CSS.317.1 Algorithmic Game Theory

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CHAPTER 1

Introduction to Equilibriums

Chapter 2

Two Player Games

Chapter 3

Related Complexity Classes

CHAPTER 4

Dynamics and Coarse Correlated Equilibrium

Chapter 5

Potential Games

5.1 Best Response Dynamics

The existence of a Nash equilibrium is clearly a desirable property of a strategic game. In this chapter and the next we discuss some natural classes of games that do have a Nash equilibrium. The *Best-Response-Dynamics* is a straightforward procedure by which players search for a pure Nash equilibrium (PNE) of a game.

Algorithm 1: Best-Response-Dynamics (BRD)

begin
  for $t = 1, \ldots, T$ **do**
    if $t = 1$ **then**
      Each player plays an arbitrary pure strategy
    else
      Pick a player $i \in [n]$
      $s_i^t \longleftarrow \arg\min_{s_i \in S_i} c_i(s_i, s_{-i}^{t-1})$
      $s_j^t \longleftarrow s_j^{t-1}\ \forall\ j \in [n],\ j \neq i$

Note:-

Best-response dynamics can only halt at a PNE and it cycles in any game without one. It can also cycle in games that have a PNE. For example consider the following 2 player.

5.2 Network (Atomic) Congestion Games

Definition 5.2.1: Network (Atomic) Congestion Games

A network (atomic) congestion game or in short NCG consists of the following:

- A directed graph $G = (V, E)$.
- N players where each player $i \in [n]$ has some source-sink pair $(s_i, t_i) \in V \times V$ associated with it.
- Edge cost functions $c_e : [n] \longrightarrow \mathbb{R}$ for each edge $e \in E$.
- Player $i \in [N]$ has strategy set S_i = Set of all $s_i \rightsquigarrow t_i$ paths in G. $S = \bigtimes_{i=1}^{N} S_i$.
- For a strategy profile $f \in S$ (often called *flow*), let $n_e(f) = |\{i : e \in f_i\}|$. Then the cost to player i of strategy profile f is $C_i(f) = \sum_{e \in s_i} c_e(n_e(f))$.

So we can define (atomic) NCG by the tuple

$$\Big(G = (V, E), N, \{(s_i, t_i) \mid i \in [N]\}, \{c_e : [N] \to \mathbb{R}_{\geq 0} \mid e \in E\}\Big)$$

Note that unlike the last few lectures where we've been talking about utility-maximization games, this is a cost-minimization game. But of course we could just let a player's utility be the negative of its cost and everything would work as you expect.

Lemma 5.2.1

Every NCG has a PNE.

Proof: Given a strategy profile $f \in S$, we will define a potential function $\Phi : S \to \mathbb{R}_{\geq 0}$ with the property that if f is not an equilibrium then $\exists\, f' \in S$ such that $\Phi(f) > \Phi(f')$. Thus if $f^* \in S$ minimizes Φ then f^* must be a PNE.

Consider the potential function $\Phi : S \to \mathbb{R}_{\geq 0}$:

$$\Phi(s) = \sum_{e \in E} \sum_{i=1}^{n_e(f)} c_e(i)$$

Now it is enough to calculate the change in potential when a player deviates to any other strategy since for $f, f' \in S$

$$\Phi(f) - \Phi(f') = \sum_{i=0}^{N-1} \Phi(f^{(i)}) - \Phi(f^{(i+1)})$$

where $f^{(i)} = (f_1', f_2', \dots, f_i', f_{i+1}, \dots, f_N)$ and for $f^{(0)} = f$. Now for any strategy profile $f \in S$ if the player i deviates to the strategy $f_i' \in S_i$ then

$$\begin{aligned}
C_i(f) - C_i(f_i', f_{-i}) &= \left[\sum_{e \in f_i \cap f_i'} c_e(n_e(f)) + \sum_{e \in f_i \setminus f_i'} c_e(n_e(f))\right] - \left[\sum_{e \in f_i \cap f_i'} c_e(n_e(f_i', f_{-i})) + \sum_{e \in f_i' \setminus f_i} c_e(n_e(f_i', f_{-i}))\right] \\
&= \sum_{e \in f_i \cap f_i'} \underbrace{c_e(n_e(f)) - c_e(n_e(f_i', f_{-i}))}_{=0} + \sum_{e \in f_i \setminus f_i'} c_e(n_e(f)) - \sum_{e \in f_i' \setminus f_i} c_e(n_e(f_i', f_{-i})) \\
&= \sum_{e \in f_i \setminus f_i'} c_e(n_e(f)) - \sum_{e \in f_i' \setminus f_i} c_e(n_e(f) + 1)
\end{aligned}$$

Therefore the change in the potential is

$$\begin{aligned}
\Phi(f) - \Phi(f_i', f_{-i}) &= \sum_{e \in E} \sum_{i=1}^{n_e(f)} c_e(i) - \sum_{e \in E} \sum_{i=1}^{n_e(f_i', f_{-i})} c_e(i) \\
&= \sum_{e \in E} \left[\sum_{i=1}^{n_e(f)} c_e(i) - \sum_{i=1}^{n_e(f_i', f_{-i})} c_e(i)\right] \\
&= \sum_{e \in f_i \setminus f_i'} c_e(n_e(f)) - \sum_{e \in f_i' \setminus f_i} c_e(n_e(f) + 1) \\
&= C_i(f) - C_i(f_i', f_{-i})
\end{aligned}$$

So the change in potential is exactly equal to the change in the cost of the player who deviates. Therefore if f is not a PNE then $\exists\, i \in [N]$ such that $\exists\, f_i' \in S_i$ such that $c_i(f) - c_i(f_i', f_{-i}) > 0$ and therefore $\Phi(f) - \Phi(f_i', f_{-i}) > 0$. Hence every NCG has a PNE. ∎

5.3 Potential Games

Definition 5.3.1: Potential Game

A game Γ is a potential game if there exists a potential function $\Phi : S \longrightarrow \mathbb{R}_{\geq 0}$ where S is the set of strategy profiles such that $\forall\, s \in S$ and $s_i' \in S_i$ $C_i(s) - C_i(s_i', s_{-i}) = \Phi(s) - \Phi(s_i', s_{-i})$

In the proof of Theorem 5.2.1 we showed that every NCG is a potential game. Now we will show that every potential game has a PNE.

