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Course: Probability Theory

Assignment - 1
Date: August 28, 2024

Problem 1

(a) Prove that if $A_1, A_2, \ldots, A_n$ are events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = S_1 - S_2 + S_3 - \ldots + (-1)^{n-1} S_n$$

where

$$\begin{aligned} S_1 &= \sum_i \mathbb{P}(A_i) \\ S_2 &= \sum_{i<j} \mathbb{P}\left(A_i \cap A_j\right) \\ S_3 &= \sum_{i<j<k} \mathbb{P}\left(A_i \cap A_j \cap A_k\right) \\ &\ldots \\ S_n &= \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) \end{aligned}$$

This is also known as the *inclusion-exclusion* principle.

(b) *Bonferroni inequalities* state that the sum of the first terms in the right-hand side of the identity we proved above is alternately an upper bound and a lower bound for the left-hand side. i.e., for odd $k \leq n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq S_1 - S_2 + \ldots + S_k$$

and for even $k \leq n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq S_1 - S_2 + \ldots - S_k$$

Note that from what we showed above Bonferroni inequality holds with equality for $k = n$.

Prove Bonferroni inequalities. Observe that the case of $k = 1$ is what you know as the *union bound* or Boole's inequality.

Solution:

(a)

□

Problem 2

Prove or disprove the following:

- The conditional independence of A and B given C implies A and B are independent.
- Independence of A and B implies the conditional independence of A and B given C.

If you disproved either of the claims above, for which events C is it then the case that the following statement holds: for all events A and B, the events A and B are conditionally independent given C if and only if A and B are independent.

Solution:

□

Problem 3

Let $A_1, A_2, \ldots$ be a sequence of events. Define

$$B_n = \bigcup_{m=n}^{\infty} A_m \quad C_n = \bigcap_{m=n}^{\infty} A_m$$

Clearly $C_n \subseteq A_n \subseteq B_n$. Also, the sequences $\{B_n\}$ and $\{C_n\}$ are decreasing respectively. Let

$$B = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m \quad C = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} A_m$$

The events B and C are denoted by $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ respectively. Show that

(a) $B = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$.

(b) $C = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$.

We say that a sequence $\{A_n\}$ converges to a limit A if B and C are the same set A. We denote this by $A_n \to A$. Suppose this is the case, then show that

(c) A is an event.

(d) $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

Solution:

(a) Let $\omega \in B$. Then $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m$. Hence $\omega \in \bigcup_{m\geq n} A_m$ forall $n \in \mathbb{N}$. Hence $\omega \in A_k$ for some $k \in \mathbb{N}$. Let k_1 be the least number such that $\omega \in A_{k_1}$. Then we also have $\omega \in B_{k_1+1}$. So we have some $k_2 \geq k_1 + 1$ such that $\omega \in A_{k_2}$. Then $\omega \in B_{k_2+1}$. So there exists $k_3 \geq k_2 + 1$ such that $\omega \in A_{k_3}$. Continuing like this at i^{th} step we have some $k_{i+1} \geq k_i + 1$ such that $\omega \in A_{k_{i+1}}$ and so on. So now we got an strictly increasing infinite sequence of positive integers $\{k_1, k_2, k_3 \ldots, k_i, \ldots\}$ such that $\omega \in A_{k_j}$ for all $j \in \mathbb{N}$. Hence $\omega \in \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$. Hence

$$B \subseteq \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$$

Now let $\omega \in \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$. Let $\{s_n\}_{n\in\mathbb{N}}$ be the strictly increasing sequence of positive integers such that $\omega \in A_{s_n}$. Hence for all $m \in \mathbb{N}$ we have $\omega \in B_m$ because $\exists n \in \mathbb{N}$ such that $s_n > m$ and $\omega \in A_{s_m} \implies \omega \in B_m$. Therefore $\omega \in \bigcap_{m=1}^{\infty} B_m$. Therefore we have

$$\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\} \subseteq B$$

Hence we have $B = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$.

