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Course: Mathematical Foundations for Computer Sciences

Assignment - 4
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> **Problem 1**
> Let $n > 0$. Consider a directed graph on partitions of n (with the parts sorted by size). There is a (directed) edge from a partition to another if the latter can be obtained from the former by moving a "unit" from a part to the one immediately after, while maintaining the order. For example, there is an edge from the partition $25 = 10+6+4+3+2$ to $25 = 10+5+5+3+2$ as a "unit" is moved from 6 to 4 and it does not change the order of the parts, but no edge vice-versa. Similarly, there is an edge from the partition $25 = 10+6+4+3+2$ to $25 = 10+6+4+3+1+1$. Characterize all the sink partitions (those without any outgoing edges) that can be by starting from the partition $n = n$ (with only one part) and following these edges.

Solution: For any partition $n = k_1 + \cdots + k_m$ where $k_1 \geq \ddots \geq k_m$ we represent the partition by the tuple $(k_1, \dots, k_m)$. We call the alteration happened at i on the partition $(k_1, \dots, k_m)$ if there is an edge from $(k_1, \dots, k_m)$ to $(k_1, \dots, k_{i-1}, k_i - 1, k_{i+1} + 1, k_{i+2}, \dots, k_m)$.

Lemma 1. *A partition does not have any edge going out if and only if any two consecutive partitions have difference at most 1*

Proof: Consider a partition of n. Let $(k_1, \dots, k_m)$ is the partition of n. Now $\exists\ i \in [m-1]$ such that $k_i - k_{i+1} \geq 2 \iff k_i - 1 \geq k_{i+1} + 1 \iff k_{i-1} \geq k_i > k_i - 1 \geq k_{i+1} + 1 > k_{i+3} \iff \exists$ edge going out from the partition to the partition $(k_1, \dots, k_{i-1}, k_i - 1, k_{i+1} + 1, k_{i+2}, \dots, k_m)$. $\square$

Lemma 2. *A partition where there exists an element which appears more than 2 times is not reachable from n.*

Proof: Let there be a partition of n which is reachable from n and there exists an element which appears more than 2 times. Now in the path from n to that partition consider the partition where first time when that element appeared more than 2 times. Let the partition be $(k_1, \dots, k_m)$. Then the parent of that partition didn't have all k_i, k_{i+1}, k_{i+2} equal. Then alteration from the parent of that partition happened at any of the $\{i-1, i, i+1, i+2\}$ places.

- **Case 1:** If alteration happened at $i-1$ then before alteration the parent of the partition had $(k_1, \dots, k_{i-1}+1, k_i - 1, k_{i+1}, k_{i+2}, \dots, k_m)$ and $k_i - 1 < k_{i+1}$. This is not possible. So this is case is not possible.
- **Case 2:** If alteration happened at i then before alteration the parent of the partition had $(k_1, \dots, k_i + 1, k_{i+1} - 1, k_{i+2}, \dots, k_m)$ and $k_{i+1} - 1 < k_{i+2}$. This is not possible. So this is case is not possible.
- **Case 3:** If alteration happened at $i+1$ then before alteration the parent of the partition had $(k_1, \dots, k_i, k_{i+1} + 1, k_{i+2} - 1, \dots, k_m)$ and $k_i < k_{i+1} + 1$. This is not possible. So this is case is not possible.
- **Case 4:** If alteration happened at $i+2$ then before alteration the parent of the partition had $(k_1, \dots, k_i, k_{i+1}, k_{i+2} + 1, k_{i+3} - 1, \dots, k_m)$ and $k_{i+1} < k_{i+2} + 1$. This is not possible. So this is case is not possible.

Therefore none of the cases are possible. Hence the partition has no parent. Hence contradiction. Therefore we have the lemma. $\square$

Hence we get that if a partition is reachable then no element in that partition is repeated more than 2 times.

