

# Bounding PoA using Linear and Quadratic Programming

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# Introduction

- **Pure Nash Equilibria:** A strategy profile  $s \in S$  of a cost-minimization game  $\Gamma$  is a *Pure Nash Equilibrium* if for every player  $i \in [n]$  and for all  $s'_i \in S_i$ ,  
 $u_i(s) \geq u_i(s'_i, s_{-i})$ .
- **Mixed Nash Equilibria:** A mixed strategy profile  $\sigma \in \Sigma$  of a cost-minimization game  $\Gamma$  is a *Mixed Nash Equilibria* if for every player  $i \in [n]$  and for all  $s'_i \in S_i$ ,  
 $\mathbb{E}_{s \sim \sigma} [u_i(s)] \geq \mathbb{E}_{s \sim \sigma} [u_i(s'_i, s_{-i})]$
- **Coarse Correlated Equilibria:** A distribution  $\mu$  over  $S$  of a cost-minimization game  $\Gamma$  is a *Coarse Correlated Equilibria* if for every player  $i \in [n]$  and for all  $s'_i \in S_i$ ,  $\mathbb{E}_{s \sim \mu} [u_i(s)] \geq \mathbb{E}_{s \sim \mu} [u_i(s'_i, s_{-i})]$

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$\text{PNE} \subseteq \text{MNE} \subseteq \text{CCE}$ .

# Lagrangian Duality

Given convex problem:

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & h_i(x) \leq 0 \quad \forall i \in [m], \\ & l_j(x) = 0 \quad \forall j \in [r]\end{array}$$

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Define Lagrangian  $\mathcal{L}(x, u, v) = f(x) + \sum_{i=1}^m u_i h_i(x) + \sum_{j=1}^r v_j l_j(x)$ . Define

$$g(u, v) = \inf_x \mathcal{L}(x, u, v)$$

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The dual of the convex problem:

$$\begin{array}{ll}\text{maximize} & g(u, v) \\ \text{subject to} & u \geq 0\end{array}$$

# Fenchel Duality

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a convex function. Then the convex conjugate of  $f$  is the function

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle y, x \rangle - f(x) \}$$

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## Theorem (Fenchel Duality)

Let  $f : X \rightarrow \mathbb{R}, g : Y \rightarrow \mathbb{R}$  are two convex functions and  $A : X \rightarrow Y$  any bounded linear map. Suppose

$$p^* = \inf_{x \in X} \{ f(x) + g(Ax) \} \quad \text{and} \quad d^* = \sup_{y \in Y} \{ -f^*(A^*y) - g^*(-y) \}$$

where  $A^*$  is the adjoint of  $A$ . Then  $p^* \geq d^*$



# Weighted Congestion Games

# Definitions

- $\mathcal{N}$ : Set of players
- $\mathcal{E}$ : The ground set of resources
- For each player  $j \in \mathcal{N}$ , let  $S_j \subseteq 2^{\mathcal{E}}$  be the set of strategies available to player  $j$ .  
Let  $S = \prod_{j \in \mathcal{N}} S_j$ .
- For each  $j \in \mathcal{N}$  and each  $e \in \mathcal{E}$  there is a weight of the resource  $w_{ej} \in \mathbb{R}^+$ .
- For each  $e \in \mathcal{E}$  the cost of resource  $e$  is an affine function  $C_e : \mathbb{R} \rightarrow \mathbb{R}$  where  $c_e(x) = a_e \cdot x + b_e$
- For any strategy profile  $f \in S$ , the cost of player  $j$  is  $\text{Cost}(f)_j = \sum_{e \in f_j} w_{ej} \cdot c_e(l_e(f))$

where  $l_e(f) = \sum_{j': e \in f_{j'}} w_{ej'}$  is the load on resource  $e$ . Do

$$\text{Cost}(f) = \sum_{j \in \mathcal{N}} \sum_{e \in f_j} w_{ej} \cdot c_e(l_e(f)) = \sum_{e \in \mathcal{E}} a_e \cdot l_e(f) + b_e \cdot l_e(f)$$

## Convex program of WCG

### Setting up the variables

For any player  $j \in \mathcal{N}$  and  $f_j \in S_j$  let  $L_{j,f_j} = \sum_{e \in f_j} w_{ej} \cdot c_e(w_{ej})$  i.e. the cost incurred by player  $j$  when it plays strategy  $f_j$ .

