Report: Polyhedral Combinatorics, Matroids and Derandomization of Isolation Lemma

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Chapter 1

Some Basics of Graph Theory

First we will introduce some graph properties and results which will help us in later chapters.

1.1 Incidence Matrix

Definition 1.1.1: Incidence Matrix

For an undirected graph $G = (V, E)$ the Incidence Matrix, M of G is the $|V| \times |E|$ matrix where for every $v \in V$ and $e \in E$, the entry $M[v, e] = 1$ if the edge e is incident on v and otherwise 0

Theorem 1.1.1

If $G = (V, E)$ is an undirected graph with $|V| = n$ then G is connected if and only if $\mathrm{Rank}(M) = n - 1$ over $\mathbb{F}_2$.

Proof: ■

Corollary 1.1.2

If $G = (V, E)$ is an undirected graph with k connected components then $\mathrm{Rank}(M) = n - k$

Proof: content... ■

Definition 1.1.2: Fundamental Cycles

Theorem 1.1.3

The Incidence vectors of the fundamental cycles for a spanning tree in the graph forms a basis of the null space of the incidence matrix

Proof: content... ■

1.2 Matching

Theorem 1.2.1 Hall's Condition

content...

Proof: content... ■

Lemma 1.2.2

Every Regular bipartite graph is union of perfect matchings.

Proof: We will induct on degree. A regular bipartite graph satisfies Hall's Condition. Therefore it has a perfect matching. So we will obtain a new regular graph of lower degree after removing the perfect matching. By induction hypothesis it must a union of perfect matchings. Hence we get that the original graph was in fact union of perfect matchings. ■

1.3 Nice Cycles and Circulation

Let $G = (V, E)$ be a graph with a perfect matching.

Definition 1.3.1: Nice Cycle

A cycle C in G is a nice cycle if it has even length and the subgraph $G - C$ still has a perfect matching

In other words a nice cycle can be obtained from the symmetric difference of two perfect matchings.

Now suppose we have a weight function $w\colon E \to \mathbb{R}$ on the edges of a graph G. Let we have an even length cycle $C = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{2k-2}} v_{2k} \xrightarrow{e_{2k-1}} v_0$ in G for some $k \in \mathbb{N}$.

Definition 1.3.2: Circulation of Cycle

For a weight assignment w on the edges the circulation $c_w(C)$ of an even length cycle is defined the alternating sum of the edge weights of C i.e.

$$c_w(C) = \left|\sum_{i=0}^{2k-1} (-1)^i w(e_i)\right|$$

The definition of circulations is independent of the edge we start with because we take the absolute value of the alternating sum. Below we show a property for cycles in a graph having nonzero circulations lead to a unique minimum weight perfect matching.

Lemma 1.3.1 [DKR09, Lemma 3.2]

Let G be a graph with a perfect matching, and let w be a weight function such that all nice cycles in G have nonzero circulations. Then the minimum perfect matching is unique i.e. w is isolating

Proof: Suppose not, then we have two minimum weight perfect matchings M_1 and M_2 with minimum weight w.r.t w. Now we take their disjoint union $M_1 \sqcup M_2$ i.e. if there is an common edge then we take two copies of that edge connecting same two vertices. Now it is a cycle cover of the vertices with nice cycles except the one's with copies.

Consider any one nice cycle from the cycle cover. We will form a new perfect matching M. Since the circulation of an nice cycle is nonzero either the part of it which is in M_1 is lighter or the part of it which is in M_2 is lighter. Either way we take the lighter part in M and we do this for all . So we take the part from M_1 from this cycle. Now we do this for all the nice cycles in the cycle cover. Now for the cycles with two copies of same edge we take one of them into M. Now since $M_1 \neq M_2$ there exists at least one edge in M_1 which is not in M_2 and one edge in M_2 which is not in M_1.

