CSS.201.1 Algorithms

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CHAPTER 1

Linear Programming

1.1 Introduction

Definition 1.1.1: Linear Program

A linear programming problem asks for a vector $x \in \mathbb{R}^d$ that maximizes or minimizes a given linear function, among all vectors x that satisfy given set of linear inequalities.

The general form of a maximization linear programming problem is the following: given $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $a_i \in \mathbb{R}^n$ for each $i \in [m]$ then

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & a_i^T x \leq b_i \quad \forall\, i \in [p], \\ & a_i^T x = b_i \quad \forall\, i \in \{p+1, \dots, p+q\}, \\ & a_i^T x \geq b_i \quad \forall\, i \in \{p+q+1, \dots, m\}, \\ & x_j \geq 0 \quad \forall j \in [k], \\ & x_j \leq 0 \quad \forall j \in [\{k+1, \dots, k+l\} \quad \text{(Some } x_j\text{'s are free)} \end{aligned}$$

The similar goes for minimization linear programming problem. For maximization problem we can always write the LP in the form

$$\begin{aligned} \text{maximize} \quad & c^T \hat{x} \\ \text{subject to} \quad & \hat{a}_i^T x \leq b_i' \quad \forall\, i \in [m], \\ & x_j' \geq 0 \quad \forall j \in [n] \end{aligned}$$

And then the LP is said to be in the ***canonical form***. What we can do is the following:

- For $i \in \{p+q+1, \dots, m\}$, we can replace $a_i^T x \leq b_i$ with $-a_i^T x \geq -b_i$
- For $i \in \{p+1, \dots, p+q\}$, we can replace with two constraints $a_i^T x \geq b_i$ and $a_i^T x \leq b_i$
- For $j \in \{k+1 \dots, k+l\}$, we can replace $x_j \leq 0$ with $-x_j \geq 0$
- For $j \in \{k+l+1 \dots, n\}$, we can replace the free x_j's with $x_j^+ - x_j^-$ all the equations where $x_j^+, x_j^- \geq 0$

This way we can always get a LP of that form. Now we can replace the $\hat{a}_i$ for $i \in [m]$ with a matrix $A \in \mathbb{R}^{m \times n}$ and replace the constraint $\hat{a}_i^T x \leq b_i', \forall\, i \in [m]$ with $Ax \leq b$

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0 \end{aligned}$$

1.2 Geometry of LP

Definition 1.2.1: Feasible Point and Region

A point $x \in \mathbb{R}^n$ is *feasible* with respect to some LP if it satisfies all the linear constraints. The set of all feasible points is called the *feasible region* for that LP.

Feasible region of a LP has a particularly nice geometric structure. Before that we will first introduce some geometric terminologies used in the linear programming context:

Definition 1.2.2: Hyperplane, Polyhedron, Polytope

- **Line**: The set $\{x + \lambda d, \lambda \in \mathbb{R}\}$ is line for any $x, d \in \mathbb{R}^n$.
- **Hyperplane**: The set $\{x \in \mathbb{R}^n : a^x = b\}$ is a hyperplane for any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- **Hyperspace**: The set $\{x \in \mathbb{R}^n : a^x \leq b\}$ is a hyperspace or half-space for any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- **Polyhedron**: A polyhedron is the intersection of a finite set of half-spaces i.e. the set $\{x \in \mathbb{R}^n : Ax \leq b\}$ for any $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^m$.
- **Polytope**: A bounded polyhedron is called a polytope.

Now it is not hard to verify that any polyhedron is a convex set i.e. if a polyhedron contains two points then it contains the entire line segment joining those two points.

Lemma 1.2.1

Polyhedron is a convex set

Hence the feasible region of a LP creates a polyhedron in $\mathbb{R}^n$. And $c^T x$ is the hyperplane normal to the vector c and the objective of the LP is by moving the plane normal to the vector c for which point in the polyhedron the hyperplane $c^T x$ has the highest value. Since polyhedron can be unbounded there may not exists any point x where $c^T x$ is maximum.

Suppose we have a LP

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \end{aligned}$$

Let P be the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. Then given $x^* \in P$ if any constraint $a_i^T x^* = b_i$ then this constrain is said to be *tight* or *binding* or *active* at x^*. Now two constraints $a_i^T x \leq b_i$ and $a_j^T x \leq b_j$ are said to be linearly independent if a_i and a_j are linearly independent.

Definition 1.2.3: Basic Solution and Basic Feasible Solution

$x^* \in \mathbb{R}^n$ is a basic solution if n linearly independent constraints are active at x^* (Doesn't need to be feasible).

$x^* \in \mathbb{R}^n$ is a basic feasible solution if x^* is a basic solution and $x^* \in P$. The basic feasible solutions are also called *corners* of a polyhedron.

Theorem 1.2.2

Given a LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0 \end{aligned}$$

Let P is the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$. Suppose P is non-empty and has at least one basic feasible solution then either the optimal value is $-\infty$ or there is an optimal basic feasible solution.

Theorem 1.2.3

If polyhedron P does not contain a line it contains at least one basic feasible solution (Hence if P is bounded it contains at least one basic feasible solution).