(b) Let $\omega \in \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$. Hence there exists $n_0 \in \mathbb{N}$ such that $\omega \in A_n$ for all $n > n_0$. Therefore $\omega \in C_n$ for all $n > n_0$. Since $C = \bigcup_{n=1}^{\infty} C_n$ we have $\omega \in C$. So we have

$$\{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\} \subseteq C$$

Now suppose $\omega \in C$. So $\exists\ n \in \mathbb{N}$ such that $\omega \in C_n$. Since $C_n = \bigcap_{m\geq n} A_m$ we have $\omega \in A_m$ for all $m \geq n$. Hence $\omega \in A_m$ for all but finitely many values of n. So

$$\omega \in \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$$

Therefore we get $C = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$.

(c)

(d)

□

Problem 4

10% of the surface of a sphere is coloured white, the rest is black. Show that, irrespective of the manner in which the colours are distributed, it is possible to inscribe a cube in S with all its vertices black.
Hint: For a given distribution of colors, select the cube"uniformly randomly" (you should make this more concrete). First note that it is enough to prove that there is a non-zero probability with which all the vertices of this random cube are colored black (why?). Now try to use the union bound from Problem 1(b) above to show this.

Solution: To show that there exists a cube in S with all its vertices black it is enough to show that if a random cube is chosen in S the probability of all vertices black is greater than 0. Now we have

$$\mathop{\mathbb{P}}_{\substack{C:\, cube \\ C \text{ is in } S}}[\text{All vertices of } C \text{ is black}] = 1 - \mathop{\mathbb{P}}_{\substack{C:\, cube \\ C \text{ is in } S}}[\text{At least one of the vertices of } C \text{ is white}]$$

So its is enough to show that $\mathop{\mathbb{P}}_{\substack{C:\, cube \\ C \text{ is in } S}}[\text{At least one of the vertices of } C \text{ is white}] < 1$. Now we also have

$$\mathop{\mathbb{P}}_{\substack{C:\, cube \\ C \text{ is in } S}}[\text{At least one of the vertices of } C \text{ is white}] = \mathop{\mathbb{P}}_{\substack{X_i \in S \\ \forall\, i \in [8]}}[\exists\, i \in [8]\ X_i \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}]$$

Now by Union Bound we have

$$\begin{aligned} &\mathop{\mathbb{P}}_{\substack{X_i \in S \\ \forall\, i \in [8]}}[\exists\, i \in [8]\ X_i \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}] \\ &\qquad \leq \sum_{j=1}^{8} \mathop{\mathbb{P}}_{\substack{X_i \in S \\ \forall\, i \in [8]}}\left[X_j \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}\right] \end{aligned}$$

So now showing

$$\sum_{j=1}^{8} \mathop{\mathbb{P}}_{\substack{X_i \in S \\ \forall\, i \in [8]}}\left[X_j \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}\right] < 1$$

is enough. Now for any $j \in [8]$,

$$\mathop{\mathbb{P}}_{\substack{X_i \in S \\ \forall\, i \in [8]}}\left[X_j \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}\right] = \mathop{\mathbb{P}}_{\substack{X_i \in S \\ \forall\, i \in [8]}}\left[X_j \text{ is colored white}\right] = \frac{1}{10}$$

The last equality because X_j is colored white if it is a point picked from the 10% area of the sphere which is colored white and the probability of that is $\frac{1}{10}$.Therefore we have

$$\sum_{j=1}^{8} \mathop{\mathbb{P}}_{\substack{X_i \in S \\ \forall\, i \in [8]}}\left[X_j \text{ is colored white} \mid X_1, \dots, X_8 \text{ forms a cube}\right] = \sum_{j=1}^{8} \frac{1}{10} = \frac{8}{10} < 1$$

Therefore we have $\mathop{\mathbb{P}}_{\substack{C:\, cube \\ C \text{ is in } S}}[\text{At least one of the vertices of } C \text{ is white}] < 1 \implies \mathop{\mathbb{P}}_{\substack{C:\, cube \\ C \text{ is in } S}}[\text{All vertices of } C \text{ is black}] > 0$. Which means there exists a cube in S with all vertices black

□