Report: Polyhedral Combinatorics, Matroids and Derandomization of Isolation Lemma

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Chapter 1

Some Basics of Graph Theory

First we will introduce some graph properties and results which will help us in later chapters.

1.1 Incidence Matrix

Definition 1.1.1: Incidence Matrix

For an undirected graph $G = (V, E)$ the Incidence Matrix, M of G is the $|V| \times |E|$ matrix where for every $v \in V$ and $e \in E$, the entry $M[v, e] = 1$ if the edge e is incident on v and otherwise 0

Theorem 1.1.1

If $G = (V, E)$ is an undirected graph with $|V| = n$ then G is connected if and only if $\text{Rank}(M) = n - 1$ over $\mathbb{F}_2$.

Proof: ■

Corollary 1.1.2

If $G = (V, E)$ is an undirected graph with k connected components then $\text{Rank}(M) = n - k$

Proof: content... ■

Definition 1.1.2: Fundamental Cycles

Theorem 1.1.3

The Incidence vectors of the fundamental cycles for a spanning tree in the graph forms a basis of the null space of the incidence matrix

Proof: content... ■

1.2 Matching

Theorem 1.2.1 Hall's Condition

content...

Proof: content... ■

Lemma 1.2.2

Every Regular bipartite graph is union of perfect matchings.

Proof: We will induct on degree. A regular bipartite graph satisfies Hall's Condition. Therefore it has a perfect matching. So we will obtain a new regular graph of lower degree after removing the perfect matching. By induction hypothesis it must a union of perfect matchings. Hence we get that the original graph was in fact union of perfect matchings. ■

1.3 Nice Cycles and Circulation

Let $G = (V, E)$ be a graph with a perfect matching.

Definition 1.3.1: Nice Cycle

A cycle C in G is a nice cycle if it has even length and the subgraph $G - C$ still has a perfect matching

In other words a nice cycle can be obtained from the symmetric difference of two perfect matchings.

Now suppose we have a weight function $w\colon E \to \mathbb{R}$ on the edges of a graph G. Let we have an even length cycle $C = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{2k-2}} v_{2k} \xrightarrow{e_{2k-1}} v_0$ in G for some $k \in \mathbb{N}$.

Definition 1.3.2: Circulation of Cycle

For a weight assignment w on the edges the circulation $c_w(C)$ of an even length cycle is defined the alternating sum of the edge weights of C i.e.

$$c_w(C) = \left| \sum_{i=0}^{2k-1} (-1)^i w(e_i) \right|$$

The definition of circulations is independent of the edge we start with because we take the absolute value of the alternating sum. Below we show a property for cycles in a graph having nonzero circulations lead to a unique minimum weight perfect matching.

Chapter 2

Matorids

2.1 Matroids

Definition 2.1.1: Matroid

A matroid $M = (E, \mathcal{I})$ has a ground set E and a collection I of subsets of E called the *Independent Sets* st

1. Downward Closure: If $Y \in \mathcal{I}$ then $\forall\, X \subseteq Y, X \in \mathcal{I}$.
2. Extension Property: If $X, Y \in \mathcal{I}$, $|X| < |Y|$ then $\exists\, e \in Y - X$ such that $X \cup \{e\}$ also written as $X + e \in \mathcal{I}$

Observation. *A maximal independent set in a matroid is also a maximum independent set. All maximal independent sets have the same size.*

Base: Maximal Independent sets are called bases.
Rank of $S \in I$: We define the rank function of a matroid $r : \mathcal{P}(E) \to \mathbb{Z}$ where $r(S) = \max\{|X| \colon X \subseteq S, X \in I\}$ We def
Rank of a Matroid: Size of the base.
Span of $S \in I$: $\{e \in E \colon rank(S) = rank(S + e)\}$

2.2 Examples of Matroids

Uniform Matroid: It is denoted as $U_{k,n}$ where $E = [n]$ and $I = \{X \subseteq E \mid |X| \leq k\}$.
Free Matroid: When $k = n$ we take all possible subsets of E into I. This matroid is called Free Matroid i.e. $U_{n,n}$
Partition Matroid: Given $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_l$ where $\{E_1, \ldots, E_l\}$ is a partition of E and $k_1, \ldots, k_l \in \mathbb{N} \cup \{0\}$

$$I = \{X \subseteq E \colon |X \cap E_i| \leq k_i \ \forall\, i \in [l]\}$$

then $M = (E, I)$ is a partition matroid.

Note:-

If the E_i's are not a partition then suppose E_1, E_2 has nonempty partition then we will not have a matroid.
For example: $E_1 = \{1, 2\}$, $E_2 = \{2, 3\}$ and $k_1 = k_2 = 1$ then $X = \{1, 3\}$ is independent but $Y = \{2\} \subsetneq X$ is not a matroid.

Linear Matroid: Given a $m \times n$ matrix denote its columns as $A_1, \ldots, A_n$. Then

$$I = \{X \subseteq [n] \colon \text{Columns corresponding to } X \text{ are linearly independent}\}$$

Here if the underlying field is $\mathbb{F}_2$ then it is called *Binary Matroid* and for $\mathbb{F}_3$ it is called *Ternary Matroid.*
Representable Matroid: A matroid with which we can associate a linear matroid is called a representable matroid.
Eg: $U_{2,3}$. It can be represented by the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, over $\mathbb{F}_2$. Over $\mathbb{F}_3$ it is same as $U_{3,3}$.

Note:-

There are matroids which are not representable as linear matroids in some field. There are matroids which are not representable on any field as well.

Lemma 2.2.1

$U_{2,4}$ is not representable over $\mathbb{F}_2$ but representable over $\mathbb{F}_3$

Regular Matroid: There are the matroids which are representable over all fields.

Lemma 2.2.2

Regular Matroids are precisely those which can be represented over $\mathbb{R}$ by a Totally Uni-modular matrix

Graphic Matroid / Cyclic Matroid: For a graph $G = (V, E)$ the graphic matroid $M_G = (E, I)$ where

$$I = \{F \subseteq E : F \text{ is acyclic}\}$$

Hence I is the collection of forests of G. It follows the downward closure trivially. For extension property let $k = |F_1| < |F_2| = l$ and then there are $n - k$ and $n - l$ components. So $n - k > n - l$. So $\exists$ an edge in F_2 which joins 2 components in F_1.

Lemma 2.2.3

A subset of columns is linearly independent iff the corresponding edges don't contain a cycle in the incidence matrix

Lemma 2.2.4

Graphic Matroids are Regular Matroids

Proof Idea: Use Incidence Matrix. ■

Matching Matroids: We can try to define it like this but it will not work:

Problem 2.1

Is the following a matroid: E = Edges of a graph and $I = \{F \subseteq E : F \text{ is a matching}\}$

Solution: It is not a matroid since maximal matchings can not be extended to a maximum matching.

■

Correct way will be: For a graph $G = (V, E)$ the ground set = V and

$$I = \{S \subseteq V : \exists \text{a matching that matches all vertices in } S\}$$

The downward closure property trivially holds. For extension property is $|S| < |S'|$ then there exists another vertex in S' which is not matched with S, so we can add that vertex to S.

2.3 Circuits

Assume we have a matroid $M = (E, I)$.

Definition 2.3.1: Circuit

A minimal dependent set C such that $\forall\, e \in C, C - e$ is an independent set.

Theorem 2.3.1

Let $S \in I$. $S + e \notin I$. Then $\exists!\ C \subseteq S + e$.

Proof: Given $S + e \notin I$. Take the set Σ where $T \in \Sigma$ if $t \notin I$ and $T \subseteq S + e$. Σ is nonempty since $S + e \in \Sigma$. Now under the ordering of inclusion T has a minimal element. Hence this minimal element is the desired circuit C which is minimal dependent set contained in $S + e$.

Now suppose it is not unique. Let $C_1, C_2 \subseteq S + e$ be circuits. Suppose $f \in C_1 - C_2$. Then $S - e + f$ will still be dependent since $C_2 \subseteq S - e + f$. Now by definition we get that $C_1 - f$ is independent. Therefore we extend $C_1 - f$ to an independent set by adding the elements of S till we reach same size as $|S|$. Now $e \in C$ since C_1 was formed because of addition of e. Hence if we extend $C_1 - f$ till same cardinality as S we will add all the edges of S not in $C_1 - f$ except f since adding f will make C be a dependent subset of an independent set which is not possible. Hence $C_1 - f$ will be extended to $S - f + e$. Therefore $S + e - f$ is independent which contradicts our previous conclusion that $S + e - f$ is dependent. Hence contradiction. ■

2.4 Axiom Systems for a Matroid

2.5 Finding Max Weight Base

The problem is given a matroid $M = (E, I)$ and a weight function $W : E \to \mathbb{R}$ find the maximum weight base of the matroid. We will solve this using basic greedy algorithm.

2.5.1 Algorithm

Algorithm 1: Algorithm for Finding Max Weight Base

Input: A matroid $M = (E, I)$ is given as an input as an oracle and a weight function $W : E \to \mathbb{R}$.
Output: Find the maximum weight base of the matroid

```
1 begin
2     Assume w(1) ≥ ··· ≥ w(n)
3     S ← ∅
4     I ← {S}
5     for i = 1 to n do
6         if S + i ∈ I then
7             S ← S + i
8     return S
```

2.5.2 Correctness Analysis and Characterization

Theorem 2.5.1

The above algorithm outputs a maximum weight base iff M is a matroid

Proof: $\Leftarrow$:
Let M be a matroid. We will prove that this greedy algorithm works by inducting on i. At any iteration i we need to prove the following claim:
Claim: At any iteration i there is a max weight base B_i such that $S_i \subseteq B_i$ and $B_i \setminus S_i \subseteq \{i + 1, \ldots, n\}$.
Proof: Base case: $S = \emptyset$. So for base case the statement is true trivially. Assume that the statement is true up to $(i - 1)$ iterations.

Now $S_{i-1} \subseteq B_{i-1}$ where B_{i-1} is a maximum weight base and $B_{i-1} - S_{i-1} \subseteq \{i, \ldots, n\}$. Now three cases arise:

Case 1: If $i \in B_{i-1}$ then $S_{i-1} + i \subseteq B_{i-1}$. Therefore $S_{i-1} + i$ is independent. So now $B_i = B_{i-1}$ and $S_i = S_{i-1} + i$ and $B_i - S_i \subseteq \{i+1, \ldots, n\}$.

Case 2: If $i \notin B_{i-1}$ and $S_{i-1} + i \notin I$. Then $S_i = S_{i-1}$ and $B_i = B_{i-1}$. And $B_i - S_i \subseteq \{i+1, \ldots, n\}$.

