Soham Chatterjee
Email: sohamc@cmi.ac.in
Course: Quantum Information Theory

Assignment - 2.2 : Quantum Foundations
Roll: BMC202175
Date: February 29, 2024

For all the questions

- $[k] := \{1, 2, \dots, k\}$ where $k \in \mathbb{N}$.
- $\mathscr{L}(\mathcal{H}) :=$ Linear operators on $\mathcal{H}$
- $\mathscr{R}(\mathcal{H}) :=$ Self-adjoint or hermitian operators on $\mathcal{H}$
- $\mathscr{P}(\mathcal{H}) :=$ Positive semi-definite operators on $\mathcal{H}$
- $\mathscr{D}(\mathcal{H}) :=$ Density operators on $\mathcal{H}$

Problem 1
For $T : \mathcal{H} \to \mathcal{H}$, prove that

$$\sum_{i=1}^{d} \langle e_i | Te_i \rangle = \sum_{i=1}^{d} \langle f_i | Tf_i \rangle$$

if $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ and $\{|f_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ are ONB.

Solution: Let $S : \mathcal{H} \to \mathcal{H}$ where it maps the basis vectors from $|e_i\rangle \to |f_i\rangle$. Then $S|e_i\rangle = |f_i\rangle$. Hence S is an unitary matrix since

$$\langle e_j | S^\dagger S | e_i \rangle = \langle f_j | f_i \rangle = \delta_{ji} \quad \text{and} \quad \langle f_j | SS^\dagger | f_i \rangle = \langle e_j | e_i \rangle = \delta_{ji}$$

Hence

$$\sum_{i=1}^{d} \langle f_i | Tf_i \rangle = \sum_{i=1}^{d} \langle e_i | S^\dagger TS | e_i \rangle = tr(S^\dagger TS) = tr(SS^\dagger T) = tr(T) = \sum_{i=1}^{d} \langle e_i | Te_i \rangle$$

Therefore we have

$$\sum_{i=1}^{d} \langle e_i | Te_i \rangle = \sum_{i=1}^{d} \langle f_i | Tf_i \rangle$$

$\square$

Problem 2
If $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \le i \le d\}$ and $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \le i \le d\}$ are ONB, then $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \le i, j \le d\} \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ is ONB

Solution: Let $|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then $|\psi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle$ where $\alpha_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \le i \le d\}$ is ONB for $\mathcal{H}_1$. Hence

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle \otimes |\phi\rangle$$

Now $|\phi\rangle = \sum_{i=1}^{d} \beta_i |f_i\rangle$ where $\beta_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \le i \le d\}$ is ONB for $\mathcal{H}_2$. Hence

$$\forall\, i \in [d] \;\; |e_i\rangle \otimes |phi\rangle = \sum_{j=1}^{d} \beta_j |e_i\rangle \otimes |f_j\rangle$$

Thereofore we get

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{d} \beta_j |e_i\rangle \otimes |f_j\rangle = \sum_{1\leq i,j\leq d} \alpha_i\beta_j |e_i\rangle \otimes |f_j\rangle$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i,j \leq d\}$ is a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Now for any $i1, i2, j1, j2 \in [d]$

$$(\langle e_{i1}| \otimes \langle f_{j1}|)(|e_{i2}\rangle \otimes |f_{j2}\rangle) = \langle e_{i1}|e_{i2}\rangle \langle f_{j1}|f_{j2}\rangle = \delta_{i1,i2}\,\delta_{j1,j2}$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i,j \leq d\}$ is orthonormal. Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i,j \leq d\}$ is a ONB for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

□

Problem 3

Let $\{|g_k\rangle \mid 1 \leq k \leq d_2\} \subseteq \mathcal{H}_2$ be ONB. For $T \in \mathscr{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, let $tr_2(T) \in \mathscr{L}(\mathcal{H}_1)$ denote the operator satisfying

$$\langle u|\, tr_2(T)\, |v\rangle = \sum_k \langle u \otimes g_k|\, T\, |v \otimes g_k\rangle$$

for any choice $|u\rangle, |v\rangle \in \mathcal{H}_1$. Prove that $\sum_k \langle u \otimes g_k|\, T\, |v \otimes g_k\rangle$ is invariant with choice of ONB $\{|g_k\rangle \mid 1 \leq k \leq K\} \subseteq \mathcal{H}_2$. Prove that $tr_2(T)$ is well defined. Specify $tr_2(T)$ by choosing ONB on $\mathcal{H}_1$, $\mathcal{H}_2$ respectively.

Solution:

- Let $\{|f_k\rangle \mid 1 \leq k \leq d_2\} \subseteq \mathcal{H}_2$ be another ONB. Suppose $S : \mathcal{H}_2 \to \mathcal{H}$ be a map such that $S|g_k\rangle = |f_k\rangle$. As we previously showed in Problem 1, S is unitary. Then for all $k \in [d_2]$ we have

$$|f_k\rangle = \sum_{i=1}^{d_2} w_{i,k} |e_i\rangle$$

where $w_{i,k} \in \mathbb{C}$. Hence

$$\langle f_i|\, S^\dagger S\, |f_j\rangle = \sum_{k=1}^{d_2} w_{i,k}^* w_{j,k} = \delta_{i,j}$$

Now for any $|u\rangle, |v\rangle \in \mathcal{H}_1$ we have

$$\begin{aligned}
\langle u|\, tr_2(T)\, |v\rangle_{\{|f_k\rangle\}} &= \langle u| \left[\sum_{k=1}^{d_2} (I \otimes \langle f_k|) T (I \otimes |f_k\rangle)\right] |v\rangle \\
&= \langle u| \left[\sum_{k=1}^{d_2} \left(I \otimes \left(\sum_{i=1}^{d_2} w_{i,k}^* \langle g_i|\right)\right) T \left(I \otimes \left(\sum_{j=1}^{d_2} w_{j,k} |g_j\rangle\right)\right)\right] |v\rangle \\
&= \sum_{k=1}^{d_2}\sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \langle u| \left[w_{i,k}^* w_{j,k} (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \langle u| \left[\left(\sum_{k=1}^{d_2} w_{i,k}^* w_{j,k}\right) (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \langle u| \left[\delta_{i,j} (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2} \langle u| \left[(I \otimes \langle g_i|) T (I \otimes |g_i\rangle)\right] |v\rangle \\
&= \langle u|\, tr_2(T)\, |v\rangle_{\{|g_k\rangle\}}
\end{aligned}$$

Hence $\sum_k \langle u \otimes g_k|\, T\, |v \otimes g_k\rangle$ is invariant.

- Let $T = \sum\limits_{i,j} \gamma_{i,j} A_i \otimes B_j$ where $\gamma_{i,j} \in \mathbb{C}$ and $A_i \in \mathscr{L}(\mathcal{H}_1)$, $B_j \in \mathscr{L}(\mathcal{H}_2)$. Then

$$\begin{aligned}
tr_2(T) = tr_2\left(\sum_{i,j} \gamma_{i,j} A_i \otimes B_j\right) &= \sum_{i,j} \gamma_{i,j} tr_2(A_i \otimes B_j) \\
&= \sum_{i,j} \gamma_{i,j} A_i tr(B_j) \\
&= \sum_{i,j} A_i \sum_k \langle g_k | B_j | g_k \rangle \\
&= \sum_{i,j} \sum_k \gamma_{i,j} (I \otimes \langle g_k |) A_i \otimes B_j (I \otimes | g_K \rangle) \\
&= \sum_k (I \otimes \langle g_k |) \left[\sum_{i,j} \gamma_{i,j} A_i \otimes B_j\right] (I \otimes | g_K \rangle) \\
&= \sum_k (I \otimes \langle g_k |) T (I \otimes | g_k \rangle)
\end{aligned}$$

So if we get two different representations of T, $T = \sum\limits_{i,j} \gamma_{i,j} A_i \otimes B_j$ and $T = \sum\limits_{i,j} \gamma'_{i,j} A'_i \otimes B'_j$ still for each of them $tr_2(T)$ will be same since it doesn't matter which representation is finally taken. Therefore $tr_2(T)$ is well defined.

- Now take $|u\rangle, |v\rangle \in \mathcal{H}_1$. Then

$$\begin{aligned}
\langle u | tr_2(T) | v \rangle = \langle u | &\left[\sum_k (I \otimes \langle g_k |) T (I \otimes | g_k \rangle)\right] | v \rangle \\
&= \sum_k \langle u | (I \otimes \langle g_k |) T (I \otimes | g_k \rangle) | v \rangle = \sum_k \langle u \otimes g_k | T | v \otimes g_k \rangle
\end{aligned}$$

□

Problem 4 Mark Wilde: Exercise 3.3.3

Show that the Pauli matrices are all Hermitian, unitary, they square to the identity, and their eigenvalues are ± 1

Solution: Pauli matrices are

$$X|0\rangle = |1\rangle, X|1\rangle = |0\rangle \qquad Y|0\rangle = -i|1\rangle, Y|1\rangle = i|0\rangle \qquad Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle$$

Therefore we have

$$X = |1\rangle\langle 0| + |0\rangle\langle 1| \qquad Y = i[|0\rangle\langle 1| - |1\rangle\langle 0|] \qquad Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Hence

$$\begin{aligned}
X^\dagger &= (|1\rangle\langle 0|)^\dagger + (|0\rangle\langle 1|)^\dagger = |0\rangle\langle 1| + |1\rangle\langle 0| = X \\
Y^\dagger &= (i|0\rangle\langle 1|)^\dagger + (-i|1\rangle\langle 0|)^\dagger = -i|1\rangle\langle 0| + i|0\rangle\langle 1| = Y \\
Z^\dagger &= (|0\rangle\langle 0|)^\dagger - (|1\rangle\langle 1|)^\dagger = |0\rangle\langle 0| - |1\rangle\langle 1| = Z
\end{aligned}$$

Therefore they are Hermitian.

