Soham Chatterjee
Email: soham.chatterjee@tifr.res.in
Course: Mathematical Foundations for Computer Sciences

Assignment - 2
Dept: STCS
October 12, 2024

Problem 1

Let $m, n > 0$ be given and let S be a subset of $[m] \times [n]$. We say S is downward closed if for all $i \leq i' \in [m]$ and $j \leq j' \in [n]$, we have $(i', j') \in S$ only if $(i, j) \in S$. How many downward closed sets are there?

Solution: S is downward closed if $\forall\ i \leq i'$ and $j \leq j'$, $(i, j) \in S$ then $(i', j') \in S$. We we define a new order $\preccurlyeq$ where $(a, b) \preccurlyeq (c, d)$ where $a, c \in [n]$ and $b, d \in [m]$ if $a \leq c$ and $b \leq d$. Therefore if $\forall\ (i, j) \preccurlyeq (a, b)$, $(i, j) \in S$ then $(a, b) \in S$. Hence S is uniquely defined if we can find the maximal elements with respect to this order since all other elements of S is $\preccurlyeq$ to one of the maximal elements of S. So we have to count how many ways we can select the maximal elements of S.

Now with respect to the first coordinate take the right most maximal element, (n_1, m_1) where $n_1 \in [n]$ and $m_1 \in [m]$. Then all other maximal elements has first coordinate less than n_1. Now all other maximal elements also has second coordinate greater than m_1 because if any element of S, (a, b) has second coordinate $b \leq m_1$ then $(a, b) \preccurlyeq (n_1, m_1)$. So now we take the second right most maximal element (n_2, m_2). We have $n_2 < n_1$ and $m_2 > m_1$. Again all other maximal elements apart from right most and second right most element has first coordinate less than n_2 and second coordinate greater than m_2 by same argument as before. Continuing like this we get that the maximal elements of S with respect to the defined order $\preccurlyeq$ from right most to left most has strictly increasing second coordinate.

Now the maximal elements of S defines an unique path from the coordinate $(n, 0)$ to $(0, m)$. We also get the maximal elements of a set from each path from $(n, 0)$ to $(0, m)$ which enters the $[n] \times [m]$ grid uniquely. For any path from $(n, 0)$ to $(0, m)$ which enters the $[n] \times [m]$ consists of up movement and left movement and in each movement we move by 1 unit. So from such a path we take the coordinates where we change from up movements to left movements i.e. the path has a staircase like structure and we take the coordinates where we take am ascending step. Hence from each such path we get uniquely the maximal elements of a downward closed set. So it suffices to calculate all possible such paths.

Now among all the paths from $(n, 0)$ to $(0, m)$ with up and left movements there is only one path which does not enter the $[n] \times [m]$ grid. This is first takes all the left movements and reaches $(0, 0)$ then takes all the up movements to reach $(0, m)$. All the other paths enter the grid $[n] \times [m]$. So we will subtract 1 from the total number of paths from $(n, 0)$ to $(0, m)$.

Now to reach from $(n, 0)$ to $(0, m)$ there is in total n left movements and m up movements. A path from $(n, 0)$ to $(0, m)$ is like a $n + m$ ordered pair where each element is up or left. Now once we select first on which positions we will put the up movements in the $n + m$ ordered pair then rest of the positions we can fill up by left movement. So number of paths is equal to in how many ways we can choose the m positions among the $n + m$ positions to put the up movements. This we can do in $\binom{m+n}{m}$ ways. Therefore the total number of paths from $(n, 0)$ to $(0, m)$ with up and left movements is $\binom{m+n}{m}$. Therefore total number of downward closed sets is $\binom{n+m}{m} - 1$. ■

Problem 2

Call an operator $\theta \in L(V)$ unitary if for all $v \in V$, we have $\|\theta(v)\| = \|v\|$ and positive if it is self-adjoint and for all $v \in V$, we have $\langle \theta(v), v \rangle \geq 0$.

- **Polar Decomposition.** Show that for all $\theta \in L(V)$, there exists a unitary $\mu \in L(V)$ and positive $\pi \in L(V)$ such that $\theta = \mu \circ \pi$.

  Hint: Start by showing $\theta^\dagger \circ \theta$ is positive and use the Spectral Theorems.