Case 3: If $i \notin B_{i-1}$ but $S_{i-1} + i \in I$. Then $S_i = S_{i-1} + i$. Now S_i can be extended to a B' by adding all but one element of B_{i-1}. So $|B'| = |B_{i-1}|$. Let the element which is not added is $j \in B_{i-1}$. So $B' = B_{i-1} + i - j$.

$$wt(B') = Wt(B_{i-1}) - wt() + wt(i)$$

But we have $wt(i) \geq wt(j)$. So $wt(B') \geq wt(B_{i-1})$. Now since B_{i-1} has maximum weight we have $wt(B') = wt(B_{i-1})$. Then our $B_i = B'$. So $B_i - S_i \subseteq \{i+1, \ldots, n\}$.

Hence the claim is true for the ith stage as well. Therefore the claim is true.

■

Therefore using the claim, after the algorithm finished we have no elements left to check, so the current set has the maximum weight which is also an independent set. So the algorithm successfully returns a maximum weight base.
$\Rightarrow$:
Assume M is not a matroid.

■

2.6 Some Matroid Properties

2.6.1 Strong Base Exchange Property

2.6.2 Exchange Graph of a Matroid wrt $S \in I$

Chapter 3

Matroid Intersection

3.1 Matroid Intersection

3.1.1 Using Matroid Intersection to Solve Problems

Bipartite Matching

Colorful Spanning Tree

Min-Max Relation for Colorful Spanning Tree

Arborescence

3.1.2 Algorithm

Chapter 4

Matroid Union

Chapter 5

Basic Linear Programming

Chapter 6

Isolation Lemma

We say a weight assignment is isolating if there exists unique minimum weight subset. Isolation Lemma plays an important role in randomized computation. It was first introduced by [VV86] not under that name. In many areas it is a big open question how to derandomize isolation lemma efficiently. In this report we have derandomization of isolation lemma in some of those settings.

Theorem 6.1 Isolation Lemma

Fix a finite set $S \subseteq \mathbb{R}$ be a finite set and let $T_1, \ldots, T_k \in 2^{[n]}$. For each $i \in [n]$ independently assign a uniformly random weight from S. Let

$$w(T_i) = \sum_{x \in T_i} w(i)$$

Then we have

$$Pr[\exists!\ T_i \text{ of minimum weight}] \geq 1 - \frac{n}{|S|}$$

Proof: Suppose E_i be the event that

$$\min\{w(T_j) : i \notin T_j\} = \min\{w(T_j) : i \in T_j\}$$

Claim 1: If none of the E_i occur, then the minimum weight is unique.
Proof: We will proof the contrapositive statement. Suppose T_i and T_j are distinct minimum-weight sets. Then there exists at least one element $x \in T_i$ such that $x \notin T_j$. Since T_i and T_j have minimum weights, the even E_x occurs. ■

Proving this now notice that

$$Pr\left[\bigcap_{i \in [n]} \neg E_i\right] = 1 - Pr\left[\bigcap_{i \in [n]} E_i\right] \geq 1 - \sum_{i \in [n]} Pr[E_i]$$

Therefore in the following claim we will bound $Pr[E_i]$.

Claim 2: $Pr[E_i] \leq \frac{1}{|S|}$
Proof: Now fix all weights, $w(j)$ except the weight $w(i)$. Let

$$L = \min\{w(T_j - i) \mid i \notin T_j\} = \min\{w(T_j) \mid i \notin T_j\}$$
$$R = \min\{w(T_j - i) \mid i \in T_j\}$$

Since

$$E_i : \quad \min\{w(T_j) : i \notin T_j\} = \min\{w(T_j) : i \in T_j\}$$

we have that

$$E_i \text{ occurs} \iff L = R + wt(i)$$

Hence there is at most one option for $wt(i)$ to choose from S so that this equation holds. Therefore $Pr[E_i] = \underset{w \in S}{Pr}[L = R + w \wedge w = wt(i)] \leq \frac{1}{|S|}$ ■

Therefore by claim 2 we have $\forall\ i \in [n], Pr[E_i] \leq \frac{1}{|S|}$. Hence

$$Pr\left[\bigcap_{i \in [n]} \neg E_i\right] \geq 1 - \sum_{i \in [n]} Pr[E_i] \geq 1 - \frac{n}{|S|}$$

■

Now we will show an example of when a weight assignment becomes isolating.

Lemma 6.2 [DKR09, Lemma 3.2]

Let G be a graph with a perfect matching, and let w be a weight function such that all nice cycles in G have nonzero circulations. Then the minimum perfect matching is unique i.e. w is isolating

Proof: Suppose not, then we have two minimum weight perfect matchings M_1 and M_2 with minimum weight w.r.t w. Now we take their disjoint union $M_1 \sqcup M_2$ i.e. if there is an common edge then we take two copies of that edge connecting same two vertices. Now it is a cycle cover of the vertices with nice cycles except the one's with copies.

Consider any one nice cycle from the cycle cover. We will form a new perfect matching M. Since the circulation of an nice cycle is nonzero either the part of it which is in M_1 is lighter or the part of it which is in M_2 is lighter. Either way we take the lighter part in M and we do this for all . So we take the part from M_1 from this cycle. Now we do this for all the nice cycles in the cycle cover. Now for the cycles with two copies of same edge we take one of them into M. Now since $M_1 \neq M_2$ there exists at least one edge in M_1 which is not in M_2 and one edge in M_2 which is not in M_1. Hence $M_1 \sqcup M_2$ has at least one nice cycle, hence the way we constructed $w(M) < w(M_i)$ for some $i \in \{1, 2\}$ which contradicts the minimality of both M_1 and M_2 ■

Chapter 7

Perfect Matching Polytope

7.1 Matching Polytope

7.2 Perfect Matching Polytope

Definition 7.2.1: Perfect Matching Polytope

Let $G = (V, E)$ be a graph. For any perfect matching M of G, consider the incidence vector $x^M = (x_e)_{e \in E} \in \mathbb{R}^E$ given by

$$c_e^M = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{o/w} \end{cases}$$

For any perfect matching M of G this vector x^M is called as a *Perfect Matching Point.* The bipartite perfect matching polytope of the graph G is defined to the convex hull of all its perfect matching points,

$$PM(G) = \text{Conv}\{x^M \mid M \text{ is a perfect matching in } G\}$$

7.3 Bipartite Perfect Matching Polytope

It also defined like the perfect matching polytope where we just take the graph to be a bipartite graph. The following lemma form [LP86] gives a simple description of the perfect matching polytope of a bipartite graph G

Theorem 7.3.1 [LP86]

Let $G = (V, E)$ be a bipartite graph and $x = (x_e)_{e \in E} \in \mathbb{R}^E$. Then $x \in PM(G)$ if and only if

$$\sum_{e \in \delta(v)} x_e = 1 \quad v \in V,$$
$$x_e \geq 0 \quad e \in E$$

where for any $v \in V$, $\delta(v)$ denotes the set of edges incident on the vertex v.

Chapter 8

Bipartite Perfect Matching

8.1 A RNC Algorithm for Search-PM

We will recall the RNC algorithm of Mulmuley, Vazirani and Vazirani [MVV87] for the construction of a perfect matching (Search-PM). Though the algorithm works for any graph, we will only consider the bipartite graphs here.

Let $G = (V, E)$ be a bipartite graph with vertex partitions $L = \{u_1, \ldots, u_{\frac{n}{2}}\}$ and $R = \{v_1, \ldots, v_{\frac{n}{2}}\}$ and a weight function w. Consider the following $\frac{n}{2} \times \frac{n}{2}$ matrix A associated with G,

$$A(i,j) = \begin{cases} 2^{w(e)} & \text{if } e = (u_i, u_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

The algorithm then computes the determinant of A. Now

$$\begin{aligned} \det(A) &= \sum_{\pi \in S_{\frac{n}{2}}} sgn(\pi) \prod_{i=1}^{\frac{n}{2}} A(i, \pi(i)) \\ &= \sum_{M:\text{PM in } G} sgn(M) 2^{w(M)} \end{aligned}$$

The second equation holds since the product $\prod_{i=1}^{\frac{n}{2}} A(i, \pi(i))$ is nonzero if and only if the permuation π corresponds to a perfect matching. Here $sgn(M)$ is the sign of the corresponding permutation.

If the graph G does not have a perfect matching then we have $\det(A) = 0$. However if the graph G has perfect matchings, $\det(A)$ maight equal to 0 due to cancellation due to $sgn(M)$. To avoid such cancellations, one needs to design the weight function w cleverly. In particular if G has a perfect matching and w is isolating then the minimum weight perfect matching can not be canceled with other terms as there is unique minimum weight perfect matching.

Given an isolating weight assignment for G, one can construct the minimum weight perfect matching in NC. Let M^* be the unique minimum weight perfect matching in G w.r.t w. First we find $w(M^*)$ be finding out the highest power of 2 that divides $\det(A)$ since after $2^{w(M^*)}$ will not divide the monomial corresponding to M^* and that monomial survives in $\det(A)$. So more that after $2^{w(M^*)}$ it will not divide $\det(A)$. Now choose an $(u_i, u_j) = e \in E$ then compute the determinant of the minor $A_{i,j}$ which is basically associated with $G - e$. If the highest power of 2 that divides $\det(A_{i,j})$ is larger than the $2^{w(M^*)}$ then $e \in M^*$. Doing this in parallel for each edge, we can find all the edges in M^*.

By Theorem 6.1 we have an isolating weight with high probability. Moreover the weights chosen by the isolation lemma are polynomially bounded. Therefore the entries of in matrix A have polynomially many bits. Hence it suffices to compute the determinant in NC^2 [Ber84]. Also the construction is in NC^2. There fore this yields an RNC-algorithm for Search-PM.

8.2 A Quasi-NC Algorithm using Isolation

Let $G = (V, E)$ be given bipartite graph. In the following discussion we will assume that G has perfect matchings. Our goal is to isolate one of the perfect matchings in G by any appropriate weight function. We will also show that if G does

not have any perfect matchings then our algorithm will detect this. In the following discussion we will prove this theorem

Theorem 8.2.1 [FGT16, Theorem 3.1]
For bipartite graphs, PM and SEARCH-PM are in QUASI-NC^2

We will construct an isolating weight function for bipartite graphs. The idea is to create a weight function which ensures nonzero circulations for a small set of cycles in a black-box way i.e. without having being able to compute the set efficiently. Then we will show that if we construct a smaller graph wrt this weight function then we don't have those small cycles with nonzero circulations then we have the number of cycles with twice the size of the previous ones are polynomially bounded. Then we proceed to create a new weight function which will give nonzero circulations to all the cycles with twice the size. And this way we will continue. This same type of idea we will repeatedly use with necessary modifications in chapter 10 and chapter 12.

The idea above to create a weight function which gives nonzero circulation to every nice cycles in G actually works because then we have unique perfect matching by Lemma 6.2

8.2.1 Isolating Small Cycles

The following lemma describes a standard trick to create a weight function for a small set of cycles in graph.

