# CSS.317.1 Algorithmic Game Theory

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## 5.1 Best Response Dynamics

The existence of a Nash equilibrium is clearly a desirable property of a strategic game. In this chapter and the next we discuss some natural classes of games that do have a Nash equilibrium. The *Best-Response-Dynamics* is a straightforward procedure by which players search for a pure Nash equilibrium (PNE) of a game.

**Algorithm 1:** Best-Response-Dynamics (BRD)

**begin**
  **for** $t = 1, \ldots, T$ **do**
    **if** $t = 1$ **then**
      Each player plays an arbitrary pure strategy
    **else**
      Pick a player $i \in [n]$
      $s_i^t \longleftarrow \arg\min_{s_i \in S_i} c_i(s_i, s_{-i}^{t-1})$
      $s_j^t \longleftarrow s_j^{t-1} \ \forall\, j \in [n],\ j \neq i$

**Note:-**

Best-response dynamics can only halt at a PNE and it cycles in any game without one. It can also cycle in games that have a PNE. For example consider the following 2 player.

## 5.2 Network (Atomic) Congestion Games

**Definition 5.1: Network (Atomic) Congestion Games**

A network (atomic) congestion game or in short NCG consists of the following:

- A directed graph $G = (V, E)$.
- $N$ players where each player $i \in [n]$ has some source-sink pair $(s_i, t_i) \in V \times V$ associated with it.
- Edge cost functions $c_e : [n] \longrightarrow \mathbb{R}$ for each edge $e \in E$.
- Player $i \in [N]$ has strategy set $S_i$ = Set of all $s_i \rightsquigarrow t_i$ paths in $G$. $S = \bigtimes_{i=1}^{N} S_i$.
- For a strategy profile $f \in S$ (often called *flow*), let $n_e(f) = |\{i : e \in f_i\}|$. Then the cost to player $i$ of strategy profile $f$ is $C_i(f) = \sum_{e \in S_i} c_e(n_e(f))$.

So we can define (atomic) NCG by the tuple

$$\Big(G = (V, E), N, \{(s_i, t_i) \mid i \in [N]\}, \{c_e : [N] \to \mathbb{R}_{\geq 0} \mid e \in E\}\Big)$$

Note that unlike the last few lectures where we've been talking about utility-maximization games, this is a cost-minimization game. But of course we could just let a player's utility be the negative of its cost and everything would work as you expect.

**Lemma 5.2.1**

Every NCG has a PNE.

***Proof:*** Given a strategy profile $f \in S$, we will define a potential function $\Phi : S \to \mathbb{R}_{\geq 0}$ with the property that if $f$ is not an equilibrium then $\exists\, f' \in S$ such that $\Phi(f) > \Phi(f')$. Thus if $f^* \in S$ minimizes $\Phi$ then $f^*$ must be a PNE.

Consider the potential function $\Phi : S \to \mathbb{R}_{\geq 0}$:

$$\Phi(s) = \sum_{e \in E} \sum_{i=1}^{n_e(f)} c_e(i)$$

Now it is enough to calculate the change in potential when a player deviates to any other strategy since for $f, f' \in S$

$$\Phi(f) - \Phi(f') = \sum_{i=0}^{N-1} \Phi(f^{(i)}) - \Phi(f^{(i+1)})$$

where $f^{(i)} = (f'_1, f'_2, \ldots, f'_i, f_{i+1}, \ldots, f_N)$ and for $f^{(0)} = f$. Now for any strategy profile $f \in S$ if the player $i$ deviates to the strategy $f'_i \in S_i$ then

$$\begin{aligned}
C_i(f) - C_i(f'_i, f_{-i}) &= \left[\sum_{e \in f_i \cap f'_i} c_e(n_e(f)) + \sum_{e \in f_i \setminus f'_i} c_e(n_e(f))\right] - \left[\sum_{e \in f_i \cap f'_i} c_e(n_e(f'_i, f_{-i})) + \sum_{e \in f'_i \setminus f_i} c_e(n_e(f'_i, f_{-i}))\right] \\
&= \sum_{e \in f_i \cap f'_i} \underbrace{c_e(n_e(f)) - c_e(n_e(f'_i, f_{-i}))}_{=0} + \sum_{e \in f_i \setminus f'_i} c_e(n_e(f)) - \sum_{e \in f'_i \setminus f_i} c_e(n_e(f'_i, f_{-i})) \\
&= \sum_{e \in f_i \setminus f'_i} c_e(n_e(f)) - \sum_{e \in f'_i \setminus f_i} c_e(n_e(f) + 1)
\end{aligned}$$

