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Course: Expander Graphs and Applications

Assignment - 1
Roll: BMC202175
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Problem 1 Problem 4.9 (The Replacement Product): Pseudorandomness By Salil Vadhan

Given a D_1-regular graph G_1 on N_1 vertices and a D_2-regular graph G_2 on D_1 vertices consider the following graph $G_1 ⓡ G_2$ on vertex set $[N_1] \times [D_1]$: vertex (u, i) is connected to (v, j) iff

(a) $u = v$ and (i, j) is an edge in G_2 or,

(b) v is the i'th neighbour of u in G_1 and u is the jth neighbour of v.

That is, we "replace" each vertex v in G_1 with a copy of G_2, associating edge incident to v with one vertex of G_2.

1. Prove that there is a function g such that if G_1 has spectral expansion $\gamma_1 > 0$ and G_2 has spectral expansion $\gamma_2 > 0$ (and both graphs are undirected) then $G_1 ⓡ G_2$ has spectral expansion $g(\gamma_1, \gamma_2, D_2) > 0$.

   [Hint: Note that $(G_1 ⓡ G_2)^3$ has $G_1 ⓩ G_2$ as a subgraph]

2. Show how to convert an explicit construction of constant degree (spectral) expanders into an explicit construction of degree 3 (spectral) expanders.

3. Without using Theorem 4.14, prove an analogye of Part 1 for edge expansion. That is, there is a function h such that if G_1 is an $\left(\frac{N_1}{2}, \epsilon_1\right)$ edge expander and G_2 is a $\left(\frac{D_1}{2}, \epsilon_2\right)$ edge expander then $G_1 ⓡ G_2$ is a $\left(\frac{N_1 D_1}{2}, h(\epsilon_1, \epsilon_2, D_2)\right)$ edge expander where $h(\epsilon_1, \epsilon_2, D_2) > 0$ if $\epsilon_1, \epsilon_2 > 0$.

   [Hint: Given any set S of vertices of $G_1 ⓡ G_2$, partition S into the clouds that are more than "half-full" and those that are not]

4. Prove that the functions $g(\gamma_1, \gamma_2, D_2)$ and $h(\epsilon_1, \epsilon_2, D_2)$ must depend on D_2 by showing that $G_1 ⓡ G_2$ cannot be a $\left(\frac{N_1 D_1}{2}, \epsilon\right)$ edge expander if $\epsilon > \frac{1}{D_1 + 1}$ and $N_1 \geq 2$

Solution:

□

Problem 2 Problem 4.10 (Unbalanced Vertex Expanders and Data Structures): Pseudorandomness By Salil Vadhan

Consider a $(K, (1 - \epsilon)D)$ bipartite vertex expander G with N left vertices, M right vertices and left degree D.

1. For a set S of left vertices, a $y \in N(S)$ is called a *unique* neighbor of S if y is incident to exactly one edge from S. Prove that every left-set S of size at most K has at least $(1 - 2\epsilon)D|S|$ unique neighbors.

2. For a set S of size at most $\frac{K}{2}$, prove that at most $\frac{|S|}{2}$ vertices outside S have at least δD neighbors in $N(S)$ for $\delta = O(\epsilon)$.

Solution:

□

Problem 3 Problem 5.5 (LDPC Codes): Pseudorandomness By Salil Vadhan

Given a D_1-regular graph G_1 on N_1 vertices and a D_2-regular graph G_2 on D_1 vertices consider the following graph $G_1 \textcircled{r} G_2$ on vertex set $[N_1] \times [D_1]$: vertex (u, i) is connected to (v, j) iff

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   [Hint: Given any set S of vertices of $G_1 \textcircled{r} G_2$, partition S into the clouds that are more than "half-full" and those that are not]

4. Prove that the functions $g(\gamma_1, \gamma_2, D_2)$ and $h(\epsilon_1, \epsilon_2, D_2)$ must depend on D_2 by showing that $G_1 \textcircled{r} G_2$ cannot be a $\left(\frac{N_1 D_1}{2}, \epsilon\right)$ edge expander if $\epsilon > \frac{1}{D_1 + 1}$ and $N_1 \geq 2$

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