CSS.102.1 Mathematical Foundations of Computer Science

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TIFR 2024, Aug-Dec

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Contents

Chapter 1
Linear Algebra

Chapter 2

Combinatorics

2.1 Twelve Problems: n Balls in m Bins

Theorem 2.1.1

	≤ 1 balls/bin $(m \geq n)$	≥ 1 balls/bin $(m \leq n)$	Unrestricted
Identical Balls, Identical Bins	1	$P(n,m)$	$\sum\limits_{i=1}^{m} P(n,i)$
Identical Balls, Distinguishable Bins	$\binom{m}{n}$	$\binom{m-1}{n-1}$	$\binom{n+m-1}{m-1}$
Distinguishable Balls, Identical Bins	1	$S_2(n,m)$	$\sum\limits_{i=1}^{m} S_2(ni)$
Distinguishable Balls, Distinguishable Bins	$\binom{m}{n}n!$	$S_2(n,m)m!$	m^n

Proof:

■

2.2 Stirling Numbers

2.2.1 Stirling Number of Second Kind

Definition 2.2.1: Stirling Number of The Second Kind

It is the number of ways to partition the set $[n]$ into m nonempty parts.

Clearly if we take the n balls to be the set $[n]$ the balls become distinguishable and each partition is bin and the order order of the partition doesn't matter the bins are identical. So the it becomes the number of ways n distinguishable balls divided into m identical bins.

Now we will see some recursion relations of the Stirling number of the first kind.

Lemma 2.2.1

$S_2(n,m) = S_2(n-1,m-1) + mS_2(n-1,m)$

Combinatorial Proof: We have the balls $[n]$. Then there are two cases. The bin containing ball '1' can has only 1 ball or it can have ≥ 2 balls.

For the first case the bin containing ball '1' has only one balls. So the rest of the $n-1$ balls are divided into the rest of the $m-1$ bins. The number of ways this is done is $S_2(n-1, m-1)$.

For the second case the bin containing ball '1' has at least 2 balls. In that case apart from the ball '1' all the other balls are filled into m identical bins where each bin has at least 1 ball. So we can think this scenario in other way that is first we fill bins with all the balls except '1' and then we choose where to put the ball '1'. So the number of ways the balls, $\{2, 3, \ldots, n\}$ i.e. $n-1$ distinguishable balls can be divided into m bins is $S_2(n-1, m)$. Now there are m choices for the ball '1' to be partnered up. Hence for this case there are $mS_2(n-1, m)$ many ways.

Therefore the total number of ways the n distinguishable balls can be divided into m bins so that each bin has at least 1 ball is $S_(n-1, m-1) + mS_2(n-1, m)$. Therefore we get $S_2(n, m) = S_2(n-1, m-1) + mS_2(n-1, m)$. ■

Theorem 2.2.2

$S_2(n+1, m+1) = \sum_{i=m}^{n} \binom{n}{i} S_2(i, m)$

Combinatorial Proof: On the *LHS* we are counting the number of ways to partition $[n+1]$ into $m+1$ parts.

For the *RHS* let's focus on the element $n+1$. So we drop the element from $[n+1]$ in the $(m+1)^{th}$ part. The $(m+1)^{th}$ block can have k elements from $[n]$ which are partnered by $n+1$ where $0 \leq k \leq n-m$. We have $k \leq n-m$ since all the other m parts have at least 1 element that leaves us $n-m$ elements to choose. So there are $\binom{n}{k}$ ways to choose the k elements. The remaining $n-k$ elements are divided into m parts which can be done in $S_2(n-k, m)$ many choices. So in total we have $\sum +k = 0^{n-m} S_2(n-k, m)$ ways to divide $[n+1]$ into $m+1$ parts. Therefore we have

$$S_2(n+1, m+1) = \sum_{i=0}^{n-m} \binom{n}{i} S_2(n-i, m) = \sum_{i=0}^{n-m} \binom{n}{n-i} S_2(n-i, m) = \sum_{i=m}^{n} \binom{n}{i} S_2(i, m)$$

