$$

Therefore the pattern is

$$x^n D^n = D(D-1)(D-2)\cdots(D-\overline{n-1})$$

Similarly for

$$a_n (ax+b)^n D^n + a_{n-1}(ax+b)^{n-1} D^{n-1} + \cdots + a_n y = X$$

putting $ax + b = e^z$, we get

$$(ax+b)^n D^n = a^n D(D-1)(D-2)\cdots(D-\overline{n-1})$$

4 Orthogonal Trajectory

4.1 Angle between two curves

Angle between two curves is the angle between their tangents at the common point of intersection. If θ is the angle between the two curves, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of the tangent to the curves at the point of intersection. For $\theta = \frac{\pi}{2}, 1 + m_1 m_2 = 0 \implies m_1 m_2 = -1$

$$\left(\frac{dy}{dx}\right)_I \left(\frac{dy}{dx}\right)_{II} = -1$$

Two curves intersect orthogonally iff product of their slopes is -1 at all points of intersection.

4.1.1 Cartesian Form

$$y = f(x) \text{ or } f(x, y) = c$$

Steps

1 Find $\frac{dy}{dx}$.

2 Eliminate the constant.

3 Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

4.1.2 Polar Form

$$r = f(\theta) \text{ or } f(\theta, r) = c$$

Steps

1 Find $\frac{dr}{d\theta}$.

2 Eliminate the constant.

3 Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

Note: If the constant is only multiplied then take log on both the sides and differentiate for ease.

4.1.3 Some standard results

1. $r = a(1 + \cos\theta) \rightleftarrows r = b(1 - \sin\theta)$
2. $r^n = a^n \cos\theta \rightleftarrows r = b^n \sin\theta$
3. $r^n \cos\theta = a^n \rightleftarrows r^n \sin\theta = b^n$
4. $r = a\theta \rightleftarrows r = be^{-\frac{\theta^2}{2}}$

5 Differential Equation of Ist order and Ist degree

Every differential equation of Ist order and Ist degree can be solved by either by exact or integrating factor.

5.1 Exact Differential Equation

Definition 5.1 *A differential equation of the form*

$$Mdx + Ndy = 0$$

is called an exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$$\begin{aligned} & f(x, y) = 0 \\ \implies & df = 0 \\ \implies & \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \end{aligned}$$

Comparing with $Mdx + Ndy = 0$

$$M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y}$$
$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}.$$

Assuming F has continous IInd order partial derivatives. Therefore $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
The required solution is

$$\underbrace{\int M dx}_{\text{keep } y \text{ as constant}} + \underbrace{\int N dy}_{\text{terms in } N \text{ free from } y} = c$$

1. Solve $(x^2 + y^2)dx + 2xydy = 0$

   $M = x^2 + y^2, \quad N = 2xy$

   $$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2y \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

   Therefore the given differential equation is exact. Now the solution is given by

   $$\begin{aligned} &= \int M dx + \int N dy = c \\ &= \int_{y \text{ is constant}} (x^2 + y^2)dx + \int 0dy = c \\ &= \frac{x^3}{3} + xy^2 = c \end{aligned}$$

2. The solution of the differential equation

   $$(x + 2y + 3)dx + (2x + y + 4)dy = 0$$

   represents which conic ?

   $$\begin{aligned} &M = x + 2y + 3, \quad N = 2x + y + 4 \\ &\frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 2 \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \end{aligned}$$

   Therefore the required solution is

   $$\begin{aligned} &\int (x + 2y + 3)dx + \int (y + 4)dy = c \\ &\therefore \frac{x^2}{2} + 2xy + 3x + \frac{y^2}{2} + 4y = c \\ &\implies x^2 + 4xy + y^2 + 6x + 8y = k \end{aligned}$$

   is the required solution Here $a = 1, b = 1, h = 2, f = 3, g = 4, c = k$ Hence

   $$\begin{aligned} \Delta &= \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 4 & 3 & k \end{vmatrix} \\ &= k - 9 - 2(2k - 12) + 4(2) = 23 - 3k \neq 0 \text{ if } k = 0 \\ &\text{and} \end{aligned}$$
   $$h^2 - ab = 4 - 1 = 3 > 0 \implies \text{Hyperbola}$$

Note: The general equation of the form

$$ax^2 + by^2 + 2hxy + fx + gy + c = 0$$

represents a conic:

1. If $a = b$ and $h = 0$
   - Circle
2. If $\Delta = 0$
   - Pair of Straight lines
3. If $\Delta \neq 0$
   - If $h^2 - ab > 0$: Hyperbola
   - If $h^2 - ab = 0$: Parabola
   - If $h^2 - ab < 0$: Ellipse

where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

5.2 Integrating Factor

Sometimes the given differential equation is not an exact differential equation, then to make it exact we multiply that equation by a function if x and y, which is called **Integrating Factor**.