Lemma 3. *A sink partition reachable from n can have at most one element which appears twice.*

Proof: Suppose that is not the case. Let $(k_1, \ldots, k_m)$ be a reachable sink partition of n such that there are at least two elements which appears twice in the partition. Let $i, j \in [m-1]$ such that $i \neq j$ and $i+1 < j$ be the closest indexes such that $k_i = k_{i+1}$ and $k_j = k_{j+1}$. Let $k_i = a$ and $k_j = b$ Now by Lemma 1 $k_t - k_{t+1} \leq 1$ for all $t \in [m-1]$. Since i, j are closest such that $k_i = k_{i+1}$ and $k_j = k_{j+1}$ we have

$$k_{i+2} = a+1, k_{i+3} = a+2, \ldots, k_{j-1} = a + (j-i-2), k_j = a + (j-i-1) = b$$

For brevity denote $j - i + 1 = l$. Then $j = l + 1 + i$. Certainly by Lemma 2 $k_{i-1} > k_i$ and $k_{j+2} < k_{j+1}$. Call i, j as i_1 and j_1.

Now since the partition is reachable from n there is a path from n to $(k_1, \ldots, k_m)$. Consider the partition $(k_{1,1}, \ldots, k_{l_1,1})$ where first time the above event occurred i.e. $k_{t,1} = k_t$ for all $t \in \{i_1, \ldots, j_1 + 1\}$. Here $l_1 \leq m$ as with alterations number of partitions always stays same or increase. Since $(k_1, \ldots, k_m)$ is a reachable sink partition $(k_{1,1}, \ldots, k_{l,1})$ is also reachable. Consider the parent of this partition. From the parent alteration can happen only on the positions $\{i_1 - 1, \ldots, j_1 + 1\}$. Alteration at any of the positions in $\{i_1 - 1, \ldots, j_1 + 1\}$ means the parent of the partition had 2 indexes i_2, j_2 with $i_2 + 1 < j_2$ which were closer to each other than i_1, j_1 and also the elements at those two indexes are different and they appeared twice. So we get a new partition $(k'_{1,1}, \ldots, k'_{l,1})$ such that there exists indexes i_2, j_2 such that $i_2 + 1 < j_2$ and $k'_{i_2} = k'_{i_2+1}$, $k'_{j_2} = k'_{j_2+1}$.

Again consider the partition $(k_{1,2}, \ldots, k_{l_2,2})$ where first time the above event occurred i.e. $k_{t,2} = k'_{t,1}$ for all $t \in \{i_2, \ldots, j_2 + 1\}$. Consider the parent of this partition. From the parent alteration can happen only on the positions $\{i_2 - 1, \ldots, j_2 + 1\}$. Alteration at any of the positions in $\{i_2 - 1, \ldots, j_2 + 1\}$ means the parent of the partition had 2 indexes i_3, j_3 with $i_3 + 1 < j_3$ which were closer to each other than i_2, j_2 and also the elements at those two indexes are different and they appeared twice. So we get a new partition $(k'_{1,2}, \ldots, k'_{l_3,2})$ such that there exists indexes i_3, j_3 such that $i_3 + 1 < j_3$ and $k'_{i_3} = k'_{i_3+1}$, $k'_{j_3} = k'_{j_3+1}$.

Continuing like this finally we get a partition $(k'_{1,s-1}, \ldots, k'_{l_{s-1},s-1})$ such that there exists $j \in [l_{s-1}]$ such that

$$k'_{j,s-1} = k'_{j+1,s-1} = k'_{j+2,s-1} + 1 = k'_{j+3,s-1} + 1$$

And from this partition $(k_{1,s}, \ldots, k_{l_s,s})$ we take the partition where first time the above event occurred i.e. $k_{t,s} = k'_{t,s-1}$ for all $t \in \{j-1, j, j+1, j+2, j+3\}$. Now for brevity denote $(k_{1,s}, \ldots, k_{l_s,s})$ as $(k_1, \ldots, k_l)$ where $k_i = k_{i+1} = k_{i+2} + 1 = k_{i+3} + 1$ for $i \in [l]$. Now we will analyze case wise:

- **Case 1:** If alteration happened at $i-1$ then before alteration the parent of the partition had $(k_1, \ldots, k_{i-1} + 1, k_i - 1, k_{i+1}, k_{i+2}, \ldots, k_l)$ and $k_i - 1 < k_{i+1}$. This is not possible. So this is case is not possible.

- **Case 2:** If alteration happened at i then before alteration the parent of the partition had $(k_1, \ldots, k_i + 1, k_{i+1} - 1, k_{i+2}, \ldots, k_l)$ and $k_{i+1} - 1 = k_{i+2} =, k_{i+3}$. By Lemma 9 this partition should not be reachable. Hence contradition. This case is not possible.