■

Algebraic Proof: We will prove by Induction.

$$S_2(n+1,m+1) = S_2(n,m) + (m+1)S_2(n,m+1)$$

$$= \sum_{i=m-1}^{n-1}\binom{n-1}{i}S_2(i,m-1) + (m+1)\sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m)$$

$$= \sum_{i=m-1}^{n-1}\binom{n-1}{i}S_2(i,m-1) + m\sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m) + \sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m)$$

$$= \sum_{i=m}^{n}\binom{n-1}{i-1}S_2(i-1,m-1) + m\sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m) + \sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m)$$

$$= \sum_{i=m}^{n}\binom{n-1}{i-1}S_2(i-1,m-1) + m\sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m) + \sum_{j=m}^{n-1}\left[\binom{n}{j} - \binom{n-1}{j-1}\right]S_2(j,m)$$

$$= \sum_{i=m}^{n}\binom{n-1}{i-1}S_2(i-1,m-1) + m\sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m) + \sum_{j=m}^{n-1}\binom{n}{j}S_2(j,m) - \sum_{j=m}^{n-1}\binom{n-1}{j-1}S_2(j,m)$$

$$= \sum_{i=m}^{n}\binom{n-1}{i-1}S_2(i-1,m-1) + m\sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m) + \sum_{j=m}^{n-1}\binom{n}{j}S_2(j,m) - \sum_{j=m}^{n-1}\binom{n-1}{j-1}\Big[S_2(j-1,m-1) + mS_2(j-1,m)\Big]$$

$$= S_2(n-1,m-1) + \cancel{\sum_{i=m}^{n-1}\binom{n-1}{i-1}S_2(i-1,m-1)} + m\sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m) + \sum_{j=m}^{n-1}\binom{n}{j}S_2(j,m) - \cancel{\sum_{j=m}^{n-1}\binom{n-1}{j-1}S_2(j-1,m-1)}$$

$$- m\sum_{j=m}^{n-1}\binom{n-1}{j-1}S_2(j-1,m)$$

$$= S_2(n-1,m-1) + m\sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m) + \sum_{j=m}^{n-1}\binom{n}{j}S_2(j,m) - m\sum_{j=m+1}^{n-1}\binom{n-1}{j-1}S_2(j-1,m)$$

$$= S_2(n-1,m-1) + m\sum_{j=m}^{n-1}\binom{n-1}{j}S_2(j,m) + \sum_{j=m}^{n-1}\binom{n}{j}S_2(j,m) - m\sum_{j=m}^{n-2}\binom{n-1}{j}S_2(j,m)$$

$$= S_2(n-1,m-1) + mS_2(n-1,m)\sum_{j=m}^{n-1}\binom{n}{j}S_2(j,m)$$

$$= S_2(n,m) + \sum_{j=m}^{n-1}\binom{n}{j}S_2(j,m) = \sum_{j=m}^{n}\binom{n}{j}S_2(j,m)$$

■

2.2.2 Stirling Number of First Kind

Definition 2.2.2: Stirling Number of The First Kind

It is the number of permutations of $[n]$ with exactly m cycles. The *signed version* of Stirling number of the first kind is $(-1)^{n-m}S_1(n,m)$.

Now we will see some recursion relations of the Stirling number of the first kind.

Lemma 2.2.3

$S_1(n,m) = S_1(n-1,m-1) + (n-1)S_1(n-1,m)$

Combinatorial Proof: The *LHS* is the number of permutations of $[n]$ into m cycles by definition.

In the *RHS* we can break the permutations into two different kinds: permutations where $1 \mapsto 1$ and permutations where $1 \not\mapsto 1$. For the permutations $1 \mapsto 1$ this alone forms a cycle. So the rest of the $n-1$ elements have to be permuted into $m-1$ cycles. Hence the number of such permutations is $S_1(n-1, m-1)$.