Therefore the change in the potential is

$$\begin{aligned}
\Phi(f) - \Phi(f'_i, f_{-i}) &= \sum_{e \in E} \sum_{i=1}^{n_e(f)} c_e(i) - \sum_{e \in E} \sum_{i=1}^{n_e(f'_i, f_{-i})} c_e(i) \\
&= \sum_{e \in E} \left[\sum_{i=1}^{n_e(f)} c_e(i) - \sum_{i=1}^{n_e(f'_i, f_{-i})} c_e(i)\right] \\
&= \sum_{e \in f_i \setminus f'_i} c_e(n_e(f)) - \sum_{e \in f'_i \setminus f_i} c_e(n_e(f) + 1) \\
&= C_i(f) - C_i(f'_i, f_{-i})
\end{aligned}$$

So the change in potential is exactly equal to the change in the cost of the player who deviates. Therefore if $f$ is not a PNE then $\exists\, i \in [N]$ such that $\exists\, f'_i \in S_i$ such that $c_i(f) - c_i(f'_i, f_{-i}) > 0$ and therefore $\Phi(f) - \Phi(f'_i, f_{-i}) > 0$. Hence every NCG has a PNE. ■

## 5.3 Potential Games

**Definition 5.2: Potential Game**

A game $\Gamma$ is a potential game if there exists a potential function $\Phi : S \to \mathbb{R}_{\geq 0}$ where $S$ is the set of strategy profiles such that $\forall\, s \in S$ and $s'_i \in S_i$ $C_i(s) - C_i(s'_i, s_{-i}) = \Phi(s) - \Phi(s'_i, s_{-i})$

In the proof of Theorem 5.2.1 we showed that every NCG is a potential game. Now we will show that every potential game has a PNE.

**Theorem 5.3.1**

Every potential game has a Pure Nash Equilibrium

***Proof:*** For a potential game $\Gamma$ let $\Phi$ is the potential function for $\Gamma$. Then $C_i(s) - C_i(s'_i, s_{-i}) = \Phi(s) - \Phi(s'_i, s_{-i})$. Now consider the strategy profile $s = \arg\min_{s \in S} \Phi(s)$. If any player had incentive to deviate there would be a strategy profile with smaller potential which is not possible by the definition of $s$. Therefore $s$ also has the minimum cost. Therefore $s$ is PNE. ■

**Lemma 5.3.2**

Best Response Dynamics cannot cycle in a potential game.

***Proof:*** In each iteration of the BRD every time any player deviates to play a best response the potential must decrease. Hence BRD cannot cycle. ■

Suppose there exists a time $T$ such that every player was chosen in the BRD to choose their best response in the Best response algorithm. Then:

**Lemma 5.3.3**

Let $s^* \in S$ be the strategy profile at time $t$. If $s^*$ is the strategy profile after $T$ further steps of BRD then $s^*$ is a PNE.

***Proof:*** Since in every $T$ steps every player has the option to deviate to another strategy but chose not to. Therefore for each player $i \in [N]$, for all $s'_i \in S_i$, $C_i(s) \leq C_i(s'_i, s_{-i})$. Therefore clearly $s^*$ is a PNE. ■

**Lemma 5.3.4**

Let $s^{\in} S$ be the strategy profile after $T|S|$ steps of BRD. Then $s^*$ is a PNE.

***Proof:*** Since BRD cannot cycle, $\exists\, s \in S$ that must have persisted fro $T$ time steps. Therefore by the previous lemma this must be a PNE. ■

**Theorem 5.3.5**

In a finite potential game from an arbitrary initial outcome the Best Response Dynamics converges to a PNE if $\exists\, T \in \mathbb{N}$ such that in every $T$ steps of BRD every player is chosen at least once.

Since every (Atomic) NCG is a potential game we have the following corollary:

**Corollary 5.3.6**

In an (Atomic) NCG, BRD converges to a PNE if $\exists\, T \in \mathbb{N}$ such that in every $T$ steps of BRD every player is chosen at least once. or "every player is chosen infinitely often".

### 5.3.1 General Congestion Games

General Congestion Games are generalized version of (atomic) NCG. We will show that they are also potential game.