- **Singular Value Decomposition.** Let $n = \dim V$. Show that, for all $\theta \in L(V)$, there exists two orthonormal basis $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$ of V and "singular values" $s_1, \ldots, s_n$

such that, for all $v \in V$, we have:

$$\theta(v) = \sum_{i=1}^{n} \langle v, b_i \rangle \cdot s_i \cdot a_i$$

Solution:

- Let $\dim V = n$. We assume that θ is a nonzero operator. Since otherwise we can take μ to be the identity operator and π to be the zero operator. Consider the operator $\theta^\dagger \circ \theta \in L(V)$. Now for any $v \in V$,

$$\langle \theta^\dagger \circ \theta(v) \rangle = \left\langle \theta(v), \left(\theta^\dagger\right)^\dagger (v) \right\rangle = \langle \theta(v), \theta(v) \rangle = \langle v, \theta^\dagger \circ (\theta(v)) \rangle = \langle v, \theta^\dagger \circ \theta(v) \rangle$$

Hence $\theta^\dagger \circ \theta$ is self-adjoint. Now for any $v \in V$ we also have

$$\langle \theta^\dagger \circ \theta(v), v \rangle = \langle \theta(v), \theta(v) \rangle \geq 0$$

Hence $\theta^\dagger \circ \theta$ is also positive. Therefore by spectral theorem there exists an orthonormal eigen basis $B = \{b_1, \ldots, b_n\}$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $\theta^\dagger \circ \theta(b_i) = \lambda_i b_i$. Since $\theta^\dagger \circ \theta$ is positive all the eigenvalues are non-negative and since θ is nonzero operator not all eigenvalues are zero.

Now take the set of vectors $B' = \left\{ \frac{1}{\sqrt{\lambda_i}} \theta(b_i) \colon \lambda_i \neq 0 \right\}$. This set is orthonormal since for $i, j \in [n]$ and $i \neq j$ and $\lambda_i, \lambda_j \neq 0$ we have

$$\left\langle \frac{1}{\sqrt{\lambda_i}} \theta(b_i), \frac{1}{\sqrt{\lambda_j}} \theta(b_j) \right\rangle = \frac{1}{\sqrt{\lambda_i \lambda_j}} \langle \theta(b_i), \theta(b_j) \rangle = \frac{1}{\sqrt{\lambda_i \lambda_j}} \langle \theta^\dagger \circ \theta(b_i), b_j \rangle = 0$$

and for $i = j$ we have

$$\left\langle \frac{1}{\sqrt{\lambda_i}} \theta(b_i), \frac{1}{\sqrt{\lambda_i}} \theta(b_i) \right\rangle = \frac{1}{\sqrt{\lambda_i \lambda_i}} \langle \theta(b_i), \theta(b_i) \rangle = \frac{1}{\lambda_i} \langle \theta^\dagger \circ \theta(b_i), b_i \rangle = \frac{1}{\lambda_i} \lambda_i \langle b_i, b_i \rangle = 1$$

Now B' can be extended to a orthonormal basis $B'' = \{b''_i \colon i \in [n]\}$ of V using Gram–Schmidt procedure. For simplicity let first k many vectors of B had nonzero eigenvalues and the vectors $b''_{k+1}, \ldots, b''_n$ are the new orthonormal added to B' by Gram-Schmidt. Hence for $i \in [k]$ $b''_i = \frac{1}{\sqrt{\lambda_i}} \theta(b_i)$. So we define the operator $\mu \in L(V)$ such that for any $i \in [n]$

$$\mu(b_i) = b''_i$$

Now also define another operator $\pi \in L(V)$ where $\pi(b_i) = \sqrt{\lambda_i} b_i$ for all $i \in [n]$. Both μ and π are defined on basis so they are unique.

We claim $\theta = \mu \circ \pi$. If we show that for any $i \in [n]$ $\theta(b_i) = \mu \circ \pi(b_i)$ we are done since B is a basis of V. Now if $\lambda_i \neq 0$ then

$$\mu \circ \pi(b_i) = \mu(\sqrt{\lambda_i} b_i) = \sqrt{\lambda_i} \mu(b_i) = \sqrt{\lambda_i} b''_i = \sqrt{\lambda_i} \frac{1}{\sqrt{\lambda_i}} \theta(b_i) = \theta(b_i)$$

When $\lambda_i = 0$ then we have

$$\mu \circ \pi(b_i) = \mu(\sqrt{\lambda_i} b_i) = \sqrt{\lambda_i} \mu(b_i) = 0 \cdot \mu(b_i) = 0$$

and on the other hand we have

$$\langle \theta(b_i), \theta(b_i) \rangle = \langle \theta^\dagger \circ \theta(b_i) \rangle = \langle \lambda_i b_i, b_i r \rangle = 0$$

Hence we have for all $i \in [n]$, $\theta(b_i) = \mu \circ \pi(b_i)$. Hence $\theta = \mu \circ \pi$.