Theorem 5.3.1

Every potential game has a Pure Nash Equilibrium

Proof: For a potential game Γ let Φ is the potential function for Γ. Then $C_i(s) - C_i(s_i', s_{-i}) = \Phi(s) - \Phi(s_i', s_{-i})$. Now consider the strategy profile $s = \arg\min_{s \in S} \Phi(s)$. If any player had incentive to deviate there would be a strategy profile with smaller potential which is not possible by the Definition of s. Therefore s also has the minimum cost. Therefore s is PNE. ■

Lemma 5.3.2

Best Response Dynamics cannot cycle in a potential game.

Proof: In each iteration of the BRD every time any player deviates to play a best response the potential must decrease. Hence BRD cannot cycle. ■

Suppose there exists a time T such that every player was chosen in the BRD to choose their best response in the Best response algorithm. Then:

Lemma 5.3.3

Let $s^* \in S$ be the strategy profile at time t. If s^* is the strategy profile after T further steps of BRD then s^* is a PNE.

Proof: Since in every T steps every player has the option to deviate to another strategy but chose not to. Therefore for each player $i \in [N]$, for all $s_i' \in S_i$, $C_i(s) \leq C_i(s_i', s_{-i})$. Therefore clearly s^* is a PNE. ■

Lemma 5.3.4

Let $s^{\in}S$ be the strategy profile after $T|S|$ steps of BRD. Then s^* is a PNE.

Proof: Since BRD cannot cycle, $\exists\, s \in S$ that must have persisted fro T time steps. Therefore by the previous lemma this must be a PNE. ■

Theorem 5.3.5

In a finite potential game from an arbitrary initial outcome the Best Response Dynamics converges to a PNE if $\exists\, T \in \mathbb{N}$ such that in every T steps of BRD every player is chosen at least once.

Since every (Atomic) NCG is a potential game we have the following corollary:

Corollary 5.3.6

In an (Atomic) NCG, BRD converges to a PNE if $\exists\, T \in \mathbb{N}$ such that in every T steps of BRD every player is chosen at least once. or “every player is chosen infinitely often".

5.3.1 General Congestion Games

General Congestion Games are generalized version of (atomic) NCG. We will show that they are also potential game.

Definition 5.3.2: General Congestion Games

A basic Definition general Congestion Games or CG consists of the following:

$$\left(E, N, \{S_i \mid i \in [N]\}, \{c_e : [N] \to \mathbb{R}_{\geq 0} \mid e \in E\}\right)$$

- A base set E of congestible elements.
- N players.
- For each player $i \in [N]$ a finite set of strategies S_i where $S_i \subseteq 2^E$. $S = \bigtimes_{i=1}^{N} S_i$.
- Cost functions $c_e : [N] \to \mathbb{R}$ for each element $e \in E$.
- For a strategy profile $s \in S$ (often called *flow*), let $n_e(s) = |\{i\colon e \in s_i\}|$. Then the cost to player i of strategy profile s is $C_i(s) = \sum_{e \in s_i} c_e(n_e(s))$.

Consider the function $\Phi : S \to \mathbb{R}_{\geq 0}$ where for any strategy profile $s \in S$,

$$\Phi(s) = \sum_{e \in E} \sum_{i=1}^{n_e(s)} c_e(i)$$

that is the same function as the potential function in the case of NCG. This is also a potential function for general CG's which makes general CG's are also potential game.

5.3.2 Max Cut Game

Definition 5.3.3: Max Cut Game

A max cut game consists of the following:

1. An undirected weighted graph, $G = (V, E)$ and $w : E \to \mathbb{R}$.
2. N players.
3. For each player $i \in [N]$, has 2 strategies: $S_i = \{L, R\}$. $S = \bigtimes_{i=1}^{N} S_i$.
4. Utility functions $u_i : S \to \mathbb{R}_{\geq 0}$ for each player $i \in [N]$. For any strategy profile $s \in S$, $u_i(s) = \sum_{\substack{e=\{i,j\} \\ s_i \neq s_j}} w_e$

So like general congestion games we can denote a Max Cut game by the tuple (G, w, N). The max cut game is also a potential game. Consider the potential function $\Phi : S \to \mathbb{R}_{\geq 0}$ where for any strategy profile $s \in S$,

$$\Phi(s) = \sum_{\substack{e=\{i,j\} \\ s)i \neq s_j}} w_e$$

With this function we can prove that the Max Cut game is indeed a potential game and henceforth there exists a PNE.

5.4 Class: PLS

Definition 5.4.1: PLS (Polynomial Local Search)

A local search problem L has a set of problem instances $D_L \subseteq \Sigma^*$ where any $I \in D_L$ is a particular problem instance. For each instance $I \in D_L$ there exists a finite solution set $F_L(I) \subseteq \Sigma^*$. Let R_L be the relation that models L i.e.

$$R_L := \{(I, s) \mid I \in D_L, s \in F_L(I)\}$$

Then R_L is in PLS if:

(i) The size of every solution $s \in F_L(I)$ for any $I \in D_L$ is polynomially bounded in the size of I.

(ii) The problem instances $I \in D_L$ and the solutions $s \in F_L(I)$ are polynomial time verifiable.

