Report: Polyhedral Combinatorics, Matroids and Derandomization of Isolation Lemma

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2024

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Chapter 1

Introduction

1.1 Some Basics of Graph Theory

Definition 1.1.1: Incidence Matrix

For an undirected graph $G = (V, E)$ the Incidence Matrix, M of G is the $|V| \times |E|$ matrix where for every $v \in V$ and $e \in E$, the entry $M[v, e] = 1$ if the edge e is incident on v and otherwise 0

Theorem 1.1.1

If $G = (V, E)$ is an undirected graph with $|V| = n$ then G is connected if and only if $\text{Rank}(M) = n - 1$

Proof: ■

Corollary 1.1.2

If $G = (V, E)$ is an undirected graph with k connected components then $\text{Rank}(M) = n - k$

Definition 1.1.2: Fundamental Cycles

Theorem 1.1.3

The Incidence vectors of the fundamental cycles for a spanning tree in the graph forms a basis of the null space of the incidence matrix

Chapter 2

Matorids

2.1 Matroids

Definition 2.1.1: Matroid

A matroid $M = (E, \mathcal{I})$ has a ground set E and a collection I of subsets of E called the *Independent Sets* st

1. Downward Closure: If $Y \in \mathcal{I}$ then $\forall\ X \subseteq Y$, $X \in \mathcal{I}$.
2. Extension Property: If $X, Y \in \mathcal{I}$, $|X| < |Y|$ then $\exists\ e \in Y - X$ such that $X \cup \{e\}$ also written as $X + e \in \mathcal{I}$

Observation. *A maximal independent set in a matroid is also a maximum independent set. All maximal independent sets have the same size.*

Base: Maximal Independent sets are called bases.
Rank of $S \in I$: We define the rank function of a matroid $r : \mathcal{P}(E) \to \mathbb{Z}$ where $r(S) = \max\{|X| \colon X \subseteq S, X \in I\}$ We def
Rank of a Matroid: Size of the base.
Span of $S \in I$: $\{e \in E \colon rank(S) = rank(S + e)\}$

2.2 Examples of Matroids

Uniform Matroid: It is denoted as $U_{k,n}$ where $E = [n]$ and $I = \{X \subseteq E \mid |X| \leq k\}$.
Free Matroid: When $k = n$ we take all possible subsets of E into I. This matroid is called Free Matroid i.e. $U_{n,n}$
Partition Matroid: Given $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_l$ where $\{E_1, \ldots, E_l\}$ is a partition of E and $k_1, \ldots, k_l \in \mathbb{N} \cup \{0\}$

$$I = \{X \subseteq E \colon |X \cap E_i| \leq k_i\ \forall\ i \in [l]\}$$

then $M = (E, I)$ is a partition matroid.

Note:-

If the E_i's are not a partition then suppose E_1, E_2 has nonempty partition then we will not have a matroid.
For example: $E_1 = \{1, 2\}$, $E_2 = \{2, 3\}$ and $k_1 = k_2 = 1$ then $X = \{1, 3\}$ is independent but $Y = \{2\} \subsetneq X$ is not a matroid.

Linear Matroid: Given a $m \times n$ matrix denote its columns as $A_1, \ldots, A_n$. Then

$$I = \{X \subseteq [n] \colon \text{Columns corresponding to } X \text{ are linearly independent}\}$$

Here if the underlying field is $\mathbb{F}_2$ then it is called *Binary Matroid* and for $\mathbb{F}_3$ it is called *Ternary Matroid*.
Representable Matroid: A matroid with which we can associate a linear matroid is called a representable matroid.

Eg: $U_{2,3}$. It can be represented by the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, over $\mathbb{F}_2$. Over $\mathbb{F}_3$ it is same as $U_{3,3}$.

Note:-

There are matroids which are not representable as linear matroids in some field. There are matroids which are not representable on any field as well.

Lemma 2.2.1

$U_{2,4}$ is not representable over $\mathbb{F}_2$ but representable over $\mathbb{F}_3$

Regular Matroid: There are the matroids which are representable over all fields.

Lemma 2.2.2

Regular Matroids are precisely those which can be represented over $\mathbb{R}$ by a Totally Uni-modular matrix

Graphic Matroid / Cyclic Matroid: For a graph $G = (V, E)$ the graphic matroid $M_G = (E, I)$ where

$$I = \{F \subseteq E \colon F \text{ is acyclic}\}$$

Hence I is the collection of forests of G. It follows the downward closure trivially. For extension property let $k = |F_1| < |F_2| = l$ and then there are $n - k$ and $n - l$ components. So $n - k > n - l$. So $\exists$ an edge in F_2 which joins 2 components in F_1.

Lemma 2.2.3

A subset of columns is linearly independent iff the corresponding edges don't contain a cycle in the incidence matrix

Lemma 2.2.4

Graphic Matroids are Regular Matroids

Proof Idea: Use Incidence Matrix. ■

Matching Matroids: We can try to define it like this but it will not work:

Problem 2.1

Is the following a matroid: $E =$ Edges of a graph and $I = \{F \subseteq E \colon F \text{ is a matching}\}$

Solution: It is not a matroid since maximal matchings can not be extended to a maximum matching.

