Bounding PoA using LP, QP and Fenchel Duality

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Introduction

The Wildcat theme is a Beamer theme for Northwestern University, but which can be modified easily with different colors, fonts, and even background patterns.

The theme is inspired by the Metropolis theme by Matthias Vogelgesang. It incorporates the Northwestern University facet design pattern, but otherwise has a clean, simple look, and relatively few bells and whistles. It is licensed under the GNU GENERAL PUBLIC LICENSE.

Weighted Congestion Games

Definitions

- $\mathcal{N}$: Set of players
- $\mathcal{E}$: The ground set of resources
- For each player $j \in \mathcal{N}$, let $S_j \subseteq 2^{\mathcal{E}}$ be the set of strategies available to player j. Let $S = \bigtimes_{j \in \mathcal{N}} S_j$.
- For each $j \in \mathcal{N}$ and each $e \in \mathcal{E}$ there is a weight of the resource $w_{ej} \in \mathbb{R}^+$.
- For each $e \in \mathcal{E}$ the cost of resource e is an affine function $C_e : \mathbb{R} \to \mathbb{R}$ where $c_e(x) = a_e \cdot x + b_e$
- For any strategy profile $f \in S$, the cost of player j is **Cost**$(f)_j = \sum_{e \in f_j} w_{ej} \cdot c_e(l_e(f))$ where $l_e(f) = \sum_{j' : e \in f_{j'}} w_{ej'}$ is the load on resource e. Do

$$\text{Cost}(f) = \sum_{j \in \mathcal{N}} \sum_{e \in f_j} w_{ej} \cdot c_e(l_e(f)) = \sum_{e \in \mathcal{E}} a_e \cdot l_e(f) + b_e \cdot l_e(f)$$

Convex program of WCG
Setting up the variables

For any player $j \in \mathcal{N}$ and $f_j \in S_j$ let $L_{j,f_j} = \sum_{e \in f_j} w_{ej} \cdot c_e(w_{ej})$ i.e. the cost incurred by player j when it plays strategy f_j.

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- $\mathbf{x_{j,f_j}}$:= Variable for player j playing strategy f_j for all $j \in \mathcal{N}$ and $f_j \in S_j$

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- x_{j,f_j} := Variable for player j playing strategy f_j for all $j \in \mathcal{N}$ and $f_j \in S_j$
- $\mathbf{y_e}$:= Variable for the load on resource e for all $e \in \mathcal{E}$

Convex program of WCG
Quadratic Program

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} x_{j,f_j} \cdot L_{j,f_j} + \sum_{e \in \mathcal{E}} a_e \cdot y_e^2 \\
\text{subject to} \quad & \sum_{f_j \in S_j} x_{j,f_j} \leq 1 \qquad \forall j \in \mathcal{N}, \\
& \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} \sum_{e \in f_j} w_{ej} \cdot x_{j,f_j} \leq y_e \qquad \forall e \in \mathcal{E}, \\
& x_{j,f_j} \geq 0 \qquad \forall j \in \mathcal{N},\ f_j \in S_j
\end{aligned}
$$

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$$\text{subject to} \quad \boxed{\sum_{f_j \in S_j} x_{j,f_j} \leq 1 \quad \forall j \in \mathcal{N},}$$

$$\sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} \sum_{e \in f_j} w_{ej} \cdot x_{j,f_j} \leq y_e \quad \forall e \in \mathcal{E},$$

$$f_j \in S_j$$

This constraint makes sure only one strategy is played by each player.

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\text{subject to} \quad & \sum_{f_j\in S_j} x_{j,f_j} \leq 1 \quad \forall j \in \mathcal{N}, \\
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& x_{j,f_j} \geq 0 \quad \forall j \in \mathcal{N},\ f_j \in S_j
\end{aligned}
$$

This constraint makes sure that the load on each resource is at least sum of the weights of the players using that resource.

