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Course: Mathematical Foundations for Computer Sciences

Assignment - 2
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Problem 1

Let $m, n > 0$ be given and let S be a subset of $[m] \times [n]$. We say S is downward closed if for all $i \leq i' \in [m]$ and $j \leq j' \in [n]$, we have $(i', j') \in S$ only if $(i, j) \in S$. How many downward closed sets are there?

Solution:

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Problem 2

Call an operator $\theta \in L(V)$ unitary if for all $v \in V$, we have $\|\theta(v)\| = \|v\|$ and positive if it is self-adjoint and for all $v \in V$, we have $\langle \theta(v), v \rangle \geq 0$.

- **Polar Decomposition.** Show that for all $\theta \in L(V)$, there exists a unitary $\mu \in L(V)$ and positive $\pi \in L(V)$ such that $\theta = \mu \circ \pi$.

  Hint: Start by showing $\theta^\dagger \circ \theta$ is positive and use the Spectral Theorems.

- **Singular Value Decomposition.** Let $n = \dim V$. Show that, for all $\theta \in L(V)$, there exists two orthonormal basis $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ of V and "singular values" $s_1, \dots, s_n$ such that, for all $v \in V$, we have:

$$\theta(v) = \sum_{i=1}^{n} \langle v, b_i \rangle \cdot s_i \cdot a_i$$

Solution:

- Let $\dim V = n$. We assume that θ is a nonzero operator. Since otherwise we can take μ to be the identity operator and π to be the zero operator. Consider the operator $\theta^\dagger \circ \theta \in L(V)$. Now for any $v \in V$,

$$\langle \theta^\dagger \circ \theta(v) \rangle = \left\langle \theta(v), \left(\theta^\dagger\right)^\dagger (v) \right\rangle = \langle \theta(v), \theta(v) \rangle = \langle v, \theta^\dagger \circ (\theta(v)) \rangle = \langle v, \theta^\dagger \circ \theta(v) \rangle$$

  Hence $\theta^\dagger \circ \theta$ is self-adjoint. Now for any $v \in V$ we also have

$$\langle \theta^\dagger \circ \theta(v), v \rangle = \langle \theta(v), \theta(v) \rangle \geq 0$$

  Hence $\theta^\dagger \circ \theta$ is also positive. Therefore by spectral theorem there exists an orthonormal eigen basis $B = \{b_1, \dots, b_n\}$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ such that $\theta^\dagger \circ \theta(b_i) = \lambda_i b_i$. Since $\theta^\dagger \circ \theta$ is positive all the eigenvalues are non-negative and since θ is nonzero operator not all eigenvalues are zero.

  Now take the set of vectors $B' = \left\{ \frac{1}{\sqrt{\lambda_i}} \theta(b_i) \colon \lambda_i \neq 0 \right\}$. This set is orthonormal since for $i, j \in [n]$ and $i \neq j$ and $\lambda_i, \lambda_j \neq 0$ we have

$$\left\langle \frac{1}{\sqrt{\lambda_i}} \theta(b_i), \frac{1}{\sqrt{\lambda_j}} \theta(b_j) \right\rangle = \frac{1}{\sqrt{\lambda_i \lambda_j}} \langle \theta(b_i), \theta(b_j) \rangle = \frac{1}{\sqrt{\lambda_i \lambda_j}} \langle \theta^\dagger \circ \theta(b_i), b_j \rangle = 0$$

  and for $i = j$ we have

$$\left\langle \frac{1}{\sqrt{\lambda_i}} \theta(b_i), \frac{1}{\sqrt{\lambda_i}} \theta(b_i) \right\rangle = \frac{1}{\sqrt{\lambda_i \lambda_i}} \langle \theta(b_i), \theta(b_i) \rangle = \frac{1}{\lambda_i} \langle \theta^\dagger \circ \theta(b_i), b_i \rangle = \frac{1}{\lambda_i} \lambda_i \langle b_i, b_i \rangle = 1$$

