



Lecture
Notes

2021

Analysis 2

Lecturer: Upendra Kulkarni

Scribe: Soham Chatterjee

Introduction

This is the lecture notes scribed by me. If you find any mistakes in the notes please email me at sohamc@cmi.ac.in.

The whole course is taken by Prof. Upendra Kulkarni, online. If you want the lecture videos then you can find them in [this link](#). Sir mainly followed Prof. Pramath Sastry's Notes (<https://www.cmi.ac.in/~pramath/teaching.html#ANA2>). You can find all the assignments problems in the following [drive link](#). Through out the course the book we followed is Principles of Mathematical Analysis by Walter Rudin.

Contents

Chapter 1	Normed Linear Space	Page 4
1.1	Defination	4
1.2	Open and Closed Ball	6
1.3	Limit of a Sequence	7
1.4	Continuity	7

Chapter 1

Normed Linear Space

Definition 1.0.1: Limit of Sequence in \mathbb{R}

Let $\{s_n\}$ be a sequence in \mathbb{R} . We say

$$\lim_{n \rightarrow \infty} s_n = s$$

where $s \in \mathbb{R}$ if \forall real numbers $\epsilon > 0 \exists$ natural number N such that for $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Want to generalize this to a sequence in \mathbb{R}^n i.e. $s_n \in \mathbb{R}^n \forall n \in \mathbb{N}$. Now the $s - s_n$ makes no sense. So it is useful to have a notion of whether vectors are big or small. We have magnitude of a vector. So lets revisit this

Definition 1.0.2: Limit of Sequence in \mathbb{R}^n

Let $\{s_n\}$ be a sequence in \mathbb{R}^n . We say

$$\lim_{n \rightarrow \infty} s_n = s$$

where $s \in \mathbb{R}^n$ if \forall real numbers $\epsilon > 0 \exists$ natural number N such that for $n > N$

$$\|s - s_n\| < \epsilon$$

The same definition works if we interpret $\|v\| = \text{length of the vector } v$.

From school, for $v = v_1, v_2, \dots, v_n$ we had

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

But it will be useful to have a more general notion of length (of which the above will be an example)

1.1 Defination

Definition 1.1.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over \mathbb{R} (or \mathbb{C}). A norm on V is function $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying

- ① $\|x\| = 0 \iff x = 0 \forall x \in V$
- ② $\|\lambda x\| = |\lambda| \|x\| \forall \lambda \in \mathbb{R}(\text{or } \mathbb{C}), x \in V$
- ③ $\|x + y\| \leq \|x\| + \|y\| \forall x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.

• Same definition works with V a vector space over \mathbb{C} (again $\|\cdot\| \rightarrow \mathbb{R}_{\geq 0}$) where ② becomes $\|\lambda x\| = |\lambda| \|x\| \forall \lambda \in \mathbb{C}, x \in V$, where for $\lambda = a + ib, |\lambda| = \sqrt{a^2 + b^2}$

Example 1.1.1 (p -Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geq 0}$. Define for $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_m|^p \right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \dots + |x_m|$ is clearly a norm by usual triangle inequality.

Special Case $p \rightarrow \infty$ (\mathbb{R}^m with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_m|\}$

For $m = 1$ these p -norms are nothing but $|x|$. Now exercise

Question 1

Prove that triangle inequality is true if $p \geq 1$ for p -norms. (What goes wrong for $p < 1$?)

Solution: For Property ③ for norm-2

When field is \mathbb{R} :

We have to show

$$\begin{aligned} \sum_i (x_i + y_i)^2 &\leq \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2} \right)^2 \\ \Rightarrow \sum_i (x_i^2 + 2x_i y_i + y_i^2) &\leq \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2 \right] \left[\sum_i y_i^2 \right]} + \sum_i y_i^2 \\ \Rightarrow \left[\sum_i x_i y_i \right]^2 &\leq \left[\sum_i x_i^2 \right] \left[\sum_i y_i^2 \right] \end{aligned}$$