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- $x_{j,f_j} :=$  Variable for player  $j$  playing strategy  $f_j$  for all  $j \in \mathcal{N}$  and  $f_j \in S_j$

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- $y_e :=$  Variable for the load on resource  $e$  for all  $e \in \mathcal{E}$

## Convex program of WCG

### Quadratic Program

$$\begin{aligned} &\text{minimize} && \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} x_{j,f_j} \cdot L_{j,f_j} + \sum_{e \in \mathcal{E}} a_e \cdot y_e^2 \\ &\text{subject to} && \sum_{f_j \in S_j} x_{j,f_j} \leq 1 \quad \forall j \in \mathcal{N}, \\ &&& \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} \sum_{e \in f_j} w_{ej} \cdot x_{j,f_j} \leq y_e \quad \forall e \in \mathcal{E}, \\ &&& x_{j,f_j} \geq 0 \quad \forall j \in \mathcal{N}, f_j \in S_j \end{aligned}$$

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subject to

$$\sum_{f_j \in S_j} x_{j,f_j} \leq 1 \quad \forall j \in \mathcal{N},$$

$$\sum_{i \in \mathcal{N}} \sum_{f_i \in S_i} \sum_{e \in E_i} w_{ej} \cdot x_{j,f_j} \leq y_e \quad \forall e \in \mathcal{E},$$

This constraint makes sure only one strategy is played by each player.  $f_j \in S_j$

## Convex program of WCG Quadratic Program

$$\text{minimize} \quad \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} x_{j,f_j} \cdot L_{j,f_j} + \sum_{e \in \mathcal{E}} a_e \cdot y_e^2$$

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$$x_{j,f_j} \geq 0 \quad \forall j \in \mathcal{N}, f_j \in S_j$$

This constraint makes sure that the load on each resource is at least sum of the weights of the players using that resource.



## Dual Program

We denote the dual variables by  $\{\mu_j\}_{j \in \mathcal{N}}$ ,  $\{\Phi_e\}_{e \in \mathcal{E}}$  and  $\{\Psi_e\}_{e \in \mathcal{E}}$ . Then we use the Fenchel Duality to obtain the dual of the convex program.

$$\begin{aligned} & \text{maximize} && \sum_{j \in \mathcal{N}} \mu_j - \sum_{e \in \mathcal{E}} \frac{1}{4a_e} \cdot \Phi_e^2 \\ & \text{subject to} && \mu_j - \sum_{e \in f_j} w_{e,j} \cdot \Psi_e \leq L_{j,f_j} \quad \forall j \in \mathcal{N}, f_j \in S_j, \\ & && \Psi_e \leq \Phi_e \quad \forall e \in \mathcal{E}, \\ & && \mu_j \geq 0 \quad \forall j \in \mathcal{N}, \\ & && \Phi_e \geq 0 \quad \forall e \in \mathcal{E} \end{aligned}$$

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## Remark

We can take  $\Phi_e = \Psi_e$  for all  $e \in \mathcal{E}$  as from every CCE we will assign  $\Phi_e$  and  $\Psi_e$  to be the same value

## $(1 + \frac{1}{\delta})$ -Approximate Solution from Primal

Consider the following changed primal program:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{\delta} \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} x_{j,f_j} \cdot L_{j,f_j} + \sum_{e \in \mathcal{E}} a_e \cdot y_e^2 \\ \text{subject to} \quad & \sum_{f_j \in S_j} x_{j,f_j} \leq 1 \quad \forall j \in \mathcal{N}, \\ & \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} \sum_{e \in f_j} w_{ej} \cdot x_{j,f_j} \leq y_e \quad \forall e \in \mathcal{E}, \\ & x_{j,f_j} \geq 0 \quad \forall j \in \mathcal{N}, f_j \in S_j \end{aligned}$$

If  $\delta = 1$  we get our original program. For any  $\delta > 0$  we get a  $(1 + \frac{1}{\delta})$ -approximate solution.

## Dual don't need to change

Taking the dual of the new program we get the following:

$$\begin{aligned} & \text{maximize} && \sum_{j \in \mathcal{N}} \mu_j - \sum_{e \in \mathcal{E}} \frac{1}{4a_e} \cdot \Phi_e^2 \\ & \text{subject to} && \mu_j - \sum_{e \in f_j} w_{e,j} \cdot \Phi_e \leq \frac{L_{j,f_j}}{\delta} \quad \forall j \in \mathcal{N}, f_j \in S_j, \\ & && \mu_j \geq 0 \quad \forall j \in \mathcal{N}, \\ & && \Phi_e \geq 0 \quad \forall e \in \mathcal{E} \end{aligned}$$

So instead if we work with the old dual program and scale our variables  $\mu_j$ ,  $\Phi_e$  and  $\Psi_e$  by  $\frac{1}{\delta}$  we still get a feasible solution to the new dual program.

