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Course: Probability Theory

Assignment - 1
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Problem 1

(a) Prove that if $A_1, A_2, \dots, A_n$ are events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = S_1 - S_2 + S_3 - \dots + (-1)^{n-1} S_n$$

where

$$\begin{aligned}
S_1 &= \sum_i \mathbb{P}(A_i) \\
S_2 &= \sum_{i<j} \mathbb{P}\left(A_i \cap A_j\right) \\
S_3 &= \sum_{i<j<k} \mathbb{P}\left(A_i \cap A_j \cap A_k\right) \\
&\dots \\
S_n &= \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n)
\end{aligned}$$

This is also known as the *inclusion-exclusion* principle.

(b) *Bonferroni inequalities* state that the sum of the first terms in the right-hand side of the identity we proved above is alternately an upper bound and a lower bound for the left-hand side. i.e., for odd $k \le n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le S_1 - S_2 + \dots + S_k$$

and for even $k \le n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \ge S_1 - S_2 + \dots - S_k$$

Note that from what we showed above Bonferroni inequality holds with equality for $k = n$.

Prove Bonferroni inequalities. Observe that the case of $k = 1$ is what you know as the *union bound* or Boole's inequality.

Solution:

(a) We will prove it using induction on n. For base case $t = 1$. Then $\mathbb{P}[A_1] = S_1 = \sum\limits_i \mathbb{P}[A_i] = \mathbb{P}[A_1]$. Hence for base case it holds. Now let this is true for $t = n$. For $t = n + 1$

$$\mathbb{P}\left(\bigcup_{i=1}^{k+1} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{k} A_i\right) + \mathbb{P}\left(A_{k+1} \setminus \bigcup_{i=1}^{k} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{k} A_i\right) + \mathbb{P}(A_{k+1}) - \mathbb{P}\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

Now using inductive hypothesis we have

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right) &= \sum_{t=1}^{k} (-1)^{t-1} \sum_{J \subseteq [k],\, |J| = t} \mathbb{P}\left[\bigcap_{i \in J} (A_i \cap A_{k+1})\right] \\
&= \sum_{t=1}^{k} (-1)^{t-1} \sum_{J \subseteq [k],\, |J| = t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i \in J} A_i\right)\right]
\end{aligned}$$

Therefore we have

$$
\begin{aligned}
&\mathbb{P}\left(\bigcup_{i=1}^{k} A_i\right) + \mathbb{P}(A_{k+1}) - \mathbb{P}\left(\bigcup_{i=1}^{k}(A_i \cap A_{k+1})\right) \\
&= \mathbb{P}\left(\bigcup_{i=1}^{k} A_i\right) + \mathbb{P}(A_{k+1}) - \left[\sum_{t=1}^{k}(-1)^{t-1} \sum_{J\subseteq[k],\ |J|=t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i\in J} A_i\right)\right]\right] \\
&= \sum_{t=1}^{k}(-1)^{t-1} \sum_{T\subseteq[k],\ |T|=t} \mathbb{P}\left[\bigcap_{i\in T} A_i\right] + \mathbb{P}[A_{k+1}] + \sum_{t=1}^{k}(-1)^{t} \sum_{J\subseteq[k],\ |J|=t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i\in J} A_i\right)\right] \\
&= \sum_{i=1}^{k+1} \mathbb{P}[A_i] + \sum_{t=1}^{k}(-1)^{t} \sum_{T\subseteq[k],\ |T|=t+1} \mathbb{P}\left[\bigcap_{i\in T} A_i\right] + \sum_{t=1}^{k}(-1)^{t} \sum_{J\subseteq[k],\ |J|=t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i\in J} A_i\right)\right] \\
&= \sum_{i=1}^{k+1} \mathbb{P}[A_i] + \sum_{t=1}^{k}(-1)^{t} \left(\sum_{T\subseteq[k],\ |T|=t+1} \mathbb{P}\left[\bigcap_{i\in T} A_i\right] + \sum_{J\subseteq[k],\ |J|=t} \mathbb{P}\left[A_{k+1} \cap \left(\bigcap_{i\in J} A_i\right)\right]\right) \\
&= \sum_{i=1}^{k+1} \mathbb{P}[A_i] + \sum_{t=1}^{k}(-1)^{t} \sum_{T\subseteq[k+1],\ |T|=t+1} \mathbb{P}\left[\bigcap_{i\in T} A_i\right] \\
&= \sum_{t=1}^{k+1}(-1)^{t-1} \sum_{T\subseteq[k+1],\ |T|=t} \mathbb{P}\left[\bigcap_{i\in T} A_i\right]
\end{aligned}
$$

(b) Suppose $\omega \in \Omega$. We will count the contribution of ω to the probability $\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right]$. Now if $\omega \notin \bigcup_{i=1}^{n} A_i$ then ω has no contribution to the probability $\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right]$. Now we also have $\omega \notin \bigcap_{i\in J} A_i$ for all $J \subseteq [n]$. Hence ω has no contribution to any of the S_i for all $i \in [n]$.

