

Nisan's Pseudorandom Generator for RL

$$\text{BPL} \subseteq \text{SC} = \text{DTISP}(\text{poly}(n), \log^2(n))$$

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Pseudorandomness Course (CSS.413.1) Presentation, STCS

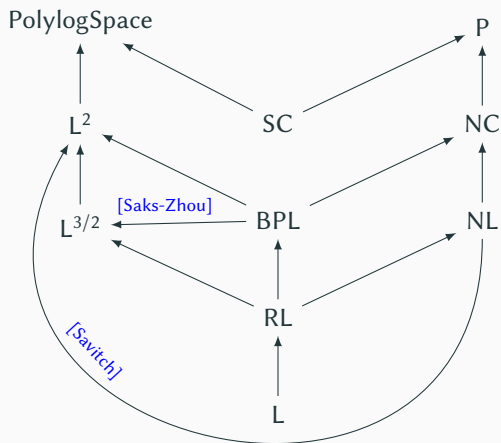
Complexity Classes

- **L**: Deterministic Logarithmic Space.
- **L^α**, $\alpha > 0$: Set of problems decidable in $O(\log^\alpha n)$ space deterministically.
- **NL**: Nondeterministic Logarithmic Space.
- **RL**: Randomized Logarithmic Space with One-sided error $\frac{1}{3}$.
- **BPL**: Randomized Logarithmic Space with Two-sided error $\frac{1}{3}$.
- **SC**: Steve's Class or **DTISP**(poly(n), poly(log n)) i.e. set of problems decidable deterministically in polynomial time and polylog space.
- **NC**: Nick's Class i.e. set of problems decidable in circuits of polynomial size and polylog depth and bounded fan-in.

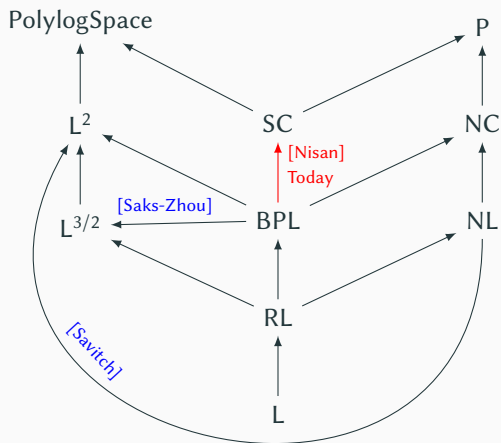
Remark

Don't confuse **SC** with $P \cap \text{PolylogSpace}$!

Complexity Classes Zoo



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Pseudorandom Generator

Definition (Pseudorandom Generator)

A map $\mathcal{G} : \{0, 1\}^l \rightarrow \{0, 1\}^n$, where $n \geq l$ is called a PRG for a class \mathcal{C} with a parameter $\epsilon > 0$ if for any $f \in \mathcal{C}$,

$$\left| \mathbb{P}_{y \in \{0, 1\}^n} [f(y) = 1] - \mathbb{P}_{x \in \{0, 1\}^l} [f(\mathcal{G}(x)) = 1] \right| \leq \epsilon$$

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- Here l is called the **seed-length** of the PRG.
- $n - l$ is called the **stretch** of the PRG.
- We call \mathcal{G} , ϵ -fools \mathcal{C} .
- Typically, we want $n \gg l$ and \mathcal{G} to be efficiently computable.

Finite State Automata

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- Each random bit is used to make a transition between two configurations.
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Therefore the configuration graph of T on input x represents a finite state automata with N states.

Computational Tableau of BPL machine

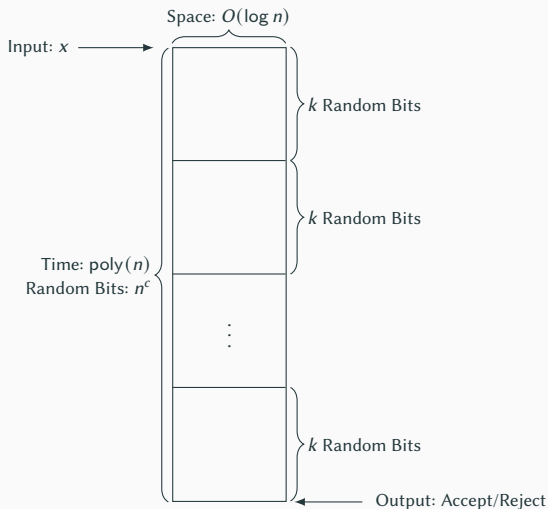


Figure 1: Computational Tableau of BPL machine T .

