Soham Chatterjee
Email: sohamc@cmi.ac.in
Course: Quantum Information Theory

Assignment - 2.2 : Quantum Foundations
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For all the questions

- $[k] := \{1, 2, \dots, k\}$ where $k \in \mathbb{N}$.
- $\mathscr{L}(\mathcal{H}) :=$ Linear operators on $\mathcal{H}$
- $\mathscr{R}(\mathcal{H}) :=$ Self-adjoint or hermitian operators on $\mathcal{H}$
- $\mathscr{P}(\mathcal{H}) :=$ Positive semi-definite operators on $\mathcal{H}$
- $\mathscr{D}(\mathcal{H}) :=$ Density operators on $\mathcal{H}$

Problem 1

For $T : \mathcal{H} \to \mathcal{H}$, prove that

$$\sum_{i=1}^{d} \langle e_i | Te_i \rangle = \sum_{i=1}^{d} \langle f_i | Tf_i \rangle$$

if $\{|e_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ and $\{|f_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ are ONB.

Solution: Let $S : \mathcal{H} \to \mathcal{H}$ where it maps the basis vectors from $|e_i\rangle \to |f_i\rangle$. Then $S|e_i\rangle = |f_i\rangle$. Hence S is an unitary matrix since

$$\langle e_j | S^\dagger S |e_i\rangle = \langle f_j | f_i\rangle = \delta_{ji} \quad \text{and} \quad \langle f_j | SS^\dagger |f_i\rangle = \langle e_j | e_i\rangle = \delta_{ji}$$

Hence

$$\sum_{i=1}^{d} \langle f_i | Tf_i \rangle = \sum_{i=1}^{d} \langle e_i | S^\dagger TS |e_i\rangle = tr(S^\dagger TS) = tr(SS^\dagger T) = tr(T) = \sum_{i=1}^{d} \langle e_i | Te_i \rangle$$

Therefore we have

$$\sum_{i=1}^{d} \langle e_i | Te_i \rangle = \sum_{i=1}^{d} \langle f_i | Tf_i \rangle$$

$\square$

Problem 2

If $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \leq i \leq d\}$ and $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \leq i \leq d\}$ are ONB, then $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\} \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ is ONB

Solution: Let $|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then $|\psi\rangle = \sum\limits_{i=1}^{d} \alpha_i |e_i\rangle$ where $\alpha_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \leq i \leq d\}$ is ONB for $\mathcal{H}_1$. Hence

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle \otimes |\phi\rangle$$

Now $|\phi\rangle = \sum\limits_{i=1}^{d} \beta_i |f_i\rangle$ where $\beta_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \leq i \leq d\}$ is ONB for $\mathcal{H}_2$. Hence

$$\forall\, i \in [d] \;\; |e_i\rangle \otimes |phi\rangle = \sum_{j=1}^{d} \beta_j |e_i\rangle \otimes |f_j\rangle$$

Thereofore we get

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{d} \beta_j |e_i\rangle \otimes |f_j\rangle = \sum_{1\le i,j\le d} \alpha_i\beta_j |e_i\rangle \otimes |f_j\rangle$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \le i, j \le d\}$ is a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Now for any $i1, i2, j1, j2 \in [d]$

$$(\langle e_{i1}| \otimes \langle f_{j1}|)(|e_{i2}\rangle \otimes |f_{j2}\rangle) = \langle e_{i1}|e_{i2}\rangle \langle f_{j1}|f_{j2}\rangle = \delta_{i1,i2}\,\delta_{j1,j2}$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \le i, j \le d\}$ is orthonormal. Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \le i, j \le d\}$ is a ONB for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

□

Problem 3

Let $\{|g_k\rangle \mid 1 \le k \le d_2\} \subseteq \mathcal{H}_2$ be ONB. For $T \in \mathscr{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, let $tr_2(T) \in \mathscr{L}(\mathcal{H}_1)$ denote the operator satisfying

$$\langle u|\, tr_2(T)\, |v\rangle = \sum_k \langle u \otimes g_k|\, T\, |v \otimes g_k\rangle$$

for any choice $|u\rangle, |v\rangle \in \mathcal{H}_1$. Prove that $\sum_k \langle u \otimes g_k|\, T\, |v \otimes g_k\rangle$ is invariant.