For permutations where $1 \not\mapsto 1$ take any permutation σ. We will consider the permutation σ' on the elements $\{2, \dots, n\}$ where if $\sigma(k) = 1$ then $\sigma'(k) = \sigma \circ \sigma(k)$ and otherwise for all $k \in \{2, \dots n\}$, $\sigma'(k) = \sigma(k)$. So σ' is now a permutation of $\{2, \dots, n\}$. For all such permutations where $1 \not\mapsto 1$ we get a new unique permutation σ'. So the number of cycles in σ is same as σ'. Hence it is enough to for now count the number of permutations of $\{2, \dots, n\}$ into m cycles is $S_1(n-1, m)$. Now for any such permutation π we can create new $n-1$ many permutations where $\forall\ i \in \{2, \dots, n\}$ where $\pi_i(i) = 1$, $\pi_i(1) = \pi(i)$. In this way for each permutation we get $n-1$ new permutations. Hence the number of permutations where $1 \not\mapsto 1$ is $(n-1)S_2(n-1, m)$.

Hence total number of permutations of $[n]$ into m cycles is $S_1(n-1, m-1) + (n-1)S_1(n-1, m)$. Therefore we get the lemma. ∎

Lemma 2.2.4

$S_1(n, m)\binom{m}{k} = \sum\limits_{j=k}^{n+k-m} \binom{n}{j} S_1(j, k) S_1(n-j, m-k)$

Combinatorial Proof: In *LHS*, $S_1(n, m)$ is the number of permutations on $[n]$ with exactly m cycles. Hence $S_1(n, m)\binom{m}{k}$ is the number of ways to choose k cycles among the m cycles from permutations on $[n]$ with exactly m cycles. This is same as first constructing the chosen k cycles with some elements of $[n]$ and then with the rest of elements construct the rest $m-k$ cycles.

In *RHS* first we select j elements for the k cycles from n in $\binom{n}{j}$ ways. Then for the chosen j elements we create k cycles in $S_1(j, k)$ ways. So the number of ways we can create k cycles by j elements from $[n]$ is $\binom{n}{j}S_1(j, k)$ ways. Now for the rest of the elements we create the rest $m-k$ cycles which we can do in $S_1(n-j, m-k)$. Therefore the number of ways to construct k cycles and with the rest of the elements construct the remaining $m-k$ cycles with elements from $[n]$ is $\sum\limits_{j=k}^{n+k-m} \binom{n}{j} S_1(j, k) S_1(n-j, m-k)$. Therefore we have

$$S_1(n, m)\binom{m}{k} = \sum_{j=k}^{n+k-m} \binom{n}{j} S_1(j, k) S_1(n-j, m-k)$$

∎

Theorem 2.2.5

$S_1(n+1, m+1) = \sum\limits_{j=m}^{n} \binom{j}{m} S_1(n, j)$.

Combinatorial Proof: Consider the permutations on $[n]$ which has at least m cycles. So take a permutation σ which has j cycles where $m \le j \le n$. So for any cycle consider the smallest element in that cycle to be the leading element. So let the permutation is

$$\sigma = (a_1 \dots a_{\ell_1})(a_{\ell_1+1} \dots a_{\ell_2}) \dots (a_{\ell_{j-1}+1} \dots a_j)$$

Now among these j cycles we choose m cycles in $\binom{j}{m}$ ways. Let the first m cycles are chosen. Then we create the last $(m+1)^{th}$ cycle using the $n+1$ in the following way

$$\begin{pmatrix} n+1 & a_{\ell_m}+1 & \dots & a_{\ell_{m+1}} & a_{\ell_{m+1}}+1 & \dots & a_j \end{pmatrix}$$

Hence for each chosen set of m cycles we can join the rest of the cycles and $n+1$ to get the $(m+1)^{th}$ cycle. So now the number of permutations on $[n]$ with j cycles is $S_1(n, j)$. Then we can choose the m cycles among j cycles in $\binom{j}{m}$ ways. So the number of permutations on $[n+1]$ with $m+1$ cycles is $\sum\limits_{j=m}^{n} \binom{j}{m} S_1(n,j)$. Therefore we have