Now we will show that μ is unitary and π is positive. Now π is diagonalizable with respect to an orthonormal eigen basis with all its eigenvalues are non-negative. Hence π is positive. So only thing remains is to show that μ is unitary. Let for any $v \in V$, $v = \sum_{i=1}^{n} a_i b_i$ where $a_i \in \mathbb{C}$. Then we have

$$\left\langle \sum_{i=1}^{n} a_i b_i, \sum_{i=1}^{n} a_i b_i \right\rangle = \sum_{i=1}^{n} |a_i|^2 \langle b_i, b_i \rangle = \sum_{i=1}^{n} |a_i|^2$$

On the other hand we have

$$\left\langle \mu\left(\sum_{i=1}^{n} a_i b_i\right), \mu\left(\sum_{i=1}^{n} a_i b_i\right) \right\rangle = \left\langle \sum_{i=1}^{n} a_i \mu(b_i), \sum_{i=1}^{n} a_i \mu(b_i) \right\rangle = \sum_{i=1}^{n} |a_i|^2 \langle b_i, b_i \rangle = \sum_{i=1}^{n} |a_i|^2$$

Hence μ is unitary. Therefore there exists an unitary operator $\mu \in L(V)$ and a positive operator $\pi \in L(V)$ such that $\theta = \mu \circ \pi$.

- By the above proof of polar decomposition there exists an unitary operator $\mu \in L(V)$ and positive operator $\pi \in L(V)$ such that $\theta = \mu \circ \pi$. We also get an orthonormal eigenbasis $B = \{b_1, \dots, b_n\}$ of $\theta^\dagger \circ \theta$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ and another orthonormal basis $B'' = \{b_1'', \dots, b_n''\}$ where $\mu(b_i) = b_i''$ and $\pi(b_i) = \sqrt{\lambda_i} b_i$. Let $v \in V$. Then $v = \sum_{i=1}^{n} \langle v, b_i \rangle b_i$. Then

$$\theta(v) = \mu \circ \pi \left(\sum_{i=1}^{n} \langle v, b_i \rangle b_i \right) = \sum_{i=1}^{n} \langle v, b_i \rangle \mu(\sqrt{\lambda_i} b_i) = \sum_{i=1}^{n} \langle v, b_i \rangle \cdot \sqrt{\lambda_i} \cdot \mu(b_i) = \sum_{i=1}^{n} \langle v, b_i \rangle \cdot \sqrt{\lambda_i} \cdot b_i''$$

Hence here $A = B''$ and the singular values are eigenvalues of vectors in B.

■

Problem 3

The following pattern is well known:

$$A = \begin{bmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ 1 & 3 & 3 & 1 & & & & \\ 1 & 4 & 6 & 4 & 1 & & & \\ 1 & 5 & 10 & 10 & 5 & 1 & & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For all $n > 0$, consider the n-th sub-triangle B_n of A defined as follows:

$$B_1 = \begin{bmatrix} 1 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 & 1 \\ & 2 \end{bmatrix} \qquad B_3 = \begin{bmatrix} 1 & 2 & 1 \\ & 3 & 3 \\ & & 6 \end{bmatrix} \qquad B_4 = \begin{bmatrix} 1 & 3 & 3 & 1 \\ & 4 & 6 & 4 \\ & & 10 & 10 \\ & & & 20 \end{bmatrix} \qquad \cdots$$

The triangles B_n have the property that for all $i \le j < n$, it holds that $(B_n)_{i,j} = (B_n)_{i+1,j+1} - (B_n)_{i,j+1}$. For all $n > 0$, find the largest number of ones in a matrix of size n that has entries in $\{0, 1\}$ and satisfied the foregoing property modulo 2.

