Soham Chatterjee
Email: soham.chatterjee@tifr.res.in
Course: Mathematical Foundations for Computer Sciences

Assignment - 2
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Problem 1

Let V be a vector space over $\mathbb{R}$. Show that the set $V_{\mathbb{C}} = V \times V$ with the operations below is a vector space over $\mathbb{C}$

$$(v_1, v_2) + (v_1', v_2') = (v_1 + v_1', v_2 + v_2')$$
$$(a + bi) \cdot (v_1, v_2) = (av_1 - bv_2, bv_1 + av_2)$$

This is called complexification and (v_1, v_2) is often denoted as $v_1 + v_2 i$. Show that:

- If B is a basis of V, it is also a basis of $V_{\mathbb{C}}$.
- For $\theta \in L(V)$, define the complexified operator $\theta_{\mathbb{C}} \in L(V_{\mathbb{C}})$ so that $\theta_{\mathbb{C}}(v_1 + v_2 i) = \theta(v_1) + \theta(v_2)i$. Show that for any basis B of V, we have $[\theta_{\mathbb{C}}]_B = [\theta]_B$
- For all $\lambda \in \mathbb{R}$, λ is an eigenvalue of θ if and only if it is an eigenvalue of $\theta_{\mathbb{C}}$. For $\lambda \in \mathbb{C}$, λ is an eigenvalue of $\theta_{\mathbb{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $\theta_{\mathbb{C}}$ and they have the same multiplicity. Conclude that every real operator over an odd dimensional real vector space has an eigenvalue.

Solution:

- B is a basis of V. Let $\dim V = n$. Suppose $B = \{b_1, \dots, b_n\}$. We want to show B is also a basis of $V_{\mathbb{C}}$ i.e. the set $B' = \{(b_i, 0) \colon i \in [n]\}$ is a basis of $V_{\mathbb{C}}$. So we have to show $\langle B_{\mathbb{C}} \rangle = V_{\mathbb{C}}$. From now if B is a basis of V then by $B_{\mathbb{C}}$ we denote the set $\{(b, 0) \colon b \in B\}$.

  Now $\forall\ i \in [n]$, $(b_i, 0) \in V_{\mathbb{C}}$. Therefore $B_{\mathbb{C}} \subseteq V_{\mathbb{C}}$. Hence $\langle B_{\mathbb{C}} \rangle \subseteq V_{\mathbb{C}}$. Now we have to show that $\langle B_{\mathbb{C}} \rangle \supseteq V_{\mathbb{C}}$. So suppose $(v_1, v_2) \in V_{\mathbb{C}}$. Then $v_1, v_2 \in V$. Hence $\exists!\ \{a_{1,i}\}_{i \in [n]}$ and $\{a_{2,i}\}_{i \in [n]}$ such that

  $$v_1 = \sum_{i=1}^{n} a_{1.i} b_i, \qquad v_2 = \sum_{i=1}^{n} a_{2,i} b_i$$

  Now for any $v \in V$, $(a + bi) \cdot (v, 0) = (av, bv)$. Therefore we have

  $$\sum_{i=1}^{n} \left(a_{1,i} + a_{2,i} i\right)(b_i, 0) = \sum_{i=1}^{n} \left(a_{1,i} b_i, a_{2,i} b_i\right) = \left(\sum_{i=1}^{n} a_{1,i} b_i, \sum_{i=1}^{n} a_{2,i} b_i\right) = (v_1, v_2)$$

  Therefore $(v_1, v_2) \in \langle B_{\mathbb{C}} \rangle$. Hence

  $$V_{\mathbb{C}} \subseteq \langle B_{\mathbb{C}} \rangle \implies V_{\mathbb{C}} = \langle B_{\mathbb{C}} \rangle$$

