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Course: Expander Graphs and Applications

Assignment - 1
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Problem 1 Problem 4.9 (The Replacement Product): Pseudorandomness By Salil Vadhan

Given a D_1-regular graph G_1 on N_1 vertices and a D_2-regular graph G_2 on D_1 vertices consider the following graph $G_1 \textcircled{r} G_2$ on vertex set $[N_1] \times [D_1]$: vertex (u, i) is connected to (v, j) iff

(a) $u = v$ and (i, j) is an edge in G_2 or,

(b) v is the i'th neighbour of u in G_1 and u is the jth neighbor of v.

That is, we "replace" each vertex v in G_1 with a copy of G_2, associating edge incident to v with one vertex of G_2.

1. Prove that there is a function g such that if G_1 has spectral expansion $\gamma_1 > 0$ and G_2 has spectral expansion $\gamma_2 > 0$ (and both graphs are undirected) then $G_1 \textcircled{r} G_2$ has spectral expansion $g(\gamma_1, \gamma_2, D_2) > 0$.

   [Hint: Note that $(G_1 \textcircled{r} G_2)^3$ has $G_1 \textcircled{z} G_2$ as a subgraph]

2. Show how to convert an explicit construction of constant degree (spectral) expanders into an explicit construction of degree 3 (spectral) expanders.

3. Without using Theorem 4.14, prove an analogue of Part 1 for edge expansion. That is, there is a function h such that if G_1 is an $\left(\frac{N_1}{2}, \epsilon_1\right)$ edge expander and G_2 is a $\left(\frac{D_1}{2}, \epsilon_2\right)$ edge expander then $G_1 \textcircled{r} G_2$ is a $\left(\frac{N_1 D_1}{2}, h(\epsilon_1, \epsilon_2, D_2)\right)$ edge expander where $h(\epsilon_1, \epsilon_2, D_2) > 0$ if $\epsilon_1, \epsilon_2 > 0$.

   [Hint: Given any set S of vertices of $G_1 \textcircled{r} G_2$, partition S into the clouds that are more than "half-full" and those that are not]

4. Prove that the functions $g(\gamma_1, \gamma_2, D_2)$ and $h(\epsilon_1, \epsilon_2, D_2)$ must depend on D_2 by showing that $G_1 \textcircled{r} G_2$ cannot be a $\left(\frac{N_1 D_1}{2}, \epsilon\right)$ edge expander if $\epsilon > \frac{1}{D_1 + 1}$ and $N_1 \geq 2$

Solution:

1. Let A_1 and A_2 denote the normalized adjacency matrices of G_1 and G_2 respectively. The degree of the new graph $G_1 \textcircled{r} G_2$ is $D_2 + 1$. Now denote $B \triangleq I_{N_1} \otimes A_2$ and A be a $N_1 \cdot D_1 \times N_1 \cdot D_1$ matrix where

$$A[(u,i),(v,j)] = \begin{cases} 1 & \text{when } i\text{th neighbor of } u \text{ is } v \text{ and } j\text{th neighbor of } v \text{ is } u \text{ in } G_1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore the adjacency matrix of the graph $G_1 \textcircled{r} G_1$ is $A + D_2 B$. Therefore the normalized adjacency matrix, M

$$M \triangleq \frac{A + D_2 B}{D_2 + 1}$$

Now notice the graph $(G_1 \textcircled{r} G_2)^3$ contains the graph $G_1 \textcircled{z} G_2$ as a subgraph. Hence

$$M^3 = \left[\frac{A + D_2 B}{D_2 + 1}\right]^3 = \frac{D_2^2}{(D_2+1)^3} BAB + \left[1 - \frac{D_2^2}{(D_2+1)^3}\right] C$$

for some matrix C. Lets denote $p := \frac{D_2^2}{(D_2+1)^3}$. Then $M^3 = pBAB + (1-p)C$. Hence for any $v \perp u$ where u is the uniform vector we have

$$\|M^3 v\| \leq p\|BABv\| + (1-p)\|Cv\|$$

Now we can think as C is a normalized adjacency matrix of an undirected graph. Hence for all $v \perp u$ we have $\|Cv\| \leq \|v\|$. Now we know for all $v \perp u$

$$\|BABv\| \leq (\lambda_1 + \lambda_2 + \lambda_2^2)\|v\|$$

where $\lambda_1 = 1 - \gamma_1$ and $\lambda_2 = 1 - \gamma_2$. Hence

$$\|M^3 v\| \leq p(\lambda_1 + \lambda_2 + \lambda_2^2)\|v\| + (1-p)\|v\| = [p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)]\|v\|$$

