Report: Polyhedral Combinatorics, Matroids and Derandomization of Isolation Lemma

Instructor: Rohit Gurjar

Soham Chatterjee

sohamchatterjee999@gmail.com
Website: sohamch08.github.io

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Chapter 1

Perfect Matching Polytope

1.1 Matching Polytope

1.2 Perfect Matching Polytope

Definition 1.2.1: Perfect Matching Polytope

Let $G = (V, E)$ be a graph. For any perfect matching M of G, consider the incidence vector $x^M = (x_e)_{e \in E} \in \mathbb{R}^E$ given by

$$c_e^M = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{o/w} \end{cases}$$

For any perfect matching M of G this vector x^M is called as a *Perfect Matching Point*. The bipartite perfect matching polytope of the graph G is defined to the convex hull of all its perfect matching points,

$$PM(G) = \text{Conv}\{x^M \mid M \text{ is a perfect matching in } G\}$$

1.3 Bipartite Perfect Matching Polytope

It also defined like the perfect matching polytope where we just take the graph to be a bipartite graph. The following lemma form [LP86] gives a simple description of the perfect matching polytope of a bipartite graph G

Lemma 1.3.1 [LP86]
Let $G = (V, E)$ be a bipartite graph and $x = (x_e)_{e \in E} \in \mathbb{R}^E$. Then $x \in PM(G)$ if and only if

$$\sum_{e \in \delta(v)} x_e = 1 \quad v \in V,$$

$$x_e \geq 0 \quad e \in E$$

where for any $v \in V$, $\delta(v)$ denotes the set of edges incident on the vertex v.

Chapter 2

Bipartite Perfect Matching

2.1 Matching and Complexity

2.2 A RNC Algorithm for Search-PM

2.3 A Quasi-NC Algorithm using Isolation

We will construct an isolating weight function for bipartite graphs. The idea is to create a weight function which ensures nonzero circulations for a small set of cycles in a black-box way i.e. without having being able to compute the set efficiently. Then we will show that if we construct a smaller graph wrt this weight function then we don't have those small cycles with nonzero circulations then we have the number of cycles with twice the size of the previous ones are polynomially bounded. Then we proceed to create a new weight function which will give nonzero circulations to all the cycles with twice the size. And this way we will continue. This same type of idea we will repeatedly use with necessary modifications in **??** and **??**.

2.3.1 Isolation in Bipartite Graphs

The following lemma describes a standard trick to create a weight function for a small set of cycles in graph.

Lemma 2.3.1 [CRS93]
Let G be a graph with n vertices. Then for any number s, one can construct a set of $O(n^2 s)$ weight assignments with weights bounded by $O(n^2 s)$, such that for any set of s cycles, one of the weight assignments gives nonzero circulation to each of the s cycles.

Proof: Let us first assign exponentially large weights. Let $e_1, e_2, \ldots, e_m$ be some enumeration of the edges of G. Define a weight function w by $w(e_i) = 2^{i-1}$ for $i \in [m]$. Then clearly every cycle has a nonzero circulation. However we want to achieve this with small weights. ■

2.3.2 Union of Minimum Weight Perfect Matchings

2.3.3 Constructing Weight Assignment

Chapter 3

Bibliography

[CRS93] Suresh Chari, Pankaj Rohatgi, and Aravind Srinivasan. Randomness-Optimal Unique Element Isolation, with Applications to Perfect Matching and Related Problems. In *Proceedings of the twenty-fifth annual ACM symposium on Theory of computing*, STOC '93, page 458–467. ACM Press, June 1993.

[LP86] László Lovász and M.D. Plummer. *Matching Theory*. Number 121 in North-Holland Mathematics Studies. North-Holland, Amsterdam, 1986.