Now suppose $\omega \in \bigcup_{i=1}^{n} A_i$. Let ω is in exactly $t \le n$ events among $A_1, \dots, A_n$. WLOG assume those events are $A_1, \dots, A_t$. Now ω is counted once in $\bigcup_{i=1}^{n} A_i$. In case of S_i with $i \le t$, for all $J \subseteq [t]$ with $|J| = i$, $\omega \in \bigcap_{j\in J} A_j$. Hence in case of S_i, ω is counted $\binom{t}{i}$ times. Hence for $k \le t$, in $S_1 - S_2 + \cdots + (-1)^k S_k$, ω is counted $\sum_{i=1}^{k}(-1)^{i-1}\binom{t}{i}$ many times. We are only considering the case $k \le t$ because when $k > t$ we have $\binom{t}{i} = 0$ for $i > k$ and therefore the sum only reduces to $k \le t$ case. Now two cases arise:

- $k = t$: Then $\sum_{i=1}^{k}(-1)^{i-1}\binom{t}{i} = \sum_{i=1}^{t}(-1)^{i-1}\binom{t}{i} = 1 - (1-1)^t = 1$. That is, ω is counted the same times on both sides.
- $k < t$: Now ω counted in LHS is once but ω counted in RHS is $\sum_{i=1}^{k}(-1)^{i-1}\binom{t}{i}$. So we will show $\sum_{i=1}^{k}(-1)^{i-1}\binom{t}{i}$ is ≥ 1 when k is odd and ≤ 1 when k is even. Or equivalently we will show

$$
f(k) = 1 - \sum_{i=1}^{k}(-1)^{i-1}\binom{t}{i} = \sum_{i=0}^{k}(-1)^{i}\binom{t}{i} = \begin{cases} \ge 0 & \text{when } k \text{ is even} \\ \le 0 & \text{when } k \text{ is odd} \end{cases}
$$

Claim: $f(k) = \sum_{i=0}^{k}(-1)^i\binom{t}{i} = (-1)^k\binom{t-1}{k}$

Proof: We will prove by induction. For base case $k = 0$ we have $\binom{t}{0} = 1 = (-1)^o\binom{t-1}{0}$. So the base case holds. Let this is true for k. Now we have to prove for $k+1$. We have for any $m, r \in \mathbb{N}$ with $m \geq r$ such that

$$\binom{m}{r} = \binom{m-1}{r} + \binom{m-1}{r-1}$$

So therefore

$$\begin{aligned}\sum_{i=0}^{k+1}(-1)^i\binom{t}{i} &= \sum_{i=0}^{k}(-1)^i\binom{t}{i} + (-1)^{k+1}\binom{t}{k+1}\\ &= (-1)^k\binom{t-1}{k} + (-1)^{k+1}\binom{t-1}{k+1} + (-1)^{k+1}\binom{t-1}{k}\\ &= (-1)^{k+1}\binom{t-1}{k+1}\end{aligned}$$

Hence by Mathematical Induction it is true for all k. $\square$

Therefore we get $f(k) = (-1)^k\binom{t-1}{k}$. Hence if k is even $f(k) = (-1)^k\binom{t-1}{k} = \binom{t-1}{k} \geq 0$ and when k is odd $f(k) = (-1)^k\binom{t-1}{k} = -\binom{t-1}{k} \leq 0$. Since ω is arbitrary and for odd k, ω is counted more in RHS so we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq S_1 - S_2 + \ldots + S_k$$

and for even k, ω is counted lesser times in RHS than LHS so we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq S_1 - S_2 + \ldots - S_k$$

Therefore we have the Bonferroni Inequalities.

$\square$

Problem 2

Prove or disprove the following:

- The conditional independence of A and B given C implies A and B are independent.
- Independence of A and B implies the conditional independence of A and B given C.

