CSS.201.1 Algorithms

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2.1 Twelve Problems: n Balls in m Bins

Theorem 2.1.1

	≤ 1 balls/bin $(m \geq n)$	≥ 1 balls/bin $(m \leq n)$	Unrestricted
Identical Balls, Identical Bins	1	$P(n, m)$	$\sum\limits_{i=1}^{m} P(n, i)$
Identical Balls, Distinguishable Bins	$\binom{m}{n}$	$\binom{m-1}{n-1}$	$\binom{n+m-1}{m-1}$
Distinguishable Balls, Identical Bins	1	$S_2(n, m)$	$\sum\limits_{i=1}^{m} S_2(ni)$
Distinguishable Balls, Distinguishable Bins	$\binom{m}{n} n!$	$S_2(n, m)m!$	m^n

Proof:

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2.2 Stirling Numbers

2.2.1 Stirling Number of Second Kind

Definition 2.2.1: Stirling Number of The Second Kind

It is the number of ways to partition the set $[n]$ into m nonempty parts.

Clearly if we take the n balls to be the set $[n]$ the balls become distinguishable and each partition is bin and the order order of the partition doesn't matter the bins are identical. So the it becomes the number of ways n distinguishable balls divided into m identical bins.

Lemma 2.2.1

$S_2(n, m) = S_2(n-1, m-1) + mS_2(n-1, m)$

Combinatorial Proof: We have the balls $[n]$. Then there are two cases. The bin containing ball '1' can has only 1 ball or it can have ≥ 2 balls.

For the first case the bin containing ball '1' has only one balls. So the rest of the $n-1$ balls are divided into the rest of the $m-1$ bins. The number of ways this is done is $S_2(n-1, m-1)$.

For the second case the bin containing ball '1' has at least 2 balls. In that case apart from the ball '1' all the other balls are filled into m identical bins where each bin has at least 1 ball. So we can think this scenario in other way that is first we fill bins with all the balls except '1' and then we choose where to put the ball '1'. So the number of ways the balls, $\{2, 3, \dots, n\}$ i.e. $n-1$ distinguishable balls can be divided into m bins is $S_2(n-1, m)$. Now there are m choices for the ball '1' to be partnered up. Hence for this case there are $mS_2(n-1, m)$ many ways.

Therefore the total number of ways the n distinguishable balls can be divided into m bins so that each bin has at least 1 ball is $S_(n-1, m-1) + mS_2(n-1, m)$. Therefore we get $S_2(n, m) = S_2(n-1, m-1) + mS_2(n-1, m)$. ■

Lemma 2.2.2

$$S_2(n+1, m+1) = \sum_{i=m}^{n} \binom{n}{i} S_2(i, m)$$

Combinatorial Proof: On the *LHS* we are counting the number of ways to partition $[n+1]$ into $m+1$ parts.

For the *RHS* let's focus on the element $n+1$. So we drop the element from $[n+1]$ in the $(m+1)^{th}$ part. The $(m+1)^{th}$ block can have k elements from $[n]$ which are partnered by $n+1$ where $0 \leq k \leq n-m$. We have $k \leq n-m$ since all the other m parts have at least 1 element that leaves us $n-m$ elements to choose. So there are $\binom{n}{k}$ ways to choose the k elements. The remaining $n-k$ elements are divided into m parts which can be done in $S_2(n-k, m)$ many choices. So in total we have $\sum +k = 0^{n-m} S_2(n-k, m)$ ways to divide $[n+1]$ into $m+1$ parts. Therefore we have

$$S_2(n+1, m+1) = \sum_{i=0}^{n-m} \binom{n}{i} S_2(n-i, m) = \sum_{i=0}^{n-m} \binom{n}{n-i} S_2(n-i, m) = \sum_{i=m}^{n} \binom{n}{i} S_2(i, m)$$

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Algebraic Proof: We will prove by Induction.

$$\begin{aligned}
S_2(n+1, m+1) &= S_2(n, m) + (m+1)S_2(n, m) \\
&= \sum_{i=m-1}^{n-1} \binom{n-1}{i} S_2(i, m-1) + (m+1) \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) \\
&= \sum_{i=m-1}^{n-1} \binom{n-1}{i} S_2(i, m-1) + m \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) + \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) \\
&= \sum_{i=m}^{n-1} \binom{n-1}{i-1} S_2(i-1, m-1) + m \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) + \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) \\
&= \sum_{i=m}^{n-1} \binom{n-1}{i-1} S_2(i-1, m-1) + m \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) + \sum_{j=m}^{n-1} \left[\binom{n}{j} - \binom{n-1}{j-1} \right] S_2(j, m) \\
&= \sum_{i=m}^{n-1} \binom{n-1}{i-1} S_2(i-1, m-1) + m \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) + \sum_{j=m}^{n-1} \binom{n}{j} S_2(j, m) - \sum_{j=m}^{n-1} \binom{n-1}{j-1} S_2(j, m) \\
&= \sum_{i=m}^{n-1} \binom{n-1}{i-1} S_2(i-1, m-1) + m \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) + \sum_{j=m}^{n-1} \binom{n}{j} S_2(j, m) - \sum_{j=m}^{n-1} \binom{n-1}{j-1} \Big[S_2(j-1, m-1) + mS_2(j-1, m) \Big] \\
&= \cancel{\sum_{i=m}^{n-1} \binom{n-1}{i-1} S_2(i-1, m-1)} + m \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) + \sum_{j=m}^{n-1} \binom{n}{j} S_2(j, m) - \cancel{\sum_{j=m}^{n-1} \binom{n-1}{j-1} S_2(j-1, m-1)} - m \sum_{j=m}^{n-1} \binom{n-1}{j-1} S_2(j - \\
&= m \sum_{j=m}^{n-1} \binom{n-1}{j} S_2(j, m) + \sum_{j=m}^{n-1} \binom{n}{j} S_2(j, m) - m \sum_{j=m}^{n-1} \binom{n-1}{j-1} S_2(j-1, m)
\end{aligned}$$

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2.2.2 Stirling Number of First Kind

Definition 2.2.2: Stirling Number of The First Kind

It is the number of ways to partition the set $[n]$ into m nonempty parts.