**Definition 5.3: General Congestion Games**

A basic definition general Congestion Games or CG consists of the following:

$$\left(E, N, \{S_i \mid i \in [N]\}, \{c_e : [N] \to \mathbb{R}_{\geq 0} \mid e \in E\}\right)$$

- A base set $E$ of congestible elements.
- $N$ players.
- For each player $i \in [N]$ a finite set of strategies $S_i$ where $S_i \subseteq 2^E$. $S = \bigtimes_{i=1}^{N} S_i$.
- Cost functions $c_e : [N] \to \mathbb{R}$ for each element $e \in E$.
- For a strategy profile $s \in S$ (often called *flow*), let $n_e(s) = |\{i\colon e \in s_i\}|$. Then the cost to player $i$ of strategy profile $s$ is $C_i(s) = \sum_{e \in S_i} c_e(n_e(s))$.

Consider the function $\Phi : S \to \mathbb{R}_{\geq 0}$ where for any strategy profile $s \in S$,

$$\Phi(s) = \sum_{e \in E} \sum_{i=1}^{n_e(s)} c_e(i)$$

that is the same function as the potential function in the case of NCG. This is also a potential function for general CG's which makes general CG's are also potential game.

### 5.3.2 Max Cut Game

**Definition 5.4: Max Cut Game**

A max cut game consists of the following:

1. An undirected weighted graph, $G = (V, E)$ and $w : E \to \mathbb{R}$.
2. $N$ players.
3. For each player $i \in [N]$, has 2 strategies: $S_i = \{L, R\}$. $S = \bigtimes_{i=1}^{N} S_i$.
4. Utility functions $u_i : S \to \mathbb{R}_{\geq 0}$ for each player $i \in [N]$. For any strategy profile $s \in S$, $u_i(s) = \sum_{\substack{e=\{i,j\} \\ s_i \neq s_j}} w_e$

The max cut game is also a potential game. Consider the potential function $\Phi : S \to \mathbb{R}_{\geq 0}$ where for any strategy profile $s \in S$,

$$\Phi(s) = \sum_{\substack{e=\{i,j\} \\ s)i \neq s_j}} w_e$$

With this function we can prove that the Max Cut game is indeed a potential game and henceforth there exists a PNE.

## 5.4 Class: PLS

**Definition 5.5: PLS (Polynomial Local Search)**

A local search problem $L$ has a set of problem instances $D_L \subseteq \Sigma^*$ where any $I \in D_L$ is a particular problem instance. For each instance $I \in D_L$ there exists a finite solution set $F_L(I) \subseteq \Sigma^*$. Let $R_L$ be the relation that models $L$ i.e.

$$R_L := \{(I, s) \mid I \in D_L, s \in F_L(I)\}$$

Then $R_L$ is in PLS if:

(i) The size of every solution $s \in F_L(I)$ for any $I \in D_L$ is polynomially bounded in the size of $I$.

(ii) The problem instances $I \in D_L$ and the solutions $s \in F_L(I)$ are polynomial time verifiable.

(iii) There is a polynomial time computable function $C_L : X \to \mathbb{R}_{\geq 0}$ that returns for each $I \in D_L$ and each $s \in F_L(I)$ the cost where $X := \bigcup\limits_{I \in D_L} \{I\} \times F_L(I)$.

(iv) There is a polynomial time computable function $N : (I, s) \mapsto S$ where $S \subseteq F_L(I)$ i.e. returns the set of neighbors for each $I \in D_L$ and each $s \in F_L(I)$.

Note that for each $I \in D_L$ and each $s \in F_L(I)$ using (iii) and (iv) we can find a neighboring solutions of lower cost of $s$ or determine $s$ is locally minimal. The problem we want to focus is to find a locally minimal cost solution given an instance $I$ of $L$.

**Definition 5.6: PLS-Reductions**

fgsd

**Theorem 5.4.1**

The Max Cut Game is PLS-complete

**Theorem 5.4.2**

General Congestion Games are PLS-complete

# Chapter 6

# Efficiency of Equilibria

Here we are going to leave aside for now the question of how a game arrived at an equilibrium and instead we will study '*quality of equilibria*'. We want to study how close to optimal the equilibria of a game are. But for that we have to define this 'closeness' and 'optimal' by introducing cost to every strategy and we basically want to find a equilibria which very close to the minimum cost strategy profile.

## 6.1 Cost Minimization Games

**Definition 6.1: Cost Minimization Games**

It is a game with $n$ players $[n]$, with their strategy sets $S_1, \ldots, S_n$ where $S = \bigtimes_{i=1}^{n} S_i$ and a cost function $C_i : S \to \mathbb{R}$ for each $i \in [n]$.

