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Course: Expander Graphs and Applications

Assignment - 1
Roll: BMC202175
Date: March 5, 2024

Problem 1 Problem 4.9 (The Replacement Product): Pseudorandomness By Salil Vadhan

Given a D_1-regular graph G_1 on N_1 vertices and a D_2-regular graph G_2 on D_1 vertices consider the following graph $G_1 \circledR G_2$ on vertex set $[N_1] \times [D_1]$: vertex (u, i) is connected to (v, j) iff

(a) $u = v$ and (i, j) is an edge in G_2 or,

(b) v is the i'th neighbour of u in G_1 and u is the jth neighbor of v.

That is, we "replace" each vertex v in G_1 with a copy of G_2, associating edge incident to v with one vertex of G_2.

1. Prove that there is a function g such that if G_1 has spectral expansion $\gamma_1 > 0$ and G_2 has spectral expansion $\gamma_2 > 0$ (and both graphs are undirected) then $G_1 \circledR G_2$ has spectral expansion $g(\gamma_1, \gamma_2, D_2) > 0$.

   [Hint: Note that $(G_1 \circledR G_2)^3$ has $G_1 ⓩ G_2$ as a subgraph]

2. Show how to convert an explicit construction of constant degree (spectral) expanders into an explicit construction of degree 3 (spectral) expanders.

3. Without using Theorem 4.14, prove an analogue of Part 1 for edge expansion. That is, there is a function h such that if G_1 is an $\left(\frac{N_1}{2}, \epsilon_1\right)$ edge expander and G_2 is a $\left(\frac{D_1}{2}, \epsilon_2\right)$ edge expander then $G_1 \circledR G_2$ is a $\left(\frac{N_1 D_1}{2}, h(\epsilon_1, \epsilon_2, D_2)\right)$ edge expander where $h(\epsilon_1, \epsilon_2, D_2) > 0$ if $\epsilon_1, \epsilon_2 > 0$.

   [Hint: Given any set S of vertices of $G_1 \circledR G_2$, partition S into the clouds that are more than "half-full" and those that are not]

4. Prove that the functions $g(\gamma_1, \gamma_2, D_2)$ and $h(\epsilon_1, \epsilon_2, D_2)$ must depend on D_2 by showing that $G_1 \circledR G_2$ cannot be a $\left(\frac{N_1 D_1}{2}, \epsilon\right)$ edge expander if $\epsilon > \frac{1}{D_1 + 1}$ and $N_1 \geq 2$

Solution:

1. Let A_1 and A_2 denote the normalized adjacency matrices of G_1 and G_2 respectively. The degree of the new graph $G_1 \circledR G_2$ is $D_2 + 1$. Now denote $B \triangleq I_{N_1} \otimes A_2$ and A be a $N_1 \cdot D_1 \times N_1 \cdot D_1$ matrix where

$$A[(u,i),(v,j)] = \begin{cases} 1 & \text{when } i\text{th neighbor of } u \text{ is } v \text{ and } j\text{th neighbor of } v \text{ is } u \text{ in } G_1 \\ 0 & \text{otherwise} \end{cases}$$

   Therefore the adjacency matrix of the graph $G_1 \circledR G_1$ is $A + D_2 B$. Therefore the normalized adjacency matrix, M

$$M \triangleq \frac{A + D_2 B}{D_2 + 1}$$

   Now notice the graph $(G_1 \circledR G_2)^3$ contains the graph $G_1 ⓩ G_2$ as a subgraph. Hence

$$M^3 = \left[\frac{A + D_2 B}{D_2 + 1}\right]^3 = \frac{D_2^2}{(D_2+1)^3} BAB + \left[1 - \frac{D_2^2}{(D_2+1)^3}\right] C$$

   for some matrix C. Lets denote $p := \frac{D_2^2}{(D_2+1)^3}$. Then $M^3 = pBAB + (1-p)C$. Hence for any $v \perp u$ where u is the uniform vector we have

$$\|M^3 v\| \leq p\|BABv\| + (1-p)\|Cv\|$$

Now we can think as C is a normalized adjacency matrix of an undirected graph. Hence for all $v \perp u$ we have $\|Cv\| \le \|v\|$. Now we know for all $v \perp u$

$$\|BABv\| \le (\lambda_1 + \lambda_2 + \lambda_2^2)\|v\|$$

where $\lambda_1 = 1 - \gamma_1$ and $\lambda_2 = 1 - \gamma_2$. Hence

$$\|M^3 v\| \le p(\lambda_1 + \lambda_2 + \lambda_2^2)\|v\| + (1-p)\|v\| = [p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)]\|v\|$$

Now

$$\begin{aligned} 1 - [p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)] &= 1 - (1-p) - p(\lambda_1 + \lambda_2 + \lambda_2^2) \\ &= p - p(\lambda_1 + \lambda_2 + \lambda_2^2) \\ &= p[1 - (\lambda_1 + \lambda_2 + \lambda_2^2)] \end{aligned}$$