Lemma 8.2.2 [CRS93]
Let G be a graph with n vertices. Then for any number s, one can construct a set of $O(n^2 s)$ weight assignments with weights bounded by $O(n^2 s)$, such that for any set of s cycles, one of the weight assignments gives nonzero circulation to each of the s cycles.

Proof: Let us first assign exponentially large weights. Let $e_1, e_2, \ldots, e_m$ be some enumeration of the edges of G. Define a weight function w by $w(e_i) = 2^{i-1}$ for $i \in [m]$. Then clearly every cycle has a nonzero circulation. However we want to achieve this with small weights.

We consider the weight assignment modulo small numbers i.e. the weight function is $\{w \bmod j \mid 2 \leq j \leq t\}$ for some appropriately chosen t. We want to show that for any fixed set of s cycles $\{C_1, \ldots, C_s\}$ one of these assignments will work when t is chosen large enough.

Now we want

$$\exists\, j \leq t,\ \forall\, i \leq s,\ c_w(S_i) \neq 0 \iff \exists\, j \leq t,\ \prod_{i=1}^{s} c_w(C_i) \neq 0 \bmod j$$

In other words we want

$$lcm(2, 3, \ldots, t) \nmid \prod_{i=1}^{t} c_w(C_i)$$

Hence if we take t such that $lcm(2, 3, \ldots, t) > \prod_{i=1}^{t} c_w(C_i)$ then we are done.

Now the product $\prod_{i=1}^{t} c_w(C_i)$ is bounded by $2^{n^2 s}$. This is because with exponential weights like in the RNC algorithm of section 8.1 we have an isolating perfect matching so we need weights less than that and therefore the new weights are bounded by the exponential weights for which weight of a cycle can at most be 2^{n^2} and since there are s many cycle we have the bound $2^{n^2 s}$. So if we have t such that $lcm(2, 3, \ldots, t) > 2^{n^2 s}$ then we are done. Now $lcm(2, 3, \ldots, t) > 2^t$ for $t \geq 7$. Thus choosing $t = n^2 s$ suffices. Clearly the weights are bounded by $t = n^2 s$. ■

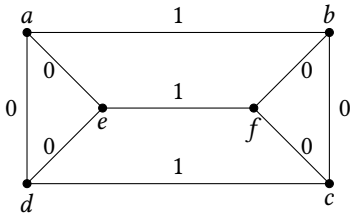
8.2.2 Union of Minimum Weight Perfect Matchings

Let us assign a weight function for bipartite graph G which gives nonzero circulations to all small cycles. Consider a new graph G_1 which obtained by the union of minimum weight perfect matchings in G. Out hope is that G_1 is significantly smaller than G.

Note:-

We don't know if G_1 can be efficiently created from G as determinant of the bi-adjacency matrix with weights in the like in the RNC algorithm of section 8.1 be zero and therefore we can not use that way to obtain perfect matchings. We will show we don't need to construct G_1. It is just used n the argument. Our final weight assignment will be completely black-box

We will also show by the following lemma that why this technique only works for biparitte graphs, not in general graphs i.e. G_1 constructed from minimum weight perfect matchings in G contains no other prefect matching than these. For general graph this does not hold:



In this graph we will denote the edge connected vertices a, b to be e_{ab}. And this way we will denote all the edges. Then the minimum weight perfect matchings have weight 1 and they are

$$\{e_{ad}, e_{bc}, e_{ef}\}, \quad \{e_{ac}, e_{bf}, e_{cd}\}, \quad \{e_{de}, e_{cf}, e_{ab}\}$$

Then their union has the perfect matching $\{e_{ab}, e_{cd}, e_{ef}\}$ which has weight 3 and not a minimum weight perfect matching.

The fact that G_1 has only minimum weight perfect matching is equivalent to saying that every nice cycle has zero circulation. The following lemma proves even stronger statement that every cycles has zero circulation (not necessarily nice cycles.)

Lemma 8.1 [FGT16, Lemma 3.2]

Let $G = (V, E)$ be a bipartite graph with weight function w. et C be a cycle in G such that $c_w(C) \neq 0$. Let E_1 be the union of all minimum weight perfect matchings in G. Then the graph $G_1 = (V, E_1)$ does not contain C.

Proof: Consider the perfect matching polytope of G, $PM(G)$. Let the weight of the minimum weight perfect matching in G be q. Let $x_1, x_2, \ldots, x_t$ be all the minimum weight perfect matching points of G i.e. corners of $PM(G)$ corresponding to weight q. Consider the average point $x \in PM(G)$ of these perfect matching points, $x = \frac{x_1+x_2+\cdots+x_t}{t}$. Clearly we have $w(x) = q$. And since each edge of G_1 participates in a minimum weight perfect matching for $x = (x_e)_{e\in E}$ we have $x_e \neq 0$ $\forall\, e \in E$.

Now consider a cycle C with $c_w(C) \neq 0$. Suppose $C = (e_1, e_2, \ldots, e_k)$ and all the edges of C are in E_1. We will show that if we move from point x along the cycle C we reach a point in $PM(G)$ with a weight smaller than q.

Consider the point y defined as

$$\forall\, e \in E, \quad y_e = \begin{cases} x_e + (-1)^i \epsilon \& \text{if } e = e_i \text{ for some } i \in [k] \\ x_e \qquad\qquad\qquad\qquad\qquad\qquad \text{o/w} \end{cases}$$

for some $\epsilon \neq 0$. Clearly $x - y$ has nonzero coordinates only on the edges of the cycle C, by alternating between ϵ and $-\epsilon$. Hence

$$w(x) - w(y) = w(x - y) = \pm c_w(C) \neq 0$$

Now we take ϵ in the following way:

- Take $sgn(\epsilon)$ such that $w(x - y) > 0$.
- Take ϵ small enough such that $y_e \geq 0$ $\forall\, e \in E$.

After choosing such ϵ since $w(x) - w(y) = w(x - y) > 0$ we have $q = w(x) > w(y)$. Now we will show that $y \in PM(G)$. To show that we will sow that y fulfills the conditions of Theorem 7.3.1. Now the second condition that $y_e \geq 0$ for all $e \in E$ is already satisfied by the choice of ϵ. So we only need to show that for any $v \in V$

$$\sum_{e\in\delta(v)} y_e = 1$$

To show this we consider 2 cases:

Case 1: $v \notin C$. Then $\forall\, e \in \delta(v)$ we have $e \notin C$. So $y_e = x_e$. Since $x \in PM(G)$ we have

$$\sum_{e\in\delta(v)} x_e = 1 \implies \sum_{e\in\delta(v)} y_e = 1$$

Case 2: $v \in C$. Let e_j and e_{j+1} are the edges incident on v in C. Then

$$y_{e_j} = x_{e_j} + (-1)^j\epsilon \qquad \text{and} \qquad y_{e_{j+1}} = x_{e_{j+1}} + (-1)^{j+1}\epsilon \qquad \forall\, e \in \delta(v) - \{e_j, e_{j+1}\},\ y_e = x_e$$

So

$$\begin{aligned}\sum_{e\in\delta(v)} y_e &= \left[\sum_{e\in\delta(v)-\{e_j,e_{j+1}\}} x_e\right] + \left[x_{e_j} + (-1)^j\epsilon\right] + \left[x_{e_{j+1}} + (-1)^{j+1}\epsilon\right] \\ &= \left[\sum_{e\in\delta(v)} x_e\right] + (-1)^j\epsilon + (-1)^{j+1}\epsilon = \sum_{e\in\delta(v)} x_e = 1\end{aligned}$$

So the point y satisfies the property $\sum_{e\in\delta(v)} = 1$ forall $v \in V$. Hence $y \in PM(G)$. Now since $w(y) < q$ there must be a corner point of the polytope which corresponds to a perfect matching in G with weight less than q. This contradicts the minimality of q. Hence C is not in G_1. ■

This technique of moving along the cycle has been used by Mahajan and Varadarajan in [MV00]. Now We will show another proof of this lemma by Rao, Shpilka and Wigderson in [GG17].

Alternate Proof *[GG17, Proof of Lemma 6]:* Let G' be the multigraph obtained by taking disjoint union of all minimum weight perfect matchings (i.e. if an edge appears in k many minimum weight perfect matchings of G then G' contains k copies of the edge.

Note:-

G' is a regular graph since it is a disjoint union of matchings and matchings are regular graph of degree 1.

Suppose there exists a cycle C of non zero circulation in G_1. Since the cycle is in G_1 then this cycle is also in G'. WLOG assume that the sum of the weights of the odd edges of C is larger than the sum of the weights of the even edges. Then we can remove a single copy of each odd edges of C from G' and add a single copy of each even edges of C to G' and we call this new graph G''

Suppose q be the minimum weight of a matching in G. Suppose G has d minimum weight matchings. Then sum of the weights of the edges in G' is qd. However, the total weight of all edges in in G'' is lower than the total weight of all edges in G'. We know that G'' is a regular bipartite graph of degree d and therefore by Lemma 1.2.2 it is an union of d perfect matchings.

If we decompose G'' into d perfect matchings, it is impossible that they all have weight at least q as G'' has total weight less than qd. Therefore G'' has a matching of weight less than q, which contradicts the minimality of q. ■

A consequence of this lemma is that G_1 has no other perfect matchings than the ones used to define G_1 cause if M_0 and M_1 be two different perfect matchings in G_1 then $M_0 \triangle M_1$ forms a set of nice cycles and by the Lemma 8.1 the circulations all of these cycles are 0 and therefore M_0 and M_1 have same weight and hence they both are minimum weight perfect matchings.

Corollary 8.2.3

Let $G = (V, E)$ be a bipartite graph with weight function w. Let E_1 be the union of all minimum weight perfect matchings in G. Then every perfect matching in the graph $G_1 = (V, E_1)$ has the same weight, the minimum weight of any perfect matching in G.

8.2.3 Bounding Number of Cycles with Length Twice Large of The Smallest Cycle

By our weight function in Lemma 8.2.2 each small cycles in G has a nonzero circulation. Hence, by Lemma 8.1 G_1 has no small cycles. Now we want to repeat this procedure with G_1 with a new weight function. G_1 has no small cycles. Hence, we look at slightly larger cycles (twice larger) and we will argue that their number remains polynomially bounded.

Teo and Koe in [TK92] showed that the number of the shortest cycles in a graph with m edges is bounded by m^2. In te following lemma we extend their argument and give a bound on the number of cycles that have length at most twice the length of the shortest cycles.

Lemma 8.2.4 [FGT16, Lemma 3.4]
Let $G = (V, E)$ be a graph with n nodes that has no cycles of length $\leq r$. Let $r' = 2r$ if r is even and $r' = 2r - 2$ otherwise. Then H has $\leq n^4$ cycles of length $\leq r'$.

