

Nisan's Pseudorandom Generator for RL

BPL \subseteq SC = DTISP($\text{poly}(n)$, $\log^2(n)$)

Soham Chatterjee

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Pseudorandomness Course (CSS.413.1) Presentation, STCS

Pseudorandom Generator

Definition (Pseudorandom Generator)

A map $\mathcal{G} : \{0, 1\}^l \rightarrow \{0, 1\}^n$, where $n \geq l$ is called a PRG for a class C with a parameter $\epsilon > 0$ if for any $f \in C$,

$$\left| \mathbb{P}_{y \in \{0,1\}^n} [f(y) = 1] - \mathbb{P}_{x \in \{0,1\}^l} [f(\mathcal{G}(x)) = 1] \right| \leq \epsilon$$

- Here l is called the seed-length of the PRG.
- $n - l$ is called the stretch of the PRG.
- We call \mathcal{G} , ϵ -fools C .
- Typically, we want $n \gg l$ and \mathcal{G} to be efficiently computable.

Finite State Automata

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Therefore the configuration graph of T on input x represents a finite state automata with $N = \text{poly}(n)$ states.

Computational Tableau of BPL machine

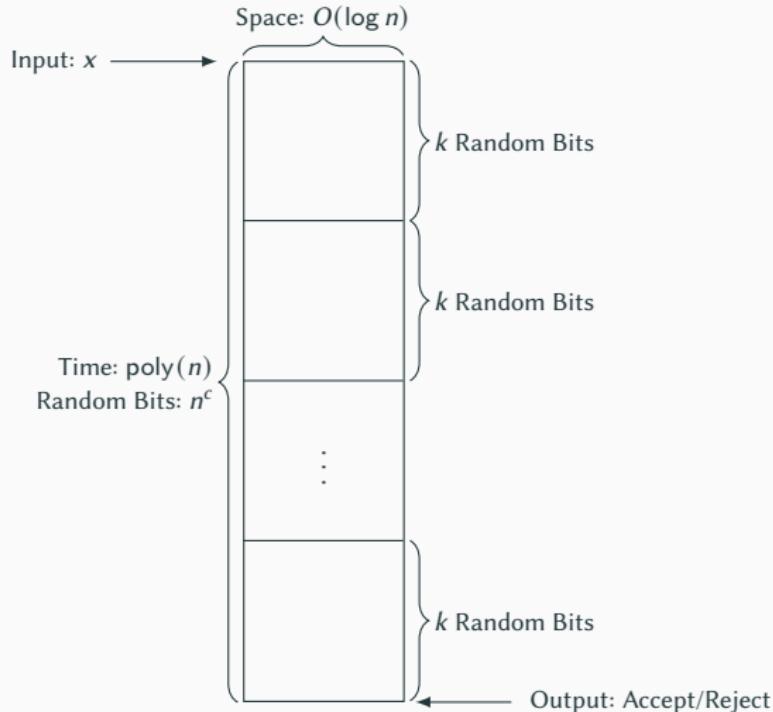


Figure 1: Computational Tableau of BPL machine T .

Dividing BPL Computation into Blocks

- Let the BPL machine T run in time n^c using $n^{c'}$ random bits and $O(\log n)$ space on input x of length n .
- Let $k \ll n^c$ be a parameter to be fixed later.
- Divide the computation of T into $t = n^c/k$ blocks, where each block uses k random bits.
- We can treat each block as a separate BPL machine T_i (in some sense) which takes input as the final configuration of T_{i-1} and k random bits.

FSA for Computation Blocks

- We can think of T as a finite state automata with N states.
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$$\mathbb{P}_{r \in \{0, 1\}^{k \times t}} [T(x, r) = \text{Acc}] = \sum_{j: \text{Accepting state}} M^t[1, j]$$

Goal: Approximate M^t using PRG.

Approximate Automata Matrix

Suppose we have a pseudorandom generator $\mathcal{G} : \{0, 1\}^k \rightarrow \{0, 1\}^{k \cdot t}$. Let Q be a finite state automata with N states and its matrix be M as defined above.

- From definition of M , $M^t[i, j] = \mathbb{P}_{r_1, \dots, r_t \in \{0, 1\}^k} [Q(i; r_1 \dots; r_t) = j]$
- Using \mathcal{G} , let $M_{\mathcal{G}}[i, j] = \mathbb{P}_{r \in \{0, 1\}^k} [Q(i; \mathcal{G}(r)) = j]$
- Want to construct \mathcal{G} such that

$$\|M^t - M_{\mathcal{G}}\| < \epsilon$$

for small ϵ with high probability.

Matrix Norm

For any vector $v \in \mathbb{R}^N$, define $\|v\| = \sum_{i \in [N]} |v(i)|$. Then for any matrix $M \in \mathbb{R}^{N \times N}$ define

$$\|M\| = \sup_{0 \neq v \in \mathbb{R}^N} \frac{\|Mv\|}{\|v\|}$$

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Properties:

- $\|M\| = \max_{i \in [N]} \sum_{j \in [N]} |M[i, j]|$
- $\|M + N\| \leq \|M\| + \|N\|$
- $\|MN\| \leq \|M\| \cdot \|N\|$

Universal Hash Family

Definition (Universal Hash Family (Carter-Wegman))

$\mathcal{H} = \{h : \{0, 1\}^k \rightarrow \{0, 1\}^m\}$ is a *universal hash family* if for any $x_1 \neq x_2 \in \{0, 1\}^k$ and $y_1, y_2 \in \{0, 1\}^m$,

- $\mathbb{P}_{h \in \mathcal{H}} [h(x_1) = y_1] = \frac{1}{2^m}$
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- For our purpose, we have $k = m$.
- We can construct such a family with $|\mathcal{H}| = 2^{O(k)}$ where

$$\mathcal{H} = \{a \cdot x + b \mid a, b \in \{0, 1\}^k\}$$

over $GF(2^k)$.

