

### Problem 1

- (a) Prove that if  $A_1, A_2, \dots, A_n$  are events, then

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = S_1 - S_2 + S_3 - \dots + (-1)^{n-1} S_n$$

where

$$S_1 = \sum_i \mathbb{P}(A_i)$$

$$S_2 = \sum_{i < j} \mathbb{P}(A_i \cap A_j)$$

$$S_3 = \sum_{i < j < k} \mathbb{P}(A_i \cap A_j \cap A_k)$$

...

$$S_n = \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n)$$

This is also known as the *inclusion-exclusion* principle.

- (b) *Bonferroni inequalities* state that the sum of the first terms in the right-hand side of the identity we proved above is alternately an upper bound and a lower bound for the left-hand side. i.e., for odd  $k \leq n$ ,

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \leq S_1 - S_2 + \dots + S_k$$

and for even  $k \leq n$

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \geq S_1 - S_2 + \dots - S_k$$

Note that from what we showed above Bonferroni inequality holds with equality for  $k = n$ .

Prove Bonferroni inequalities. Observe that the case of  $k = 1$  is what you know as the *union bound* or Boole's inequality.

### Solution:

- (a) We will prove it using induction on  $n$ . For base case  $t = 1$ . Then  $\mathbb{P}[A_1] = S_1 = \sum_i \mathbb{P}[A_i] = \mathbb{P}[A_1]$ . Hence for base case it holds. Now let this is true for  $t = n$ . For  $t = n + 1$

$$\mathbb{P}\left(\bigcup_{i=1}^{k+1} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^k A_i\right) + \mathbb{P}\left(A_{k+1} \setminus \bigcup_{i=1}^k A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^k A_i\right) + \mathbb{P}(A_{k+1}) - \mathbb{P}\left(\bigcup_{i=1}^k (A_i \cap A_{k+1})\right)$$

Now using inductive hypothesis we have

$$\begin{aligned} \mathbb{P}\left(\bigcup_{i=1}^k (A_i \cap A_{k+1})\right) &= \sum_{t=1}^k (-1)^{t-1} \sum_{J \subseteq [k], |J|=t} \mathbb{P}\left[\bigcap_{i \in J} (A_i \cap A_{k+1})\right] \\ &= \sum_{t=1}^k (-1)^{t-1} \sum_{J \subseteq [k], |J|=t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i \in J} A_i\right)\right] \end{aligned}$$

Therefore we have

$$\begin{aligned}
& \mathbb{P}\left(\bigcup_{i=1}^k A_i\right) + \mathbb{P}(A_{k+1}) - \mathbb{P}\left(\bigcup_{i=1}^k (A_i \cap A_{k+1})\right) \\
&= \mathbb{P}\left(\bigcup_{i=1}^k A_i\right) + \mathbb{P}(A_{k+1}) - \left[ \sum_{t=1}^k (-1)^{t-1} \sum_{J \subseteq [k], |J|=t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i \in J} A_i\right)\right] \right] \\
&= \sum_{t=1}^k (-1)^{t-1} \sum_{T \subseteq [k], |T|=t} \mathbb{P}\left[\bigcap_{i \in T} A_i\right] + \mathbb{P}[A_{k+1}] + \sum_{t=1}^k (-1)^t \sum_{J \subseteq [k], |J|=t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i \in J} A_i\right)\right] \\
&= \sum_{i=1}^{k+1} \mathbb{P}[A_i] + \sum_{t=1}^k (-1)^t \sum_{T \subseteq [k], |T|=t+1} \mathbb{P}\left[\bigcap_{i \in T} A_i\right] + \sum_{t=1}^k (-1)^t \sum_{J \subseteq [k], |J|=t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i \in J} A_i\right)\right] \\
&= \sum_{i=1}^{k+1} \mathbb{P}[A_i] + \sum_{t=1}^k (-1)^t \left( \sum_{T \subseteq [k], |T|=t+1} \mathbb{P}\left[\bigcap_{i \in T} A_i\right] + \sum_{J \subseteq [k], |J|=t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i \in J} A_i\right)\right] \right) \\
&= \sum_{i=1}^{k+1} \mathbb{P}[A_i] + \sum_{t=1}^k (-1)^t \sum_{T \subseteq [k+1], |T|=t+1} \mathbb{P}\left[\bigcap_{i \in T} A_i\right] \\
&= \sum_{t=1}^{k+1} (-1)^{t-1} \sum_{T \subseteq [k+1], |T|=t} \mathbb{P}\left[\bigcap_{i \in T} A_i\right]
\end{aligned}$$

□

### Problem 2

Prove or disprove the following:

- The conditional independence of  $A$  and  $B$  given  $C$  implies  $A$  and  $B$  are independent.
- Independence of  $A$  and  $B$  implies the conditional independence of  $A$  and  $B$  given  $C$ .