Now

$$\begin{aligned}
X^\dagger X = XX^\dagger = X^2 &= [|1\rangle\langle 0| + |0\rangle\langle 1|][|1\rangle\langle 0| + |0\rangle\langle 1|] \\
&= |1\rangle\langle 0|1\rangle\langle 0| + |1\rangle\langle 0|0\rangle\langle 1| + |0\rangle\langle 1|1\rangle\langle 0| + |0\rangle\langle 1|0\rangle\langle 1| \\
&= |1\rangle\langle 1| + |0\rangle\langle 0| = I
\end{aligned}$$

$$
\begin{aligned}
Y^\dagger Y = Y^\dagger = Y^2 &= [i(|0\rangle\langle 1| - |1\rangle\langle 0|)]\,[i(|0\rangle\langle 1| - |1\rangle\langle 0|)] \\
&= -\left[|0\rangle\langle 1|0\rangle\langle 1| - |0\rangle\langle 1|1\rangle\langle 0| - |1\rangle\langle 0|0\rangle\langle 1| + |1\rangle\langle 0|1\rangle\langle 0|\right] \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = I
\end{aligned}
$$

$$
\begin{aligned}
Z^\dagger Z = Z^\dagger = Z^2 &= [|0\rangle\langle 0| - |1\rangle\langle 1|]\,[|0\rangle\langle 0| - |1\rangle\langle 1|] \\
&= |0\rangle\langle 0|0\rangle\langle 0| - |0\rangle\langle 0|1\rangle\langle 1| - |1\rangle\langle 1|0\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 1| \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = I
\end{aligned}
$$

Therefore X, Y, Z are unitary and they square to the identity.

Since $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$ we have

$$
X\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \quad X\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
$$

So the for the eigenvalue 1 the corresponding eignevector is $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and for the eigenvalue -1 the corresponding eigenvalue is $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Since $Y|0\rangle = -i|1\rangle$ and $Y|1\rangle = i|0\rangle$ we have

$$
Y\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\left(-i|1\rangle + i^2|0\rangle\right) = -\frac{1}{\sqrt{2}}(i|1\rangle + |0\rangle)
$$

$$
Y\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = \frac{1}{\sqrt{2}}\left(-i|1\rangle - i^2|0\rangle\right) = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)
$$

So the for the eigenvalue 1 the corresponding eigenvector is $|0\rangle - i|1\rangle$ and for the eigenvalue -1 the corresponding eigenvalue is $|0\rangle + i|1\rangle$.

Since $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$. So the for the eigenvalue 1 the corresponding eigenvector is $|0\rangle$ and for the eigenvalue -1 the corresponding eigenvalue is $|1\rangle$.

$\square$

Problem 5

For $S, T \in \mathscr{L}(\mathcal{H})$, show that

$$
tr(T) = tr(T^\dagger)^*, \qquad tr(ST) = tr(TS)
$$

[Recall T^+ denotes adjoint of T]. For $|x\rangle, |y\rangle \in \mathcal{H}$ show

$$
tr(|x\rangle\langle y|\,T) = tr(T\,|x\rangle\langle y|) = \langle y|Tx\rangle
$$

Solution:

- $tr(T)$ is the summation of the diagonal entries of T. Now $T^\dagger = (T^t)^*$. Now the diagonal elements of T remains in the same same position even after transpose. Hence the diagonal elements of $T^\dagger$ are the complex conjugate of the diagonal elements of T. Hence sum of the diagonal entries of $T^\dagger$ will also be the complex conjugate of the sum of the diagonal entries of T. Therefore we get

$$
tr(T) = tr(T^\dagger)^*
$$

- Let $\dim \mathcal{H} = d$. Suppose $\{|e_k\rangle \mid k \in [d]\} \subseteq \mathcal{H}$ be an ONB of $\mathcal{H}$

$$\begin{aligned}
tr(ST) &= \sum_{k=1}^{d} \langle e_k| ST |e_k\rangle = \sum_{k=1}^{d} \langle e_k| SIT |e_k\rangle \\
&= \sum_{k=1}^{d} \langle e_k| S \left[\sum_{i=1}^{d} |e_i\rangle \langle e_i|\right] |e_k\rangle \\
&= \sum_{k-1}^{d} \sum_{i=1}^{d} \langle e_k| S |e_i\rangle \langle e_i| T |e_k\rangle \\
&= \sum_{i=1}^{d} \sum_{k=1}^{d} \langle e_i| T |e_k\rangle \langle e_k| S |e_i\rangle \\
&= \sum_{i=1}^{d} \langle e_i| T \left[\sum_{k=1}^{d} |e_k\rangle \langle e_k|\right] S |e_i\rangle \\
&= \sum_{i=1}^{d} \langle e_i| TIS |e_i\rangle = \sum_{i=1}^{d} \langle e_i| TS |e_i\rangle = tr(TS)
\end{aligned}$$

- Let $\{|e_i\rangle \mid 1 \le i \le d\}$ is an ONB of $\mathcal{H}$. Then $|x\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle$ and $|y\rangle = \sum_{i=1}^{d} \beta_i |e_i\rangle$ where $\alpha_i, \beta_i \in \mathbb{C}$ for all $i \in [d]$. Now

$$\begin{aligned}
tr(|x\rangle \langle y| T) &= \sum_{k=1}^{d} \langle e_i|x\rangle \langle y| T |e_i\rangle \\
&= \sum_{k=1}^{d} \langle y| T |e_i\rangle \langle e_i|x\rangle \\
&= \langle y| T \left[\sum_{k=1}^{d} |e_i\rangle \langle e_i|\right] |x\rangle \\
&= \langle y| TI |x\rangle = \langle y| T |x\rangle
\end{aligned}$$

Now

$$tr(|x\rangle \langle y| T) = tr([|x\rangle \langle y|]T) = tr(T[|x\rangle \langle y|])$$

Therefore we have

$$tr(|x\rangle \langle y| T) = tr(T |x\rangle \langle y|) = \langle y|Tx\rangle$$

$\square$

Problem 6

Suppose $\mathcal{H}$ is finite dimensional complex inner product spacewith $\dim(\mathcal{H}) = d$. Show complex dimensionality of $\mathscr{L}(\mathcal{H})$ is d^2, real dimensionality of $\mathscr{R}(\mathcal{H})$ is d^2.

Suppose $\mathcal{H}$ is a real inner product space of dim d, show $\mathscr{L}(\mathcal{H})$ has dimension d^2 and the space of all symmetric operators is a real vector space of dimension $\frac{d(d+1)}{2}$.

Solution:

- Suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is an ONB of $\mathcal{H}$. Let $T \in \mathscr{L}(\mathcal{H})$. Then for all $i \in [d]$

$$T |e_i\rangle = \sum_{j=1}^{d} \alpha_{i,j} |e_j\rangle$$

where $\alpha_{i,j} \in \mathbb{C}$. Hence, the map T is uniquely decided by the numbers $\alpha_{i,j} \in \mathbb{C}$ for all $i, j \in [d]$. Hence, there are d^2 many numbers which uniquely decides T. Therefore $\dim(\mathscr{L}(\mathcal{H})) = d^2$.

- Now let $T \in \mathscr{R}(\mathcal{H})$. Then $T^\dagger = T$. Again suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ is an ONB of $\mathcal{H}$. Let (i,j)th element of T is denoted by $t_{i,j}$. Then for all $i \in [d]$, $T_{i,i} \in \mathbb{R}$ since $T^\dagger = T$. Now for all off diagonal entries $t_{j,i} = t^*_{i,j}$. So there are $\frac{n^2-n}{2}$ many complex numbers which decides T uniquely apart from the n real entries in the diagonal. Now for each $i,j \in [d]$ let $t_{i,j} = x_{i,j} + iy_{i,j}$ where $x_{i,j}, y_{i,j} \in \mathbb{R}$. Therefore,

$$t_{j,i} = t^*_{i,j} = x_{i,j} - iy_{i,j}$$

So for each off-diagonal entries there are corresponding 2 real numbers. And there are total $\frac{d^2-d}{2}$ many off-diagonal entries which participates in uniquely deciding T. Hence there are total

$$2 \times \frac{d^2-d}{2} + d = d^2$$

real numbers which uniquely decides T. Hence $\dim(\mathscr{R}(\mathcal{H})) = d^2$.

- Suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ is a basis of $\mathcal{H}$. Let $T \in \mathscr{L}(\mathcal{H})$. Then for all $i \in [d]$

$$T|e_i\rangle = \sum_{j=1}^{d} \alpha_{i,j} |e_j\rangle$$

where $\alpha_{i,j} \in \mathbb{R}$. Hence, the map T is uniquely decided by the numbers $\alpha_{i,j} \in \mathbb{C}$ for all $i,j \in [d]$. Since there are d^2 many numbers which uniquely decides T, $\dim(\mathscr{L}(\mathcal{H})) = d^2$.