5.2.1 Properties

1. **If the given differential equation is homogeneous, then**

$$\textbf{I.F} = \frac{1}{Mx + Ny}$$

provided $Mx + Ny \neq 0$**.**

Solve $(x^2 + y^2)dx - (xy)dy = 0$

$M = x^2 + y^2, \quad N = -xy$

$$\frac{\partial M}{\partial y} = 2y \neq -y = \frac{\partial N}{\partial x}$$

Therefore not exact. To make it exact we need to find an I.F. Since M and N are both homogeneous function with $n = 2$,

$$\text{I.F} = \frac{1}{Mx + Ny} = \frac{1}{x^3 + xy^2 - xy^2} = \frac{1}{x^3}$$

. Multiplying by I.F, we get

$$\left(\frac{1}{x} + \frac{y^2}{x^3}\right) dx + \left(-\frac{y}{x^2}\right) dy = 0$$

The solution is give by:

$$\int \left(\frac{1}{x} + \frac{y^2}{x^3}\right) dx + 0 = c \implies \log x - \frac{1}{2}\left(\frac{y^2}{x^2}\right) = c$$

. Hence the solution is $\log x - \frac{1}{2}\left(\frac{y^2}{x^2}\right) = c$

2. **If the differential equation is of the form**

$$f_1(xy)ydx + f_2(xy)xdy = c$$

then the I.F $= \dfrac{1}{Mx - Ny}$

Solve $(x^2y^3+xy^2+y)dx+(x^3y^2-x^2y+x)dy=0$
$M=(x^2y^2+xy+1), \quad N=(x^2y^2-xy+1)$

It can be rewritten as

$$(x^2y^2+xy+1)ydx+(x^2y^2-xy+1)xdy=0$$

I.F $=\dfrac{1}{x^3y^3+x^2y^2+xy-(x^3y^3-x^2y^2+xy)}=\dfrac{1}{2x^2y^2}$ The solution is

$$\int\left(y+\frac{1}{x}+\frac{1}{x^2y}\right)dx-\int\frac{1}{y}dy=c=xy+\log x-\frac{1}{xy}-\log y=c$$

Hence the solution is $xy-\dfrac{1}{xy}+\log\left(\dfrac{x}{y}\right)=c$

3. For the differential equation

$$Mdx+Ndy=0$$

if it is not exact then if $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial x}-\dfrac{\partial N}{\partial y}\right)$ **is a function of** x **say** $f(x)$**,**

$$\textbf{I.F}=e^{\int f(x)dx}$$

4. For the differential equation

$$Mdx+Ndy=0$$

if it is not exact then if $-\dfrac{1}{M}\left(\dfrac{\partial M}{\partial x}-\dfrac{\partial N}{\partial y}\right)$ **is a function of** y **say** $g(y)$**,**

$$\textbf{I.F}=e^{\int g(y)dy}$$

Example: $(x^2+y^2+x)dx+xydy=0$

$$\frac{\partial M}{\partial y}=2y\neq y=\frac{\partial N}{\partial x}$$

$\dfrac{1}{xy}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=\dfrac{1}{x}$ which is a function if x, hence I.F $=e^{\int\frac{1}{x}dx}=x$. Therefore the required solution is

$$\int(x^3+xy^2+x^2)dx+0=c \implies \frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{x^3}{3}=c$$

5. If the differential equation $Mdx+Ndy=0$ **is of the form**

$$\underbrace{x^{\alpha}y^{\beta}(mydx+nxdy)}_{\textbf{I.F}\,=\,x^{km-1-\alpha}y^{kn-1-\beta}}+\underbrace{x^{\alpha_1}y^{\beta_1}(m_1xdy+n_1xdy)}_{\textbf{I.F}=x^{k_1m_1-1-\alpha_1}y^{k_1n_1-1-\beta_1}}=0$$