# Setting the Dual Variables

Let  $\sigma$  is any CCE of the game. Set

- $\mu_j = \frac{1}{\delta} \cdot \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)]$
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$$\begin{aligned} \text{Cost}_j(f_j, \theta_{-j}) &\leq \sum_{e \in f_j} w_{e,j} \cdot (a_e(l_e(\theta) + w_{e,j}) + b_e) \\ &= \sum_{e \in f_j} w_{e,j} (a_e \cdot w_{e,j} + b_e) + \sum_{e \in f_j} w_{e,j} \cdot a_e \cdot l_e(\theta) \\ &= L_{j,f_j} + \sum_{e \in f_j} w_{e,j} \cdot a_e \cdot l_e(\theta) \end{aligned}$$

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### Remark

It is a feasible solution to the dual program.

## Bound on PoA : I

$$\begin{aligned}\sum_{e \in \mathcal{E}} \frac{1}{a_e} \cdot a_e^2 \cdot \mathbb{E}_{f \sim \sigma} [l_e(f)]^2 &= \sum_{e \in \mathcal{E}} a_e \cdot \mathbb{E}_{f \sim \sigma} [l_e(f)]^2 \\ &\leq \mathbb{E}_{f \sim \sigma} \left[ \sum_{e \in \mathcal{N}} a_e \cdot l_e^2(f) \right] && \text{[Jensen]} \\ &\leq \mathbb{E}_{f \sim \sigma} \left[ \sum_{e \in \mathcal{N}} \text{Cost}_j(f) \right] = \sum_{j \in \mathcal{N}} \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)]\end{aligned}$$



## Bound on PoA : II

$$\begin{aligned}\text{Primal-Sol} &\geq \sum_{j \in \mathcal{N}} \frac{1}{\delta} \cdot \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)] - \sum_{e \in \mathcal{E}} \frac{1}{\delta^2} \cdot \frac{1}{4} a_e \cdot \mathbb{E}_{f \sim \sigma} [l_e(f)]^2 \\ &\geq \frac{1}{\delta} \sum_{j \in \mathcal{N}} \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)] - \frac{1}{4 \cdot \delta^2} \cdot \sum_{e \in \mathcal{E}} \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)] \\ &= \frac{4\delta - 1}{4\delta^2} \sum_{e \in \mathcal{E}} \mathbb{E}_{f \sim \sigma} [\text{Cost}_j(f)]\end{aligned}$$

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Primal is  $(1 + \frac{1}{\delta})$ -approximate solution to the optimal solution. So we get a bound of  $(1 + \frac{1}{\delta}) \frac{4\delta^2}{4\delta - 1}$  bound on PoA. Take  $\delta = \frac{1+\sqrt{5}}{4}$  you will get a bound of  $1 + \phi$  where  $\phi$  is the golden ratio.

Questions?

# Simultaneous Second-Price Auctions

# Definition

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GOAL: Maximize the social welfare of the players  $V(b) = \sum_{j \in \mathcal{N}} v_j(W_j(b))$

# Property of Biddings

## Theorem

$\forall j \in \mathcal{N}, \forall T \subseteq \mathcal{M}, \forall b \in \mathbb{R}_{\geq 0}^{m \times n}, \exists b_j(T) \in \mathbb{R}_{\geq 0}^m$  such that

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\}$$

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Let  $T = \{1, \dots, i\}$ . Take  $b_{ij}^* = v_j(1, 2, \dots, i) - v_j(1, 2, \dots, i-1)$ . Take  $b_j(T) = b_j^*$

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$\forall j \in \mathcal{N}, \forall T \subseteq \mathcal{M}, \forall b \in \mathbb{R}_{\geq 0}^{m \times n}, \exists b_j(T) \in \mathbb{R}_{\geq 0}^m$  such that

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\}$$

Let  $T = \{1, \dots, i\}$ . Take  $b_{ij}^* = v_j(1, 2, \dots, i) - v_j(1, 2, \dots, i-1)$ . Take  $b_j(T) = b_j^*$

Observe:  $\sum_{i \in T'} b_{ij}^* \leq v_j(T')$  for all  $T' \subseteq T$  by submodularity and for  $T = T'$  its equality.

## Proof of Theorem

$$\begin{aligned} u_j(b_j(T), b_{-j}) &= v_j(T^*) - \sum_{i \in T^*} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \\ &\geq v_j(T^*) - \sum_{i \in T^*} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} + \left[ \sum_{i \in T \setminus T^*} b_{ij}^* - \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \right] \\ &\geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \end{aligned}$$



# LP Formulation

- $x_{j,T} :=$  Variable for player  $j$  winning item  $T$ .