Now B' can be extended to a orthonormal basis $B'' = \{b''_i : i \in [n]\}$ of V using Gram–Schmidt procedure. For simplicity let first k many vectors of B had nonzero eigenvalues and the vectors $b''_{k+1}, \ldots, b''_n$ are the new orthonormal added to B' by Gram-Schmidt. Hence for $i \in [k]$ $b''_i = \frac{1}{\sqrt{\lambda_i}}\theta(b_i)$. So we define the operator $\mu \in L(V)$ such that for any $i \in [n]$

$$\mu(b_i) = b''_i$$

Now also define another operator $\pi \in L(V)$ where $\pi(b_i) = \sqrt{\lambda_i} b_i$ for all $i \in [n]$. Both μ and π are defined on basis so they are unique.

We claim $\theta = \mu \circ \pi$. If we show that for any $i \in [n]$ $\theta(b_i) = \mu \circ \pi(b_i)$ we are done since B is a basis of V. Now if $\lambda_i \neq 0$ then

$$\mu \circ \pi(b_i) = \mu(\sqrt{\lambda_i} b_i) = \sqrt{\lambda_i}\mu(b_i) = \sqrt{\lambda_i} b''_i = \sqrt{\lambda_i}\frac{1}{\sqrt{\lambda_i}}\theta(b_i) = \theta(b_i)$$

When $\lambda_i = 0$ then we have

$$\mu \circ \pi(b_i) = \mu(\sqrt{\lambda_i} b_i) = \sqrt{\lambda_i}\mu(b_i) = 0 \cdot \mu(b_i) = 0$$

and on the other hand we have

$$\langle \theta(b_i), \theta(b_i) \rangle = \langle \theta^\dagger \circ \theta(b_i) \rangle = \langle \lambda_i b_i, b_i r \rangle = 0$$

Hence we have for all $i \in [n]$, $\theta(b_i) = \mu \circ \pi(b_i)$. Hence $\theta = \mu \circ \pi$.

Now we will show that μ is unitary and π is positive. Now π is diagonalizable with respect to an orthonormal eigen basis with all its eigenvalues are non-negative. Hence π is positive. So only thing remains is to show that μ is unitary. Let for any $v \in V$, $v = \sum_{i=1}^{n} a_i b_i$ where $a_i \in \mathbb{C}$. Then we have

$$\left\langle \sum_{i=1}^{n} a_i b_i, \sum_{i=1}^{n} a_i b_i \right\rangle = \sum_{i=1}^{n} |a_i|^2 \langle b_i, b_i \rangle = \sum_{i=1}^{n} |a_i|^2$$

On the other hand we have

$$\left\langle \mu\left(\sum_{i=1}^{n} a_i b_i\right), \mu\left(\sum_{i=1}^{n} a_i b_i\right) \right\rangle = \left\langle \sum_{i=1}^{n} a_i \mu(b_i), \sum_{i=1}^{n} a_i \mu(b_i) \right\rangle = \sum_{i=1}^{n} |a_i|^2 \langle b_i, b_i \rangle = \sum_{i=1}^{n} |a_i|^2$$

Hence μ is unitary. Therefore there exists an unitary operator $\mu \in L(V)$ and a positive operator $\pi \in L(V)$ such that $\theta = \mu \circ \pi$.

- By the above proof of polar decomposition there exists an unitary operator $\mu \in L(V)$ and positive operator $\pi \in L(V)$ such that $\theta = \mu \circ \pi$. We also get an orthonormal eigenbasis $B = \{b_1, \ldots, b_n\}$ of $\theta^\dagger \circ \theta$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ and another orthonormal basis $B'' = \{b''_1, \ldots, b''_n\}$ where $\mu(b_i) = b''_i$ and $\pi(b_i) = \sqrt{\lambda_i} b_i$. Let $v \in V$. Then $v = \sum_{i=1}^{n} \langle v, b_i \rangle b_i$. Then

$$\theta(v) = \mu \circ \pi\left(\sum_{i=1}^{n} \langle v, b_i \rangle b_i\right) = \sum_{i=1}^{n} \langle v, b_i \rangle \mu(\sqrt{\lambda_i} b_i) = \sum_{i=1}^{n} \langle v, b_i \rangle \cdot \sqrt{\lambda_i} \cdot \mu(b_i) = \sum_{i=1}^{n} \langle v, b_i \rangle \cdot \sqrt{\lambda_i} \cdot b''_i$$