Equate both Integrating factors and find k **and** k_1**, i.e Solve for** k **and** k_1 **the system of linear equations:**

$$km-1-\alpha=k_1m_1-1-\alpha_1$$
$$kn-1-\beta=k_1n_1-1-\beta_1$$

6. Method by Inspection
Solve $(x-x^2y)dx-ydy=0$

$$\begin{aligned} xdx-ydy&=x^2ydy\\ \frac{xdx-ydy}{x^2}&=ydy\\ d\left(\frac{y}{x}\right)&=ydy\\ \frac{y}{x}&=\frac{y^2}{2}+c \end{aligned}$$

Note

I If the given differential equation contains $(xdy - ydx)$ as a term, then its multiplication with

i $\frac{1}{x^2}$ gives $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

ii $\frac{1}{y^2}$ gives $\frac{xdy - ydx}{y^2} = -d\left(\frac{y}{x}\right)$

iii $\frac{1}{xy}$ gives $\frac{xdy - ydx}{xy} = \frac{dy}{y} - \frac{dx}{x} = d\left(\log\frac{y}{x}\right)$

iv $\frac{1}{x^2+y^2}$ gives $\frac{xdy - ydx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

v $\frac{1}{x\sqrt{x^2-y^2}}$ gives $\frac{xdy - ydx}{x\sqrt{x^2-y^2}} = d\left(\sin^{-1}\frac{y}{x}\right)$

II If the given differential equation contains $(xdy - ydx)$

i $\frac{1}{xy}$ gives $\frac{xdy + ydx}{xy} = \frac{dy}{y} + \frac{dx}{x} = d\left(\log xy\right)$

ii $\frac{1}{(xy)^n}$ gives $\frac{xdy + ydx}{(xy)^n} = \frac{d(xy)}{(xy)^n} + \frac{dx}{x} = d\left(\frac{-1}{(n-1)(xy)^{n-1}}\right)$

6 Linear Differential Equation of first order

Definition 6.1 *A differential equation of the form*

$$\frac{dy}{dx} + Py = Q$$

where P and Q both are function of only x or constants is called Linear Differential Equation In this case $I.F = e^{\int Pdx}$ and the solution of this equation is

$$y(I.F) = \int Q(I.F)\,dx + C$$

OR

Definition 6.2 *A differential equation of the form*

$$\frac{dx}{dy} + Py = Q$$

where P and Q both are function of only y or constants is called Linear Differential Equation In this case $I.F = e^{\int Pdx}$ and the solution of this equation is

$$x(I.F) = \int Q(I.F)\,dy + C$$

Definition 6.3 Bernouli's Equation: *A differential equation of the form*

$$\frac{dx}{dy} + Py = Qy^n \quad (n \neq 0, 1) \tag{1}$$

where P and Q both are function of only y or constants is called Bernouli's Equation

If $n = 0$ Equation (1) becomes Linear Differential Equation of first order.

$$\frac{dy}{dx} + Py = Q$$

If $n = 1$ Then equation (1) directly converts to separable variable. Otherwise, dividing by y^n in equation (1) we get,

$$\begin{aligned} &\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P = Q \\ &\text{Let } \frac{1}{y^{n-1}} = v \\ \implies &(1-n)\frac{1}{y^n}\frac{dy}{dx} = \frac{dv}{dx} \\ \implies &\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx} \\ \implies &\frac{1}{(1-n)}\frac{dv}{dx} + Pv = Q \\ \implies &\frac{dv}{dx} + (1-n)Pv = (1-n)Q \end{aligned}$$

which is the called reducible linear equation.

6.1 Separable Variables

6.1.1 Cases

1. A differential equation of the form
$$\frac{dy}{dx} = \frac{f_1(x)}{f_2(y)}$$
i.e
$$f_1(y)dy = f_2(x)dx$$
Integrate and solve for y.

2. A differential equation of the form
$$\frac{dy}{dx} = f(ax + by + c)$$
In this case let $ax + by + c = v$,
$$\begin{aligned} \implies &a + b\frac{dy}{dx} = \frac{dv}{dx} \\ \implies &\frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right) \\ \implies &\frac{1}{b}\left(\frac{dv}{dx} - a\right) = f(v) \\ \implies &\frac{dv}{dx} = bf(v) + a \\ \implies &\frac{dv}{bf(v) + a} = dx \end{aligned}$$
Integrate and find the solution.
Note: If the differential equation is of the form
$$\frac{dy}{dx} = f(ax + by)$$
then substitute $ax + by = v$.