- **Case 3:** If alteration happened at $i+1$ then before alteration the parent of the partition had $(k_1, \ldots, k_i, k_{i+1} + 1, k_{i+2} - 1, \ldots, k_l)$ and $k_i < k_{i+1} + 1$. This is not possible. So this is case is not possible.

- **Case 4:** If alteration happened at $i+2$ then before alteration the parent of the partition had $(k_1, \ldots, k_i, k_{i+1}, k_{i+2} + 1, k_{i+3} - 1, \ldots, k_l)$ and $k_i = k_{i+1} = k_{i+2} + 1$. By Lemma 9 this partition should not be reachable. Hence contradition. This case is not possible.

- **Case 4:** If alteration happened at $i+3$ then before alteration the parent of the partition had $(k_1, \ldots, k_i, k_{i+1}, k_{i+2}, k_{i+3} + 1, k_{i+4} - 1, \ldots, k_l)$ and $k_{i+2} < k_{i+3} + 1$. This is not possible. So this is case is not possible.

Therefore none of the cases is possible. Hence contradiction. Therefore $(k_1, \ldots, k_m)$ is not a reachable sink partition of n. Therefore we have the lemma. $\square$

Now we will show every partition with at most one element appearing twice is reachable from n.

Lemma 4. *For a partition $(r, r-1, \ldots, r-k)$ is reachable from n for any r and k which follows $n = \sum\limits_{i=0}^{k}(r-i)$, $k < r$*

Proof: For any r we will prove this inducting on k. For $k = 0$ we have $n = r$ then we dont have to do any alteration. Hence the base case follows. Suppose for $k-1$ this is true. Now we have $n = \sum\limits_{i=0}^{k}(r-i)$. Now by inductive hypothesis we can reach $(r, r-1, \ldots, r-(k-1))$ starting from $\sum\limits_{i=0}^{k-1}(r-i)$. We will do the same alterations to reach $(r, r-1, \ldots, r-(k-1))$ but the starting number will be $\sum\limits_{i=0}^{k}(r-i)$. So after the same alterations we will reach $(2r-k, r-1, \ldots, r-(k-1))$. Now we will do the following operations

$$\begin{aligned}(2r-k, r-1, \ldots, r-(k-1)) &\to (2r-k-1, r-1+1, \ldots, r-(k-1)) \to \\ \cdots &\to (2r-k-1, r-1, \ldots, r-(k-1)+1) \to (2r-k-1, r-1, \ldots, r-(k-1), 1)\end{aligned}$$

Now afterwards at any stage $i \in [r-k-1]$ we do the operation

$$\begin{aligned}&(2r-k-1-i, r-1, \ldots, r-(k-1), i) \to (2r-k-1-i-1, r-1+1, \ldots, r-(k-1), i) \to \\ &\cdots \to (2r-k-1-i-1, r-1, \ldots, r-(k-1)+1, i) \to (2r-k-1-i-1, r-1, \ldots, r-(k-1), i+1)\end{aligned}$$

We keep doing this operation for $r-k-1$ many times and finally we reach $(r, r-1, \ldots, r-(k-1), r-k)$. Hence by mathematical induction this is true for all k, $k < r$. Hence we can reach a partition $(r, r-1, \ldots, r-k)$ from n for any r and k which follows $n = \sum\limits_{i=0}^{k}(r-i)$, $k < r$. □

Lemma 5. *For a partition $(r, r-1, \ldots, r-k, r-k)$ is reachable from n for any r and k which follows $n = \sum\limits_{i=0}^{k}(r-i) + (r-k)$, $k < r$*

Proof: We first apply the same operations as to reach $(r, r-1, \ldots, r-k)$ but we start from $\sum\limits_{i=0}^{k}(r-i) + (r-k)$ as described in Lemma 4. Then we reach the partition $(2r-k, r-1, \ldots, r-k)$. Now again we do the same operations as in case of the inductive step of the proof of Lemma 4. This propagates the $r-k$ by 1 element at a time. This finally gives the partition $(r, r-1, \ldots, r-k, r-k)$. Hence we can reach $(r, r-1, \ldots, r-k, r-k)$ from n. □