(iii) There is a polynomial time computable function $C_L : \Sigma^* \times \Sigma^* \to \mathbb{R}_{\geq 0}$ that returns for each $I \in D_L$ and each $s \in F_L(I)$ the cost $C_L(I, s)$.

(iv) There is a polynomial time computable function $N : (I, s) \mapsto S$ where $S \subseteq F_L(I)$ i.e. returns the set of neighbors of lower cost for each $I \in D_L$ and each $s \in F_L(I)$ or states the s is locally optimal.

Note that for each $I \in D_L$ and each $s \in F_L(I)$ using (iii) and (iv) we can find a neighboring solutions of lower cost of s or determine s is locally minimal. The problem we want to focus is to find a locally minimal cost solution given an instance I of L.

Goal. *Find a locally minimal cost solution given an instance I of L.*

Finding PNE in a Congestion Game (PNE-CG) is in PLS. Also finding PNE in a Max Cut Game is in PLS.

Lemma 5.4.1

PNE-CG $\in$ PLS

Proof: We will show that computing PNE in congestion games is an instance of PLS by fist describing conditions needed for PLS. Suppose $\left(E, N, \{S_i \mid i \in [N]\}, \{c_e : [N] \to \mathbb{R}_{\geq 0} \mid e \in E\}\right)$ be any instance of CG.

(i) Any strategy profile is a solution for a specific instance of congestion games. Now each element of strategy profile by Definition of CG is a subset of the base set E. Therefore size of any strategy profile is polynomially bounded by the size of the instance of the CG.

(ii) For solution for the CG has to be valid strategy profile. So given any N-tuple of subsets of E we can check if the i^{th} element of the tuple is in S_i for all $i \in [N]$ in polynomial time.

(iii) For any strategy profile $s \in S$ we return the potential function for the CG as the cost function $\Phi(s) = \sum_{e \in E} \sum_{j=1}^{n_e(s)} c_e(i)$. This function is a polynomial time computable function.

(iv) Given a strategy profile s it checks if it is a PNE by checking if we can switch the strategy of agent i from s_i to s_i' such that $\Phi(s) > \Phi(s_i', s_{-i})$ for all agents. If reduces then returns those strategy profiles. This is also polynomial time computable.

Therefore PNE-CG is in PLS. ■

Definition 5.4.2: PLS-Reductions

A local search problem L_1 is PLS-reducible to another local search problem L_2, denoted by $L_1 \leq_{\text{PLS}} L_2$ if there are two polynomial time computable functions $f : D_{L_1} \to D_{L_2}$ and $g : (I_1, s_2) \mapsto s_1$ where I_1, D_{L_1}, $s_2 \in F_{L_2}(f(I_1))$ and $s_1 \in F_{L_1}(I_1)$ such that:

(i) If I_1 is an instance of L_1 then $f(I_1)$ is an instance of L_2

(ii) If s_2 is an solution of $f(I_1)$ of L_2 then $g(I_1, s_2)$ is a solution for I_1 of L_1

(iii) If s_2 is a local optimum for instance $f(I_1)$ of L_2 then $f(I_1, s_2)$ has to be a local optimum for instance I_1 of L_1.

Theorem 5.4.2

The Max Cut Game is PLS-complete

We will not prove this but using this theorem we will prove the next theorem about general congestion games.

Theorem 5.4.3

General Congestion Games are PLS-complete

Proof: We already shown that general congestion games are in PLS. For completeness we will show a PLS-reduction from the Max Cut Game. Let $\Gamma = (G, w, N)$ be an instance of a max cut game. We will create a congestion game $\Gamma' = \Big(E', N, \{S_i \mid i \in [N]\}, \{c_e : [N] \to \mathbb{R}_{\geq 0} \mid e \in E\}\Big)$ with N players.

- E': For each edge $e \in E$ add two elements e_L and e_R to E'. So $E' = \{e_L, e_R \mid e \in E\}$.
- S_i: Each player $i \in [N]$ has two strategies $S_i = \{\{e_L \mid e \text{ incident on } i\}, \{e_R \mid e \text{ incident on } i\}\}$. Thus if $e = \{i, j\} \in E$ then the elements e_L, e_R can be used by exactly 2 players i and j in Γ'.
- c_e : Define $c_{e_L}(1) = c_{e_R}(1) = 0$ and $c_{e_L}(2) = c_{e_R}(2) = w(e)$.

This defines a general congestion games. Now consider a strategy profile s in Γ'. Then $C_i(s) = \sum\limits_{\substack{j:\{i,j\}\in E\\ s_j \neq s_i}} e_{i,j}$. Therefore the local minimum in the Γ' gives a local minimum in Γ. Hence max cut game is PLS-reducible to general congestion game. ■

Chapter 6

Efficiency of Equilibria

Here we are going to leave aside for now the question of how a game arrived at an equilibrium and instead we will study '*quality of equilibria*'. We want to study how close to optimal the equilibria of a game are. But for that we have to define this 'closeness' and 'optimal' by introducing cost to every strategy and we basically want to find a equilibria which very close to the minimum cost strategy profile.

6.1 Cost Minimization Games

Definition 6.1.1: Cost Minimization Games

It is a game with n players $[n]$, with their strategy sets $S_1, \dots, S_n$ where $S = \bigtimes_{i=1}^{n} S_i$ and a cost function $C_i : S \to \mathbb{R}$ for each $i \in [n]$.

There is an objective function $f : S \to \mathbb{R}$ with which the different strategy profiles are compared. There are many common choices for f. Conventionally the concepts PNE, MNE, CE, CCE are defined for utility-maximization games with all of its inequalities reversed. But the two Definitions are completely equivalent.

