Report: Matroids and Derandomization of Isolation Lemma

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Chapter 1

Introduction

1.1 Matroids

Definition 1.1.1: Matroid

A matroid $M = (E, \mathcal{I})$ has a ground set E and a collection I of subsets of E called the *Independent Sets* st

1. Downward Closure: If $Y \in \mathcal{I}$ then $\forall\, X \subseteq Y$, $X \in \mathcal{I}$.
2. Extension Property: If $X, Y \in \mathcal{I}$, $|X| < |Y|$ then $\exists\, e \in Y - X$ such that $X \cup \{e\}$ also written as $X + e \in \mathcal{I}$

Observation. *A maximal independent set in a matroid is also a maximum independent set. All maximal independent sets have the same size.*

Base: Maximal Independent sets are called bases.
Rank of $S \in I$: We define the rank function of a matroid $r : \mathcal{P}(E) \to \mathbb{Z}$ where $r(S) = \max\{|X| : X \subseteq S, X \in I\}$ We def
Rank of a Matroid: Size of the base.
Span of $S \in I$: $\{e \in E : rank(S) = rank(S + e)\}$

1.2 Examples of Matroids

1.2.1 Uniform Matroid

It is denoted as $U_{k,n}$ where $E = [n]$ and $I = \{X \subseteq E \mid |X| \leq k\}$.
Free Matroid: When $k = n$ we take all possible subsets of E into I. This matroid is called Free Matroid i.e. $U_{n,n}$

1.2.2 Partition Matroid

Given $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_l$ where $\{E_1, \ldots, E_l\}$ is a partition of E and $k_1, \ldots, k_l \in \mathbb{N} \cup \{0\}$

$$I = \{X \subseteq E : |X \cap E_i| \leq k_i \;\forall\, i \in [l]\}$$

then $M = (E, I)$ is a partition matroid.

Note:-

If the E_i's are not a partition then suppose E_1, E_2 has nonempty partition then we will not have a matroid.
For example: $E_1 = \{1, 2\}$, $E_2 = \{2, 3\}$ and $k_1 = k_2 = 1$ then $X = \{1, 3\}$ is independent but $Y = \{2\} \subsetneq X$ is not a matroid.

1.2.3 Linear Matroid

Given a $m \times n$ matrix denote its columns as $A_1, \dots, A_n$. Then

$$I = \{X \subseteq [n] \colon \text{Columns corresponding to } X \text{ are linearly independent}\}$$

Here if the underlying field is $\mathbb{F}_2$ then it is called *Binary Matroid* and for $\mathbb{F}_3$ it is called *Ternary Matroid*.

1.2.4 Representable Matroid

A matroid with which we can associate a linear matroid is called a representable matroid.

Eg: $U_{2,3}$. It can be represented by the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, over $\mathbb{F}_2$. Over $\mathbb{F}_3$ it is same as $U_{3,3}$.

Note:-

There are matroids which are not representable as linear matroids in some field. There are matroids which are not representable on any field as well.

Lemma 1.2.1

$U_{2,4}$ is not representable over $\mathbb{F}_2$ but representable over $\mathbb{F}_3$

1.2.5 Regular Matroid

There are the matroids which are representable over all fields.

Lemma 1.2.2

Regular Matroids are precisely those which can be represented over $\mathbb{R}$ by a Totally Uni-modular matrix

1.2.6 Graphic Matroid / Cyclic Matroid

For a graph $G = (V, E)$ the graphic matroid $M_G = (E, I)$ where

$$I = \{F \subseteq E \colon F \text{ is acyclic}\}$$

Hence I is the collection of forests of G. It follows the downward closure trivially. For extension property let $k = |F_1| < |F_2| = l$ and then there are $n - k$ and $n - l$ components. So $n - k > n - l$. So $\exists$ an edge in F_2 which joins 2 components in F_1.

Lemma 1.2.3

A subset of columns is linearly independent iff the corresponding edges don't contain a cycle in the incidence matrix

Lemma 1.2.4

Graphic Matroids are Regular Matroids

Proof Idea: Use Incidence Matrix. ■

1.2.7 Matching Matroids

We can try to define it like this but it will not work:

Problem 1.1

Is the following a matroid: $E =$ Edges of a graph and $I = \{F \subseteq E : F \text{ is a matching}\}$

Solution: It is not a matroid since maximal matchings can not be extended to a maximum matching.

■

Correct way will be: For a graph $G = (V, E)$ the ground set $= V$ and

$$I = \{S \subseteq V : \exists \text{a matching that matches all vertices in } S\}$$

The downward closure property trivially holds. For extension property is $|S| < |S'|$ then there exists another vertex in S' which is not matched with S, so we can add that vertex to S.

1.3 Circuits

Assume we have a matroid $M = (E, I)$.

Definition 1.3.1: Circuit

A minimal dependent set C such that $\forall\, e \in C$, $C - e$ is an independent set.

Theorem 1.3.1

Let $S \in I$. $S + e \notin I$. Then $\exists!\, C \subseteq S + e$.

Proof. Given $S + e \notin I$. Take the set Σ where $T \in \Sigma$ if $t \notin I$ and $T \subseteq S + e$. Σ is nonempty since $S + e \in \Sigma$. Now under the ordering of inclusion T has a minimal element. Hence this minimal element is the desired circuit C which is minimal dependent set contained in $S + e$.

Now suppose it is not unique. Let $C_1, C_2 \subseteq S + e$ be circuits. Suppose $f \in C_1 - C_2$. Then $S - e + f$ will still be dependent since $C_2 \subseteq S - e + f$. Now by definition we get that $C_1 - f$ is independent. Therefore we extend $C_1 - f$ to an independent set by adding the elements of S till we reach same size as $|S|$. Now $e \in C$ since C_1 was formed because of addition of e. Hence if we extend $C_1 - f$ till same cardinality as S we will add all the edges of S not in $C_1 - f$ except f since adding f will make C be a dependent subset of an independent set which is not possible. Hence $C_1 - f$ will be extended to $S - f + e$. Therefore $S + e - f$ is independent which contradicts our previous conclusion that $S + e - f$ is dependent. Hence contradiction. ■

Chapter 2

Axiom systems for a matroi

Chapter 3

Bibliography