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$$\begin{aligned} & \text{maximize} && \sum_{T \subseteq \mathcal{M}} \sum_{j \in \mathcal{N}} x_{j,T} \cdot v_j(T) \\ & \text{subject to} && \sum_{j \in \mathcal{N}} \sum_{i \in T} x_{j,T} \leq 1 \quad \forall i \in \mathcal{M}, \\ & && \sum_{T \subseteq \mathcal{M}} x_{j,T} \leq 1 \quad \forall j \in \mathcal{N}, \\ & && x_{j,T} \geq 0 \quad \forall j \in \mathcal{N}, T \subseteq \mathcal{M} \end{aligned}$$

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This constraint makes sure no item is over-allocated  
i.e. each item is sold to only one player.

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This constraint makes sure each agent receives exactly one set from  $2^{\mathcal{M}}$ .

# Dual Program

$$\begin{aligned} &\text{minimize} && \sum_{j \in \mathcal{N}} y_j + \sum_{i \in \mathcal{M}} z_i \\ &\text{subject to} && y_j + \sum_{i \in T} z_i \geq v_j(T) \quad \forall j \in \mathcal{N}, T \subseteq \mathcal{M}, \\ &&& z_i \geq 0 \quad \forall i \in \mathcal{M}, \\ &&& y_j \geq 0 \quad \forall j \in \mathcal{N} \end{aligned}$$

## Setting the Dual Variables

Given a CCE  $\sigma$  of the game, we set the dual variables as follows:

- $y_j = \mathbb{E}_{b \sim \sigma} [u_j(b)]$  for all  $j \in \mathcal{N}$ .

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Since  $\sigma$  is an CCE

$$\mathbb{E}_{b \sim \sigma} [u_j(b)] \geq \mathbb{E}_{b \sim \sigma} [u_j(b_j(T), b_{-j})] \quad \forall T \subseteq \mathcal{M}$$



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By the theorem

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N}} \{b_{ij'}\}$$

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So  $\mathbb{E}_{b \sim \sigma} [u_j(b)] \geq v_j(T) - \sum_{i \in T} \mathbb{E}_{b \sim \sigma} \left[ \max_{j' \in \mathcal{N}} \{b_{ij'}\} \right]$ . So it is feasible solution to the dual program.

## Bound on PoA

$$\begin{aligned}\text{Primal-Sol} &\leq \sum_{j \in \mathcal{N}} \mathbb{E}_{b \sim \sigma} [u_j(b)] + \sum_{i \in \mathcal{M}} \mathbb{E}_{b \sim \sigma} \left[ \max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\ &= \mathbb{E}_{b \sim \sigma} \left[ \sum_{j \in \mathcal{N}} u_j(b) \right] + \mathbb{E}_{b \sim \sigma} \left[ \sum_{i \in \mathcal{M}} \max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\ &\leq 2 \cdot \mathbb{E}_{b \sim \sigma} [V(b)]\end{aligned}$$

## Bound on PoA

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So we get a bound of 2.

Questions?

# Facility Location Games

# Definition

- $\mathcal{M}$ : Set of  $m$  clients (Indexed by  $i$ )
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- Each client  $i \in \mathcal{M}$  has some value  $\pi_i \geq 0$  for the service money he is willing to pay.
- There is a cost  $c(l, i)$  for serving the client  $i \in \mathcal{M}$  from the location  $l \in \mathcal{L}$

## More Definitions

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- $V(s)$ : Social welfare of the strategy profile  $s$ ,  $W(s) = \sum_{j \in \mathcal{N}} u_j(s) + \sum_{i \in \mathcal{M}} D_i(s)$

# Choosing Prices

## Theorem

*For any strategy profile  $s$ , for any client  $i$  and supplier  $j$ ,  $SP(i) = j$*

$$(i) \quad c(s_j, i) = \min_{j' \in \mathcal{N}} c(s_{j'}, i)$$

$$(ii) \quad p_s(i, j) = \max \left\{ c(s_j, i), \min_{l \in \mathcal{K}(s) \setminus \{s_j\}} c(l, i) \right\}$$

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$$P_j(i, l, s_{-j}) = \begin{cases} \min_{l' \in \mathcal{K}(s) \setminus \{s_j\}} c(l', i) - c(l, i) & \text{If } c(l, i) \leq c(l', i) \\ 0 & \text{Otherwise} \end{cases}$$

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$$W(s) = \sum_{j \in \mathcal{N}} u_j(s) + \sum_{i \in \mathcal{M}} D_i(s) = \sum_{i \in \mathcal{M}} \pi_i - c(s_{SP(i)}, i)$$

## LP Formulation

- $x_{ijl} :=$  Variable indicating if the supplier  $j$  serves the client  $i$  from location  $l$ .
- $x_{jl} :=$  Variable indicating if the supplier  $j$  opens a facility at location  $l$ .