Now we know

$$\lambda_1 + \lambda_2 + \lambda_2^2 < 1 \iff 0 < 1 - (\lambda_1 + \lambda_2 + \lambda_2^2) < 1 \qquad \text{and} \qquad 0 < p < 1$$

Then $0 < p[1 - (\lambda_1 + \lambda_2 + \lambda_2^2)] < 1$. Hence

$$0 < p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p) < 1$$

Therefore for all $v \perp u$,

$$\|Mv\| \le \left[p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)\right]^{\frac{1}{3}} \|v\|$$

Now

$$\begin{aligned} \left[p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)\right]^{\frac{1}{3}} &= \left[1 - p[1 - (\lambda_1 + \lambda_2 + \lambda_2^2)]\right]^{\frac{1}{3}} \\ &\le 1 - \frac{1}{3} p[1 - (\lambda_1 + \lambda_2 + \lambda_2^2)] < 1 \end{aligned}$$

So

$$g(\gamma_1, \gamma_2, D_2) = 1 - \left[p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)\right]^{\frac{1}{3}} > 0$$

2.

□

Problem 2 Problem 4.10 (Unbalanced Vertex Expanders and Data Structures): Pseudorandomness By Salil Vadhan

Consider a $(K, (1-\epsilon)D)$ bipartite vertex expander G with N left vertices, M right vertices and left degree D.

1. For a set S of left vertices, a $y \in N(S)$ is called a *unique* neighbor of S if y is incident to exactly one edge from S. Prove that every left-set S of size at most K has at least $(1-2\epsilon)D|S|$ unique neighbors.
2. For a set S of size at most $\frac{K}{2}$, prove that at most $\frac{|S|}{2}$ vertices outside S have at least δD neighbors in $N(S)$ for $\delta = O(\epsilon)$.

Solution:

1. Let U be the set of unique neighbors in $N(S)$. Denote $T = \Gamma(S) - U$. Then we have $|U \cup T| \ge (1-\epsilon)D|S|$. Now we will count the number of edges between S and $\Gamma(S)$. From each vertex in S there are D edges going out. Hence total $D|S|$ many edges are going out from S. Now in $\Gamma(S)$ for each vertex in U there is exactly

one edge coming from S and for each edge in T there are at least 2 edges coming from S. Hence there are at least $|U| + 2|T|$ many edges are coming towards $\Gamma(S)$. Hence we have:

$$\begin{aligned} |U| + 2|T| \leq D|S| &\iff |U| + 2(|\Gamma(S)| - |U|) \leq D|S| \\ &\iff |U| \geq 2|\Gamma(S)| - D|S| \geq (1-\epsilon)D|S| - D|S| = (1-2\epsilon)D|S| \end{aligned}$$

Hence there are at least $(1-2\epsilon)D|S|$ unique neighbors.

2.

□

Problem 3 Problem 5.5 (LDPC Codes): Pseudorandomness By Salil Vadhan

Given a D_1-regular graph G_1 on N_1 vertices and a D_2-regular graph G_2 on D_1 vertices consider the following graph $G_1 Ⓡ G_2$ on vertex set $[N_1] \times [D_1]$: vertex (u, i) is connected to (v, j) iff

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1. Prove that there is a function g such that if G_1 has spectral expansion $\gamma_1 > 0$ and G_2 has spectral expansion $\gamma_2 > 0$ (and both graphs are undirected) then $G_1 Ⓡ G_2$ has spectral expansion $g(\gamma_1, \gamma_2, D_2) > 0$.

   [Hint: Note that $(G_1 Ⓡ G_2)^3$ has $G_1 Ⓩ G_2$ as a subgraph]

2. Show how to convert an explicit construction of constant degree (spectral) expanders into an explicit construction of degree 3 (spectral) expanders.

3. Without using Theorem 4.14, prove an analogue of Part 1 for edge expansion. That is, there is a function h such that if G_1 is an $\left(\frac{N_1}{2}, \epsilon_1\right)$ edge expander and G_2 is a $\left(\frac{D_1}{2}, \epsilon_2\right)$ edge expander then $G_1 Ⓡ G_2$ is a $\left(\frac{N_1 D_1}{2}, h(\epsilon_1, \epsilon_2, D_2)\right)$ edge expander where $h(\epsilon_1, \epsilon_2, D_2) > 0$ if $\epsilon_1, \epsilon_2 > 0$.

   [Hint: Given any set S of vertices of $G_1 Ⓡ G_2$, partition S into the clouds that are more than "half-full" and those that are not]

4. Prove that the functions $g(\gamma_1, \gamma_2, D_2)$ and $h(\epsilon_1, \epsilon_2, D_2)$ must depend on D_2 by showing that $G_1 Ⓡ G_2$ cannot be a $\left(\frac{N_1 D_1}{2}, \epsilon\right)$ edge expander if $\epsilon > \frac{1}{D_1+1}$ and $N_1 \geq 2$

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