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Course: Mathematical Foundations for Computer Sciences

Assignment - 3
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Problem 1

Let $n > 0$. Consider a directed graph on partitions of n (with the parts sorted by size). There is a (directed) edge from a partition to another if the latter can be obtained from the former by moving a "unit" from a part to the one immediately after, while maintaining the order. For example, there is an edge from the partition $25 = 10 + 6 + 4 + 3 + 2$ to $25 = 10 + 5 + 5 + 3 + 2$ as a "unit" is moved from 6 to 4 and it does not change the order of the parts, but no edge vice-versa. Similarly, there is an edge from the partition $25 = 10 + 6 + 4 + 3 + 2$ to $25 = 10 + 6 + 4 + 3 + 1 + 1$. Characterize all the sink partitions (those without any outgoing edges) that can be by starting from the partition $n = n$ (with only one part) and following these edges.

Solution: $\blacksquare$

Problem 2

Consider a tournament with k teams. Each team gets a rank at the end of the tournament, possibly allowing for ties. For example, if there are 5 teams, team 3 may be first, teams 1, 4 maybe tied second, and teams 2, 5, may be tied third. Compute an explicit formula for the exponential generating function of $R(k)$ for the number of rankings, assuming that $R(0) = 1$.

Solution:

$\blacksquare$

Problem 3

Let $(U, \leq)$ be poset and M be a square matrix with rows and columns indexed by U such that the (x, y)-th entry of M is $\mathbb{1}(x \leq y)$.

- Show that M is invertible.
- Compute the inverse of M when $(U, \leq)$ is the divisibility poset for positive integers at most n (for some fixed n) and use it to derive the Möbius inversion formula.

Solution: $\blacksquare$

Problem 4

Let $k, p > 0$ be integers where p is prime. A seller has kp marbles, where marble $i \in [kp]$ costs ₹i, that he is trading. Call a trade "divisible" if both the cost and the number of marbles sold is a multiple of p (selling no marbles is one such trade). How many divisible trades are there? Write your answer in terms of p-th roots of unity.

Solution:

$\blacksquare$

Problem 5

A graph is a thunderstorm graph if every connected component is either a cycle (clouds) or a path (lightning bolts) or isolated vertices (raindrops). Compute an explicit formula, i.e., a formula without summation signs, for the exponential generating function of:

- The number of connected thunderstorm graphs on the vertex set $[n]$.

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Solution:

$\blacksquare$