Hence here $A = B''$ and the singular values are eigenvalues of vectors in B.

$\blacksquare$

Problem 3

The following pattern is well known:

$$A = \begin{bmatrix} 1 &&&&&&& \\ 1 & 1 &&&&&& \\ 1 & 2 & 1 &&&&& \\ 1 & 3 & 3 & 1 &&&& \\ 1 & 4 & 6 & 4 & 1 &&& \\ 1 & 5 & 10 & 10 & 5 & 1 && \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For all $n > 0$, consider the n-th sub-triangle B_n of A defined as follows:

$$B_1 = \begin{bmatrix} 1 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 & 1 \\ & 2 \end{bmatrix} \qquad B_3 = \begin{bmatrix} 1 & 2 & 1 \\ & 3 & 3 \\ && 6 \end{bmatrix} \qquad B_4 = \begin{bmatrix} 1 & 3 & 3 & 1 \\ & 4 & 6 & 4 \\ && 10 & 10 \\ &&& 20 \end{bmatrix} \qquad \cdots$$

The triangles B_n have the property that for all $i \leq j < n$, it holds that $(B_n)_{i,j} = (B_n)_{i+1,j+1} - (B_n)_{i,j+1}$. For all $n > 0$, find the largest number of ones in a matrix of size n that has entries in $\{0, 1\}$ and satisfied the foregoing property modulo 2.

Solution: For all $n > 0$ we have

$$(B_n)_{i,j} = (B_n)_{i+1,j+1} - (B_n)_{i,j+1} \iff (B_n)_{i,j} + (B_n)_{i,j+1} = (B_n)_{i+1,j+1}$$

First we will prove an upper bound on the number of 1's in the triangle. [I got this bound statement from a reddit post[1]]

Lemma 1. *The number of* 1*'s in the resulting matrix of size $n > 0$ for any $n \in \mathbb{N}$ is at most* $\frac{n^2+n+1}{3}$

Proof: We will prove this inductively. For base case $n = 1$ we have the number of 1's is 1. and $\frac{1^2+1+1}{3} = 1$. Hence the base case follows.

Now suppose this is true for $n = 1, \ldots, k$. For $n = k + 1$ we will consider two cases: the first row as either $\leq \frac{2k+2}{3}$ many 1's or $> \frac{2k+2}{3}$ many 1's.

Suppose the first row has $\leq \frac{2k+2}{3}$ many 1's. Then from next row on wards there are k rows and these k rows can have at most $\frac{k^2+k+1}{3}$ many $1'$s by Induction Hypothesis. Therefore

$$\#1'\text{s} = \frac{k^2+k+1}{3} + \frac{2k+2}{3} = \frac{k^2+3k+3}{3} = \frac{(k^2+2k+1)+(k+1)+1}{3} = \frac{(k+1)^2+(k+1)+1}{3}$$

Therefore the statement is followed.

