Non Linear Dynamics Assignment-1

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Problem 1

1. **Show that the system is linear.**
   As the power of highest order derivative in this polynomial equation is one, hence the equation is linear.

2. **Let b = 0. Show that the phase-space trajectories are elliptical without explicitly solving for x(t).**
   If the equation is differentiated with respect to time and replacing $\dot{x} = \frac{dx}{dt}$, we get similar equation for velocity.

$$\frac{d^2\dot{x}}{dt^2} + \omega_0^2\dot{x} = 0$$

   But the given initial conditions, when applied to each equation, result in a phase difference between x and $\dot{x}$, because of which the trajectory is elliptical.

3. **Solve for x(t) when $b > 2\omega_0$ i.e., when the system is overdamped.**
   Replacing x with Ae^{mx} we get resultant equation

$$m^2 + bm + \omega_0^2 = 0$$

   Solving this, as $b > 2\omega_0$ we get two real roots which are

$$m_1 = \frac{-b + \sqrt{b^2 - 4\omega_0^2}}{2}$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4\omega_0^2}}{2}$$

   The general solution is given by

$$x(t) = A_1e^{m_1t} + A_2e^{m_2t}$$

4. **Solve for x(t) when $b = 2\omega_0$ i.e when the system is critically damped.**
   As $b = 2\omega_0$ we get same value of m $= \frac{-b}{2} = -\omega_0$. By properties of homogeneous second order differential equation, in this case we get an additional linear multiple of time.

$$x(t) = (A_1 + A_2t)e^{-\omega_0t}$$

5. **Part (e)**
   Assuming initial conditions $x_{od}(0) = a$ and $\dot{x}_{od}(0) = c$, then

$$A_1 + A_2 = a$$

$$m_1A_1 + m_2A_2 = c$$

Solving for A_1 and A_2 we get

$$A_1 - A_2 = \frac{2c + ab}{\sqrt{b^2 - 4\omega_0^2}}$$

Taking the right hand limit at $b = 2\omega_0$,

$$x_{od} = \lim_{b \to 2\omega_0^+} A_1 e^{m_1 t} + A_2 e^{m_2 t}$$

$$x_{od} = e^{-\omega_0 t} \lim_{b \to 2\omega_0^+} A_1 e^{\frac{\sqrt{b^2 - 4\omega_0^2}}{2} t} + A_2 e^{\frac{-\sqrt{b^2 - 4\omega_0^2}}{2} t}$$

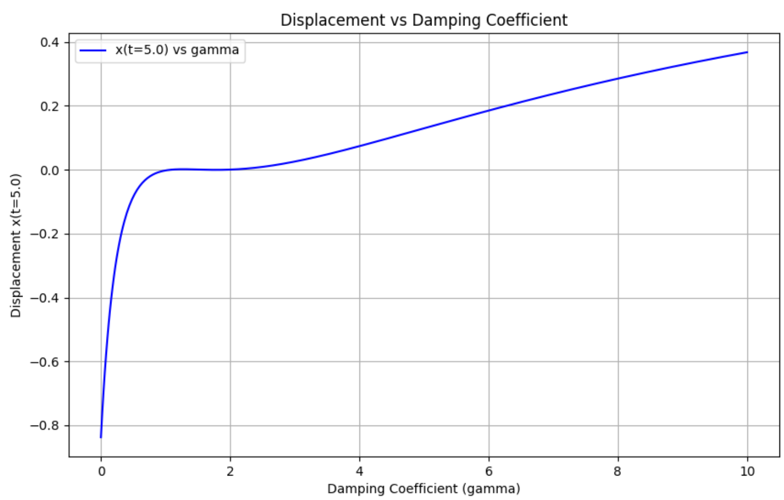
Using Taylor expansion till first order as $\sqrt{b^2 - 4\omega_0^2}$ is very small at this limit,

$$x_{od} = e^{-\omega_0 t}(A_1 + A_2 + (A_1 - A_2)\sqrt{b^2 - 4\omega_0^2 t})$$

Putting value of $A_1 - A_2$ as derived above, we get

$$x_{od} = (A_1' + A_2' t)e^{-\omega_0 t} = x_{cd}$$

Hence it is continuous. Also quite evident by simulation graph below.

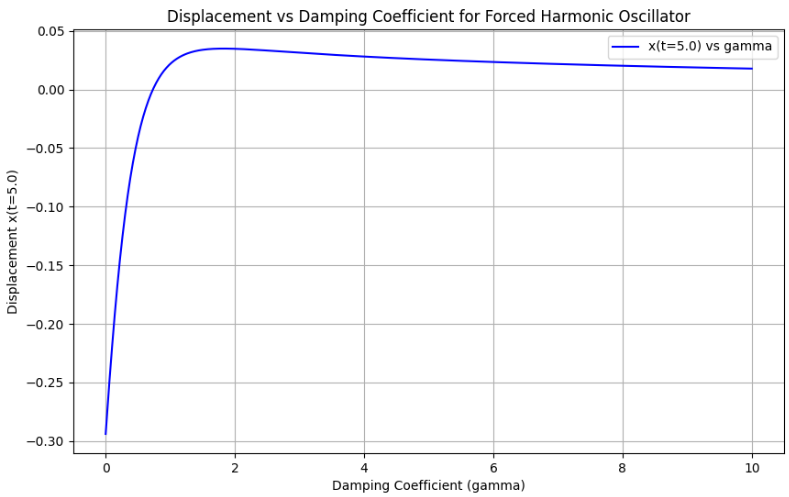


6. **Part (f)**
For $b > 2\omega_0$ case, the taylor expansion would not be restricted to linear time dependence terms and would extend to higher powers as well, that is

$$x_{od} = x_{cd} + \epsilon$$

where $\epsilon > 0$, hence $x_{od} > x_{cd} \, \forall \, t > 0$.

7. **Part (g)**: Graph plotted is -



Problem 2

1. **Part (a)** :
   Basic intution behind this is that you have only one undetermined coefficient, after solving the equation which can be found out using intial conditions. Also there is existence and uniqueness theorem in mathematics which gurantees a unique solution for given differential equation in its domain.

2. **Part (b)** :
   The solution is not unique as it is not a linear differential equation, as highest order derivative has power 3. It has two solutions $x = 0$ and $x^{\frac{2}{3}} = \frac{2t}{3}$. Also $x^{\frac{1}{3}}$ is not Lipchitz continuous, which explains existence of more than one solution.

3. **Part (c)** :
   Solving the equation,
   $$\int_0^x \frac{dx}{1+x^2} = \int_0^t dt$$
   gives
   $$x(t) = \tan t$$
   Hence after 4 seconds, particle will be at $x = tan(4) = 1.158$.

Problem 3

As it is a homogeneous linear equation, solution would be of form $x = e^{mt}$, solving it gives $m = k$ hence solution is $x = Ae^{kt}$. This is not a chaotic system, as the trajectory is not much sensitive to initial conditions. Starting from any position will result in exponentially diverging motion only. Also motion of this particle is predictble easily at every point of space and time, without any stochastic components.