6.2 Homogeneous Differential Equation

Definition 6.4 *An equation of the form*

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$$

is called homogenous equation in x and y of degree n. The above equation can be written as

$$f(x,y) = x^n \left(a_0 + a_1 \frac{y}{x} + a_2 \left(\frac{y}{x}\right)^2 + \cdots + a_n \left(\frac{y}{x}\right)^n \right)$$

$$\therefore f(x,y) = x^n F\left(\frac{y}{x}\right)$$

where n is the degree of the function.

Theorem 6.1 Euler's Theorem *If $f(x,y)$ is an homogenous function in x and y of degree n, then*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Corollary 6.1.1 $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \textbf{degree}\left(\dfrac{f(u)}{f'(u)}\right)$ *where the f is f in the equation*

$$f(u) = g(x,y,z) \text{ or } f(u) = g(x,y)$$

and g is homogenous function and **degree** *is degree of g.*

Definition 6.5 *A differential equation of the form*

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$$

where $f_1(x,y)$ and $f_2(x,y)$ are both homogenous function of x, y of degree n

$$\left.\begin{aligned} \frac{dy}{dx} &= \frac{x^n F_1\left(\frac{y}{x}\right)}{x^n F_2\left(\frac{y}{x}\right)} \\ \frac{dy}{dx} &= \frac{F_1\left(\frac{y}{x}\right)}{F_2\left(\frac{y}{x}\right)} \\ \frac{dy}{dx} &= F\left(\frac{y}{x}\right) \end{aligned}\right\} \implies \begin{aligned} \frac{y}{x} &= v \\ \therefore y &= vx \\ \therefore \frac{dy}{dx} &= \frac{dv}{dx} + v \\ v + \frac{dv}{dx} &= F(v) \\ \frac{dv}{dx} &= F(v) - v \end{aligned}$$

6.3 Non-Homogeneous Differential Equation

Definition 6.6 *A differential equation of the form*

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \tag{2}$$

is called a non homogenous differential equation.

1 If $\dfrac{a}{a'} \neq \dfrac{a}{a'}$ replace

$$\begin{aligned} x &= X + h & dx &= dX \\ y &= Y + k & dy &= dY \end{aligned}$$

in equation (2), then In this case

$$\begin{aligned} \frac{dy}{dx} &= \frac{a(X+h) + b(Y+k) + c}{a'(X+h) + b'(Y+k) + c')} \\ &= \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \end{aligned} \tag{3}$$

Put

$$\begin{aligned} ah + bk + c &= 0 \\ a'h + b'k + c' &= 0 \end{aligned}$$

and solve for (h,k). Equation (3) becomes

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

which is a homogenous differential equation.

6.4 Oblique Trajectory

Consider the curve $F : f(x, y) = 0$. For finding the Oblique Trajectory to the curve F at an angle θ. To find the family of curves oblique to the family of curves $f(x, y) = c$, find the derivative of the curve. Let

$$\frac{dy}{dx} = p = g(x, y)$$

be the derivative of the family of curves F. Then replace p by

$$\frac{p + p\tan\theta}{1 - \tan\theta}$$

and eliminate any constant before solving

$$\frac{\frac{dy}{dx} + \tan\theta}{1 - \frac{dy}{dx}\tan\theta} = g(x, y)$$

7 Linear Differential Equation of second order

The general form of equation of snd order is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \tag{4}$$

where P, Q, R are the function of only x. Let $y = u$ be one integral part of complementary function

$$\text{Let } y = uv \tag{5}$$

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \text{ and} \tag{6}$$

$$\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2} + \frac{du}{dx}\frac{dv}{dx} + \frac{d^2v}{dx^2} \tag{7}$$

Put the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation (4). Simplifying the equation, we get

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u} \tag{8}$$

Let $\frac{dv}{dx} = p$. Therefore equation (8)

$$\frac{dp}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right)p = \frac{R}{u}$$

which is linear in p.

