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Course: Quantum Information Theory

Assignment - 2.2 : Quantum Foundations
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For all the questions

- $[k] := \{1, 2, \ldots, k\}$ where $k \in \mathbb{N}$.
- $\mathscr{L}(\mathcal{H}) :=$ Linear operators on $\mathcal{H}$
- $\mathscr{R}(\mathcal{H}) :=$ Self-adjoint or hermitian operators on $\mathcal{H}$
- $\mathscr{P}(\mathcal{H}) :=$ Positive semi-definite operators on $\mathcal{H}$
- $\mathscr{D}(\mathcal{H}) :=$ Density operators on $\mathcal{H}$

Problem 1

For $T : \mathcal{H} \to \mathcal{H}$, prove that

$$\sum_{i=1}^{d} \langle e_i | T e_i \rangle = \sum_{i=1}^{d} \langle f_i | T f_i \rangle$$

if $\{|e_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ and $\{|f_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ are ONB.

Solution: Let $S : \mathcal{H} \to \mathcal{H}$ where it maps the basis vectors from $|e_i\rangle \to |f_i\rangle$. Then $S|e_i\rangle = |f_i\rangle$. Hence S is an unitary matrix since

$$\langle e_j | S^\dagger S | e_i \rangle = \langle f_j | f_i \rangle = \delta_{ji} \quad \text{and} \quad \langle f_j | S S^\dagger | f_i \rangle = \langle e_j | e_i \rangle = \delta_{ji}$$

Hence

$$\sum_{i=1}^{d} \langle f_i | T f_i \rangle = \sum_{i=1}^{d} \langle e_i | S^\dagger T S | e_i \rangle = tr(S^\dagger T S) = tr(S S^\dagger T) = tr(T) = \sum_{i=1}^{d} \langle e_i | T e_i \rangle$$

Therefore we have

$$\sum_{i=1}^{d} \langle e_i | T e_i \rangle = \sum_{i=1}^{d} \langle f_i | T f_i \rangle$$

$\square$

Problem 2

If $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \leq i \leq d\}$ and $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \leq i \leq d\}$ are ONB, then $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\} \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ is ONB

Solution: Let $|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then $|\psi\rangle = \sum\limits_{i=1}^{d} \alpha_i |e_i\rangle$ where $\alpha_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \leq i \leq d\}$ is ONB for $\mathcal{H}_1$. Hence

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle \otimes |\phi\rangle$$

Now $|\phi\rangle = \sum\limits_{i=1}^{d} \beta_i |f_i\rangle$ where $\beta_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \leq i \leq d\}$ is ONB for $\mathcal{H}_2$. Hence

$$\forall\, i \in [d] \;\; |e_i\rangle \otimes |phi\rangle = \sum_{j=1}^{d} \beta_j |e_i\rangle \otimes |f_j\rangle$$

Thereofore we get

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{d} \beta_j |e_i\rangle \otimes |f_j\rangle = \sum_{1\leq i,j\leq d} \alpha_i\beta_j |e_i\rangle \otimes |f_j\rangle$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i,j \leq d\}$ is a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Now for any $i1, i2, j1, j2 \in [d]$

$$(\langle e_{i1}| \otimes \langle f_{j1}|)(|e_{i2}\rangle \otimes |f_{j2}\rangle) = \langle e_{i1}|e_{i2}\rangle \langle f_{j1}|f_{j2}\rangle = \delta_{i1,i2}\,\delta_{j1,j2}$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i,j \leq d\}$ is orthonormal. Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i,j \leq d\}$ is a ONB for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

$\square$

Problem 3

Let $\{|g_k\rangle \mid 1 \leq k \leq d_2\} \subseteq \mathcal{H}_2$ be ONB. For $T \in \mathscr{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, let $tr_2(T) \in \mathscr{L}(\mathcal{H}_1)$ denote the operator satisfying

$$\langle u|\, tr_2(T)\, |v\rangle = \sum_k \langle u \otimes g_k|\, T\, |v \otimes g_k\rangle$$

for any choice $|u\rangle, |v\rangle \in \mathcal{H}_1$. Prove that $\sum_k \langle u \otimes g_k|\, T\, |v \otimes g_k\rangle$ is invariant.

