CSS.414.1: Polynomial Methods in Combinatorics

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TIFR 2024, Aug-Dec

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2024

Contents

1 Introduction and Targets

The content of this course will be the followings:

- Polynomial Methods in Combinatorics/Geometry

  1. Kakeya/Nikodym Problem over finite fields
  2. Joints Problem
  3. Combinatorial Nullstellensatz (CN)
  4. CN proof of Cauchy-Devenport, Erdös-Heilbronn Conjecture

- Polynomial Methods in Algebraic Algorithms

  1. Noisy Polynomial Interpolation (Sudan, Guruswami-Sudan)
  2. Multiplicative noise (Von zur Gathen-Shparlinski)
  3. Coppersmith's Problem (Given an univariate $f(x)\mathbb{Z}[x]$, compute all 'small' integer roots modulo a composite)

- Polynomial Methods in Circuit Complexity

  1. Razborov-Smolensky (Lower Bound for constant depth AND, OR, NOT, $\bmod p$ gates)
  2. Algorithmic consequences (all pairs shortest paths)
  3. Upper bounds on matrix rigidity (Alman-Williams '2015, Dvir-Edelman '2017)

- Polynomial in Property Testing: Polischuk-Speilman Lemma/Variants

- Weil Bounds (Stepanov, Schmidtm Bombieri)

- Rational Approximations of Algebraic Numbers (Thue[1907] - Siegel - Roth[1954])

2 Joints Problem

3 Combinatorial Nullstellensatz

3.1 Chevally-Warning Theorem

4 Sum Sets

4.1 Sum Sets over Finite Fields

4.1.1 Cauchy-Davenport Theorem

4.2 Restricted Sum Sets

4.2.1 Erdös-Heilbronn Conjecture

5 Arithmetic Progression Free Sets in $\mathbb{F}_3^n$

5.1 3AP Free sets in $\mathbb{F}_q$

6 3-Tensors and Slice Rank

6.1 Rank

6.2 Generalization to 3-Dimension

6.3 Slice Rank of Diagonal 3D Tensor

7 Kakeya and Nikodym Problem

Definition 7.1: Kakeya Sets

In a finite field $\mathbb{F}_q$, $K \subseteq \mathbb{F}^n$ is a Kakeya Set if $\forall\, a \in \mathbb{F}^n$, $\exists\, b \in \mathbb{F}^n$ such that

$$L_{a,b} = \{b + at : t \in \mathbb{F}_q\} \subseteq K$$

i.e. informally it has a line in every direction

Now notice that we can take the whole $\mathbb{F}_q^n$ as the Kakeya Set. We can also remove a point from $\mathbb{F}_q^n$ and it will still be a Kakeya Set. Having defined the Kakeya sets the biggest question which is studied is:

Question 1

How small can a Kakeya Set be?

7.1 Lower Bound on Nikodym Sets

7.2 Lower Bound on Kakeya Sets

7.2.1 Hasse Derivative

8 Razborov Smolensky Lower Bound