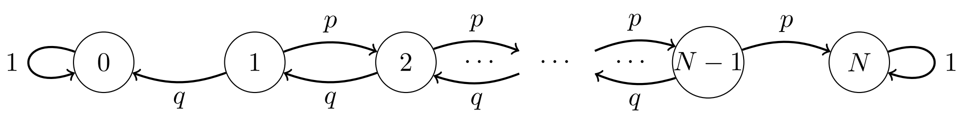
MTL106
Probability and Stochastic Processes

Assignment 2
Ruin to Returns

Deadline : 31st October 2024

Consider a gambler betting on the outcome of a sequence of independent biased coin tosses. If the coin comes up heads with probability p, she adds one dollar to her purse; if the coin lands tails up with probability $q = 1 - p$, she loses one dollar. If she ever reaches a fortune of N dollars, she will stop playing. If her purse is ever empty, then she must stop betting.



1. Let τ denote the time at which the gambler is either ruined (wealth is 0) or wins the game (wealth is N).

   (a) What is the probability that the gambler wins the game given that she starts with an initial wealth of k dollars? [0.5 pt]

   (b) If we let $N \to \infty$ and let the player gamble forever unless she is ruined, what is the probability that she amasses an infinite amount of wealth? [0.5 pt]

   (c) What is the expected value of the number of rounds the gambler plays until she is ruined or wins the game when she starts from an initial wealth of k dollars? $(0 \leq k \leq N)$ [1 pt]

2. The gambler is tired of placing \$1 bets and decides to play aggressively. If she currently has a wealth of $k < N/2$, she bets the entire amount, and either gets ruined with probability q or doubles her wealth with probability p. If she has $k \geq N/2$, she only bets the difference $N - k$. She will win the game with wealth N with probability p, or she will lose her bet of $N - k$ with probability q and continues betting.

   (a) What is the probability of the gambler winning the betting game when she starts with an initial wealth of k? $(0 \leq k \leq N)$ [1 pt]

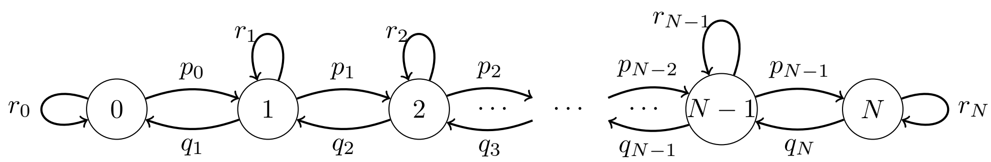
   (b) What is the expected duration of the above game? (The game ends when the gambler is ruined or wins the game) [1 pt]

   Note : For the above 2 problems, you may assume that the ratio of the initial wealth to the maximum wealth k/N has a finite binary decimal expansion. i.e., $k/N = \sum (1/2)^{n_k}$ where the finite increasing sequence $\{n_k\}$ represents the positions of the 1s in the binary representation of k/N

3. The gambler is having a streak of bad luck. She starts with an initial fortune of k and wins or loses each bet with probability p and q, respectively. But her curse is that if she ever has a wealth of m, she is not able to attain a wealth higher than $m+W$. That is, after attaining a wealth of m, if she manages to amass her wealth to $m+W$, she will always lose in the next round. Noticing this, she decides to count her losses and stop playing when her wealth reaches t. How many rounds is she expected to gamble for? $(0 < t < k < N - W)$ [2 pts]

   Note : In this part, it can be shown that the answer can be expressed as an irreducible fraction $\frac{a}{b}$, where p and q are integers and $b \not\equiv 0 (mod\ M)$. Output the integer equal to $a.b^{-1} mod\ M$. In other words, output such an integer x that $0 \leq x < M$ and $x.b \equiv a\ (mod\ M)$. Let $M = 10^9 + 7$.

4. Consider the variations in the price of a particular stock. This can be modeled as a markov chain with a finite state space $\{0, 1, 2, \ldots, N\}$. At the end of each time step, the price of the stock can increase or decrease by at most 1.



   The transition probabilities can be specified by $\{(p_k, r_k, q_k)\}_{k=0}^{N}$, where $p_k + r_k + q_k = 1$ for each k and

   - p_k is the probability of moving from k to $k+1$ when $0 \leq k < n$,
   - q_k is the probability of moving from k to $k-1$ when $0 < k \leq n$,
   - r_k is the probability of remaining at k when $0 \leq k \leq n$,
   - $q_0 = p_n = 0$.

   (a) What is the stationary distribution π across the prices of the stock? What is the expected price of the stock in the steady state? [1 pt]

   (b) What is the expected amount of time it takes for the price of the stock to reach b given that it's initial price is a? $(0 \leq a \leq b \leq N)$ [1 pt]

Remarks

- Clearly show all your calculations wherever necessary and comment your code for readability.
- Any instance of copying from each other/the internet will be penalized heavily. All code will be checked for plagiarism.
- All your code should run within 10 seconds to pass all the testcases.

Submission Format

Complete the implementation of the functions given in the starter code. Do not modify any of the arguments. Submit a pdf file outlining your solution to all of the questions. Do NOT forget to map your solutions to the respective question. For the coding part, submit your code in a .zip file named `EntryNumber_mtl106_a2` containing all the files.