CSS.201.1 Algorithms

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Chapter 1

Maximum Flow

1.1 Flow

Suppose we are given a directed graph $G = (V, E)$ with a source vertex s and a target vertex t. And additionally for every edge $e \in E$ we are given a number $c_e \in \mathbb{Z}_0$ which is called the capacity of the edge.

Definition 1.1.1: Flow

An $s - t$ flow is a function $f : E \longrightarrow \mathbb{R}_0$ which satisfies the following:

① $\forall\, e \in E,\, f(e) \le c_e$

② $\forall\, v \in V \setminus \{s, t\},\ \sum\limits_{e \in in(v)} f(e) = \sum\limits_{e \in out(v)} f(e)$

Also the value of a flow f is denoted by $|f| := \sum\limits_{e \in out(s)} f(e)$.

Before proceeding into the setup and the problem first we will assume some things

Assumption.

- *$in(s) = \emptyset$ i.e. there is no edge into s.*
- *$out(t) = \emptyset$ i.e. there is no edge out of t.*
- *There are no parallel edges*

Lemma 1.1.1

For any flow f, $|f| = \sum\limits_{e \in in(t)} f(e)$

Proof: We have for every edge $e \in E$, $\exists\, v \in V$ such that $e \in in(v)$ and $\exists\, u \in V$ such that $e \in out(u)$. Hence we get

$$\sum_{e \in E} f(e) = \sum_{v \in V} \sum_{e \in in(v)} f(e) = \sum_{v \in V} \sum)e \in out(v) f(e) \implies \sum_{v \in V} \left[\sum_{e \in in(v)} f(e) - \sum_{e \in out(v)} f(e) \right] = 0$$

Now we know $\forall\, v \in V \setminus \{s, t\}$. $\sum\limits_{e \in in(v)} f(e) = \sum\limits_{e \in out(v)} f(e)$. Therefore we get

$$\sum_{v \in V} \left[\sum_{e \in in(v)} f(e) - \sum_{e \in out(v)} f(e) \right] = 0 \implies \sum_{v \in \{s,t\}} \left[\sum_{e \in in(v)} f(e) - \sum_{e \in out(v)} f(e) \right] = 0 \implies \sum_{e \in out(s)} f(e) - \sum_{e \in in(t)} f(e)$$

Hence we have $|f| = \sum\limits_{e \in in(t)} f(e)$. ∎

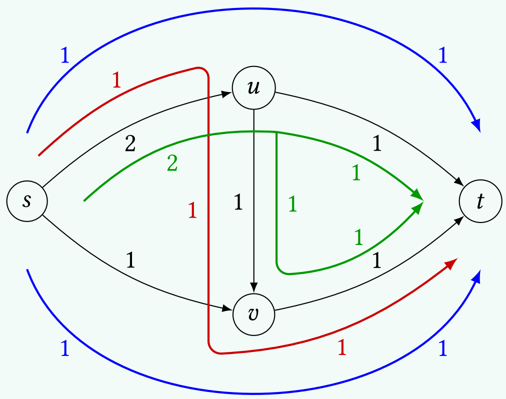
MAX FLOW

Input: A directed graph $G = (V, E)$ with source vertex s and target vertex t and for all edge $e \in E$ capacity of the edge $c_e \in \mathbb{Z}_+$

Question: Given such a graph and its capacities find an $s - t$ flow which has the maximum value

Example 1.1.1

Consider the following directed graph with capacities: $V = \{s, t, u, v\}$, $c_{s,u} = 2, c_{s,v} = c_{u,t} = c_{v,t} = c_{u,v} = 1$. Firstly the following function: $f' : f'(s,u) = 2 = f(u,t)$. It is not a flow since $f(u,t) = 2 > 1 = c_{u,t}$. Now we define three different flow functions:



- f: $f(s,u) = f(u,v) = f(v,t) = 1$ and otherwise 0. Therefore $|f| = 1$
- g: $g(s,u) = g(u,t) = 1$, $g(s,v) = g(v,t) = 1$ and otherwise 0. Therefore $|g| = 2$
- h: $h(s,u) = 2$, $h(u,t) = h(u,v) = h(v,t) = 1$ and otherwise 0. Therefore $|h| = 2$

Notice here g and h has the maximum flow value.

1.2 Ford-Fulkerson Algorithm

Definition 1.2.1: Residual Graph

Given a directed graph $G = (V, E)$ and capacities C_e for all $e \in E$ and an $s - t$ flow f the residual graph $G_f = (V, E_f)$ has the edges with the following properties:

1. If $(u,v) \in E$ and $f(u,v) > 0$ then $(v,u) \in E_f$ and $c^f_{v,u} = f(u,v)$. Such an edge is called a *backward* edge.
2. If $(u,v) \in E$ and $f(u,v) < c_{u,v}$ then $(u,v) \in E_f$ and $c^f_{u,v} = c_{u,v} - f(u,v)$. It is called *forward* edge.