■

Correct way will be: For a graph $G = (V, E)$ the ground set $= V$ and

$$I = \{S \subseteq V \colon \exists \text{a matching that matches all vertices in } S\}$$

The downward closure property trivially holds. For extension property is $|S| < |S'|$ then there exists another vertex in S' which is not matched with S, so we can add that vertex to S.

2.3 Circuits

Assume we have a matroid $M = (E, I)$.

Definition 2.3.1: Circuit

A minimal dependent set C such that $\forall\, e \in C$, $C - e$ is an independent set.

Theorem 2.3.1

Let $S \in I$. $S + e \notin I$. Then $\exists!\, C \subseteq S + e$.

Proof: Given $S + e \notin I$. Take the set Σ where $T \in \Sigma$ if $t \notin I$ and $T \subseteq S + e$. Σ is nonempty since $S + e \in \Sigma$. Now under the ordering of inclusion T has a minimal element. Hence this minimal element is the desired circuit C which is minimal dependent set contained in $S + e$.

Now suppose it is not unique. Let $C_1, C_2 \subseteq S + e$ be circuits. Suppose $f \in C_1 - C_2$. Then $S - e + f$ will still be dependent since $C_2 \subseteq S - e + f$. Now by definition we get that $C_1 - f$ is independent. Therefore we extend $C_1 - f$ to an independent set by adding the elements of S till we reach same size as $|S|$. Now $e \in C$ since C_1 was formed because of addition of e. Hence if we extend $C_1 - f$ till same cardinality as S we will add all the edges of S not in $C_1 - f$ except f since adding f will make C be a dependent subset of an independent set which is not possible. Hence $C_1 - f$ will be extended to $S - f + e$. Therefore $S + e - f$ is independent which contradicts our previous conclusion that $S + e - f$ is dependent. Hence contradiction. ■

2.4 Axiom Systems for a Matroid

2.5 Finding Max Weight Base

The problem is given a matroid $M = (E, I)$ and a weight function $W : E \to \mathbb{R}$ find the maximum weight base of the matroid. We will solve this using basic greedy algorithm.

2.5.1 Algorithm

Algorithm 1: Algorithm for Finding Max Weight Base

Input: A matroid $M = (E, I)$ is given as an input as an oracle and a weight function $W : E \to \mathbb{R}$.
Output: Find the maximum weight base of the matroid

begin
  Assume $w(1) \geq \cdots \geq w(n)$
  $S \leftarrow \emptyset$
  $I \leftarrow \{S\}$
  for $i = 1$ *to* n **do**
    if $S + i \in I$ **then**
      $S \leftarrow S + i$
  return S

2.5.2 Correctness Analysis and Characterization

Theorem 2.5.1

The above algorithm outputs a maximum weight base iff M is a matroid

Proof: $\Leftarrow$:
Let M be a matroid. We will prove that this greedy algorithm works by inducting on i. At any iteration i we need to prove the following claim:

Claim: At any iteration i there is a max weight base B_i such that $S_i \subseteq B_i$ and $B_i \setminus S_i \subseteq \{i+1,\dots,n\}$.
Proof: Base case: $S=\emptyset$. So for base case the statement is true trivially. Assume that the statement is true up to $(i-1)$ iterations.

Now $S_{i-1} \subseteq B_{i-1}$ where B_{i-1} is a maximum weight base and $B_{i-1} - S_{i-1} \subseteq \{i,\dots,n\}$. Now three cases arise:

Case 1: If $i \in B_{i-1}$ then $S_{i-1}+i \subseteq B_{i-1}$. Therefore $S_{i-1}+i$ is independent. So now $B_i = B_{i-1}$ and $S_i = S_{i-1}+i$ and $B_i - S_i \subseteq \{i+1,\dots,n\}$.

Case 2: If $i \notin B_{i-1}$ and $S_{i-1}+i \notin I$. Then $S_i = S_{i-1}$ and $B_i = B_{i-1}$. And $B_i - S_i \subseteq \{i+1,\dots,n\}$.

Case 3: If $i \notin B_{i-1}$ but $S_{i-1}+i \in I$. Then $S_i = S_{i-1}+i$. Now S_i can be extended to a B' by adding all but one element of B_{i-1}. So $|B'| = |B_{i-1}|$. Let the element which is not added is $j \in B_{i-1}$. So $B' = B_{i-1}+i-j$.

$$wt(B') = Wt(B_{i-1}) - wt() + wt(i)$$

But we have $wt(i) \geq wt(j)$. So $wt(B') \geq wt(B_{i-1})$. Now since B_{i-1} has maximum weight we have $wt(B') = wt(B_{i-1})$. Then our $B_i = B'$. So $B_i - S_i \subseteq \{i+1,\dots,n\}$.

Hence the claim is true for the ith stage as well. Therefore the claim is true.

■

Therefore using the claim, after the algorithm finished we have no elements left to check, so the current set has the maximum weight which is also an independent set. So the algorithm successfully returns a maximum weight base.
$\Rightarrow$:
Assume M is not a matroid.

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2.6 Some Matroid Properties

2.6.1 Strong Base Exchange Property

2.6.2 Exchange Graph of a Matroid wrt $S \in I$

Chapter 3

Basic Linear Programming

3.1 Totally Unimodular Matrix

3.2 Polytope, Face, Vertex

Chapter 4

Perfect Matching Polytope

Chapter 5

Bipartite Perfect Matching

Chapter 6

Bibliography