  Hence B is also a basis of $V_{\mathbb{C}}$

- By the above part we know if B is a basis of V then $B_{\mathbb{C}}$ is basis of $V_{\mathbb{C}}$. Now if $\theta \in L(V)$ then $\theta_{\mathbb{C}} \in L(V_{\mathbb{C}})$ such that $\theta_{\mathbb{C}}(v_1 + v_2 i) = \theta(v_1) + \theta(v_2)i$. So for any $b + 0i \in B_{\mathbb{C}}$ we have

  $$\theta_{\mathbb{C}}(b + 0i) = \theta(b) + \theta(0)i = \theta(b) + 0i$$

  Let b_j be the j^{th} vector of B. $\exists!\ a_{j,l}$ forall $l \in [n]$ such that $\theta\left(b_j\right) = \sum\limits_{l=1}^{n} a_{j,l} b_l$. Then $[\theta]_B = \left(a_{j,l}\right)_{1 \leq j,l \leq n}$. Then

  $$\theta_C\left(b_j\right) = \theta(b) + 0i = \sum_{l=1}^{n} a_{j,l} b_l + 0i = \sum_{l=1}^{n} \left(a_{j,l} + 0i\right)(b_l + 0i) = \sum_{l=1}^{n} a_{j,l}(b_l + 0i)$$

  Therefore $[\theta_{\mathbb{C}}]_B = \left(a_{j,l}\right)_{1 \leq j,l \leq n}$. Therefore $[\theta_{\mathbb{C}}]_B = [\theta]_B$.

- Let $\lambda \in \mathbb{R}$ is an eigenvalue of $\theta \in L(V)$. Suppose $v \in V$, $v \neq 0$ be eigenvector corresponding to λ. Then in $V_{\mathbb{C}}$ we have the vector $v + 0i$. Then

$$\theta_{\mathbb{C}}(v + 0i) = \theta(v) + \theta(0)i = \lambda v + 0i = \lambda v + \lambda \cdot 0i = \lambda(v + 0i)$$

Hence λ is also an eigenvalue of $\theta_{\mathbb{C}}$. Now suppose $\lambda \in \mathbb{R}$ is an eigenvalue of $\theta_{\mathbb{C}}$. Then suppose $v_1 + v_2 i \in V_{\mathbb{C}}$, $v_1 + v_2 \neq 0$ be an eigenvector corresponding to λ. Now

$$\theta_{\mathbb{C}}(v_1 + v_2 i) = \theta(v_1) + \theta(v_2)i,\ \theta_{\mathbb{C}}(v_1 + v_2 i) = \lambda(v_1 + v_2 i) = \lambda v_1 + \lambda v_2 i \implies \theta(v_1) + \theta(v_2)i = \lambda v_1 + \lambda v_2 i$$

Hence we get $\theta(v_1) = \lambda v_1$ and $\theta(v_2) = \lambda v_2$. Since $v_1 + v_2 i \neq 0$, either $v_1 \neq 0$ or $v_2 \neq 0$. So there exists at least one eigenvector for λ in V.

Suppose $\lambda \in \mathbb{C}$. Now we know $\overline{\overline{\lambda}} = \lambda$. So showing if λ is eigenvalue of $\theta_{\mathbb{C}} \implies \overline{\lambda}$ is eigenvalue of $\theta_{\mathbb{C}}$ is enough since then replacing $\overline{\lambda}$ in place of λ we get that if $\overline{lm}$ is eigenvalue of $\theta_{\mathbb{C}} \implies \overline{\overline{\lambda}} = \lambda$ is eigenvalue of $\theta_{\mathbb{C}}$. Now suppose $v_1 + v_2 i \in V_{\mathbb{C}}$, $v_1 + v_2 i \neq 0$ be eigenvector corresponding to λ. Let $\lambda = a + bi$ where $a, b \in \mathbb{R}$. Then

$$\lambda(v_1 + v_2 i) = (a + bi)(v_1 + v_2 i) = (av_1 - bv_2, bv_1 + av_2) = \theta(v_1) + \theta(v_2)i$$

Hence we have $\theta(v_1) = av_1 - bv_2$ and $\theta(v_2) = bv_1 + av_2$. Hence

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Problem 2

Solution: ■

Problem 3

Solution:

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Problem 4

Solution: ■

Problem 5

Solution:

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