If you disproved either of the claims above, for which events  $C$  is it then the case that the following statement holds: for all events  $A$  and  $B$ , the events  $A$  and  $B$  are conditionally independent given  $C$  if and only if  $A$  and  $B$  are independent.

### Solution:

1. We will disprove both of the statements by constructing a counter example.

- Consider we have two decks of cards. Now in the from the first deck we pick a card. If it is a face card then we pick a card uniformly from all non-face cards in the second deck. And if the picked card from the first deck is a non-face card then we pick a card uniformly at random from all non-numbered cards in the second deck. Here the aces comes into both non-numbered cards and non-face cards. So now let
  - $A$  be the event of picking 'King' in the first deck
  - $B$  be the event of picking 'Ace' in the second deck
  - $C$  be the event of picking 'Jack' in the first deck

Now  $\mathbb{P}[A | C] = 0$  and  $\mathbb{P}[B | C] = \frac{4}{40} = \frac{1}{10}$  and

$$\mathbb{P}[A \cap B | C] = \mathbb{P}[\text{Picking 'King', 'Ace'} | \text{Picking 'Jack' in first deck}] = 0 = \mathbb{P}[A | C] \mathbb{P}[B | C]$$

So  $A, B$  are independent conditioned on  $C$ . Now  $\mathbb{P}[A] = \frac{4}{52} = \frac{1}{13}$ ,  $\mathbb{P}[B] = \frac{12}{52} \frac{4}{40} + \frac{40}{52} \frac{4}{16} = \frac{3}{130} + \frac{5}{26} = \frac{14}{65}$ .

$$\mathbb{P}[A \cap B] = \frac{4}{52} \frac{4}{40} = \frac{3}{130} \neq \mathbb{P}[A] \mathbb{P}[B]$$

- Let us have two unbiased 6-faced dice. We throw both the dice. Let
  - $A$  be the event that first dice outcome is 2
  - $B$  be the event that second dice outcome is 5.
  - $C$  be the event that the sum of first dice outcome and second dice outcome is 6

Then  $\mathbb{P}[A] = \mathbb{P}[B] = \frac{1}{6}$ . And  $\mathbb{P}[A \cap B] = \frac{1}{36}$  since  $(2, 5)$  is one outcome of all 36 possible outcomes. Hence  $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$ . So  $A, B$  are independent events. Certainly  $\mathbb{P}[C] > 0$ . Then  $\mathbb{P}[A | C], \mathbb{P}[B | C] \neq 0$ . But the  $\mathbb{P}[A \cap B | C] = 0$  since  $2 + 5 \neq 6$ . Hence  $\mathbb{P}[A \cap B | C] \neq \mathbb{P}[A | C]\mathbb{P}[B | C]$ .

2.

□

### Problem 3

Let  $A_1, A_2, \dots$  be a sequence of events. Define

$$B_n = \bigcup_{m=n}^{\infty} A_m \quad C_n = \bigcap_{m=n}^{\infty} A_m$$

Clearly  $C_n \subseteq A_n \subseteq B_n$ . Also, the sequences  $\{B_n\}$  and  $\{C_n\}$  are decreasing respectively. Let

$$B = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m \quad C = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} A_m$$

The events  $B$  and  $C$  are denoted by  $\limsup_{n \rightarrow \infty} A_n$  and  $\liminf_{n \rightarrow \infty} A_n$  respectively. Show that

- $B = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$ .
- $C = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$ .

We say that a sequence  $\{A_n\}$  converges to a limit  $A$  if  $B$  and  $C$  are the same set  $A$ . We denote this by  $A_n \rightarrow A$ . Suppose this is the case, then show that

- $A$  is an event.
- $\mathbb{P}(A_n) \rightarrow \mathbb{P}(A)$ .