- **Pure Nash Equilibria**: A strategy profile $s \in S$ of a cost-minimization game Γ is a *Pure Nash Equilibrium* if for every player $i \in [n]$ and for all $s_i' \in S_i$, $C_i(s) \le C_i(s_i', s_{-i})$.
- **Mixed Nash Equilibria**: A mixed strategy profile $\sigma \in \Sigma$ of a cost-minimization game Γ is a *Mixed Nash Equilibria* if for every player $i \in [n]$ and for all $s_i' \in S_i$, $\underset{s\sim\sigma}{\mathbb{E}}[C_i(s)] \le \underset{s\sim\sigma}{\mathbb{E}}[C_i(s_i', s_{-i})]$
- **Correlated Equilibria**: A distribution μ over S of a cost-minimization game Γ is a *Correlated Equilibria* if for every player $i \in [n]$ and for all $s_i' \in S_i$, $\underset{s\sim\mu}{\mathbb{E}}[C_i(s) \mid s_i] \le \underset{s\sim\mu}{\mathbb{E}}[C_i(s_i', s_{-i}) \mid s_i]$
- **Coarse Correlated Equilibria**: A distribution μ over S of a cost-minimization game Γ is a *Coarse Correlated Equilibria* if for every player $i \in [n]$ and for all $s_i' \in S_i$, $\underset{s\sim\mu}{\mathbb{E}}[C_i(s)] \le \underset{s\sim\mu}{\mathbb{E}}[C_i(s_i', s_{-i})]$

6.2 Pareto Optimality

Definition 6.2.1: Pareto Optimal Strategy

Given a game Γ, a strategy profile $s \in S$ is pareto optimal also denoted by PO if $\nexists\, s' \in S$ such that

$$\forall\, i \in [n],\ c_i(s') \le c_i(s) \qquad \exists\, i \in [n]\ c_i(s') < c_i(s)$$

or equivalently for all $s' \in S$, either $\forall\, i \in [n]$, $c_i(s) = c_i(s')$ or $\exists\, i \in [n]$, $c_i(s') > c_i(s)$.

Economists call Pareto Optimality "efficiency". PO induces a partial order over the set of all strategy profiles. Let $s, s' \in S$. We say that $s >_p s'$ if $\forall\, i \in [n],\ c_i(s') \leq c_i(s)$ and $\exists\, i \in [n]\ c_i(s') < c_i(s)$.

To introduce a total order we can think of social welfare function for example:

(1) Utilitarian Social Welfare: For any $s \in S$, $C(s) = \sum_{i=1}^{n} c_i(s)$

(2) Nash Social Welfare: For any $s \in S$, $C(s) = \sum_{i=1}^{n} c_i(s)$

(3) Egalitarian Social Welfare: For any $s \in S$, $C(s) = \max_{i=1}^{n} c_i(s)$

This allows us to quantitatively see how good or bad a equilibrium is by comparing two strategy profiles. Typically by "social welfare" we mean utilitarian social welfare. We will focus on calculating utilitarian social welfare from now on.

6.3 Price of Anarchy

For a game Γ we also want to know how bad is the social welfare at equilibrium compared to the best possible social welfare. This ratio is know as Price of Anarchy.

Definition 6.3.1: Price of Anarchy

We denote it by PoA. For a game Γ:

$$\begin{aligned} \mathrm{PoA}(\Gamma) &= \frac{\text{Social welfare of "worst equilibrium"}}{\text{Optimal social welfare}} \\ &= \frac{\max\left\{\sum_{i=1}^{n} C_i(s) : s \in S \text{ is an PNE}\right\}}{\min\left\{\sum_{i=1}^{n} C_i(s) : s \in S\right\}} \end{aligned}$$

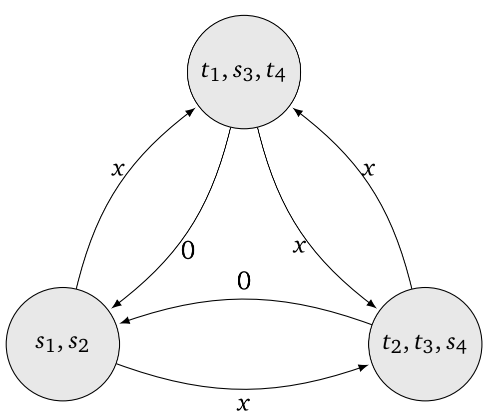
6.3.1 PoA of Network (Atomic) Congestion Games

Theorem 6.3.1

The PoA in network congestion games with affine cost functions is $\frac{5}{2}$.

Proof: We will prove this in two stages. First we will show that the lower bound for PoA for NCG is at least $\frac{5}{2}$ by constructing an example. Then we will show the upper bound.

Lower Bound $\geq \frac{5}{2}$:



Consider the NCG with 4 players drawn on the left.

The optimal cost for this NCG is when every player uses the single-edge $s_i \to t_i$ paths. And the cost for each such path is 1 as none of these 4 edges is used by more than 1 player. So the optimal value is 4. Note that this is also a PNE.

Now we will calculate the worst PNE. Every player uses the 2-edge $s_i \to t_i$ paths. Then the costs for each players are

1: 3, 2: 3, 3: 2, 4: 2

Therefore total cost is 10. Hence PoA $\geq \frac{10}{4} = \frac{5}{2}$.

Upper Bound $\leq \frac{5}{2}$:

Lemma 6.3.2

For any NCG with N players if s is any strategy profile and s^* is a PNE then $C(s^*) \leq \frac{5}{2}C(s)$.

