Nisan's Pseudorandom Generator for RL

$\mathsf{BPL} \subseteq \mathsf{SC} = \mathsf{DTISP}(\mathrm{poly}(n), \log^2(n))$
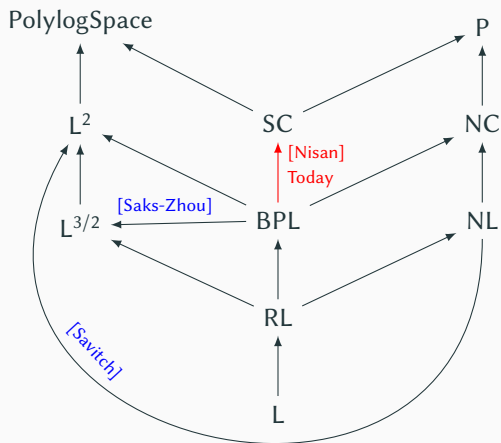
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December 2, 2025

Complexity Classes

- L: Deterministic Logarithmic Space.

- L^α, $\alpha > 0$: Set of problems decidable in $O(\log^\alpha n)$ space deterministically.

- NL: Nondeterministic Logarithmic Space.

- RL: Randomized Logarithmic Space with One-sided error $\frac{1}{3}$.

- BPL: Randomized Logarithmic Space with Two-sided error $\frac{1}{3}$.

- SC: Steve's Class or $DTISP(\text{poly}(n), \text{poly}(\log n))$ i.e. set of problems decidable deterministically in polynomial time and polylog space.

- NC: Nick's Class i.e. set of problems decidable in circuits of polynomial size and polylog depth and bounded fan-in.

Remark

Don't confuse SC with P $\cap$ PolylogSpace!

Definition (Pseudorandom Generator)

A map $\mathcal{G} : \{0, 1\}^l \to \{0, 1\}^n$, where $n \geq l$ is called a PRG for a class $\mathcal{C}$ with a parameter $\epsilon > 0$ if for any $f \in \mathcal{C}$,

$$\left| \mathbb{P}_{y \in \{0,1\}^n}[f(y) = 1] - \mathbb{P}_{x \in \{0,1\}^l}[f(\mathcal{G}(x)) = 1] \right| \leq \epsilon$$

- Here l is called the **seed-length** of the PRG.
- $n - l$ is called the **stretch** of the PRG.
- We call $\mathcal{G}$, ϵ-fools $\mathcal{C}$.
- Typically, we want $n >> l$ and $\mathcal{G}$ to be efficiently computable.

Finite State Automata

Let T be a BPL machine which uses n^c random bits on inputs of length n and runs in polynomial time and uses $S = O(\log n)$ space.

- There are at most $N := 2^{O(S)} = \text{poly}(n)$ configurations of T.
- Each random bit is used to make a transition between two configurations.
- The starting configuration is fixed for any input.
- Input x is accepted if T reaches a state representing acceptance.

Therefore the configuration graph of T on input x represents a finite state automata with N states.

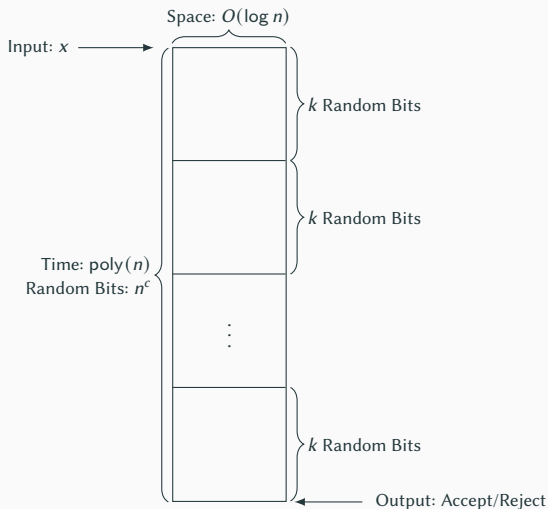# Computational Tableau of BPL machine

Figure 1: Computational Tableau of BPL machine T.

Dividing BPL Computation into Blocks

- Let the BPL machine T run in time $T(n) = \text{poly}(n)$ using n^c random bits and $O(\log n)$ space on input x of length n.
- Let $k << n^c$ be a parameter to be fixed later.
- Divide the computation of T into $t = n^c/k$ blocks, where each block uses k random bits.
- We can treat each block as a separate BPL machine T_i (in some sense) which takes input as the final configuration of T_{i-1} and k random bits.

