
CSS.307.1: ALGEBRA, NUMBER THEORY AND COMPUTATION

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TIFR 2025, Aug-Dec

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CHAPTER 2

FAST PARALLEL GCD PAGE 7

Polynomial Arithmetic

1.1 Multiplication

1.2 Fast Division

POLYNOMIAL DIVISION

Input: $f, g \in \mathbb{F}[X]$, $\deg(f, g) \leq d$

Output: Quotient and remainder when f is divided by g .

Suppose $\deg f = a$ and $\deg g = b$. Let $(q, r) \in \mathbb{F}[X]$ are the quotient and remainder when f is divided by g i.e. $f = qg + r$. Therefore $\deg q = a - b$ and $m := \deg r < b$.

We can follow the long division algorithm to find (q, r) . This algorithm takes $O(a - b) = O(d)$ many iteration to find q . And in each iteration we subtract a polynomial from another polynomial by multiplying one of them with power of x . For the multiplying with power x is just shifting of the coefficients. For the subtraction of polynomials it takes $O(d)$ time. Therefore each iteration of the algorithm takes $O(d)$ time complexity. Therefore the long division algorithm takes $O(d^2)$ time complexity.

If we can obtain q from f, g then we can get r by following the equation $r = f - qg$.

1.2.1 Reversal of Polynomials

Idea. Reversal of Polynomials i.e. if $f \in \mathbb{F}[X]$ such that $f = f_0 + f_1X + \dots + f_aX^a$ then

$$\text{rev}(f) = f_0X^a + f_1X^{a-1} + \dots + f_a = f\left(\frac{1}{X}\right)X^a$$

Note:-

We have $\deg f \geq \deg(\text{rev}(f))$. Degree of $\text{rev}(f)$ can be strictly lesser than the degree of f . For example if $f_0 = 0$ and $f_1 \neq 0$, since $\text{rev}(f) = X^a f\left(\frac{1}{X}\right)$ the degree of $\text{rev}(f)$ is $a - 1$.

So using reversal we will review the equation $f = qg + r$:

$$\begin{aligned} f &= qg + r \\ \iff X^a f\left(\frac{1}{X}\right) &= X^a \left[q\left(\frac{1}{X}\right)g\left(\frac{1}{X}\right) + r\left(\frac{1}{X}\right) \right] \\ \iff X^a f\left(\frac{1}{X}\right) &= X^a q\left(\frac{1}{X}\right)g\left(\frac{1}{X}\right) + X^a r\left(\frac{1}{X}\right) \\ \iff \text{rev}(f) &= \text{rev}(q)\text{rev}(g) + X^{a-m}\text{rev}(r) \end{aligned}$$

Now we know $a \geq b > m \implies a - m \geq b - m > 0$. Therefore $X^{a-m}\text{rev}(r)$ is multiple of some nontrivial power of X . Now also we have

$$a - m > a - b = \deg q \geq \deg(\text{rev}(q))$$

Therefore we have

$$\text{rev}(f) \equiv \text{rev}(q)\text{rev}(g) \bmod X^{a-m}$$

Since $a - m \geq a - b + 1$ we have

$$\text{rev}(q) \bmod X^{a-m} \equiv \text{rev}(q) \bmod X^{a-b+1} \equiv \text{rev}(q)$$

Therefore we have

$$\text{rev}(f) \equiv \text{rev}(q)\text{rev}(g) \bmod X^{a-b+1}$$

Hence it suffices to recover $\text{rev}(q)$ in order to recover q from here. So the problem now reduced to finding a solution $h \in \mathbb{F}[X]$ for the system $\tilde{f} - h\tilde{g} \equiv 0 \bmod X^N$.

1.2.2 Find solution of $\tilde{f} - h\tilde{g} \equiv 0 \bmod X^N$

SOLVE $\tilde{f} - h\tilde{g} \equiv 0 \bmod X^N$

Input: $\tilde{f}, \tilde{g} \in \mathbb{F}[X]$, $\deg(f, g) \leq d$, $\tilde{f}(0), \tilde{g}(0) \neq 0$ with $N \in \mathbb{N}$

Output: Find solution h for the equation $\tilde{f} - h\tilde{g} \equiv 0 \bmod X^N$

Lemma 1.2.1

There is an unique $h \in \mathbb{F}[X]$ satisfying $\tilde{f} - h\tilde{g} \equiv 0 \bmod X^N$.