Solution: For all $n > 0$ we have

$$(B_n)_{i,j} = (B_n)_{i+1,j+1} - (B_n)_{i,j+1} \iff (B_n)_{i,j} + (B_n)_{i,j+1} = (B_n)_{i+1,j+1}$$

First we will prove an upper bound on the number of 1's in the triangle. [I got this bound statement from a reddit post[1]]

Lemma 1. *The number of* 1*'s in the resulting matrix of size* $n > 0$ *for any* $n \in \mathbb{N}$ *is at most* $\frac{n^2+n+1}{3}$

Proof: We will prove this inductively. For base case $n = 1$ we have the number of 1's is 1. and $\frac{1^2+1+1}{3} = 1$. Hence the base case follows.

Now suppose this is true for $n = 1, \ldots, k$. For $n = k + 1$ we will consider two cases: the first row as either $\le \frac{2k+2}{3}$ many 1's or $> \frac{2k+2}{3}$ many 1's.

Suppose the first row has $\le \frac{2k+2}{3}$ many 1's. Then from next row on wards there are k rows and these k rows can have at most $\frac{k^2+k+1}{3}$ many $1'$s by Induction Hypothesis. Therefore

$$\#1'\text{s} = \frac{k^2+k+1}{3} + \frac{2k+2}{3} = \frac{k^2+3k+3}{3} = \frac{(k^2+2k+1)+(k+1)+1}{3} = \frac{(k+1)^2+(k+1)+1}{3}$$

Therefore the statement is followed.

Suppose the first row has $> \frac{2k+2}{3}$ i.e. $\ge \frac{2k+3}{3}$ many 1's. Now in the second row each 1 is originated from a 0 and a 1 in the first row. Each 0 in the first row gives at most two 1's in the second row. Therefore

$$\#1'\text{s in second row} \le 2 \times \#0'\text{s in first row}$$

Hence

$$\begin{aligned}
\#1'\text{s in first two rows} &= \#1'\text{s in first row} + \#1'\text{s in second row} \\
&\le \#1'\text{s in first row} + 2 \times \#0'\text{s in first row} \\
&= 2(k+1) - \#1'\text{s in first row} \le 2(k+1) - \frac{2k+3}{3} = \frac{4k+3}{3}
\end{aligned}$$

Now from third row on wards there are $k-1$ rows and by inductive hypothesis there can be at most $\frac{(k-1)^2+(k-1)+1}{3} = \frac{k^2-k+1}{3}$ many $1'$s. Now if $3 \mid k$ then from third row on wards there are at most $\frac{k^2-k}{3}$ many 1's are there. Therefore

$$\begin{aligned}
\#1'\text{s} &= \#1'\text{s from third row on wards} + \#1'\text{s in first two row} \\
&\le \frac{k^2-k}{3} + \frac{4k+3}{3} = \frac{k^2-k+4k+3}{3} \\
&= \frac{(k^2+2k+1)+(k+1)+1}{3} = \frac{(k+1)^2+(k+1)+1}{3}
\end{aligned}$$

If $3 \nmid k$ then from third row on wards we keep the bound on the number of 1's to be $\frac{(k-1)^2+(k-1)+1}{3} = \frac{k^2-k+1}{3}$. But now $\frac{4k+3}{3}$ is not an integer. So the number of $1'$ in the first two rows is at most $\frac{4k+2}{3}$. Hence we have

$$\begin{aligned}
\#1'\text{s} &= \#1'\text{s from third row on wards} + \#1'\text{s in first two row} \\
&\le \frac{k^2-k+1}{3} + \frac{4k+2}{3} = \frac{k^2-k+1+4k+2}{3} \\
&= \frac{(k^2+2k+1)+(k+1)+1}{3} = \frac{(k+1)^2+(k+1)+1}{3}
\end{aligned}$$

Hence for both cases we have the total number of 1's is at most $\frac{(k+1)^2+(k+1)+1}{3}$. Hence by Mathematical Induction the number of 1's in the resulting matrix of size $n > 0$ for any $n \in \mathbb{N}$ is at most $\frac{n^2+n+1}{3}$. ■

Having this bound on the number of 1's we will now show an instance to achieve this number for any $n > 0$. So we will show instances where for any $n > 0$ from any i^{th} row on wards the bound $\left\lfloor \frac{i^2+i+1}{3} \right\rfloor$ is achieved for all $i \in [n]$.

[1] https://www.reddit.com/r/mathriddles/comments/ojpqgg/binary_pascal_triangle/

Consider the sequence $\{0,1,1\}$. We put them in that order circularly. i.e.