If you disproved either of the claims above, for which events C is it then the case that the following statement holds: for all events A and B, the events A and B are conditionally independent given C if and only if A and B are independent.

Solution:

1. We will disprove both of the statements by constructing a counter example.
   - Consider we have two decks of cards. Now in the from the first deck we pick a card. It it is a face card then we pick a card uniformly from all non-face cards in the second deck. And if the picked card from the first deck is a non-face card then we pick a card uniformly at random from all non-numbered cards in the second deck. Here the aces comes into both non-numbered cards and non-face cards. So now let
     - A be the event of picking 'King' in the first deck
     - B be the event of picking 'Ace' in the second deck

- C be the event of picking 'Jack' in the first deck

Now $\mathbb{P}[A \mid C] = 0$ and $\mathbb{P}[B \mid C] = \frac{4}{40} = \frac{1}{10}$ and

$$\mathbb{P}[A \cap B \mid C] = \mathbb{P}[\text{Picking ('King','Ace')} \mid \text{Picking 'Jack' in first deck}] = 0 = \mathbb{P}[A \mid C]\mathbb{P}[B \mid C]$$

So A, B are independent conditioned on C. Now $\mathbb{P}[A] = \frac{4}{52} = \frac{1}{13}$, $\mathbb{P}[B] = \frac{12}{52}\frac{4}{40} + \frac{40}{52}\frac{4}{16} = \frac{3}{130} + \frac{5}{26} = \frac{14}{65}$. But $\mathbb{P}[A \cap B] = \frac{4}{52}\frac{4}{40} = \frac{3}{130} \neq \mathbb{P}[A]\mathbb{P}[B]$. So they are not independent without conditioning on C.

- Let we have two unbiased 6–faced dice. We throw both the dice. Let
  - A be the event that first dice outcome is 2
  - B be the event that second dice outcome is 5.
  - C be the event that the sum of first dice outcome and second dice outcome is 6

  Then $\mathbb{P}[A] = \mathbb{P}[B] = \frac{1}{6}$. And $\mathbb{P}[A \cap B] = \frac{1}{36}$ since $(2, 5)$ is one outcome of all 36 possible outcomes. Hence $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. So A, B are independent events. Certainly $\mathbb{P}[C] > 0$. Then $\mathbb{P}[A \mid C]$, $\mathbb{P}[B \mid C] \neq 0$. But the $\mathbb{P}[A \cap B \mid C] = 0$ since $2 + 5 \neq 6$. Hence $\mathbb{P}[A \cap B \mid C] \neq \mathbb{P}[A \mid C]\mathbb{P}[B \mid C]$. Hence they are not independent conditioning on C.

2. If we take $C = \Omega$ then for any two events A, B, $\mathbb{P}[A \mid C] = \mathbb{P}[A]$ and $\mathbb{P}[B \mid C] = \mathbb{P}[B]$. Therefore in that case A, B are independent if and only if A, B are independent conditioned on C.

□

Problem 3

Let $A_1, A_2, \ldots$ be a sequence of events. Define

$$B_n = \bigcup_{m=n}^{\infty} A_m \quad C_n = \bigcap_{m=n}^{\infty} A_m$$

Clearly $C_n \subseteq A_n \subseteq B_n$. Also, the sequences $\{B_n\}$ and $\{C_n\}$ are decreasing respectively. Let

$$B = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m \quad C = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} A_m$$

The events B and C are denoted by $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ respectively. Show that

(a) $B = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$.

(b) $C = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$.

We say that a sequence $\{A_n\}$ converges to a limit A if B and C are the same set A. We denote this by $A_n \to A$. Suppose this is the case, then show that

(c) A is an event.

(d) $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

Solution:

