

For all the questions

- $[k] := \{1, 2, \dots, k\}$ where $k \in \mathbb{N}$.
- $\mathcal{L}(\mathcal{H}) :=$ Linear operators on \mathcal{H}
- $\mathcal{R}(\mathcal{H}) :=$ Self-adjoint or hermitian operators on \mathcal{H}
- $\mathcal{P}(\mathcal{H}) :=$ Positive semi-definite operators on \mathcal{H}
- $\mathcal{D}(\mathcal{H}) :=$ Density operators on \mathcal{H}

Problem 1

For $T : \mathcal{H} \rightarrow \mathcal{H}$, prove that

$$\sum_{i=1}^d \langle e_i | T e_i \rangle = \sum_{i=1}^d \langle f_i | T f_i \rangle$$

if $\{|e_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ and $\{|f_i\rangle \in \mathcal{H} \mid 1 \leq i \leq d\}$ are ONB.

Solution: Let $S : \mathcal{H} \rightarrow \mathcal{H}$ where it maps the basis vectors from $|e_i\rangle \rightarrow |f_i\rangle$. Then $S|e_i\rangle = |f_i\rangle$. Hence S is an orthonormal matrix since

$$\langle e_j | S^\dagger S | e_i \rangle = \langle f_j | f_i \rangle = \delta_{ji} \quad \text{and} \quad \langle f_j | S S^\dagger | f_i \rangle = \langle e_j | e_i \rangle = \delta_{ji}$$

Hence

$$\sum_{i=1}^d \langle f_i | T f_i \rangle = \sum_{i=1}^d \langle e_i | S^\dagger T S | e_i \rangle = \text{tr}(S^\dagger T S) = \text{tr}(S S^\dagger T) = \text{tr}(T) = \sum_{i=1}^d \langle e_i | T e_i \rangle$$

Therefore we have

$$\sum_{i=1}^d \langle e_i | T e_i \rangle = \sum_{i=1}^d \langle f_i | T f_i \rangle$$

□

Problem 2

If $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \leq i \leq d\}$ and $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \leq i \leq d\}$ are ONB, then $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\} \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ is ONB

Solution: Let $|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then $|\psi\rangle = \sum_{i=1}^d \alpha_i |e_i\rangle$ where $\alpha_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \leq i \leq d\}$ is ONB for \mathcal{H}_1 . Hence

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^d \alpha_i |e_i\rangle \otimes |\phi\rangle$$

Now $|\phi\rangle = \sum_{i=1}^d \beta_i |f_i\rangle$ where $\beta_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \leq i \leq d\}$ is ONB for \mathcal{H}_2 . Hence

$$\forall i \in [d] \quad |e_i\rangle \otimes |\phi\rangle = \sum_{j=1}^d \beta_j |e_i\rangle \otimes |f_j\rangle$$

Therefore we get

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^d \alpha_i |e_i\rangle \otimes |\phi\rangle = \sum_{i=1}^d \alpha_i \sum_{j=1}^d \beta_j |e_i\rangle \otimes |f_j\rangle = \sum_{1 \leq i,j \leq d} \alpha_i \beta_j |e_i\rangle \otimes |f_j\rangle$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\}$ is a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Now for any $i1, i2, j1, j2 \in [d]$

$$(\langle e_{i1} | \otimes \langle f_{j1} |)(|e_{i2}\rangle \otimes |f_{j2}\rangle) = \langle e_{i1} | e_{i2}\rangle \langle f_{j1} | f_{j2}\rangle = \delta_{i1,i2} \delta_{j1,j2}$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\}$ is orthonormal. Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \leq i, j \leq d\}$ is a ONB for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

□

Problem 3

Let $\{|g_k\rangle \mid 1 \leq k \leq d_2\} \subseteq \mathcal{H}_2$ be ONB. For $T \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, let $\text{tr}_2(T) \in \mathcal{L}(\mathcal{H}_1)$ denote the operator satisfying

$$\langle u | \text{tr}_2(T) | v \rangle = \sum_k \langle u \otimes g_k | T | v \otimes g_k \rangle$$

for any choice $|u\rangle, |v\rangle \in \mathcal{H}_1$. Prove that $\sum_k \langle u \otimes g_k | T | v \otimes g_k \rangle$ is invariant.