Lemma 6. *For a partition $(r, r-1, \ldots, r-l, r-l, \ldots, r-k)$ is reachable from n for any r, k, l where $n = \sum\limits_{i=0}^{k}(r-i) + (r-l)$, $r > k > l$*

Proof: For a partition $(r + (k+1-l), r-1, \ldots, r-l, r-l-1, \ldots, r-k, r-k-1)$ is reachable from n for any r, k, l where $n = \sum\limits_{i=0}^{k}(r-i) + (r-l)$, $r > k > l$ since

$$\sum_{i=0}^{k}(r-i) + (r-l) = \sum_{i=0}^{k+1}(r-i) + (r-l) - (r-k-1) = \sum_{i=0}^{k+1}(r-i) + (k+1-l)$$

following the operations in the proof of Lemma 4. Now we will do the following operation

$$\begin{aligned}
&(r+(k+1-l), r-1, \ldots, r-k-1) \to (r+(k+1-l)-1, r-1+1, \ldots, r-k-1) \to \\
&\cdots \to (r+(k+1-l)-1, r-1, \ldots, r-k, r-k) \cdots \to (r+(k+1-l)-2, r-1, \ldots, r-k+1, r-k) \\
&\cdots \xrightarrow{i \text{ times}} (r+(k+1-l)-i, r-1, \ldots, r-k+i-1, r-k+i-1 \ldots, r-k) \\
&\cdots \xrightarrow{(k+1-l) \text{ times}} (r, r-1, \ldots, r-k+(k+1-l-1), r-k+(k+1-l-1) \ldots, r-k) \\
&= (r, \ldots, r-l, r-l, \ldots, r-k)
\end{aligned}$$

Hence $(r+(k+1-l), r-1, \ldots, r-l, r-l-1, \ldots, r-k, r-k-1)$ is reachable from n. □

Therefore with all these lemmas we obtain that if a partition is reachable from n then the partition is of the form $(r, r-1, \ldots, r-k)$ or $(r, r-1, \ldots, r-l, r-l, r-l-1, \ldots, r-k)$ where $r > k \geq l$. Notice if $r-k > 1$ then we can also break $r-k$ into smaller partitions. Hence if a partition is sink partition then it must end with 1. Therefore the sink partitions are of the form $(r, r-1, \ldots, r-l, r-l, r-l-1, \ldots, 1)$ where $n = \sum\limits_{i=0}^{r} -1(r-i) + (r-l)$ for some $r > l$. ■

Problem 3

Let $(U, \leq)$ be poset and M be a square matrix with rows and columns indexed by U such that the (x, y)-th entry of M is $\mathbb{1}(x \leq y)$.

- Show that M is invertible.
- Compute the inverse of M when $(U, \leq)$ is the divisibility poset for positive integers at most n (for some fixed n) and use it to derive the Möbius inversion formula.

Solution:

- Consider the directed graph $G = (V, E)$ with vertices $V = U$ and for any $x, y \in U$, $x \neq y$ the directed edge $(x, y) \in E$ if $x \leq y$. Then the adjacency matrix of G is the matrix $M - I$. Since U is a poset the graph G is a directed acyclic graph. Hence in $(M-I)^k$ for any $x, y \in U$, $x \neq y$ the entry $(M-I)^k(x, y)$ is the number of paths from $x \rightsquigarrow y$. Since the maximum length of a path in G is at most $|U|-1$ there is no path of length $|U|$ between any two vertices. Hence $(M-I)^{|U|} = 0$. Now $(M-I)^{|U|} = 0$ implies that the minimal polynomial of M is a divisor of $(x-1)^{|U|}$. Therefore the only eigenvalue of M is 1. Hence $\det(M) = 1$. Therefore M is invertible.
- Consider the $n \times n$ matrix S where for all $d, m \in [n]$ such that

$$S(d, n) = \mathbb{1}(d \mid m) \mu\left(\frac{m}{d}\right)$$

  We will show that $MS = I$.

  Now for any $m \in [n]$ we have

$$MS(m, m) = \sum_{k=1}^{n} M(m, k) S(k, m) = \sum_{\substack{k \in [n] \\ m | k, k | m}} \mu\left(\frac{m}{k}\right) = \mu\left(\frac{m}{m}\right) = 1$$

  Hence MS has 1's on the diagonals.

