Super-Polynomial Lower Bound of TSP Extended Formula

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Introduction

Definition (Travelling Salesman)

Given a graph $G = (V, E)$, $S \subseteq V$ and weights $w : E \to \mathbb{R}$ find minimum weight cycle which visits every vertex of S exactly once.

We will focus on $S = V$.

- We know Traveling Salesman Problem is NP-complete.
- In [Yannkakis, 1988, STOC] he proved every symmetric LP for the TSP has expnential size.
- Here we will show TSP admits no polynomial-size LP.
- This proof also shows unconditional super-polynomial lower bound on the number of inequalities.
- Therefore it is impossible to prove $\mathrm{P} = \mathrm{NP}$ by means of a polynomial size LP.

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Preliminaries

Definitions

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} = conv(V)$ is a polytope with $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$ and $V \subseteq \mathbb{R}^d$. We will consider V as the characteristic vector for all hamiltonian paths.

Definition (Extension Polytope)

An extension of P is a polytope $Q \subseteq \mathbb{R}^{d+e}$ such that there is a linear map $\pi : \mathbb{R}^{d+e} \to \mathbb{R}^d$ such that $\pi(Q) = P$.

Definition (Extended Formula)

An EF Q is an extension of P is a linear system in variable s (x, y) such that

$$x \in P \iff \exists y \, (x, y) \in Q$$

Extension Complexity

Extension complexity of P is the minimum size EF of P where size of a polytope is the number inequalities. We denote by $xc(P)$.

Lemma

Let P, Q and F be polytopes. Then the following holds:

(i) *If F is an extension of P then $xc(F) \geq xc(P)$.*

(ii) *If F is a face of Q then $xc(Q) \geq xc(F)$.*

Slack Matrix

Definition

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} = conv(V)$ is a polytope with $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$ and $V \subseteq \mathbb{R}^d$. Let $V = \{v_1, \ldots, v_n\}$. Then $S \in \mathbb{R}_0^{m \times n}$ is called the slack matrix of P wrt $Ax \leq b$ and V where

$$S(i,j) = b_i - A_i v_j$$

Some times we may refer to the submatrix of slack matrix induced by rows corresponding to facets as the slack matrix of P denoted by $S(P)$.

Some Polytopes

- $TSP(n)$ is the traveling salesman polytope for $K_n = (V_n, E_n)$. Let $C \subseteq E_n$ denotes a tour of K_n. Then χ^C denotes the characteristic vector of C. Then

$$TSP(n) := conv\{\chi^C \mid C \subseteq E_n \text{ is a tour of } K_n\}$$

- Given $G = (V, E)$, for any $S \subseteq V$, χ^S denote characteristic vector of S. Then the independent set polytope

$$IND(G) := conv\{\chi^S \mid S \text{ is independent set of } G\}$$

- The correlation polytope $COR(n)$ is

$$COR(n) := conv\{bb^T \mid b \in \{0, 1\}^n\}$$

Proof Flow

Theorem

$xc(TSP(n)) = 2^{\Omega\left(n^{\frac{1}{4}}\right)}$

Step 1: First we will prove $xc(COR(n)) = 2^{\Omega(n)}$

Step 2: For all n, $\exists$ graph G_n with n vertices such that $xc(IND(G_n)) \geq xc(COR(n'))$ where $n' = n^{\frac{1}{d}}$ for some $d > 1$.

Step 3: For any n-vertex graph G, $IND(G)$ is linear projection of a face of $TSP(k)$ where $k = O(n^2)$.

Covering Bound of Matrix and Non-negative Factorization

Covering Bound of Matrix with Rectangles

- Let $M \in \{0,1\}^{n \times n}$ matrix.
- A monochromatic rectangle R in M means a submatrix N of M whose all entries are 1.
- A collection of rectangles $\mathcal{C}$ covers M if their union covers all the nonzero entries of M.
- $|\mathcal{C}|$ is called a covering bound of M. $Cov(X) = \min\{|\mathcal{C}| : \mathcal{C} \text{ covers } M\}$

Covering Bound of Simple Matrix

Consider A matrix X of dimension $2^n \times 2^n$ where the rows and columns are indexed by strings from $\{0,1\}^n$. Let $X(a,b) = (1 - a^T b)^2$ where $a, b \in \{0,1\}^n$.

Theorem (Yannkakis, 1988, STOC)

Every monochromatic rectangle cover of suppmat(X) has size $2^{\Omega(n)}$ i.e.

$$Cov(suppmat(X)) \geq 2^{\Omega(n)}$$

Non-negative Factorization

- A rank r non-negative factorization of a matrix M is a factorization $M = TU$ where T, U are non-negative matrix with r columns and r rows respectively.
- Non-negative rank of M is the minimum rank of a non-negative factorization of M. Denote it by $\mathrm{rank}_+(M)$.

Theorem (Factorization Theorem)

For a polytope $P = \{x \mid Ax \leq b\}$ where S is the slack matrix of P the following are equivalent:

(i) S has non-negative rank at most r.

(ii) P has an extension of size at most r.

(iii) P has an EF of size at most r.

We get $xc(P) = \mathrm{rank}_+(S)$.