Solution: Let $\{|f_k\rangle \mid 1 \le k \le d_2\} \subseteq \mathcal{H}_2$ be another ONB. Suppose $S : \mathcal{H}_2 \to \mathcal{H}$ be a map such that $S\,|g_k\rangle = |f_k\rangle$. As we previously showed in Problem 1, S is unitary. Then for all $k \in [d_2]$ we have

$$|f_k\rangle = \sum_{i=1}^{d_2} w_{i,k} |e_i\rangle$$

where $w_{i,k} \in \mathbb{C}$. Hence

$$\langle f_i|\, S^\dagger S\, |f_j\rangle = \sum_{k=1}^{d_2} w_{i,k}^* w_{j,k} = \delta_{i,j}$$

Now for any $|u\rangle, |v\rangle \in \mathcal{H}_1$ we have

$$\begin{aligned}
\langle u|\, tr_2(T)\, |v\rangle_{\{|f_k\rangle\}} &= \langle u| \left[\sum_{k=1}^{d_2} (I \otimes \langle f_k|) T (I \otimes |f_k\rangle)\right] |v\rangle \\
&= \langle u| \left[\sum_{k=1}^{d_2} \left(I \otimes \left(\sum_{i=1}^{d_2} w_{i,k}^* \langle g_i|\right)\right) T \left(I \otimes \left(\sum_{j=1}^{d_2} w_{j,k} |g_j\rangle\right)\right)\right] |v\rangle \\
&= \sum_{k=1}^{d_2}\sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \langle u| \left[w_{i,k}^* w_{j,k} (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \langle u| \left[\left(\sum_{k=1}^{d_2} w_{i,k}^* w_{j,k}\right) (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \langle u| \left[\delta_{i,j} (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2} \langle u| \left[(I \otimes \langle g_i|) T (I \otimes |g_i\rangle)\right] |v\rangle \\
&= \langle u|\, tr_2(T)\, |v\rangle_{\{|g_k\rangle\}}
\end{aligned}$$

Hence $\sum_k \langle u \otimes g_k|\, T\, |v \otimes g_k\rangle$ is invariant.

□

Problem 4 Mark Wilde: Exercise 3.3.3

Show that the Pauli matrices are all Hermitian, unitary, they square to the identity, and their eigenvalues are ± 1

Solution: Pauli matrices are

$$X|0\rangle = |1\rangle, X|1\rangle = |0\rangle \qquad Y|0\rangle = -i|1\rangle, Y|1\rangle = i|0\rangle \qquad Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle$$

Therefore we have

$$X = |1\rangle\langle 0| + |0\rangle\langle 1| \qquad Y = i[|0\rangle\langle 1| - |1\rangle\langle 0|] \qquad Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Hence

$$\begin{aligned}
X^\dagger &= (|1\rangle\langle 0|)^\dagger + (|0\rangle\langle 1|)^\dagger = |0\rangle\langle 1| + |1\rangle\langle 0| = X \\
Y^\dagger &= (i|0\rangle\langle 1|)^\dagger + (-i|1\rangle\langle 0|)^\dagger = -i|1\rangle\langle 0| + i|0\rangle\langle 1| = Y \\
Z^\dagger &= (|0\rangle\langle 0|)^\dagger - (|1\rangle\langle 1|)^\dagger = |0\rangle\langle 0| - |1\rangle\langle 1| = Z
\end{aligned}$$

Therefore they are Hermitian.

