

CS345 Assignment-1

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1 Faster algorithm for Non-Dominated Points in plane

In the lecture, two approaches were discussed to evaluate non-dominated points in a plane. The first approach was **output sensitive** and worked in a time-complexity of $O(N * h)$ where h is the number of dominated points in the **plane consisting of N points**. This algorithm was $O(N)$ in best case but $O(N^2)$ in the worst case.

We then improved this approach to $O(N \log N)$. This new algorithm was based on **divide-and-conquer paradigm**, but there were scenarios where the first algorithm could outperform the second algorithm. In this assignment, we devise an algorithm with a **time complexity of $O(N \log h)$** which is faster than both these algorithms, specifically in cases where the number of non-dominated points are very few. As mentioned in the problem statement, it can be shown that if n points are selected randomly uniformly from a unit square, then the expected (average) number of non-dominated points is just $O(\log n)$.

2 Sketch of Algorithm

2.1 Assumptions

- We assume that no two points in the plane have **same x-coordinate** or **the same y-coordinate**.
- There will be only one point on the vertical line passing through the median of x- coordinate of the given points.
- We define the x-median to be the median of x coordinates of all given points-in case there are even number of points, we take the floor of the actual median.
- The set of points S received as an argument to the function is has a non-zero size.

2.2 Intuition

We begin by computing the x-median of the input points, and splitting the given sample space into two halves, via a vertical line passing through the x-median [**Divide Step**]. The median can be computed using an $O(N)$ algorithm, as discussed in class. Since we are splitting on the basis of x-median, the sizes of the two halves **would differ by at most one**. Let's call these partitions **left set (LS)** and **right set (RS)**.

We invoke a recursive call to the divide function for the right set. Among all those points, we choose the point with maximum value of y-coordinate. Let the y-coordinate of this point be y_{max} .

We collect all the points in the left set whose y-coordinate is greater than y_{max} . Unless this set is empty, we invoke another recursive call for this left set.

3 Pseudocode

Algorithm 1: Faster Algorithm for non-dominated points

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1 Function fasterNDP( $S$ )
2    $x_m \leftarrow \text{getmax}_x(S)$ ,  $y_m \leftarrow \text{getmax}_y(S)$  /* Can be done in  $O(N)$  time */
3    $p_{max} \leftarrow (x_m, y_m)$ 
4    $\text{finalNDP} \leftarrow \emptyset$  /* List to store all non-dominated points */
   /* Base Case: If  $x_m$  and  $y_m$  correspond to a single point in the set, then there
   are no more non-dominated points, since this point encompasses all others.
   */
5   if  $p_{max} \in S$  then
6      $\text{finalNDP.push}(p_{max})$ 
7     return  $\text{finalNDP}$ 
8    $(LS, RS) \leftarrow \text{SplitByMedian}(S)$  /* Can be done in  $O(N)$  time */
9    $\text{rightNDP} \leftarrow \text{fasterNDP}(RS)$  /* Recursive call on the right set */
10   $y_{max} \leftarrow \text{getmax}_y(\text{rightNDP})$  /* Can be done in  $O(N)$  time */
11   $\text{reduced}_{LS} \leftarrow \emptyset$  /* List to store the candidate points from the left set */
   /* We iterate on the left set and add only those points whose y-coordinate is
   greater than  $y_{max}$  */
12  for  $\text{point } p \in LS$  do
13    if  $p.y > y_{max}$  then
14       $\text{reduced}_{LS}.push(p)$ 
15   $\text{finalNDP.push}(\text{rightNDP})$ 
16  if  $\text{reduced}_{LS}.size > 0$  then
17     $\text{leftNDP} \leftarrow \text{fasterNDP}(\text{reduced}_{LS})$  /* Recursive call on the left set */
18     $\text{finalNDP.push}(\text{leftNDP})$ 
19  return  $\text{finalNDP}$ 

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4 Proof of Correctness

First we consider the base case. Since we assumed that x-coordinate and y-coordinate of all points are unique, if there exists a point whose both x-coordinate and y-coordinate are maximum among all points in that current set, it dominates all other points in that set (since, it encompasses all other points). Hence it is the only non-dominated point in the given set and we return it as an answer and exit from the recursive call.

Next, we need to prove that set of points rightNDP obtained in recursive call on right set (RS) are non-dominated points. Since their x-coordinates are greater than all the points in the left set (LS), they are non-dominated with respect to the points in the left set, and we already know, they are non-dominated among themselves. Thus we add rightNDP into the final list of non-dominated points.

