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Course: Algorithms

Assignment - 1
Dept: STCS
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Problem 1 P3 (15 marks)

Solve the recurrences:

(i) $T(n) = 2T(n/2) + n\log n$,

(ii) $T(n) = 7T(n/3) + n^2$,

(iii) $T(n) = \sqrt{n}T(\sqrt{n}) + n$.

Solution:

□

Problem 2 P4 (5 marks)

Give the best upper bounds you can on the nth Fibonacci number F_n, where $F_n = F_{n-1} + F_{n-2}$ and $F_1 = F_2 = 1$ are conditionally independent given C if and only if A and B are independent.

Solution:

□

Problem 3 P5 (10 marks)

Consider two sets A and B, each having n integers in the range from 0 to $10n$. We wish to compute the Cartesian sum of A and B, defined by

$$C = \{x + y : x \in A, y \in B\}$$

Note that the integers in C are in the range 0 to $20n$. We want to find the elements in C and the number of times each element of C is realized as a sum of elements in A and B. Give an algorithm that solves the problem in $O(n\log n)$ time, and prove correctness.

Solution:

□

Problem 4 P6 (20 marks)

Define $[n] := \{1, 2, \ldots, n\}$. You are given n, and oracle access to a function $f : [n]\times[n] \to [n]\times[n]$ that takes as input two positive integers of value at most n, and returns two positive integers of value at most n. Let $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ be the first and second coordinates of $f(x_1, x_2)$, respectively. You are also told that f_i is monotone nondecreasing in coordinate i when coordinate $3-i$ is kept fixed, and monotone nonincreasing in coordinate $3-i$ when coordinate i is kept fixed. That is, given $x_1 \le x_1' \in [n]$ and $x_2 \le x_2' \in [n]$, $f_1(x_1, x_2) \le f_1\left(x_1', x_2\right)$, and $f_1(x_1, x_2) \ge f_1\left(x_1, x_2'\right)$. Similarly, $f_2(x_1, x_2) \ge f_2\left(x_1', x_2\right)$, and $f_2(x_1, x_2) \le f_2\left(x_1, x_2'\right)$.

The problem is to find a fixed point of the function, i.e., values $x_1, x_2 \in [n]$ so that $f(x_1, x_2) = (x_1, x_2)$. Give an algorithm that given n and oracle access to such a function f, finds a fixed point of f in time $O(\text{poly}(\log n))$. You must also give a proof of correctness, and running time analysis.

Solution:

□

Problem 5 P7 (15 marks)

A palindrome is a nonempty string over some alphabet that reads the same forward and backward. Examples of palindromes are all strings of length 1, civic, racecar, and aibohphobia. Give an efficient algorithm, with proof of correctness and run-time analysis, to find the longest palindrome that is a subsequence of a given input string. For example, given the input string character, your algorithm should return carac.

Solution:

□

Problem 6 P8 (25 marks)

The purpose of this question is to extend the closest-points algorithm seen in the first lecture, to give an $O\left(n\log^2 n\right)$ algorithm for finding the closest pair of points in 3 dimensions. All points in this question are in $\mathbb{R}^3$.

(a) (5 marks) Prove that, if all points are at least distance δ apart, a cube with each dimension of size 2δ contains at most a constant (say k) number of points.

(b) (10 marks) You are now given 2 sets of points S_1 and S_2, each containing n points. The distance between any pair of points in S_1 is at least δ, and further, each point in S_1 has z-coordinate in $[0,\delta]$. Similarly, the distance between any pair of points in S_2 is at least δ, and each point in S_2 has z-coordinate in $[-\delta,0]$.

Extend the algorithm discussed in class to give an $O(n\log n)$-time algorithm for finding the closest pair of points in $S_1 \cup S_2$. Note that, by the first part of the question, any cube with each dimension at most 2δ, contains at most $2k$ points from $S_1 \cup S_2$.

(c) (10 marks) Given a set S of n points in $\mathbb{R}^3$, now give an $O\left(n\log^2 n\right)$-time algorithm to find the closest pair of points.

Solution:

□

Problem 7 P9 (10 marks)

This problem relates to one of the questions asked in class. For any $p,q \geq 1$, and any points x, y, and $z \in \mathbb{R}^2$, prove or disprove the following:

$$\|x-y\|_p \leq \|x-z\|_p \Leftrightarrow \|x-y\|_q \leq \|x-z\|_q$$

That is, prove or disprove that y is closer to x than z in the L_p distance metric if and only if it is closer to x in the L_q distance metric As usual, $\|x-y\|_p = ((x_1-y_1)^p + (x_2-y_2)^p)^{1/p}$.

Solution:

□