Partially Symmetric Function: $f: \mathbb{F}_2^n \to \mathbb{F}_2$ is J-partially symmetric for some $J \subseteq [n]$ if $\forall\, \pi \in S_J$ $f(x) = f(\pi x)$

f is t-symmetric or $(n-t)$-~~co~~symmetric if $\exists\, J \subseteq [n]$ with $|J| \geq t$ such that f is J-symmetric

▶ How to define the influence:

Take a subset $J \subseteq [n]$. See the similarity with Junta Case:

Junta: Want to Test $\to$ f is J-Junta

$\Leftrightarrow$ J has only the depending variable

$\Leftrightarrow$ $\bar{J}$ has no effect on the output

$\Leftrightarrow$ If changing assignment in $\bar{J}$ for any x changes output

So Want to see $\to$ Fraction of entries in table of output of f need to be changed to make f J-Junta

$$= \Pr_{x \in \mathbb{F}_2^n,\, y \in \mathbb{F}_2^{\bar{J}}} \left[f(x) \neq f(x_J\, y_{\bar{J}}) \right] =: \mathrm{Inf}_f(J)$$

Partially Symmetric: Want to Test $\to$ f is J-partially Symmetric

$\Leftrightarrow$ Permuting variables of J has no effect on output

So want to see $\longrightarrow$ Fraction of entries in table of outputs of f need to be changed to make f J-partially symmetric

$$= \Pr_{x \in \mathbb{F}_2^n,\, \pi \in S_J} \left[f(x) \neq f(\pi x) \right] =: \mathrm{SymInf}_f(J)$$

▶ How to Test (What is the Intuition):

Note the similarity between K-Junta Testing and K-cosymmetry testing

K-Junta Testing

i> K-Junta => K element set with high influence
-> K variables each with high influence

ii> Take a Random partition to have those variables in different parts with h.p.

iii> Disjoint Union of parts is K-Junta
<=> ∃ K parts which are 1-Junta ie High Influence

K-Cosymmetry Testing

i> K-cosymmetry => K element set with high sym-influence
=> K variables each with high sym-influence

ii> Take a Random partition to have those variables in different parts with h.p.

iii> Problem: For any partition it is not possible to break a permutation into disjoint permutions following the partitions.

So Disjoint union of parts is K-cosymmetric
⇎ ∃ K parts which have high sym-influence

Solution: If I fix a partition I can break the permution into small permutations each of which corresponds to a part and acts on that part ∪ Fixed Part.

So select a partition as work space. Then break the permutation into a set of smaller permutations each of which corresponds to different part in the partition and works on Associated part ∪ Fixed Part

▸ Algorithm:

1. Create a Random Partition $\mathcal{I}$ of $[n]$ into $r = \Theta(\frac{k^2}{\epsilon^2})$ parts
2. $J := \emptyset$
3. Pick a Random workspace $W \in \mathcal{I}$
4. If $|W| < \frac{n}{2r}$: Return Fail
5. For $i=1$ to $\Theta(k/\epsilon)$ do: $\longrightarrow$ Want to do a little more than k to know if asymmetric variables $> k$
6. $\quad I := \text{Find-Asymmetric-Set}(f, \mathcal{I}, J, W)$ $\longrightarrow$ Finds a ~~set~~ part containing asymmetric variables
7. $\quad\quad$ If $I \neq \emptyset$:

   $\quad\quad\quad J = J \cup I$

8. $\quad\quad$ If J is union of $> k$ part: Return Reject $\rightarrow$ This is why we need to loop a little more than k
9. Return Accept

- **How to find $x^0, \dots, x^t$:**

Suppose we found till i

Define $\mathcal{J}^+ = \{ I \in \mathcal{I}' - w : |x^i_I| > |x^t_I| \}$ → Need to decrease wt of I to ensure $x^{i+1}_I = x^t_I$

$\mathcal{J}^- = \{ I \in \mathcal{I}' - w : |x^i_I| < |x^t_I| \}$ → Need to increase wt of I to ensure $x^{i+1}_I = x^t_I$

Note: We don't need to worry for $I \in \mathcal{I}$ st $|x^i_I| = |x^t_I|$ as they can be permuted to have $x^i_I = x^t_I$ with a permutation within I.
So let $\mathcal{J}^+ \cup \mathcal{J}^- \neq \emptyset$

Case 1: $\mathcal{J}^+ \neq \emptyset$, $\mathcal{J}^- \neq \emptyset$ Note $\max\{|x^i_w|, W - |x^i_w|\} \geq \lceil \frac{|W|}{2} \rceil$

If $|x^i_w| \geq \lceil \frac{|W|}{2} \rceil$ then $\exists I \in \mathcal{J}^-$ then with a permutation send some wt from x^i_w to x^i_I to have $x^{i+1}_I = x^t_I$ as $|I| \leq \lceil \frac{|W|}{4} \rceil$

If $|x^i_w| < \lceil \frac{|W|}{2} \rceil$ then $\exists I \in \mathcal{J}^+$ then with a permutation send some wt from x^i_I to x^i_w to have $x^{i+1}_I = x^t_I$ as $|I| \leq \lceil \frac{|W|}{4} \rceil$

Case 2: $\mathcal{J}^+ = \emptyset$ or $\mathcal{J}^- = \emptyset$. WLOG $\mathcal{J}^+ = \emptyset$ and $\mathcal{J}^- \neq \emptyset$

$$|x^i| = |x^t| \iff |x^i_w| - |x^t_w| = -\sum_{I \in \mathcal{I}' - w} |x^i_I| - |x^t_I| = -\sum_{I \in \mathcal{J}^-} |x^i_I| - |x^t_I| = \sum_{I \in \mathcal{J}^-} |x^t_I| - |x^i_I|$$

Choose any $I \in \mathcal{J}^-$ and decrease ham wt of x^i_w to send wt to x^i_I to ensure $x^{i+1}_I = x^t_I$

Do same for $\mathcal{J}^- = \emptyset$, $\mathcal{J}^+ \neq \emptyset$.

So Find-Asymmetric-Set $(f, \mathcal{I}, J, W)$:

Generate ~~x~~ $x \in_R \mathbb{F}_2^n$, $\pi \in_R S_J$

If $f(x) \neq f(\pi x)$:

Find $x = x^0, \ldots, x^t = \pi x$ from x to πx

Perform binary search over $x^0, \ldots, x^t$ to find i such that $f(x^{i-1}) \neq f(x^i)$

Return the part $I \in \mathcal{I} - W$ such that $x^{i-1}_I \neq x^i_I$

Return ϕ.