Dual Program

We denote the dual variables by $\{\mu_j\}_{j\in\mathcal{N}}$, $\{\Phi_e\}_{e\in\mathcal{E}}$ and $\{\Psi_e\}_{e\in\mathcal{E}}$. Then we use the Fenchel Duality to obtain the dual of the convex program.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j\in\mathcal{N}} \mu_j - \sum_{e\in\mathcal{E}} \frac{1}{4a_e} \cdot \Phi_e^2 \\
\text{subject to} \quad & \mu_j - \sum_{e\in f_j} w_{e,j} \cdot \Psi_e \leq L_{j,f_j} \quad && \forall j \in \mathcal{N}, f_j \in S_j, \\
& \Psi_e \leq \Phi_e && \forall e \in \mathcal{E}, \\
& \mu_j \geq 0 && \forall j \in \mathcal{N}, \\
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& \mu_j \ge 0 & \forall j \in \mathcal{N}, \\
& \Phi_e \ge 0 & \forall e \in \mathcal{E}
\end{aligned}$$

Remark

We can take $\Phi_e = \Psi_e$ for all $e \in \mathcal{E}$ as from every CCE we will assign Φ_e and Ψ_e to be the same value

$\left(1+\frac{1}{\delta}\right)$-Approximate Solution from Primal

Consider the following changed primal program:

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{\delta} \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} x_{j,f_j} \cdot L_{j,f_j} + \sum_{e \in \mathcal{E}} a_e \cdot y_e^2 \\
\text{subject to} \quad & \sum_{f_j \in S_j} x_{j,f_j} \leq 1 \qquad \forall j \in \mathcal{N}, \\
& \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} \sum_{e \in f_j} w_{ej} \cdot x_{j,f_j} \leq y_e \qquad \forall e \in \mathcal{E}, \\
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$$

If $\delta = 1$ we get our original program. For any $\delta > 0$ we get a $\left(1+\frac{1}{\delta}\right)$-approximate solution.

Dual don't need to change

Taking the dual of the new program we get the following:

$$\begin{aligned}
\text{maximize} \quad & \sum_{j \in \mathcal{N}} \mu_j - \sum_{e \in \mathcal{E}} \frac{1}{4a_e} \cdot \Phi_e^2 \\
\text{subject to} \quad & \mu_j - \sum_{e \in f_j} w_{e,j} \cdot \boldsymbol{\Phi_e} \leq \frac{\mathbf{L_{j,f_j}}}{\boldsymbol{\delta}} && \forall j \in \mathcal{N}, f_j \in S_j, \\
& \mu_j \geq 0 && \forall j \in \mathcal{N}, \\
& \Phi_e \geq 0 && \forall e \in \mathcal{E}
\end{aligned}$$

So instead if we work with the old dual program and scale our variables μ_j, Φ_e and Ψ_e by $\frac{1}{\delta}$ we still get a feasible solution to the new dual program.

Setting the Dual Variables

Let σ is any CCE of the game. Set

- $\mu_j = \frac{1}{\delta} \cdot \mathbb{E}_{f \sim \sigma}[\mathrm{Cost}_j(f)]$
- $\Phi_e = \frac{1}{\delta} \cdot a_e \cdot \mathbb{E}_{f \sim \sigma}[l_e(f)]$

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$$\begin{aligned}
\text{Cost}_j(f_j, \theta_{-j}) &\leq \sum_{e \in f_j} w_{e,j} \cdot (a_e(l_e(\theta) + w_{e,j}) + b_e) \\
&= \sum_{e \in f_j} w_{e,j}(a_e \cdot w_{e,j} + b_e) + \sum_{e \in f_j} w_{e,j} \cdot a_e \cdot l_e(\theta) \\
&= L_{j,f_j} + \sum_{e \in f_j} w_{e,j} \cdot a_e \cdot l_e(\theta)
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&= L_{j,f_j} + \sum_{e \in f_j} w_{e,j} \cdot a_e \cdot l_e(\theta)
\end{aligned}
$$

Remark

It is a feasible solution to the dual program.

Bound on PoA : I

$$
\begin{aligned}
\sum_{e\in\mathcal{E}} \frac{1}{a_e} \cdot a_e^2 \cdot \mathop{\mathbb{E}}_{f\sim\sigma}[l_e(f)]^2 &= \sum_{e\in\mathcal{E}} a_e \cdot \mathop{\mathbb{E}}_{f\sim\sigma}[l_e(f)]^2 \\
&\leq \mathop{\mathbb{E}}_{f\sim\sigma}\left[\sum_{e\in\mathcal{N}} a_e \cdot l_e^2(f)\right] && \text{[Jensen]} \\
&\leq \mathop{\mathbb{E}}_{f\sim\sigma}\left[\sum_{e\in\mathcal{N}} \mathrm{Cost}_j(f)\right] = \sum_{j\in\mathcal{N}} \mathop{\mathbb{E}}_{f\sim\sigma}[\mathrm{Cost}_j(f)]
\end{aligned}
$$

