Bounding PoA using Linear and Quadratic Programming

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Introduction

- **Pure Nash Equilibria**: A strategy profile $s \in S$ of a game Γ is a *Pure Nash Equilibrium* if for every player $i \in [n]$ and for all $s'_i \in S_i$, $u_i(s) \geq u_i(s'_i, s_{-i})$.
- **Mixed Nash Equilibria**: A mixed strategy profile $\sigma \in \Sigma$ of a game Γ is a *Mixed Nash Equilibria* if for every player $i \in [n]$ and for all $s'_i \in S_i$, $\mathop{\mathbb{E}}_{s\sim\sigma}[u_i(s)] \geq \mathop{\mathbb{E}}_{s\sim\sigma}[u_i(s'_i, s_{-i})]$
- **Coarse Correlated Equilibria**: A distribution μ over S of a game Γ is a *Coarse Correlated Equilibria* if for every player $i \in [n]$ and for all $s'_i \in S_i$, $\mathop{\mathbb{E}}_{s\sim\mu}[u_i(s)] \geq \mathop{\mathbb{E}}_{s\sim\mu}[u_i(s'_i, s_{-i})]$

PNE $\subseteq$ MNE $\subseteq$ CCE.

Lagrangian Duality

Given convex problem:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h_i(x) \leq 0 \quad \forall\, i \in [m], \\ & l_j(x) = 0 \quad \forall\, j \in [r] \end{aligned}$$

Define Lagrangian $\mathcal{L}(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j l_j(x)$. Define

$$g(u, v) = \inf_x \mathcal{L}(x, u, v)$$

The dual of the convex problem:

$$\begin{aligned} \text{maximize} \quad & g(u, v) \\ \text{subject to} \quad & u \geq 0 \end{aligned}$$

Fenchel Duality

Let $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Then the convex conjugate of f is the function

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle y, x\rangle - f(x)\}$$

Theorem (Fenchel Duality)

Let $f : X \to \mathbb{R}, g : Y \to \mathbb{R}$ are two convex functions and $A : X \to Y$ any bounded linear map. Suppose

$$p^* = \inf_{x \in X}\{f(x) + g(Ax)\} \quad \text{and} \quad d^* = \sup_{y \in Y}\{-f^*(A^*y) - g^*(-y)\}$$

where A^* is the adjoint of A. Then $p^* \geq d^*$

Weighted Congestion Games

Definitions

- $\mathcal{N}$: Set of players
- $\mathcal{E}$: The ground set of resources
- For each player $j \in \mathcal{N}$, let $S_j \subseteq 2^{\mathcal{E}}$ be the set of strategies available to player j. Let $S = \bigtimes_{j \in \mathcal{N}} S_j$.
- For each $j \in \mathcal{N}$ and each $e \in \mathcal{E}$ there is a weight of the resource $w_{ej} \in \mathbb{R}^+$.
- For each $e \in \mathcal{E}$ the cost of resource e is an affine function $C_e : \mathbb{R} \to \mathbb{R}$ where $c_e(x) = a_e \cdot x + b_e$
- For any strategy profile $f \in S$, the cost of player j is **Cost**$(f)_j = \sum_{e \in f_j} w_{ej} \cdot c_e(l_e(f))$ where $l_e(f) = \sum_{j' : e \in f_{j'}} w_{ej'}$ is the load on resource e. Do

$$\text{Cost}(f) = \sum_{j \in \mathcal{N}} \sum_{e \in f_j} w_{ej} \cdot c_e(l_e(f)) = \sum_{e \in \mathcal{E}} a_e \cdot l_e(f) + b_e \cdot l_e(f)$$

Convex program of WCG
Setting up the variables

For any player $j \in \mathcal{N}$ and $f_j \in S_j$ let $L_{j,f_j} = \sum_{e \in f_j} w_{ej} \cdot c_e(w_{ej})$ $L_{j,f_j} = \sum_{e \in f_j} w_{ej} \cdot c_e(w_{ej})$ i.e. the cost incurred by player j when it plays strategy f_j.

- $x_{j,f_j} :=$ Variable for player j playing strategy f_j for all $j \in \mathcal{N}$ and $f_j \in S_j$
- $y_e :=$ Variable for the load on resource e for all $e \in \mathcal{E}$

Convex program of WCG
Quadratic Program

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j\in\mathcal{N}} \sum_{f_j\in S_j} x_{j,f_j} \cdot L_{j,f_j} + \sum_{e\in\mathcal{E}} a_e \cdot y_e^2 \\
\text{subject to} \quad & \boxed{\sum_{f_j\in S_j} x_{j,f_j} \le 1 \quad \forall j \in \mathcal{N},} \\
& \boxed{\sum_{j\in\mathcal{N}} \sum_{f_j\in S_j} \sum_{e\in f_j} w_{ej} \cdot x_{j,f_j} \le y_e \quad \forall e \in \mathcal{E},} \\
& f_j \in S_j
\end{aligned}
$$

This constraint makes sure only one strategy is played by each player.

