Expander Graphs and It's Application

Instructor: Partha Mukhopadhyay

Scribe: Soham Chatterjee

sohamchatterjee999@gmail.com
Website: sohamch08.github.io

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CHAPTER 1

Introduction to Expanders

Consider you have many computers and you want to create a network among them. You want to have well connectivity so that you can communicate from any computer to any computer. But since adding all possible connections results in high cost you want to use minimum number of connections but still want to achieve well connected. Similarly you don't want to connect too many computers to one single computer to achieve that. Also if any connection fail you don't want to loose any computer.

So your connection graph should be symmetric. So the properties we need:

1. "Well Connected"
2. "Nicely Symmetric"
3. They should be sparse
4. Low degree graph

Because of this we study expander graphs to achieve all these 4 properties

Observation. *Complete graph follows first 2 properties but not the last 2 properties.*

Remark: Expander graphs are very good approximation of complete graphs

Now the big question comes.

Question 1. *Do they exists? How to construct them explicitly?*

Using probabilistic method one can show they exists in large number. Though constructions of expander graphs are highly non trivial. They often use: Group theory, Number Theory, Representation Theory, Combinatorics.

Through out this we will study expander with keeping these two question in mind. Their study helps us in a lot of fields:

- Derandomization of Randomized Algorithms
- Error Reduction in Randomized Algorithms
- Circuit Complexity
- Error Correcting Codes

- Counting
- Space Complexity
- PCP

Assumptions: By default we will assume our graphs $G = (V, E)$ are undirected and they are d-regular.

1.1 Vertex Expander

By the term “Expander Graphs" we can guess something should expand. So one example of expanders is we expander a set of vertices $S \subseteq V$.

Definition 1.1.1 (Vertex Boundary)**.** *$\forall\ S \subseteq V$, the boundary of S is $N(S) = \{v \in S \mid \exists\ u \in S, (v, u) \in E\}$*

Definition 1.1.2 (Vertex Expander)**.** *A graph G is (δ, ϵ)-vertex expander if $\forall\ S \subseteq V$, $|S| \leq \delta|V|$, $\frac{|N(S)|}{|V|} \geq \epsilon$*

In general many times we will take δ to be $\frac{1}{2}$.

1.2 Spectral Expander

First we take the adjacency matrix of G and then normalize it. Let A_G denote that matrix. So for every row of A_G it is the corresponding row of the adjacency matrix then divided by d as G is $d-$regular graph. So

$$A_G[i, j] = \begin{cases} \frac{1}{d} & \text{when } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

We can think of A_G as a probability distribution over the vertices.

Observation 1. *The rows as well as columns sum up to 1. Such matrix is called doubly stochastic.*

Observation 2. *A_G is a real symmetric matrix*

Lemma 1.2.1. *A_G has real eigenvalues*

Proof: Let λ be eigenvalue and the corresponding nonzero eigenvector is v. Then $A_G v = \lambda v$. Now

$$\lambda(v^\dagger v) = v^\dagger \lambda v = v^\dagger A v = (A^\dagger v)^\dagger v = (Av)^\dagger v = \lambda^\dagger (v^\dagger v)$$

So we get $\lambda = \lambda^\dagger$. Hence λ is real. ■

We define the vector $u := \left(\frac{1}{n} \cdots \frac{1}{n}\right)^T$ is uniform vector

Observation 3. *$A_G u = u$. u eigenvector with eigenvalue 1*

Lemma 1.2.2. *$\forall\ i \in [n]\ |\lambda_i| \leq 1$*

Proof: Let λ is an eigenvalue and corresponding nonzero eigenvector x. Let $|x_j|$ has the maximum absolute value among all the entries of x. Then jth entry of λx is λx_j. jth entry of $A_G x$ is $\sum\limits_{i=1}^{n} A_G[j, i] x_i$. So

$$\left|\sum_{i=1}^{n} A_G[j, i] x_i\right| = |\lambda| \cdot |x_j|$$

Now

$$|x_j| \geq \left| \sum_{i=1}^{n} A_G[j,i] x_i \right| = |\lambda| \cdot |x_j| \implies |\lambda| \leq 1$$

■

Remark: We will denote the eigenvalues of A_G in this following manner

$$1 = |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$$

Definition 1.2.1 (Spectral Expander). *$G = (V, E)$ is λ-spectral expander if $\lambda_2 \leq \lambda$. The spectral expansion is $1 - \lambda_2 \geq 1 - \lambda$, also called spectral gap.*

Remark: We are interested when λ is constant.

CHAPTER 2

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