Bound on PoA : II

$$\begin{aligned}
\text{Primal-Sol} &\geq \sum_{j \in \mathcal{N}} \frac{1}{\delta} \cdot \mathop{\mathbb{E}}_{f \sim \sigma} [\mathrm{Cost}_j(f)] - \sum_{e \in \mathcal{E}} \frac{1}{\delta^2} \cdot \frac{1}{4} a_e \cdot \mathop{\mathbb{E}}_{f \sim \sigma} [l_e(f)]^2 \\
&\geq \frac{1}{\delta} \sum_{j \in \mathcal{N}} \mathop{\mathbb{E}}_{f \sim \sigma} [\mathrm{Cost}_j(f)] - \frac{1}{4 \cdot \delta^2} \cdot \sum_{e \in \mathcal{E}} \mathop{\mathbb{E}}_{f \sim \sigma} [\mathrm{Cost}_j(f)] \\
&= \frac{4\delta - 1}{4\delta^2} \sum_{e \in \mathcal{E}} \mathop{\mathbb{E}}_{f \sim \sigma} [\mathrm{Cost}_j(f)]
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&\geq \frac{1}{\delta} \sum_{j \in \mathcal{N}} \mathop{\mathbb{E}}_{f \sim \sigma}[\text{Cost}_j(f)] - \frac{1}{4 \cdot \delta^2} \cdot \sum_{e \in \mathcal{E}} \mathop{\mathbb{E}}_{f \sim \sigma}[\text{Cost}_j(f)] \\
&= \frac{4\delta - 1}{4\delta^2} \sum_{e \in \mathcal{E}} \mathop{\mathbb{E}}_{f \sim \sigma}[\text{Cost}_j(f)]
\end{aligned}$$

Primal is $\left(1 + \frac{1}{\delta}\right)$-approximate solution to the optimal solution. So we get a bound of $\left(1 + \frac{1}{\delta}\right) \dfrac{4\delta^2}{4\delta - 1}$ bound on PoA. Take $\delta = \frac{1+\sqrt{5}}{4}$ you will get a bound of $1 + \Phi$ where Φ is the golden ratio.

Questions?

Simultaneous Second-Price Auctions

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- Let $W_j(b)$ denote the set of items won by player $j \in \mathcal{N}$ when the bids are b.
- Let $p(i, b)$ is the second highest bid for item i when the bids are b.
- Let $u_j(b)$ be the utility of player j when the bids are b. Then $u_j(b) = v_j(W_j(b)) - \sum\limits_{i \in W_j(b)} p(i, b)$.

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GOAL: Maximize the social welfare of the players $V(b) = \sum\limits_{j \in \mathcal{N}} v_j(W_j(b))$

Property of Biddings

Theorem

$\forall\, j \in \mathcal{N}, \forall\, T \subseteq \mathcal{M}, \forall\, b \in \mathbb{R}_{\geq 0}^{m \times n}, \exists\, b_j(T) \in \mathbb{R}_{\geq 0}^{m}$ *such that*

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\}$$

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Let $T = \{1, \ldots, i\}$. Take $b^*_{ij} = v_j(1, 2, \ldots, i) - v_j(1, 2, \ldots, i-1)$. Take $b_j(T) = b^*_j$

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Observe: $\sum_{i \in T'} b^*_{i,j} \leq v_j(T')$ for all $T' \subseteq T$ by submodularity and for $T = T'$ its equality.

Proof of Theorem

$$\begin{aligned}
u_j(b_j(T), b_{-j}) &= v_j(T^*) - \sum_{i \in T*} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \\
&\geq v_j(T^*) - \sum_{i \in T*} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} + \left[\sum_{i \in T \setminus T*} b^*_{i,j} - \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \right] \\
&\geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\}
\end{aligned}$$

LP Formulation

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This constraint makes sure no item is over-allocated
i.e. each item is sold to only one player.

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& x_{j,T} \geq 0 \quad \forall j \in \mathcal{N},\ T \subseteq \mathcal{M}
\end{aligned}
$$

This constraint makes sure each agent receives exactly one set from $2^{\mathcal{M}}$.