- Let $T \in \mathscr{R}(\mathcal{H})$. Then $T^t = T$. Again suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ is an basis of $\mathcal{H}$. Let (i,j)th element of T is denoted by $T_{i,j}$. Now for all off diagonal entries $T_{j,i} = T_{i,j}$. So there are $\frac{d^2-d}{2}$ many real numbers which decides T uniquely apart from the d entries in the diagonal. Therefore, there are total $\frac{d^2-d}{2}$ many off-diagonal entries which participates in uniquely deciding T. Hence there are total

$$\frac{d^2-d}{2} + d = \frac{d^2+d}{2} = \frac{d(d+1)}{2}$$

real numbers which uniquely decides T. Hence $\dim(\mathscr{R}(\mathcal{H})) = \frac{d(d+1)}{2}$.

□

Problem 7

Show that $\mathscr{D}(\mathcal{H})$ is a convex subset of the real vector space of all Hermitian operators on $\mathcal{H}$. Show that the extreme points of $\mathscr{D}(\mathcal{H})$ are pure states, i.e. rank 1 projection operators.

Solution:

- Let $\rho_1\rho_2 \in \mathscr{D}(\mathcal{H})$. Suppose $\lambda \in [0,1]$. Then denote

$$\rho = \lambda\rho_1 + (1-\lambda)\rho_2$$

Now $\rho \in \mathscr{D}(\mathcal{H})$ if $\rho \geq 0$ and $tr(\rho) = 1$. Now

$$tr(\rho) = tr(\lambda\rho_1 + (1-\lambda)\rho_2) = \lambda tr(\rho_1) + (1-\lambda)tr(\rho_2) = \lambda + (1-\lambda) = 1$$

Hence we only need to show that for all $|u\rangle \in \mathcal{H}$ we have $\langle u|\rho|u\rangle \geq 0$. Now

$$\langle u|\rho|u\rangle = \langle u|(\lambda\rho_1 + (1-\lambda)\rho_2)ketu = \lambda\langle u|\rho_1|u\rangle + (1-\lambda)\langle u|\rho_2|u\rangle$$

Now since $\rho_1, \rho_2 \in \mathscr{D}(\mathcal{H})$ we have $\langle u|\rho_1|u\rangle \geq 0$, $\langle u|\rho_2|u\rangle \geq 0$. We also have $\lambda \geq 0$. Hence, $1-\lambda \geq 0$. Therefore, $= \lambda\langle u|\rho_1|u\rangle + (1-\lambda)\langle u|\rho_2|u\rangle \geq 0$. Therefore we got $\langle u|\rho|u\rangle \geq 0$. Since $|u\rangle$ is any arbitrary vector of $\mathcal{H}$, we have $\rho \geq 0$. Hence, $rho \in \mathscr{D}(\mathcal{H})$. Therefore $\mathscr{D}(\mathcal{H})$ is convex.

- $\rho \in \mathscr{D}(\mathcal{H})$ is an extreme point if a strict convex combination $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ with $\rho_1, \rho_2 \in \mathscr{D}(\mathcal{H})$ and $\lambda \in (0,1)$ is possible only if $\rho = \rho_1 = \rho_2$. Suppose ρ is not a pure state. Then ρ represents an ensemble $\{p_X(x), |x\rangle \mid x \in X\} \subseteq \mathcal{H}$. By assumption $|X| \geq 2$. Then

$$\rho = \sum_{x\in X} p_X(x)\,|x\rangle\,\langle x|$$

Since $|X| \geq 2$ we can partition X into two disjoint sets U, V such that $U \sqcup V = X$. Then take $a = \sum\limits_{x\in U} p_X(x)$. Certainly $a \in (0,1)$. Then take the ensembles $\mathcal{F}_1 = \left\{\frac{p_X(x)}{a}, |x\rangle \mid x \in U\right\}$ and $\mathcal{F}_2 = \left\{\frac{p_X(x)}{1-a}, |x\rangle \mid x \in V\right\}$. Certainly

$$\sum_{x\in U} \frac{p_X(x)}{a} = \frac{1}{a}\sum_{x\in U} p_X(x) = \frac{1}{a}\,a = 1$$

and

$$\sum_{x\in V} \frac{p_X(x)}{1-a} = \frac{1}{1-a}\sum_{x\in V} p_X(x) = \frac{1}{1-a}\left[1 - \sum_{x\in U} p_X(x)\right] = \frac{1}{1-a}[1-a] = 1$$

Hence $\mathcal{F}_1$ and $\mathcal{F}_2$ are in fact ensembles. Then their corresponding density matrices are

$$\rho_1 = \frac{1}{a}\sum_{x\in U} p_X(x)\,|x\rangle\,\langle x| \qquad \rho_2 = \frac{1}{1-a}\sum_{x\in V} p_X(x)\,|x\rangle\,\langle x|$$

Then we have

$$\begin{aligned} a\rho_1 + (1-a)\rho_2 &= a\left[\frac{1}{a}\sum_{x\in U} p_X(x)\,|x\rangle\,\langle x|\right] + (1-a)\left[\frac{1}{1-a}\sum_{x\in V} p_X(x)\,|x\rangle\,\langle x|\right] \\ &= \sum_{x\in U} p_X(x)\,|x\rangle\,\langle x| + \sum_{x\in V} p_X(x)\,|x\rangle\,\langle x| \\ &= \sum_{x\in X} p_X(x)\,|x\rangle\,\langle x| = \rho \end{aligned}$$

We can write ρ as a strictly convex sum of two density operators but none of them are equal to ρ. Hence contradiction. Hence ρ is a pure state, a rank 1 operator.

For the other direction let ρ is a pure state. Suppose

$$\rho = \lambda\rho_1 + (1-\lambda)\rho_2$$

with $\rho_1, \rho_2 \in \mathscr{D}(\mathcal{H})$ and $\lambda \in (0,1)$. Since $\rho = |\phi\rangle\,\langle\phi|$ for some $|\phi\rangle \in \mathcal{H}$ we have $\rho^2 = \rho$. Then by Cauchy-Schwartz Inequality we have

$$\begin{aligned} tr(\rho) &= \mathrm{tr}(\rho^2) \\ &= \lambda^2 tr(\rho_1^2) + 2\lambda(1-\lambda)tr(\rho_1\rho_2) + (1-\lambda)^2 tr(\rho_2^2) \\ &\leq \lambda^2 tr(\rho_1^2) + (1-\lambda)^2 tr(\rho_2^2) + 2\lambda(1-\lambda)\sqrt{tr(\rho_1^2)tr(\rho_2^2)} \\ &= \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\sqrt{1\times 1} \\ &= \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = 1 \end{aligned}$$

But we know $tr(\rho) = 1$. Hence equality is attained. Therefore, in Cauchy-Schwartz Inequality equality is attained. Therefore $tr(\rho_1\rho_2) = \sqrt{tr(\rho_1^2)tr(\rho_2^2)}$ i.e. $\rho_1 = \rho_2$. Hence ρ is a extreme point. Therefore, pure states are extreme points in $\mathscr{D}(\mathcal{H})$.

□

Problem 8

Show that if $\dim(\mathcal{H}) = d$, then $\mathscr{D}(\mathcal{H})$ can be embedded into a real vector space of dimension $n = d^2 - 1$

Solution: In any operator of $\mathscr{D}(\mathcal{H})$ there are ^{2} entries in the matrix of the operator. But density operator also has one extra condition which is its trace equal's to 1. Hence sum of the diagonal entries is 1. Hence for the diagonal entries it is enough to know about the $d-1$ entries instead of the all d entries because the last entry will be decided by the other $d-1$ entries as their sum is 1. Except the diagonal there are total $d^2 - d$ many off diagonal entries. Hence to uniquely characterize a operator in $\mathscr{D}(\mathcal{H})$ at most $(d^2 - d) + (d - 1) = d^2 - 1$ many numbers are needed. Therefore the set of operators, $\mathscr{D}(\mathcal{H})$ can be embedded into a real vector space of dimension $n = d^2 - 1$.

□

Problem 9

Prove the Singular value decomposition theorem stated in class.

Solution: We will first state the singular value decomposition theorem then we will prove it.
Singular Value Decomposition Theorem: Suppose $T : \mathcal{H} \to \mathcal{H}$ and $\dim(\mathcal{H}) = s$ then $\exists\, U, V \in \mathscr{L}(H)$ which are unitary and diagonal $D \in \mathscr{L}(\mathcal{H})$ with non-negative entries so that

$$T = UDV$$

Proof: Suppose we have an ONB of $\mathcal{H}$, $\{|e_i\rangle \mid i \in [d]\}$ of $\mathcal{H}$. Let's denote $S = T^\dagger T$. Now S is hermitian. Hence by spectral theorem there are unitary matrix W and a diagonal matrix Λ such that

$$S = W\Lambda W^\dagger$$

and also we get an orthonormal eigen basis $\{|v_i\rangle \mid i \in [d]\} \subseteq \mathcal{H}$ of $\mathcal{H}$ with corresponding eigenvalues $\{\lambda_i \mid i \in [d]\}$ of S which are the diagonal entries of Λ.