### Solution:

- Let  $\omega \in B$ . Then  $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m$ . Hence  $\omega \in \bigcup_{m \geq n} A_m$  for all  $n \in \mathbb{N}$ . Hence  $\omega \in A_k$  for some  $k \in \mathbb{N}$ . Let  $k_1$  be the least number such that  $\omega \in A_{k_1}$ . Then we also have  $\omega \in B_{k_1+1}$ . So we have some  $k_2 \geq k_1 + 1$  such that  $\omega \in A_{k_2}$ . Then  $\omega \in B_{k_2+1}$ . So there exists  $k_3 \geq k_2 + 1$  such that  $\omega \in A_{k_3}$ . Continuing like this at  $i^{th}$  step we have some  $k_{i+1} \geq k_i + 1$  such that  $\omega \in A_{k_{i+1}}$  and so on. So now we got an strictly increasing infinite sequence of positive integers  $\{k_1, k_2, k_3, \dots, k_i, \dots\}$  such that  $\omega \in A_{k_j}$  for all  $j \in \mathbb{N}$ . Hence  $\omega \in \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$ . Hence

$$B \subseteq \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$$

Now let  $\omega \in \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$ . Let  $\{s_n\}_{n \in \mathbb{N}}$  be the strictly increasing sequence of positive integers such that  $\omega \in A_{s_n}$ . Hence for all  $m \in \mathbb{N}$  we have  $\omega \in B_m$  because  $\exists n \in \mathbb{N}$  such that  $s_n > m$  and  $\omega \in A_{s_n} \implies \omega \in B_m$ . Therefore  $\omega \in \bigcap_{m=1}^{\infty} B_m$ . Therefore we have

$$\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\} \subseteq B$$

Hence we have  $B = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$ .

- (b) Let  $\omega \in \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$ . Hence there exists  $n_0 \in \mathbb{N}$  such that  $\omega \in A_n$  for all  $n > n_0$ . Therefore  $\omega \in C_n$  for all  $n > n_0$ . Since  $C = \bigcup_{n=1}^{\infty} C_n$  we have  $\omega \in C$ . So we have

$$\{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\} \subseteq C$$

Now suppose  $\omega \in C$ . So  $\exists n \in \mathbb{N}$  such that  $\omega \in C_n$ . Since  $C_n = \bigcap_{m \geq n} A_m$  we have  $\omega \in A_m$  for all  $m \geq n$ . Hence  $\omega \in A_n$  for all but finitely many values of  $n$ . So  $\omega \in \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$ . Hence we get

$$C \subseteq \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$$

Therefore we get  $C = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$ .

- (c) For all  $n \in \mathbb{N}$   $B_n$  is the countable union of events. So  $B_n$  is an event for all  $n \in \mathbb{N}$ . And similarly  $\forall n \in \mathbb{N}$ ,  $C_n$  is the countable intersection of events. Therefore  $C_n$  is also an event. Now since  $B$  is just countable intersection of all  $B_n$ 's and each  $B_n$  is event we have that  $B$  is also an event. And similarly since  $C$  is just the countable union of all  $C_n$ 's and each  $C_n$  is an event we have that  $C$  is also an event. Now given that  $B = C = A$ . Therefore  $A$  is also an event.
- (d) Since for each  $n \in \mathbb{N}$  we have that  $C_n \subseteq A_n \subseteq B_n$ . Therefore

$$\mathbb{P}[C_n] \leq \mathbb{P}[A_n] \leq \mathbb{P}[B_n]$$

Hence we have

$$\lim_{n \rightarrow \infty} \mathbb{P}[C_n] \leq \lim_{n \rightarrow \infty} \mathbb{P}[A_n] \leq \lim_{n \rightarrow \infty} \mathbb{P}[B_n]$$

Now we will analyze  $\lim_{n \rightarrow \infty} \mathbb{P}[B_n]$  and  $\lim_{n \rightarrow \infty} \mathbb{P}[C_n]$ . Now we have

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots \supseteq B_n \supseteq \dots \quad \text{and} \quad C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots \subseteq C_n \subseteq \dots$$

$$\mathbb{P}[B] = \mathbb{P}\left[\bigcap_{n=1}^{\infty} B_n\right] = \mathbb{P}\left[\lim_{k \rightarrow \infty} \bigcap_{n=1}^k B_n\right] = \lim_{k \rightarrow \infty} \mathbb{P}\left[\bigcap_{n=1}^k B_n\right] = \lim_{k \rightarrow \infty} \mathbb{P}[B_k]$$

Similarly we have

$$\mathbb{P}[C] = \mathbb{P}\left[\bigcup_{n=1}^{\infty} C_n\right] = \mathbb{P}\left[\lim_{k \rightarrow \infty} \bigcup_{n=1}^k C_n\right] = \lim_{k \rightarrow \infty} \mathbb{P}\left[\bigcup_{n=1}^k C_n\right] = \lim_{k \rightarrow \infty} \mathbb{P}[C_k]$$