Suppose the first row has $> \frac{2k+2}{3}$ i.e. $\geq \frac{2k+3}{3}$ many 1's. Now in the second row each 1 is originated from a 0 and a 1 in the first row. Each 0 in the first row gives at most two 1's in the second row. Therefore

$$\#1'\text{s in second row} \leq 2 \times \#0'\text{s in first row}$$

Hence

$$\begin{aligned} \#1'\text{s in first two rows} &= \#1'\text{s in first row} + \#1'\text{s in second row} \\ &\leq \#1'\text{s in first row} + 2 \times \#0'\text{s in first row} \\ &= 2(k+1) - \#1'\text{s in first row} \leq 2(k+1) - \frac{2k+3}{3} = \frac{4k+3}{3} \end{aligned}$$

[1] https://www.reddit.com/r/mathriddles/comments/ojpqgg/binary_pascal_triangle/

Now from third row on wards there are $k-1$ rows and by inductive hypothesis there can be at most $\frac{(k-1)^2+(k-1)+1}{3} = \frac{k^2-k+1}{3}$ many $1'$s. Now if $3 \mid k$ then from third row on wards there are at most $\frac{k^2-k}{3}$ many 1's are there. Therefore

$$\begin{aligned}
\#1'\text{s} &= \#1'\text{s from third row on wards} + \#1'\text{s in first two row} \\
&\leq \frac{k^2-k}{3} + \frac{4k+3}{3} = \frac{k^2-k+4k+3}{3} \\
&= \frac{(k^2+2k+1)+(k+1)+1}{3} = \frac{(k+1)^2+(k+1)+1}{3}
\end{aligned}$$

If $3 \nmid k$ then from third row on wards we keep the bound on the number of 1's to be $\frac{(k-1)^2+(k-1)+1}{3} = \frac{k^2-k+1}{3}$. But now $\frac{4k+3}{3}$ is not an integer. So the number of $1'$ in the first two rows is at most $\frac{4k+2}{3}$. Hence we have

$$\begin{aligned}
\#1'\text{s} &= \#1'\text{s from third row on wards} + \#1'\text{s in first two row} \\
&\leq \frac{k^2-k+1}{3} + \frac{4k+2}{3} = \frac{k^2-k+1+4k+2}{3} \\
&= \frac{(k^2+2k+1)+(k+1)+1}{3} = \frac{(k+1)^2+(k+1)+1}{3}
\end{aligned}$$

Hence for both cases we have the total number of 1's is at most $\frac{(k+1)^2+(k+1)+1}{3}$. Hence by Mathematical Induction the number of 1's in the resulting matrix of size $n > 0$ for any $n \in \mathbb{N}$ is at most $\frac{n^2+n+1}{3}$. ■

Having this bound on the number of 1's we will now show an instance to achieve this number for any $n > 0$. So we will show instances where for any $n > 0$ from any i^{th} row on wards the bound $\left\lfloor \frac{i^2+i+1}{3} \right\rfloor$ is achieved for all $i \in [n]$.

Consider the sequence $\{0, 1, 1\}$. We put them in that order circularly. i.e.

Starting with 0:	0	1	1	0	1	1	0	1	1 $\cdots$
Starting with first 1:	1	1	0	1	1	0	1	1	0 $\cdots$
Starting with second 1:	1	0	1	1	0	1	1	0	1 $\cdots$

Let S_n^0 denote the n-length sequence starting with 0, S_n^1 denote n-length sequence starting with 1 and S_n^2 denote n-length sequence starting with 1. Now for any $j \in \mathbb{F}_3$ and $i \in [n]$, $S_n^j(i)$ denote the i^{th} element in S_n^j. And in general for any $i > 0$, $i \in \mathbb{N}$ the i^{th} element of the sequence starting with 0 is by $S^0(i)$, for i^{th} element of the sequence starting with first 1 denoted by $S^1(i)$ and for i^{th} element of the sequence starting with second 1 denoted by $S^2(i)$. Now we have the following relation

Lemma 2. *Then for any $j \in \mathbb{F}_3$ and for any $i > 0$ and $i \in \mathbb{N}$*

$$S^j(i) + S^j(i+1) \equiv S^{j+2}(i) \pmod 2$$

Since $j \in \mathbb{F}_3$ we take $j+2 \bmod 2$.