  Now take any $m_1, m_2 \in [n]$ such that $m_1 \neq m_2$. We will show that $MS(m_1, m_2) = 0$. Now if $m_1 \nmid m_2$ then $\nexists\, k \in [n]$ such that $m_1 \mid k$ and $k \mid m_2$. Therefore

$$MS(m_1, m_2) = \sum_{\substack{k \in [n] \\ m_1 | k, k | m_2}} \mu\left(\frac{m_2}{k}\right) = 0$$

Now assume $m_1 \mid m_2$. Now $m_1 \neq m_2$ hence $m_2 \neq 1$. Let $d = \frac{m_2}{m_1}$. Then we have

$$MS(m_1, m_2) = \sum_{\substack{k \in [n] \\ m_1 | k, k | m_2}} \mu\left(\frac{m_2}{k}\right) = \sum_{k | d} \mu\left(\frac{d}{k}\right)$$

Lemma 7. *For any positive integer n if $n \neq 1$ then $\sum\limits_{d|n} \mu\left(\frac{n}{d}\right) = 0$*

Proof: Let $n \neq 1$ and $n = \prod\limits_{i=1}^{k} p_i^{e_i}$ be the prime factorization of n where for all $i \in [k]$, $e_i \neq 0$. Now for any d, where $d \mid n$,

$$\mu\left(\frac{n}{d}\right) \neq 0 \iff \exists\ S \subseteq [k],\ \frac{n}{d} = \prod_{i \in S} p_i$$

Now $\mu\left(\frac{n}{d}\right) = 1$ iff $\exists\ S \subseteq [k],\ \frac{n}{d} = \prod\limits_{i \in S} p_i$ and $|S| \equiv 0 \bmod 2$ and $\mu\left(\frac{n}{d}\right) = -1$ iff $\exists\ S \subseteq [k],\ \frac{n}{d} = \prod\limits_{i \in S} p_i$ and $|S| \equiv 1 \bmod 2$. Therefore we have

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) = \sum_{S \subseteq [k]} (-1)^{|S|} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i = (1 + (-1))^k = 0$$

So we have the lemma. □

Therefore from the lemma we have that $\sum\limits_{k|d} \mu\left(\frac{d}{k}\right) = 0$. Therefore for $m_1 \neq m_2 \in [n]$ with $m_1 \mid m_2$ we have $MS(m_1, m_2) = 0$. Hence $MS = I$. Therefore S is the inverse of M.

Now we will prove the Möbius Inversion Formula. Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ be two functions such that $g(k) = \sum\limits_{d|k} f(d)$. Now define the column vectors F and G where for any $i \in [n]$, $F(i) = f(i)$ and $G(i) = g(i)$. Then for any $k \in [n]$ we have

$$M^T F(k) = \sum_{d|k} f(d) = g(k) \implies M^T F = G$$

Therefore

$$F = (M^T)^{-1} G = S^T G$$

Now

$$S^T G(k) = \sum_{d \in [n]} S^T(k, d) g(d) = \sum_{d \in [n]} S(d, k) g(d) = \sum_{d|k} \mu\left(\frac{k}{d}\right) g(d)$$

Therefore $\sum\limits_{d|k} \mu\left(\frac{k}{d}\right) g(d) = f(k)$ for all $k \in [n]$. Therefore we have the Möbius Formula.

■

[Me and Soumyadeep came up with the solution together and I discussed solution of second part with Aakash]

> **Problem 4**
>
> Let $k, p > 0$ be integers where p is prime. A seller has kp marbles, where marble $i \in [kp]$ costs ₹i, that he is trading. Call a trade "divisible" if both the cost and the number of marbles sold is a multiple of p (selling no marbles is one such trade). How many divisible trades are there? Write your answer in terms of p-th roots of unity.