Proof: Let $\deg \tilde{f} = k$ and $\deg \tilde{g} = l$. Then Suppose $\tilde{f} = \sum_{i=0}^k \tilde{f}_i X^i$ and $\tilde{g} = \sum_{i=0}^l \tilde{g}_i X^i$. Then we can write the equation $\tilde{f} - h\tilde{g} \equiv 0 \bmod X^N$ as a linear system like the following:

$$\begin{bmatrix} \tilde{g}_0 & & & & \\ \tilde{g}_1 & \tilde{g}_0 & & & \\ \tilde{g}_2 & \tilde{g}_1 & \tilde{g}_0 & & \\ \vdots & & & \ddots & \\ & & & & \tilde{g}_{k-l} \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{k-l} \end{bmatrix} = \begin{bmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \vdots \\ \tilde{f}_k \end{bmatrix}$$

Lets call the matrix G . Since $\tilde{g}_0 \neq 0$ the G has nonzero elements in the diagonal. Since the G is lower triangular the determinant of the G is nonzero. Therefore there exists unique solution for h . ■

But we don't know how to find inverse of G in near linear time. So we cannot find h like this.

Idea. Find a power series solution for $h = \frac{\tilde{f}}{\tilde{g}} \bmod X^N$ in $\mathbb{F}[[X]] \supseteq \mathbb{F}[X]$ since in $\mathbb{F}[[X]]$ inverse of \tilde{g} exists

Lemma 1.2.2

For every power series $P = \sum_{i=0}^{\infty} P_i X^i \in \mathbb{F}[[X]]$, P has a multiplicative inverse iff $P_0 \neq 0$.

Since we are dealing with the equation $\tilde{f} - h\tilde{g} \equiv 0 \bmod X^N$ and $\tilde{g}(0) \neq 0$ there exists a power series solution for h . We will see two algorithms to find $h \in \mathbb{F}[[X]]$.

1.2.2.1 Algorithm I

$$\frac{\tilde{f}(X)}{\tilde{g}(X)} \bmod X^N = \frac{\tilde{f}(X)}{\tilde{g}(X)} \frac{\tilde{g}(-X)}{\tilde{g}(-X)} \bmod X^N = \frac{\tilde{f}(X)\tilde{g}(-X)}{\tilde{g}(X)\tilde{g}(-X)} \bmod X^N$$

Now $\tilde{g}(X)\tilde{g}(-X)$ is an even function. Therefore $\exists G \in \mathbb{F}[X]$ and $\deg G \leq d$ such that $G(X^2) = \tilde{g}(X)\tilde{g}(-X)$. Now we can also decompose $\tilde{f}(X) = \tilde{f}_0(X^2) + X\tilde{f}_1(X^2)$ and $\tilde{g}(-X) = \tilde{g}_0(X^2) + X\tilde{g}_1(X^2)$. Then we have

$$\begin{aligned} \tilde{f}(X)\tilde{g}(-X) &= (\tilde{f}_0(X^2) + X\tilde{f}_1(X^2))(\tilde{g}_0(X^2) + X\tilde{g}_1(X^2)) \\ &= \underbrace{[\tilde{f}_0(X^2)\tilde{g}_0(X^2) + X^2\tilde{f}_1(X^2)\tilde{g}_1(X^2)]}_{F_0(X^2)} + X \underbrace{[\tilde{f}_1(X^2)\tilde{g}_0(X^2) + \tilde{f}_0(X^2)\tilde{g}_1(X^2)]}_{F_1(X^2)} \\ &= F_0(X^2) + XF_1(X^2) \end{aligned} \quad [\deg F_i \leq d \ \forall i \in \{0, 1\}]$$

Therefore we have

$$\begin{aligned} \frac{\tilde{f}(X)}{\tilde{g}(X)} \bmod X^N &= \frac{F_0(X^2)}{G(X^2)} + X \frac{F_1(X^2)}{G(X^2)} \bmod X^N \\ &= \underbrace{\frac{F_0(X^2)}{G(X^2)} \bmod X^N}_{\frac{F_0(Z)}{G(Z)} \bmod Z^{\frac{N}{2}} \Big|_{Z=X^2}} + X \underbrace{\frac{F_1(X^2)}{G(X^2)} \bmod X^N}_{\frac{F_1(Z)}{G(Z)} \bmod Z^{\frac{N}{2}} \Big|_{Z=X^2}} \end{aligned}$$