Now if λ is an eigenvalue of S with corresponding eigenvector $|v\rangle$ then

$$S\,|v\rangle = \lambda\,|v\rangle \implies \lambda\,\langle v|v\rangle = \langle v|\,Sv = (Sv)^\dagger\,|v\rangle = \lambda^\dagger\,\langle v|v\rangle$$

Therefore the eigenvalues are real. Also since

$$\lambda\,\langle v|v\rangle = \langle v|\,S\,|v\rangle = \langle v|\,T^\dagger T\,|v\rangle = (T\,|v\rangle)^\dagger(T\,|v\rangle) \geq 0$$

we have $\lambda \geq 0$ since $\langle v|v\rangle \geq 0$. Therefore eigenvalues of S are real and non-negative. Therefore entries of Λ are non-negative.

Let us denote the ith eigenvalue of Λ as λ_i. So now take $\sigma_i =\triangleq \sqrt{\lambda_i}$. Now create the diagonal matrix Σ with ith eigenvalue of Σ is σ_i. Define the vectors $|u_i\rangle = \frac{1}{\sigma_i} T\,|v_i\rangle$ for all $i \in [d]$. Then

$$\langle u_i|u_j\rangle = \frac{1}{\sigma_i\sigma_j}\,\langle v_i|\,T^\dagger T\,|v_j\rangle = \frac{1}{\sigma_i\sigma_j}\,\langle v_i|\,S\,|v_j\rangle = \frac{1}{\sigma_i\sigma_j}\,\langle v_i|\,\lambda_j\,|v_j\rangle = \frac{\lambda_j}{\sigma_i\sigma_j}\,\langle v_i|v_j\rangle = \delta_{i,j}$$

Hence $\{|u_i\rangle \mid i \in [d]\}$ also forms an orthonormal basis.

Now we have two maps $V : \mathcal{H} \to \mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ which send the orthonormal basis $\{|e_i\rangle \mid i \in [d]\}$ to $\{|v_i\rangle \mid i \in [d]\}$ by $V\,|e_i\rangle = |v_i\rangle$ and $\{|e_i\rangle \mid i \in [d]\}$ to $\{|u_i\rangle \mid i \in [d]\}$ by $U\,|e_i\rangle = |u_i\rangle$. Therefore we have the matrix of U, V are orthonormal. By definition of u_i we have $Tv_i = \sigma_i u_i$ for all $i \in [d]$. Hence we have

$$TV = U\Sigma \implies T = U\Sigma V^\dagger$$

□

Problem 10

Suppose $|\psi\rangle_{AR_1} \in \mathcal{H}_A \otimes \mathcal{H}_{R_1}$, $|\psi\rangle_{AR_2} \in \mathcal{H}_A \otimes \mathcal{H}_{R_2}$ are purifications of $\rho_A \in \mathscr{D}(\mathcal{H}_A)$ and $\dim(\mathcal{H}_{R_2}) \geq \dim(\mathcal{H}_{R_1})$, then show that there exists an isometry $V : \mathcal{H}_{R_1} \to \mathcal{H}_{R_2}$ such that

$$|\psi\rangle_{AR_2} = (I \otimes V) |\psi\rangle_{AR_1}$$

Solution: Let $\dim(\mathcal{H}_A) = d$, $\dim(\mathcal{H}_{R_1}) = d_1$ and $\dim(\mathcal{H}_{R_2}) = d_2$. Then suppose $\{|a_i\rangle \mid i \in [d]\}$is an ONB and eigenbasis of $\mathcal{H}_A$ with corresponding eigenvalues are $\{\alpha_i \mid i \in [d]\}$. Also suppose $\{|e_i\rangle \mid i \in [d_1]\}$ is an ONB of $\mathcal{H}_{R_1}$ and $\{|f_i\rangle \mid i \in [d_2]\}$ is an ONB of $\mathcal{H}_{R_2}$. We can express

$$\rho_A = \sum_{i=1}^{d} \alpha_i |a_i\rangle \langle a_i|$$

Now we will prove a lemma:
Lemma: Suppose $d_1 < d$. Take $\min\{d, d_2\} = N$. Then by Schmidt Decomposition

$$|\psi\rangle_{AR_1} = \sum_{i=1}^{d_1} \lambda_i |a_i\rangle \otimes |e_i\rangle \qquad |\psi\rangle_{AR_2} = \sum_{j=1}^{N} \gamma_j |a_j\rangle_A \otimes |f_j\rangle_{R_2}$$

where $\lambda_i, \gamma_j \in \mathbb{C}$. Then

$$\{|a_i\rangle \mid i \in [d_1], \lambda_i \neq 0\} = \{|a_j\rangle \mid j \in [N], \gamma_j \neq 0\}$$

Proof: Suppose not.

$$\begin{aligned} tr_{R_1}(|\psi\rangle \langle\psi|_{AR_1}) &= tr_{R_1}\left[\left(\sum_{i=1}^{d_1} \lambda_i |a_i\rangle_A \otimes |e_i\rangle_{R_1}\right)\left(\sum_{j=1}^{d_1} \lambda_j^* \langle a_j|_A \otimes \langle e_j|_{R_1}\right)\right] \\ &= tr_{R_1}\left[\sum_{i=1}^{d_1}\sum_{j=1}^{d_1} \lambda_i\lambda_j^* |a_i\rangle \langle a_j|_A \otimes |e_i\rangle \langle e_j|_{R_1}\right] \\ &= \sum_{i=1}^{d_1}\sum_{j=1}^{d_1} \lambda_i\lambda_j^* |a_i\rangle \langle a_j|_A\, tr(|e_i\rangle \langle e_i|_{R_1}) = \sum_{i=1}^{d_1} |\lambda_i|^2 |a_i\rangle \langle a_j|_A = \rho_A \end{aligned}$$

Similarly we get

$$\begin{aligned} tr_{R_2}(|\psi\rangle \langle\psi|_{AR_2}) &= tr_{R_2}\left[\left(\sum_{i=1}^{N} \gamma_i |a_i\rangle_A \otimes |e_i\rangle_{R_2}\right)\left(\sum_{j=1}^{N} \gamma_j^* \langle a_j|_A \otimes \langle e_j|_{R_2}\right)\right] \\ &= tr_{R_2}\left[\sum_{i=1}^{N}\sum_{j=1}^{N} \gamma_i\gamma_j^* |a_i\rangle \langle a_j|_A \otimes |e_i\rangle \langle e_j|_{R_2}\right] \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \gamma_i\gamma_j^* |a_i\rangle \langle a_j|_A\, tr(|e_i\rangle \langle e_j|_{R_2}) = \sum_{i=1}^{N} |\gamma_i|^2 |a_i\rangle \langle a_i|_A = \rho_A \end{aligned}$$

Hence if there is an vector $|a\rangle$ in one of the sets $\{|a_i\rangle \mid i \in [d_1], \lambda_i \neq 0\}$ and $\{|a_j\rangle \mid j \in [N], \gamma_j \neq 0\}$ since $|a_i\rangle \langle a_i|$ for all $i \in [d]$ it will not be possible to satisfy

$$\sum_{i=1}^{d_1} |\lambda_i|^2 |a_i\rangle \langle a_j| = \rho_A = \sum_{i=1}^{N} |\gamma_i|^2 |a_i\rangle \langle a_i|$$

since one of the LHS and RHS as the term $|a\rangle \langle a|$ where the other don't. Hence contradiction. So we have

$$\{|a_i\rangle \mid i \in [d_1], \lambda_i \neq 0\} = \{|a_j\rangle \mid j \in [N], \gamma_j \neq 0\}$$

□

Now take $\min\{d, d_1\} = K$ and $\min\{d, d_2\} = L$. Therefore $K \leq L$ since $d_1 \leq d_2$. Then by Schmidt Decomposition there exists $\lambda_j \in \mathbb{C}\ \forall\ j \in [k]$ and $\gamma_j \in \mathbb{C}, \forall\ j \in [L]$ such that

$$|\psi\rangle_{AR_1} = \sum_{i=1}^{K} \lambda_i |a_i\rangle_A \otimes |e_i\rangle_{R_1} \qquad \text{and} \qquad |\psi\rangle_{AR_2} = \sum_{i=1}^{L} \gamma_i |a_i\rangle_A \otimes |f_i\rangle_{R_2}$$

Then if $K, L = d$ and $K < d$ both cases we have the same $|a_i\rangle$'s appearing in the representation of $|\psi\rangle_{AR_1} = \sum_{i=1}^{K} \lambda_i |a_i\rangle_A \otimes |e_i\rangle_{R_1}$ as well as $|\psi\rangle_{AR_2} = \sum_{i=1}^{L} \gamma_i |a_i\rangle_A \otimes |f_i\rangle_{R_2}$. Hence in the schmidt decomposition of $|\psi_{AR_1}\rangle$ and $|\psi_{AR_2}\rangle$ the number of nonzero Schmidt Coefficients is always same and they take the same take the same orthonormal vectors from the given orthonormal basis of $\mathcal{H}_A$. So WLOG we can assume the first K vectors in the Schmidt Decomposition of $|\psi\rangle_{AR_2}$ here is the vectors having nonzero Schmidt Coefficients and instead of writing L we can write K as well. This will help us create the isometry $V : \mathcal{H}_{R_1} \to \mathcal{H}_{R_2}$.