Algorithm 1: FORD-FULKERSON

Input: Directed graph $G = (V, E)$, source s, target t and edge capacities C_e for all $e \in E$
Output: Flow f with maximum value

```
1 begin
2     for e ∈ E do
3         f(e) = 0
4     while ∃ s ⇝ t path P in G_f do
5         δ ⟵ min_{e∈P}{c_e^f} for e = (u,v) ∈ P do
6             if e is Forward Edge then
7                 f(u,v) ⟵ f(u,v) + δ
8             else
9                 f(u,v) ⟵ f(v,u) − δ
```

We call one iteration of the While loop at line 4 *Flow Augmentation*.

Lemma 1.2.1

At any iteration the f' obtained after the flow augmentation of the flow f is a valid flow

Proof: At any iteration let P be the path from $s \rightsquigarrow t$ and $\delta = \min_{e\in P} c_f(e)$. Let f' be the new function such that for each $(u,v) \in P$ if (u,v) is forward edge in G_f then $f'(u,v) = f(u,v) + \delta$ and if (u,v) is backward edge in G_f then $f'(v,u) = f(v,u) - \delta$ and for other edges $e \in E \setminus P$, $f'(e) = f(e)$.

Now since $\delta = \min_{e\in P} c_f(e)$, $c_f(e) \geq \delta$ for all $e \in P$. Hence if (u,v) is backward edge then $(v,u) \in E$ and $c_f(u,v) = f(u,v)$. Hence $f'(v,u) = f(v,u) - \delta \geq 0$. Therefore for all $e \in E$, $f'(e) \geq 0$.

Now first we will show $f'(e) \leq c_e$ for all $e \in E$. If $(u,v) \in P$ is a forward edge then $(u,v) \in E$ and $c_f(u,v) = c_{u,v} f(u,v)$. Therefore $f'(u,v) = f(u,v) + \delta \leq f(u,v) + c_{u,v} - f(u,v) = c_{u,v}$. Now if $(u,v) \in P$ is a backward edge then $(v,u) \in E$ and $c_f(u,v) = f(u,v)$. Therefore $f'(u,v) = f(u,v) - \delta \leq f(u,v) \leq c_{u,v}$. For other edges $e \in E \setminus P$, $f'(e) = f(u) \leq c_e$. Therefore $f'(e) \leq c_e$ for all $e \in E$

Now we will prove for all $v \in V \setminus \{s,t\}$, $\sum_{e\in in(v)} f'(e) = \sum_{e\in out(v)} f'(e)$. If v is not in the path P in G_f then, $f'(e) = f(e)$ for all edges $e \in in(v) \cup out(v)$. Hence the condition is satisfied for such vertices. Suppose v is in the path P. Then there are two edges e_1 and e_2 in P which are incident on e. If both are forward edges or both are backward edges then one of them is in $in(v)$ and other one is in $out(v)$. WLOG suppose $e_1 \in in(v)$ and $e_2 \in out(v)$ we have

$$\sum_{e\in in(v)} f'(e) = \sum_{e\in in(v)\setminus\{e_1\}} f(e) + f(e_1) \pm \delta = \sum_{e\in out(v)\setminus\{e_2\}} f(e) + f(e_2) \pm \delta = \sum_{e\in out(v)} f'(e)$$

If one of e_1, e_2 forward edge and other one is backward edge then either $e_1, e_2 \in in(v)$ (when e_1 is forward and e_2 is backward) or $e_1, e_2 \in out(v)$ (when e_1 is backward and e_2 is forward). Now if $e_1, e_2 \in in(v)$, $f'(e_1) + f'(e_2) = f(e_1) + \delta + f(e_2) - \delta = f(e_1) + f(e_2)$ and if $e_1, e_2 \in out(v)$ then $f'(e_1) + f'(e_2) = f(e_1) - \delta + f'(e_2) + \delta = f(e_1) + f(e_2)$. Hence

$$\sum_{e\in in(v)} f'(e) = \sum_{e\in in(v)} f(e) = \sum_{e\in out(v)} f(e) = \sum_{e\in out(v)} f'(e)$$