(a) Let $\omega \in B$. Then $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} A_m$. Hence $\omega \in \bigcup_{m \geq n} A_m$ forall $n \in \mathbb{N}$. Hence $\omega \in A_k$ for some $k \in \mathbb{N}$. Let k_1 be the least number such that $\omega \in A_{k_1}$. Then we also have $\omega \in B_{k_1+1}$. So we have some $k_2 \geq k_1 + 1$ such that $\omega \in A_{k_2}$. Then $\omega \in B_{k_2+1}$. So there exists $k_3 \geq k_2 + 1$ such that $\omega \in A_{k_3}$. Continuing like this at i^{th} step we have some $k_{i+1} \geq k_i + 1$ such that $\omega \in A_{k_{i+1}}$ and so on. So now we got an strictly increasing infinite sequence of positive integers $\{k_1, k_2, k_3 \ldots, k_i, \ldots\}$ such that $\omega \in A_{k_j}$ for all $j \in \mathbb{N}$. Hence $\omega \in \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$. Hence

$$B \subseteq \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}$$

Now let $\omega \in \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$. Let $\{s_n\}_{n\in\mathbb{N}}$ be the strictly increasing sequence of positive integers such that $\omega \in A_{s_n}$. Hence for all $m \in \mathbb{N}$ we have $\omega \in B_m$ because $\exists n \in \mathbb{N}$ such that $s_n > m$ and $\omega \in A_{s_m} \implies \omega \in B_m$. Therefore $\omega \in \bigcap\limits_{m=1}^{\infty} B_m$. Therefore we have

$$\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\} \subseteq B$$

Hence we have $B = \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$.

(b) Let $\omega \in \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$. Hence there exists $n_0 \in \mathbb{N}$ such that $\omega \in A_n$ for all $n > n_0$. Therefore $\omega \in C_n$ for all $n > n_0$. Since $C = \bigcup\limits_{n=1}^{\infty} C_n$ we have $\omega \in C$. So we have

$$\{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\} \subseteq C$$

Now suppose $\omega \in C$. So $\exists\ n \in \mathbb{N}$ such that $\omega \in C_n$. Since $C_n = \bigcap\limits_{m\geq n} A_m$ we have $\omega \in A_m$ for all $m \geq n$. Hence $\omega \in A_m$ for all but finitely many values of n. So $\omega \in \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$. Hence we get

$$C \subseteq \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$$

Therefore we get $C = \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$.

(c) For all $n \in \mathbb{N}$ B_n is the countable union of events. So B_n is an event for all $n \in \mathbb{N}$. And similarly $\forall\ n \in \mathbb{N}$, C_n is the countable intersection of events. Therefore C_n is also an event. Now since B is just countable intersection of all B_n's and each B_n is event we have that B is also an event. And similarly since C is just the countable union of all C_n's and each C_n is an event we have that C is also an event. Now given that $B = C = A$. Therefore A is also an event.

(d) Since for each $n \in \mathbb{N}$ we have that $C_n \subseteq A_n \subseteq B_n$. Therefore

$$\mathbb{P}[C_n] \leq \mathbb{P}[A_n] \leq \mathbb{P}[B_n]$$

Hence we have

$$\lim_{n\to\infty} \mathbb{P}[C_n] \leq \lim_{n\to\infty} \mathbb{P}[A_n] \leq \lim_{n\to\infty} \mathbb{P}[B_n]$$

Now we will analyze $\lim\limits_{n\to\infty} \mathbb{P}[B_n]$ and $\lim\limits_{n\to\infty} \mathbb{P}[C_n]$. Now we have

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \supseteq B_n \supseteq \cdots \quad \text{and} \quad C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots \subseteq C_n \subseteq \cdots$$

$$\mathbb{P}[B] = \mathbb{P}\left[\bigcap_{n=1}^{\infty} B_n\right] = \mathbb{P}\left[\lim_{k\to\infty} \bigcap_{n=1}^{k} B_n\right] = \lim_{k\to\infty} \mathbb{P}\left[\bigcap_{n=1}^{k} B_n\right] = \lim_{k\to\infty} \mathbb{P}[B_k]$$

Similarly we have

$$\mathbb{P}[C] = \mathbb{P}\left[\bigcup_{n=1}^{\infty} C_n\right] = \mathbb{P}\left[\lim_{k\to\infty} \bigcup_{n=1}^{k} C_n\right] = \lim_{k\to\infty} \mathbb{P}\left[\bigcup_{n=1}^{k} C_n\right] = \lim_{k\to\infty} \mathbb{P}[C_k]$$

Hence we get $\lim\limits_{n\to\infty} \mathbb{P}[B_n] = \mathbb{P}[B]$ and $\lim\limits_{n\to\infty} \mathbb{P}[C_n] = \mathbb{P}[C]$. Since $B = C$ we have

$$\lim_{n\to\infty} \mathbb{P}[B_n] = \mathbb{P}[B] = \mathbb{P}[C] = \lim_{n\to\infty} \mathbb{P}[C_n]$$