Hence $M_1 \sqcup M_2$ has at least one nice cycle, hence the way we constructed $w(M) < w(M_i)$ for some $i \in \{1, 2\}$ which contradicts the minimality of both M_1 and M_2 ■

Chapter 2

Perfect Matching Polytope

2.1 Matching Polytope

2.2 Perfect Matching Polytope

Definition 2.2.1: Perfect Matching Polytope

Let $G = (V, E)$ be a graph. For any perfect matching M of G, consider the incidence vector $x^M = (x_e)_{e \in E} \in \mathbb{R}^E$ given by

$$c_e^M = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{o/w} \end{cases}$$

For any perfect matching M of G this vector x^M is called as a *Perfect Matching Point*. The bipartite perfect matching polytope of the graph G is defined to the convex hull of all its perfect matching points,

$$PM(G) = \text{Conv}\{x^M \mid M \text{ is a perfect matching in } G\}$$

2.3 Bipartite Perfect Matching Polytope

It also defined like the perfect matching polytope where we just take the graph to be a bipartite graph. The following lemma form [LP86] gives a simple description of the perfect matching polytope of a bipartite graph G

Theorem 2.3.1 [LP86]
Let $G = (V, E)$ be a bipartite graph and $x = (x_e)_{e \in E} \in \mathbb{R}^E$. Then $x \in PM(G)$ if and only if

$$\sum_{e \in \delta(v)} x_e = 1 \quad v \in V,$$
$$x_e \geq 0 \quad e \in E$$

where for any $v \in V$, $\delta(v)$ denotes the set of edges incident on the vertex v.

Chapter 3

Bipartite Perfect Matching

3.1 Matching and Complexity

3.2 A RNC Algorithm for Search-PM

3.3 A Quasi-NC Algorithm using Isolation

Let $G = (V, E)$ be given bipartite graph. In the following discussion we will assume that G has perfect matchings. Our goal is to isolate one of the perfect matchings in G by any appropriate weight function. We will also show that if G does not have any perfect matchings then our algorithm will detect this.

We will construct an isolating weight function for bipartite graphs. The idea is to create a weight function which ensures nonzero circulations for a small set of cycles in a black-box way i.e. without having being able to compute the set efficiently. Then we will show that if we construct a smaller graph wrt this weight function then we don't have those small cycles with nonzero circulations then we have the number of cycles with twice the size of the previous ones are polynomially bounded. Then we proceed to create a new weight function which will give nonzero circulations to all the cycles with twice the size. And this way we will continue. This same type of idea we will repeatedly use with necessary modifications in **??** and **??**.

The idea above to create a weight function which gives nonzero circulation to every nice cycles in G actually works because then we have unique perfect matching by Lemma 1.3.1

3.3.1 Isolating Small Cycles

The following lemma describes a standard trick to create a weight function for a small set of cycles in graph.

Lemma 3.3.1 [CRS93]
Let G be a graph with n vertices. Then for any number s, one can construct a set of $O(n^2 s)$ weight assignments with weights bounded by $O(n^2 s)$, such that for any set of s cycles, one of the weight assignments gives nonzero circulation to each of the s cycles.

Proof: Let us first assign exponentially large weights. Let $e_1, e_2, \ldots, e_m$ be some enumeration of the edges of G. Define a weight function w by $w(e_i) = 2^{i-1}$ for $i \in [m]$. Then clearly every cycle has a nonzero circulation. However we want to achieve this with small weights.

We consider the weight assignment modulo small numbers i.e. the weight function is $\{w \bmod j \mid 2 \le j \le t\}$ for some appropriately chosen t. We want to show that for any fixed set of s cycles $\{C_1, \ldots, C_s\}$ one of these assignments will work when t is chosen large enough.

Now we want

$$\exists\, j \le t,\ \forall\, i \le s,\ c_w(S_i) \ne 0 \iff \exists\, j \le t,\ \prod_{i=1}^{s} c_w(C_i) \ne 0 \bmod j$$

In other words we want

$$lcm(2,3,\ldots,t) \nmid \prod_{i=1}^{t} c_w(C_i)$$

Hence if we take t such that $lcm(2,3,\ldots,t) > \prod_{i=1}^{t} c_w(C_i)$ then we are done.

Now the product $\prod_{i=1}^{t} c_w(C_i)$ is bounded by $2^{n^2 s}$. This is because with exponential weights like in the RNC algorithm we have an isolating perfect matching so we need weights less than that and therefore the new weights are bounded by the exponential weights for which weight of a cycle can at most be 2^{n^2} and since there are s many cycle we have the bound $2^{n^2 s}$. So if we have t such that $lcm(2,3,\ldots,t) > 2^{n^2 s}$ then we are done. Now $lcm(2,3,\ldots,t) > 2^t$ for $t \geq 7$. Thus choosing $t = n^2 s$ suffices. Clearly the weights are bounded by $t = n^2 s$. ■

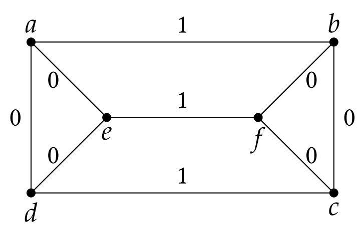
3.3.2 Union of Minimum Weight Perfect Matchings

Let us assign a weight function for bipartite graph G which gives nonzero circulations to all small cycles. Consider a new graph G_1 which obtained by the union of minimum weight perfect matchings in G. Out hope is that G_1 is significantly smaller than G.