Now we recurse. So the algorithm is

Algorithm 1: Solve $\tilde{f} - h\tilde{g} \equiv 0 \bmod X^N$

Input: $\tilde{f}, \tilde{g} \in \mathbb{F}[X]$, $\deg(f, g) \leq d$, $\tilde{f}(0), \tilde{g}(0) \neq 0$ with $N \in \mathbb{N}$

Output: Find solution h for the equation $\tilde{f} - h\tilde{g} \equiv 0 \bmod X^N$

1 **begin**

2 Construct G, F_0, F_1

3 Compute $\frac{F_0(Z)}{G(Z)} \bmod Z^{\frac{N}{2}}, \frac{F_1(Z)}{G(Z)} \bmod Z^{\frac{N}{2}}$

4 Set $Z \leftarrow X^2$ and combine and return

Now we have that

$$\frac{F_1(Z)}{G(Z)} \bmod Z^{\frac{N}{2}} = \frac{F_1(Z) \bmod Z^{\frac{N}{2}}}{G(Z) \bmod Z^{\frac{N}{2}}} \bmod X^{\frac{N}{2}}$$

Hence the degree got reduced by half. So in the recursion step we can reduce the degree with this.

Time Complexity: If $T(N)$ is the total time taken while solving for modulo X^N then we have the recursion relation

$$T(N) \leq 2T\left(\frac{N}{2}\right) + 10M(N)$$

Hence the total running time of this algorithm is $T(N) = M(N) \log N = N \text{poly}(\log N)$

1.2.2.2 Algorithm II

Here we can divide h into two parts with each part of degrees $< \frac{N}{2}$. Then $h(X) = h_0(X) + X^{\frac{N}{2}}h_1(X)$ where $\deg h_0 < \frac{N}{2}$ and $\deg h_1 < \frac{N}{2}$. Then we have

$$\tilde{f} - (h_0 + h_1X^{\frac{N}{2}})\tilde{g} \equiv 0 \bmod X^N \implies (\tilde{f} - h_0\tilde{g}) - X^{\frac{N}{2}}h_1\tilde{g} \equiv 0 \bmod X^N$$

Hence we have $\tilde{f} - h_0\tilde{g} \equiv 0 \bmod X^{\frac{N}{2}}$. Therefore we have

$$X^{\frac{N}{2}} \mid \tilde{f} - h_0\tilde{g} \implies \tilde{f} - h_0\tilde{g} = X^{\frac{N}{2}}$$

Hence

$$X^{\frac{N}{2}}p - X^{\frac{N}{2}}h_1\tilde{g} \equiv 0 \bmod X^N \implies p - h_1\tilde{g} \equiv 0 \bmod X^{\frac{N}{2}}$$

Therefore we have the following algorithm

Algorithm 2: Solve $\tilde{f} - h\tilde{g} \equiv 0 \pmod{X^N}$

Input: $\tilde{f}, \tilde{g} \in \mathbb{F}[X]$, $\deg(f, g) \leq d$, $\tilde{f}(0), \tilde{g}(0) \neq 0$ with $N \in \mathbb{N}$

Output: Find solution h for the equation $\tilde{f} - h\tilde{g} \equiv 0 \pmod{X^N}$

```

1 begin
2   Construct  $h_0, h_1$ 
3   Solve  $\tilde{f} - h_0\tilde{g} \equiv 0 \pmod{X^{\frac{N}{2}}}$ 
4    $R \leftarrow \frac{\tilde{f} - h_0\tilde{g}}{X^{\frac{N}{2}}}$ 
5   Solve  $R - h_1\tilde{g} \equiv 0 \pmod{X^{\frac{N}{2}}}$ 
6   Output  $h_0 + X^{\frac{N}{2}}h_1$ 

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Time Complexity: If $T(N)$ is the total time taken while solving for modulo X^N then we have the recursion relation

$$T(N) \leq 2T\left(\frac{N}{2}\right) + O(M(N))$$

Hence the total running time of this algorithm is $T(N) = O(M(N) \log N) = Npoly(\log N)$

CHAPTER 2

Fast Parallel GCD