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Course: Probability Theory

**Assignment - 2**
Dept: STCS
Date: October 4, 2024

**Problem 1**

We know that independent random variables are uncorrelated. Argue that uncorrelated jointly Gaussian random variables are independent.
Hint: do this for two random variables first. For $n$ random variables, you might find it easier to use the characteristic function.

***Solution:*** Let $\overline{Z} = (Z_1, \dots, Z_n)^T$ be the $n$ uncorrelated jointly Gaussian random variables. Let $K$ be the covarince matrix of $Z$. ■

**Problem 2**

(i) * Let $X$ and $Y$ be independent random variables. $X_1 \sim N(0,1)$; and $Y = +1$ with probability $p$ and $Y = -1$ with probability $1 - p$. We define $X_2 = YX_1$. Is $X_2$ Gaussian? Are $X_1, X_2$ jointly Gaussian? Justify your answers.

[See Example 3.3.4 from [G] for a solution]

(ii) Repeat (i) if $X_1 \sim N(m,1)$ and $m > 0$

***Solution:***

■

**Problem 3** [G] Exercise 3.8

(a) Let $[K] = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$. Show that 1 and $\frac{1}{2}$ are eigenvalues of $[K]$ and find the normalized eigenvectors. Express $[K]$ ad $[Q\Lambda Q^{-1}]$, where $[\Lambda]$ is diagonal and $[Q]$ is orthonormal.

(b) Let $[K'] = \alpha[K]$ for real $\alpha \neq 0$. Find the eigenvalues and eigenvectors of $[K']$. Don not use brute force - think!

(c) Find the eigenvalues and eigenvectors of $[K^m]$, where $[K^m]$ is the $mth$ power of $[K]$.

***Solution:*** ■

**Problem 4**

We derived the p.d.f. of a jointly Gaussian random vector $X = AW$, where $A$ is an $n \times n$ matrix. We used the fact $A$ is invertible. How would you precisely describe the distribution of $X$ if $A$ is not invertible? Describe the underlying geometry of the distribution of $X$. Use the following $A$ as an example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

***Solution:*** ■

**Problem 5** [G] Problem 3.9

Let $X$ and $Y$ be jointly Gaussian with means $m_X$, $m_Y$, variances $\sigma_X^2, \sigma_Y^2$, and normalized covariance $\rho$. Find the conditional density $f_{X|Y}(x \mid y)$.

***Solution:*** ■

In the next two problems we will use a common model for communication systems. The transmitted signal $\vec{X}$ is a Gaussian random vector of size m (vector since there are several, say $m$, transmit antennas and each component of the vector stands for the input to a separate antenna). The signal goes over a linear and additive Gaussian noise channel and is picked up by a receiver which also has $n$ antennas. The received vector of length $n$ has the form.

$$\vec{Y} = H\vec{X} + \vec{Z}, \tag{1}$$

where $H$ is a constant $n \times m$ vector and $\vec{Z}$ is a Gaussian random vector of size $n$ and independent of $\vec{X}$.

**Problem 6**

Let us first consider the simpler case of $m = 1$ and $n = 2$. So $X$ is a scalar random variable. Let $X$ have the standard normal distribution $N(0, 1)$. The received signals are

$$Y_i = h_i X + Z_i, \qquad i = 1, 2,$$

where $Z_i \sim N(0, \sigma^2)$ are i.i.d and independent of $X$. And $h_i'$s are constants which represent the channel "gains" from the transmit antenna to the receive antennas.

(a) Find the conditional joint distribution of $Y_1, Y_2$ conditioned on $X = x$.

(b) Find the conditional joint distribution of $X$ conditioned on $Y_1 = y_1$, $Y_2 = y_2$.

(c) Using (b), what is your estimate of the transmitted signal $X$ if you are told that the receive antennas observed $Y_1 = y_1$, $Y_2 = y_2$. **Interpret your results**. Does your answer make intuitive sense? What happens to the estimate when the noise variance $\sigma^2$ becomes small? or large?

***Solution:*** ■

**Problem 7**

Now consider the general model in (1) for general $n, m$. Let $\vec{X} \sim N(\vec{0}, K_X)$, $\vec{Z} \sim N(\vec{0}, K_Z)$ and $\vec{Z}$ is independent of $\vec{X}$.

(a) Show that $\vec{U} = (\vec{X}, \vec{Y})$ is jointly Gaussian. You may use any of the equivalent definitions we saw in class

(b) Find a simple condition on $H, K_X, K_Z$ so that $K_U$ is invertible.

(c) What is the conditional distribution of the input $\vec{X}$ given the output $\vec{Y} = \vec{y}$.

***Solution:*** ■