Solution: Consider the polynomial $f(t,c) = \sum_{i=1}^{kp}(1+tc^i)$ where in i^{th} term $(1+tc^i)$ the power of c represent the cost of i^{th} coin and t represents if the i^{th} is taken into consideration. Therefore for any t^kc^l the $s(k,l) \coloneqq \mathit{Coeff}(t^kc^l)$ represent the number of ways to pick k coins such that the sum their costs is l. Hence

$$f(t,c) = \sum_{i=1}^{kp}(1+tc^i) = \sum_{k,l} s(k,l)t^kc^l$$

Hence we have to find the coefficients $s(k,l)$ for which $p \mid k$ and $p \mid l$. In order to do that we will use the following lemma:

Lemma 8. *Let $f(x) = \sum_{i=0}^{n} a_i x^i$ where $a_i \in \mathbb{R}$ is a polynomial with $a_n \neq 0$. Then*

$$\frac{1}{p}\sum_{i=0}^{p-1} f(\zeta_p^i) = \sum_{k=0,\ p|k}^{n} a_k$$

where ζ_p is the p^{th} root of unity.

Proof: For any $i \in \{0,1,\dots,p-1\}$, Therefore we have

$$\sum_{i=0}^{p-1} f(\zeta_p^i) = \sum_{i=0}^{p-1}\sum_{k=0}^{n} a_k\zeta_p^{ik} = \sum_{k=0}^{n} a_k \left[\sum_{i=0}^{p-1}\zeta_p^{ki}\right]$$

Now for $k \equiv 0 \bmod p$ we have $\sum_{i=0}^{p-1}\zeta_p^{ki} = \sum_{i=0}^{p-1} 1 = p$ and for $k \not\equiv 0 \bmod p$ we have

$$\sum_{i=0}^{p-1}\zeta_p^{ki} = \sum_{k=0}^{p-1}\zeta_p^{i} = \frac{z^p-1}{z-1} = 0$$

Hence we have

$$\sum_{i=0}^{p-1} f(\zeta_p^i) = \sum_{k=0}^{n} a_k \left[\sum_{i=0}^{p-1}\zeta_p^{ki}\right] = \sum_{k=0,\ p|k}^{n} pa_k = p\sum_{k=0,\ p|k}^{n} a_k$$

Therefore $\frac{1}{p}\sum_{i=0}^{p-1} f(\zeta_p^i) = \sum_{k=0,\ p|k}^{n} a_k$. □

We will use this lemma to first find the sum of coefficients in $f(t,c)$ for which the power of c is divisible by p then that becomes a polynomial over t where we take the sum of coefficient for which the power of t is divisible by p.

So consider the polynomial $g(t) = \frac{1}{p}\sum_{i=0}^{p-1} f(t,\zeta_p^i)$. We have another lemma

Lemma 9. $f(t,\zeta_p^i) = \left[\prod_{i=0}^{p-1}(1+t\zeta_p^i)\right]^k$

Proof: We will show that

$$\prod_{i=0}^{p-1}\left(1+t\zeta_p^{lp+i}\right) = \prod_{i=0}^{p-1}(1+t\zeta_p^i)$$

for any $l \in \{0,\dots,k-1\}$. We have

$$\zeta_p^{lp+i} = \left(\zeta_p^p\right)^l \times \zeta_p^i = \zeta_p^i$$

Therefore

$$\prod_{i=0}^{p-1}\left(1+t\zeta_p^{lp+i}\right)=\prod_{i=0}^{p-1}\left(1+t\zeta_p^{i}\right)$$

Hence we have

$$f(t,\zeta_p^i)=\prod_{l=0}^{k-1}\prod_{i=0}^{p-1}\left(1+t\zeta_p^{lp+i}\right)=\prod_{l=0}^{k-1}\prod_{i=0}^{p-1}\left(1+t\zeta_p^{i}\right)=\left[\prod_{i=0}^{p-1}(1+t\zeta_p^i)\right]^k$$

Hence we have the lemma. □

Hence we have

$$g(t)=\frac{1}{p}\sum_{i=0}^{p-1}f(t,\zeta_p^i)=\frac{1}{p}\left[f(t,1)+\sum_{i=1}^{p-1}\left[\prod_{i=0}^{p-1}(1+t\zeta_p^i)\right]^k\right]=\frac{1}{p}\left[(1+t)^{kp}+(p-1)\left[\prod_{i=0}^{p-1}(1+t\zeta_p^i)\right]^k\right]$$