Solution: Let $\{|f_k\rangle \mid 1 \leq k \leq d_2\} \subseteq \mathcal{H}_2$ be another ONB. Suppose $S : \mathcal{H}_2 \to \mathcal{H}$ be a map such that $S|g_k\rangle = |f_k\rangle$. As we previously showed in Problem 1, S is unitary. Then for all $k \in [d_2]$ we have

$$|f_k\rangle = \sum_{i=1}^{d_2} w_{i,k} |e_i\rangle$$

where $w_{i,k} \in \mathbb{C}$. Hence

$$\langle f_i|\, S^\dagger S\, |f_j\rangle = \sum_{k=1}^{d_2} w_{i,k}^* w_{j,k} = \delta_{i,j}$$

Now for any $|u\rangle, |v\rangle \in \mathcal{H}_1$ we have

$$\begin{aligned}
\langle u|\, tr_2(T)\, |v\rangle_{\{|f_k\rangle\}} &= \langle u| \left[\sum_{k=1}^{d_2} (I \otimes \langle f_k|) T (I \otimes |f_k\rangle)\right] |v\rangle \\
&= \langle u| \left[\sum_{k=1}^{d_2} \left(I \otimes \left(\sum_{i=1}^{d_2} w_{i,k}^* \langle g_i|\right)\right) T \left(I \otimes \left(\sum_{j=1}^{d_2} w_{j,k} |g_j\rangle\right)\right)\right] |v\rangle \\
&= \sum_{k=1}^{d_2} \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} \langle u| \left[w_{i,k}^* w_{j,k} (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} \langle u| \left[\left(\sum_{k=1}^{d_2} w_{i,k}^* w_{j,k}\right) (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} \langle u| \left[\delta_{i,j} (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2} \langle u| \left[(I \otimes \langle g_i|) T (I \otimes |g_i\rangle)\right] |v\rangle \\
&= \langle u|\, tr_2(T)\, |v\rangle_{\{|g_k\rangle\}}
\end{aligned}$$

Hence $\sum_k \langle u \otimes g_k|\, T\, |v \otimes g_k\rangle$ is invariant.

$\square$

Problem 4

Show that the Pauli matrices are all Hermitian, unitary, they square to the identity, and their eigenvalues are ± 1

Solution: Fuck you arun

$\square$

Problem 5 Mark Wilde: Exercise 3.3.3

For $S, T \in \mathscr{L}(\mathcal{H})$, show that

$$tr(T) = tr(T^{+}), \qquad tr(ST) = tr(TS)$$

[Recall T^{+} denotes adjoint of T]. For $|x\rangle, |y\rangle \in \mathcal{H}$ show

$$tr(|x\rangle\langle y| T) = tr(T|x\rangle\langle y|) = \langle y|Tx\rangle$$

Problem 6

Suppose $\mathcal{H}$ is finite dimensional complex inner product spacewith $\dim(\mathcal{H}) = d$. Show complex dimensionality of $\mathscr{L}(\mathcal{H})$ is d^2, real dimensionality of $\mathscr{R}(\mathcal{H})$ is d^2.

Suppose $\mathcal{H}$ is a real inner product space of dim d, show $\mathscr{L}(\mathcal{H})$ has dimension d and the space of all symmetric operators is a real vector space of dimension $\frac{d(d+1)}{2}$.

Solution: Suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is an ONB of $\mathcal{H}$. Let $T \in \mathscr{L}(\mathcal{H})$. Then for all $i \in [d]$

$$T|e_i\rangle = \sum_{j=1}^{d} \alpha_{i,j} |e_j\rangle$$

where $\alpha_{i,j} \in \mathbb{C}$. Hence the map T is uniquely decided by the numbers $\alpha_{i,j} \in \mathbb{C}$ for all $i, j \in [d]$. Hence there are d^2 many numbers which uniquely decides T. Therefore $\dim(\mathscr{L}(\mathcal{H})) = d^2$.