Proof: Let

$$C = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{l-2}} v_{l-1} \xrightarrow{e_{l-1}} v_1$$

be a cycle of length $l \leq r'$ in G. Now we successively choose 4 nodes in C (u_0, u_1, u_2, u_3) where

$$u_i = v_{\lfloor \frac{il}{4} \rfloor} \quad \forall\, i \in \{0, 1, 2, 3\}$$

Now observe the distance between two successive nodes is $\leq \lfloor \frac{l}{4} \rfloor \leq \frac{r}{2}$. and therefore distance between the nodes u_3 and u_0 is also $\leq \lfloor \frac{l}{4} \rfloor \leq \frac{r}{2}$.

Since any node of C can be chosen as a starting point u_0, the four nodes (u_0, u_1, u_2, u_3) associated with C are not uniquely defined. But by the following claim we will show they uniquely define C.

Claim: Cycle C is the only cycle in G of length $\leq r'$ that is associated with (u_0, u_1, u_2, u_3).
Proof: Suppose $C' \neq C$ be a cycle of length $\leq r'$ such that both C and C' are associated with same (u_0, u_1, u_2, u_3). Consider the paths between two successive nodes in both C and C'. Since $C \neq C'$ there exists a path p and p' following C and C' respectively connecting two same successive nodes in both C and C' such that $p \neq p'$. Now p and p' form a cycle in H of length

$$|p| + |p'| \leq \frac{r}{2} + \frac{r}{2} = r$$

which is not possible as there are no cycles of length $\leq r$ in G. Hence, contradiction. ■

Hence by the claim each tuple of 4 nodes uniquely defines a cycle C in H. There are $\leq n^4$ ways to choose 4 nodes and their order. Hence the number of cycles of length $\leq r'$ is at most n^4. ■

Lemma 8.2.4 suggests that we continue form G_1 and in each successive round, we double the length of cycles and adapt the weight function to give nonzero circulations to these slightly longer cycles (twice larger). By Lemma 8.1 these cycles will disappear from the new graph G_2 obtained by taking only minimum weight perfect matching from G_1. This way in $\log n$ rounds we reach a graph with no cycles i.e. with a unique perfect matching. In the following section we show how to construct the weight assignment.

8.2.4 Constructing Weight Assignment

Let $G = (V, E) = G_0$ be bipartite graph with n nodes that has a perfect mathcing. Define $k = \lfloor \log n \rfloor - 1$ which is the number of successive rounds we will need. Note that the shortest cycles in G have length 4. Then suppose we have defined subgraphs w_i and G_i for $0 \leq i \leq k$ when define

G_{i+1} : The union of minimum weight perfect matchings in G_i according to weight w_i

w_{i+1} : A weight function on G_{i+1} such that all cycles in G_{i+1} of length $2^{(i+1)+2}$ have nonzero circulations by Lemma 8.2.2

By the definition of G_i any two perfect matchings in G_i have the same weight, not only according to w_i but also w_j for all $j < i$ for any $i \in [k]$. By Lemma 8.1 graph G_i does not have any cycles of length $\leq 2^{i+1}$ for each $i \in [k]$. In particular G_k does not have any cycles since $2^{k+1} \geq n$. Therefore, G_k has a unique perfect matching.

Our final weight function w will be a combination of $w_0, \ldots, w_{k-1}$. We combine them in a way that the weight assignment in a later round does not interfere with the order of perfect matchings given by earlier round weights. Let B be a number greater thatn the weight of any edge under any of these weight assignments. Then define

$$w = w_0 B^{k-1} + w_1 B^{k-2} + \cdots + w_{k-1} B^0$$

In the definition of w, the precedence decreases from w_0 to w_{k-1} since once two perfect matchings have same minimum weight with restpect to w_i, w_i doesn't participate in the calculations for w_j in G_j with $j > i$. Therefore, for any two perfect matchings M_1 and M_2 in G_0 we have $w(M_1) < w(M_2)$ if and only if there exists an $0 \leq i \leq k - 1$ such that

$$w_j(M_1) = w_j(M_2), \text{ for } j < i, \quad w_i(M_1) < w_i(M_2)$$

As a consequence, the prefect matchings left in G_i have a strictly smaller weight with respect to w than the ones in G_{i-1} that did not make it to G_i.

Lemma 8.2.5 [FGT16, Lemma 3.5]

For any $i \in [k]$ let M_1 be a perfect matching in G_i and M_2 be a perfect matching in G_{i-1} which is not in G_i. Then $w(M_1) < w(M_2)$

Proof: Since M_1 and M_2 are prefect matchings in G_{i-1} we have $w_j(M_1) = w_j(M_2)$ for all $j < i-1$. From the definition of G_i and Corollary 8.2.3 we have $w_{i-1}(M_1) < w_{i-1}(M_2)$. Hence we have $w(M_1) < w(M_2)$. ■

Therefore by the lemma it follows that the unique perfect matching in G_k has strictly smaller weight with respect to w than all other perfect matchings.

Corollary 8.2.6

The weight assignment w defined

$$w = w_0 B^{k-1} + w_1 + B^{k-2} + \cdots + w_{k-1} B^0$$

is isolating for G_0.

Now all it remains is to bound the values of the weight assigned. First we will look at the number of the cycles which need to be assigned a nonzero circulations in each round. So in first round, we give nonzero circulations to every cycle of length 4. Clearly the number of such cycles is $\leq n^4$. By Lemma 8.2.2 there are a set of $O(n^6)$ weight assignments with weights bounded by $O(n^6)$ as $s \leq n^4$. So G_1 does not have any cycles of length 4. In i-th round we have the graph G_i that does not have any cycles of length $\leq 2^{i+1}$. Now by Lemma 8.2.4 the number of cycles of length $\leq 2^{i+2}$ is bounded by n^4. So by Lemma 8.2.2 we give a set of $O(n^6)$ weight assignments with weights bounded by $O(n^6)$ whcih gives nonzero circulations to all $\leq n^4$ cycles of length $\leq 2^{i+2}$. Now the length of the largest cycle can be at most n. Hence we only have to iterate like this $O(\log n)$ times to have all cycles.

Recall that the number B used in defining w is the highest weight assigned by any w_i. Hence $B = O(n^6)$ as for all each round each set of weight assignments have their weights bounded by $O(n^6)$. Therefore the weihts in the assignment w are bounded by $B^k = O(n^{6 \log n})$. Or we can say the weights have $O(\log^2 n)$ bits.

For each w_i there are at most $O(n^6)$ possibilities. We don't know which one would work. Therefore we try all of them and take all possible combinations to create w and then we try all of them. In total we need to try $O(n^{6 \log n})$ weight assignments. This can be done in parallel. Hence clearly every weight assignment can be constructed in Quasi-NC^1. Therefore we proved:

Theorem 8.2.7 [FGT16, Lemma 3.7]

In Quasi-NC^1 one can construct a set of $O(n^{6 \log n})$ integer weight functions on $\left[\frac{n}{2}\right] \times \left[\frac{n}{2}\right]$ where the weights have $O(\log^2 n)$ bits, such that for any given bipartite graphs with n nodes, one of the weight functions is isolating.

With this construction of weight assignments, we can decide the existsence of a perfect matching in a bipartite graph in Quasi-NC^2 as follows: We take the biadjacency matrix A from section 8.1 ehich has entry $2^{w(e)}$ for edge e. We compute $\det(A)$ for each of the constructed weight funcions in parallel. If the given graph has a perfect matching, then one of the weight functions isolates a perfect matching. As we discussed in section 8.1 for this weight function $\det(A)$ will be nonzero. When there is no perfect matching, then $\det(A)$ will be zero for any weight function.

As our weight function have $O(\log^2 n)$ bits, the determinant entries have quasi-polynomial bits. The deteminant can still be computed in parallel with circuits of quasi polynomial of size $2^{O(\log^2 n)}$ by the algorithm of Berkowitz [Ber84]. As we need to compute $2^{O(\log^2 n)}$-many determinants in parallel, our algorithm is in Quasi-NC^2 with circuit size $2^{O(\log^2 n)}$.

To construct a perfect matching we follow the algorithm of Mulmuley in [MVV87] from section 8.1 with each of our weight functions. For a weight function w which is isolating, the algorithm outputs the unique minimum weight perfect matching M. If we have a weight function w' which is not isolating, still $\det(A)$ might be non-zero with respect to w' then the algorithm computes a set of edges M' that might or might not be a perfect matching. However, it is easy to verify if M' is indeed a perfect matching and in this case, we will output M'. As the algorithm involves computation of

similar determinants as in the decision algorithm, it is in Quas-NC^2 with circuit size $2^{O(\log^2 n)}$. This finishes the proof of the Theorem 8.2.1.

8.3 RNC Algorithm with Fewer Random Bits

We can also present the bipartite matching algorithm in section 8.2 in an alternate way i.e. instead of Quasi-NC we wil get an RNC circuit but with only oly-logarithmically many, namely $O(\log^2 n)$ random bits.

Note:-

For complete derandomization, it would suffice to bring the number of random bits down to $O(\log n)$. Then there are only polynomially many random strings which can all be tested in NC. Hence this method is only a log-factor away from derandomization

8.3.1 Decision Version

There are two reasons that we need quasi-polynomially large circuits:

(i) We need to try quasi-polynomially many different weight assignments.

(ii) Each weight assignment has quasi-polynomially large weights

For the first problem we modify the Lemma 8.2.2 to get a random weight assignment which works with high probability.

Lemma 8.3.1 [CRS93, KS01]

Let G be a graph with n nodes and $s \geq 1$. There is a random weight assignment w which uses $O(\log ns)$ random bits and assigns weights bounded by $O(n^3 s \log ns)$ i.e. with $O(\log ns)$ bits such that for any set of s, cycles w gives nonzero circulation to each of the s cycles with probability at least $1 - \frac{1}{n}$

Proof: We will follow the process of Lemma 8.2.2 and give exponential weights modulo small numbers. Here we will use prime numbers as moduli. Recall that the weight function w defined by $w(e_i) = 2^{i-1}$. Let us choose a random prime p among the first t primes. We take our random weight assignment to be w mod p. We want to show that with high probability this weight function gives nonzero circulation to every cycle in $\{C_1, \ldots, C_s\}$ In other words

$$\prod_{i=1}^{s} c_w(C_i) \not\equiv 0 \bmod p$$

as the product is bounded by $2^{n^2 s}$ it has at most $n^2 s$ prime factors. Let us choose $t = n^3 s$. This would mean that a random prime works with probability at least $\left(1 - \frac{1}{n}\right)$. As the $t-$th prime can only be as large as $2t \log t$, the weights are bounded by $2t \log t = O(n^3 s \log ns)$ and hense $O(\log ns)$ bits. A prime with $O(\log ns)$ bits can be constructed using $O(\log ns)$ random bits (see [KS01]) ∎

We will do the same as in subsection 8.2.4. However we use the random scheme Lemma 8.3.1 to choose each of the weight functions $w_0, w_1, \ldots, w_{k-1}$ independently. The probability that all of them provide nonzero circulations on their respective set of cycles $\geq \left(1 - \frac{1}{n}\right)^k \geq 1 - \frac{k}{n} \geq 1 - \frac{\log n}{n}$ using the union bound.

