

Bounding PoA using LP, QP and Fenchel Duality

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April 2025

Introduction

- **Pure Nash Equilibria:** A strategy profile $s \in S$ of a cost-minimization game Γ is a *Pure Nash Equilibrium* if for every player $i \in [n]$ and for all $s'_i \in S_i$,
 $u_i(s) \geq u_i(s'_i, s_{-i})$.
- **Mixed Nash Equilibria:** A mixed strategy profile $\sigma \in \Sigma$ of a cost-minimization game Γ is a *Mixed Nash Equilibria* if for every player $i \in [n]$ and for all $s'_i \in S_i$,
 $\mathbb{E}_{s \sim \sigma} [u_i(s)] \geq \mathbb{E}_{s \sim \sigma} [u_i(s'_i, s_{-i})]$
- **Coarse Correlated Equilibria:** A distribution μ over S of a cost-minimization game Γ is a *Coarse Correlated Equilibria* if for every player $i \in [n]$ and for all $s'_i \in S_i$, $\mathbb{E}_{s \sim \mu} [u_i(s)] \geq \mathbb{E}_{s \sim \mu} [u_i(s'_i, s_{-i})]$

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$\text{PNE} \subseteq \text{MNE} \subseteq \text{CCE} \subseteq \text{CCE}$.

Fenchel and Lagrangian Duality

Given convex problem:

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & h_i(x) \leq 0 \quad \forall i \in [m], \\ & l_j(x) = 0 \quad \forall j \in [r]\end{array}$$

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Define Lagrangian $\mathcal{L}(x, u, v) = f(x) + \sum_{i=1}^m u_i h_i(x) + \sum_{j=1}^r v_j l_j(x)$. Define

$$g(u, v) = \inf_x \mathcal{L}(x, u, v)$$

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The dual of the convex problem:

$$\begin{array}{ll}\text{maximize} & g(u, v) \\ \text{subject to} & u \geq 0\end{array}$$

Weighted Congestion Games

Definitions

- \mathcal{N} : Set of players
- \mathcal{E} : The ground set of resources
- For each player $j \in \mathcal{N}$, let $S_j \subseteq 2^{\mathcal{E}}$ be the set of strategies available to player j .
Let $S = \bigtimes_{j \in \mathcal{N}} S_j$.
- For each $j \in \mathcal{N}$ and each $e \in \mathcal{E}$ there is a weight of the resource $w_{ej} \in \mathbb{R}^+$.
- For each $e \in \mathcal{E}$ the cost of resource e is an affine function $C_e : \mathbb{R} \rightarrow \mathbb{R}$ where $c_e(x) = a_e \cdot x + b_e$
- For any strategy profile $f \in S$, the cost of player j is $\text{Cost}(f)_j = \sum_{e \in f_j} w_{ej} \cdot c_e(l_e(f))$

where $l_e(f) = \sum_{j': e \in f_{j'}} w_{ej'}$ is the load on resource e . Do

$$\text{Cost}(f) = \sum_{j \in \mathcal{N}} \sum_{e \in f_j} w_{ej} \cdot c_e(l_e(f)) = \sum_{e \in \mathcal{E}} a_e \cdot l_e(f) + b_e \cdot l_e(f)$$

Convex program of WCG

Setting up the variables

For any player $j \in \mathcal{N}$ and $f_j \in S_j$ let $L_{j,f_j} = \sum_{e \in f_j} w_{ej} \cdot c_e(w_{ej})$ i.e. the cost incurred by player j when it plays strategy f_j .

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- $x_{j,f_j} :=$ Variable for player j playing strategy f_j for all $j \in \mathcal{N}$ and $f_j \in S_j$

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- $x_{j,f_j} :=$ Variable for player j playing strategy f_j for all $j \in \mathcal{N}$ and $f_j \in S_j$
- $y_e :=$ Variable for the load on resource e for all $e \in \mathcal{E}$

