

---

# ALGEBRA AND COMPUTATION

*Instructor: Amit Kumar Sinhababu and Sumanta Ghosh*

---

SCRIBE: SOHAM CHATTERJEE

SOHAMCHATTERJEE999@GMAIL.COM

WEBSITE: SOHAMCH08.GITHUB.IO

# CONTENTS

<b>CHAPTER 1</b>	<b>INTRODUCTION</b>	<b>PAGE 2</b>
<b>CHAPTER 2</b>	<b>INTEGER AND POLYNOMIAL ADDITION</b>	<b>PAGE 3</b>
<b>CHAPTER 3</b>	<b>INTEGER AND POLYNOMIAL MULTIPLICATION</b>	<b>PAGE 4</b>
<b>CHAPTER 4</b>	<b>POLYNOMIAL EVALUATION</b>	<b>PAGE 5</b>
4.1	Introduction	5
4.2	Single Point Evaluation	6
4.2.1	Horner's Method	6
4.3	Fast Multi-point Evaluation	6
<b>CHAPTER 5</b>	<b>POLYNOMIAL INTERPOLATION</b>	<b>PAGE 8</b>
<b>CHAPTER 6</b>	<b>BIBLIOGRAPHY</b>	<b>PAGE 9</b>

# **Chapter 1**

## **Introduction**

## **Chapter 2**

# **Integer and Polynomial Addition**

# Chapter 3

## Integer and Polynomial Multiplication

### Definition 3.1: Multiplication Time Function: $M(n)$

The function  $M : \mathbb{N} \rightarrow \mathbb{R}_+$  for any commutative ring  $R[x]$  is called multiplication time function for if polynomials in  $R[x]$  of degree less than  $n$  can be multiplied using at most  $M(n)$  operations in  $R$ .

Similarly we can define the function  $M$  as above for multiplication time for  $\mathbb{Z}$  if two integers of length  $n$  bits can be multiplied using at most  $M(n)$  operations

**Assumption 3.0.1.** *content...*

**Proof of Claim c:** *ontent...* ■

# Chapter 4

## Polynomial Evaluation

### 4.1 Introduction

We will consider the following situation:  $R$  is a commutative ring as always and  $f \in R[x]$  where  $\deg(f) = d$ . We also have  $k$  points  $u_0, \dots, u_{k-1} \in R$ . Now we want to discuss here the fast algorithms of finding out  $(f(u_0), \dots, f(u_{k-1}))$ . So we basically want the evaluation map

$$\begin{aligned}\varphi : R[x] / \langle m \rangle &\rightarrow R^n \\ f &\rightarrow (f(u_0), \dots, f(u_{k-1}))\end{aligned}$$

which is a ring homomorphism. If  $R$  is a field then  $R[x]$  is a vector space over  $R$  and the  $\phi$  is an isomorphism. Formally we want to solve the following two problems with fast algorithms:

#### Problem 4.1: Single Point evaluation

Given  $f \in R[x]$  with  $\deg(f) = d$  and  $\alpha \in R$  compute  $f(\alpha)$

#### Problem 4.2: Multi-Point evaluation

Given  $f \in R[x]$  with  $\deg(f) = d$  and  $u_0, \dots, u_{n-1} \in R$  compute  $f(u_0), \dots, f(u_{n-1})$

## 4.2 Single Point Evaluation

### 4.2.1 Horner's Method

#### Theorem 4.2.1 Horner's Method

Given a polynomial  $f(x) = \sum_{i=0}^d a_i x^i$  where  $a_i \in R$  for all  $i \in [n]$  and a point  $\alpha \in R$  using only  $O(d)$  many additions and multiplications.

**Proof:** Consider the following algorithm:

---

#### Algorithm 1: Horner's Method

---

```
1 begin  
2    $p(x) = a_0 + x(a_1 + x(a_2 + (x(\cdots + x(a_n) \cdots))))$ 
```

---

Clearly we are using only  $d$  many additions and  $d$  many multiplications. So overall we need  $2d = O(d)$  ring operations to evaluate the polynomial. The following lower bound results we obtain. ■

This is the minimal number of additions and multiplications for any algorithm to evaluate a polynomial.

#### Theorem 4.2.2 [OST13]

Any algorithm to evaluate an arbitrary degree  $d$  polynomial  $f \in R[x]$  at any point  $\alpha \in R$  must use at least  $n$  additions

#### Theorem 4.2.3 [Pan66]

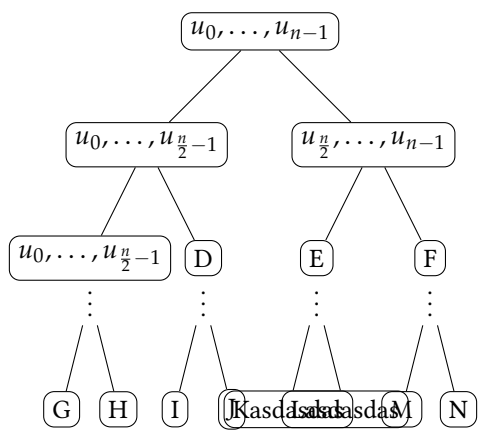
Any algorithm to evaluate an arbitrary degree  $d$  polynomial  $f \in R[x]$  at any point  $\alpha \in R$  without initial conditioning of coefficients has at least  $n$  multiplications and at least  $n$  additions.

#### Theorem 4.2.4 [Pan66],[Mot55]

Any degree  $d$  real polynomial can be evaluated using  $\left\lfloor \frac{d}{2} \right\rfloor + 2$  multiplications and  $d$  additions.

## 4.3 Fast Multi-point Evaluation

A trivial algorithm for using  $O(d^2)$  ring operations is to apply [Horner's Method](#) for each point and since it takes  $O(d)$  operations for each point we can find the evaluations at all  $d$  points in  $O(d^2)$  many ring operations. But we want to get close to linear operations. Since Horner's rules uses lowest number of ring operations doesn't mean for  $d$  points  $O(d^2)$  is lowest. There is a fast algorithm to evaluate the polynomial at all  $d$  points using  $O(M(d) \log d)$  operations.





## **Chapter 5**

# **Polynomial Interpolation**

# Chapter 6

## Bibliography

- [Mot55] T. S. Motzkin. Evaluation of polynomials and evaluation of rational functions. *Bulletin of the American Mathematical Society*, 61:163, 1955.
- [OST13] A. OSTROWSKI. On two problems in abstract algebra connected with horner’s rule. In RICHARD von MISES, editor, *Studies in Mathematics and Mechanics*, pages 40–48. Academic Press, 2013.
- [Pan66] Victor Y. Pan. Methods of computing values of polynomials. *Russian Mathematical Surveys*, 21:105–136, 1966.