Dual Program

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in \mathcal{N}} y_j + \sum_{i \in \mathcal{M}} z_i \\
\text{subject to} \quad & y_j + \sum_{i \in T} z_i \geq v_j(T) && \forall\, j \in \mathcal{N},\ T \subseteq \mathcal{M}, \\
& z_i \geq 0 && \forall\, i \in \mathcal{M}, \\
& y_j \geq 0 && \forall\, j \in \mathcal{N}
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Since σ is an CCE

$$\mathop{\mathbb{E}}_{b\sim\sigma}[u_j(b)] \geq \mathop{\mathbb{E}}_{b\sim\sigma}\left[u_j(b_j(T), b_{-j})\right] \qquad \forall\, T \subseteq \mathcal{M}$$

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- $z_i = \mathop{\mathbb{E}}_{b\sim\sigma}\left[\max_{j\in\mathcal{N}} b_{ij}\right]$ for all $i \in \mathcal{M}$.

Since σ is an CCE

$$\mathop{\mathbb{E}}_{b\sim\sigma}[u_j(b)] \geq \mathop{\mathbb{E}}_{b\sim\sigma}\left[u_j(b_j(T), b_{-j})\right] \qquad \forall\, T \subseteq \mathcal{M}$$

By the theorem

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i\in T} \max_{j'\in\mathcal{N}\setminus\{j\}} \{b_{ij'}\} \geq v_j(T) - \sum_{i\in T} \max_{j'\in\mathcal{N}} \{b_{ij'}\}$$

Setting the Dual Variables

Given a CCE σ of the game, we set the dual variables as follows:

- $y_j = \mathop{\mathbb{E}}_{b\sim\sigma}[u_j(b)]$ for all $j \in \mathcal{N}$.
- $z_i = \mathop{\mathbb{E}}_{b\sim\sigma}\left[\max_{j\in\mathcal{N}} b_{ij}\right]$ for all $i \in \mathcal{M}$.

Since σ is an CCE

$$\mathop{\mathbb{E}}_{b\sim\sigma}[u_j(b)] \geq \mathop{\mathbb{E}}_{b\sim\sigma}\left[u_j(b_j(T), b_{-j})\right] \qquad \forall\, T \subseteq \mathcal{M}$$

By the theorem

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i\in T} \max_{j'\in\mathcal{N}\setminus\{j\}} \{b_{ij'}\} \geq v_j(T) - \sum_{i\in T} \max_{j'\in\mathcal{N}} \{b_{ij'}\}$$

So $\mathop{\mathbb{E}}_{b\sim\sigma}[u_j(b)] \geq v_j(T) - \sum_{i\in T} \mathop{\mathbb{E}}_{b\sim\sigma}\left[\max_{j'\in\mathcal{N}} \{b_{ij'}\}\right]$. So it is feasible solution to the dual program.

Bound on PoA

$$
\begin{aligned}
\text{Primal-Sol} &\leq \sum_{j \in \mathcal{N}} \mathop{\mathbb{E}}_{b \sim \sigma} [u_j(b)] + \sum_{i \in \mathcal{M}} \mathop{\mathbb{E}}_{b \sim \sigma} \left[\max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\
&= \mathop{\mathbb{E}}_{b \sim \sigma} \left[\sum_{j \in \mathcal{N}} u_j(b) \right] + \mathop{\mathbb{E}}_{b \sim \sigma} \left[\sum_{i \in \mathcal{M}} \max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\
&\leq 2 \cdot \mathop{\mathbb{E}}_{b \sim \sigma} [V(b)]
\end{aligned}
$$

Bound on PoA

$$\begin{aligned}
\text{Primal-Sol} &\leq \sum_{j \in \mathcal{N}} \mathop{\mathbb{E}}_{b \sim \sigma} [u_j(b)] + \sum_{i \in \mathcal{M}} \mathop{\mathbb{E}}_{b \sim \sigma} \left[\max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\
&= \mathop{\mathbb{E}}_{b \sim \sigma} \left[\sum_{j \in \mathcal{N}} u_j(b) \right] + \mathop{\mathbb{E}}_{b \sim \sigma} \left[\sum_{i \in \mathcal{M}} \max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\
&\leq 2 \cdot \mathop{\mathbb{E}}_{b \sim \sigma} [V(b)]
\end{aligned}$$

So we get a bound of 2.

Questions?

Facility Location Games