Super-Polynomial Lower Bound of TSP Extended Formula

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May 2025

Introduction

Definition (Travelling Salesman)

Given a graph $G = (V, E)$, $S \subseteq V$ and weights $w : E \to \mathbb{R}$ find minimum weight cycle which visits every vertex of S exactly once.

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We will focus on $S = V$.

- We know Traveling Salesman Problem is NP-complete.
- In [Yannkakis, 1988, STOC] he proved every symmetric LP for the TSP has expnential size.
- Here we will show TSP admits no polynomial-size LP.
- This proof also shows unconditional super-polynomial lower bound on the number of inequalities.
- Therefore it is impossible to prove $\mathrm{P} = \mathrm{NP}$ by means of a polynomial size LP.

Definitions

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Extension complexity of P is the minimum size EF of P where size of a polytope is the number inequalities. We denote by $xc(P)$.

Some Polytopes

- $TSP(n)$ is the traveling salesman polytope for $K_n = (V_n, E_n)$. Let $C \subseteq E_n$ denotes a tour of K_n. Then χ^C denotes the characteristic vector of C. Then

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- Given $G = (V, E)$, for any $S \subseteq V$, χ^S denote characteristic vector of S. Then

$$IND(G) := conv\{\chi^S \mid S \text{ is independent set of } G\}$$

- The correlation polytope $COR(n)$ is

$$COR(n) := conv\{bb^T \mid b \in \{0, 1\}^n\}$$