Factorization and Covering Bound Relation

For any matrix $M \in \mathbb{R}^{m \times n}$ let $suppmat(M) \in \{0,1\}^{m \times n}$ is a matrix where the $(i,j)^{th}$ element is 1 if $M(i,j) \neq 0$ and otherwise 0.

Theorem (Yannkakis, 1988, STOC)

Let M be any matrix with non-negative real entries. Then

$$\mathrm{rank}_+(M) \geq Cov(suppmat(M))$$

Correlation Polytope Lower Bound

Polytope equations

Consider the inner product of two matrices $A, B \in \mathbb{R}^{m\times n}$ be $\langle A, B\rangle = Tr(A^TB)$
For all $a \in \{0,1\}^n$ there are some $b \in \{0,1\}^n$ such that

$$\langle 2\text{diag}(a) - aa^T, bb^T\rangle = 1$$

$$\begin{aligned} 1 - \langle \text{diag}(a) - aa^T, bb^T\rangle &= 1 - 2\langle \text{diag}(a), bb^T\rangle + \langle aa^T, bb^T\rangle \\ &= 1 - 2a^Tb + (a^Tb)^2 = (1 - a^Tb)^2 \end{aligned}$$

Remark

Because of above prove for all $b \in COR(n)$, for all $a \in \{0,1\}^n$, $\langle 2\text{diag}(a) - aa^T, bb^T\rangle \leq 1$.

Hence let A, b be such that $COR(n) = \{x \mid Ax \leq b\}$ where (A, b) includes these inequities. So the slack matrix S of $COR(n)$ contains X.

Lower Bound

Let S is the slack matrix of $COR(n)$. Then S contains the matrix X.

- By Factorization Theorem $xc(COR(n)) = \mathrm{rank}_+(S)$.
- Since X is submatrix of S we have $\mathrm{rank}_+(S) \geq \mathrm{rank}_+(X)$.
- By Covering-Factorization Relation $\mathrm{rank}_+(X) \geq Cov(suppmat(X)) \geq 2^{\Omega(n)}$.

Theorem

$xc(COR(n)) = 2^{\Omega(n)}$.

Independent Set Polytope Lower Bound

New Graph Construction

Let fix an n. Now consider the complete graph K_n. Now we will construct a graph $H_n = (V_n, E_n)$ with $O(n^2)$ vertices.

- Each vertex $i \in K_n$ there is a 2-clique on $i, \hat{i}$ in H_n.
- Each edge $(i, j) \in K_n$
    - There is a 4-clique on the vertices $\{ij, \hat{i}j, i\hat{j}, \hat{i}\hat{j}\}$.
    - The additional edges

$$
\begin{array}{cccc}
(i, \hat{i}j) & (\hat{i}, ij) & (j, i\hat{j}) & (\hat{j}, ij) \\
(i, \hat{i}\hat{j}) & (\hat{i}, i\hat{j}) & (j, \hat{i}\hat{j}) & (\hat{j}, \hat{i}j)
\end{array}
$$

Let F is the face of $IND(H_n)$ containing independent sets which have exactly one vertex from each vertex-clique and one vertex from each edge-clique

$COR(n)$ Inside Independent Set Polytope

Take the linear map $\pi : \mathbb{R}^{V_n} \to \mathbb{R}^{n\times n}$. Let $\pi(x) = y$. Then

$$y_{ij} = y_{ji} = x_{ij}$$

- S is independent set of H_n. χ^S is the characteristic vector.
- Define $b \in \{0,1\}^n$ where $b_i = 1$ iff $ii \in S$ otherwise 0

Observe: For edge $(i,j) \in K_n$, $ij \in S \iff ii, jj \in S$.
Then $\pi(\chi^S) = bb^T$. So $\pi(F) \subseteq COR(n)$

- $b \in \{0,1\}^n$. Consider bb^T.
- S contains a vertex ii if $b_i = 1$ and S contains $\hat{i}$ if $b_i = 0$.

$\chi^S \in F$. So $\pi(\chi^S) = bb^T$. So

$$\pi(F) = COR(n)$$

So $COR(n)$ is a face of $IND(H_n)$.

Lower Bound

Above H_n has $2n + \binom{n}{2}$ vertices.

- For any n consider p to be the maximum such that $2p + \binom{p}{2} \leq n$.
- Take the graph H_p and add $n - 2p - \binom{p}{2}$ isolated vertices to construct G_n.
- $IND(H_p)$ isomorphic to $IND(G_n)$

$$xc(IND(G_n)) = xc(IND(H_p)) \geq xc(COR(p)) \geq 2^{\Omega(p)} = 2^{\Omega\left(n^{\frac{1}{2}}\right)}$$

Theorem

For all $n \in \mathbb{N}$ there exists graph G_n, $xc(IND(G_n)) = 2^{\Omega\left(n^{\frac{1}{2}}\right)}$

TSP Polytope Lower Bound

Theorem (Yannkakis, 1988, STOC)

Every p-vertex graph G, IND(G) is the linear projection of a face of TSP(n) with $n = O(p^2)$.

Therefore

$$xc(TSP(n)) \geq xc(IND(G_p)) = 2^{\Omega\left(p^{\frac{1}{2}}\right)} = 2^{\Omega\left(n^{\frac{1}{4}}\right)}$$

Thank You