Note:-

We don't know if G_1 can be efficiently created from G as determinant of the bi-adjacency matrix with weights in the like in the RNC algorithm be zero and therefore we can not use that way to obtain perfect matchings. We will show we don't need to construct G_1. It is just used n the argument. Our final weight assignment will be completely black-box

We will also show by the following lemma that why this technique only works for biparitte graphs, not in general graphs i.e. G_1 constructed from minimum weight perfect matchings in G contains no other prefect matching than these. For general graph this does not hold:



In this graph we will denote the edge connected vertices a, b to be e_{ab}. And this way we will denote all the edges. Then the minimum weight perfect matchings have weight 1 and they are

$$\{e_{ad}, e_{bc}, e_{ef}\}, \quad \{e_{ac}, e_{bf}, e_{cd}\}, \quad \{e_{de}, e_{cf}, e_{ab}\}$$

Then their union has the perfect matching $\{e_{ab}, e_{cd}, e_{ef}\}$ which has weight 3 and not a minimum weight perfect matching.

The fact that G_1 has only minimum weight perfect matching is equivalent to saying that every nice cycle has zero circulation. The following lemma proves even stronger statement that every cycles has zero circulation (not necessarily nice cycles.)

Lemma 3.3.2 [FGT16, Lemma 3.2]

Let $G = (V, E)$ be a bipartite graph with weight function w. et C be a cycle in G such that $c_w(C) \neq 0$. Let E_1 be the union of all minimum weight perfect matchings in G. Then the graph $G_1 = (V, E_1)$ does not contain C.

Proof: Consider the perfect matching polytope of G, $PM(G)$. Let the weight of the minimum weight perfect matching in G be q. Let $x_1, x_2, \ldots, x_t$ be all the minimum weight perfect matching points of G i.e. corners of $PM(G)$ corresponding to weight q. Consider the average point $x \in PM(G)$ of these perfect matching points, $x = \frac{x_1+x_2+\cdots+x_t}{t}$. Clearly we have $w(x) = q$. And since each edge of G_1 participates in a minimum weight perfect matching for $x = (x_e)_{e\in E}$ we have $x_e \neq 0 \; \forall\, e \in E$.

Now consider a cycle C with $c_w(C) \neq 0$. Suppose $C = (e_1, e_2, \ldots, e_k)$ and all the edges of C are in E_1. We will show that if we move from point x along the cycle C we reach a point in $PM(G)$ with a weight smaller than q.

Consider the point y defined as

$$\forall\, e \in E, \quad y_e = \begin{cases} x_e + (-1)^i \epsilon\text{\&if } e = e_i \text{ for some } i \in [k] \\ x_e \qquad\qquad\qquad\qquad\qquad\qquad \text{o/w} \end{cases}$$

for some $\epsilon \neq 0$. Clearly $x - y$ has nonzero coordinates only on the edges of the cycle C, by alternating between ϵ and $-\epsilon$. Hence

$$w(x) - w(y) = w(x - y) = \pm c_w(C) \neq 0$$

Now we take ϵ in the following way:

- Take $sgn(\epsilon)$ such that $w(x - y) > 0$.
- Take ϵ small enough such that $y_e \geq 0 \; \forall \; e \in E$.

After choosing such ϵ since $w(x) - w(y) = w(x - y) > 0$ we have $q = w(x) > w(y)$. Now we will show that $y \in PM(G)$. To show that we will sow that y fulfills the conditions of Theorem 2.3.1. Now the second condition that $y_e \geq 0$ for all $e \in E$ is already satisfied by the choice of ϵ. So we only need to show that for any $v \in V$

$$\sum_{e \in \delta(v)} y_e = 1$$

To show this we consider 2 cases:

Case 1: $v \notin C$. Then $\forall \; e \in \delta(v)$ we have $e \notin C$. So $y_e = x_e$. Since $x \in PM(G)$ we have

$$\sum_{e \in \delta(v)} x_e = 1 \implies \sum_{e \in \delta(v)} y_e = 1$$

Case 2: $v \in C$. Let e_j and e_{j+1} are the edges incident on v in C. Then

$$y_{e_j} = x_{e_j} + (-1)^j \epsilon \qquad \text{and} \qquad y_{e_{j+1}} = x_{e_{j+1}} + (-1)^{j+1} \epsilon \qquad \forall \; e \in \delta(v) - \{e_j, e_{j+1}\},\; y_e = x_e$$