Now

$$\rho_A = \sum_{i=1}^{d} \alpha_i |a_i\rangle \langle a_i|$$

where $\alpha_i \in \mathbb{R}$ and $\alpha_i \geq 0$ for all $i \in [d]$. Now

$$\begin{aligned}
tr_{R_1}(|\psi\rangle\langle\psi|_{AR_1}) &= tr_{R_1}\left[\left(\sum_{i=1}^{d_1} \lambda_i |a_i\rangle_A \otimes |e_i\rangle_{R_1}\right)\left(\sum_{j=1}^{d_1} \lambda_j^* \langle a_j|_A \otimes \langle e_j|_{R_1}\right)\right] \\
&= tr_{R_1}\left[\sum_{i=1}^{d_1}\sum_{j=1}^{d_1} \lambda_i\lambda_j^* |a_i\rangle\langle a_j|_A \otimes |e_i\rangle\langle e_j|_{R_1}\right] \\
&= \sum_{i=1}^{d_1}\sum_{j=1}^{d_1} \lambda_i\lambda_j^* |a_i\rangle\langle a_j|_A\, tr(|e_i\rangle\langle e_i|_{R_1}) = \sum_{i=1}^{d_1} |\lambda_i|^2 |a_i\rangle\langle a_j|_A = \rho_A
\end{aligned}$$

Similarly we get

$$\begin{aligned}
tr_{R_2}(|\psi\rangle\langle\psi|_{AR_2}) &= tr_{R_2}\left[\left(\sum_{i=1}^{K} \gamma_i |a_i\rangle_A \otimes |e_i\rangle_{R_2}\right)\left(\sum_{j=1}^{K} \gamma_j^* \langle a_j|_A \otimes \langle e_j|_{R_2}\right)\right] \\
&= tr_{R_2}\left[\sum_{i=1}^{K}\sum_{j=1}^{K} \gamma_i\gamma_j^* |a_i\rangle\langle a_j|_A \otimes |e_i\rangle\langle e_j|_{R_2}\right] \\
&= \sum_{i=1}^{K}\sum_{j=1}^{K} \gamma_i\gamma_j^* |a_i\rangle\langle a_j|_A\, tr(|e_i\rangle\langle e_j|_{R_2}) = \sum_{i=1}^{K} |\gamma_i|^2 |a_i\rangle\langle a_i|_A = \rho_A
\end{aligned}$$

From this we get $\lambda_i = \gamma_i = \sqrt{\alpha_i}$. So we take the isometry V to be

$$V = \sum_{i=1}^{K} |f_i\rangle_{R_1} \langle e_i|_{R_2}$$

Clearly $V^\dagger V = I$ and

$$\begin{aligned}
(I \otimes V)|\psi_{AR_1}\rangle = (I \otimes V)\left[\sum_{i=1}^{K} \lambda_i |a_i\rangle \otimes |e_i\rangle\right] &= \sum_{i=1}^{K} \lambda_i I |a_i\rangle \otimes V|e_i\rangle \\
&= \sum_{i=1}^{K} \lambda_i |a_i\rangle \otimes \left[\sum_{i=1}^{K} |f_i\rangle_{R_2} \langle e_i|_{R_1}\right] |e_i\rangle = \sum_{i=1}^{K} \gamma_i |a_i\rangle \otimes |f_i\rangle = |\psi\rangle_{AR_2}
\end{aligned}$$

□

Problem 11 Mark Wilde: Exercise 3.6.5

Show that the Bell states form an orthonormal basis:

$$\langle \Phi^{z_1 x_1} | \Phi^{z_2 x_2} \rangle = \delta_{z_1,z_2}\, \delta_{x_1,x_2}$$

Solution: By definition we have

$$|\Phi^{z,x}\rangle = (Z^z \otimes I)(X^x \otimes I)\,|\Phi^+\rangle = (Z^z X^x \otimes I)\,|\Phi^+\rangle$$

We have $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Therefore,

$$\begin{aligned}
\langle \Phi^{z_1 x_1} | \Phi^{z_2 x_2} \rangle &= \langle \Phi^+ |\, (X^{x_1} Z^{z_1} \otimes I)(Z^{z_2} X^{x_2} \otimes I)\,|\Phi^+\rangle \\
&= \langle \Phi^+ |\, (X^{x_1} Z^{z_1} Z^{z_2} X^{x_2}) \otimes I\,|\Phi^+\rangle \\
&= \langle \Phi^+ |\, (X^{x_1} Z^{z_1 \oplus z_2} X^{x_2}) \otimes I\,|\Phi^+\rangle \\
&= \frac{1}{2}\Big[\, \langle 0|\, X^{x_1} Z^{z_1\oplus z_2} X^{x_2}\,|0\rangle \langle 0|0\rangle + \langle 1|\, X^{x_1} Z^{z_1\oplus z_2} X^{x_2}\,|1\rangle \langle 1|1\rangle \,\Big] \\
&= \frac{1}{2}\Big[\, \langle 0|\, X^{x_1} Z^{z_1\oplus z_2} X^{x_2}\,|0\rangle + \langle 1|\, X^{x_1} Z^{z_1\oplus z_2} X^{x_2}\,|1\rangle \,\Big]
\end{aligned}$$

Now we will do case wise analysis.

Case 1: $z_1 \oplus z_2 = 0$

Then $Z^{z_1\oplus z_2} = I$. Therefore

$$X^{x_1} Z^{z_1\oplus z_2} X^{x_2} = X^{x_1} X^{x_2} = X^{x_1 \oplus x_2}$$

Hence

$$\frac{1}{2}\Big[\, \langle 0|\, X^{x_1\oplus x_2}\,|0\rangle + \langle 1|\, X^{x_1\oplus x_2}\,|1\rangle \,\Big] = \delta_{0,x_1\oplus x_2} = \delta_{x_1,x_2} = \delta_{x_1,x_2} \times 1 = \delta_{0,z_1\oplus z_2}\delta_{x_1,x_2} = \delta_{z_1,z_2}\delta_{x_1,x_2}$$

Case 2: $z_1 \oplus z_2 = 1$

So now $Z^{z_1\oplus z_2} = Z$. Now we will analyze all possible cases

Case 2.I: $x_1 = 0, x_2 = 0$

Then $X^{x_1} Z^{z_1\oplus z_2} X^{x_2} = Z$. Hence

$$\begin{aligned}
&\frac{1}{2}\Big[\, \langle 0|\, Z\,|0\rangle + \langle 1|\, Z\,|1\rangle \,\Big] \\
&= \frac{1}{2}\Big[\, \langle 0|0\rangle - \langle 1|1\rangle \,\Big] \\
&= 0 = \delta_{z_1,z_2} = \delta_{z_1,z_2} \times 1 = \delta_{z_1,z_2}\delta_{x_1,x_2}
\end{aligned}$$

Case 2.II: $x_1 = 0, x_2 = 1$

Now $X^{x_1} Z^{z_1\oplus z_2} X^{x_2} = ZX$. So

$$\frac{1}{2}\Big[\, \langle 0|\, ZX\,|0\rangle + \langle 1|\, ZX\,|1\rangle \,\Big] = \frac{1}{2}\Big[\, \langle 0|\, Z\,|1\rangle + \langle 1|\, Z\,|0\rangle \,\Big] = \frac{1}{2}\Big[-\langle 0|1\rangle + \langle 1|0\rangle \,\Big] = 0 = \delta_{z_1,z_2}\delta_{x_1,x_2}$$

Case 2.III: $x_1 = 1, x_2 = 0$

Now $X^{x_1} Z^{z_1\oplus z_2} X^{x_2} = XZ$. So

$$\frac{1}{2}\Big[\, \langle 0|\, XZ\,|0\rangle + \langle 1|\, ZX\,|1\rangle \,\Big] = \frac{1}{2}\Big[\, \langle 0|\, X\,|0\rangle - \langle 1|\, X\,|1\rangle \,\Big] = \frac{1}{2}\Big[\, \langle 0|1\rangle - \langle 1|0\rangle \,\Big] = 0 = 0 \times 0 = \delta_{z_1,z_2}\delta_{x_1,x_2}$$

Case 2.IV: $x_1 = 1, x_2 = 1$

Now $X^{x_1} Z^{z_1 \oplus z_2} X^{x_2} = XZX$. So

$$\frac{1}{2}\Big[\langle 0| XZX |0\rangle + \langle 1| ZX |1\rangle \Big] = \frac{1}{2}\Big[\langle 0| XZ |1\rangle + \langle 1| XZ |0\rangle \Big] = \frac{1}{2}\Big[-\langle 0| X |1\rangle + \langle 1| X |0\rangle \Big]$$
$$= \frac{1}{2}\Big[-\langle 0|0\rangle + \langle 1|1\rangle \Big] = \frac{1}{2}[-1+1] = 0 = 0 \times 1 = \delta_{z_1,z_2}\delta_{x_1,x_2}$$

Therefore we can say

$$\langle \Phi^{z_1 x_1} | \Phi^{z_2 x_2} \rangle = \delta_{z_1,z_2}\, \delta_{x_1,x_2}$$

Hence Bell States form an orthonormal basis.