Convex program of WCG

Quadratic Program

$$\begin{aligned} &\text{minimize} && \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} x_{j,f_j} \cdot L_{j,f_j} + \sum_{e \in \mathcal{E}} a_e \cdot y_e^2 \\ &\text{subject to} && \sum_{f_j \in S_j} x_{j,f_j} \leq 1 \quad \forall j \in \mathcal{N}, \\ &&& \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} \sum_{e \in f_j} w_{ej} \cdot x_{j,f_j} \leq y_e \quad \forall e \in \mathcal{E}, \\ &&& x_{j,f_j} \geq 0 \quad \forall j \in \mathcal{N}, f_j \in S_j \end{aligned}$$

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subject to

$$\sum_{f_j \in S_j} x_{j,f_j} \leq 1 \quad \forall j \in \mathcal{N},$$

$$\sum_{i \in \mathcal{N}} \sum_{f_i \in S_i} \sum_{e \in E_i} w_{ej} \cdot x_{j,f_j} \leq y_e \quad \forall e \in \mathcal{E},$$

This constraint makes sure only one strategy is played by each player. $f_j \in S_j$

Convex program of WCG Quadratic Program

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$$x_{j,f_j} \geq 0 \quad \forall j \in \mathcal{N}, f_j \in S_j$$

This constraint makes sure that the load on each resource is at least sum of the weights of the players using that resource.

Dual Program

We denote the dual variables by $\{\mu_j\}_{j \in \mathcal{N}}$, $\{\Phi_e\}_{e \in \mathcal{E}}$ and $\{\Psi_e\}_{e \in \mathcal{E}}$. Then we use the Fenchel Duality to obtain the dual of the convex program.

$$\begin{aligned} & \text{maximize} && \sum_{j \in \mathcal{N}} \mu_j - \sum_{e \in \mathcal{E}} \frac{1}{4a_e} \cdot \Phi_e^2 \\ & \text{subject to} && \mu_j - \sum_{e \in f_j} w_{e,j} \cdot \Psi_e \leq L_{j,f_j} \quad \forall j \in \mathcal{N}, f_j \in S_j, \\ & && \Psi_e \leq \Phi_e \quad \forall e \in \mathcal{E}, \\ & && \mu_j \geq 0 \quad \forall j \in \mathcal{N}, \\ & && \Phi_e \geq 0 \quad \forall e \in \mathcal{E} \end{aligned}$$

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Remark

We can take $\Phi_e = \Psi_e$ for all $e \in \mathcal{E}$ as from every CCE we will assign Φ_e and Ψ_e to be the same value

$(1 + \frac{1}{\delta})$ -Approximate Solution from Primal

Consider the following changed primal program:

$$\begin{aligned} &\text{minimize} && \frac{1}{\delta} \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} x_{j,f_j} \cdot L_{j,f_j} + \sum_{e \in \mathcal{E}} a_e \cdot y_e^2 \\ &\text{subject to} && \sum_{f_j \in S_j} x_{j,f_j} \leq 1 \quad \forall j \in \mathcal{N}, \\ &&& \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} \sum_{e \in f_j} w_{ej} \cdot x_{j,f_j} \leq y_e \quad \forall e \in \mathcal{E}, \\ &&& x_{j,f_j} \geq 0 \quad \forall j \in \mathcal{N}, f_j \in S_j \end{aligned}$$

If $\delta = 1$ we get our original program. For any $\delta > 0$ we get a $(1 + \frac{1}{\delta})$ -approximate solution.

Dual don't need to change

Taking the dual of the new program we get the following:

$$\begin{aligned} \text{maximize} \quad & \sum_{j \in \mathcal{N}} \mu_j - \sum_{e \in \mathcal{E}} \frac{1}{4a_e} \cdot \Phi_e^2 \\ \text{subject to} \quad & \mu_j - \sum_{e \in f_j} w_{e,j} \cdot \Phi_e \leq \frac{L_{j,f_j}}{\delta} \quad \forall j \in \mathcal{N}, f_j \in S_j, \\ & \mu_j \geq 0 \quad \forall j \in \mathcal{N}, \\ & \Phi_e \geq 0 \quad \forall e \in \mathcal{E} \end{aligned}$$

So instead if we work with the old dual program and scale our variables μ_j , Φ_e and Ψ_e by $\frac{1}{\delta}$ we still get a feasible solution to the new dual program.