$$\begin{aligned}
\therefore \text{I.F} &= e \text{ to the power } \int\left(\frac{2}{u}\frac{du}{dx} + P\right)dx \\
&= e \text{ to the power } \int\left(\frac{2}{u}du + Pdx\right) \\
&= e \text{ to the power } \left(\log u^2 + \int Pdx\right) \\
&= u^2 e^{\int Pdx}
\end{aligned}$$

The required solution is

$$\begin{aligned}
p{\cdot}u^2 e^{\int Pdx} &= \int \frac{R}{u}u^2 e^{\int Pdx}dx + c_1 \\
\therefore p &= u^{-2}e^{-\int Pdx}\int \frac{R}{u}u^2 e^{\int Pdx}dx + c_1 u^{-2}e^{-\int Pdx} \\
\therefore \frac{dv}{dx} &= u^{-2}e^{-\int Pdx}\int \frac{R}{u}u^2 e^{\int Pdx}dx + c_1 u^{-2}e^{-\int Pdx} \\
\therefore v &= \int\left(u^{-2}e^{-\int Pdx}\int \frac{R}{u}u^2 e^{\int Pdx}dx + c_1 u^{-2}e^{-\int Pdx}\right)dx + c_2
\end{aligned}$$

Therefore the required solution is

$$y = \underbrace{c_2 u + c_1 u \int \left(u^{-2} e^{-\int P dx}\right) dx}_{\text{complementary function}} + \underbrace{\int \left(u^{-2} e^{-\int P dx} \int R u e^{\int P dx}\right) dx}_{\text{particular function}}$$

Note: The second integral part of the C.F $= u \int \left(u^{-2} e^{-\int P dx}\right) dx$

$$= u \int \left(\frac{e^{-\int P dx}}{u^2}\right) dx$$

$$\text{P.I} = \int \left(\frac{e^{-\int P dx}}{u^2} \left(\int R u e^{\int P dx}\right) dx\right) dx$$

7.1 Examples

1 Let $y = x$ be one integral part of the C.F of the differential equation Find the P.I and other integral part.

$$x^2 \frac{d^2 y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x) = x^3$$

Soln: The given equation can be written as

$$\frac{d^2 y}{dx^2} - 2\frac{(1+x)}{x}\frac{dy}{dx} + 2\frac{(1+x)}{x^2} = x$$

where $P = \dfrac{-2(1+x)}{x}, Q = \dfrac{2(1+x)}{x^2}, u = x$ and $R = x$, other part of the integral is given by

$$u \int \frac{e^{-P dx}}{u^2} dx = x \int \frac{e^{\int \left(2 + \frac{2}{x}\right) dx}}{x^2} dx = x \int \frac{e^{2x} x^2}{x^2} dx = \frac{x e^2}{2}$$

2 Let $y = xv$ be a solution to the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 3x\frac{dy}{dx} + 3y = 0$$

If $v(0) = 0, v(1) = 2$, then find $v(-2)$

Soln:

$$\frac{d^2 y}{dx^2} + \left(-\frac{3}{x}\right)\frac{dy}{dx} + \left(\frac{3}{x^2}\right) y = 0$$

Clearly $P = -\dfrac{3}{x}, Q = \dfrac{3}{x^2}, R = 0, u = x$, we know that

$$\boldsymbol{v} = c_1 \int \left(\frac{e^{-\int P dx}}{u^2}\right) dx + c_2$$

$$= c_1 \int \left(\frac{x^3}{x^2}\right) dx + c_2$$

$$= \frac{c_1 x^2}{2} + c_2$$

Therfore $v = \dfrac{c_1 x^2}{2} + c_2$

$$v(0) = 0 \quad \Longrightarrow \frac{c_1(0)}{2} + c_2 = 0 \quad \Longrightarrow c_2 = 0$$

$$v(1) = 1 \quad \Longrightarrow \frac{c_1(1)}{2} + c_2 = 1 \quad \Longrightarrow \frac{c_1}{2} + c_2 = 1$$

Therefore $c_1 = 2, c_2 = 0$ implies $\boldsymbol{v = x^2}$, $v(-2) = (-2)^2 = 4$.