Now

$$\begin{aligned}
X^\dagger X = XX^\dagger = X^2 &= [|1\rangle\langle 0| + |0\rangle\langle 1|][|1\rangle\langle 0| + |0\rangle\langle 1|] \\
&= |1\rangle\langle 0|1\rangle\langle 0| + |1\rangle\langle 0|0\rangle\langle 1| + |0\rangle\langle 1|1\rangle\langle 0| + |0\rangle\langle 1|0\rangle\langle 1| \\
&= |1\rangle\langle 1| + |0\rangle\langle 0| = I
\end{aligned}$$

$$\begin{aligned}
Y^\dagger Y = Y^\dagger = Y^2 &= [i(|0\rangle\langle 1| - |1\rangle\langle 0|)][i(|0\rangle\langle 1| - |1\rangle\langle 0|)] \\
&= -[|0\rangle\langle 1|0\rangle\langle 1| - |0\rangle\langle 1|1\rangle\langle 0| - |1\rangle\langle 0|0\rangle\langle 1| + |1\rangle\langle 0|1\rangle\langle 0|] \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = I
\end{aligned}$$

$$\begin{aligned}
Z^\dagger Z = Z^\dagger = Z^2 &= [|0\rangle\langle 0| - |1\rangle\langle 1|][|0\rangle\langle 0| - |1\rangle\langle 1|] \\
&= |0\rangle\langle 0|0\rangle\langle 0| - |0\rangle\langle 0|1\rangle\langle 1| - |1\rangle\langle 1|0\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 1| \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = I
\end{aligned}$$

Therefore X, Y, Z are unitary and they square to the identity.

Since $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$ we have

$$X\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \qquad X\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

So the for the eigenvalue 1 the corresponding eignevector is $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and for the eigenvalue -1 the corresponding eigenvalue is $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Since $Y|0\rangle = -i|1\rangle$ and $Y|1\rangle = i|0\rangle$ we have

$$\begin{aligned}
Y\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) &= \frac{1}{\sqrt{2}}\left(-i|1\rangle + i^2|0\rangle\right) = -\frac{1}{\sqrt{2}}(i|1\rangle + |0\rangle) \\
Y\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) &= \frac{1}{\sqrt{2}}\left(-i|1\rangle - i^2|0\rangle\right) = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)
\end{aligned}$$

So the for the eigenvalue 1 the corresponding eigenvector is $|0\rangle - i|1\rangle$ and for the eigenvalue -1 the corresponding eigenvalue is $|0\rangle + i|1\rangle$.

Since $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$. So the for the eigenvalue 1 the corresponding eigenvector is $|0\rangle$ and for the eigenvalue -1 the corresponding eigenvalue is $|1\rangle$.

$\square$

Problem 5

For $S, T \in \mathscr{L}(\mathcal{H})$, show that

$$tr(T) = tr(T^{\dagger})^*, \qquad tr(ST) = tr(TS)$$

[Recall T^{+} denotes adjoint of T]. For $|x\rangle, |y\rangle \in \mathcal{H}$ show

$$tr(|x\rangle\langle y| T) = tr(T|x\rangle\langle y|) = \langle y|Tx\rangle$$

Solution:

- $tr(T)$ is the summation of the diagonal entries of T. Now $T^{\dagger} = (T^t)^*$. Now the diagonal elements of T remains in the same same position even after transpose. Hence the diagonal elements of $T^{\dagger}$ are the complex conjugate of the diagonal elements of T. Hence sum of the diagonal entries of $T^{\dagger}$ will also be the complex conjugate of the sum of the diagonal entries of T. Therefore we get

$$tr(T) = tr(T^{\dagger})^*$$

- Let $\dim \mathcal{H} - d$. Suppose $\{|e_k\rangle \mid k \in [d]\} \subseteq \mathcal{H}$ be an ONB of $\mathcal{H}$