Proof: For each player $i \in [N]$ we have

$$C_i(s^*) \leq C_i(s_i, s^*_{-i}) = \sum_{e\in s_i \cap s_i^*} c_e(n_e(s^*)) + \sum_{e\in s_i \setminus s_i^*} c_e(n_e(s^*)+1) \leq \sum_{e\in s_i} c_e(n_e(s^*)+1)$$

Now summing over all players we get the total cost

$$\begin{aligned}
C(s^*) = \sum_{i\in[N]} c_i(s^*) &= \sum_{i\in[N]}\sum_{e\in s_i} c_e(n_e(s^*)+1) \\
&= \sum_{e\in E} c_e(n_e(s^*)+1)n_e(s) \\
&= \sum_{e\in E} \Big(a_e(n_e(s^*)+1)+b_e\Big)n_e(s) && [\text{Let } c_e(i) = a_e i + b_e,\ a_e, b_e \geq 0] \\
&= \sum_{e\in E} a_e n_e(s)(n_e(s^*)+1) + b_e n_e(s) \\
&\leq \sum_{e\in E} a_e\left(\frac{5}{3}n_e^2(s) + \frac{1}{3}n_e^2(s^*)\right) + b_e n_e(s) && \left[x, y \in \mathbb{Z}, x(y+1) \leq \frac{5}{3}x^2 + \frac{1}{3}y^2\right] \\
&= \sum_{e\in E} n_e(s)\left(\frac{5}{3}a_e n_e(s) + b_e\right) + \frac{1}{3}a_e n_e^2(s^*) \\
&\leq \frac{5}{3}\sum_{e\in E} n_e(s)(a_e n_e(s) + b_e) + \frac{1}{3}\sum_{e\in E} n_e(s^*)(a_e n_e(s^*) + b_e) \\
&= \frac{5}{3}C(s) + \frac{1}{3}C(s^*)
\end{aligned}$$

Therefore we get

$$C(s^*) \leq \frac{5}{3}C(s) + \frac{1}{3}C(s^*) \implies \frac{2}{3}C(s^*) \leq \frac{5}{3}C(s) \implies C(s^*) \leq \frac{5}{2}C(s)$$

Hence we have the lemma. ■

Using the lemma for any PNE $s^* \in S$ and any strategy profile $s \in S$ we have $C(s^*) \leq \frac{5}{2}C(s)$. Therefore the worst cost PNE and the optimal strategy profile also follows the inequality. Therefore the PoA is at most $\frac{5}{2}$. Therefore we get the upper bound also. ■

6.4 Global Connection Games

Definition 6.4.1: Global Connection Games

A Global Connection Games or in short GCG consists of the following:

$$\Big(G=(V,E),N,\{(s_i,t_i)\mid i\in[N]\},\{c_e\mid e\in E,c_e\geq 0\}\Big)$$

- A directed graph $G=(V,E)$.
- N players where each player $i\in[n]$ has some source-sink pair $(s_i,t_i)\in V\times V$ associated with it.
- Edge costs c_e for each edge $e\in E$ where $c_e\geq 0$.
- Player $i\in[N]$ has strategy set $S_i=$ Set of all $s_i\rightsquigarrow t_i$ paths in G. $S=\bigtimes_{i=1}^{N}S_i$.
- For a strategy profile $f\in S$, let $n_e(f)=|\{i\colon e\in f_i\}|$. Then the cost to player i of strategy profile f is $C_i(f)=\sum\limits_{e\in s_i}\frac{c_e}{n_e(f)}$ i.e. the cost of edge is divides equally among all the players using that edge.

Therefore for any strategy profile the total cost is

$$C(f)=\sum_{i\in[N]}C_i(f)=\sum_{i\in[N]}\sum_{e\in s_i}\frac{c_e}{n_e(s)}=\sum_{\substack{e\in E\\ n_e(f)>0}}c_e$$

Lemma 6.4.1

GCG is a potential game

Proof: Consider the function $\Phi:S\to\mathbb{R}_{\geq 0}$ where for any strategy profile $f\in S$

$$\Phi(f)=\sum_{e\in E}\sum_{j=1}^{n_e(f)}\frac{c_e}{j}=\sum_{\substack{e\in E\\ n_e(f)\geq 1}}c_eH_{n_e(s)}$$

where H_n is the n^{th} harmonic number. Then we have for any $f\in S$ and $f_i'\in S_i$

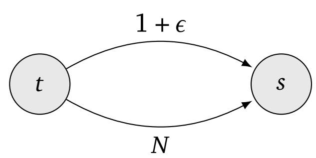
$$\begin{aligned}
\Phi(f)-\Phi(f_i',f_{-i}) &= \sum_{e\in E}\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{e\in E}\sum_{j=1}^{n_e(f_i',f_{-i})}\frac{c_e}{j}=\sum_{e\in E}\left[\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{k=1}^{n_e(f_i',f_{-i})}\frac{c_e}{k}\right]\\
&= \sum_{e\in E\setminus f_i\cup f_i'}\left[\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{k=1}^{n_e(f_i',f_{-i})}\frac{c_e}{k}\right]+\sum_{e\in f_i\cap f_i'}\left[\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{k=1}^{n_e(f_i',f_{-i})}\frac{c_e}{k}\right]+\sum_{e\in f_i\setminus f_i'}\left[\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{k=1}^{n_e(f_i',f_{-i})}\frac{c_e}{k}\right]\\
&\qquad +\sum_{e\in f_i'\setminus f_i}\left[\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{k=1}^{n_e(f_i',f_{-i})}\frac{c_e}{k}\right]\\
&= \underbrace{\sum_{e\in f_i\cap f_i'}\left[\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{k=1}^{n_e(f_i',f_{-i})}\frac{c_e}{k}\right]}_{0}+\sum_{e\in f_i\setminus f_i'}\left[\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{k=1}^{n_e(f_i',f_{-i})}\frac{c_e}{k}\right]+\sum_{e\in f_i'\setminus f_i}\left[\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{k=1}^{n_e(f_i',f_{-i})}\frac{c_e}{k}\right]\\
&= \sum_{e\in f_i\cap f_i'}\left[\sum_{j=1}^{n_e(f)}\frac{c_e}{j}-\sum_{k=1}^{n_e(f_i',f_{-i})}\frac{c_e}{k}\right]+\sum_{e\in f_i\setminus f_i'}\frac{c_e}{n_e(f)}-\sum_{e\in f_i'\setminus f_i}\frac{c_e}{n_e(f_i',f_{-i})}=C_i(f)-C_i(f_i',f_{-i})
\end{aligned}$$

Therefore GCG is a potential game. ■

Theorem 6.4.2
The PoA for any GCG with N players is N.