Starting with 0:	0	1	1	0	1	1	0	1	1	$\cdots$
Starting with first 1:	1	1	0	1	1	0	1	1	0	$\cdots$
Starting with second 1:	1	0	1	1	0	1	1	0	1	$\cdots$

Let S_n^0 denote the n-length sequence starting with 0, S_n^1 denote n-length sequence starting with 1 and S_n^2 denote n-length sequence starting with 1. Now for any $j \in \mathbb{F}_3$ and $i \in [n]$, $S_n^j(i)$ denote the i^{th} element in S_n^j. And in general for any $i > 0$, $i \in \mathbb{N}$ the i^{th} element of the sequence starting with 0 is by $S^0(i)$, for i^{th} element of the sequence starting with first 1 denoted by $S^1(i)$ and for i^{th} element of the sequence starting with second 1 denoted by $S^2(i)$. Now we have the following relation

Lemma 2. *Then for any $j \in \mathbb{F}_3$ and for any $i > 0$ and $i \in \mathbb{N}$*

$$S^j(i) + S^j(i+1) \equiv S^{j+2}(i) \pmod 2$$

Since $j \in \mathbb{F}_3$ we take $j+2 \bmod 2$.

Proof: For $j = 0$ we have

$$S^0(i) = \begin{cases} 0 & \text{If } i \equiv 1 \pmod 3 \\ 1 & \text{Otherwise} \end{cases}, \quad S^1(i) = \begin{cases} 0 & \text{If } i \equiv 0 \pmod 3 \\ 1 & \text{Otherwise} \end{cases}, \quad S^2(i) = \begin{cases} 0 & \text{If } i \equiv 2 \pmod 3 \\ 1 & \text{Otherwise} \end{cases}$$

Now we will analyze case wise:

- **Case 1:** $i \equiv 0 \pmod 3$: Then $S^0(i) = 1$, $S^1(i) = 0$ and $S^2(i) = 1$ Therefore $S^0(i+1) = 0$, $S^1(i+1) = 1$, and $S^2(i+1) = 1$. Therefore we have $S^0(i) + S^1(i+1) = 1 + 0 = 1 = S^2(i)$, $S^1(i) + S^1(i+1) = 0 + 1 = 1 = S^0(i)$ and $S^2(i) + S^2(i+1) = 1 + 1 \equiv 0 = S^1(i) \bmod 2$.
- **Case 2:** $i \equiv 1 \pmod 3$: Then $S^0(i) = 0$, $S^1(i) = 1$ and $S^2(i) = 1$ Therefore $S^0(i+1) = 1$, $S^1(i+1) = 1$, and $S^2(i+1) = 0$. Therefore we have $S^0(i) + S^1(i+1) = 0 + 1 = 1 = S^2(i)$, $S^1(i) + S^1(i+1) = 1 + 1 = \equiv 0 = S^0(i) \pmod 2$ and $S^2(i) + S^2(i+1) = 1 + 0 = 1 = S^1(i)$.
- **Case 3:** $i \equiv 2 \pmod 3$: Then $S^0(i) = 1$, $S^1(i) = 1$ and $S^2(i) = 0$ Therefore $S^0(i+1) = 1$, $S^1(i+1) = 0$, and $S^2(i+1) = 1$. Therefore we have $S^0(i) + S^1(i+1) = 1 + 1 \equiv 0 = S^2(i) \pmod 2$, $S^1(i) + S^1(i+1) = 1 + 0 = 1 = S^0(i)$ and $S^2(i) + S^2(i+1) = 0 + 1 = 0 = S^1(i)$.

Hence we have for all $i > 0$ and $i \in \mathbb{N}$ and for all $j \in \mathbb{F}_3$ we have $S^j(i) + S^j(i+1) \equiv S^{j+2}(i) \pmod 2$. ■

Now we will count the number of 1's in S_n^j for any $j \in \mathbb{F}_3$. First we define the following function $f\mathbb{F}_3^2 \to \mathbb{F}_3$ where we give the values of at all possible inputs by the table below:

$f(i,j)$	$j=0$	$j=1$	$j=1$
$i=0$	0	0	0
$i=1$	0	1	1
$i=2$	1	2	1

Lemma 3. *Let $n = 3k + i$ where $i \in \{0,1,2\}$ and $k \in \mathbb{N}$. Then number of $1's$ in S_n^j for any $j \in \mathbb{F}_3$ is $2k + f(i,j)$.*

Proof: For any $i > 0$, $i \in \mathbb{N}$ and for any $j \in \mathbb{F}_3$ in the block $S^j(i), S^j(i+1), S^j(i+2)$ there is exactly two 1's and one 0 since in the sequence $0,1,1$ comes circularly again and again and any 3 consecutive element is just one time appearance of the sequence. Therefore for 3-block there are two 1's. Since $n = 3k + i$, $S^j(3k)$ has $2k$ many 1's. Now we will analyze case wise:

- **Case 1** $i = 0$: Then $n = 3k$. Hence we already know we have $2k$ many 1's. And since $f(0,j) = 0$ for all $j \in \mathbb{F}_3$ we have $2k + f(i,j)$ many 1's.