Proof: For $j = 0$ we have

$$S^0(i) = \begin{cases} 0 & \text{If } i \equiv 1 \pmod 3 \\ 1 & \text{Otherwise} \end{cases}, \quad S^1(i) = \begin{cases} 0 & \text{If } i \equiv 0 \pmod 3 \\ 1 & \text{Otherwise} \end{cases}, \quad S^2(i) = \begin{cases} 0 & \text{If } i \equiv 2 \pmod 3 \\ 1 & \text{Otherwise} \end{cases}$$

Now we will analyze case wise:

- **Case 1:** $i \equiv 0 \pmod 3$: Then $S^0(i) = 1$, $S^1(i) = 0$ and $S^2(i) = 1$ Therefore $S^0(i+1) = 0$, $S^1(i+1) = 1$, and $S^2(i+1) = 1$. Therefore we have $S^0(i) + S^1(i+1) = 1 + 0 = 1 = S^2(i)$, $S^1(i) + S^1(i+1) = 0 + 1 = 1 = S^0(i)$ and $S^2(i) + S^2(i+1) = 1 + 1 \equiv 0 = S^1(i) \bmod 2$.

- **Case 2:** $i \equiv 1 \pmod 3$: Then $S^0(i) = 0$, $S^1(i) = 1$ and $S^2(i) = 1$ Therefore $S^0(i+1) = 1$, $S^1(i+1) = 1$, and $S^2(i+1) = 0$. Therefore we have $S^0(i) + S^1(i+1) = 0 + 1 = 1 = S^2(i)$, $S^1(i) + S^1(i+1) = 1 + 1 = \equiv 0 = S^0(i) \pmod 2$ and $S^2(i) + S^2(i+1) = 1 + 0 = 1 = S^1(i)$.

- **Case 3:** $i \equiv 2 \pmod 3$: Then $S^0(i) = 1$, $S^1(i) = 1$ and $S^2(i) = 0$ Therefore $S^0(i+1) = 1$, $S^1(i+1) = 0$, and $S^2(i+1) = 1$. Therefore we have $S^0(i) + S^1(i+1) = 1 + 1 \equiv 0 = S^2(i) \pmod 2$, $S^1(i) + S^1(i+1) = 1 + 0 = 1 = S^0(i)$ and $S^2(i) + S^2(i+1) = 0 + 1 = 0 = S^1(i)$.

Hence we have for all $i > 0$ and $i \in \mathbb{N}$ and for all $j \in \mathbb{F}_3$ we have $S^j(i) + S^j(i+1) \equiv S^{j+2}(i) \pmod 2$. ∎

Now we will count the number of 1's in S_n^j for any $j \in \mathbb{F}_3$. First we define the following function $f\mathbb{F}_3^2 \to \mathbb{F}_3$ where we give the values of at all possible inputs by the table below:

$f(i,j)$	$j=0$	$j=1$	$j=1$
$i=0$	0	0	0
$i=1$	0	1	1
$i=2$	1	2	1

Lemma 3. *Let $n = 3k + i$ where $i \in \{0,1,2\}$ and $k \in \mathbb{N}$. Then number of $1's$ in S_n^j for any $j \in \mathbb{F}_3$ is $2k + f(i,j)$.*

Proof: For any $i > 0$, $i \in \mathbb{N}$ and for any $j \in \mathbb{F}_3$ in the block $S^j(i), S^j(i+1), S^j(i+2)$ there is exactly two 1's and one 0 since in the sequence $0, 1, 1$ comes circularly again and again and any 3 consecutive element is just one time appearance of the sequence. Therefore for 3-block there are two 1's. Since $n = 3k + i$, $S^j(3k)$ has $2k$ many 1's. Now we will analyze case wise:

- **Case 1** $i = 0$: Then $n = 3k$. Hence we already know we have $2k$ many 1's. And since $f(0,j) = 0$ for all $j \in \mathbb{F}_3$ we have $2k + f(i,j)$ many 1's.