Proof: Like the case for NCG we will prove this in two stages. First we will show that the lower bound for PoA for GCG with N players is at least N by constructing an example. Then we will show the upper bound.

Lower Bound $\geq N$:



Consider the GCG with 5 players drawn on the left where for each player $i \in [N]$, $s_i = s$ and $t_i = t$.

The optimal cost for this GCG is when every player uses the edge with cost $1+\epsilon$. Hence the total cost is $1+\epsilon$.

Now we will calculate the worst PNE. Every player uses the edge with cost N. Therefore total cost is N. Hence PoA $\geq \frac{N}{1+\epsilon} \geq N$.

Upper Bound $\leq N$:

Let Γ be any GCG. Suppose s^* be any PNE of Γ and s_{OPT} is the optimal strategy profile.

Claim 6.4.3
$C_i(s^*) \leq \text{cost}(s_{\text{OPT}_i})$ where $\text{cost}(s_{\text{OPT}_i}) = \sum\limits_{e \in s_{\text{OPT}_i}} c_e$.

Proof: Suppose not. Then $C_i(s^*) > \text{cost}(s_{\text{OPT}_i})$. Now we have

$$C_i(s_{\text{OPT}_i}, s^*_{-i}) = \sum_{e \in s_{\text{OPT}_i}} \frac{c_e}{n_e(s_{\text{OPT}_i}, s^*_{-i})} \leq \sum_{e \in s_{\text{OPT}_i}} c_e = \text{cost}(s_{\text{OPT}_i})$$

Therefore we get $C_i(s^*) > C_i(s_{\text{OPT}_i}, s^*_{-i})$. But s^* is a PNE. Hence contradiction $\lightning$ ■

Therefore we have

$$C(s^*) = \sum_{i \in [N]} C_i(s^*) \leq \sum_{i \in [N]} \text{cost}(s_{\text{OPT}_i}) \leq \sum_{i \in [N]} \sum_{e \in s_{\text{OPT}_i}} c_e \leq \sum_{i \in [N]} \sum_{e \in s_{\text{OPT}_i}} N \cdot \frac{c_e}{n_e(s_{\text{OPT}})} = N \sum_{i \in [N]} C_i(s_{\text{OPT}}) = N \cdot C(s_{\text{OPT}})$$

Hence we get that $\text{PoA}(\Gamma) \leq N$. ■

6.5 Price of Stability

In the case of GCG we can see that always comparing the worst Pure Nash Equilibria with the optimal strategy may lead to very large value. So instead sometimes we prefer to compare the best PNE and the optimal strategy.

Definition 6.5.1: Price of Stability

We denote it by PoS. For a game Γ:

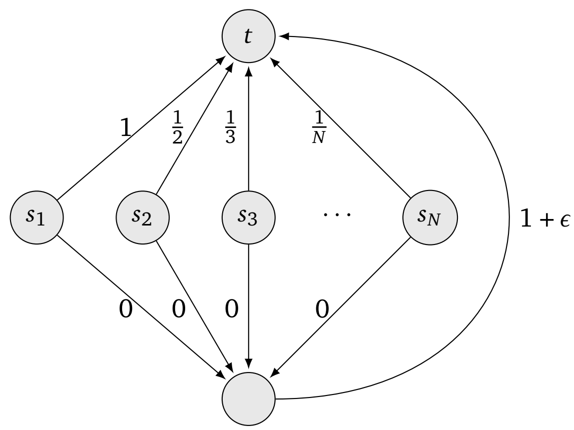
$$\text{PoS}(\Gamma) = \frac{\text{Social welfare of “best equilibrium"}}{\text{Optimal social welfare}} = \frac{\min\left\{\sum\limits_{i=1}^{n} C_i(s) : s \in S \text{ is an PNE}\right\}}{\min\left\{\sum\limits_{i=1}^{n} C_i(s) : s \in S\right\}}$$

6.5.1 PoS of Global Connection Games

Lemma 6.5.1

In a GCG with N players the PoS is at least H_N.

Proof: We will prove this using an example of GCG.



Consider the GCG with N players drawn on the left where for each player $i \in [N]$, $t_i = t$. The optimal cost for this GCG is when every player goes to t using the edge with 0 weight and then the edge with cost $1+\epsilon$. Hence the total optimal cost is $1+\epsilon$.

Now we will calculate the best PNE. The only PNE is when every player uses the direct edge with cost $\frac{1}{i}$ for each player $i \in [N]$. Therefore total cost is H_N. Hence PoS is $\frac{H_N}{1+\epsilon} \geq H_N$. ■

Lemma 6.5.2

The PoS of any GCG with N players is at most H_N.