Property of Universal Hash Family

Definition $((\epsilon, A, B)\text{-good hash function})$

Let $A \subseteq \{0, 1\}^k$, $B \subseteq \{0, 1\}^m$, $\epsilon > 0$, $h : \{0, 1\}^k \rightarrow \{0, 1\}^m$ is said to be (ϵ, A, B) -good if

$$\left| \mathbb{P}_{x \in \{0,1\}^k} [x \in A \wedge h(x) \in B] - \alpha \cdot \beta \right| \leq \epsilon$$

where $\alpha = \frac{|A|}{2^k}$ and $\beta = \frac{|B|}{2^m}$.

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Lemma

If \mathcal{H} is a universal hash family, then for any $A \subseteq \{0, 1\}^k$, $B \subseteq \{0, 1\}^m$, $\epsilon > 0$,

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For $k = m$, by union bound an uniformly random h is $\left(\frac{\epsilon}{k^2}, A, B\right)$ -good for all A, B .

Nisan's Generator

Let \mathcal{H} be an universal hash family from $\{0, 1\}^k$ to $\{0, 1\}^k$. For any integer $m \geq 0$ define the function $\mathcal{G}_m: \{0, 1\}^k \times \mathcal{H}^m \rightarrow \{0, 1\}^{k \cdot 2^m}$ recursively as follows:

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- $\mathcal{G}_0(x) = x$
- $\mathcal{G}_m(x, h_1, \dots, h_m) = (\mathcal{G}_{m-1}(x, h_1, \dots, h_{m-1}), \mathcal{G}_{m-1}(h_m(x), h_1, \dots, h_{m-1}))$

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For example:

$$\mathcal{G}_1(x, h) = (x, h(x)), \quad \mathcal{G}_2(x, h_1, h_2) = (x, h_1(x), h_2(x), h_1 \cdot h_2(x))$$

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$$\mathcal{G}_3(x, h_1, h_2, h_3) = (x, h_1(x), h_2(x), h_1 \cdot h_2(x), h_3(x), h_1 \cdot h_3(x), h_2 \cdot h_3(x), h_1 \cdot h_2 \cdot h_3(x))$$

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- We want $k \cdot 2^m = n^c \implies m = \log t$.
- This gives a stretch of $k \cdot (t - 1)$ bits.

Proof Flow

Let h_1, \dots, h_s be some fixed hash functions from \mathcal{H} . Define the matrix

$$M_{h_1, \dots, h_s}[i, j] = \mathbb{P}_{x \in \{0,1\}^k} [Q(i; \mathcal{G}_s(x, h_1, \dots, h_s)) = j]$$

- Using h_1, \dots, h_s we had 2^s many transitions in Q . So we should compare M_{h_1, \dots, h_s} with M^{2^s} .

Goal: For ‘good’ choice of h_1, \dots, h_m , $\|M^{2^m} - M_{h_1, \dots, h_m}\| < \epsilon$

Approach:

Step 1: Suppose we have h_1, \dots, h_{s-1} . We will find $h_s \in \mathcal{H}$ such that for all $i, j \in [N]$,

$$\left\| M_{h_1, \dots, h_{s-1}}^2 - M_{h_1, \dots, h_s} \right\| \leq \delta$$

Step 2: Using above property show for all $s \in [m]$,

$$\left\| M_{h_1, \dots, h_s} - M^{2^s} \right\| \leq (2^s - 1) \delta$$

Find good h_s from h_1, \dots, h_{s-1}

Suppose we have $h_1, \dots, h_{s-1} \in \mathcal{H}$ such that,

$$\left\| M_{h_1, \dots, h_{s-1}} - M^{2^{s-1}} \right\| \leq (2^{s-1} - 1)\delta$$

If we can find h_s such that $\|M_{h_1, \dots, h_{s-1}}^2 - M_{h_1, \dots, h_s}\| \leq \delta$ then we are done.

Algorithm: Go over all $h \in \mathcal{H}$ and all $i, j \in [N]$:

Step 1: Compute

- $p_1 = M_{h_1, \dots, h_{s-1}, h}[i, j]$
- $p_2 = \sum_{l \in [N]} M_{h_1, \dots, h_{s-1}}[i, l] \cdot M_{h_1, \dots, h_{s-1}}[l, j]$

Step 2: Check if $|p_1 - p_2| > \frac{\delta}{N}$ go to next h else return h .

Algorithm returns *good* h_s

Claim

If h_s is returned by the above algorithm, then

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$$\|M_{h_1, \dots, h_s} - M^{2^s}\| \leq \|M_{h_1, \dots, h_s} - M_{h_1, \dots, h_{s-1}}^2\| + \|M_{h_1, \dots, h_{s-1}}^2 - M^{2^s}\|$$

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Thank You