So in other words prove $\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$ where

$$\langle x, y \rangle = \sum_i x_i y_i$$

Note:-

- $\|x\|^2 = \langle x, x \rangle$
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle \cdot, \cdot \rangle$ is \mathbb{R} -linear in each slot i.e.

$$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$

Here in $\langle x, y \rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned} \langle x - \lambda y, x - \lambda y \rangle &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\ &= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\ &= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle \end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is \mathbb{C} :

Modify the definition by

$$\langle x, y \rangle = \sum_i \bar{x}_i y_i$$

Then we still have $\langle x, x \rangle \geq 0$

1.2 Open and Closed Ball

Definition 1.2.1: Open and Closed Ball in Normed Linear Space

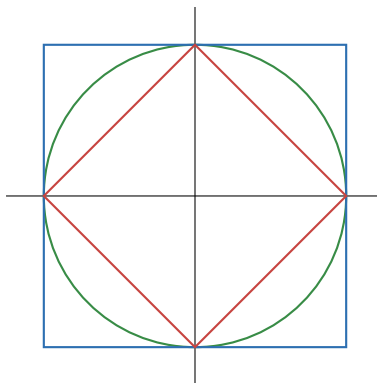
An open Ball of radius r with center x in Normed Linear Space V is the set

$$\{y \in V \mid \|x - y\| < r\} = B_r(x)$$

and Closed ball is the set

$$\{y \in V \mid \|x - y\| \leq r\} = \overline{B_r(x)}$$

Now take $B_r(0)$ w.r.t $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$. Now imagine a sequence converging to origin. So if I



draw an ordinary circle around the origin then no matter how small the circle the points of the sequence are eventually land inside the circle. If instead of that circle can same be said for diamond w.r.t norm 2. Then i can take circle that is inside that diamond. Same is true for ∞ -norm. Hence convergence with respect to all norm 1 and norm 2 and even ∞ results for convergence.

Now there is no reason why we can not consider a norm on an infinite dimensional vector space. It will work. Perhaps i can define only for some sequences where the norm converges.

Example 1.2.1

Suppose for set of all bounded infinite sequences a vector space because every number in a vector is less than some number so if you add two vectors then add the bound and if you scale then scale the bound. Now the ∞ norm works on that.

Now suppose you take all continuous real valued functions on closed interval $[0, 1]$, such a function is bounded and this is a vector space and we can define ∞ -norm even for that because for all f in this space attains its maximum value so just take that maximum value. Its an extremely infinite dimensional space.

Note:-

\mathbb{R}^∞ is the space of all sequences.

Question 2

Modify the above proof for field \mathbb{C}

Question 3

Show that the following are normed linear spaces.

- (a) l^∞ = Set of all bounded infinite sequences (x_1, x_2, \dots) $x_i \in \mathbb{R}$ with norm $\|x\| = \sup |x_i|$
- (b) $C[0, 1]$ = Set of all continuous functions $[0, 1] \rightarrow \mathbb{R}$ with norm $\|f\| = \sup_{x \in [0, 1]} |f(x)|$

1.3 Limit of a Sequence

Definition 1.3.1: Limit of Sequence in Normed Linear Space

A sequence $\{s_n\}$ in a normed linear space V converge to s means \forall real number $\epsilon > 0 \exists$ natural number N such that for $\forall n > N$ $\|s - s_n\| < \epsilon$

1.4 Continuity

Definition 1.4.1: Continuity in Normed Linear Space

Let S be a subset of V and $f : S \rightarrow W$ where V, W are normed linear space. f is continuous at $v \in V$ means $\forall \epsilon > 0, \exists \delta > 0$, st whenever $\|x - v\| < \delta$ for $x \in S$ one has $\|f(x) - f(v)\| < \epsilon$

Distance in a normed linear space for $x, y \in V$ is

$$d(x, y) = \|x, y\|$$

Hence properties of this d are

- ① $d(x, y) = 0 \iff x = y$
- ② $d(\lambda x, \lambda y) = |\lambda|d(x, y)$ for any scalar λ
- ③ $d(u, v) + d(u, v) \geq d(u, w)$