Hence f' is a valid flow. ■

Lemma 1.2.2

At any iteration Given G_f if the flow, f' obtained after flow augmentation of f by δ then

$$|f'| = |f| + \delta$$

Proof: Since we augment flow along an $s \rightsquigarrow t$ path, the first edge of the path is always in $out(s)$. Let the first edge is $e = (s,u)$. Now e has to be a forward edge because otherwise $(u,s) \in E$ and then there is an incoming edge in G which is not possible. Hence

$$|f'| = \sum_{e\in out(s)} f'(e) = \sum_{e\in out(s)\setminus\{e\}} f(e) + f'(e) = \sum_{e\in out(s)\setminus\{e\}} f(e) + f(e) + \delta = \sum_{e\in out(s)} f(e) + \delta = |f| + \delta$$

Hence we have the lemma. ■

Lemma 1.2.3

At every iteration of the Ford-Fulkerson Algorithm the flow values and the residual capacities of the residual graph are non-negative integers.

Proof: Initial flow and the residual capacities are non-negative integers. Let till i^{th} iteration the flow values and the residual capacities were non-negative integers. Let the flow after i^{th} iteration was f. Hence $\forall\, e \in E$, $f(e) \in \mathbb{Z}_0$. Therefore in the G_f for all $e \in E_f$, $c_f(e) \in \mathbb{Z}_0$. Hence $\delta \in \mathbb{Z}_0$. Therefore $\forall\, e \in E$, $f'(e) \in \mathbb{Z}_0$. And therefore for all $e \in E_{f'}$ where $G_{f'}$ is the residual graph of the flow f', $c_{f'}(e) \in \mathbb{Z}_0$. Hence by mathematical induction the lemma follows. ■

At any iteration let P be the path from $s \rightsquigarrow t$. Then for all $e \in P$, $c_f(e) > 0$. Therefore $\delta = \min_{e\in P} c_f(e) \geq 1$. Therefore the algorithm must stop in at most $\sum_{e\in out(s)} c_e$ since we can have the value of a flow to be at max the value of the sum of capacities of edges in $out(s)$ and therefore we can increase the flow at max that many times.

Lemma 1.2.4

If f is a max flow then there is no $s \rightsquigarrow t$ path in G_f.

Proof: Suppose there is an $s \rightsquigarrow t$ path P in G_f. We will show that then f is not a max flow following the algorithm. Then $\forall\ e \in P$, $c_f(e) > 0$. Hence $\delta = \min_{e\in P} c_f(e) \geq 1$. Now after the flow augmentation process of f by δ we get a new valid flow f' by Lemma 1.2.1 and by Lemma 1.2.2 we have $|f'| = |f| + \delta > ||f|$. Hence f is not a maximum flow. Hence contradiction. Therefore there is no $s \rightsquigarrow t$ path in G_f. ■

1.2.1 Max Flow Min Cut

Definition 1.2.2: Cut Set

For a graph $G = (V, E)$ and a subset $A \subseteq V$, the cut $(A, V \setminus A)$ is a bipartition of V where the edges E_A of the graph $G_A = (A, V \setminus A, E_A)$ is the set $E_A = E \cap (A \times (V \setminus A))$.

Now if s, t are two vertices of G then an $s - t$ *Cut* $(A, V \setminus A)$ is a cut such that $s \in A$ and $t \in V \setminus A$.

Now we define for a cut $(A, V \setminus A)$ the *Capacity of the Cut* $(A, V \setminus A) = \sum_{e\in E_A} c_e$. For an $s - t$ cut $(A, V \setminus A)$ we denote the capacity of the cut by $cap(A)$ A *Min* $s - t$ *Cut* is a $s - t$ cut of minimum capacity. Then we have the following relation between cut and flow.

Lemma 1.2.5

Given a graph $G = (V, E)$, s, t, $c_e \in \mathbb{Z}_0$ for all $e \in E$ for any flow f and a $s - t$ cut $(A, V \setminus A)$

$$|f| \leq cap(A)$$

Proof: Given f and the $s - t$ cut $(A, V \setminus A)$ we have

$$\begin{aligned}
|f| &= \sum_{e\in out(s)} f(e) \\
&= \sum_{v\in A}\left[\sum_{e\in out(v)} f(e) - \sum_{e\in in(v)} f(e)\right] \\
&= \sum_{\substack{e=(u,v),\\ u\in A, v\notin A}} f(e) - \sum_{\substack{e=(u,v),\\ u\notin A, v\in A}} f(e) && \text{[Edges for both endpoints in } A \text{ are canceled out]} \\
&= \sum_{e\in out(A)} f(e) - \sum_{e\in in(A)} f(e) \\
&\leq \sum_{e\in out(A)} f(e) \leq \sum_{e\in out(A)} c_e = cap(A)
\end{aligned}$$

Hence we have the lemma. ■

Having this lemma we have have for any flow f and $s - t$ cut $(A, V \setminus A)$ we have

$$|f| \leq cap(A) \implies \max_f |f| \leq \min_{s-t \text{ cut } (A, V\setminus A)} cap(A)$$

So we have the following theorem that the value of maximum flow is equal to the capacity of minimum cut.