- We can think of T as a finite state automata with N states.

- Each state makes 2^k transitions where each transion corresponds to a choice of k random bits.

- Let Q be this automata. For transition we write $Q(i; r) = j$ for any $r \in \{0, 1\}^k$ if Q goes from state i to state j when fed r.

- Let M be the matrix where $M[i, j] = \underset{r \in \{0,1\}^k}{\mathbb{P}} [Q(i; r) = j]$

$$\underset{r \in \{0,1\}^{k \times t}}{\mathbb{P}} [T(x, r) = \text{Acc}] = \sum_{j:\text{Accepting state}} M^t[1, j]$$

Goal: Approximate M^t using PRG.

Approximate Automata Matrix

Suppose we have a pseudorandom generator $\mathcal{G} : \{0, 1\}^k \rightarrow \{0, 1\}^{k \cdot t}$. Let Q be a finite state automata with N states and its matrix be M as defined above.

- From definition of M, $M^t[i, j] = \underset{r_1, \ldots, r_t \in \{0,1\}^k}{\mathbb{P}} [Q(i; r_1 \ldots ; r_t) = j]$
- Using $\mathcal{G}$, let $M_\mathcal{G}[i, j] = \underset{r \in \{0,1\}^k}{\mathbb{P}} [Q(i; \mathcal{G}(r)) = j]$
- Want to construct $\mathcal{G}$ such that

$$\|M^t - M_\mathcal{G}\| < \epsilon$$

for small ϵ.

Then if T decides a language with error probability at most $\frac{1}{3}$, using $\mathcal{G}$ we can calculate the $\sum_{j:\text{Accepting state}} M_\mathcal{G}[1, j]$ and decide the language if it is at least $\frac{2}{3} - \epsilon$.

For any vector $v \in \mathbb{R}^N$, define $\|v\| = \sum_{i \in [N]} |v(i)|$. Then for any matrix $M \in \mathbb{R}^{N \times N}$ define $\|M\| = \sup\limits_{0 \neq v \in \mathbb{R}^N} \frac{\|Mv\|}{\|v\|}$

Properties:

- $\|M\| \leq \max\limits_{i \in [N]} \sum_{j \in [N]} |M[i,j]|$

- $\|M + N\| \leq \|M\| + \|N\|$

- $\|MN\| \leq \|M\| \cdot \|N\|$

- For our use-case M is row-stochastic, so $\|M\| = 1$

Definition (Universal Hash Family (Carter-Wegman))

$\mathcal{H} = \{h : \{0,1\}^k \to \{0,1\}^m\}$ is a *universal hash family* if for any $x_1 \neq x_2 \in \{0,1\}^k$ and $y_1, y_2 \in \{0,1\}^m$,

- $\underset{h \in \mathcal{H}}{\mathbb{P}}[\, h(x_1) = y_1 \,] \;=\; \frac{1}{2^m}$

- $\underset{h \in \mathcal{H}}{\mathbb{P}}[\, h(x_1) = y_1 \;\wedge\; h(x_2) = y_2 \,] \;=\; \frac{1}{2^{2m}}$

- For our purpose, we have $k = m$.
- We can construct such a family with $|\mathcal{H}| = 2^{O(k)}$ where each $h \in \mathcal{H}$ can be represented using $O(k)$ bits and evaluated in $\mathsf{poly}(k)$ time over $GF(2^k)$.

Property of Universal Hash Family

Definition ((ϵ, A, B)-good hash function)

Let $A \subseteq \{0, 1\}^k$, $B \subseteq \{0, 1\}^m$, $\epsilon > 0$, $h : \{0, 1\}^k \rightarrow \{0, 1\}^m$ is said to be
(ϵ, A, B)-good if

$$\left| \mathop{\mathbb{P}}_{x \in \{0,1\}^k} [x \in A \wedge h(x) \in B] - \alpha \cdot \beta \right| \leq \epsilon$$

where $\alpha = \frac{|A|}{2^k}$ and $\beta = \frac{|B|}{2^m}$.