Dividing BPL Computation into Blocks

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- Divide the computation of T into $t = n^c/k$ blocks, where each block uses k random bits.
- We can treat each block as a separate BPL machine T_i (in some sense) which takes input as the final configuration of T_{i-1} and k random bits.

FSA for Computation Blocks

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$$\mathbb{P}_{r \in \{0, 1\}^{k \times t}} [T(x, r) = \text{Acc}] = \sum_{j: \text{Accepting state}} M^t[1, j]$$

Goal: Approximate M^t using PRG.

Approximate Automata Matrix

Suppose we have a pseudorandom generator $\mathcal{G} : \{0, 1\}^k \rightarrow \{0, 1\}^{k \cdot t}$. Let Q be a finite state automata with N states and its matrix be M as defined above.

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- Want to construct \mathcal{G} such that

$$\|M^t - M_{\mathcal{G}}\| < \epsilon$$

for small ϵ .

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Then if T decides a language with error probability at most $\frac{1}{3}$, using \mathcal{G} we can calculate the $\sum_{j: \text{Accepting state}} M_{\mathcal{G}}[1, j]$ and decide the language if it is at least $\frac{2}{3} - \epsilon$.

Matrix Norm

For any vector $v \in \mathbb{R}^N$, define $\|v\| = \sum_{i \in [N]} |v(i)|$. Then for any matrix $M \in \mathbb{R}^{N \times N}$ define $\|M\| = \sup_{0 \neq v \in \mathbb{R}^N} \frac{\|Mv\|}{\|v\|}$

Properties:

- $\|M\| \leq \max_{i \in [N]} \sum_{j \in [N]} |M[i, j]|$
- $\|M + N\| \leq \|M\| + \|N\|$
- $\|MN\| \leq \|M\| \cdot \|N\|$

Universal Hash Family

Definition (Universal Hash Family (Carter-Wegman))

$\mathcal{H} = \{h : \{0, 1\}^k \rightarrow \{0, 1\}^m\}$ is a *universal hash family* if for any $x_1 \neq x_2 \in \{0, 1\}^k$ and $y_1, y_2 \in \{0, 1\}^m$,

- $\mathbb{P}_{h \in \mathcal{H}} [h(x_1) = y_1] = \frac{1}{2^m}$
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- For our purpose, we have $k = m$.
- We can construct such a family with $|\mathcal{H}| = 2^{O(k)}$ where

$$\mathcal{H} = \{a \cdot x + b \mid a, b \in \{0, 1\}^k\}$$

over $GF(2^k)$.

Property of Universal Hash Family

Definition $((\epsilon, A, B)$ -good hash function)

Let $A \subseteq \{0, 1\}^k$, $B \subseteq \{0, 1\}^m$, $\epsilon > 0$, $h : \{0, 1\}^k \rightarrow \{0, 1\}^m$ is said to be (ϵ, A, B) -good if

$$\left| \mathbb{P}_{x \in \{0, 1\}^k} [x \in A \wedge h(x) \in B] - \alpha \cdot \beta \right| \leq \epsilon$$

where $\alpha = \frac{|A|}{2^k}$ and $\beta = \frac{|B|}{2^m}$.

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Lemma (Proved in Appendix)

If \mathcal{H} is a universal hash family, then for any $A \subseteq \{0, 1\}^k$, $B \subseteq \{0, 1\}^m$, $\epsilon > 0$,

$$\mathbb{P}_{h \in \mathcal{H}} [h \text{ is not } (\epsilon, A, B)\text{-good}] \leq \frac{\alpha \cdot \beta}{2^k \epsilon^2}$$

Nisan's Generator

Let \mathcal{H} be an universal hash family from $\{0, 1\}^k$ to $\{0, 1\}^k$. For any integer $m \geq 0$ define the function $\mathcal{G}_m: \{0, 1\}^k \times \mathcal{H}^m \rightarrow \{0, 1\}^{k \cdot 2^m}$ recursively as follows:

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- $\mathcal{G}_0(x) = x$
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For example:

$$\mathcal{G}_1(x, h) = (x, h(x)), \quad \mathcal{G}_2(x, h_1, h_2) = (x, h_1(x), h_2(x), h_1 \circ h_2(x))$$

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- We want $k \cdot 2^m = n^c \implies m = \log t$.
- This gives a stretch of $k \cdot (t - 1)$ bits.