Theorem 1.2.6 Max Flow Min Cut

Given a graph $G = (V, E)$, s, t, $c_e \in \mathbb{Z}_0$ for all $e \in E$. Then the following are equivalent:

(1) f is a maximum flow.

(2) There is no $s \rightsquigarrow t$ path in G_f

(3) There exists an $s - t$ cut of capacity $|f|$

Proof:

(1) $\implies$ (2): This is by Lemma 1.2.4.

(2) $\implies$ (3): We are given a flow f such that there is no $s \rightsquigarrow t$ path in G_f. We will construct a $s - t$ cut which has the capacity $|f|$. Now take A to be all the vertices reachable from s in G_f. This is a valid $s - t$ cut since $s \in A$ and as there is no $s \rightsquigarrow t$ path in G_f, $t \notin A$. Now

$$|f| = \sum_{e \in out(A)} f(e) - \sum_{e \in in(A)} f(e)$$

Now $\forall\ e = (u, v) \in E$ where $u \in A$ and $v \notin A$ we have $c_{u,v} = f(u, v) \implies c_{u,v} - f(u, v) = 0$ since otherwise $c_{u,v} - f(u, v) \neq 0 \implies c_{u,v} > f(u, v) \implies (u, v) \in E_f$ and therefore v is reachable from s but $v \notin A$, contradiction. Therefore (u, v) is a backward edge and hence . Now $\forall\ e = (u, v) \in E$ where $u \notin A$ and $v \in A$ we have $f(u, v) = 0$ since otherwise $f(u, v) > 0 \implies (v, u) \in E_f$ and therefore u is reachable from s but $u \notin A$, contradiction. Hence we have

$$|f| = \sum_{e \in out(A)} f(e) - \sum_{e \in in(A)} f(e) = \sum_{e \in out(A)} c_e = cap(A)$$

(3) $\implies$ (1): Now by Lemma 1.2.5 we have for any flow f and $s - t$ cut

$$|f| \leq cap(A) \implies \max_f |f| \leq \min_{s-t\ cut\ (A, V \setminus A)} cap(A)$$

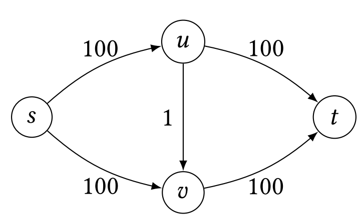
Now given f there exists an $s - t$ cut of capacity $|f|$. Hence f is a max flow.

■

Hence at the end of the Ford-Fulkerson Algorithm let the flow returned by the algorithm is f. The algorithm terminates when there is no $s \rightsquigarrow t$ path in G_f. Hence by Max Flow Min Cut Theorem we have f is a maximum flow. This completes the analysis of the Ford-Fulkerson Algorithm.

Since the capacities of the edges can be very large we want an algorithm return the maximum flow with running time $poly(n, m, \log c_e)$ where n is the number of vertices and m is number of edges and $\log c_e$ basically means number of bits at most needed to represent the capacities.

But Ford-Fulkerson algorithm takes does not run in $poly(n, m, \log c_e)$ instead $poly(n, m, c_e)$ as the while loop in the algorithm takes $poly(c_e)$ many iterations. For example in the following graph: it takes around 100 steps



and in general Ford-Fulkerson takes $O(|f_{\max}|)$ time. For this reason we will now discuss a modification of the Ford-Fulkerson Algorithm which takes $poly(n, m, \log c_e)$ time, Edmonds-Karp Algorithm.

1.2.2 Edmonds-Karp Algorithm

To get a $poly(n, m, \log c_e)$ time algorithm we will always pick the shortest $s \rightsquigarrow t$ path in the residual graph. This algorithm is known as the Edmonds-Karp Algorithm

Suppose f_i be the total flow after i^{th} iteration. And G_{f_i} be the residual graph with respect f_i. Then $f_0(e) = 0$ for all $e \in E$ and $G_{f_0} = G$. Also suppose $dist_i(v)$ = Shortest $s \rightsquigarrow v$ path distance in the residual graph G_{f_i}. Hence $dist_i(s) = 0$ for all i and $dist_i(t) = \infty$ at the end of the algorithm.