Now instead of combining them to form a single weight assignment like in subsection 8.2.4 we use a different variable for each weight assignment. We modify the construction of matrix A from section 8.1. Let $L = \{u_1, \ldots, u_{\frac{n}{2}}\}$ and $R = \{v_1, \ldots, v_{\frac{n}{2}}\}$ be the vertex partition of G. For variables $x_0, x_1, \ldots, x_{k-1}$ define an $\frac{n}{2} \times \frac{n}{2}$ matrix A by

$$A(i,j) = \begin{cases} \prod_{i=0}^{k-1} x_i^{w_i(e)} & \text{if } e = (u_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

And therefore we have

$$\det(A) = \sum_{M:\text{pm in } G} sgn(M) \prod_{i=0}^{k-1} x_i^{w_i(M)}$$

where $sgn(M)$ is the sign of the corresponding permutation. From the construction of weight assignments it follows that if a graph has a perfect matching then lexicographically minimum term in $\det(A)$ with respect to the exponents of variables $x_0, \ldots, x_{k-1}$ in this precedence order, comes from a unique perfect matching. Therefore we have the following lemma:

Lemma 8.3.2

$\det(A) \neq 0 \iff G$ has a perfect matching

Since each w_i needs to give nonzero circulations to n^4 cycles the weights obtained by the scheme of Lemma 8.3.1 will be bounded by $O(n^7 \log n)$. This means the weight of a matching will be bounded by $O(n^8 \log n)$. Hence $\det(A)$ is a polynomial of individual degree $O(n^8 \log n)$ with $\log n$) variables. To test if $\det(A)$ is nonzero one can apply the standard randomized polynomial identity test

Theorem 8.3.3 Schwartz-Zippel Lemma: [Sch80, Zip79]

Let $P(x_1, \ldots, x_n)$ be a nonzero polynomial of n variables with degree d over field $\mathbb{F}$. Let S be a finite subset of $\mathbb{F}$ with at least d elements in it. Then we have:

$$\Pr_{\alpha_1,\ldots,\alpha_n \in S}[p(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0] \leq \frac{d}{|S|}$$

Hence if we plug in random values for variables x_i independently from $\{1, 2, \ldots, n^9\}$ and if $\det(A) \neq 0$ then the evaluation is nonzero with high probability.

Number of Random Bits: For a weight assignment w_i we need $O(\log ns)$ random bits from Lemma 8.3.1 where $s = n^4$ by Lemma 8.2.4. Thud the number of random bits required for all $w_i's$ togather is $O(k \log n) = O(\log^2 n)$. Finally, we need to plug in $O(\log n)$ random bits for each x_i. This again requires $O(\log^2 n)$ random bits, as discussed above.

Complexity: The weight construction involves taking exponential modulo small primes by Lemma 8.3.1. Primality testing can be done by brute force algorithm in NC^2, as the numbers involved have $O(\log n)$ bits. Thus the weight assignments can be constructed in NC^2. Moreover, the determinant with polynomially bounded entries can be computed in NC^2 [Ber84].

In summery we get the following theorem,

Theorem 8.3.4 [FGT16, Theorem 4.1]

For bipartite graphs, there is an RNC^2-algorithm for PM which uses $O(\log^2 n)$ random bits.

8.3.2 Search Version

Here we get a similar algorithm for SEARCH-PM using also only $O(\log^2 n)$ random bits. This result in [FGT16] improves the RNC algorithm of Goldwasser and Grossman [GG17] based on an earlier version of [FGT16] that uses $O(\log^4 n)$ random bits. Their RNC algorithm has an additional property that it is *pseudo-deterministic* i.e. it outputs the same perfect matching for almost all choice of random bits. The following algorithm does not have that property.

Theorem 8.3.5

For bipartite graphs, there is an RNC^3 algorithm for SEARCH-PM which uses $O(\log^2 n)$ random bits

Proof: Suppose again $G = (V, E)$ be a bipartite graph with vertex partitions $L = \{u_1, \ldots, u_{\frac{n}{2}}\}$ and $R = \{v_1, \ldots, v_{\frac{n}{2}}\}$. Now we will have the weight functions $w_0, \ldots, w_{k-1}$ same as in subsection 8.3.1 by the Lemma 8.3.1. In this case we will combine the weight functions like in subsection 8.2.4 to obtain w. Let M^* be the unique minimum weight perfect matching in G with respect to the combined weight assignment w.

Now in subsection 8.2.4 of the QUASI-NC algorithm we had a sequence of graph $G_1, G_2, \ldots, G_k$ with $G = G_0$ where we took G_{i+1} to be the graph constructed by taking only the minimum weight perfect matchings in G_i with respect to w_i. Now instead of computing $G_1, \ldots, G_k$ in $O(\log^2 n)$ random bits (which is not clear anyway) we will compute a sequence of graphs $H_1, \ldots, H_k$ where H_i will be a subgraph of G_i for each $i \in [k]$. And also each H_i will contain M^* (obviously since we have to find M^*). Hence, once we have $H_k = G_k$ we are done.

So we start with $H_0 = G_0 = G$ and let $0 \leq i < k$. We will describe the ith round. So suppose we have constructed $H_i = (V, E_i)$. Now we want to compute H_{i+1}. Now an edge of E_i will appear in H_{i+1} only if it participates in a matching

M with $w_i(M) = w_i(M^*)$. Therefore, H_{i+1} is a subgraph of G_{i+1} as H_i is a subgraph of G_{i+1}. Now for an edge e denote the product $\mathbb{X}_i^{w(e)}$:

$$\mathbb{X}_i^{w(e)} = \prod_{j=0}^{k-1-i} x_{i+j}^{w_i(e)}$$

For a matching M the term $\mathbb{X}_i^{w(M)}$ is defined similarly. Now let $N(e)$ denote the set of edges which are neighbors of an edge in G_i i.e. all edges $e' \neq e$ such that $e' \cap e \neq \emptyset$. Also for an edge $e \in E_i$, define the $\frac{n}{2} \times \frac{n}{2}$ matrix A_e as:

$$A_e(k,l) = \begin{cases} \mathbb{X}_i^{w(e')} & \text{If } e' = (u_k, v_l) \in E_i - N(e) \\ 0 & \text{otherwise} \end{cases}$$

Now the matrix has 0 entry for each neighboring edge of e. Thus its determinant is a sum over all perfect matchings which contain e. That is

$$\det(A_e) = \sum_{M:\text{PM in } H_i} sgn(M)\, \mathbb{X}_{i+1}^{w(M)}$$

Consider the coefficient c_e of $x_i^{w_i(M^*)}$ in $\det(A_e)$,

$$c_e = \sum_{\substack{M:\text{PM in } H_i \\ w_i(M)=w_i(M^*), e\in M}} sgn(M)\, \mathbb{X}_{i+1}^{w_i(M)}$$

Now define H_{i+1} to be the union of all the edges e for which $c_e \neq 0$.
Claim: $M^* \subseteq E_{i+1}$ i.e. all edges of M^* appears in H_{i+1}.
Proof: For any edge $e \in M^*$, the polynomial c_e will contain the term $\mathbb{X}_{i+1}^{w_i(M^*)}$. As the matching M^* is isolated in H_i with respect to the weight vector $\{w_{i+1}, \ldots, w_{k-1}\}$ the polynomial c_e is nonzero. ∎

For construction of H_{i+1} we need to test if c_e is nonzero for each edge $e \in E_i$. As argued above in the decision part (in subsection 8.3.1) the degree of c_e is $O(n^7 \log^2 n)$. We apply the standard identity testing Theorem **??** and we plug in random values for the variables $x_{i+1}, \ldots, x_{k-1}$ independently from $[n^{11}]$. The probability that the evaluation will be nonzero is at least $1 - O\left(\frac{\log^2 n}{n^3}\right)$.

To compute the evaluation, we plug in values of $x_{i+1}, \ldots, x_{k-1}$ in $\det(A_e)$ in $\det(A_e)$ and find the coefficient of $x_i^{w_i(M^*)}$. This can be done in NC^2 by [BCP83, Corollary 4.4]. For all the edges we use the same random values for the variables $x_{i+1}, \ldots, x_{k-1}$ in each identity test. The probability that e test works successfully for each edge is at least $1 - O\left(\frac{\log^2 n}{n}\right)$ by the union bound. We continue this for k rounds to find H_i, which is a perfect matching.

We need again $O(\log^2 n)$ random bits for the weight assignments $w_0, \ldots, w_{k-1}$ and the values for the $x_i's$. Note that we use the same random bits for x_i in all k rounds. This decreases the success probability, which is now at least $1 - O\left(\frac{\log^3 n}{n}\right)$ by the union bound.

In NC^2 we can construct the weight asssignments and compute the determinants in each round. As we have $k = O(\log n)$ rounds, the overall complexity becomes NC^3.

∎

Chapter 9

Matroid Polytope

Chapter 10

Linear Matroid Intersection

Chapter 11

Matroid Matching

Chapter 12

Fractional Matroid Matching

Fractional Matroid Matchings generalizes the case for Matroid Matching or Matroid Parity problem with allowing fractional solutions for the polytope which we will show below. We start with the same kind of state like Matroid Parity Problem

12.1 Fractional Matroid Matchings Polytope

Let $M = (E, \mathcal{I})$ is a matroid with ground set E of even cardinality and with elements E is partitioned into lines or pairs. Let L is the set of lines. Let $r : \mathcal{P}(E) \to \mathbb{Z}$ be the rank function and $sp : \mathcal{P}(E) \to \mathcal{P}(E)$ be the span function. Assume that $\forall\, l \in L$, $r(L) = 2$. With this setting (same as matroid parity problem) we now define the polytope following [VV92]

Definition 12.1.1: Fractional Matroid Matching Polytope

Let $\mathscr{L}$ denote the lattice of flats in M wtih $S_1 \wedge S_2 = S_1 \cap S_2$ and $S_1 \vee S_2 = sp(S_1 \cup S_2)$ and for each line $l \in L$ let $a_l : \mathscr{L} \to \{0, 1, 2\}$ be the function $a_l(S) = r(sp(l) \cap S)$. Now for any $S \in \mathscr{L}$ and $x \in \mathbb{R}_+^{|L|}$ let $a(S) \cdot x$ denote the vector $(a(S) \cdot x)_l = a_l(S)x_l$ for any $l \in L$. Then the set

$$FP(M) = \{x \in \mathbb{R}_+^{|L|} \mid : a(S) \cdot x \leq r(S) \text{ for each } S \in \mathscr{L}\}$$

is fractional matroid matching polytope for M and each vector $x \in FP(M)$ is called a fractional matroid matching.

