Matroids: Combinatorial Optimization

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1 Introduction

Definition 1.1: Matroid

A matroid $M = (E, I)$ has a ground set E and a collection I of subsets of E called the *Independent Sets* st

1. Downward Closure: If $Y \in I$ then $\forall X \subseteq Y$, $X \in I$.
2. Extension Property: If $x, Y \in I$, $|X| < |Y|$ then $\exists\, e \in Y - X$ such that $X \cup \{e\}$ also written as $X + e \in I$

Observation. *A maximal independent set in a matroid is also a maximum independent set. All maximal independent sets have the same size.*

Base: Maximal Independent sets are called bases.
Rank of $S \in I$: $\max\{|X| : X \subseteq S, X \in I\}$
Rank of a Matroid: Size of the base.
Span of $S \in I$: $\{e \in E : rank(S) = rank(S + e)\}$

2 Types of Matroids

2.1 Uniform Matroid:

It is denoted as $U_{k,n}$ where $E = [n]$ and $I = \{X \subseteq E \mid |X| \le k\}$.
Free Matroid: When $k = n$ we take all possible subsets of E into I. This matroid is called Free Matroid i.e. $U_{n,n}$

2.2 Partition Matroid:

Given $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_l$ where $\{E_1, \dots, E_l\}$ is a partition of E and $k_1, \dots, k_l \in \mathbb{N} \cup \{0\}$

$$I = \{X \subseteq E : |X \cap E_i| \le k_i \;\forall\, i \in [l]\}$$

then $M = (E, I)$ is a partition matroid.

Note:-

If the E_i's are not a partition then suppose E_1, E_2 has nonempty partition then we will not have a matroid.
For example: $E_1 = \{1, 2\}$, $E_2 = \{2, 3\}$ and $k_1 = k_2 = 1$ then $X = \{1, 3\}$ is independent but $Y = \{2\} \subsetneq X$ is not a matroid.

2.3 Linear Matroid:

Given a $m \times n$ matrix denote its columns as $A_1, \dots, A_n$. Then

$$I = \{X \subseteq [n] : \text{Columns corresponding to } X \text{ are linearly independent}\}$$

Here if the underlying field is $\mathbb{F}_2$ then it is called *Binary Matroid* and for $\mathbb{F}_3$ it is called *Ternary Matroid*.

2.4 Representable Matroid:

A matroid with which we can associate a linear matroid is called a representable matroid.

Eg: $U_{2,3}$. It can be represented by the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, over $\mathbb{F}_2$. Over $\mathbb{F}_3$ it is same as $U_{3,3}$.

Note:-

There are matroids which are not representable as linear matroids in some field. There are matroids which are not representable on any field as well.

Lemma 2.1

$U_{2,4}$ is not representable over $\mathbb{F}_2$ but representable over $\mathbb{F}_3$

2.5 Regular Matroid:

There are the matroids which are representable over all fields.

Lemma 2.2

Regular Matroids are precisely those which can be represented over $\mathbb{R}$ by a Totally Uni-modular matrix

2.6 Graphic Matroid / Cyclic Matroid:

For a graph $G = (V, E)$ the graphic matroid $M_G = (E, I)$ where

$$I = \{F \subseteq E \colon F \text{ is acyclic}\}$$

Hence I is the collection of forests of G. It follows the downward closure trivially. For extension property let $k = |F_1| < |F_2| = l$ and then there are $n - k$ and $n - l$ components. SO $n - k > n - l$. So $\exists$ an edge in F_2 which joins 2 components in F_1.

Lemma 2.3

A subset of columns is linearly independent iff the corresponding edges don't contain a cycle in the incidence matrix

Lemma 2.4

Graphic Matroids are Regular Matroids

Proof Idea: Use Incidence Matrix. ■

2.7 Matching Matroids

We can try to define it like this but it will not work:

Problem 1

Is the following a matroid: $E =$ Edges of a graph and $I = \{F \subseteq E \colon F \text{ is a matching}\}$

Solution: It is not a matroid since maximal matchings can not be extended to a maximum matching.

■

Correct way will be: For a graph $G = (V, E)$ the ground set $= V$ and

$$I = \{S \subseteq V \colon \exists \text{a matching that matches all vertices in } S\}$$

The downward closure property trivially holds. For extension property is $S| < |S'|$ then there exists another vertex in S' which is not matched with S, so we can add that vertex to S.

3 Circuits

Assume we have a matroid $M = (E, I)$.

Definition 3.1: Circuit

A minimal dependent set C such that $\forall\, e \in C$, $C - e$ is an independent set.

Theorem 3.1

Let $S \in I$. $S + e \notin I$. Then $\exists!\, C \subseteq S + e$.

4 Finding Max Weight Base

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6.3 Min-Max Relation for Colorful Spanning Tree

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