

Universal Optimality of Dijkstra Algorithm

Using Fibonacci-Like Priority Queue with Working Sets

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Introduction

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- Recently Duan, Mao, Shu and Yin in [Dua+23] solved SSSP for undirected graphs with expected time $O(m\sqrt{\log n \log \log n})$
- This year Stefansson, Biggar and Johansson gave a fixed-parameter linear algorithm with running time $O(m + n \log w)$ for the single-source shortest path problem (SSSP) on directed graphs where fixed parameter over nesting width (w).

Universal Optimality

- Let \mathcal{A} is the set of all correct algorithms.
- $\mathcal{G}_{n,m}$ is the set of all graphs with n vertices and m edges.
- \mathcal{W}_G is the set of all possible weights for a graph $G \in \mathcal{G}_{n,m}$.

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A correct algorithm A^* is *existentially optimal* if

$$\forall n, m : \sup_{\substack{G \in \mathcal{G}_{n,m} \\ w \in \mathcal{W}_G}} A^*(G, w) \leq \alpha(n, m) \inf_{A \in \mathcal{A}} \sup_{\substack{G \in \mathcal{G}_{n,m} \\ w \in \mathcal{W}_G}} A(G, w)$$

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But this is not good. It is just saying A^* may take as much time as it takes in a star-graph or more complicated one.

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In this work we focus solely on α being a constant i.e.
 $\alpha(n, m) = O(1)$.

Dijkstra Algorithm

Algorithm 1: DIJKSTRA(G, s, w)

$F \leftarrow \emptyset$, INSERT(F, s), $dist(s) \leftarrow 0$

while $F \neq \emptyset$ **do**

$u \leftarrow$ EXTRACTMIN(F)

for $e = (u, v) \in E$ **do**

 INSERT(F, v)

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Dijkstra solves three problems:

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- Build Shortest Path Tree
- Sorts vertices by Shortest Distance (DO)

Exploration Tree and DO

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- Let T be the exploration tree. Let \prec be the final distance ordering of the vertices.
- Then for every edge $(u, v) \in T$, $u \prec v$.

Order of Vertices by a Tree

- Let T be any tree in G . An order of T is a total order of $V(T)$ such that for every edge $(u, v) \in E(T)$ we have $u \prec v$ in the order.

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Theorem

For any graph G , L is an order of G iff there exists non-negative weights w such that

1. *For every two nodes $u \neq v$, $d_w(s, u) \neq d_w(s, v)$.*
2. *$u \prec_L v$ if and only if $d_w(s, u) < d_w(s, v)$.*

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- $OPT(G)$ be the number of total steps taken by an optimal correct algorithm for the graph.
- Since $OPT(G) = \Omega(m)$, $OPT_Q(G) + n + m = O(OPT(G))$.

Dijkstra Induced Interval Set

Let an interval of time for any vertex $v \in V(G)$ is the set $[l_v, r_v]$ where l_v is the time when v was first discovered and added to the heap and r_v is the time when v was extracted from the heap.

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A run of Dijkstra induces intervals for each vertex $v \in V$ with the operations INSERT and EXTRACTMIN.

An interval set \mathcal{I} is collection of intervals for each vertex. It is called Dijkstra Induced when all the intervals for each vertex in \mathcal{I} is induced by a run of Dijkstra on some (G, w) .

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$$W_{v,t} = \{[l_u, r_u] \in \mathcal{I} : l_v \leq l_u \leq t \leq r_u\}$$

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- Working set of v , $W_v = W_{v,t^*}$ such that $t^* = \arg \max_t |W_{v,t}|$.

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- Working set of v , $W_v = W_{v,t^*}$ such that $t^* = \arg \max_t |W_{v,t}|$.
- The cost of a vertex $v \in V(G)$ is $Cost(v) = \log |W_v|$. And so $Cost(\mathcal{I}) = \sum_{v \in V(G)} \log |W_v|$.

Fibonacci-Like Priority Queue with Working Set Property

A Fibonacci-like priority queue is a priority queue made using a Fibonacci Heap. Fibonacci-Like Priority Queue with Working Set Property is a data structure if it satisfies the amortized time complexity for any sequence of operations as follows:

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Fact

There is a Fibonacci-Like Priority Queue with Working Set Property for Dijkstra. We will use this data structure in every argument from now on by default.

Time Complexity of Dijkstra

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- Hence total time taken by all DECREASEKEY calls is $O(m)$.
- Total time taken by all EXTRACTMIN calls is

$$\begin{aligned}\sum_{v \in V(G)} O(1 + \log |W_v|) &= O\left(n + \sum_{v \in V(G)} \log |W_v|\right) \\ &= O(n + \text{Cost}(\mathcal{I}))\end{aligned}$$

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We'll show $\text{OPT}_Q(G) = \Omega(\text{Cost}(\mathcal{I}))$.

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- Let $C_L \in \{-1, 0, 1\}^*$ be the sequence of answers of comparisons made by A on (G, w_L) . Then $\mathcal{C} : \mathcal{W} \rightarrow \{-1, 0, 1\}^*$, $\mathcal{C}(w_L) = C_L$ is a ternary prefix free code.