Setting the Dual Variables

Let σ is any CCE of the game. Set

- $\mu_j = \frac{1}{\delta} \cdot \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)]$
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$$\begin{aligned} \text{Cost}_j(f_j, \theta_{-j}) &\leq \sum_{e \in f_j} w_{e,j} \cdot (a_e(l_e(\theta) + w_{e,j}) + b_e) \\ &= \sum_{e \in f_j} w_{e,j} (a_e \cdot w_{e,j} + b_e) + \sum_{e \in f_j} w_{e,j} \cdot a_e \cdot l_e(\theta) \\ &= L_{j,f_j} + \sum_{e \in f_j} w_{e,j} \cdot a_e \cdot l_e(\theta) \end{aligned}$$

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Remark

It is a feasible solution to the dual program.

Bound on PoA : I

$$\begin{aligned}\sum_{e \in \mathcal{E}} \frac{1}{a_e} \cdot a_e^2 \cdot \mathbb{E}_{f \sim \sigma} [l_e(f)]^2 &= \sum_{e \in \mathcal{E}} a_e \cdot \mathbb{E}_{f \sim \sigma} [l_e(f)]^2 \\ &\leq \mathbb{E}_{f \sim \sigma} \left[\sum_{e \in \mathcal{N}} a_e \cdot l_e^2(f) \right] && \text{[Jensen]} \\ &\leq \mathbb{E}_{f \sim \sigma} \left[\sum_{e \in \mathcal{N}} \text{Cost}_j(f) \right] = \sum_{j \in \mathcal{N}} \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)]\end{aligned}$$

Bound on PoA : II

$$\begin{aligned}\text{Primal-Sol} &\geq \sum_{j \in \mathcal{N}} \frac{1}{\delta} \cdot \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)] - \sum_{e \in \mathcal{E}} \frac{1}{\delta^2} \cdot \frac{1}{4} a_e \cdot \mathbb{E}_{f \sim \sigma} [l_e(f)]^2 \\ &\geq \frac{1}{\delta} \sum_{j \in \mathcal{N}} \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)] - \frac{1}{4 \cdot \delta^2} \cdot \sum_{e \in \mathcal{E}} \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)] \\ &= \frac{4\delta - 1}{4\delta^2} \sum_{e \in \mathcal{E}} \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)]\end{aligned}$$

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Primal is $(1 + \frac{1}{\delta})$ -approximate solution to the optimal solution. So we get a bound of $(1 + \frac{1}{\delta}) \frac{4\delta^2}{4\delta - 1}$ bound on PoA. Take $\delta = \frac{1+\sqrt{5}}{4}$ you will get a bound of $1 + \phi$ where ϕ is the golden ratio.

Questions?

Simultaneous Second-Price Auctions

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- Let $p(i, b)$ is the second highest bid for item i when the bids are b .
- Let $u_j(b)$ be the utility of player j when the bids are b . Then
$$u_j(b) = v_j(W_j(b)) - \sum_{i \in W_j(b)} p(i, b).$$

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GOAL: Maximize the social welfare of the players $V(b) = \sum_{j \in \mathcal{N}} v_j(W_j(b))$

Property of Biddings

Theorem

$\forall j \in \mathcal{N}, \forall T \subseteq \mathcal{M}, \forall b \in \mathbb{R}_{\geq 0}^{m \times n}, \exists b_j(T) \in \mathbb{R}_{\geq 0}^m$ such that

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\}$$

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Let $T = \{1, \dots, i\}$. Take $b_{ij}^* = v_j(1, 2, \dots, i) - v_j(1, 2, \dots, i-1)$. Take $b_j(T) = b_j^*$

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Let $T = \{1, \dots, i\}$. Take $b_{ij}^* = v_j(1, 2, \dots, i) - v_j(1, 2, \dots, i-1)$. Take $b_j(T) = b_j^*$

Observe: $\sum_{i \in T'} b_{ij}^* \leq v_j(T')$ for all $T' \subseteq T$ by submodularity and for $T = T'$ its equality.

Proof of Theorem

$$\begin{aligned}u_j(b_j(T), b_{-j}) &= v_j(T^*) - \sum_{i \in T^*} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \\&\geq v_j(T^*) - \sum_{i \in T^*} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} + \left[\sum_{i \in T \setminus T^*} b_{ij}^* - \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \right] \\&\geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\}\end{aligned}$$

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This constraint makes sure no item is over-allocated
i.e. each item is sold to only one player.