Hence we get  $\lim_{n \rightarrow \infty} \mathbb{P}[B_n] = \mathbb{P}[B]$  and  $\lim_{n \rightarrow \infty} \mathbb{P}[C_n] = \mathbb{P}[C]$ . Since  $B = C$  we have

$$\lim_{n \rightarrow \infty} \mathbb{P}[B_n] = \mathbb{P}[B] = \mathbb{P}[C] = \lim_{n \rightarrow \infty} \mathbb{P}[C_n]$$

And since  $A = B = C$  we have  $\mathbb{P}[B] = \mathbb{P}[A] = \mathbb{P}[C]$ . Hence

$$\lim_{n \rightarrow \infty} \mathbb{P}[C_n] \leq \lim_{n \rightarrow \infty} \mathbb{P}[A_n] \leq \lim_{n \rightarrow \infty} \mathbb{P}[B_n] \implies \mathbb{P}[A] = \mathbb{P}[B] \leq \lim_{n \rightarrow \infty} \mathbb{P}[A_n] \leq \mathbb{P}[C] = \mathbb{P}[A]$$

Therefore  $\lim_{n \rightarrow \infty} \mathbb{P}[A_n] = \mathbb{P}[A]$

□

#### Problem 4

10% of the surface of a sphere is colored white, the rest is black. Show that, irrespective of the manner in which the colors are distributed, it is possible to inscribe a cube in  $S$  with all its vertices black.

**Hint:** For a given distribution of colors, select the cube “uniformly randomly” (you should make this more

concrete). First note that it is enough to prove that there is a non-zero probability with which all the vertices of this random cube are colored black (why?). Now try to use the union bound from Problem 1(b) above to show this.

**Solution:** To show that there exists a cube in  $S$  with all its vertices black it is enough to show that if a random cube is chosen in  $S$  the probability of all vertices black is greater than 0. Now we have

$$\mathbb{P}_{\substack{C: \text{cube} \\ C \text{ is in } S}} [\text{All vertices of } C \text{ is black}] = 1 - \mathbb{P}_{\substack{C: \text{cube} \\ C \text{ is in } S}} [\text{At least one of the vertices of } C \text{ is white}]$$

So its is enough to show that  $\mathbb{P}_{\substack{C: \text{cube} \\ C \text{ is in } S}} [\text{At least one of the vertices of } C \text{ is white}] < 1$ . Now we also have

$$\mathbb{P}_{\substack{C: \text{cube} \\ C \text{ is in } S}} [\text{At least one of the vertices of } C \text{ is white}] = \mathbb{P}_{\substack{X_i \in S \\ \forall i \in [8]}} [\exists i \in [8] X_i \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}]$$

Now by Union Bound we have

$$\begin{aligned} \mathbb{P}_{\substack{X_i \in S \\ \forall i \in [8]}} [\exists i \in [8] X_i \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}] \\ \leq \sum_{j=1}^8 \mathbb{P}_{\substack{X_i \in S \\ \forall i \in [8]}} [X_j \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}] \end{aligned}$$

So now showing

$$\sum_{j=1}^8 \mathbb{P}_{\substack{X_i \in S \\ \forall i \in [8]}} [X_j \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}] < 1$$

is enough. Now for any  $j \in [8]$ ,

$$\mathbb{P}_{\substack{X_i \in S \\ \forall i \in [8]}} [X_j \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}] = \mathbb{P}_{\substack{X_i \in S \\ \forall i \in [8]}} [X_j \text{ is colored white}] = \frac{1}{10}$$

The last equality because  $X_j$  is colored white if it is a point picked from the 10% area of the sphere which is colored white and the probability of that is  $\frac{1}{10}$ . Therefore we have

$$\sum_{j=1}^8 \mathbb{P}_{\substack{X_i \in S \\ \forall i \in [8]}} [X_j \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}] = \sum_{j=1}^8 \frac{1}{10} = \frac{8}{10} < 1$$

Therefore we have  $\mathbb{P}_{\substack{C: \text{cube} \\ C \text{ is in } S}} [\text{At least one of the vertices of } C \text{ is white}] < 1 \implies \mathbb{P}_{\substack{C: \text{cube} \\ C \text{ is in } S}} [\text{All vertices of } C \text{ is black}] > 0$ . Which means there exists a cube in  $S$  with all vertices black

□