Lemma (Proved in Appendix)

If $\mathcal{H}$ is a universal hash family, then for any $A \subseteq \{0, 1\}^k$, $B \subseteq \{0, 1\}^m$, $\epsilon > 0$,

$$\mathop{\mathbb{P}}_{h \in \mathcal{H}} [h \text{ is not } (\epsilon, A, B)\text{-good}] \leq \frac{\alpha \cdot \beta}{2^k \epsilon^2}$$

Nisan's Generator

Let $\mathcal{H}$ be an universal hash family from $\{0,1\}^k$ to $\{0,1\}^k$. For any integer $m \geq 0$ define the function $\mathcal{G}_m \colon \{0,1\}^k \times \mathcal{H}^m \to \{0,1\}^{k \cdot 2^m}$ recursively as follows:

- $\mathcal{G}_0(x) = x$
- $\mathcal{G}_m(x, h_1, \ldots, h_m) = (\mathcal{G}_{m-1}(x, h_1, \ldots, h_{m-1}), \mathcal{G}_{m-1}(h_m(x), h_1, \ldots, h_{m-1}))$

For example:

$$\mathcal{G}_1(x, h) = (x, h(x)), \quad \mathcal{G}_2(x, h_1, h_2) = (x, h_1(x), h_2(x), h_1 \circ h_2(x))$$

$$\mathcal{G}_3(x, h_1, h_2, h_3) = (x, h_1(x), h_2(x), h_1 \circ h_2(x),$$
$$h_3(x), h_1 \circ h_3(x), h_2 \circ h_3(x), h_1 \circ h_2 \circ h_3(x))$$

- We want $k \cdot 2^m = n^c \implies m = \log t$.
- This gives a stretch of $k \cdot (t - 1)$ bits.

Let $h_1, \ldots, h_s$ be some fixed hash functions from $\mathcal{H}$. Define the matrix

$$M_{h_1,\ldots,h_s}[i,j] = \mathop{\mathbb{P}}_{x \in \{0,1\}^k} [Q(i; \mathcal{G}_s(x, h_1, \ldots, h_s)) = j]$$

- Using $h_1, \ldots, h_s$ we had 2^s many transitions in Q. So we should compare $M_{h_1,\ldots,h_s}$ with M^{2^s}.

Goal: For 'good' choice of $h_1, \ldots, h_m$, $\|M^{2^m} - M_{h_1,\ldots,h_m}\| < \epsilon$

Approach:

Step 1: Suppose we have $h_1, \ldots, h_{s-1}$. We will find $h_s \in \mathcal{H}$ such that for all $i, j \in [N]$,

$$\left\| M^2_{h_1,\ldots,h_{s-1}} - M_{h_1,\ldots,h_s} \right\| \le \delta$$

Step 2: Using above property will show for all $s \in [m]$,

$$\left\| M_{h_1,\ldots,h_s} - M^{2^s} \right\| \le (2^s - 1)\delta$$

Find good h_s from $h_1, \ldots, h_{s-1}$

Suppose we have $h_1, \ldots, h_{s-1} \in \mathcal{H}$ such that,

$$\left\| M_{h_1, \ldots, h_{l_{s-1}}} - \mathcal{M}^{2^{s-1}} \right\| \leq (2^{s-1} - 1)\delta$$

If we can find h_s such that $\| M^2_{h_1, \ldots, h_{s-1}} - M_{h_1, \ldots, h_s} \| \leq \delta$ then we are done.

Algorithm (FIND): Go over all $h \in \mathcal{H}$ and all $i, j \in [N]$:

Step 1: Compute

- $p_1 = M_{h_1, \ldots, h_{s-1}, h}[i, j]$
- $p_2 = \sum_{l \in [N]} M_{h_1, \ldots, h_{s-1}}[i, l] \cdot M_{h_1, \ldots, h_{s-1}}[l, j]$

Step 2: Check if $|p_1 - p_2| > \frac{\delta}{N}$ go to next h else return h.

> **Remark**
>
> To compute $M_{h_1, \ldots, h_{s-1}, h}[i, j]$ it goes over all $r \in \{0, 1\}^k$ and compute $\mathcal{G}_s(r; h)1, \ldots, h_s)$ and counts how many r gives $Q(i; \mathcal{G}_s(r, h_1, \ldots, h_s)) = j$ by simulating T and return $count/2^k$.