And since $A = B = C$ we have $\mathbb{P}[B] = \mathbb{P}[A] = \mathbb{P}[C]$. Hence

$$\lim_{n\to\infty} \mathbb{P}[C_n] \leq \lim_{n\to\infty} \mathbb{P}[A_n] \leq \lim_{n\to\infty} \mathbb{P}[B_n] \implies \mathbb{P}[A] = \mathbb{P}[B] \leq \lim_{n\to\infty} A_n \leq \mathbb{P}[C] = \mathbb{P}[A]$$

Therefore $\lim\limits_{n\to\infty} \mathbb{P}[A_n] = \mathbb{P}[A]$

$\square$

Problem 4

10% of the surface of a sphere is colored white, the rest is black. Show that, irrespective of the manner in which the colors are distributed, it is possible to inscribe a cube in S with all its vertices black.
Hint: For a given distribution of colors, select the cube"uniformly randomly" (you should make this more concrete). First note that it is enough to prove that there is a non-zero probability with which all the vertices of this random cube are colored black (why?). Now try to use the union bound from Problem 1(b) above to show this.

Solution: To show that there exists a cube in S with all its vertices black it is enough to show that if a random cube is chosen in S the probability of all vertices black is greater than 0. Now we have

$$\underset{\substack{C:\,cube\\ C \text{ is in } S}}{\mathbb{P}}[\text{All vertices of } C \text{ is black}] = 1 - \underset{\substack{C:\,cube\\ C \text{ is in } S}}{\mathbb{P}}[\text{At least one of the vertices of } C \text{ is white}]$$

So its is enough to show that $\underset{\substack{C:\,cube\\ C \text{ is in } S}}{\mathbb{P}}[\text{At least one of the vertices of } C \text{ is white}] < 1$. Now we also have

$$\underset{\substack{C:\,cube\\ C \text{ is in } S}}{\mathbb{P}}[\text{At least one of the vertices of } C \text{ is white}] = \underset{\substack{X_i \in S\\ \forall\, i \in [8]}}{\mathbb{P}}[\exists\, i \in [8]\ X_i \text{ is colored white} \mid X_1, \ldots, X_8 \text{ forms a cube}]$$

Now by Union Bound we have

$$\begin{aligned}\underset{\substack{X_i \in S\\ \forall\, i \in [8]}}{\mathbb{P}}&[\exists\, i \in [8]\ X_i \text{ is colored white} \mid X_1, \ldots, X_8 \text{ forms a cube}]\\ &\le \sum_{j=1}^{8} \underset{\substack{X_i \in S\\ \forall\, i \in [8]}}{\mathbb{P}}\left[X_j \text{ is colored white} \mid X_1, \ldots, X_8 \text{ forms a cube}\right]\end{aligned}$$

So now showing

$$\sum_{j=1}^{8} \underset{\substack{X_i \in S\\ \forall\, i \in [8]}}{\mathbb{P}}\left[X_j \text{ is colored white} \mid X_1, \ldots, X_8 \text{ forms a cube}\right] < 1$$

is enough. Now for any $j \in [8]$,

$$\underset{\substack{X_i \in S\\ \forall\, i \in [8]}}{\mathbb{P}}\left[X_j \text{ is colored white} \mid X_1, \ldots, X_8 \text{ forms a cube}\right] = \underset{\substack{X_i \in S\\ \forall\, i \in [8]}}{\mathbb{P}}\left[X_j \text{ is colored white}\right] = \frac{1}{10}$$

The last equality because X_j is colored white if it is a point picked from the 10% area of the sphere which is colored white and the probability of that is $\frac{1}{10}$.Therefore we have

$$\sum_{j=1}^{8} \underset{\substack{X_i \in S\\ \forall\, i \in [8]}}{\mathbb{P}}\left[X_j \text{ is colored white} \mid X_1, \ldots, X_8 \text{ forms a cube}\right] = \sum_{j=1}^{8} \frac{1}{10} = \frac{8}{10} < 1$$

Therefore we have $\underset{\substack{C:\,cube\\ C \text{ is in } S}}{\mathbb{P}}[\text{At least one of the vertices of } C \text{ is white}] < 1 \implies \underset{\substack{C:\,cube\\ C \text{ is in } S}}{\mathbb{P}}[\text{All vertices of } C \text{ is black}] > 0$. Which means there exists a cube in S with all vertices black

$\square$