- **Case 2** $i = 1$: We have $S^0(n) = 0$ and $S^1(n) = S^2(n) = 1$. Hence for S^0 we see no extra 1 at n^{th} position. Hence number of 1's in S_n^0 is $2k+1 = 2k+f(1,0)$. For S^1 we see an extra 1 at n^{th} position. We also have $f(1,j) = 1$ for $j = 1,2$. Therefore number of 1's in S_n^1 or S_n^2 is $2k+1 = 2k+f(1,j)$ for $j = 1,2$. Therefore for $i = 1$ number of 1's in S_n^j is $2k+f(1,j)$ for $j \in \mathbb{F}_3$.
- **Case 3** $i = 2$: We have $S^2(n) = 1$ and $S^0(n) = S^1(n) = 1$. And by case 2 analysis we have $2k$ many 1's in S_{n-1}^0 and $2k+1$ many 1's in both S_{n-1}^1 and S_{n-1}^2. For S^0 there is 1 at n^{th} position. Therefore we see an extra 1. Hence there are total $2k+1$ many 1's. We also have $f(2,0) = 1$. Hence there are $2k+f(2,0)$ many 1's in S_n^0. For S_n^1 there is 1 at n^{th} position. Therefore we see an extra 1. Therefore there are total $(2k+1)+1 = 2k+2$ many 1's in S_n^1. We also have $f(2,1) = 2$. Therefore we have $2k+f(2,1)$ many 1's in S_n^1. Now for S_n^2 there is 0 at n^{th} position. Therefore we have no extra 1. So the number of 1's in S_n^2 is same as S_{n-1}^2 which is $2k+1$. We have $f(2,2) = 1$. So we have $2k+f(2,2)$ many 1's in S_n^2. Therefore we have for $i = 2$ number of 1's in S_n^j is $2k+f(2,j)$ for $j \in \mathbb{F}_3$.

Hence by analyzing all possible cases we get that for $n = 3k+i$ where $k \in \mathbb{N}$ and $i \in \{0,1,2\}$ then number of 1's in S_n^j is $2k+f(i,j)$. ■

With all these setup for any $n > 0$ and $n \in \mathbb{N}$ we define the $0-1$ matrix M_n to be the following

$$M_n = \begin{bmatrix} S_n^l & & & & \\ & S_{n-1}^{l-1} & & & \\ & & S_{n-2}^{l-2} & & \\ & & & \ddots & \\ & & & & S_1^1 \end{bmatrix}$$

Where $l = n \bmod 3$ and we do the subtraction by 1 in modulo 3. So basically in M_n the k^{th} row has $k-1$ leading 0's then $S_{n-k+1}^{n-k+1 \bmod 3}$ for all $k \in [n]$. Also observe that if we remove the first row and first column from M_n we get M_{n-1}. Now by Lemma 2 M_n follows the rule that $(M_n)_{i,j} + (M_n)_{i,j+1} = (M_n)_{i+1,j+1}$ Now we will show that the total number of 1's in M_n is actually $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor$.

Lemma 4. *The total number of* 1*'s in* M_n *is* $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor$.