Note: To find the one integral part of C.F of the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

1. If $P + Qx = 0$, then $y = x$ is one integral part of the C.F.
2. If $2 + 2Px + Qx^2 = 0$, then $y = x^2$ is one integral part of the C.F.
3. If $m(m-1) + Pmx + Qx^2 = 0$, then $y = x^m$ is one integral part of the C.F.
4. If $1 + P + Q = 0$, then $y = e^x$ is one integral part of the C.F.
5. If $1 - P + Q = 0$, then $y = e^{-x}$ is one integral part of the C.F.
6. If $m^2 + Pm + Q = 0$, then $y = e^{mx}$ is one integral part of the C.F.

7.2 Removal of first derivative

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \tag{9}$$

Let y = uv be the solution of the equation 9 Therefore equation 9 reduces to

$$\frac{d^2v}{dx^2} + Xv = Y$$

where

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2, \qquad Y = Re^{\frac{1}{2}\int Pdx} \qquad u = e^{-\frac{1}{2}\int Pdx}$$

Ex. If $y = v\sec x$ is a solution of $y'' - (2\tan x)y' + 5y = 0, -\frac{\pi}{2} < x < \frac{\pi}{2}$, satisfying $y(0) = 0, y'(0) = \sqrt{6}$, then $v\left(\frac{\pi}{6\sqrt{6}}\right) = ?$

Soln: $P = -2\tan x, \quad Q = 5, \quad R = 0, \quad u = \sec x$ By eliminating first derivative equation in the question the equation reduces to

$$\frac{d^2v}{dx^2} + Xv = Y$$

where

$$\begin{aligned} X &= Q - \left(\frac{1}{2}\right)\frac{dP}{dx} - \frac{1}{4}P^2 & Y &= Re^{-\int Pdx} \\ &= 5 + \left(\frac{1}{2}\right)2\sec^2 x - \left(\frac{1}{4}\right)4\tan^2 x & &= 0 \\ &= 5 + 1 = 6 & &= 0 \end{aligned}$$

that is

$$\frac{d^2v}{dx^2} + 6v = 0$$

. The auxillary equation for this differential equation is $m^2 + 6 = 0 \implies m = \pm\sqrt{6}\iota$ Therefore the solution is

$$y = v\sec x = (c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x)\sec x$$

The boundary value conditions are

$$\begin{aligned} &y(0) = 0 & y'(0) &= \sqrt{6} \\ \implies &c_1 = 0 \quad \sec x\tan x(c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x) + \sec x(-\sqrt{6}c_1\sin\sqrt{6}x + \sqrt{6}c_2\cos\sqrt{6}x)\Big|_{x=0} & &= \sqrt{6} \\ \implies & & 0 + c_2\sqrt{6} &= \sqrt{6} \end{aligned}$$

Therefore $y = \sec x\sin\sqrt{6}x$ and $v = \sin\sqrt{6}x$ which implies $v\left(\frac{\pi}{6\sqrt{6}}\right) = \sin\left(\sqrt{6}\frac{\pi}{6\sqrt{6}}\right) = \mathbf{\frac{1}{2}}$

Note: If algebraic function is given apply this section's starting formulae and if trigonometric function is given then use Removal of first derivative concept

8 General Theory of Linear Differential Equation of Higher Order

The general linear differential equation of the nth order is of the form

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = Q \tag{10}$$

where $a_0(x), a_1(x), \cdots, a_n(x)$ and $Q(x)$ are continous function of x over the interval $I = [a, b]$ and $a_0(x) \neq 0$. Equation (10) can be rewritten as

$$(a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = Q \tag{11}$$

8.1 Classification of L.D.E

Homogenenous and Non-Homogeneous differential equation

The differential equations (10) and (11) are said to be homogeneous differential equations if $Q(x) = 0$, else they are said to be non-homogeneous differential equations. $(Q(x) \neq 0)$

Variable coefficents and Constant coefficents

If $a_0(x), a_1(x), \cdots, a_{n-1}(x), a_n(x)$ are **all** constants, then that differential equation is called L.D.E with constant coefficents otherwise it's called L.D.E with variable coefficents.