- **Case 2** $i = 1$: We have $S^0(n) = 0$ and $S^1(n) = S^2(n) = 1$. Hence for S^0 we see no extra 1 at n^{th} position. Hence number of 1's in S_n^0 is $2k+1 = 2k + f(1,0)$. For S^1 we see an extra 1 at n^{th} position. We also have $f(1,j) = 1$ for $j = 1, 2$. Therefore number of 1's in S_n^1 or S_n^2 is $2k+1 = 2k + f(1,j)$ for $j = 1, 2$. Therefore for $i = 1$ number of 1's in S_n^j is $2k + f(1,j)$ for $j \in \mathbb{F}_3$.

- **Case 3** $i = 2$: We have $S^2(n) = 1$ and $S^0(n) = S^1(n) = 1$. And by case 2 analysis we have $2k$ many 1's in S_{n-1}^0 and $2k+1$ many 1's in both S_{n-1}^1 and S_{n-1}^2. For S^0 there is 1 at n^{th} position. Therefore we see an extra 1. Hence there are total $2k+1$ many 1's. We also have $f(2,0) = 1$. Hence there are $2k + f(2,0)$ many 1's in S_n^0. For S_n^1 there is 1 at n^{th} position. Therefore we see an extra 1. Therefore there are total $(2k+1) + 1 = 2k + 2$ many 1's in S_n^1. We also have $f(2,1) = 2$. Therefore we have $2k + f(2,1)$ many 1's in S_n^1. Now for S_n^2 there is 0 at n^{th} position. Therefore we have no extra 1. So the number of 1's in S_n^2 is same as S_{n-1}^2 which is $2k+1$. We have $f(2,2) = 1$. So we have $2k + f(2,2)$ many 1's in S_n^2. Therefore we have for $i = 2$ number of 1's in S_n^j is $2k + f(2,j)$ for $j \in \mathbb{F}_3$.

Hence by analyzing all possible cases we get that for $n = 3k + i$ where $k \in \mathbb{N}$ and $i \in \{0,1,2\}$ then number of 1's in S_n^j is $2k + f(i,j)$. ∎

With all these setup for any $n > 0$ and $n \in \mathbb{N}$ we define the $0-1$ matrix M_n to be the following

$$M_n = \begin{bmatrix} S_n^l & & & & \\ & S_{n-1}^{l-1} & & & \\ & & S_{n-2}^{l-2} & & \\ & & & \ddots & \\ & & & & S_1^1 \end{bmatrix}$$

Where $l = n \bmod 3$ and we do the subtraction by 1 in modulo 3. So basically in M_n the k^{th} row has $k-1$ leading 0's then $S_{n-k+1}^{n-k+1 \bmod 3}$ for all $k \in [n]$. Also observe that if we remove the first row and first column from M_n we get M_{n-1}. Now by Lemma 2 M_n follows the rule that $(M_n)_{i,j} + (M_n)_{i,j+1} = (M_n)_{i+1,j+1}$ Now we will show that the total number of 1's in M_n is actually $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor$.

Lemma 4. *The total number of* 1*'s in* M_n *is* $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor$.

Proof: We will prove this inductively on n. For $n = 1$ we have $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor = 1$ which is true since $M_1 = [S_1^1] = [1]$. Hence the base case follows. Let this is true for $n = 1, \ldots, l-1$. Now $n = l$ we will analyze case wise. Now we have

$$\left\lfloor \frac{l^2+l+1}{3} \right\rfloor = \begin{cases} 3k^2 + k & \text{When } l = 3k \\ 3(k^2+k)+1 & \text{When } l = 3k+1 \\ 3k^2+5k+2 & \text{When } l = 3k+2 \end{cases}$$

Now if we ignore the first row and first column we have M_{l-1}. By inductive hypothesis M_{l-1} has $\left\lfloor \frac{(l-1)^2+(l-1)+1}{3} \right\rfloor$ many 1's

- **Case 1 $\boldsymbol{l = 3k}$**: Then $l - 1 = 3(k-1)+2$. Then we have

  $$\left\lfloor \frac{(l-1)^2+(l-1)+1}{3} \right\rfloor = 3(k-1)^2 + 5(k-1) + 2 = 3(k^2-2k+1) + 5k - 5 + 2 = 3k^2 - k$$

  And by Lemma 3 in S_l^0 there are $2k + f(0,0) = 2k$. Hence total number of 1's is

  $$3k^2 - k + 2k = 3k^2 + k = \left\lfloor \frac{l^2+l+1}{3} \right\rfloor$$

  Hence this case follows.