Claim

There exists an $h_s \in \mathcal{H}$ such that for all $i, j \in [N]$,

$$\left| M_{h_1,\ldots,h_{s-1},h_s}[i,j] - M^2_{h_1,\ldots,h_{s-1}}[i,j] \right| \le \frac{\delta}{N}$$

Let $A_{i,j}$ be the set of $r \in \{0,1\}^k$ such that $Q(i; \mathcal{G}_{s-1}(r, h_1, \ldots, h_{s-1})) = j$. So $M_{h_1,\ldots,h_{s-1}}[i,j] = \rho(A_{i,j})$ where $\rho(A) = |A|/2^k$.

- For any $i, j \in [N]$,

$$M^2_{h_1,\ldots,h_{s-1}}[i,j] = \sum_{l \in [N]} \rho(A_{i,l}) \cdot \rho(A_{l,j})$$

- For any $h \in \mathcal{H}$,

$$M_{h_1,\ldots,h_{s-1},h}[i,j] = \sum_{l \in [N]} \mathop{\mathbb{P}}_{r \in \{0,1\}^k} \left[r \in A_{i,l} \wedge h(r) \in A_{l,j} \right]$$

For a random $h \in \mathcal{H}$ with probability at least $1 - \frac{N^4}{2^k \delta^2} \geq 1 - \frac{1}{2n^3}$,

$$\left| \mathbb{P}_{r \in \{0,1\}^k} [r \in A_{i,l} \wedge h(r) \in A_{l,j}] - \rho(A_{i,l}) \cdot \rho(A_{l,j}) \right| \leq \frac{\delta}{N^2}$$

So by Union Bound random $h \in \mathcal{H}$, $(\frac{\delta}{N^2}, A, B)$ -good for all A, B with probability at least $\frac{1}{2}$.

$$\left| \mathcal{M}^2_{h_1,\ldots,h_{s-1}}[i,j] - M_{h_1,\ldots,h_{s-1},h}[i,j] \right|$$

$$\leq \sum_{l \in [N]} \left| \mathbb{P}_{r \in \{0,1\}^k} [r \in A_{i,l} \wedge h(r) \in A_{l,j}] - \rho(A_{i,l}) \cdot \rho(A_{l,j}) \right|$$

$$\leq N \cdot \frac{\delta}{N^2} = \frac{\delta}{N}$$

Claim

If h_s is returned by the above algorithm, then

$$\left\| M_{h_1,\ldots,h_s} - M^{2^s} \right\| \leq (2^s - 1)\delta$$

We have $\| M^2_{h_1,\ldots,h_{s-1}} - M_{h_1,\ldots,h_s} \| \leq \delta$.

$$\| M_{h_1,\ldots,h_s} - M^{2^s} \| \leq \| M_{h_1,\ldots,h_s} - M^2_{h_1,\ldots,h_{s-1}} \| + \| M^2_{h_1,\ldots,h_{s-1}} - M^{2^s} \|$$

$$\| M^2_{h_1,\ldots,h_{s-1}} - M^{2^s} \| \leq \| M_{h_1,\ldots,h_{s-1}} \| \cdot \| M_{h_1,\ldots,h_{s-1}} - M^{2^{s-1}} \|$$
$$+ \| M_{h_1,\ldots,h_{s-1}} - M^{2^{s-1}} \| \cdot \| M^{2^{s-1}} \|$$
$$\leq 1 \cdot (2^{s-1} - 1)\delta + (2^{s-1} - 1)\delta \cdot 1 = (2^s - 2)\delta$$

- Set $k = \log(N) = O(\log n)$. So $t \approx n^c$.

- Set $m = \log t = O(\log n)$.

- Want $(2^m - 1)\delta = \epsilon \implies \delta = \frac{\epsilon}{2^m}$

Final Algorithm

- Compute $h_1, \ldots, h_m$ one by one using the algorithm FIND.
- Compute $A[i, j] = M_{h_1, \ldots, h_m}[i, j]$ for all $i, j \in [N]$.
- Compute $\sum_{j:\text{Accepting state}} A[1, j]$ and accept if this is at least $\frac{2}{3} - \epsilon$ else reject.

Space: The only place where more than $O(\log n)$ space is needed is to store the value of $h_1, \ldots, h_m$. And each h_i can be stored in $O(k) = O(\log n)$ space. So total space used is $O(\log^2 n)$.