□

Problem 12 Mark Wilde: Exercise 3.7.11

Show that the set of states $\{|\Phi^{x,z}\rangle_{AB}\}_{x,z=0}^{d-1}$ forms a complete, orthonormal basis:

$$\langle \Phi^{x_1,z_1} | \Phi^{x_2,z_2} \rangle = \delta_{x_1,x_2}\, \delta_{z_1,z_2} \qquad \sum_{x,z=0}^{d} |\Phi^{x,z}\rangle \langle \Phi^{x,z}| = I_{AB}$$

Solution: We have

$$X(x)\,|j\rangle = |j \oplus x\rangle \qquad Z(z)\,|j\rangle = \exp\left\{\frac{2\pi i z j}{d}\right\} |j\rangle$$

Therefore

$$X(x)Z(z)\,|j\rangle = \exp\left\{\frac{2\pi i z j}{d}\right\} |j \oplus x\rangle$$

Now

$$|\Phi^{x,z}\rangle_{AB} = (X_A(x)Z_A(z) \otimes I_B)\,|\Phi\rangle_{AB} \qquad |\Phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A\,|j\rangle_B$$

Therefore we have

$$|\Phi^{x,z}\rangle_{AB} = X_A(x)Z_A(z) \otimes I_B\,|\Phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} X_A(x)Z_A(z)\,|j\rangle_A \otimes |j\rangle_B = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp\left\{\frac{2\pi i j z j}{d}\right\} |j \oplus x\rangle_A \otimes |j\rangle_B$$

Hence

$$\begin{aligned}
\langle \Phi^{x_1,z_1} | \Phi^{x_2,z_2} \rangle &= \frac{1}{d}\left[\sum_{j=0}^{d-1} \exp\left\{-\frac{2\pi i z_1 j}{d}\right\} \langle j \oplus x_1|_A \otimes \langle j|_B\right]\left[\sum_{k=0}^{d-1} \exp\left\{\frac{2\pi i z_2 k}{d}\right\} |k \oplus x_2\rangle_A \otimes |k\rangle_B\right] \\
&= \frac{1}{d}\left[\sum_{j=0}^{d-1}\sum_{k=1}^{d-1} \exp\left\{\frac{2\pi i}{d}(k z_2 - j z_1)\right\} \langle j \oplus x_1|_A\,|k \oplus x_2\rangle_A\,\langle j|_B\,|k\rangle_B\right] \\
&= \frac{1}{d}\left[\sum_{j=0}^{d-1} \exp\left\{\frac{2\pi i j}{d}(z_2 - z_1)\right\} \langle j \oplus x_1|_A\,|j \oplus x_2\rangle_A\right] \\
&= \frac{1}{d}\left[\delta_{x_1,x_2} \sum_{i=0}^{d-1} \exp\left\{\frac{2\pi i j}{d}(z_2 - z_1)\right\}\right] \\
&= \begin{cases} \dfrac{1}{d}\, d\, \delta_{x_1,x_2} = \delta_{x_1,x_2} & \text{when } z_1 = z_2 \\[2ex] \dfrac{1}{d}\,\delta_{x_1,x_2}\, \dfrac{1-\exp\left\{\frac{2\pi i d}{d}\right\}}{1-\exp\left\{\frac{2\pi i}{d}\right\}} = \dfrac{1}{d}\,\delta_{x_1,x_2}\, \dfrac{1-\exp\{2\pi i d\}}{1-\exp\left\{\frac{2\pi i}{d}\right\}} = 0 & \text{when } z_1 \neq z_2 \end{cases} \\
&= \delta_{x_1,x_2}\delta_{z_1,z_2}
\end{aligned}$$

Also

$$\begin{aligned}
\sum_{x,z=0}^{d} |\Phi^{x,z}\rangle\langle\Phi^{x,z}| &= \frac{1}{d}\sum_{x,z=0}^{d}\left[\sum_{j=0}^{d-1}\exp\left\{\frac{2\pi izj}{d}\right\}|j\oplus x\rangle_A\otimes|j\rangle_B\right]\left[\sum_{k=0}^{d-1}\exp\left\{-\frac{2\pi izk}{d}\right\}\langle k\oplus x|_A\otimes\langle k|_B\right]\\
&= \frac{1}{d}\sum_{x,z=0}^{d}\left[\sum_{j=0}^{d-1}\sum_{k=1}^{d-1}\exp\left\{\frac{2\pi iz}{d}(k-j)\right\}|j\oplus x\rangle_A\langle k\oplus x|_A\otimes|j\rangle_B\langle k|_B\right]\\
&= \frac{1}{d}\sum_{x=0}^{d}\sum_{j=0}^{d-1}\sum_{k=1}^{d-1}\left[\sum_{z=0}^{d}\exp\left\{\frac{2\pi iz}{d}(k-j)\right\}\right]|j\oplus x\rangle_A\langle k\oplus x|_A\otimes|j\rangle_B\langle k|_B\\
&= \frac{1}{d}\sum_{x=0}^{d}\sum_{j=0}^{d-1} d\,|j\oplus x\rangle_A\langle k\oplus x|_A\otimes|j\rangle_B\langle k|_B\\
&= \sum_{x=0}^{d}\sum_{j=0}^{d-1}|j\oplus x\rangle_A\langle j\oplus x|_A\otimes|j\rangle_B\langle j|_B\\
&= \sum_{j=0}^{d-1}\left[\sum_{x=0}^{d}]\,|j\oplus x\rangle_A\langle j\oplus x|_A\right]\otimes|j\rangle_B\langle j|_B\\
&= \sum_{j=0}^{d-1} I_A\otimes|j\rangle_B\langle j|_B = I_A\otimes I_B = I_{A,B}
\end{aligned}$$

□

Problem 13 Mark Wilde: Exercise 4.1.5

Show that the following ensembles have the same density operator: $\left\{\left\{\frac{1}{2},|0\rangle\right\},\left\{\frac{1}{2},|1\rangle\right\}\right\}$ and $\left\{\left\{\frac{1}{2},|+\rangle\right\},\left\{\frac{1}{2},|-\rangle\right\}\right\}$

Solution: The density operator of the ensemble $\left\{\left\{\frac{1}{2},|0\rangle\right\},\left\{\frac{1}{2},|1\rangle\right\}\right\}$ is

$$\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\Big[\,|0\rangle\langle 0| + |1\rangle\langle 1|\,\Big]$$

Now

$$|+\rangle\langle +| = \frac{1}{2}\Big[\,|0\rangle + |1\rangle\,\Big]\Big[\,\langle 0| + \langle 1|\,\Big] = \frac{1}{2}\Big[\,|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\,\Big]$$

and similarly

$$|-\rangle\langle -| = \frac{1}{2}\Big[\,|0\rangle - |1\rangle\,\Big]\Big[\,\langle 0| - \langle 1|\,\Big] = \frac{1}{2}\Big[\,|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| - |1\rangle\langle 1|\,\Big]$$

The density operator of the ensemble $\left\{\left\{\frac{1}{2},|+\rangle\right\},\left\{\frac{1}{2},|-\rangle\right\}\right\}$ is

$$\begin{aligned}
\rho_2 &= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|\\
&= \frac{1}{4}\Big[\,|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\,\Big] + \frac{1}{4}\Big[\,|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| - |1\rangle\langle 1|\,\Big]\\
&= \frac{1}{4}\Big[2\,|0\rangle\langle 0| + 2\,|1\rangle\langle 1|\,\Big] = \frac{1}{2}\Big[\,|0\rangle\langle 0| + |1\rangle\langle 1|\,\Big] = \rho_1
\end{aligned}$$

Hence both the ensembles have the same density operator.

□

Problem 14

For $T \in \mathscr{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ specified through $T = \sum_{i,j} \gamma_{i,j} A_i \otimes B_j$, where $A_i \in \mathscr{L}(\mathcal{H}_A)$, $B_i \in \mathscr{L}(\mathcal{H}_B)$, $\gamma_{i,j} \in \mathbb{C}$ we define

$$f(T) = \sum_{i,j} \gamma_{i,j} tr(B_j) A_i$$

Let $g : \mathscr{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathscr{L}(\mathcal{H}_A)$ be defined as a map satisfying

$$\langle u| g(Z) |v\rangle = \sum_k \langle u \otimes e_k| Z |v \otimes e_k\rangle$$

for any ONB $\{|e_k\rangle \mid 1 \le k \le K\} \subseteq \mathcal{H}_B$.

1. Prove that $f(T) = g(T)$ and both are well defined.
2. Prove if $\{M_r \mid 1 \le r \le R\} \subseteq \mathscr{L}(\mathcal{H}_A)$ is a measurement then

$$tr(M_r g(Z) M_r^\dagger) = tr[(M_r \otimes I_B) Z (M_r^\dagger \otimes I_B)]$$

Solution:

1. Now in the proof of Problem 3 we showed that $tr_B(T)$ is well defined. Since

$$f(T) = \sum_{i,j} \gamma_{i,j} tr(B_j) A_i = \sum_{i,j} \gamma_{i,j} tr_B(A_i \otimes B_j) = tr_B\left(\sum_{i,j} \gamma_{i,j} A_i \otimes B_j\right) = tr_B(T)$$

So $f(T)$ is also well defined. Let $\{|c_l\rangle \mid 1 \le l \le L\}$ be an ONB of $\mathcal{H}_A$. Since $g(Z) \in \mathscr{L}(\mathcal{H}_A)$ we the (i,j)th element of $g(Z)$ with respect to the basis $\{|c_l\rangle \mid 1 \le l \le L\}$ is the value of $\langle c_i| g(Z) |c_j\rangle$. Since for each $i, j \in [L]$

$$\langle c_i| g(Z) |c_j\rangle = \sum_k \langle c_i \otimes e_k| Z |c_j \otimes e_k\rangle$$

So each of the values $\langle c_i| g(Z) |c_j\rangle$ are well defined. And therefore $g(Z)$ is also well defined.