Suppose $\max\limits_{v \perp u} \frac{\|M^3 v\|}{\|v\|} = \lambda$. Then we have $\lambda = (1 - g(\gamma_1, \gamma_2, D_2))^3$. Therefore we have

$$\begin{aligned}\lambda = \max_{v \perp u} \frac{\|M^3 v\|}{\|v\|} &\leq \max_{v \perp u} \frac{\|(pBAB + (1-p)C)v\|}{\|v\|} \\ &\leq \max_{v \perp u} \frac{[p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)]\|v\|}{\|v\|} = [p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)]\end{aligned}$$

Hence

$$(1 - g(\gamma_1, \gamma_2, D_2))^3 \leq= [p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)]$$

Now

$$\begin{aligned}1 - [p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)] &= 1 - (1-p) - p(\lambda_1 + \lambda_2 + \lambda_2^2) \\ &= p - p(\lambda_1 + \lambda_2 + \lambda_2^2) \\ &= p[1 - (\lambda_1 + \lambda_2 + \lambda_2^2)]\end{aligned}$$

Now we know

$$\lambda_1 + \lambda_2 + \lambda_2^2 < 1 \iff 0 < 1 - (\lambda_1 + \lambda_2 + \lambda_2^2) < 1 \qquad \text{and} \qquad 0 < p < 1$$

Then $0 < p[1 - (\lambda_1 + \lambda_2 + \lambda_2^2)] < 1$. Hence

$$0 < p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p) < 1$$

Now

$$\begin{aligned}1 - g(\gamma_1, \gamma_2, D_2) &= \left[p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)\right]^{\frac{1}{3}} \\ &= \left[1 - p[1 - (\lambda_1 + \lambda_2 + \lambda_2^2)]\right]^{\frac{1}{3}} \\ &\leq 1 - \frac{1}{3}p[1 - (\lambda_1 + \lambda_2 + \lambda_2^2)] < 1\end{aligned}$$

So

$$g(\gamma_1, \gamma_2, D_2) = 1 - \left[p(\lambda_1 + \lambda_2 + \lambda_2^2) + (1-p)\right]^{\frac{1}{3}} > 0$$

2. First we will prove some lemmas

   Lemma 1: Eigenvalues of the permutation $\sigma \in S_n$ where $\sigma = (12 \cdots n)$ are all the n-th roots of unity.

   Proof: The permutation matrix of σ is

$$P = \begin{bmatrix} 0 & 1 \\ I_{n-1} & 0 \end{bmatrix}$$

   Now by Wikipedia: Circulant Matrix Any circulant matrix looks like

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

Hence P is a circulant matrix with $c_0 = 0$, $c_1 = 1$ and for all $i \in [n] - \{1\}$, $c_i = 0$. Hence from the same reference we get that for all $j \in [n-1] \cup \{0\}$, the jth eigenvalue λ_j is

$$\lambda_j = c_0 + c_1\omega^j + c_2\omega^{2j} + \cdots + c_{n-1}\omega^{(n-1)j} = \omega^j$$

where $\omega = e^{\frac{2\pi i}{n}}$. Hence the eigenvalues of P are the n-th roots of unity. □

Lemma 2: A k-cycle graph is a $\left(k, 2, 1 - \Theta\left(\frac{1}{k^2}\right)\right)$-expander.

Proof: Let P_k denote the matrix

$$P_k = \begin{bmatrix} 0 & 1 \\ I_{k-1} & 0 \end{bmatrix}$$

The the adjacency matrix of k-cycle is just $M = P_k + P_k^T$. Since P_k is unitary matrix Let S be the matrix such that $SP_kS^\dagger$ is diagonalized. Let's denote that D. Then

$$SMS^\dagger = S(P_k + P_k^\dagger)S^\dagger = SP_kS^\dagger + SP_k^\dagger S^\dagger = D + S(SP_k)^\dagger = D + (SP_kS^\dagger)^\dagger = D + D^\dagger$$

Hence the eigenvalues of M are $2\Re(\omega^j)$ for all $j \in [n]$ where $\omega = e^{\frac{2\pi i}{k}}$

