Soham Chatterjee
Email: sohamc@cmi.ac.in
Course: Quantum Information Theory

Assignment - 2.2 : Quantum Foundations
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For all the questions $[k] := \{1, 2, \ldots, k\}$ where $k \in \mathbb{N}$

Problem 1

For $T : \mathcal{H} \to \mathcal{H}$, prove that

$$\sum_{i=1}^{d} \langle e_i \,|Te_i\rangle = \sum_{i=1}^{d} \langle f_i \,|T f_i\rangle$$

if $\{|e_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ and $\{|f_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ are ONB.

Solution: Let $S : \mathcal{H} \to \mathcal{H}$ where it maps the basis vectors from $|e_i\rangle \to |f_i\rangle$. Then $S\,|e_i\rangle = |f_i\rangle$. Hence S is an orthonormal matrix since

$$\langle e_j|\, S^\dagger S\, |e_i\rangle = \langle f_j|\, f_i\rangle = \delta_{ji} \quad \text{and} \quad \langle f_j|\, SS^\dagger\, |f_i\rangle = \langle e_j|\, e_i\rangle = \delta_{ji}$$

Hence

$$\sum_{i=1}^{d} \langle f_i|T f_i\rangle = \sum_{i=1}^{d} \langle e_i|S^\dagger TS\, |e_i\rangle = tr(S^\dagger TS) = tr(SS^\dagger T) = tr(T) = \sum_{i=1}^{d} \langle e_i\,|Te_i\rangle$$

Therefore we have

$$\sum_{i=1}^{d} \langle e_i \,|Te_i\rangle = \sum_{i=1}^{d} \langle f_i \,|T f_i\rangle$$

□

Problem 2

If $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \leq i \leq d\}$ and $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \leq i \leq d\}$ are ONB, then $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\} \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ is ONB

Solution: Let $|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then $|\psi\rangle = \sum\limits_{i=1}^{d} \alpha_i\,|e_i\rangle$ where $\alpha_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \leq i \leq d\}$ is ONB for $\mathcal{H}_1$. Hence

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i\,|e_i\rangle \otimes |\phi\rangle$$

Now $|\phi\rangle = \sum\limits_{i=1}^{d} \beta_i\,|f_i\rangle$ where $\beta_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \leq i \leq d\}$ is ONB for $\mathcal{H}_2$. Hence

$$\forall\, i \in [d] \;\; |e_i\rangle \otimes |phi\rangle = \sum_{j=1}^{d} \beta_j\,|e_i\rangle \otimes |f_j\rangle$$

Thereofore we get

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i\,|e_i\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{d} \beta_j\,|e_i\rangle \otimes |f_j\rangle = \sum_{1 \leq i,j \leq d} \alpha_i\beta_j\,|e_i\rangle \otimes |f_j\rangle$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\}$ is a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Now for any $i1, i2, j1, j2 \in [d]$

$$(\langle e_{i1}| \otimes \langle f_{j1}|)(|e_{i2}\rangle \otimes |f_{j2}\rangle) = \langle e_{i1}|e_{i2}\rangle\,\langle f_{j1}|f_{j2}\rangle = \delta_{i1,i2}\,\delta_{j1,j2}$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\}$ is orthonormal. Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\}$ is a ONB for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

□

Problem 3

Let $\{|g_k\rangle \mid 1 \leq i \leq d_2\} \subseteq \mathcal{H}_2$ be ONB. For $T \in \mathscr{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, let $tr_2(T) \in \mathscr{L}(\mathcal{H}_1)$ denote the operator satisfying

$$\langle u| \, tr_2(T) \, |v\rangle = \sum_k \langle u \otimes g_k| \, T \, |v \otimes g_k\rangle$$

for any choice $|u\rangle, |v\rangle \in \mathcal{H}_1$. Prove that $\sum_k \langle u \otimes g_k| \, T \, |v \otimes g_k\rangle$ is invariant.