Now let $T \in \mathscr{R}(\mathcal{H})$. Then $T^{\dagger} = T$. Again suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is an ONB of $\mathcal{H}$. Let (i, j)th element of T is denoted by $t_{i,j}$. Then for all $i \in [d]$, $T_{i,i} \in \mathbb{R}$ since $T^{\dagger} = T$. Now for all off diagonal entries $t_{j,i} = t^*_{i,j}$. So there are $\frac{n^2-n}{2}$ many complex numbers which decides T uniquely apart from the n real entries in the diagonal. Now for each $i, j \in [d]$ let $t_{i,j} = x_{i,j} + iy_{i,j}$ where $x_{i,j}, y_{i,j} \in \mathbb{R}$. Therefore

$$t_{j,i} = t^*_{i,j} = x_{i,j} - iy_{i,j}$$

So for each off-diagonal entries there are corresponding 2 real numbers. And there are total $\frac{n^2-n}{2}$ many off-diagonal entries which participates in uniquely deciding T. Hence there are total $2 \times \frac{n^2-n}{2} + n = n^2$ real numbers which uniquely decides T. Hence $\dim(\mathscr{R}(\mathcal{H})) = d^2$.

$\square$

Problem 7

Show that $\mathscr{D}(\mathcal{H})$ is a convex subset of the real vector space of all Hermitian operators on $\mathcal{H}$. Show that the extreme points of $\mathscr{D}(\mathcal{H})$ are pure states, i.e. rank 1 projection operators.

Problem 8

Show that if $\dim(\mathcal{H}) = d$, then $\mathscr{D}(\mathcal{H})$ can be embedded into a real vector space of dimension $n = d^2 - 1$

Problem 9

Prove the Singular value decomposition theorem stated in class.

Problem 10

Suppose $|\psi\rangle_{AR_1} \in \mathcal{H}_A \otimes \mathcal{H}_{R_1}$, $|\psi\rangle_{AR_2} \in \mathcal{H}_A \otimes \mathcal{H}_{R_2}$ are purifications of $\rho_A \in \mathscr{D}(\mathcal{H}_A)$ and $\dim(\mathcal{H}_{R_2}) \geq \dim(\mathcal{H}_{R_1})$, then show that there exists an isometry $V : \mathcal{H}_{R_1} \to \mathcal{H}_{R_2}$ such that

$$|\psi\rangle_{AR_2} = (V \otimes I) |\psi\rangle_{AR_1}$$

Problem 11 Mark Wilde: Exercise 3.6.5

Show that the Bell states form an orthonormal basis:

$$\langle \Phi^{z_1 x_1} | \Phi^{z_2 x_2} \rangle = \delta_{z_1,z_2} \, \delta_{x_1,x_2}$$

Problem 12 Mark Wilde: Exercise 3.7.11

Show that the set of states $\{|\Phi^{x,z}\rangle_{AB}\}_{x,z=0}^{d-1}$ forms a complete, orthonormal basis:

$$\langle \Phi^{x_1,z_1} | \Phi^{x_2,z_2} \rangle = \delta_{x_1,x_2} \, \delta_{z_1,z_2} \qquad \sum_{x,z=0}^{d} |\Phi^{x,z}\rangle \langle \Phi^{x,z}| = I_{AB}$$

Problem 13 Mark Wilde: Exercise 4.1.5

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

Problem 14

Show that the set of states $\{|\Phi^{x,z}\rangle_{AB}\}_{x,z=0}^{d-1}$ forms a complete, orthonormal basis:

$$\langle \Phi^{x_1,z_1} | \Phi^{x_2,z_2} \rangle = \delta_{x_1,x_2} \, \delta_{z_1,z_2} \qquad \sum_{x,z=0}^{d} |\Phi^{x,z}\rangle \langle \Phi^{x,z}| = I_{AB}$$

Problem 15 Mark Wilde: Exercise 4.1.3

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

Problem 16 Mark Wilde: Exercise 3.7.12

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

Problem 17

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

Problem 18

Show that the following ensembles have the same density operator: $\left\{\left\{\frac{1}{2}, |0\rangle\right\}, \left\{\frac{1}{2}, |1\rangle\right\}\right\}$ and $\left\{\left\{\frac{1}{2}, |+\rangle\right\}, \left\{\frac{1}{2}, |-\rangle\right\}\right\}$

Problem 19

Show that the following ensembles have the same density operator: $\left\{\left\{\frac{1}{2}, |0\rangle\right\}, \left\{\frac{1}{2}, |1\rangle\right\}\right\}$ and $\left\{\left\{\frac{1}{2}, |+\rangle\right\}, \left\{\frac{1}{2}, |-\rangle\right\}\right\}$