We take $|L| = m$ to imply that originally the ground set has $2m$ elements. Now we can also allow x to be from $\mathbb{R}^m$, not restricting only to positive vectors. This polytope is a subset of $[0, 1]^m$. We will explain the setting wtih the following example:

Example 12.1

Consider the matroid M with ground set

$$E = \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$$

where every 4 element subset of E is a base except these 4 sets

$$\{a_1, a_2, b_1, b_2\}, \quad \{a_1, a_2, c_1, c_2\}, \quad \{a_1, a_2, d_1, d_2\},$$
$$\{b_1, b_2, c_1, c_2\}, \quad \{b_1, b_2, d_1, d_2\}, \quad \{c_1, c_2, d_1, d_2\}$$

Now the lines are defined to be

$$l_1 = \{a_1, a_2\} \qquad l_2 = \{b_1, b_2\}, \qquad l_3 = \{c_1, c_2\}, \qquad l_4 = \{d_1, d_2\}$$

Now the flats of M are empty set, individual elements, every pair of elements, set consists of one element from

each of three lines, pair of line and E. Hence $FP(M)$ is the set of $x \in \mathbb{R}_+^{|L|}$ satisfying

$$2x_1 + 2x_2 \leq 3 \qquad 2x_1 + 2x_3 \leq 3 \qquad 2x_1 + 2x_4 \leq 3$$
$$2x_2 + 2x_3 \leq 3 \qquad 2x_2 + 2x_4 \leq 3 \qquad 2x_3 + 2x_4 \leq 3$$
$$2x_1 + 2x_2 + 2x_3 + 2x_4 \leq 4$$
$$2x_i \leq 2 \quad \text{for each } i \in [4]$$

Now the we show the theorem Theorem 12.1.1 which states that the fractional matroid matching polytope arises as a linear relaxation of the matroid matching problem.

Theorem 12.1.1 [VV92, Theoerm 2.1]
An integer vector $x \in \mathbb{R}_+^m$ is the incidence vector of a matroid matching iff x is a fractional matroid matching.

You can clearly see this theorem by comparing the Matroid Matching Polytope and Fractional Matroid Matching Polytope so we are omitting the proof.

Theorem 12.1.2 [GP13, Theorem 1]
The vertices of the fractional matroid matching are half-integral

12.1.1 Weighted Fractional Matroid Matching

Definition 12.1.2: Weighted Fractional Matroid Matching Problem

It is to find a fractional matroid matching x that maximizes $w \cdot x$ for a non-negative weight assignment $w : L \to \mathbb{Z}_+$.
For plain Fractional Linear Matroid Matching Problem we need to find a fractional matroid matching x which maximizes the size i.e. L_1 norm of x which is $\sum_{l \in L} |x_l|$.

Gijswijt and Pap in [GP13] gave a polynomial time algorithm for weighted fractional linear matroid matching. They also gave the following characterization for maximizing face of the polytope with respect to a weight function.

Theorem 12.1.3 [GP13, Prood of Theorem 1]
Let $L = \{l_1, \ldots, l_m\}$ be a set of lines with $l_i \subseteq \mathbb{F}^n$ and $w : L \to \mathbb{Z}$ be a weight assignment on L. Let F denote the set of fractional linear matroid matchings maximizing and $S \subseteq [m]$ such that every $x \in F$ has $y_e = 0$ for all $e \in S$. Then for some $k \leq n$, $\exists$ a $k \times m$ matrix D_F and $b_F \in \mathbb{Z}^k$ such that

- $D_F \in \{0, 1, 2\}^{k \times m}$
- The sum of entries in any column of D_F is exactly 2
- A fractional matroid matching x is in F iff $y_e = 0$ for $e \in S$ and $D_F x = b_F$.

12.2 A QUASI-NC Algorithm with Isolating Weight Assignment

In this section we will describe how we can construct an isolating weight assignment for fractional matroid matching with just the number of lines as input.

Now for a face F of a polytope, let $\mathcal{L}_F$ denote the lattice

$$\mathcal{L}_F = \{v \in \mathbb{Z}^m \mid v = \alpha(x_1 - x_2) \text{ for some } x_1, x_2 \in F \text{ and } \alpha \in \mathbb{R}\}$$

and $\lambda(\mathcal{L}_F)$ denote the length of the shortest vector of $\mathcal{L}_F$. Hence $\mathcal{L}_F$ consistes of all integral vectors parallel to the face F.

Now by Theorem 12.1.3 the face maximizing the size is described by the equation $D_F x = b_F$ where $D_F \in \{0,1,2\}^{k\times m}$ with column sum 2. Hence $\mathcal{L}_F$ is exactly the set of integral vectors in the null space of D_F. Therefore

$$\mathcal{L}_F = \{v \in \mathbb{Z}^m \mid D_F v = 0\}$$

So we will prove that the number of vectors in $\mathcal{L}_F$ with size less than twice the length of shortest vector is polynomially bounded in Subsection 1.2.2.

First we will show how we can interpret D_F an incidence matrix for a graph instead of general matrix. $D_F \in \{0,1,2\}^{k\times m}$ where every column sum is 2. Hence we can thing of a graph G_D with vertex set $[k]$ and the m edges defined as follows: for every $e \in [m]$ the e-th edge of G_D is drawn between the vertices $s,t \in [k]$ if $D_F[s,e] = D_F[t,e] = 1$ and e-th edge is a self look on the vertex $s \in [k]$ if $D_F[s,e] = 2$.

12.2.1 Alternating Circuits

First we will define some elements called alternating indicator vetors and alternating circuits following the [ST17] for the proof of Theorem 12.2.3.

Definition 12.2.1: Alternating Indicator Vector & Alternating Circuit

Let $C = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} v_2 \cdots \xrightarrow{e_{k-2}} v_{k-1} \xrightarrow{e_{k-1}} v_0$ be a closed walk of even length in a multigraph G with loops. Then the *Alternating Indicator Vector* of C denoted by $(\pm\mathbb{1})_C$ is the vector

$$(\pm\mathbb{1})_C := \sum_{i=0}^{k-1} (-1)^i \mathbb{1}_{e_i}$$

C is called *Altenating Circuit* if its alternating indicator vector is nonzero

We can use the parity for $(-1)^i$ as direction of movement in C. So a closed walk C is a alternating circuti if there exists at least one edge $e \in C$ for which C has moved through e more time in one direction than the other.

Observation. $|(\pm\mathbb{1})_C| \le |C|$ *for any even length closed walk* C.

Now we will prove a property for all alternating circuits in G_D

Lemma 12.2.1 [GOR24, Proof of Claim 1, Proof of Theorem 3.4]
For any alternating circuit C we have $D_F \cdot (\pm\mathbb{1})_C = 0$

Proof: Suppose $C = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} v_2 \cdots \xrightarrow{e_{k-2}} v_{k-1} \xrightarrow{e_{k-1}} v_0$. We denote the i-th column of D_F is denoted by D_i. Now

$$D_i \cdot (\pm\mathbb{1})_C = \sum_{j=0}^{k-1} (-1)^j D_F[i,e_j] = \sum_{e_j : i \in e_j \in C} (-1)^j D_F[i,e_j]$$

Hence only the edges in C which are incident on i contributes to the above sum. Therefore the whole sum is partitioned into distinct subparts of walks where each subpart is of the form

$$v_s \xrightarrow{e_s} \underbrace{i \xrightarrow{e_{s+1}} i \xrightarrow{e_{s+2}} \cdots \xrightarrow{e_{s+k-1}} i}_{k \text{ times}} \xrightarrow{e_{s+k}} v_t \qquad \text{where } v_s, v_t \neq i$$

i.e. the part starts from a vertex not equal to i then goes to i and after looping in i for some times the walk goes to another vertex not equal to i. We will show that for each of these parts the contribution to the sum is 0.

Now for such a subpart their contribution to the sum is

$$\delta = (-1)^s + 2\sum_{j=1}^{k-1} (-1)^{s+i} + (-1)^{s+k}$$

We will analyze case wise:

Case 1: k is odd: Then $k-1$ is even. Hence

$$\sum_{j=1}^{k-1}(-1)^{s+i} = (-1)^s \sum_{j=1}^{k-1}(-1)^i = 0$$

Therefore

$$\delta = (-1)^s + (-1)^{s+k} = (-1)^s[1 + (-1)^k] = 0$$

as k is odd.
Case 2: k is even: Then $k-2$ is even. Hence

$$\sum_{j=1}^{k-1}(-1)^{s+i} = \left[\sum_{j=1}^{k-2}(-1)^{s+i}\right] + (-1)^{s+k-1} = (-1)^{s+k-1}$$

Hence

$$\delta = (-1)^s + 2(-1)^{s+k-1} + (-1)^{s+k} = (-1)^s[1 + 2(-1)^{k-1} + (-1)^k] = 0$$

Therefore we showed that for each such parts their contribution to the sum is 0. Therefore the total sum is 0 i.e. $D_i \cdot (\pm\mathbb{1})_C = 0$. Since this is true for all $i \in [k]$ we have $D_F \cdot (\pm\mathbb{1})_C = 0$. ■

Now we will show that any vector in the lattice $\mathcal{L}_F$ can be decomposed into finite sum of alternating indicator vectors of alternating circuits with the property that for each such alternating circuit C, $|(\pm\mathbb{1})_C| = |C|$. Before that we introduce a relation between two vectors in $\mathbb{R}^m$ this will come in handy for the decomposition.

Definition 12.2.2: Conformal

For $x, y \in \mathbb{R}^m$, x is said to be conformal to y if $x_i y_i \geq 0$ and $|x_i| \leq |y_i|$ $\forall\, i \in [m]$ and it is denoted by $x \sqsubseteq y$

Lemma 12.2.2 [GOR24, Claim 1, Proof of Theorem 3.4]

For any $x \in \mathcal{L}_F$, $\exists$ alternating circuits $C_1, C_2, \ldots, C_t$ in G_D such that

$$x = \sum_{i=1}^{t}(\pm\mathbb{1})_{C_i}$$

where $\forall\, i \in [t]$, $(\pm\mathbb{1})_{C_i} \sqsubseteq x$ and $|(\pm\mathbb{1})_{C_i}| = |C_i|$.