So

$$\begin{aligned}
\sum_{e \in \delta(v)} y_e &= \left[\sum_{e \in \delta(v) - \{e_j, e_{j+1}\}} x_e \right] + \left[x_{e_j} + (-1)^j \epsilon \right] + \left[x_{e_{j+1}} + (-1)^{j+1} \epsilon \right] \\
&= \left[\sum_{e \in \delta(v)} x_e \right] + (-1)^j \epsilon + (-1)^{j+1} \epsilon = \sum_{e \in \delta(v)} x_e = 1
\end{aligned}$$

So the point y satisfies the property $\sum\limits_{e \in \delta(v)} = 1$ forall $v \in V$. Hence $y \in PM(G)$. Now since $w(y) < q$ there must be a corner point of the polytope which corresponds to a perfect matching in G with weight less than q. This contradicts the minimality of q. Hence C is not in G_1. ■

This technique of moving along the cycle has been used by Mahajan and Varadarajan in [MV00]. Now We will show another proof of this lemma by Rao, Shpilka and Wigderson in [GG17].

Alternate Proof *[GG17, Proof of Lemma 6]:* Let G' be the multigraph obtained by taking disjoint union of all minimum weight perfect matchings (i.e. if an edge appears in k many minimum weight perfect matchings of G then G' contains k copies of the edge.

Note:-

G' is a regular graph since it is a disjoint union of matchings and matchings are regular graph of degree 1.

Suppose there exists a cycle C of non zero circulation in G_1. Since the cycle is in G_1 then this cycle is also in G'. WLOG assume that the sum of the weights of the odd edges of C is larger than the sum of the weights of the even edges. Then we can remove a single copy of each odd edges of C from G' and add a single copy of each even edges of C to G' and we call this new graph G''

Suppose q be the minimum weight of a matching in G. Suppose G has d minimum weight matchings. Then sum of the weights of the edges in G' is qd. However, the total weight of all edges in in G'' is lower than the total weight of all edges in G'. We know that G'' is a regular bipartite graph of degree d and therefore by Lemma 1.2.2 it is an union of d perfect matchings.

If we decompose G'' into d perfect matchings, it is impossible that they all have weight at least q as G'' has total weight less than qd. Therefore G'' has a matching of weight less than q, which contradicts the minimality of q. ■

A consequence of this lemma is that G_1 has no other perfect matchings than the ones used to define G_1 cause if M_0 and M_1 be two different perfect matchings in G_1 then $M_0 \triangle M_1$ forms a set of nice cycles and by the Lemma 3.3.2 the circulations all of these cycles are 0 and therefore M_0 and M_1 have same weight and hence they both are minimum weight perfect matchings.

Corollary 3.3.3

Let $G = (V, E)$ be a bipartite graph with weight function w. Let E_1 be the union of all minimum weight perfect matchings in G. Then every perfect matching in the graph $G_1 = (V, E_1)$ has the same weight, the minimum weight of any perfect matching in G.

3.3.3 Constructing Weight Assignment

By our weight function in Lemma 3.3.1 each small cycles in G has a nonzero circulations. Hence by Lemma 3.3.2 G_1 has no small cycles. Now we want to repeat this procedure with G_1 with a new weight function. G_1 has no small cycles. Hence we look at slightly larger cycles (twice larger) and we will argue that their number remains polynomially bounded.

Teo and Kow in [TK92] showed that the number of shortest cycles in a graph with m edges is bounded by m^2. In te following lemma we extend their argument and give a bound on the number of cycles that have length at most twice the lengthof shortest cycles.

Lemma 3.3.4 [FGT16, Lemma 3.4]

Let $G = (V, E)$ be a graph with n nodes that has no cycles of length $\leq r$. Let $r' = 2r$ if r is even and $r' = 2r - 2$ otherwise. Then H has $\leq n^4$ cycles of length $\leq r'$.

Proof: Let

$$C = v_0 \overrightarrow{e_0} v_1 \overrightarrow{e_1} \cdots \overrightarrow{e_{l-2}} v_{l-1} \overrightarrow{e_{l-1}} v_1$$

be a cycle of length $l \leq r'$ in G. Now we successively choose ■

Chapter 4

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