Let $\{|c_l\rangle \mid 1 \le l \le L\}$ be an ONB of $\mathcal{H}_A$. Since $g(Z) \in \mathscr{L}(\mathcal{H}_A)$ we the (i,j)th element of $g(Z)$ with respect to the basis $\{|c_l\rangle \mid 1 \le l \le L\}$ is the value of $\langle c_i| g(Z) |c_j\rangle$. Now Let $T = \sum_{i,j} \gamma_{i,j} A_i \otimes B_j$ then $\forall\, m, m \in [d_A]$

$$\begin{aligned}
\langle c_m| g(T) |c_n\rangle &= \sum_k \langle c_m \otimes e_k| T |c_n \otimes e_k\rangle \\
&= \sum_k \langle c_m \otimes e_k| \left[\sum_{i,j} \gamma_{i,j} A_i \otimes B_j\right] |c_n \otimes e_k\rangle \\
&= \sum_k \sum_{i,j} \gamma_{i,j} \langle c_m| A_i |c_n\rangle \langle e_k| B_j |e_k\rangle \\
&= \sum_{i,j} \gamma_{i,j} \langle c_m| A_i |c_n\rangle \sum_k \langle e_k| B_j |e_k\rangle \\
&= \sum_{i,j} \gamma_{i,j} \langle c_m| A_i |c_n\rangle\, tr(B_j) \\
&= \langle c_m| \left[\sum_{i,j} \gamma_{i,j} tr(B_j) A_i\right] |c_n\rangle \\
&= \langle c_m| f(T) |c_n\rangle
\end{aligned}$$

Hence for all $m, n \in [d_A]$ we have

$$\langle c_m| g(T) |c_n\rangle = \langle c_m| f(T) |c_n\rangle$$

Hence $g(T) = f(T)$. Since T is any arbitrary operator in $\mathscr{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Therefore we have $g(T) = f(T)$ for all $T \in \mathscr{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

2. Let $Z = \sum_{i,j} \gamma_{i,j} A_i \otimes B_j$ where $\gamma_{i,j} \in \mathbb{C}$ and $A_i \in \mathscr{L}(\mathcal{H}_A)$, $B_j \in \mathscr{L}(\mathcal{H}_B)$. Now

$$\begin{aligned}
tr(M_r g(Z) M_r^\dagger) &= tr(M_r f(Z) M_r^\dagger) \\
&= tr\left(M_r \left[\sum_{i,j} \gamma_{i,j} tr(B_j) A_i \right] M_r^\dagger \right) \\
&= \sum_l \langle f_l| M^r \left[\sum_{i,j} \gamma_{i,j} A_i tr(B_j) \right] M_r^\dagger |f_l\rangle \\
&= \sum_l \langle f_l| M^r \left[\sum_{i,j} \gamma_{i,j} A_i \left(\sum_k \langle e_k| B_j |e_k\rangle \right) \right] M_r^\dagger |f_l\rangle \\
&= \sum_{i,j} \gamma_{i,j} \left(\sum_l \langle f_l| M^r A_i M_r^\dagger |f_l\rangle \right) \left(\sum_k \langle e_k| B_j |e_k\rangle \right) \\
&= \sum_{i,j} \gamma_{i,j} \sum_{k,l} \langle f_l| M^r A_i M_r^\dagger |f_l\rangle \langle e_k| B_j |e_k\rangle \\
&= \sum_{i,j} \gamma_{i,j} \sum_{k,l} \langle f_l \otimes e_k| (M^r A_i M_r^\dagger) \otimes B_j |f_l \otimes e_k\rangle \\
&= \sum_{i,j} \gamma_{i,j} \sum_{k,l} \langle f_l \otimes e_k| (M^r \otimes I) A_i \otimes B_j (M_r^\dagger \otimes I) |f_l \otimes e_k\rangle \\
&= \sum_{k,l} \langle f_l \otimes e_k| (M^r \otimes I) \left[\sum_{i,j} \gamma_{i,j} A_i \otimes B_j \right] (M_r^\dagger \otimes I) |f_l \otimes e_k\rangle \\
&= \sum_{k,l} \langle f_l \otimes e_k| (M^r \otimes I) f(Z) (M_r^\dagger \otimes I) |f_l \otimes e_k\rangle \\
&= tr\left[(M^r \otimes I) f(Z) (M_r^\dagger \otimes I) \right]
\end{aligned}$$

$\square$

Problem 15 Mark Wilde: Exercise 4.1.3

Prove the following equality:

$$tr(A) = \langle \Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_S$$

where A is a square operator acting on a Hilbert space $\mathcal{H}_S$, I_R is the operator acting on a Hilbert space $\mathcal{H}_R$ isormorphic to $\mathcal{H}_S$ and $|\Gamma\rangle_{RS}$ is the unnormalized maximally entangled vector

$$|\Gamma\rangle_{RS} = \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_S$$

Solution:

$$\begin{aligned}
\langle \Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_S &= \left[\sum_{i=0}^{d-1} \langle i|_R \langle i|_S \right] I_R \otimes A_S \left[\sum_{j=0}^{d-1} |j\rangle_R |j\rangle_S \right] \\
&= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \langle i|_R |j\rangle_R \langle i|_S A_S |j\rangle_S \\
&= \sum_{i=0}^{d-1} \langle i|_S A_S |i\rangle_S = tr(A)
\end{aligned}$$

$\square$

Problem 16 Mark Wilde: Exercise 3.7.12

Show that the following "transpose trick" or "ricochet" property holds for a maximally entangled state $|\Phi\rangle_{AB}$ defined as

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B$$

and any $d \times d$ matrix M

$$(M_A \otimes I_B) |\Phi\rangle_{AB} = (I_A \otimes M_B^t) |\Phi\rangle_{AB}$$

where M^t is the transpose of the operator M with respect to the basis $\{|i\rangle_B\}$ from the definition of $|\Phi\rangle_{AB}$. The implication is that some local action of Alice on $|\Phi\rangle_{AB}$ is equivalent to Bob performing the transpose of this action on his share of the state. Of course, the same equality is true for the unnormalized maximally entangled vector $|\Gamma\rangle_{AB}$ of the from Problem 15

$$(M_A \otimes I_B) |\Gamma\rangle_{AB} = (I_A \otimes M_B^t) |\Gamma\rangle_{AB}$$

Solution: $M_{i,j}$ denotes the (i,j)th entry of M and $M_{i,j}^t$ denotes the (i,j)th entry of M^t which is equal to $M_{j,i}$. Then we can say

$$M |i\rangle = \sum_{j=0}^{d-1} M_{i,j} |j\rangle$$

$$\begin{aligned}
(M_A \otimes I_B) |\Phi\rangle_{AB} &= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} M_A |i\rangle_A \otimes |i\rangle_B \\
&= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} M_{i,j} |j\rangle_A \otimes |i\rangle_B \\
&= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} M_{j,i}^t |j\rangle_A \otimes |i\rangle_B \\
&= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} |j\rangle_A \otimes M_{j,i}^t |i\rangle_B \\
&= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A \otimes \sum_{i=0}^{d-1} M_{j,i}^t |i\rangle_B \\
&= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A \otimes M_B^t |j\rangle_B \\
&= (I_A \otimes M_B^t) \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A \otimes |j\rangle_B \\
&= (I_A \otimes M_B^t) |\Phi\rangle_{AB}
\end{aligned}$$

□

Problem 17

For each of the maps

(i) State whether the map is a quantum evolution i.e. a CPTP linear map

(ii) If the answer to (i) is 'Yes' then identify the Choi-Krous Operators specifying the map

(iii) Identify the components - reference Hilbert Space, isometry - that specify the map through Shine spring citation i.e. if the map $\mathscr{E} : \mathscr{L}(\mathcal{H}) \to \mathscr{L}(\mathcal{H}_B)$ is a CPTP map. Identify a Hilbert space $\mathcal{H}_R$ and an isometry $V : \mathcal{H}_A \to \mathcal{H}_R \otimes \mathcal{H}_A \otimes \mathcal{H}_B$ so that $\mathscr{E}(S) = tr_{RA}(VSV^\dagger)$

(1) $\mathscr{E} : \mathbb{C} \to \mathscr{D}(\mathcal{H})$, where $1 \mapsto \rho$

(2) $\mathscr{E} : \mathscr{L}(\mathcal{H}_A) \to \mathscr{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $S \mapsto S \otimes \rho_B$ where $\rho_B \in \mathscr{D}(\mathcal{H}_B)$ is a fixed density operator.