Proof: We will decompose a given x into alternating indicator vectors by the following iterative algorithm:

Algorithm 2: Decomposition of a Lattice Vector

Input: $x \in \mathcal{L}_F$
Output: $\mathcal{Y} = \{y_i\}$ where $|\mathcal{Y}| < \infty$ and $\forall\, y_i \in \mathcal{Y}$ are alternating indicator vectors

```
1  begin
2      while |x| ≠ 0 do
3          y ⟵ 0, j ⟵ 1
4          Let e_0 ∈ [m] such that x_{e_0} > 0
5          y_{e_0} ⟵ 1 and let e_0-th edge in G_D be {v_0, v_1}
6          while True do
7              if ∃ e ∈ [m] such that e-th edge is {v_j, u}, |x_e| > |y_e| and (−1)^j x_e > 0 then
8                  y_e ⟵ y_e + (−1)^j, e_j ⟵ e
9                  v_{j+1} ⟵ u
10                 j ⟵ j + 1
11             else
12                 return x ∉ L_F
13             if j ≡ 0 (mod 2) and v_j = v_0 then
14                 x ⟵ x − y
15                 return y and exit inner while loop
```

Now suppose x denote the vector at some iteration of the outer while loop. Now forall $e \in [m]$, let at jth and lth iteration of the inner while loop $(-1)^j$ and $(-1)^l$ are added to y_e respectively. Now both j and l has same parity because since the edge is not changing both times $(-1)^j x_e > 0$ and $(-1)^l x_e > 0$ has to be satisfies. Therefore we get j and l have same parity. Hence for a full run of the inner while loop for any $e \in [m]$ everytime y_e is changed the same $(-1)^j$ is added. Hence whenever y_e is changed $|y_e|$ is increased. Therefore $|y|$ increases for each iteration of the inner while loop. But $|y|$ cannot exceed $|x|$ since as the addition step works when $\exists\ e \in [m]$ with $|x_e| > |y_e|$ and $(-1)^j x_e > 0$, after addition $|y_e + (-1)^j| \leq |y_e| + 1 \leq |x|$. So if we start with $|y| \leq |x|$ after addition with each iteration of inner while loop we still have $|y| \leq |x|$. Also since for every such edge we add $(-1)^j$ to y_e when $(-1)^j x_e > 0$ and $|x_e| > |y_e|$. So by the first condition we have y_e and x_e has the same sign i.e. $x_e y_e > 0$. And we also have $|x_e| \geq |y_e|$. Therefore we have $y \sqsubseteq x$.

Now since we start the algorithm by adding 1 to y_{e_0} at Line 5 we have initially $|y| > 0$ and afterwards with each iteration of the inner while loop $|y|$ increases so we have $|y| > 0$.Now if the current iteration of the inner while loop ends at Line 13 then we get a closed walk C since in each iteration of the inner while loop the algorithm follows a walk in G_D. So C is an alternating circuit and we have $|y| = |C|$. Now since $y \sqsubseteq x$ we have $|x - y| = |x| - |y| < |x|$. Since C is alternating circuit by Lemma 12.2.1 we have $y \in \mathcal{L}_F$. Hence we have $x - y \in \mathcal{L}_F$.

Now all that remains is to show that the algorithm never goes to Line 11 if $x \in \mathcal{L}_F$. The only reason the algorithm can go to Line 11 if at some iteration of the inner while loop can not find an edge $e \in [m]$ such that $|x_e| > |y_e|$ and $(-1)^j x_e > 0$ where $e \in \delta(v_j)$ where $\delta(v_j)$ denotes te set of edges incident on v_j. Suppose at this point we have the walk

$$\mathcal{P} = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{j-1}} v_j$$

Now by following the proof of Lemma 12.2.1 we have that $D_i y = 0$ forall $i \in [k]$ except $i \notin \{v_0.v_j\}$.

Claim: If j is odd then $D_{v_j} y > 0$ and if j is even then $D_{v_j} y < 0$.
Proof: We will prove the case for j is odd. The even case will follow similarly. Since j is odd we have $j - 1$ even. Therefore $x_{e_{j-1}} > 0$ as $(-1)^{j-1} x_{e_{j-1}} > 0$. Therefore for both e_0 and e_{j-1}, $x_{e_0}, x_{e_{j-1}}$ positive. Now take the partitions as described in the proof of Lemma 12.2.1. Now we analyze case wise:
Case 1: $v_0 \neq v_j$**:** Then only the partition containing v_{e_j} might contribute something nonzero because other partitions contributes 0 to the sum $D_{v_j} y$. Let the partition be

$$v_s \xrightarrow{e_{j-k-1}} \underbrace{v_j \xrightarrow{e_{j-k}} v_j \xrightarrow{e_{j-k+1}} \cdots \xrightarrow{e_{j-1}} v_j}_{k+1 \text{ times}} \qquad \text{where } v_s \neq v_j$$

Now k can not be more than 1 since if e_{j-1} and e_{j-2} are both loops in v_j then for e_{j-1}, $x_{e_{j-1}} > 0$ but $x_{e_{j-2}} < 0$ but $e_{j-1} = e_{j-2} \implies 0 < x_{e_{j-1}} = x_{e_{j-2}} < 0$. Then this partition contributes $\delta = (-1)^{j-2} + 2(-1)^{j-1} = -1 + 2 > 0$ to $D_{v_j} y$ if there is a loop and if there is no loop then it also contributes 1 as then the contribution will be only $(-1)^{j-1} = 1 > 0$.
Case 2: $v_0 = v_j$**:** If v_0 and v_j is in same partition i.e. $j = 1$ then the it contributes $(-1)^0 = 1 > 0$ to the sum. Otherwise v_0 is in different partition. By the above case we know that the value contributed by the partition containing v_j is 1. So we have to only find the contribution by the partition containing v_0. Again by same logic as the above case from v_0 at most one loop starting from v_0 and then it moves to some other vertex. Hence the sum contributed is $2(-1)^0 + (-1)^1 = 2 - 1 = 1$ if there is a loop and if there is no loop then it contributes $(-1)^0 = 1$. Hence in both cases the contribution of the partition containing v_0 to the sum $D_{v_j} y$ is 1. Hence we have $D_{v_j} y > 0$.

Hence by above we get that if j is odd then the value $D_{v_j} y > 0$. Similarly following the same process in the case of j is even we get $D_{v_j} y < 0$. ∎

Hence by the lemma we have that if j is odd $D_{v_j} y > 0$ and if j is even then $D_{v_j} y < 0$. So WLOG suppose j is odd. Now define

$$\text{supp}^+(x) = \{i \in [m] \mid x_i > 0\} \qquad \text{and} \qquad \text{supp}^-(x) = \{i \in [m] \mid x_i < 0\}$$

Then we have

$$0 = D_{v_j} x = \sum_{e \in \text{supp}^+(x) \cap \delta(v_j)} D_F[v_j, e] \cdot |x_e| - \sum_{e \in \text{supp}^-(x) \cap \delta(v_j)} D_F[v_j, e] \cdot |x_e|$$

$$0 < D_{v_j} y = \sum_{e \in \text{supp}^+(x) \cap \delta(v_j)} D_F[v_j, e] \cdot |y_e| - \sum_{e \in \text{supp}^-(x) \cap \delta(v_j)} D_F[v_j, e] \cdot |y_e|$$

Now be construction of y we have $y \sqsubseteq x \implies |x_e| \geq |y_e|, x_e y_e > 0 \forall\, e \in [m]$. The algorithm stopped at Line 13 since $\nexists\, e \in \delta(v_j)$ such that $|x_e| > |y_e|$ and $(-1)^j x_e = -x_e > 0$. Therefore $|x_e| = |y_e|$ for all $e \in \text{supp}^+(x) \cap \delta(v_j)$ otherwise the algorithm would have proceeded one more step and $\text{supp}^-(x) \cap \delta(v_j) = \text{supp}^-(y) \cap \delta(v_j)$ since for any $e \in [m]$, $x_e y_e > 0$. Therefore $\text{supp}^+(x) \cap \delta(v_j) = \text{supp}^+(y) \cap \delta(v_j)$. Now for all $e \in \text{supp}^+(x) \cap \delta(v_j)$ we have $|x_e| \geq |y_e|$. Hence we have

$$0 = D_{v_j} y \geq D_{v_j} y > 0$$

It is a contradiction. Hence if $x \in \mathcal{L}_F$ the algorithm never goes to Line 13. And therefore the algorithm successfully decomposes x into sum of alternating circuits. ■

12.2.2 Bounding vectors in $\mathcal{L}_F$ with Small Size

Theorem 12.2.3 [GOR24]

Let $D \in \{0, 1, 2\}^{pimesm}$ be a matrix such that the sum of entries of each column equals 2. Let $\mathcal{L}_D$ denote the lattice $\{v \in \mathbb{Z}^m \mid Dv = 0\}$. Then it holds that

$$|\{v \in \mathcal{L}_D \mid |v| < 2\lambda(\mathcal{L}_D)\}| \leq m^{O(1)}$$

Proof: For the given D consider the graph G_D obtained from D as explained at the start of Section 1.2. to show that the number of vectors in $\mathcal{L}_D$ with size less than twice the size of shortest vector in $\mathcal{L}_D$ we will show that for any such lattice vector there is only one alternating circuit in the decomposition of the vector in Lemma 12.2.2.

Claim: Any lattice vector $x \in \mathcal{L}_D$ with $|x| < 2\lambda(\mathcal{L}_D)$ is an alternating vector $(\pm\mathbb{1})_C$ of some alternating circuit C in G_D such that $|x| = |C|$

Proof: Suppose the contrary. Since $x \in \mathcal{L}_D$ by Lemma 12.2.2 $\exists\, C_1, \ldots, C_t$ with $t \geq 2$ such that $x = \sum_{i=1}^{t} (\pm\mathbb{1})_C$ with $(\pm\mathbb{1})_C \sqsubseteq x$ and $|(\pm\mathbb{1})_{C_i}| = |C_i|$ for all $i \in [t]$. Then we have

$$|x| = \sum_{i=1}^{t} |(\pm\mathbb{1})_{C_i}| \geq t\lambda(\mathcal{L}_D) \geq 2\lambda(\mathcal{L}_D)$$

which is a contradiction since we assumed that $|x| < 2\lambda(\mathcal{L}_D)$. Hence $t = 1$ i.e. $x = (\pm\mathbb{1})_C$ for some alternating circuit C with $|x| = |C|$. ■

Hence the Claim implies that $\lambda(\mathcal{L}_D)$ is equal to the size of the smallest alternating circuit of G_D. And it also implies that we only need to bound the number of alternating indicator vectors that correspond to alternating circuit of size at most $2\lambda(\mathcal{L}_D)$ to prove the lemma. For that by the Theorem 12.2.4 we get that the number of such alternating indicator vectors are polynomially bounded by m. ■

In the following theorem I have modified the proof of the theorem since we are not working the general setting like in [ST17]. We are basically taking node-weight for every vertex to be 1.

Theorem 12.2.4 [ST17, Lemma 5.4]

Let G be a graph on n vertices such that the size of the smallest alternating circuit λ. Then the cardinality of the set

$$\{(\pm\mathbb{1})_C \mid C \text{ is an alternating circuit in } G \text{ of size at most } 2\lambda\}$$

is at most n^{17}.