$$\begin{aligned}
tr(ST) &= \sum_{k=1}^{d} \langle e_k| ST |e_k\rangle = \sum_{k=1}^{d} \langle e_k| SIT |e_k\rangle \\
&= \sum_{k=1}^{d} \langle e_k| S \left[\sum_{i=1}^{d} |e_i\rangle\langle e_i|\right] |e_k\rangle \\
&= \sum_{k-1}^{d}\sum_{i=1}^{d} \langle e_k| S |e_i\rangle \langle e_i| T |e_k\rangle \\
&= \sum_{i=1}^{d}\sum_{k=1}^{d} \langle e_i| T |e_k\rangle \langle e_k| S |e_i\rangle \\
&= \sum_{i=1}^{d} \langle e_i| T \left[\sum_{k=1}^{d} |e_k\rangle\langle e_k|\right] S |e_i\rangle \\
&= \sum_{i=1}^{d} \langle e_i| TIS |e_i\rangle = \sum_{i=1}^{d} \langle e_i| TS |e_i\rangle = tr(TS)
\end{aligned}$$

- $tr(|x\rangle\langle y| T) = tr([|x\rangle\langle y|]T) = tr(T[|x\rangle\langle y|]) = tr(T|x\rangle\langle y|)$

□

Problem 6

Suppose $\mathcal{H}$ is finite dimensional complex inner product spacewith $\dim(\mathcal{H}) = d$. Show complex dimensionality of $\mathscr{L}(\mathcal{H})$ is d^2, real dimensionality of $\mathscr{R}(\mathcal{H})$ is d^2.

Suppose $\mathcal{H}$ is a real inner product space of dim d, show $\mathscr{L}(\mathcal{H})$ has dimension d^2 and the space of all symmetric operators is a real vector space of dimension $\frac{d(d+1)}{2}$.

Solution:

- Suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ is an ONB of $\mathcal{H}$. Let $T \in \mathscr{L}(\mathcal{H})$. Then for all $i \in [d]$

$$T|e_i\rangle = \sum_{j=1}^{d} \alpha_{i,j} |e_j\rangle$$

where $\alpha_{i,j} \in \mathbb{C}$. Hence, the map T is uniquely decided by the numbers $\alpha_{i,j} \in \mathbb{C}$ for all $i, j \in [d]$. Hence, there are d^2 many numbers which uniquely decides T. Therefore $\dim(\mathscr{L}(\mathcal{H})) = d^2$.

- Now let $T \in \mathscr{R}(\mathcal{H})$. Then $T^\dagger = T$. Again suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is an ONB of $\mathcal{H}$. Let (i,j)th element of T is denoted by $t_{i,j}$. Then for all $i \in [d]$, $T_{i,i} \in \mathbb{R}$ since $T^\dagger = T$. Now for all off diagonal entries $t_{j,i} = t_{i,j}^*$. So there are $\frac{n^2-n}{2}$ many complex numbers which decides T uniquely apart from the n real entries in the diagonal. Now for each $i, j \in [d]$ let $t_{i,j} = x_{i,j} + iy_{i,j}$ where $x_{i,j}, y_{i,j} \in \mathbb{R}$. Therefore,

$$t_{j,i} = t_{i,j}^* = x_{i,j} - iy_{i,j}$$

So for each off-diagonal entries there are corresponding 2 real numbers. And there are total $\frac{d^2-d}{2}$ many off-diagonal entries which participates in uniquely deciding T. Hence there are total

$$2 \times \frac{d^2-d}{2} + d = d^2$$

real numbers which uniquely decides T. Hence $\dim(\mathscr{R}(\mathcal{H})) = d^2$.

- Suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is a basis of $\mathcal{H}$. Let $T \in \mathscr{L}(\mathcal{H})$. Then for all $i \in [d]$

$$T|e_i\rangle = \sum_{j=1}^{d} \alpha_{i,j} |e_j\rangle$$

where $\alpha_{i,j} \in \mathbb{R}$. Hence, the map T is uniquely decided by the numbers $\alpha_{i,j} \in \mathbb{C}$ for all $i, j \in [d]$. Since there are d^2 many numbers which uniquely decides T, $\dim(\mathscr{L}(\mathcal{H})) = d^2$.