$$S_1(n+1,m+1) = \sum_{j=m}^{n} \binom{j}{m} S_1(n,j)$$

■

Algebraic Proof: First we will prove an identity of $S_1(n+1,m+1)$ then we will dive into the prove of this expression. We will show that $S_1(n+1,m+1) = \sum\limits_{k=m}^{n} \frac{n!}{k!} S_1(k,m)$. We can use induction on $n+m+2$

$$\begin{aligned} S_1(n+1,m+1) &= S_1(n,m) + nS_1(n,m+1) \\ &= S_1(n,m) + n\sum_{k=m}^{n-1} \frac{(n-1)!}{k!} S_1(k,m) \\ &= \frac{n!}{n!} S_1(n,m) + \sum_{k=m}^{n-1} \frac{n!}{k!} S_1(k,m) = \sum_{k=m}^{n} \frac{n!}{k!} S_1(k,m) \end{aligned}$$

Now we will prove this inductively.

$$\begin{aligned} \sum_{j-m}^{n} \binom{j}{m} S_1(n,j) &= \sum_{j=m}^{n} \sum_{k=m}^{n+m-j} \binom{n}{k} S_1(k,m) S_1(n-k,j-m) && \text{[Using Lemma 2.2.4]} \\ &= \sum_{k=m}^{n} \binom{n}{k} S_1(k,m) \sum_{j=m}^{n+m-k} S_1(n-k,j-m) \\ &= \sum_{k=m}^{n} \binom{n}{k} S_1(k,m) \sum_{j=0}^{n-k} S_1(n-k,j) \\ &= \sum_{k=m}^{n} \binom{n}{k} S_1(k,m)(n-k)! && \left[\text{Since } \sum_{j=0}^{n-k} S_1(n-k,j) \text{ is number of permutations on } [n-k]\right] \\ & \sum_{k=m}^{n} \frac{n!}{k!} S_1(k,m) \\ &= S_1(n+1,m+1) \end{aligned}$$

■

Now we will show you a property of the signed Stirling number of the first kind.

Theorem 2.2.6

$$S_1(n,m) = \sum_{i=m}^{n} (-1)^{i-m} \binom{i}{m} S_1(n+1,i+1)$$

Proof:

$$
\begin{aligned}
\sum_{i=m}^{n}(-1)^{i-m}\binom{i}{m}S_1(n+1,i+1) &= (-1)^{i-m}\binom{i}{m}\sum_{j=i}^{n}\binom{j}{i}S_1(n,j)\\
&= \sum_{j=m}^{n}S_1(n,j)\sum_{i=m}^{j}(-1)^{i-m}\binom{i}{m}\binom{j}{i}\\
&= \sum_{j=m}^{n}S_1(n,j)\sum_{i=m}^{j}(-1)^{i-m}\binom{j}{m}\binom{j-m}{i-m}\\
&= \sum_{j=m}^{n}\binom{j}{m}S_1(n,j)\sum_{i=0}^{j-m}(-1)^{i}\binom{j-m}{i}\\
&= \sum_{j=m+1}^{n}\binom{j}{m}S_1(n,j)\underbrace{\sum_{i=0}^{j-m}(-1)^{i}\binom{j-m}{i}}_{=0}+\binom{m}{m}S_1(n,m)(-1)^0\binom{0}{0}\\
&= S_1(n,m)
\end{aligned}
$$

■

2.2.3 Connecting the Two Stirling Numbers

Theorem 2.2.7

Let S_1 and S_2 be $k\times k$ matrix where for any $n,m\in[k]$ with $n\geq m$ we have $(S_1)_{n,m}=(-1)^{n-m}S_1(n,m)$ and $(S_2)_{n,m}=S_2(n,m)$ and 0 otherwise then $S_1S_2=I$i.e.