Proof: Like in the proof of Lemma 8.2.4 we will associate a small signature $\sigma(C)$ with each alternating circuit C in G of size at most 2λ. The signatures have the property that alternating circuits with different alternating indicator vectors are

assigned to different signatures. This will prove that the number of alternating circuits in G of size at most 2λ is at most the number of possible signatures which we will show polynomially bounded.

So let

$$C = v_0 \xrightarrow{e_0} v_1 \xrightarrow{e-1} \cdots \xrightarrow{e_{k-2}} v_{k-1} \xrightarrow{e_{k-1}} v_0$$

be an alternating circuit in G of size at most 2λ. Now for $v_i \in C$ define

$$in(v_i) \coloneqq \text{Incoming edge of } v_i \qquad out(v_i) \coloneqq \text{Ourgoing edge og } v_i$$

Now we define the signature $\sigma(C)$:

- Let $i_0 = 0$ be the first vertex in C.
- For $j \in [3]$, select $i_j = \left\lceil \frac{j\lambda}{4} \right\rceil$. Hence $\sum_{l=i_{j-1}+1}^{i_j - 1} 1 \leq \frac{\lambda}{2}$.
- Take $t = \max\{j \mid i_j < k\}$. Output

$$\sigma(C) = \left((-1)^{i_j}, in(v_{i_j}), out(v_{i_j})\right)_{j \in \{0,1,\ldots,t\}}$$

Note:-

We are considering t as an important index because size of C can be small so that $k < i_3 < 2k$ then to define i_3 properly we basically take $v_{i_j} = v_{i_j \bmod k}$. In that case the index which is still less than k helps us to consider the cycle as joining of t different paths. Also this case can only happen to i_3 since C has length at least λ so both i_1 and i_2 are same after modulo k.

The indices $i_0, i_1, \ldots, i_t$ partition C into paths:

$$\begin{aligned}
C_0 &= v_{i_0} \xrightarrow{e_{i_0}} v_{i_0+1} \xrightarrow{e_{i_0+1}} \cdots \xrightarrow{e_{i_1-1}} v_{i_1} \\
C_1 &= v_{i_1} \xrightarrow{e_{i_1}} v_{i_1+1} \xrightarrow{e_{i_1+1}} \cdots \xrightarrow{e_{i_2-1}} v_{i_2} \\
C_2 &= v_{i_2} \xrightarrow{e_{i_2}} v_{i_2+1} \xrightarrow{e_{i_2+1}} \cdots \xrightarrow{e_{i_3-1}} v_{i_3} \\
&\vdots \\
C_t &= v_{i_t} \xrightarrow{e_{i_t}} v_{i_t+1} \xrightarrow{e_{i_t+1}} \cdots \xrightarrow{e_{i_0-1}} v_{i_0}
\end{aligned}$$

So the node-weight of internal vertices for each path is C_i for $i \in \{0, 1, \ldots, t-1\}$ is at most $\frac{\lambda}{2}$.Therefore by the maximality of t we have:

$$\underbrace{\sum_{l=i_0+1}^{i_1} 1}_{\geq \frac{\lambda}{2}} + \underbrace{\sum_{l=i_1+1}^{i_2} 1}_{\geq \frac{\lambda}{2}} + \cdots + \underbrace{\sum_{l=i_{t-1}+1}^{i_t} 1}_{\geq \frac{\lambda}{2}}$$

Each sum is at least $\frac{lm}{2}$ so we have the total node-weight of internal vertices of C_t is at most $\frac{\lambda}{2}$.

Now we will count the number of possible signatures. For each $j \in \{0, 1, \ldots, t\}$ in the tuple $\left((-1)^{i_j}, in(v_{i_j}), out(v_{i_j})\right)$ there are 2 possible parity of i_j and both $in(v_{i_j})$ and $out(v_{i_j})$ has at most n^2 choices. So for each j there are $2n^4$ many possible tuples. Therefore:

For $t = 0$ there are $2n^4$ many signatures
For $t = 1$ there are $(2n^4)^2$ many signatures
For $t = 2$ there are $(2n^4)^3$ many signatures
For $t = 3$ there are $(2n^4)^4$ many signatures

Hence total number of possible signatures is $2n^4 + (2n^4)^2 + (2n^4)^3 + (2n^4)^4 < n^{17}$. So number of signatures is polynomially bounded. Now we will show that any 2 alternating circuits C, D of size at most 2λ have different signatures i.e.

$$\sigma(C) \neq \sigma(D) \iff (\pm\mathbb{1})_C \neq (\pm\mathbb{1})_D$$

Claim: C is the only alternating circuit C of size at most 2λ that is associated to the signature $\sigma(C)$.
Proof: So let C, D are two alternating circuits of size at most 2λ and $\sigma(C) = \sigma(D)$. As described above C is partitioned into paths $C_0, \ldots, C_t$ using $i_0, \ldots, i_t$ and similarly D is also partitioned into $D_0, \ldots, D_t$ using $j_0, \ldots, j_t$. Since $\sigma(C) = \sigma(D)$, i_k and j_k have same parity for all $k \in \{0, 1, \ldots, t\}$. and since $in(v_{i_k}) = in(v_{j_k})$ and $out(v_{i_k}) = out(v_{j_k})$ we have $v_{i_k} = v_{j_k}$ for all $k \in \{0, 1, \ldots, t\}$. Now let b_k denote the parity of kth tuple in $\sigma(C)$ i.e. $b_k = (-1)^{i_k} = (-1)^{j_k}$. Then we have

$$(\pm\mathbb{1})_C = \sum_{k=0}^{t} b_k (\pm\mathbb{1})_{C_k} \qquad (\pm\mathbb{1})_D = \sum_{k=0}^{t} b_k (\pm\mathbb{1})_{D_k}$$

Since we know $(\pm\mathbb{1})_C \neq (\pm\mathbb{1})_D$, $\exists\ k \in \{0, 1, \ldots, t\}$ such that $(\pm\mathbb{1})_{C_k} \neq (\pm\mathbb{1})_{D_k}$Now basically we will glue C_k and D_k togather. Now

$$C_k = \underset{\|}{v_{i_k}} = a \to b \to \cdots \to c \to d = \underset{\|}{v_{i_{k+1 \bmod t}}}$$
$$\underset{v_{j_k}}{} \qquad\qquad\qquad\qquad\qquad\qquad \underset{v_{j_{k+1 \bmod t}}}{}$$

Hence the path C_k and D_k differs in between the path $b \rightsquigarrow c$. Let P_C denotes the path from b to c following C_k and P_D denotes the path from b to c following D_k. As $(-1)^{i_k} = (-1)^{j_k}$ and $(-1)^{i_{k+1 \bmod t}} = (-1)^{j_{k+1 \bmod t}}$. Therefore we have

$$|C_k| + |D_k| \equiv 0 \bmod 2 \implies |P_C| + |P_D| \equiv 0 \bmod 2$$

So denote the cycle C to be the closed walk:

$$b \xrightarrow{f_1} \cdots \xrightarrow{f_{|P_C|}} c \xrightarrow{g_1} \cdots \xrightarrow{g_{|P_D|}} b$$

Where $f_1, \ldots, f_{|P_C|}$ are the edges of the path P_C and $g_1, \ldots, g_{|P_D|}$ are the edges of the path P_D in reverse. Hence length of the cycle C is

$$|P_C| + |P_D| = |C_k| - 2 + |D_k| - 2 < \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda$$

Therefore if B is an alternating circuit then we have an alternating circuit $(\pm\mathbb{1})_C$ of size less than λ, which is not possible. Hence all that is left is showing that $(\pm\mathbb{1})_C$ is indeed an alternating circuit i.e. $(\pm\mathbb{1})_C \neq 0$. Now:

$$\begin{aligned}
-(\pm\mathbb{1})_C &= \sum_{i=1}^{|P_C|} (-1)^i \mathbb{1}_{f_i} + \sum_{i=1}^{|P_D|} (-1)^{|P_C|+i} \mathbb{1}_{g_i} \\
&= \underbrace{\left[(-1)^0 \mathbb{1}_{out(a)} + \sum_{i=1}^{|P_C|} (-1)^i \mathbb{1}_{f_i} + (-1)^{|P_C|+1} \mathbb{1}_{in(d)} \right]}_{=(\pm\mathbb{1})_{C_k}} \\
&\qquad - \underbrace{\left[(-1)^{|P_C|+1} \mathbb{1}_{in(d)} + \sum_{i=1}^{|P_D|} (-1)^{|P_C|+i+2} \mathbb{1}_{g_i} + (-1)^{|P_C|+|P_D|+3} \mathbb{1}_{out(a)} \right]}_{=-(\pm\mathbb{1})_{D_k}}
\end{aligned}$$

Therefore we have $(\pm\mathbb{1})_C = (\pm\mathbb{1})_{D_k} - (\pm\mathbb{1})_{C_k} \neq 0$. Therefore we have $(\pm\mathbb{1})_C$ is an alternating circuit. This leads to a contradiction ∎

So there aren't two different alternating circuits of size at most 2λ with same signatures. Therefore number of alternating circuits in G of size at most 2λ is bounded by number of signatures which is at most n^{17}. ∎

12.2.3 Algorithm for Finding Isolating Weight Assignment

With this theorem we have

Theorem 12.2.5 [GTV21, Theorem 2.5]

Let k be a positive integer and $P \subseteq \mathbb{R}^m$ a polytope such that its extreme ppoints are in $\left\{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\right\}^m$ and there exists a constant $c > 1$ with

$$|\{v \in \mathcal{L}_F \colon |v| < c\lambda(\mathcal{L}_F)\}| \leq m^{O(1)}$$

for any face F of P. Then there exists an algorithm that, given k and m, outputs a set $\mathcal{W} \subseteq \mathbb{Z}^m$ of $m^{O(\log km)}$ weight assignments with weights bounded by $m^{O(\log km)}$ such that there exists at least one $w \in \mathcal{W}$ that is isolating for P, in time $polylog(km)$ using $m^{O(\log km)}$ many parallel processors.

Using this we finally have an algorithm for isolating a fractional matroid matching polytope:

Theorem 12.2.6 [GOR24, Theorem 3.1]

There exists an algorithm that given $m \in \mathbb{Z}_+$ outputs a set $\mathcal{W} \subseteq \mathbb{Z}_+^m$ of $m^{O(\log m)}$ weight assignments with weights bounded by $m^{O(\log m)}$ such that, for any fractional matroid matching polytope P of m lines, there exists at least one $w \in \mathcal{W}$ that is isolating for P, in time $polylog(m)$ usign $m^{O(\log m)}$ many parallel processors.

Chapter 13

Isolation of Paths in Layered Graph

Chapter 14

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