Time: For all $s \in [m]$, computing $M_{h_1, \ldots, h_s}[i, j]$ takes $O(2^k)$ times computation of $\mathcal{G}_s(r, h_1, \ldots, h_s)$ for all r and to check if $Q(i; \mathcal{G}_s(r, h_1, \ldots, h_s)) = j$ which takes $O(2^m \cdot mk) \cdot T(n)$ time. So FIND takes $O(N^2 \cdot 2^{2m} \cdot 2^{m+k} \cdot mk) \cdot T(n)$. Hence total time $O(N^2 \cdot 2^{2m} \cdot 2^{m+k} \cdot mk) \cdot m \cdot T(n) = \text{poly}(n)$.

Thank You

Lemma

If $\mathcal{H}$ is a universal hash family, then for any $A \subseteq \{0, 1\}^k$, $B \subseteq \{0, 1\}^m$, $\epsilon > 0$,

$$\mathbb{P}_{h \in \mathcal{H}}[h \text{ is not } (\epsilon, A, B)\text{-good}] \leq \frac{\alpha\beta(1 - \beta)}{2^k \epsilon^2}$$

Consider the matrix $M \in \{0, 1\}^{2^k \times |\mathcal{H}|}$ where $M[x, h] = 1$ if $h(x) \in B$ and 0 otherwise. For any $x_1 \neq x_2 \in \{0, 1\}^k$, $\mathbb{E}_{h \in \mathcal{H}}[M[x_1, h]] = \beta$ and $\mathbb{E}_{h \in \mathcal{H}}[M[x_1, h]M[x_2, h]] = \beta^2$

$$\mathop{\mathbb{E}}_{h \in \mathcal{H}} \left[\left(\beta - \mathop{\mathbb{E}}_{x \in A} [M[x,h]] \right)^2 \right] = \mathop{\mathbb{E}}_{x_1,x_2 \in A} \mathop{\mathbb{E}}_{h \in \mathcal{H}} \left[(\beta - M[x_1,h])(\beta - M[x_2,h]) \right]$$

$$= \mathop{\mathbb{E}}_{x_1,x_2 \in A} \left[\beta^2 - \beta \mathop{\mathbb{E}}_{h \in \mathcal{H}} [M[x_1,h]] - \beta \mathop{\mathbb{E}}_{h \in \mathcal{H}} [M[x_1,h]] \right.$$
$$\left. + \mathop{\mathbb{E}}_{h \in \mathcal{H}} [M[x_1,h] \cdot M[x_2,h]] \right]$$

$$= \mathop{\mathbb{E}}_{x_1,x_2 \in A} \left[\mathop{\mathbb{E}}_{h \in \mathcal{H}} [M[x_1,h] \cdot M[x_2,h]] - \beta^2 \right]$$

- For $x_1 \neq x_2$: $\mathop{\mathbb{E}}_{h \in \mathcal{H}} [M[x_1,h] \cdot M[x_2,h]] = \beta^2$
- For $x_1 = x_2$: $\mathop{\mathbb{E}}_{h \in \mathcal{H}} [M[x_1,h] \cdot M[x_2,h]] = \mathop{\mathbb{E}}_{h \in \mathcal{H}} [M[x_1,h]] = \beta$.

So,
$$\mathop{\mathbb{E}}_{h \in \mathcal{H}}\left[\left(\beta - \mathop{\mathbb{E}}_{x \in A}[M[x,h]]\right)^2\right] = \frac{1}{|A|}(\beta - \beta^2) = \frac{\alpha\beta(1-\beta)}{2^k}$$

Now $\mathop{\mathbb{P}}_{x \in \{0,1\}^k}[x \in A \wedge h(x) \in B] = \alpha \mathop{\mathbb{P}}_{x \in A}[h(x) \in B] = \alpha \cdot \mathop{\mathbb{E}}_{x \in A}[M[x,h]]$. So h is not (ϵ, A, B) -good iff

$$\left|\mathop{\mathbb{E}}_{x \in A}[M[x,h]] - \beta\right| \geq \frac{\epsilon}{\alpha}$$

By Markov,

$$\mathop{\mathbb{P}}_{h \in \mathcal{H}}\left[\left|\beta - \mathop{\mathbb{E}}_{x \in A}[M[x,h]]\right| \geq \frac{\epsilon}{\alpha}\right] \leq \frac{\alpha\beta(1-\beta)}{2^k\epsilon^2}$$