$$\sum_{i=m}^{n}(-1)^{n-i}S_1(n,i)S_2(i,m)=\mathbb{1}(n=m)$$

Proof: We will induct on $n+m$. Then we have

$$
\begin{aligned}
\sum_{i=m}^{n}(-1)^{n-i}S_1(n,i)S_2(i,m) &= \sum_{i=0}^{\infty}(-1)^{n-i}(S_1(n-1,i-1)+(n-1)S_1(n-1,i))S_2(i,m)\\
&= \sum_{i=0}^{\infty}(-1)^{n-i}S_1(n-1,i-1)S_2(i,m)+(n-1)\sum_{i=0}^{\infty}(-1)^{n-i}S_1(n-1,i)S_2(i,m)\\
&= \sum_{i=0}^{\infty}(-1)^{n-i}S_1(n-1,i-1)[S_2(i-1,m-1)+mS_2(i-1,m)]-(n-1)\mathbb{1}(n-1=m)\\
&= \sum_{i=0}^{\infty}(-1)^{n-i}S_1(n-1,i-1)S_2(i-1,m-1)+m\sum_{i=0}^{\infty}(-1)^{n-i}S_1(n-1,i-1)S_2(i-1,m)\\
&\qquad -(n-1)\mathbb{1}(n-1=m)\\
&= \mathbb{1}(n=m)+m\mathbb{1}(n-1=m)-(n-1)\mathbb{1}(n-1=m)\\
&= \mathbb{1}(n=m)+(m-n+1)\mathbb{1}(n-1=m)=\mathbb{1}(n=m)
\end{aligned}
$$

■

2.3 Inclusion Exclusion Principle

Theorem 2.3.1 Inclusion-Exclusion Principle

Let $A_1, \dots, A_n$ be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{J\subseteq[n], J\neq\emptyset} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|$$

Proof: We will prove this using induction on n.

$$\begin{aligned}
\left|\bigcup_{i=1}^{n} A_i\right| &= \left|\bigcup_{i=1}^{n-1} A_i\right| + |A_n| - \left|\left(\bigcup_{i=1}^{n-1} A_i\right) \cap A_n\right| \\
&= \left|\bigcup_{i=1}^{n-1} A_i\right| + |A_n| - \left|\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right| \\
&= \sum_{J\subseteq[n-1], J\neq\emptyset} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| + |A_n| - \sum_{J\subseteq[n-1], J\neq\emptyset} (-1)^{|J|+1} \left|A_n \cap \left(\bigcap_{j\in J} A_j\right)\right| \\
&= \sum_{J\subseteq[n-1], J\neq\emptyset} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| + |A_n| - \sum_{\substack{J\subseteq[n] \\ J\neq\{n\}, n\in J}} (-1)^{|J|+1} \left|A_n \cap \left(\bigcap_{j\in J-\{n\}} A_j\right)\right| \\
&= \sum_{J\subseteq[n-1], J\neq\emptyset} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| + \sum_{J=\{n\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| + \sum_{\substack{J\subseteq[n] \\ J\neq\{n\}, n\in J}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| \\
&= \sum_{\substack{J\subseteq[n] \\ J\neq\emptyset, n\notin J}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| + \sum_{J=\{n\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| + \sum_{\substack{J\subseteq[n] \\ J\neq\{n\}, n\in J}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| \\
&= \sum_{J\subseteq[n], J\neq\emptyset} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|
\end{aligned}$$

Hence by mathematical induction we have the theorem. ■

Corollary 2.3.2

If $\forall\, i \in [n]$, $A_i = \{0\}$. Then

$$1 = \sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i}$$

Proof: Using the Inclusion-Exclusion Principle we have

$$1 = \left|\bigcup_{i=1}^{n} A_i\right| = \sum_{J\subseteq[n], J\neq\emptyset} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| = \sum_{J\subseteq[n], J\neq\emptyset} (-1)^{|J|+1} = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i}$$

■

Corollary 2.3.3

There are $\sum\limits_{k=0}^{n} \binom{m}{k} (-1)^k (m-k)^n$ onto functions from $[n] \to [m]$

Theorem 2.3.4 Strong Inclusion-Exclusion

Let $f : 2^{[n]} \to \mathbb{R}$. Define $g : 2^{[n]} \to \mathbb{R}$ on a subset $T \subseteq [n]$ to be as follows

$$g(T) = \sum_{S \subseteq T} f(S) \quad T \subseteq [n]$$

Then

$$f(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} g(S)$$