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$$\text{maximize} \quad \sum_{j \in \mathcal{N}} \sum_{l \in S_j} \sum_{i \in \mathcal{M}} (\pi_i - c(l, i)) \cdot x_{ijl}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{N}} \sum_{l \in S_j} x_{ijl} \leq 1 \quad \forall i \in \mathcal{M},$$

$$\sum_{j \in \mathcal{N}} x_{jl} \leq 1 \quad \forall l \in \mathcal{L},$$

$$\sum_{k \in S_j} x_{jl} \leq 1 \quad \forall j \in \mathcal{N},$$

$$x_{ijl} \leq x_{jl} \quad \forall i \in \mathcal{M}, j \in \mathcal{N}, l \in S_j,$$

$$x_{ijl} \geq 0 \quad \forall i \in \mathcal{M}, j \in \mathcal{N}, l \in S_j$$

# Dual Program

We denote the dual variables by  $\{\alpha_j\}_{j \in \mathcal{N}}$ ,  $\{\beta_i\}_{i \in \mathcal{M}}$ ,  $\{\gamma_l\}_{l \in \mathcal{L}}$  and  $\{z_{ijl}\}_{i \in \mathcal{M}, j \in \mathcal{N}, l \in \mathcal{S}_j}$ .



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$$\begin{aligned} & \text{minimize} && \sum_{j \in \mathcal{N}} \alpha_j + \sum_{i \in \mathcal{M}} \beta_i + \sum_{l \in \mathcal{L}} \gamma_l \\ & \text{subject to} && \beta_i + z_{ijl} \geq \pi_i - c_{il} \quad \forall i \in \mathcal{M}, j \in \mathcal{N}, l \in S_j, \\ & && \gamma_l + \alpha_j \geq \sum_{i \in \mathcal{M}} z_{ijl} \quad \forall j \in \mathcal{N}, l \in S_j, \\ & && \alpha_j \geq 0 \quad \forall j \in \mathcal{N}, \\ & && \beta_i \geq 0 \quad \forall i \in \mathcal{M} \end{aligned}$$

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- $z_{ijl} = \mathbb{E}_{s \sim \sigma} [P_j(i, l, s_{-j})]$  for all  $i \in \mathcal{M}, j \in \mathcal{N}$  and  $l \in S_j$ .

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- Define  $W_l(s) = u_j(s)$  if  $l \in \mathcal{K}(s)$  and  $s_j = l$  for some  $j \in \mathcal{N}$  and otherwise 0.  
Then  $\gamma_l = \mathbb{E}_{s \sim \sigma} [W_l(s)]$  for all  $l \in \mathcal{L}$ .

# Feasibility Checking

- $\pi_i - p_s(i, SP(i)) \geq \pi_i - c(l, i)$  for any  $l \in \mathcal{L}$ . Now  $P_j(i, l, s_{-j}) \neq 0$  when  $l = SP(i)$ . Then clearly  $\pi_i - p_s(i, SP(i)) + P_j(i, SP(i), s_{-j}) = \pi_i - c(SP(i), i)$  and for other locations  $P_j(i, l, s_{-j}) = 0$ . So the first constraint is satisfied

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- If  $l \in \mathcal{K}(s)$  then  $W_l(s) = \sum_{i \in \mathcal{M}} P_j(i, l, \theta_{-j})$  for some  $j \in \mathcal{N}$  such that  $s_j = l$ . So it satisfies the second constraint. If  $l \notin \mathcal{K}(s)$ .  $u_j(s) \geq P_j(i, l, s_{-j})$  since  $\sigma$  is a CCE. So the second constraint is satisfied.

## Bound on PoA

$\sum_{j \in \mathcal{N}} \alpha_j + \sum_{i \in \mathcal{M}} \beta_i$  is the expected social welfare under the distribution  $\sigma$ .

$\sum_{l \in \mathcal{L}} W_l(s)$  is at most the social welfare since  $\sigma$  is a CCE.

So by Weak Duality

$$\text{Primal-Sol} \leq \sum_{j \in \mathcal{N}} \alpha_j + \sum_{i \in \mathcal{M}} \beta_i + \sum_{l \in \mathcal{L}} \gamma_l \leq 2 \cdot \mathbb{E}_{s \sim \sigma} [V(s)]$$