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$$x_{j,T} \geq 0 \quad \forall j \in \mathcal{N}, T \subseteq \mathcal{M}$$

This constraint makes sure each agent receives exactly one set from $2^{\mathcal{M}}$.

Dual Program

$$\begin{array}{ll}\text{minimize} & \sum_{j \in \mathcal{N}} y_j + \sum_{i \in \mathcal{M}} z_i \\ \text{subject to} & y_j + \sum_{i \in T} z_i \geq v_j(T) \quad \forall j \in \mathcal{N}, T \subseteq \mathcal{M}, \\ & z_i \geq 0 \quad \forall i \in \mathcal{M}, \\ & y_j \geq 0 \quad \forall j \in \mathcal{N}\end{array}$$

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$$\mathbb{E}_{b \sim \sigma} [u_j(b)] \geq \mathbb{E}_{b \sim \sigma} [u_j(b_j(T), b_{-j})] \quad \forall T \subseteq \mathcal{M}$$

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By the theorem

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N}} \{b_{ij'}\}$$

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So $\mathbb{E}_{b \sim \sigma} [u_j(b)] \geq v_j(T) - \sum_{i \in T} \mathbb{E}_{b \sim \sigma} \left[\max_{j' \in \mathcal{N}} \{b_{ij'}\} \right]$. So it is feasible solution to the dual program.

Bound on PoA

$$\begin{aligned}\text{Primal-Sol} &\leq \sum_{j \in \mathcal{N}} \mathbb{E}_{b \sim \sigma} [u_j(b)] + \sum_{i \in \mathcal{M}} \mathbb{E}_{b \sim \sigma} \left[\max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\ &= \mathbb{E}_{b \sim \sigma} \left[\sum_{j \in \mathcal{N}} u_j(b) \right] + \mathbb{E}_{b \sim \sigma} \left[\sum_{i \in \mathcal{M}} \max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\ &\leq 2 \cdot \mathbb{E}_{b \sim \sigma} [V(b)]\end{aligned}$$

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So we get a bound of 2.

Questions?

Facility Location Games

Definition

- \mathcal{M} : Set of m clients (Indexed by i)
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- Each client $i \in \mathcal{M}$ has some value $\pi_i \geq 0$ for the service money he is willing to pay.
- There is a cost $c(l, i)$ for serving the client $i \in \mathcal{M}$ from the location $l \in \mathcal{L}$

More Definitions

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- Total utility of the supplier $j \in \mathcal{N}$ is $u_j(s) = \sum_{i: SP(i)=j} P_j(i, s_j, s_{-j})$
- $V(s)$: Social welfare of the strategy profile s , $W(s) = \sum_{j \in \mathcal{N}} u_j(s) + \sum_{i \in \mathcal{M}} D_i(s)$

Choosing Prices

Theorem

For any strategy profile s , for any client i and supplier j , $SP(i) = j$

$$(i) \quad c(s_j, i) = \min_{j' \in \mathcal{N}} c(s_{j'}, i)$$

$$(ii) \quad p_s(i, j) = \max \left\{ c(s_j, i), \min_{l \in \mathcal{K}(s) \setminus \{s_j\}} c(l, i) \right\}$$

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$$P_j(i, l, s_{-j}) = \begin{cases} \min_{l' \in \mathcal{K}(s) \setminus \{s_j\}} c(l', i) - c(l, i) & \text{If } c(l, i) \leq c(l', i) \\ 0 & \text{Otherwise} \end{cases}$$

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$$W(s) = \sum_{j \in \mathcal{N}} u_j(s) + \sum_{i \in \mathcal{M}} D_i(s) = \sum_{i \in \mathcal{M}} \pi_i - c(s_{SP(i)}, i)$$

