# CSS.201.1 Algorithms

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# Contents

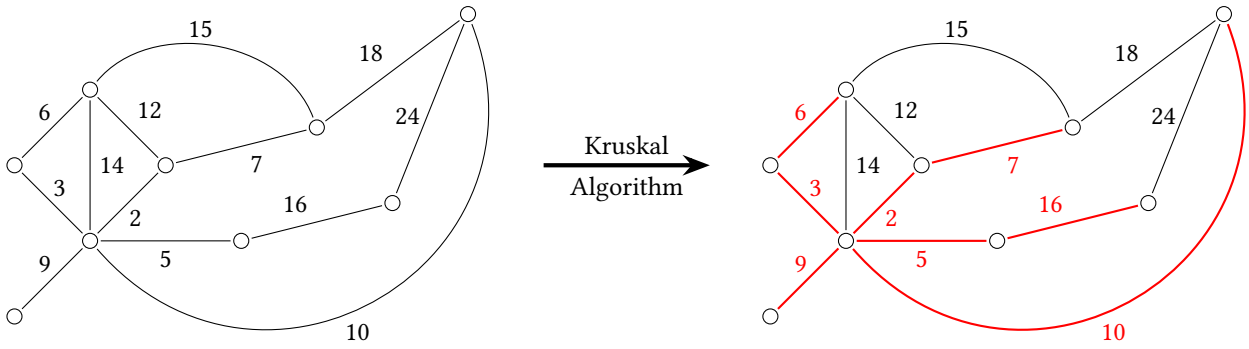
Chapter 1

# Kruskal Algorithm with Data Structure

Minimum Spanning Tree
**Input:** Weighted undirected graph $G = (V, E)$ and weights of edges $W = \{w_e \in \mathbb{Z}_0 : e \in E\}$.
**Question:** Find a spanning tree $T \subseteq E$ such that $\sum_{e \in T} w_e$ is minimum.

In this chapter we will discuss this problem. We will first discuss the Kruskal algorithm which gives a greedy solution to the problem. Then we will discuss the data structure that we can use to implement the Kruskal algorithm efficiently.

## 1.1 Kruskal Algorithm




The Kruskal algorithm uses a concept of component.

## 1.2 Data Structure 1: Array

## 1.3 Data Structure 2: Left Child Right Siblings Tree

## 1.4 Data Structure 3: Union Find

### 1.4.1 Analyzing the Union-Find Data-Structure

We call a node in the union-find data-structure a *leader* if it is the root of the (reversed) tree.

**Lemma 1.4.1**

Once a node stop being a leader (i.e. the node in top of a tree). it can never become a leader again.

***Proof:*** A node $x$ stops being a leader only because of the Union operation which made $x$ child of a node $y$ which is a leader of a tree. From this point on, the only operation that might change the parent pointer of $x$ is the Find operation which traverses through $x$. Since path-compression only change the parent pointer of $x$ to point to some other node $y$. Therefore the parent pointer of $x$ will never become equal to itself i.e. $x$ can never be a leader again. Hence once $x$ stops being a leader it can never be a leader again. ■

**Lemma 1.4.2**

Once a node stop being a leader then its rank is fixed.

***Proof:*** The rank of a node changes only by an Union operation. But the Union operation only changes the rank of nodes that are leader after the operation is done. Therefore once a node stops being a leader it's rank will not being changed by an Union operation. Hence once a node stop being a leader then its rank is fixed. ■

**Lemma 1.4.3**

Ranks are monotonically increasing in the reversed trees, as we travel from a node to the root of the tree.

***Proof:*** To show that the ranks are monotonically increasing it suffices to prove that for all edge $u \to v$ in the data structure we have $\text{rank}(u) < \text{rank}(v)$. ■

**Lemma 1.4.4**

When a node gets rank $k$ than there are at least $\geq 2^k$ elements in its subtree.

**Corollary 1.4.5**

For all vertices $v$, $v.rank \leq \lfloor \log n \rfloor$

**Corollary 1.4.6**

Height of any tree $\leq \lfloor \log_2 n \rfloor$

**Lemma 1.4.7**

The number of nodes that get assigned rank $k$ throughout the execution of the Union-Find data-structure is at most $\frac{n}{2^k}$.

Define $N(r)$ = #vertices with rank at least $k$. Then by the above lemma we have $N(r) \leq \frac{n}{2^k}$.

**Lemma 1.4.8**

The time to perform a single find operation when we perform union by rank and path compression is $O(\log n)$ time.

We will show that we can do much better. In fact we will show that for $m$ operations over $n$ elements the overall running time is $O((n+m)\log^* n)$

**Lemma 1.4.9**

During a single $\textsc{Find}(x)$ operation, the number of jumps between blocks along the search path is $O(\log^* n)$.

**Lemma 1.4.10**

At most $|Block(i)| \leq Tower(i)$ many $\textsc{Find}$ operations can pass through an element $x$ which is in the $i^{th}$ block (i.e. $\textsc{index}_B(x) = i$) before $x.parent$ is no longer in the $i^{th}$ block. That is $\textsc{index}_B(x.parent) > i$.

**Lemma 1.4.11**

There are at most $\frac{n}{Tower(i)}$ nodes that have ranks in the $i^{th}$ block throughout the algorithm execution.

**Lemma 1.4.12**

The number of internal jumps performed, inside the $i^{th}$ block, during the lifetime of $\textsc{Union-Find}$ data structure is $O(n)$.

**Theorem 1.4.13**

The number of internal jumps performed by the $\textsc{Union-Find}$ data structure overall $O(n \log^* n)$.

**Theorem 1.4.14**

The overall time spent on $m$ $\textsc{Find}$ operations, throughout the lifetime of a Union-Find data structure defined over $n$ elements is $O((n+m)\log^* n)$.