Problem 4

Show that the Pauli matrices are all Hermitian, unitary, they square to the identity, and their eigenvalues are ± 1

Problem 5 Mark Wilde: Exercise 3.3.3

For $S, T \in \mathcal{L}(\mathcal{H})$, show that

$$\text{tr}(T) = \text{tr}(T^+), \quad \text{tr}(ST) = \text{tr}(TS)$$

[Recall T^+ denotes adjoint of T]. For $|x\rangle, |y\rangle \in \mathcal{H}$ show

$$\text{tr}(|x\rangle \langle y| T) = \text{tr}(T |x\rangle \langle y|) = \langle y | T x \rangle$$

Problem 6

Suppose \mathcal{H} is finite dimensional complex inner product space with $\dim(\mathcal{H}) = d$. Show complex dimensionality of $\mathcal{L}(\mathcal{H})$ is d^2 , real dimensionality of $\mathcal{R}(\mathcal{H})$ is d^2 .

Suppose \mathcal{H} is a real inner product space of dim d , show $\mathcal{L}(\mathcal{H})$ has dimension d and the space of all symmetric operators is a real vector space of dimension $\frac{d(d+1)}{2}$.

Problem 7

Show that $\mathcal{D}(\mathcal{H})$ is a convex subset of the real vector space of all Hermitian operators on \mathcal{H} . Show that the extreme points of $\mathcal{D}(\mathcal{H})$ are pure states, i.e. rank 1 projection operators.

Problem 8

Show that if $\dim(\mathcal{H}) = d$, then $\mathcal{D}(\mathcal{H})$ can be embedded into a real vector space of dimension $n = d^2 - 1$

Problem 9

Prove the Singular value decomposition theorem stated in class.

Problem 10

Suppose $|\psi\rangle_{AR_1} \in \mathcal{H}_A \otimes \mathcal{H}_{R_1}$, $|\psi\rangle_{AR_2} \in \mathcal{H}_A \otimes \mathcal{H}_{R_2}$ are purifications of $\rho_A \in \mathcal{D}(\mathcal{H}_A)$ and $\dim(\mathcal{H}_{R_2}) \geq \dim(\mathcal{H}_{R_1})$, then show that there exists an isometry $V : \mathcal{H}_{R_1} \rightarrow \mathcal{H}_{R_2}$ such that

$$|\psi\rangle_{AR_2} = (V \otimes I) |\psi\rangle_{AR_1}$$

Problem 11 Mark Wilde: Exercise 3.6.5

Show that the Bell states form an orthonormal basis:

$$\langle \Phi^{z_1 x_1} | \Phi^{z_2 x_2} \rangle = \delta_{z_1, z_2} \delta_{x_1, x_2}$$

Problem 12 Mark Wilde: Exercise 3.7.11

Show that the set of states $\{|\Phi^{x,z}\rangle_{AB}\}_{x,z=0}^{d-1}$ forms a complete, orthonormal basis:

$$\langle \Phi^{x_1, z_1} | \Phi^{x_2, z_2} \rangle = \delta_{x_1, x_2} \delta_{z_1, z_2} \quad \sum_{x,z=0}^d |\Phi^{x,z}\rangle \langle \Phi^{x,z}| = I_{AB}$$

Problem 13 Mark Wilde: Exercise 4.1.5

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2}, |0\rangle\}, \{\frac{1}{2}, |1\rangle\}\}$ and $\{\{\frac{1}{2}, |+\rangle\}, \{\frac{1}{2}, |-\rangle\}\}$