Now the normalized adjacency matrix for the k-cycle is $\frac{1}{2}M$. Hence the eigenvalues for the normalized adjacency matrix are $\Re(\omega^j) = \cos\frac{2j\pi}{k}$ for all $j \in [k]$. Hence the second largest eigenvalue is when $j = 1$ i.e.

$$\cos\frac{2\pi}{k} \geq 1 - \frac{1}{2}\left(\frac{2\pi}{k}\right)^2 = 1 - \frac{2\pi^2}{k^2} = 1 - \frac{1}{\Theta(k^2)}$$

Therefore k-cycle is $1 - \frac{1}{\Theta(k^2)}$ expander. □

Now we will show an explicit construction of degree 3 expanders from an constant degree expanders. Let G be an (N, D, λ)-expander. Take H to be a D-cycle. Hence by the Lemma 2 we have H is a $\left(D, 2, 1 - \frac{1}{\Theta(D^2)}\right)$-expander. Take the graph $G' = G \text{ⓡ} H$. G' is a 3 regular graph. Hence G' is a $(ND, 3, \lambda')$-expander where $1 - \lambda' > 0$ by part (1). Hence G' is a degree 3 expander.

3.

4.

□

Problem 2 Problem 4.10 (Unbalanced Vertex Expanders and Data Structures): Pseudorandomness By Salil Vadhan

Consider a $(K, (1-\epsilon)D)$ bipartite vertex expander G with N left vertices, M right vertices and left degree D.

1. For a set S of left vertices, a $y \in N(S)$ is called a *unique* neighbor of S if y is incident to exactly one edge from S. Prove that every left-set S of size at most K has at least $(1-2\epsilon)D|S|$ unique neighbors.
2. For a set S of size at most $\frac{K}{2}$, prove that at most $\frac{|S|}{2}$ vertices outside S have at least δD neighbors in $N(S)$ for $\delta = O(\epsilon)$.

Solution:

1. Let U be the set of unique neighbors in $N(S)$. Denote $T = \Gamma(S) - U$. Then we have $|U \cup T| \geq (1-\epsilon)D|S|$. Now we will count the number of edges between S and $\Gamma(S)$. From each vertex in S there are D edges going out. Hence total $D|S|$ many edges are going out from S. Now in $\Gamma(S)$ for each vertex in U there is exactly one edge coming from S and for each edge in T there are at least 2 edges coming from S. Hence there are at least $|U| + 2|T|$ many edges are coming towards $\Gamma(S)$. Hence we have:

$$|U| + 2|T| \le D|S| \iff |U| + 2(|\Gamma(S)| - |U|) \le D|S|$$
$$\iff |U| \ge 2|\Gamma(S)| - D|S| \ge (1-\epsilon)D|S| - D|S| = (1-2\epsilon)D|S|$$

Hence there are at least $(1-2\epsilon)D|S|$ unique neighbors.

2.

□

Problem 3 Problem 5.5 (LDPC Codes): Pseudorandomness By Salil Vadhan

Given a D_1-regular graph G_1 on N_1 vertices and a D_2-regular graph G_2 on D_1 vertices consider the following graph $G_1 \textcircled{r} G_2$ on vertex set $[N_1] \times [D_1]$: vertex (u, i) is connected to (v, j) iff

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That is, we "replace" each vertex v in G_1 with a copy of G_2, associating edge incident to v with one vertex of G_2.

1. Prove that there is a function g such that if G_1 has spectral expansion $\gamma_1 > 0$ and G_2 has spectral expansion $\gamma_2 > 0$ (and both graphs are undirected) then $G_1 \textcircled{r} G_2$ has spectral expansion $g(\gamma_1, \gamma_2, D_2) > 0$.

   [Hint: Note that $(G_1 \textcircled{r} G_2)^3$ has $G_1 \textcircled{z} G_2$ as a subgraph]

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   [Hint: Given any set S of vertices of $G_1 \textcircled{r} G_2$, partition S into the clouds that are more than "half-full" and those that are not]

4. Prove that the functions $g(\gamma_1, \gamma_2, D_2)$ and $h(\epsilon_1, \epsilon_2, D_2)$ must depend on D_2 by showing that $G_1 \textcircled{r} G_2$ cannot be a $\left(\frac{N_1 D_1}{2}, \epsilon\right)$ edge expander if $\epsilon > \frac{1}{D_1 + 1}$ and $N_1 \ge 2$

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