By Lemma 8 $g(t)$ is actually independent of ζ_p. Therefore in $g(t)$ the coefficient of t^i is the sum of coefficients of $s(i,j)$ for all j such that $p \mid j$. Hence

$$g(t)=\sum_{i=0}^{kp}\left[\sum_{j,\ p|j}s(i,j)\right]t^i$$

Now again by Lemma 8 the sum of the coefficients of the powers of t that are multiple of p is $\frac{1}{p}\sum\limits_{i=0}^{p-1}g\left(\zeta_p^i\right)$.

$$\begin{aligned}
\frac{1}{p}\sum_{i=0}^{p-1}g\left(\zeta_p^i\right)&=\frac{1}{p}\sum_{i=0}^{p-1}\frac{1}{p}\left[(1+\zeta_p^i)^{kp}+(p-1)\left[\prod_{j=0}^{p-1}(1+\zeta_p^{i+j})\right]^k\right] && \text{[Lemma 9]}\\
&=\frac{1}{p^2}\sum_{i=0}^{p-1}(1+\zeta_p^i)^{kp}+\frac{p-1}{p^2}\sum_{i=0}^{p-1}\left[\prod_{j=0}^{p-1}(1+\zeta_p^{i+j})\right]^k\\
&=\frac{1}{p^2}\sum_{i=0}^{p-1}(1+\zeta_p^i)^{kp}+\frac{p-1}{p^2}\sum_{i=0}^{p-1}\left[\prod_{j=0}^{p-1}(1+\zeta_p^{j})\right]^k\\
&=\frac{1}{p^2}\sum_{i=0}^{p-1}(1+\zeta_p^i)^{kp}+\frac{p-1}{p^2}\,p\left[\prod_{i=0}^{p-1}(1+\zeta_p^{i})\right]^k\\
&=\frac{1}{p^2}\sum_{i=0}^{p-1}(1+\zeta_p^i)^{kp}+\frac{p-1}{p}\left[\prod_{i=0}^{p-1}(1+\zeta_p^{i})\right]^k
\end{aligned}$$

Now we will analyze case wise:

- **Case 1:** p is odd prime. Then we have

$$\left[\prod_{i=0}^{p-1}(1+\zeta_p^i)\right]^k=2^k$$

Hence we have

$$\frac{1}{p^2}\sum_{i=0}^{p-1}(1+\zeta_p^i)^{kp}+\frac{p-1}{p}\left[\prod_{i=0}^{p-1}(1+\zeta_p^i)\right]^k=\frac{1}{p^2}\sum_{i=0}^{p-1}(1+\zeta_p^i)^{kp}+\frac{p-1}{p}2^k$$

- **Case 2:** $p = 2$. Then $\zeta_2 = -1$. Hence

$$\frac{1}{2^2}\sum_{i=0}^{1}(1+(-1)^i)^{2k}+\frac{1}{2}\left[\prod_{i=0}^{1}(1+(-1)^i)\right]^k = \frac{1}{2^2}(1+(-1)^0)^{2k}+\frac{1}{2}\left[\prod_{i=0}^{1}(1+(-1)^i)\right]^k = 2^{2k-2}+0 = 4^{k-1}$$

Therefore the number of was to get both the number of marbles sold and the total cost of them divisible by p is 4^{k-1} is $p=2$ and $\frac{1}{p^2}\sum_{i=0}^{p-1}(1+\zeta_p^i)^{kp} + \frac{p-1}{p}2^k$ if p is an odd prime. ■

[I discussed with Vivek]

Problem 5

A graph is a thunderstorm graph if every connected component is either a cycle (clouds) or a path (lightning bolts) or isolated vertices (raindrops). Compute an explicit formula, i.e., a formula without summation signs, for the exponential generating function of:

- The number of connected thunderstorm graphs on the vertex set $[n]$.
- The number of thunderstorm graphs on the vertex set $[n]$.

Solution:

- Suppose $n \geq 3$. Since the graph is connected it can be either a path or a cycle. Now from a path with $[n]$ as vertices we get two permutations since if $\sigma_1 \to \sigma_2 \to \cdots \sigma_n$ is a path then the we get the permutations $(\sigma_1, \dots, \sigma_n)$ and $(\sigma_n, \dots, \sigma_1)$. From each each path we get different sets of permutations. Now number of possible permutations of $[n]$ is $n!$. Hence there are total $\frac{n!}{2}$ many paths with vertices $[n]$.

