Super-Polynomial Lower Bound of TSP Extended Formula

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May 2025

Introduction

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Definition (Travelling Salesman)

Given a graph $G = (V, E)$, $S \subseteq V$ and weights $w : E \to \mathbb{R}$ find minimum weight cycle which visits every vertex of S exactly once.

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We will focus on $S = V$.

- We know Traveling Salesman Problem is NP-complete.
- In [Yannkakis, 1988, STOC] he proved every symmetric LP for the TSP has expnential size.
- Here we will show TSP admits no polynomial-size LP.
- This proof also shows unconditional super-polynomial lower bound on the number of inequalities.
- Therefore it is impossible to prove P $=$ NP by means of a polynomial size LP.

Definitions

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} = conv(V)$ is a polytope with $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$ and $V \subseteq \mathbb{R}^d$. We will consider V as the characteristic vector for all hamiltonian paths.

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Lemma

Let P, Q and F be polytopes. Then the following holds:

(i) *If F is an extension of P then $xc(F) \geq xc(P)$.*

(ii) *If F is a face of Q then $xc(Q) \geq xc(F)$.*

Some Polytopes

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- Given $G = (V, E)$, for any $S \subseteq V$, χ^S denote characteristic vector of S. Then

$$IND(G) := conv\{\chi^S \mid S \text{ is independent set of } G\}$$

- The correlation polytope $COR(n)$ is

$$COR(n) := conv\{bb^T \mid b \in \{0,1\}^n\}$$

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Step 3: For any n-vertex graph G, $IND(G)$ is linear projection of a face of $TSP(k)$ where $k = O(n^2)$.

Correlation Polytope Lower Bound