- Let $T \in \mathscr{R}(\mathcal{H})$. Then $T^t = T$. Again suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is an basis of $\mathcal{H}$. Let (i,j)th element of T is denoted by $T_{i,j}$. Now for all off diagonal entries $T_{j,i} = T_{i,j}$. So there are $\frac{d^2-d}{2}$ many real numbers which decides T uniquely apart from the d entries in the diagonal. Therefore, there are total $\frac{d^2-d}{2}$ many off-diagonal entries which participates in uniquely deciding T. Hence there are total

$$\frac{d^2-d}{2} + d = \frac{d^2+d}{2} = \frac{d(d+1)}{2}$$

real numbers which uniquely decides T. Hence $\dim(\mathscr{R}(\mathcal{H})) = d^2$.

□

Problem 7

Show that $\mathscr{D}(\mathcal{H})$ is a convex subset of the real vector space of all Hermitian operators on $\mathcal{H}$. Show that the extreme points of $\mathscr{D}(\mathcal{H})$ are pure states, i.e. rank 1 projection operators.

Problem 8

Show that if $\dim(\mathcal{H}) = d$, then $\mathscr{D}(\mathcal{H})$ can be embedded into a real vector space of dimension $n = d^2 - 1$

Problem 9

Prove the Singular value decomposition theorem stated in class.

Problem 10

Suppose $|\psi\rangle_{AR_1} \in \mathcal{H}_A \otimes \mathcal{H}_{R_1}$, $|\psi\rangle_{AR_2} \in \mathcal{H}_A \otimes \mathcal{H}_{R_2}$ are purifications of $\rho_A \in \mathscr{D}(\mathcal{H}_A)$ and $\dim(\mathcal{H}_{R_2}) \ge \dim(\mathcal{H}_{R_1})$, then show that there exists an isometry $V : \mathcal{H}_{R_1} \to \mathcal{H}_{R_2}$ such that

$$|\psi\rangle_{AR_2} = (V \otimes I)|\psi\rangle_{AR_1}$$

Problem 11 Mark Wilde: Exercise 3.6.5

Show that the Bell states form an orthonormal basis:

$$\langle \Phi^{z_1 x_1} | \Phi^{z_2 x_2} \rangle = \delta_{z_1, z_2} \, \delta_{x_1, x_2}$$

Problem 12 Mark Wilde: Exercise 3.7.11

Show that the set of states $\{|\Phi^{x,z}\rangle_{AB}\}_{x,z=0}^{d-1}$ forms a complete, orthonormal basis:

$$\langle \Phi^{x_1,z_1} | \Phi^{x_2,z_2} \rangle = \delta_{x_1,x_2} \, \delta_{z_1,z_2} \qquad \sum_{x,z=0}^{d} |\Phi^{x,z}\rangle \langle \Phi^{x,z}| = I_{AB}$$

Problem 13 Mark Wilde: Exercise 4.1.5

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

Problem 14

Show that the set of states $\{|\Phi^{x,z}\rangle_{AB}\}_{x,z=0}^{d-1}$ forms a complete, orthonormal basis:

$$\langle \Phi^{x_1,z_1} | \Phi^{x_2,z_2} \rangle = \delta_{x_1,x_2} \, \delta_{z_1,z_2} \qquad \sum_{x,z=0}^{d} |\Phi^{x,z}\rangle \langle \Phi^{x,z}| = I_{AB}$$

Problem 15 Mark Wilde: Exercise 4.1.3

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

Problem 16 Mark Wilde: Exercise 3.7.12

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

Problem 17

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

Problem 18

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$

Problem 19

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$