  Now for each cycle of vertices $[n]$ gives n distinct permutations since $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_n \to \sigma_1$, cycle gives the permutations $(\sigma_i, \sigma_{i+1}, \dots, \sigma_n, \sigma_1, \dots, \sigma_{i-1})$ for all $i \in [n]$. For no two cycles we get same set of permutations. Now if we take a mirror reflection of a cycle we get the same cycle in the graph but a different circular permutation. Hence total number of cycles is $\frac{n!}{2n} = \frac{(n-1)!}{2}$.

  Hence there are total $\frac{n!}{2} + \frac{(n-1)!}{2}$ many thunderstorm graphs for $n \geq 3$. Now for $n = 1, 2$ there is only one graph is possible which is a path. Now define $a_n :=$ number of thunderstorm graphs on the vertex set $[n]$ for all $n \in \mathbb{N}$, $n \geq 3$. For $n = 1, 2$ we have $a_1 = 1 = a_2$. For $n = 0$ we have no graphs so $a_0 = 0$. So the exponential generating function, $g(x)$ is

$$\begin{aligned}
\sum_{i=0}^{\infty} a_n \frac{x^n}{n!} = \sum_{i=1}^{\infty} a_n \frac{x^n}{n!} & \\
&= x + \frac{x^2}{2!} + \sum_{n=3}^{\infty}\left[\frac{n!}{2} + \frac{(n-1)!}{2}\right]\frac{x^n}{n!} \\
&= x + \frac{x^2}{2!} + \frac{1}{2}\sum_{n=3}^{\infty} x^n + \frac{1}{2}\sum_{n=3}^{\infty}\frac{x^n}{n} \\
&= \frac{x}{2} - \frac{1}{2} + \frac{1}{2}\left[\sum_{n=0}^{\infty} x^n\right] + \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{x^n}{n}\right] - \frac{x}{2} - \frac{x^2}{4} \\
&= \frac{1}{2(1-x)} - \frac{1}{2}\ln(1-x) - \frac{x^2}{4} - \frac{1}{2} \qquad \left[\int \frac{dx}{1-x} = \sum_{n=0}^{\infty}\int x^n dx = \sum_{n=1}^{\infty}\frac{x^n}{n}\right]
\end{aligned}$$

- Fix n. The vertices are $[n]$. Suppose there are k connected components. Now we will count the number of thunderstorm graphs with vertices $[n]$ with k connected components. Suppose in an instance the size of the components are $c_1, \dots, c_k$. The number of ways to partition of $[n]$ into k

partitions of sizes $c_1, \ldots, c_k$ is $\frac{n!}{k! \prod_{i=1}^{k} c_i!}$. Now if we take $\frac{n!}{l!} Coeff_{x^n}\left(g^l(x)\right)$ this computes the number of thunder storm graphs with vertices $[n]$ and k components since $g(x)$ has the number of connected thunderstorm graphs of size i divided by $i!$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n!}\left[\sum_{l=1}^{n} \frac{n!}{l!} Coeff_{x^n}\left(g^l(x)\right)\right] x^n &= \sum_{n=0}^{\infty}\left[\sum_{l=1}^{n} \frac{1}{l!} Coeff_{x^n}\left(g^l(x)\right)\right] x^n \\
&= \sum_{n=0}^{\infty}\left[\sum_{l=1}^{n} Coeff_{x^n}\left(\frac{g(x)^l}{l!}\right)\right] x^n \\
&= \sum_{n=0}^{\infty} Coeff_{x^n}\left(\sum_{l=1}^{n} \frac{g(x)^l}{l!}\right) x^n \\
&= \sum_{n=0}^{\infty} Coeff_{x^n}\left(\sum_{l=1}^{\infty} \frac{g(x)^l}{l!}\right) x^n \\
&= \sum_{n=0}^{\infty} Coeff_{x^n}\left(e^{g(x)}-1\right) x^n = e^{g(x)}-1
\end{aligned}$$

Hence the exponential generating function is

$$e^{g(x)}-1 = e^{\frac{1}{2(1-x)}-\frac{1}{2}\ln(1-x)-\frac{x^2}{4}-\frac{1}{2}}-1 = \frac{e^{\frac{1}{2(1-x)}-\frac{x^2}{4}-\frac{1}{2}}}{\sqrt{1-x}}$$

■

[Me and Soumyadeep came up with the solution together]