Proof Flow

Let h_1, \dots, h_s be some fixed hash functions from \mathcal{H} . Define the matrix

$$M_{h_1, \dots, h_s}[i, j] = \mathbb{P}_{x \in \{0,1\}^k} [Q(i; \mathcal{G}_s(x, h_1, \dots, h_s)) = j]$$

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Approach:

Step 1: Suppose we have h_1, \dots, h_{s-1} . We will find $h_s \in \mathcal{H}$ such that for all $i, j \in [N]$,

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Step 2: Using above property will show for all $s \in [m]$,

$$\left\| M_{h_1, \dots, h_s} - M^{2^s} \right\| \leq (2^s - 1)\delta$$

Find good h_s from h_1, \dots, h_{s-1}

Suppose we have $h_1, \dots, h_{s-1} \in \mathcal{H}$ such that,

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If we can find h_s such that $\|M_{h_1, \dots, h_{s-1}}^2 - M_{h_1, \dots, h_s}\| \leq \delta$ then we are done.

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Step 1: Compute

- $p_1 = M_{h_1, \dots, h_{s-1}, h}[i, j]$
- $p_2 = \sum_{l \in [N]} M_{h_1, \dots, h_{s-1}}[i, l] \cdot M_{h_1, \dots, h_{s-1}}[l, j]$

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Step 2: Check if $|p_1 - p_2| > \frac{\delta}{N}$ go to next h else return h .

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Step 2: Check if $|p_1 - p_2| > \frac{\delta}{N}$ go to next h else return h .

Remark

To compute $M_{h_1, \dots, h_{s-1}, h}[i, j]$ it goes over all $r \in \{0, 1\}^k$ and compute $\mathcal{G}_s(r; h_1, \dots, h_s)$ and counts how many r gives $Q(i; \mathcal{G}_s(r, h_1, \dots, h_s)) = j$ by simulating T . and return $\text{count}/2^k$.

Claim

There exists an $h_s \in \mathcal{H}$ such that for all $i, j \in [N]$,

$$\left| M_{h_1, \dots, h_{s-1}, h_s} [i, j] - M_{h_1, \dots, h_{s-1}}^2 [i, j] \right| \leq \frac{\delta}{N}$$

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Let $A_{i,j}$ be the set of $r \in \{0, 1\}^k$ such that $Q(i; \mathcal{G}_{s-1}(r, h_1, \dots, h_{s-1})) = j$. So $M_{h_1, \dots, h_{s-1}} [i, j] = \rho(A_{i,j})$ where $\rho(A) = |A|/2^k$.

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For a random $h \in \mathcal{H}$ with probability at least $1 - \frac{N^4}{2^k \delta^2} \geq 1 - \frac{1}{2n^3}$,

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$$\begin{aligned} & \left| M_{h_1, \dots, h_{s-1}}^2[i, j] - M_{h_1, \dots, h_{s-1}, h}[i, j] \right| \\ & \leq \sum_{l \in [N]} \left| \mathbb{P}_{r \in \{0,1\}^k} [r \in A_{i,l} \wedge h(r) \in A_{l,j}] - \rho(A_{i,l}) \cdot \rho(A_{l,j}) \right| \\ & \leq N \cdot \frac{\delta}{N^2} = \frac{\delta}{N} \end{aligned}$$

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If h_s is returned by the above algorithm, then

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$$\begin{aligned} \|M_{h_1, \dots, h_{s-1}}^2 - M^{2^s}\| &\leq \|M_{h_1, \dots, h_{s-1}}\| \cdot \|M_{h_1, \dots, h_{s-1}} - M^{2^{s-1}}\| \\ &\quad + \|M_{h_1, \dots, h_{s-1}} - M^{2^{s-1}}\| \cdot \|M^{2^{s-1}}\| \end{aligned}$$

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$$\begin{aligned} \|M_{h_1, \dots, h_{s-1}}^2 - M^{2^s}\| &\leq \|M_{h_1, \dots, h_{s-1}}\| \cdot \|M_{h_1, \dots, h_{s-1}} - M^{2^{s-1}}\| \\ &\quad + \|M_{h_1, \dots, h_{s-1}} - M^{2^{s-1}}\| \cdot \|M^{2^{s-1}}\| \\ &\leq 1 \cdot (2^{s-1} - 1)\delta + (2^{s-1} - 1)\delta \cdot 1 = (2^s - 2)\delta \end{aligned}$$

Setting Parameters

- Set $k = \log(N) = O(\log n)$. So $t \approx n^c$.
- Set $m = \log t = O(\log n)$.
- Want $(2^m - 1)\delta = \epsilon \implies \delta = \frac{\epsilon}{2^m}$

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Space: The only place where more than $O(\log n)$ space is needed is to store the value of h_1, \dots, h_m . And each h_i can be stored in $O(k) = O(\log n)$ space. So total space used is $O(\log^2 n)$.