With this geometry in hand, we can easily picture two pathological cases where a given linear programming problem has no solution. The first possibility is that there are no feasible points; in this case the problem is called *infeasible*. The second possibility is that there are feasible points at which the objective function is arbitrarily large; in this case, we call the problem *unbounded*. The same polyhedron could be unbounded for some objective functions but not others, or it could be unbounded for every objective function.

Example 1.2.1

- **Maximum Matchings:** Given undirected graph $G = (V, E)$. Say variable x_e for each $e \in E$, $x_e = 1 \implies e$ in matching and $x_e = 0$ otherwise.

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e \text{ incident on } v} x_e \le 1 && \forall\, v \in V, \\ & x_e \ge 0 && \forall\, e \in E, \\ & x_e \in \{0, 1\} && \forall\, e \in E \end{aligned}$$

Observation. *M is a matching iff $\{x\colon x_e = 1$ if $e \in M, = 0$ otherwise$\}$ is a feasible solution*

- **Maximum $s - t$ Flow:** Given directed graph $G = (V, E)$ with vertices s, t and capacity c_e on edges. Say variable x_e for each edge and equal to flow on that edge. Then the LP of this problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in out(s)} x_e \\ \text{subject to} \quad & \sum_{e \in in(v)} x_e - \sum_{c \in out(v)} x_e = 0 && \forall\, v \in V, v \neq s, t, \\ & c_e \ge x_e \ge 0 && \forall\, e \in E \end{aligned}$$

We will now introduce a theorem without proof that for any LP with a polytope we can find a solution in polynomial time.

Theorem 1.2.4

Let $P = \{x \in \mathbb{R}^n \colon Ax \ge b\}$ be a polytope. Then we can find an optimal basic feasible solution for the LP $\min c^T x$ where $x \in P$ in polynomial time.

1.3 LP Integrality

For the LP for matchings in bipartite graphs $G = (L \cup R, E)$ we have:

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e \text{ incident on } v} x_e \le 1 && \forall\, v \in V, \\ & x_e \ge 0 && \forall\, e \in E \end{aligned}$$

We want $x_e \in \{0,1\}$ i.e. we want to have integral solution for this LP

Question 1.1

LP's can give fractional solutions. When is solution integral?

Sufficient Condition: Every basic feasible solution of the feasible polytope is integral i.e. x^* is basic feasible solution $\implies x^* \in \mathbb{Z}^n$. If all basic feasible solution are integral then for all $I \subseteq [m]$ with $|I| = n$, $A_I^{-1}b_I$ is integral. Let $x = A_I^{-1}b_I$ Then j^{th} component $x_j = \frac{|A_I^j|}{|A|}$ (Cramer's Rule).

1.3.1 Totally Unimodular Matrix

Definition 1.3.1: Totally Unimodular Matrix (TUM)

A matrix $A \in \{0,1,-1\}^{m\times n}$ is totally unimodular (TU) if every square submatrix of A has determinant $-1, 0, 1$.

Hence in the above LP is A is TU and b is integral then all basic feasible solutions are integral.

Lemma 1.3.1

Let A be TUM and $b \in \mathbb{Z}^n$ then $P = \{x : Ax \geq b\}$ is integral i.e. every basic feasible solution is integral.

Hence using Theorem 1.2.4 if the polytope is integral we can find optimal integral solution in polynomial time. We will now discuss properties of Totally Unimodular Matrix.

Lemma 1.3.2

$A \in \{0,1,-1\}^{m\times n}$ is TU iff the following are TU:

(i) $-A$

(ii) A^T

(iii) $\begin{bmatrix} A & e_i \end{bmatrix}, \begin{bmatrix} A & -e_i \end{bmatrix}$

(iv) $\begin{bmatrix} A & I \end{bmatrix}, \begin{bmatrix} A & -I \end{bmatrix}$

(v) $\begin{bmatrix} A & A_i \end{bmatrix}, \begin{bmatrix} A & -A_i \end{bmatrix}$ where A_i is the i^{th} column of A.

Corollary 1.3.3

If A is TUM and $a, b, c, d \in \mathbb{Z}^n$ are integer vectors then the polytope $Q = \{x \in \mathbb{R}^n : a \leq Ax \leq b, c \leq x \leq d\}$ is integral.

Proof: We can combine the four inequalities in one inequality. Consider the matrix $\begin{bmatrix} A & -A & I & -I \end{bmatrix}^T$. Then the given polytope is

$$Q = \left\{ x \in \mathbb{Z}^n : \begin{bmatrix} A \\ -A \\ I \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -a \\ d \\ -c \end{bmatrix} \right\}$$

By Lemma 1.3.2, $\begin{bmatrix} A & -A & I & -I \end{bmatrix}^T$ is a TUM since A is TUM. Therefore the polytope Q is integral. ■

The following theorem lets us to give a necessary and sufficient condition to check if a given matrix is TUM. Again we will accept the following theorem without the proof since the proof is a little nontrivial.

Theorem 1.3.4

Let $A \in \{-1, 0, 1\}^{m \times n}$. Then A is TU iff every set $S \subseteq [n]$ can be partitioned into S_1, S_2 such that

$$\sum_{i \in S_1} A_i - \sum_{i \in S_2} A_i \in \{-1, 0, 1\}^m$$

where A_i is the i^{th} column of A. C

Now using this theorem we will show that the polytope for bipartite maximum matching is integral.

1.4 Duality

CHAPTER 2

Bibliography