There is an objective function $f : S \to \mathbb{R}$ with which the different strategy profiles are compared. There are many common choices for $f$. Conventionally the concepts PNE, MNE, CE, CCE are defined for utility-maximization games with all of its inequalities reversed. But the two definitions are completely equivalent.

- **Pure Nash Equilibria**: A strategy profile $s \in S$ of a cost-minimization game $\Gamma$ is a *Pure Nash Equilibrium* if for every player $i \in [n]$ and for all $s'_i \in S_i$, $C_i(s) \leq C_i(s'_i, s_{-i})$.
- **Mixed Nash Equilibria**: A mixed strategy profile $\sigma \in \Sigma$ of a cost-minimization game $\Gamma$ is a *Mixed Nash Equilibria* if for every player $i \in [n]$ and for all $s'_i \in S_i$, $\mathop{\mathbb{E}}_{s \sim \sigma}[C_i(s)] \leq \mathop{\mathbb{E}}_{s \sim \sigma}[C_i(s'_i, s_{-i})]$
- **Correlated Equilibria**: A distribution $\mu$ over $S$ of a cost-minimization game $\Gamma$ is a *Correlated Equilibria* if for every player $i \in [n]$ and for all $s'_i \in S_i$, $\mathop{\mathbb{E}}_{s \sim \mu}[C_i(s) \mid s_i] \leq \mathop{\mathbb{E}}_{s \sim \mu}[C_i(s'_i, s_{-i}) \mid s_i]$
- **Coarse Correlated Equilibria**: A distribution $\mu$ over $S$ of a cost-minimization game $\Gamma$ is a *Coarse Correlated Equilibria* if for every player $i \in [n]$ and for all $s'_i \in S_i$, $\mathop{\mathbb{E}}_{s \sim \mu}[C_i(s)] \leq \mathop{\mathbb{E}}_{s \sim \mu}[C_i(s'_i, s_{-i})]$

## 6.2 Pareto Optimality

**Definition 6.2: Pareto Optimal Strategy**

Given a game $\Gamma$, a strategy profile $s \in S$ is pareto optimal also denoted by PO if $\nexists\, s' \in S$ such that

$$\forall\, i \in [n],\ c_i(s') \leq c_i(s) \qquad \exists\, i \in [n]\ c_i(s') < c_i(s)$$

or equivalently for all $s' \in S$, either $\forall\, i \in [n]$, $c_i(s) = c_i(s')$ or $\exists\, i \in [n]$, $c_i(s') > c_i(s)$.

Economists call Pareto Optimality “efficiency". PO induces a partial order over the set of all strategy profiles. Let $s, s' \in S$. We say that $s >_p s'$ if $\forall\, i \in [n],\ c_i(s') \leq c_i(s)$ and $\exists\, i \in [n]\ c_i(s') < c_i(s)$.

To introduce a total order we can think of social welfare function for example:

(1) Utilitarian Social Welfare: For any $s \in S$, $C(s) = \sum_{i=1}^{n} c_i(s)$

(2) Nash Social Welfare: For any $s \in S$, $C(s) = \sum_{i=1}^{n} c_i(s)$

(3) Egalitarian Social Welfare: For any $s \in S$, $C(s) = \max_{i=1}^{n} c_i(s)$

This allows us to quantitatively see how good or bad a equilibrium is by comparing two strategy profiles. Typically by “social welfare" we mean utilitarian social welfare. We will focus on calculating utilitarian social welfare from now on.

## 6.3 Price of Anarchy

For a game $\Gamma$ we also want to know how bad is the social welfare at equilibrium compared to the best possible social welfare. This ratio is know as Price of Anarchy.

**Definition 6.3: Price of Anarchy**

We denote it by PoA. For a game $\Gamma$:

$$\text{PoA}(\Gamma) = \frac{\text{Social welfare of “worst equilibrium"}}{\text{Optimal social welfare}} = \frac{\max\left\{\sum_{i=1}^{n} c_i(s) : s \in S \text{ is an MNE}\right\}}{\min\left\{\sum_{i=1}^{n} c_i(s) : s \in S\right\}}$$

### 6.3.1 PoA of Network (Atomic) Congestion Games

**Theorem 6.3.1**

The PoA in network congestion games with affine cost functions is $\frac{5}{2}$.