Proof: Suppose Γ be any GCG. Let f be any strategy profile and f^* minimizes Φ. Hence f^* is an PNE and $\Phi(f^*) \leq \Phi(f)$. Then we have

$$C(f) = \sum_{\substack{e\in E\\ n_e(f)>0}} c_e = \frac{1}{H_N}\sum_{\substack{e\in E\\ n_e(f)>0}} c_e H_N \geq \frac{1}{H_N}\sum_{\substack{e\in E\\ n_e(f)>0}} c_e H_{n_e(f)} = \frac{1}{H_N}\Phi(f) \geq \frac{1}{H_N}\phi(f^*)$$

Now $\forall\ f' \in S$ we have

$$\Phi(f') = \sum_{\substack{e\in E\\ n_e(f')\geq 1}} c_e H_{n_e(f')} \geq \sum_{\substack{e\in E\\ n_e(f')\geq 1}} c_e = C(f')$$

Therefore we get $C(f) \geq \frac{1}{H_N}C(f^*)$. Hence PoS$(\Gamma) \leq H_N$. ■

Note:-

The general form of argument for potential games goes like this: If $\alpha C(s) \leq \Phi(s) \leq \beta C(s)$, then PoS $\leq \frac{\beta}{\alpha}$

Therefore with these two lemmas we get the final theorem:

Theorem 6.5.3

The PoS of any Global Connection Games with N players is the H_N where H_N is N^{th} harmonic number.

CHAPTER 7

Price of Anarchy Bounds in Smooth Games

7.1 Facility Location Game

Definition 7.1.1: Facility Location Game

A Facility Location Game consists of

(i) There is a set L, of n locations.

(ii) SP is the set of k service providers or players.

(iii) Player $i \in SP$ has its strategy set some $S_i \subseteq L$. For player i, S_i represents the places where player i might build a facility. For some player $i \in SP$, it may also be the case that $S_i = \emptyset$, i.e. player i prefers nowhere.

(iv) A set C of m clients.

(v) Each client $j \in C$ has some value $\pi_j \geq 0$. Think of this as how much the client is willing to pay for the service that the facilities provide.

(vi) For all $l \in L$ and $j \in C$ let $c(l, j)$ is the transportation cost.

(vii) For all $i \in SP$ and $j \in C$ let $p(i, j)$ is the price of i for serving j.

Assumption 7.1. *We will have the following assumptions for the game:*

- $c(l, j) \neq c(l', j)$ *for all* $l, l' \in L$, $l \neq l'$ *and* $j \in C$.
- $\pi_j \geq c(l, j)$ *for all* $l \in L$ *and* $j \in C$.

7.1.1 Utilities: Definition

We will define the utilities of client and service providers. Let $s \in S$ be any strategy profile. If a client $j \in C$ chooses the service provider $i \in SP$ then we denote $SP(j) = i$.

So for any client $j \in C$ the utility for strategy profile s is

$$u_j(s) = \pi_j - p(SP(j), j)$$

and for any service provider $i \in SP$, the utility of i is

$$u_i(s) = \sum_{j:\ SP(j)=i} p(i, j) - c(s_i, j)$$

Now we define the utilitarian social welfare for a strategy profile s to be

$$V(s) = \sum_{i \in SP} u_i(s) + \sum_{j \in C} u_j(s) = \sum_{j \in C} \pi_j - c\left(s_{SP(j)}, j\right)$$

To make the utilities of every service provider to be non-negative we have another assumption:

Assumption 7.2. *For any strategy profile s, $\forall\, i \in SP$ and $\forall\, j \in C$, $p(i, j) \geq c(s_i, j)$*

Remark: This social welfare considers the clients as players as well but with simple strategies choosing the least price service provider

7.1.2 Choosing Prices

Note that technically prices too are chosen by the service providers. However we will show that given a strategy profile the prices are fixed at equilibrium.

Now the service provider i at location s_i can get profit from client $j \in C$ only if it's the closest i.e. transportation cost satisfies $c(s_i, j) \leq c(l, j)$ for all $j \in L$. Any client $j \in C$ chooses the service provider $i \in SP$ if the price charged by the i to j is minimum i.e. $p(i, j) = \min_{i' \in SP} p(i', j)$.

Observation. *For any client $j \in C$ and any service provider $i \in SP$ in a strategy profile s, $SP(j) = i$ if*

(i) $i \in \arg\min_{i' \in SP} p(i, j)$

(ii) $c(s_i, j) = \min_{i' \in SP} c(s_{i'}, j)$

Lemma 7.1.1

For any client $j \in C$ and any service provider $i \in SP$ in a strategy profile s, $SP(j) = i$ if

(i) $c(s_i, j) = \min_{i' \in SP} c(s_{i'}, j)$

(ii) $p(i, j) = \max \left\{ c(s_i, j), \min_{\substack{i' \in SP \\ i' \neq i'}} c(s_{i'}, j) \right\}$

Proof: The first condition directly follows from the observation. We will prove the second one. Let $\hat{i} = \arg\min_{i' \neq i} c(s_{i'}, j)$. Suppose $c(s_i, j) < c(s_{\hat{i}}, j)$. Then we have to show $p(i, j) = c(s_{\hat{i}}, j)$.

Since the service providers want to maximize their utility they want to maximize the price charged to client. Now if $p(i, j) \neq c(s_{\hat{i}}, j)$ then we can assume $p(i, j) > c(s_{\hat{i}}, j)$. If $p(i, j) > c(s_{\hat{i}}, j)$ then if $p(\hat{i}, j) = \frac{1}{2}\left(p(i, j) + c(s_{\hat{i}}, j)\right)$ then $p(\hat{i}, j) < p(i, j)$. Hence $SP(j) = \hat{i}$ but we are given $SP(j) = i$. Hence contradiction $\lightning$ Hence we have the lemma. ■

7.1.3 Potential Game

We will now show that Facility Location Game is a potential game. We will take the function V which is the total utilitarian social welfare as the potential function of the game. And we will show now that this follows the condition of potential games.