Note:-

In i^{th} iteration of the Ford-Fulkerson Algorithm or Edmonds-Karp Algorithm if P is the $s \rightsquigarrow t$ in the residual graph G_{f_i} where $e = (u, v) \in P$ and $c_{f_i}(u, v) = \delta = \min_{e \in P} c_{f_i}(e)$ then the edge (u, v) is not present in the next residual graph $G_{f_{i+1}}$. Thus at least one edge disappears in each iteration of Ford-Fulkerson or Edmonds-Karp Algorithm.

Now we will prove following two lemmas which will help us to prove that the Edmond-Karp algorithm takes $O(mn)$ iterations.

Lemma 1.2.7

At any iteration i, $\forall\, v \in V$, $dist_i(v) \leq dist_{i+1}(v)$

Proof: Suppose this is not true. Then let i be the first iteration in which there exists a vertex $v \in V$ such that $dist_i(v) > dist_{i+1}(v)$. We pick such v which minimizes $dist_{i+1}(v)$. Consider the shortest path P from $s \rightsquigarrow v$ in $G_{f_{i+1}}$. Hence length of P, $|P| = dist_{i+1}(v)$. Let (u, v) be the last edge of P.

$$P: \quad s \rightsquigarrow u \longrightarrow v$$

Then

$$dist_{i+1}(v) = dist_{i+1}(u) + 1 \geq dist_i(u) + 1$$

Here the last inequality follows because v is the vertex which has the minimum $dist_{i+1}(v)$ among all the vertices $w \in V$ which follows $dist_i(w) > dist_{i+1}(w)$. Now we will analyze case wise.

- **Case 1:** $(u, v) \in E_{f_i}$. Then
$$dist_i(v) \leq dist_i(u) + 1 \leq dist_{i+1}(v)$$
But this is not possible since $dist_i(v) > dist_{i+1}(v)$.

- **Case 2:** $(u, v) \notin E_{f_i}$. Then $(v, u) \in E_{f_i}$. Since $(u, v) \in E_{f_{i+1}}$ then we must have sent flow along (v, u). Since we take the shortest $s \rightsquigarrow t$ path in G_{f_i} in the algorithm we have $dist_i(u) = dist_i(v) + 1$. But then
$$dist_i(u) \leq dist_{i+1}(v) - 1 \implies dist_{i+1}(v) \geq dist_i(v) + 2$$
But this is not possible.

Hence contradiction ↯ Therefore for all iterations i, for all vertices $v \in V$, $dist_i(v) \leq dist_{i+1}(v)$. ■

Lemma 1.2.8

For any edge $e = (u, v) \in E$ the number of iterations where either (u, v) appears or (v, u) appears is at most $O(n)$ i.e.

$$\left|\left\{i \colon (u, v) \notin G_{f_i}, (u, v) \in G_{f_{i+1}}\right\}\right| + \left|\left\{i \colon (v, u) \notin G_{f_i}, (v, u) \in G_{f_{i+1}}\right\}\right| = O(n)$$

Proof: Following the proof of Lemma 1.2.7 in the second case we showed if $(u, v) \notin G_{f_i}$ but $(u, v) \in G_{f_{i+1}}$ then $dist_{i+1}(v) \geq dist_i(v) + 2$. Hence the distance increases by at least 2. Now this can happen at most $O(n)$ many times since $\forall\, i$, $dist_i(v) \leq n - 1$. Hence the number of iterations where either (u, v) appears or (v, u) appears is at most $O(n)$. ■

With this this lemma we will prove that the Edmonds-Karp Algorithm takes $O(mn)$ iterations.

Theorem 1.2.9

Edmonds-Karp Algorithm terminates in $O(mn)$ many iterations.

Proof: For k iterations at least k edges must disappear. Since each edge can reappear $O(n)$ times by Lemma 1.2.8, it can disappear at most $O(n)$ many times. In each iteration at least one edge disappears. Now after $O(mn)$ iterations number of disappearances is at most $O(mn)$. But after $O(mn)$ many disappearances there are no edge remaining and therefore there is no $s \rightsquigarrow t$ path. Hence the algorithm terminates. Therefore the Algorithm terminates in $O(mn)$ iterations. ■

Hence Edmond-Karp Algorithm takes $O(m^2 n) poly(\log c_e) = O\left(m^2 n \log^{O(1)}(c_e)\right)$ time since it takes $O(mn)$ iterations and in each iteration it finds the shortest $s \rightsquigarrow t$ path in G_{f_i} in $O(m)$ time and in each iteration it does addition and subtraction and finds minimum of the capacities which takes polynomial of the bits needed to represent them time.

1.3 Preflow-Push/Push-Relabel Algorithm