Proof: We will prove this inductively on n. For $n = 1$ we have $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor = 1$ which is true since $M_1 = [S_1^1] = [1]$. Hence the base case follows. Let this is true for $n = 1, \ldots, l-1$. Now $n = l$ we will analyze case wise. Now we have

$$\left\lfloor \frac{l^2+l+1}{3} \right\rfloor = \begin{cases} 3k^2+k & \text{When } l = 3k \\ 3(k^2+k)+1 & \text{When } l = 3k+1 \\ 3k^2+5k+2 & \text{When } l = 3k+2 \end{cases}$$

Now if we ignore the first row and first column we have M_{l-1}. By inductive hypothesis M_{l-1} has $\left\lfloor \frac{(l-1)^2+(l-1)+1}{3} \right\rfloor$ many 1's

- **Case 1** $\boldsymbol{l = 3k}$: Then $l-1 = 3(k-1)+2$. Then we have

$$\left\lfloor \frac{(l-1)^2+(l-1)+1}{3} \right\rfloor = 3(k-1)^2+5(k-1)+2 = 3(k^2-2k+1)+5k-5+2 = 3k^2-k$$

  And by Lemma 3 in S_l^0 there are $2k+f(0,0) = 2k$. Hence total number of 1's is

$$3k^2-k+2k = 3k^2+k = \left\lfloor \frac{l^2+l+1}{3} \right\rfloor$$

  Hence this case follows.

- **Case 2 $l = 3k + 1$**: Then $l - 1 = 3k$. Then we have

$$\left\lfloor \frac{(l-1)^2 + (l-1) + 1}{3} \right\rfloor = 3k^2 + k$$

And by Lemma 3 in S_l^1 there are $2k + f(1,1) = 2k + 1$ many 1's. Hence total number of 1's is

$$3k^2 + k + 2k + 1 = 3k^2 + 3k + 1 = \left\lfloor \frac{l^2 + l + 1}{3} \right\rfloor$$

Hence this case follows.

- **Case 3 $l = 3k + 2$**: Then $l - 1 = 3k + 1$. Then we have

$$\left\lfloor \frac{(l-1)^2 + (l-1) + 1}{3} \right\rfloor = 3k^2 + 3k + 1$$

And by Lemma 3 in S_l^2 there are $2k + f(2,2) = 2k + 1$ many 1's. Hence total number of 1's is

$$3k^2 + 3k + 1 + 2k + 1 = 3k^2 + 5k + 2 = \left\lfloor \frac{l^2 + l + 1}{3} \right\rfloor$$

Hence this case follows.

Therefore in all cases M_l has in total $\left\lfloor \frac{l^2+l+1}{3} \right\rfloor$ many 1's. Therefore by mathematical induction we have that for all $n > 0$, $n \in \mathbb{N}$ the total number of 1's in M_n is $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor$. ■

Since by Lemma 1 the maximum number of 1's we can achieve is $\left\lfloor \frac{n^2+n+1}{3} \right\rfloor$ for n size matrix this sequence of matrices has the maximum number of 1's. ■

Problem 4

Let $n > 0$ be an integer. Count the number of subsets $S \subseteq [n]$ that:

(a) satisfy $|S| \in S$.

(b) satisfy $|S| \in S$ and that for all $S' \subsetneq S$, we have $|S'| \notin S$.

Solution:

(a) Let $|S| = k$. Therefore $k \in S$. Now rest of the $k - 1$ elements are from $[n] \setminus \{k\}$. So the rest $k - 1$ elements can be chosen from $[n] \setminus \{k\}$ in $\binom{n-1}{k-1}$ ways. Therefore total number of sets $S \subseteq [n]$ that satisfy $|S| \in S$ is

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$$

Hence there are 2^{n-1} such sets are possible

(b) Let $|S| = k$. Now for all $S' \subsetneq S$, we have $|S'| \notin S$. Hence for all $m < k$, $m \notin S$. Therefore the rest of the $k-1$ elements of S are from $[n] \setminus [k]$. For this to satisfy we should have $n - k \geq k - 1 \implies \frac{n+1}{2} \geq k$. For such k the rest $k - 1$ elements can be chosen from $[n] \setminus [k]$ in $\binom{n-k}{k-1}$ ways. Hence total number of sets $S \subseteq [n]$ that satisfy the given property is

$$\sum_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n-k}{k-1}$$

Another way of counting: We call subsets $S \subseteq [n]$ which follows the property that $|S| \in S$ and that for all $S' \subsetneq S$, we have $|S'| \notin S$ to be selfish subsets of $[n]$. Let a_n denotes the number of selfish subsets of $[n]$. There can be two kinds of such subsets. Subsets which doesn't contain n and subsets which contain n. For selfish subsets which doesn't contain n are also selfish subsets of $[n-1]$. Therefore number of selfish subsets of $[n-1]$ is a_{n-1}. Hence number of selfish subsets of $[n]$ that doesn't contain n is a_{n-1}.