- **Case 2 $\boldsymbol{l = 3k+1}$**: Then $l - 1 = 3k$. Then we have

  $$\left\lfloor \frac{(l-1)^2+(l-1)+1}{3} \right\rfloor = 3k^2 + k$$

  And by Lemma 3 in S_l^1 there are $2k + f(1,1) = 2k+1$ many 1's. Hence total number of 1's is

  $$3k^2 + k + 2k + 1 = 3k^2 + 3k + 1 = \left\lfloor \frac{l^2+l+1}{3} \right\rfloor$$

  Hence this case follows.

- **Case 3 $\boldsymbol{l = 3k+2}$**: Then $l - 1 = 3k+1$. Then we have

  $$\left\lfloor \frac{(l-1)^2+(l-1)+1}{3} \right\rfloor = 3k^2 + 3k + 1$$

  And by Lemma 3 in S_l^2 there are $2k + f(2,2) = 2k+1$ many 1's. Hence total number of 1's is

  $$3k^2 + 3k + 1 + 2k + 1 = 3k^2 + 5k + 2 = \left\lfloor \frac{l^2+l+1}{3} \right\rfloor$$

  Hence this case follows.

Therefore in all cases M_l has in total $\left\lfloor \frac{l^2+l+1}{3} \right\rfloor$ many 1's. Therefore by mathematical induction we have that for all $n > 0$, $n \in \mathbb{N}$ the total number of 1's in M_n is $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor$. ■

Since by Lemma 1 the maximum number of 1's we can achieve is $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor$ for n size matrix this sequence of matrices has the maximum number of 1's. ■

Problem 4

Let $n > 0$ be an integer. Count the number of subsets $S \subseteq [n]$ that:

(a) satisfy $|S| \in S$.

(b) satisfy $|S| \in S$ and that for all $S' \subsetneq S$, we have $|S'| \notin S$.

Solution:

(a) Let $|S| = k$. Therefore $k \in S$. Now rest of the $k-1$ elements are from $[n] \setminus \{k\}$. So the rest $k-1$ elements can be chosen from $[n] \setminus \{k\}$ in $\binom{n-1}{k-1}$ ways. Therefore total number of sets $S \subseteq [n]$ that satisfy $|S| \in S$ is

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$$

Hence there are 2^{n-1} such sets are possible

(b) Let $|S| = k$. Now for all $S' \subsetneq S$, we have $|S'| \notin S$. Hence for all $m < k$, $m \notin S$. Therefore the rest of the $k-1$ elements of S are from $[n] \setminus [k]$. For this to satisfy we should have $n-k \geq k-1 \implies \frac{n+1}{2} \geq k$. For such k the rest $k-1$ elements can be chosen from $[n] \setminus [k]$ in $\binom{n-k}{k-1}$ ways. Hence total number of sets $S \subseteq [n]$ that satisfy the given property is

$$\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k}{k-1}$$

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Problem 5

A triangulation of a polygon is a partition of its area into (disjoint) triangles with the same vertex set.

- Consider a regular polygon with n sides. Show that any triangulation of this polygon has $n-2$ triangles. How many such triangulations are there?
- For what values of n is there a triangulation into isosceles triangles? How many such triangulations are there?

Use the ideas above to show that a d-dimensional polytope that is the intersection of $n-$halfspaces can be partitioned into at most n^d simplices.

Solution: ■