Theorem 7.1.2

For any strategy profile s, for all $i \in SP$, $\forall\, s_i' \in S_i$

$$V(s_i', s_{-i}) - V(s) = u_i(s_i', s_{-i}) - u_i(s)$$

Proof: We will show that $u_i(s_i' = \emptyset, s_{-i}) - u_i(s) = V(s_i' = \emptyset, s_{-i}) - V(s)$. Now we have

$$V(s) = \sum_{j \in C} \pi_j - c\left(s_{SP(j)}, j\right) = \sum_{j \in C} \pi_j - \min_{i' \in SP} c(s_{i'}, j) \qquad V(s_i', s_{-i}) = \sum_{j \in C} \pi_j - \min_{\substack{i \in SP \\ i' \neq i}} c(s_{i'}, j)$$

Therefore

$$V(s_i' = \emptyset, s_{-i}) - V(s) = \sum_{j \in C} \min_{i' \in SP} c(s_{i'}, j) - \min_{\substack{i \in SP \\ i' \neq i}} c(s_{i'}, j)$$

Now for $j \in C$ for which $SP(j) \neq i$ we have $\min_{i' \in SP} c(s_{i'}, j) = \min_{\substack{i \in SP \\ i' \neq i}} c(s_{i'}, j$. By Lemma 7.1.1 we have $\min_{\substack{i \in SP \\ i' \neq i}} c(s_{i'}, j = p(i, j)$ for $SP(j) = i$. Then we have

$$V(s_i' = \emptyset, s_{-i}) - V(s) = \sum_{j:SP(j) = i} p(i, j) - c(s_i, j) = u_i(s_i', s_{-i}) - u_i(s)$$

Therefore V is a potential function of satisfying the condition of potential game. Hence Facility Location Game is a potential game. ■

Corollary 7.1.3

The PoS of Facility Location Games is 1

Proof: Since the potential function and the utility functions are same the strategy which maximizes the utility and the PNE which has maximum utility are the same strategy profiles. Therefore PoS is 1. ■

7.2 Valid Utility Games

7.3 Smooth Games

Definition 7.3.1: (λ, μ)-smooth Cost Minimization Game

A cost minimization game is (λ, μ) smooth if for all strategy profiles $s, s' \in S$

$$\sum_{i=1}^{n} c_i(s_i', s_{-i}) \leq \lambda cost(s') + \mu cost(s)$$

Therefore (Atomic) Network Congestion Games is $\left(\frac{5}{3}, \frac{1}{3}\right)$-smooth. If a game is (λ, μ) smooth then we get an upper bound on the PoA directly.

Theorem 7.3.1

For any (λ, μ)-smooth cost-minimization game the PoA $\leq \frac{\lambda}{1-\mu}$.

Proof: Let s^* be an equilibrium and s be minimum cost strategy profile. Since the game is (λ, μ) smooth we have

$$cost(s^*) = \sum_{i=1}^{n} c_i(s^*) \leq \sum_{i=1}^{n} c_i(s_i, s_{-i}^*) \leq \lambda cost(s) + \mu cost(s^*)$$

Therefore $\frac{cost(s^*)}{cost(s)} \leq \frac{\lambda}{1-\mu}$. ■

7.3.1 Bound of PoA for CCE

We can also extend this bound of PoA for coarse correlated equilibriums too. So we have the following theorem:

Theorem 7.3.2
For a (λ, μ)-smooth cost-minimization game, the PoA for CCE's is at most $\frac{\lambda}{1-\mu}$.

Proof: We know the distribution $\mathscr{D} \in \Delta_{|S|}$ is a CCE if $\forall\, i \in [n], \forall\, s'_i \in S_i, \underset{s\sim\mathscr{D}}{\mathbb{E}}[c_i(s)] \leq \underset{s\sim\mathscr{D}}{\mathbb{E}}[c_i(s'_i, s_{-i})]$. Let s^* be the minimum cost pure strategy profile. Then we have

$$\underset{s\sim\mathscr{D}}{\mathbb{E}}[cost(s)] = \sum_{i=1}^{n} \underset{s\sim\mathscr{D}}{\mathbb{E}}[c_i(s)] \leq \sum_{i=1}^{n} \underset{s\sim\mathscr{D}}{\mathbb{E}}[c_i(s^*_i, s_{-i})] \leq \sum_{i=1}^{n} \lambda \underset{s\sim\mathscr{D}}{\mathbb{E}}[c_i(s^*)] + \mu \underset{s\sim\mathscr{D}}{\mathbb{E}}[c_i(s)] = \lambda cost(s^*) + \mu \underset{s\sim\mathscr{D}}{\mathbb{E}}[cost(s)]$$

Therefore we have $\frac{\underset{s\sim\mathscr{D}}{\mathbb{E}}[cost(s)]}{cost(s^*)} \leq \frac{\lambda}{1-\mu}$. Hence we have the upper bound PoA.

■

7.3.2 Bound of PoA for ε-PNE

Definition 7.3.2: ε − PNE

For a cost-minimization game $s \in S$ is an ε − PNE if $\forall\, i \in [n], \forall\, s'_i \in S_i$ if

$$c_i(s'_i, s_{-i}) \geq \frac{1}{1+\varepsilon} c_i(s)$$

We can extend the bound of PoA for PNE's to even approximate PNE's too but we don't get the exact $\frac{\lambda}{1-\mu}$ bound. Instead there is a

7.3.3 Utility Maximization

Definition 7.3.3: (λ, μ)-smooth Utility Maximization Game

A utility maximization game is (λ, μ) smooth if for all strategy profiles $s, s' \in S$

$$\sum_{i=1}^{n} u_i(s'_i, s_{-i}) \geq \lambda U(s') - \mu U(s)$$

Therefore any Valid Utility Game is $(1, 1,)$-smooth game. And like in the case of cost minimization game we get an upper bound on the PoA for utility maximization game.

Theorem 7.3.3
For any (λ, μ)-smooth utility maximization game the PoA $\leq \frac{\lambda}{1+\mu}$.

7.4 Load Balancing Game

Chapter 8

Introduction to Mechanism Design