Now we will count the number of selfish subsets of $[n]$ which contains n. Now we claim that there is a one-one correspondence between selfish subsets of $[n-2]$ and selfish subsets of $[n]$ that contains n. For each selfish subset P of $[n-2]$ we have $\forall\, x \in P$, $x \geq |P|$. So we create a new subset of $[n]$, $Q = \{n\} \sqcup \{x+1 \colon x \in P\}$. Now $|Q| = |P| + 1$. Since $\forall\, x \in P$, $x \geq |P| \implies x+1 \geq |P|+1$. Now since $P \subseteq [n-2]$, for all $x \in P$, $x+1 \leq n-1$. Hence we have $n \geq |Q|$. Therefore Q is a selfish subset of P which contains n. Similarly let Q be a selfish subset of $[n]$ which contains n. Now $|Q| \geq 2$ since otherwise $|Q| = 1$ then $Q = \{1\}$. Since $|Q| \geq 2$, $1 \notin Q$. So we form a new set $P = \{x-1 \colon x \in Q \setminus \{n\}\}$.Since $n \notin Q \setminus \{n\}$, for all $x \in Q \setminus \{n\}$, $|Q| \leq x \leq n-1 \implies |P| = |Q| - 1 \leq x - 1 \leq n-2$. Hence P is also a selfish subset of $[n-2]$. Therefore there is a one-one correspondence between the selfish subsets of $[n-2]$ and selfish subsets of $[n]$ that contains n. Now number of selfish subsets of $[n-2]$ is a_{n-2}. Therefore number of selfish subsets of $[n]$ that contains n is a_{n-2}.

Hence total number of selfish subsets of $[n]$ is $a_{n-1} + a_{n-2}$. Therefore we get the recursion relation $a_n = a_{n-1} + a_{n-2}$. Now for $n = 1$ there is only one selfish subset of [1]. For $n = 2$ the only possible selfish subset of [2] is $\{1\}$ since neither $\{1,2\}$ nor $\{2\}$ is a selfish subset of [2]. Hence we have $a_1 = 1$ and $a_2 = 1$. Therefore we get the recursion relation of Fibonacci sequence same initial conditions. Hence the number of selfish subsets of $[n]$ is n^{th} Fibonacci number, F_n. Therefore we also get the identity that

$$\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k}{k-1} = F_n$$

■

Problem 5

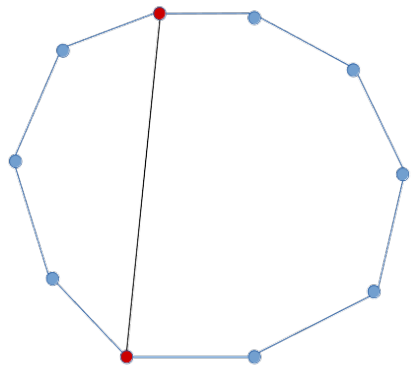
A triangulation of a polygon is a partition of its area into (disjoint) triangles with the same vertex set.

- Consider a regular polygon with n sides. Show that any triangulation of this polygon has $n-2$ triangles. How many such triangulations are there?
- For what values of n is there a triangulation into isosceles triangles? How many such triangulations are there?

Use the ideas above to show that a d-dimensional polytope that is the intersection of $n-$halfspaces can be partitioned into at most n^d simplices.

Solution:

- – We will prove this using induction on number of sides of any convex polygon. For $n = 3$ there is only one triangle. Hence the base case follows. Let this is true for $n = 3, \ldots, k-1$. For $n = k$ take any triangulation of the any polygon with n sides. There is at least one edge among all the edges of all the triangles of the triangulation which is not a side of the polygon. Let k_1 be the number of vertices on the left side of the edge and k_2 be the number of vertices on the right side of the edge.



Then we have $k_1 + k_2 + 2 = k$. Now by induction the number of triangles in any triangulation of the $k_1 + 2$-polygon bounded by the edge and the k_1 vertices to the left is $k_1 + 2 - 2 = k_1$. And the number of triangles in any triangulation of the $k_2 + 2$-polygon bounded by the edge and the k_2 vertices to the right of the edge is $k_2 + 2 - 2 = k_2$. Hence the number of triangles in the triangulation of the n sided polygon is $k_1 + k_2 = k - 2$. Therefore by mathematical induction any triangulation of the convex polygon with n sides has $n - 2$ triangles.

Therefore any triangulation of a regular polygon with n sides has $n - 2$ triangles.

–

■