For the left part, we construct a set of points reduced_{LS} whose y-coordinate is greater than y_{max} . This means these points are not dominated by any point in right set (RS). After a recursive call on this set, we get a set of non-dominated points leftNDP among the reduced_{LS} . We already proved that these points are not dominated by right set (RS) due to the condition that their y-coordinate is greater than y_{max} , and after the recursive call, we know that they are non-dominated among themselves. Hence, this set of points also holds the non-dominated conditions for the current set S . Thus we add leftNDP into the final list of non-dominated points and return this as an answer.

5 Time Complexity Analysis

We try to analyse the time complexity using induction. Let $T(h)$ denote the time-complexity of the proposed function when there are exactly "h" non-dominated points among the current set of points S, of size N. The functions $getmax_x$, $getmax_y$ and p_{max} checking condition, x-median calculation and splitting in LS and RS, filtering points on basis of y_{max} and add to final list are linear time functions in N. So, for cases where $h=1$, the time complexity is $O(N)$ (Since $\log(1)=0$).

Note that, after division of the sample space into two halves, there might be an arbitrary number of non-dominated points in the left half(let's call this h_1), and as a consequence, the number of non-dominated points in the right half would be $h-h_1$. Keeping this in mind, we would have to recur on the left half(containing $n/2$ points) and the right half(containing $n/2$ points). Thus the time complexity when there are h non-dominated points would be bounded by the maximum of all such divisions, hence the recurrence relation (equation 2).

The inductive hypothesis is assumed to be as follows :

$$\text{Induction Hypothesis : } \boxed{T(h) \leq c * (N + N * \log_2(h))} \text{ for N points.} \quad (1)$$

When left set(LS) is completely deleted by y_{max} , the size of h is 0. For such a case, $T(h)$ for any size of N is 0. For the case of $h=1$, the algorithm is simply linear time due to the p_{max} if condition.

The recurrence relation for this pseudocode can be written as :

$$\boxed{T(h) \leq \max(T(h_1) + T(h - h_1) + c_1 * N) \quad \forall \quad 0 \leq h_1 \leq h} \quad (2)$$

Please note that $T(h)$ is defined for all N points, but $T(h_1)$, $T(h - h_1)$ are defined for $N/2$ points. Value of h_1 can vary from 0 to N, thus, we need to take the max of all the right recurrences and compare it with the $T(h)$ for N points.

$$\boxed{T(h_1) \leq c * (N/2 + N/2 * \log_2(h_1)) \quad \text{and} \quad T(h - h_1) \leq c * (N/2 + N/2 * \log_2(h - h_1))} \quad (3)$$

Let us substitute these induction hypothesis for $N/2$ points into equation 2.

$$\begin{aligned} T(h) &\leq \max (T(h_1) + T(h - h_1) + c_1 * N) \quad \forall \quad 0 \leq h_1 \leq h \\ \Rightarrow T(h) &\leq \max (c*(N/2 + N/2*\log_2(h_1)) + c * (N/2 + N/2 * \log_2(h - h_1)) + c_1 * N) \quad \forall \quad 0 \leq h_1 \leq h \\ \Rightarrow T(h) &\leq \max ((c+c_1) * N + c * N/2 * (\log_2(h_1 * (h - h_1)))) \quad \forall \quad 0 \leq h_1 \leq h \end{aligned}$$

The $(c + c_1)$ component is independent of h_1 . It can be easily shown through differentiation that the max value of $h_1 * h - h_1$ is achieved at $h_1 = h/2$, which gives the value of $h_1 * h - h_1 = h^2/4$. If h is an odd number, then too the value of $h_1 * h - h_1 < h^2/4$. Thus, we can say that, $h_1 * h - h_1 \leq h^2/4$. $\log_2(x)$ is a monotonically increasing function in x (input). So we can say:

$$\begin{aligned} T(h) &\leq \max ((c+c_1) * N + c * N/2 * (\log_2(h^2/4)) \\ \Rightarrow T(h) &\leq \max ((c+c_1) * N + c * N * (\log_2(h/2)) \\ \Rightarrow T(h) &\leq \max ((c+c_1) * N + c * N * (\log_2(h) - 1)) \\ \Rightarrow T(h) &\leq \max (c_1 * N + c * N * (\log_2(h))) \end{aligned}$$

We can always choose a **c which is greater than** c_1 . On doing so, we get the desired bound:

$$\boxed{T(h) \leq c * N(1 + \log_2(h)) \quad \text{where } c \geq c_1}$$

Thus, our initial hypothesis is true; the time complexity of the proposed algorithm is **$O(N \log h)$** .