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- By Shannon's source coding theorem for symbol codes any such code has expected length $\Omega(\log(|\mathcal{W}|)) = \Omega(\log(\text{Order}(G)))$

Barrier Sequence

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Theorem

A sequence (B_1, \dots, B_k) of pairwise disjoint vertex sets is barrier sequence if and only if for all $1 \leq i \leq j \leq k$, $v \in B_j$ is not ancestor of any $u \in B_i$ in T .

Barriers Give Lower Bounds

Theorem

Let T be any spanning tree and (B_1, \dots, B_k) be a barrier sequence of T . Then $\log(\text{Order}(G)) = \Omega \left(\sum_{i=1}^k |B_i| \log |B_i| \right)$

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- Delete vertices of B_k to get T' . By induction for the barrier sequence (B_1, \dots, B_{k-1}) for T' , $\text{Order}(T') \geq |B_1|! |B_2|! \cdots |B_{k-1}|!$.

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So finally we got the result:

Result

If T is a spanning tree of G and (B_1, \dots, B_k) is a barrier sequence for T then

$$OPT_Q(G) = \Omega \left(\sum_{i=1}^k |B_i| \log |B_i| \right)$$

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Result

At any time of the algorithm the set of elements in the priority queue forms a barrier

Intersecting Coloring

Definition (Intersecting Coloring)

An intersecting coloring of \mathcal{I} with k colors is a function $C : \mathcal{I} \rightarrow [k]$ that assigns a color to every interval and additionally for every color $i \in [k]$, $\bigcap_{I \in \mathcal{I}, C(I)=i} I \neq \emptyset$.

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- Order $\{B_c\}$ by increasing order of $\{t_c\}$. WLOG $t_1 < \dots < t_k$.
- (B_1, \dots, B_k) is a barrier sequence.

Intersecting Coloring Gives Lower Bounds

Let C be an intersecting coloring of \mathcal{I} with k colors. Let (B_1, \dots, B_k) is the barrier sequence induced by C . Then let the energy of C is defined to be

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Result

If \mathcal{I} is the interval set induced by Dijkstra and C be any arbitrary intersecting coloring of \mathcal{I} then

$$OPT_Q(G) = \Omega(E(C))$$

Good Intersecting Coloring gives Optimality

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- So overall Cost of EXTRACTMIN in Dijkstra is upper bounded by $O(n + OPT_Q(G))$.
- Dijkstra achieves universal optimality for time complexity.

Good Intersecting Coloring gives Optimality

Goal: Find an intersecting coloring of \mathcal{I} , C such that $E(C) \geq \text{Cost}(\mathcal{I})$

- Then time complexity of all EXTRACTMIN operations is $O(n + \text{Cost}(\mathcal{I})) = O(n + E(C))$.
- We have $OPT_Q(G) = \Omega(E(C))$.
- So overall Cost of EXTRACTMIN in Dijkstra is upper bounded by $O(n + OPT_Q(G))$.
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We will find such a good intersecting coloring recursively.

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For working set W_x with the largest size

$$\text{Cost}(\mathcal{I}) \leq \text{Cost}(\mathcal{I} \setminus W_x) + 2|W_x| \log |W_x|$$

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$$\text{Cost}(\mathcal{I}) \leq \text{Cost}(\mathcal{I}') + 2|W_x| \log |W_x|. \text{ Hence, } E(C) \geq \text{Cost}(\mathcal{I}).$$

Thank You

Deleting Intervals from \mathcal{I}

Theorem

Let \mathcal{I} an interval set and $x \in \mathcal{I}$. $k = \max_t |\{I \in \mathcal{I} \mid t \in I\}|$. Then

$$\text{Cost}(\mathcal{I}) \leq \text{Cost}(\mathcal{I} \setminus \{x\}) + \log |W_x| + \log k$$

- Let $I_1, \dots, I_l \in \mathcal{I}$ are the only intervals which had nonempty intersection with x . So $l \leq k - 1$.
- Let t_i is starting point of I_i . WLOG assume $t_l > \dots > t_1$.
- Let W_i, W'_i are working sets of I_i before and after removing x .

Deleting Intervals from \mathcal{I}

- Let t is starting point of x . Then $W_{i,t}$ contains x, l_1, \dots, l_i . So $|W_i| \geq i + 1$.
- $|W_i| \in \{|W'_i|, |W'_i| + 1\}$ for all $i \in [l]$.

$$\begin{aligned} & \text{Cost}(\mathcal{I}) - \text{Cost}(\mathcal{I} \setminus \{x\}) - \log |W_x| \\ &= \sum_{i=1}^l \log |W_i| - \log |W'_i| \\ &\leq \sum_{i=1}^l \log(i+1) - \log i = \log(l+1) \leq \log k \end{aligned}$$

Fact

For any working set $|W_x| = k$ we have

$$\text{Cost}(\mathcal{I}) \leq \text{Cost}(\mathcal{I} \setminus W_x) + 2|W_x| \log |W_x|$$