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Time: For all $s \in [m]$, computing $M_{h_1, \dots, h_s}[i, j]$ takes $O(2^k)$ times computation of $\mathcal{G}_s(r, h_1, \dots, h_s)$ for all r and to check if $Q(i; \mathcal{G}_s(r, h_1, \dots, h_s)) = j$ which takes $O(2^m \cdot mk) \cdot T(n)$ time. So FIND takes $O(N^2 \cdot 2^{2m} \cdot 2^{m+k} \cdot mk) \cdot T(n)$. Hence total time $O(N^2 \cdot 2^{2m} \cdot 2^{m+k} \cdot mk) \cdot m \cdot T(n) = \text{poly}(n)$.

Thank You

Appendix i

Lemma

If \mathcal{H} is a universal hash family, then for any $A \subseteq \{0, 1\}^k$, $B \subseteq \{0, 1\}^m$, $\epsilon > 0$,

$$\mathbb{P}_{h \in \mathcal{H}} [h \text{ is not } (\epsilon, A, B)\text{-good}] \leq \frac{\alpha\beta(1-\beta)}{2^k\epsilon^2}$$

Consider the matrix $M \in \{0, 1\}^{2^k \times |\mathcal{H}|}$ where $M[x, h] = 1$ if $h(x) \in B$ and 0 otherwise. For any $x_1 \neq x_2 \in \{0, 1\}^k$, $\mathbb{E}_{h \in \mathcal{H}} [M[x_1, h]] = \beta$ and

$$\mathbb{E}_{h \in \mathcal{H}} [M[x_1, h]M[x_2, h]] = \beta^2$$

Appendix ii

$$\begin{aligned}\mathbb{E}_{h \in \mathcal{H}} \left[\left(\beta - \mathbb{E}_{x \in A} [M[x, h]] \right)^2 \right] &= \mathbb{E}_{x_1, x_2 \in A} \mathbb{E}_{h \in \mathcal{H}} [(\beta - M[x_1, h])(\beta - M[x_2, h])] \\ &= \mathbb{E}_{x_1, x_2 \in A} \left[\beta^2 - \beta \mathbb{E}_{h \in \mathcal{H}} [M[x_1, h]] - \beta \mathbb{E}_{h \in \mathcal{H}} [M[x_2, h]] \right. \\ &\quad \left. + \mathbb{E}_{h \in \mathcal{H}} [M[x_1, h] \cdot M[x_2, h]] \right] \\ &= \mathbb{E}_{x_1, x_2 \in A} \left[\mathbb{E}_{h \in \mathcal{H}} [M[x_1, h] \cdot M[x_2, h]] - \beta^2 \right]\end{aligned}$$

- For $x_1 \neq x_2$: $\mathbb{E}_{h \in \mathcal{H}} [M[x_1, h] \cdot M[x_2, h]] = \beta^2$
- For $x_1 = x_2$: $\mathbb{E}_{h \in \mathcal{H}} [M[x_1, h] \cdot M[x_2, h]] = \mathbb{E}_{h \in \mathcal{H}} [M[x_1, h]] = \beta$.

Appendix iii

So,

$$\mathbb{E}_{h \in \mathcal{H}} \left[\left(\beta - \mathbb{E}_{x \in A} [M[x, h]] \right)^2 \right] = \frac{1}{|A|} (\beta - \beta^2) = \frac{\alpha \beta (1 - \beta)}{2^k}$$

Now $\mathbb{P}_{x \in \{0,1\}^k} [x \in A \wedge h(x) \in B] = \alpha \mathbb{P}_{x \in A} [h(x) \in B] = \alpha \cdot \mathbb{E}_{x \in A} [M[x, h]]$. So h is not (ϵ, A, B) -good iff

$$\left| \mathbb{E}_{x \in A} [M[x, h]] - \beta \right| \geq \frac{\epsilon}{\alpha}$$

By Markov,

$$\mathbb{P}_{h \in \mathcal{H}} \left[\left| \beta - \mathbb{E}_{x \in A} [M[x, h]] \right| \geq \frac{\epsilon}{\alpha} \right] \leq \frac{\alpha \beta (1 - \beta)}{2^k \epsilon^2}$$