LP Formulation

- $x_{ijl} :=$ Variable indicating if the supplier j serves the client i from location l .
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$$\text{maximize} \quad \sum_{j \in \mathcal{N}} \sum_{l \in S_j} \sum_{i \in \mathcal{M}} (\pi_i - c(l, i)) \cdot x_{ijl}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{N}} \sum_{l \in S_j} x_{ijl} \leq 1 \quad \forall i \in \mathcal{M},$$

$$\sum_{j \in \mathcal{N}} x_{jl} \leq 1 \quad \forall l \in \mathcal{L},$$

$$\sum_{k \in S_j} x_{jl} \leq 1 \quad \forall j \in \mathcal{N},$$

$$x_{ijl} \leq x_{jl} \quad \forall i \in \mathcal{M}, j \in \mathcal{N}, l \in S_j,$$

$$x_{ijl} \geq 0 \quad \forall i \in \mathcal{M}, j \in \mathcal{N}, l \in S_j$$

Dual Program

We denote the dual variables by $\{\alpha_j\}_{j \in \mathcal{N}}$, $\{\beta_i\}_{i \in \mathcal{M}}$, $\{\gamma_l\}_{l \in \mathcal{L}}$ and $\{z_{ijl}\}_{i \in \mathcal{M}, j \in \mathcal{N}, l \in \mathcal{S}_j}$.

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$$\begin{aligned} & \text{minimize} && \sum_{j \in \mathcal{N}} \alpha_j + \sum_{i \in \mathcal{M}} \beta_i + \sum_{l \in \mathcal{L}} \gamma_l \\ & \text{subject to} && \beta_i + z_{ijl} \geq \pi_i - c_{il} \quad \forall i \in \mathcal{M}, j \in \mathcal{N}, l \in S_j, \\ & && \gamma_l + \alpha_j \geq \sum_{i \in \mathcal{M}} z_{ijl} \quad \forall j \in \mathcal{N}, l \in S_j, \\ & && \alpha_j \geq 0 \quad \forall j \in \mathcal{N}, \\ & && \beta_i \geq 0 \quad \forall i \in \mathcal{M} \end{aligned}$$

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- $z_{ijl} = \mathbb{E}_{s \sim \sigma} [P_j(i, l, s_{-j})]$ for all $i \in \mathcal{M}, j \in \mathcal{N}$ and $l \in S_j$.

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- $z_{ijl} = \mathbb{E}_{s \sim \sigma} [P_j(i, l, s_{-j})]$ for all $i \in \mathcal{M}, j \in \mathcal{N}$ and $l \in S_j$.
- Define $W_l(s) = u_j(s)$ if $l \in \mathcal{K}(s)$ and $s_j = l$ for some $j \in \mathcal{N}$ and otherwise 0.
Then $\gamma_l = \mathbb{E}_{s \sim \sigma} [W_l(s)]$ for all $l \in \mathcal{L}$.

Feasibility Checking

- $\pi_i - p_s(i, SP(i)) \geq \pi_i - c(l, i)$ for any $l \in \mathcal{L}$. Now $P_j(i, l, s_{-j}) \neq 0$ when $l = SP(i)$. Then clearly $\pi_i - p_s(i, SP(i)) + P_j(i, SP(i), s_{-j}) = \pi_i - c(SP(i), i)$ and for other locations $P_j(i, l, s_{-j}) = 0$. So the first constraint is satisfied

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- If $l \in \mathcal{K}(s)$ then $W_l(s) = \sum_{i \in \mathcal{M}} P_j(i, l, \theta_{-j})$ for some $j \in \mathcal{N}$ such that $s_j = l$. So it satisfies the second constraint. If $l \notin \mathcal{K}(s)$. $u_j(s) \geq P_j(i, l, s_{-j})$ since σ is a CCE. So the second constraint is satisfied.

Bound on PoA

$\sum_{j \in \mathcal{N}} \alpha_j + \sum_{i \in \mathcal{M}} \beta_i$ is the expected social welfare under the distribution σ .

$\sum_{l \in \mathcal{L}} W_l(s)$ is at most the social welfare since σ is a CCE.

So by Weak Duality

$$\text{Primal-Sol} \leq \sum_{j \in \mathcal{N}} \alpha_j + \sum_{i \in \mathcal{M}} \beta_i + \sum_{l \in \mathcal{L}} \gamma_l \leq 2 \cdot \mathbb{E}_{s \sim \sigma} [V(s)]$$