8.2 Termonologies

Linear Combination of functions

Let $f_1, f_2, f_3, \cdots f_n$ be n functions defined on a domain D, then the expression $c_1f_1 + c_2f_2 + c_3f_3 + \cdots c_nf_n$ is called the linear combination of those functions

Convex Combination

A convex combination is a linear combination of a type where $\sum_{i=1}^{n} c_i = 1$ and $c_i \geq 0$ for all i's

Linearly Independent functions

The n functions $f_1, f_2, \cdots f_n$ are called lineraly independent functions on a common domain D if $c_i = 0$ for all i's

Linearly Dependent functions

The n functions $f_1, f_2, \cdots f_n$ are called lineraly independent functions on a common domain D if there exist a scalar $c_i \neq 0$

8.2.1 Principle of Superposition

Consider the nth order linear differential equation (10). If $y_1, y_2, \cdots y_n$ are any n solutions of (10), then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

is also a solution of (10) if either

$$Q(x) = 0$$

OR

$$\sum_{i=1}^{n} c_i = 1$$

Note

1. If y_1, y_2 are solutions of a **homogeneous differential equation**, then $c_1y_1 + c_2y_2$, where c_1, c_2 are any scalars is also a solution of the same **homogeneous differential equation**.

2. If y_1, y_2 are solutions of a **non - homogeneous differential equation**, then $c_1y_1 + c_2y_2$, where c_1, c_2 are any scalars is also a solution of the same **non - homogeneous differential equation** if $c_1 + c_2 = 1$.

3. If y_1, y_2 are solutions of a **non - homogeneous differential equation**, then $y_1 - y_2$ is the solution of same **homogeneous differential equation**

8.3 Wronskian

Consider the second order homogenenous linear differential equation

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0 \tag{12}$$

where $a_0(x), a_1(x)$ and $a_2(x)$ are continous functions of x and $a_0 \neq 0$ for all $x \in [a, b]$, then the Wronskian

$$W(y_1, y_2) = W(x) = Ae^{\int -\left(\frac{a_1(x)}{a_0(x)}\right)dx}$$

This is also known as the **Abel's Formula**.
Let y_1, y_2 be two solutions of (12)

$$\therefore \quad a_0y_1'' + a_1y_1' + a_0y_1 = 0 \text{ and} \tag{13}$$
$$a_0y_2'' + a_1y_2' + a_0y_2 = 0 \tag{14}$$

Now $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ and $W' = y_1y_2'' - y_2y_1''$. Consider

$$\begin{aligned} a_0W' &= y_1(a_0y_2'') - y_2(a_0y_1'') \\ &= y_1(-a_1y_2' - a_2y_2) - y_2(-a_1y_1' - a_2y_1) \\ &= -a_1(y_1y_2' - y_2y_1') \\ &= -a_1W \\ \therefore \quad \frac{W'}{W} &= -\frac{a_1(x)}{a_0(x)} \\ \implies W &= Ae^{\int -\left(\frac{a_1(x)}{a_0(x)}\right)dx} \end{aligned}$$

8.4 Wronskian Properties

1. Consider the IInd order homogeneous linear differential equation

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$$

   where $a_0(x), a_1(x)$ and $a_2(x)$ are continous functions over the interval $I = [a, b]$. Then any two solutions

   (a) y_1, y_2 of the above equation are linearly independent iff $W(y_1, y_2) \neq 0$ over the interval I

   (b) y_1, y_2 of the above equation are linearly dependent iff $W(y_1, y_2) = 0$ over the interval I

2. If y_1, y_2 are two solutions of a differential equation and the differential equation is not given then

   (a) If $W(y_1, y_2) \neq 0 \implies y_1$ and y_2 are linearly independent.

   (b) If $W(y_1, y_2) = 0$, then we can't say anything.

3. Let $y_1 = e^{m_1x}, y_2 = e^{m_2x}, \cdots, y_n = e^{m_nx}$ be n linearly independent solutions of a differential equation, then their Wronskian is given by

$$W(y_1, y_2, \cdots, y_n) = e^{(m_1+m_2+\cdots+m_n)x}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ m_1 & m_2 & \cdots & m_n \\ m_1^2 & m_2^2 & \cdots & m_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ m_1^{n-1} & m_2^{n-1} & \cdots & m_n^{n-1} \end{vmatrix}$$

   This is also called as the **Vandermonde determinant**.

4. Let $y = y(x)$, then consider the differential equation $\frac{dy}{dx} = y^{\alpha}, y(b) = 0, b \in \mathbb{R}$ and $a \in (0, 1)$. This D.E has infinte number of real valued solution and and infinte linearly independent solutions. But if $y(b) = 1 b \in \mathbb{R}, a \in (0, 1)$, then number of solutions is unique. (IISc Motherfucking Banglore IITJAM 2021)