(3) $\mathscr{E} : \mathscr{L}(\mathcal{H}_A) \to \mathscr{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ where $S \mapsto tr_B(S)$

(4) $\mathscr{E} : \mathscr{L}(\mathcal{H}_A) \to \mathscr{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ where $S \mapsto VSV^{\dagger}$ where $V : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_B$ is an isometry

(5) Suppose $\mathcal{H}_x = span\{|x\rangle \mid x \in \mathcal{X}\}$ where $\mathcal{X}$ is a finite set such that

$$\langle \tilde{x}|x\rangle = \delta_{\tilde{x},x}$$

and similarly $\mathcal{H}_y = span\{|y\rangle \mid y \in \mathcal{Y}\}$ where $\mathcal{Y}$ is a finite set and

$$\langle \tilde{y}|y\rangle = \delta_{\tilde{y},y}$$

Suppose $p_{Y|X}(y|x)$, $(x,y) \in \mathcal{X} \times \mathcal{Y}$ is a stochastic matrix i.e. $\sum_{y\in\mathcal{Y}} p_{Y|X}(y|x) = 1$ for all $x \in \mathcal{X}$ and $p_{Y|X}(y|x) \geq 0$.
Then consider the maps

$$\sum_{x\in\mathcal{X}} p_X(x)\,|x\rangle\langle x| \longrightarrow \sum_{x,y} p_X(x)p_{Y|X}(y|x)\,|xy\rangle\langle xy|$$

$$\sum_{x\in\mathcal{X}} p_X(x)\,|x\rangle\langle x| \longrightarrow \sum_{y\in\mathcal{Y}} p_Y(y)\,|y\rangle\langle y|$$

where $p_Y(y) = \sum_{x\in\mathcal{X}} p_X(x)p_{Y|X}(y|x)$.

(6) Suppose $\{M_k \mid 1 \leq k \leq K\}$ form a measurement, then

$$S \mapsto \sum_{k=1}^{K} M_k S M_k^{\dagger} \otimes |k\rangle\langle k|$$

Problem 18

If $A \in \mathscr{L}(\mathcal{H}_1)$, $B \in \mathscr{L}(\mathcal{H}_2)$ and $A \geq 0$, $B \geq 0$ prove or disprove (with counter example) $A \otimes B \geq 0$.
[Recall that for any operator $S \in \mathscr{L}(\mathcal{H})$ we say $S \geq 0$ if $\langle u|\,S\,|u\rangle \geq 0 \;\forall\; |u\rangle \in \mathcal{H}$]

Solution: Let $\{|e_k\rangle \mid 1 \leq k \leq K\}$ be an ONB of $\mathcal{H}_1$ and $\{|f_l\rangle \mid 1 \leq l \leq L\}$ be an ONB of $\mathcal{H}_2$. Now given that $A \geq 0, B \geq 0$. By Problem 2 we have that $\{|e_k\rangle \otimes |f_l\rangle \mid k \in [K], l \in [L]\}$ is an ONB of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then for any $k \in [K]$ and $l \in [L]$ we have

$$\langle e_k \otimes f_l|\,A \otimes B\,|e_k \otimes f_l\rangle = \langle e_k|\,A\,|e_k\rangle \langle f_l|\,B\,|f_l\rangle \geq 0$$

Now let

$$A\,|e_i\rangle = \sum_{j=1}^{K} \alpha_{i,j}\,|e_j\rangle \qquad \text{and} \qquad B\,|f_i\rangle = \sum_{j=1}^{L} \beta_{i,j}\,|f_j\rangle$$

Given that for all $|\phi\rangle \in \mathcal{H}_1$, $\langle\phi|\,A\,|\phi\rangle \geq 0$ and for all $|\psi\rangle \in \mathcal{H}_2$, $\langle\psi|\,B\,|\psi\rangle \geq 0$. Now let $|u\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then we can write

$$|u\rangle = \sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_{k,l}\,|e_k\rangle \otimes |f_l\rangle$$

Then

$$\begin{aligned}
\langle u| A\otimes B |u\rangle &= \left[\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_{k,l}^{*}\langle e_k|\otimes\langle f_l|\right] A\otimes B \left[\sum_{k'=1}^{K}\sum_{l'=1}^{L}\gamma_{k',l'}|e_{k'}\rangle\otimes|f_{l'}\rangle\right]\\
&= \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{k'=1}^{K}\sum_{l'=1}^{L}\gamma_{k,l}^{*}\gamma_{k',l'}\langle e_k| A |e_{k'}\rangle\langle f_l| B |f_{l'}\rangle\\
&= \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{k'=1}^{K}\sum_{l'=1}^{L}\gamma_{k,l}^{*}\gamma_{k',l'}\langle e_k|\left[\sum_{j=1}^{K}\alpha_{k',j}|e_j\rangle\right]\langle f_l|\left[\sum_{j=1}^{L}\beta_{l',j}|f_j\rangle\right]\\
&= \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{k'=1}^{K}\sum_{l'=1}^{L}\gamma_{k,l}^{*}\gamma_{k',l'}\alpha_{k',k}\langle e_k|e_k\rangle\beta_{l',l}\langle f_l|f_l\rangle
\end{aligned}$$

□

Problem 19

Suppose $\mathcal{H}_A$, $\mathcal{H}_B$ are Hilbert spaces. State the definition of when a map $\mathscr{E}:\mathscr{L}(\mathcal{H}_A)\to\mathscr{L}(\mathcal{H}_B)$ is a completely positive map. Let us say a map $\mathscr{F}:\mathscr{L}(\mathcal{H}_A)\to\mathscr{L}(\mathcal{H}_B)$ is positive if whenever $X\in\mathscr{P}(\mathcal{H}_A)$ is positive semidefinite, we have $\mathscr{F}(X)\in\mathscr{P}(\mathcal{H}_B)$ is positive definite. Provide an example of $\mathcal{H}_A$, $\mathcal{H}_B$ and a map $\mathscr{F}:\mathscr{L}(\mathcal{H}_A)\to\mathscr{L}(\mathcal{H}_B)$ is positive by not completely positive.

Identify $\rho\in\mathscr{D}(\mathcal{H}_A)$, a density operator, so that $(i_R\otimes\mathscr{F})(|\phi_\rho\rangle\langle\phi_\rho|)$ is not positive semi-definite, wherein $|\phi_\rho\rangle\in\mathcal{H}_R\otimes\mathcal{H}_A$ is a purification of ρ.

Solution:

- A linear map $\mathscr{E}:\mathscr{L}(\mathcal{H}_A)\to\mathscr{L}(\mathcal{H}_B)$ is completely positive if $id_R\otimes\mathscr{E}$ is a positive map for a reference system R of arbitrary size.
- Take first $\mathcal{H}_A=\mathcal{H}_B=\mathcal{H}_R=\mathcal{H}=\mathbb{C}^2$. Then $\forall\, S\in\mathscr{L}(\mathcal{H})$, take $\mathscr{F}(S)=S^{\dagger}$. Suppose S is an positive operator. Then for any $|u\rangle\in\mathcal{H}$ we have

$$\langle u| S^{\dagger}|u\rangle = [\langle u| S|u\rangle]^{*}, \langle u| S|u\rangle\geq 0 \implies [\langle u| S|u\rangle]^{*}\geq 0$$

Hence $\langle u| S^{\dagger}|u\rangle\geq 0$. Therefore $\mathscr{F}$ is positive map.

Now take $|\phi_\rho\rangle=\frac{1}{\sqrt{2}}[|00\rangle+|11\rangle]$ where the density matrix, $\rho=\frac{1}{2}[|0\rangle\langle 0|+|1\rangle\langle 1|]$. $|\psi_\rho\rangle$ is the purification of ρ. Therefore

$$\begin{aligned}
&(i_R\otimes\mathscr{F})(|\phi_\rho\rangle\langle\phi_\rho|)\\
=&\frac{1}{2}(i_R\otimes\mathscr{F})\,[|0\rangle\langle 0|\otimes|0\rangle\langle 0|+|0\rangle\langle 1|\otimes|0\rangle\langle 1|+|1\rangle\langle 0|\otimes|1\rangle\langle 0|+|1\rangle\langle 1|\otimes|1\rangle\langle 1|]\\
=&\frac{1}{2}\,[|0\rangle\langle 0|\otimes|0\rangle\langle 0|+|0\rangle\langle 1|\otimes|1\rangle\langle 0|+|1\rangle\langle 0|\otimes|0\rangle\langle 1|+|1\rangle\langle 1|\otimes|1\rangle\langle 1|]
\end{aligned}$$

This operator has a eigenvector $\frac{1}{\sqrt{2}}[|01\rangle-|10\rangle]$ with negative eigenvalue

$$\begin{aligned}
&\left(\frac{1}{2}[|0\rangle\langle 0|\otimes|0\rangle\langle 0|+|0\rangle\langle 1|\otimes|1\rangle\langle 0|+|1\rangle\langle 0|\otimes|0\rangle\langle 1|+|1\rangle\langle 1|\otimes|1\rangle\langle 1|]\right)\frac{1}{\sqrt{2}}[|01\rangle-|10\rangle]\\
=&\frac{1}{2}\frac{1}{\sqrt{2}}[|0\rangle\langle 1|\otimes|1\rangle\langle 0|+|1\rangle\langle 0|\otimes|0\rangle\langle 1|][|01\rangle-|10\rangle]\\
=&\frac{1}{2\sqrt{2}}[|10\rangle-|01\rangle]=\left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}[|01\rangle-|10\rangle]\right)
\end{aligned}$$

Hence

Here $(i_R\otimes\mathscr{F})(|\phi_\rho\rangle\langle\phi_\rho|)$ has a negative eigenvalue, $-\frac{1}{2}$. Therefore $(i_R\otimes\mathscr{F})(|\phi_\rho\rangle\langle\phi_\rho|)$ is not positive. Hence $\mathscr{F}$ is positive but not completely positive.

□