This constraint makes sure that the load on each resource is at least sum of the weights of the players using that resource.

Dual Program

We denote the dual variables by $\{\mu_j\}_{j\in\mathcal{N}}$, $\{\Phi_e\}_{e\in\mathcal{E}}$ and $\{\Psi_e\}_{e\in\mathcal{E}}$. Then we use the Fenchel Duality to obtain the dual of the convex program.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j\in\mathcal{N}} \mu_j - \sum_{e\in\mathcal{E}} \frac{1}{4a_e} \cdot \Phi_e^2 & \\
\text{subject to} \quad & \mu_j - \sum_{e\in f_j} w_{e,j} \cdot \Psi_e \leq L_{j,f_j} & \forall j \in \mathcal{N}, f_j \in S_j, \\
& \Psi_e \leq \Phi_e & \forall e \in \mathcal{E}, \\
& \mu_j \geq 0 & \forall j \in \mathcal{N}, \\
& \Phi_e \geq 0 & \forall e \in \mathcal{E}
\end{aligned}
$$

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j\in\mathcal{N}} \mu_j - \sum_{e\in\mathcal{E}} \frac{1}{4a_e} \cdot \Phi_e^2 & \\
\text{subject to} \quad & \mu_j - \sum_{e\in f_j} w_{e,j} \cdot {\color{red}\Phi_e} \leq L_{j,f_j} & \forall j \in \mathcal{N}, f_j \in S_j, \\
& \mu_j \geq 0 & \forall j \in \mathcal{N}
\end{aligned}
$$

$\left(1+\frac{1}{\delta}\right)$-Approximate Solution from Primal

Consider the following changed primal program:

$$\begin{aligned}
\text{minimize} \quad & \frac{1}{\delta} \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} x_{j,f_j} \cdot L_{j,f_j} + \sum_{e \in \mathcal{E}} a_e \cdot y_e^2 \\
\text{subject to} \quad & \sum_{f_j \in S_j} x_{j,f_j} \leq 1 \quad \forall j \in \mathcal{N}, \\
& \sum_{j \in \mathcal{N}} \sum_{f_j \in S_j} \sum_{e \in f_j} w_{ej} \cdot x_{j,f_j} \leq y_e \quad \forall e \in \mathcal{E}, \\
& x_{j,f_j} \geq 0 \quad \forall j \in \mathcal{N},\ f_j \in S_j
\end{aligned}$$

If $\delta = 1$ we get our original program. For any $\delta > 0$ we get a $\left(1+\frac{1}{\delta}\right)$-approximate solution.

Dual don't need to change

Taking the dual of the new program we get the following:

$$\begin{aligned}
\text{maximize} \quad & \sum_{j \in \mathcal{N}} \mu_j - \sum_{e \in \mathcal{E}} \frac{1}{4a_e} \cdot \Phi_e^2 & \\
\text{subject to} \quad & \mu_j - \sum_{e \in f_j} w_{e,j} \cdot \boldsymbol{\Phi_e} \leq \frac{\mathbf{L_{j,f_j}}}{\boldsymbol{\delta}} & \forall j \in \mathcal{N}, f_j \in S_j, \\
& \mu_j \geq 0 & \forall j \in \mathcal{N}, \\
& \Phi_e \geq 0 & \forall e \in \mathcal{E}
\end{aligned}$$

So instead if we work with the old dual program and scale our variables μ_j, Φ_e and Ψ_e by $\frac{1}{\delta}$ we still get a feasible solution to the new dual program.

Setting the Dual Variables

Let σ is any CCE of the game. Set

- $\mu_j = \frac{1}{\delta} \cdot \mathbb{E}_{f \sim \sigma}[\text{Cost}_j(f)]$
- $\Phi_e = \frac{1}{\delta} \cdot a_e \cdot \mathbb{E}_{f \sim \sigma}[l_e(f)]$

$$
\begin{aligned}
\text{Cost}_j(f_j, \theta_{-j}) &\leq \sum_{e \in f_j} w_{e,j} \cdot (a_e(l_e(\theta) + w_{e,j}) + b_e) \\
&= \sum_{e \in f_j} w_{e,j}(a_e \cdot w_{e,j} + b_e) + \sum_{e \in f_j} w_{e,j} \cdot a_e \cdot l_e(\theta) \\
&= L_{j,f_j} + \sum_{e \in f_j} w_{e,j} \cdot a_e \cdot l_e(\theta)
\end{aligned}
$$

Remark

It is a feasible solution to the dual program.

Bound on PoA : I

$$
\begin{aligned}
\sum_{e\in\mathcal{E}} \frac{1}{a_e}\cdot a_e^2 \cdot \mathop{\mathbb{E}}_{f\sim\sigma}[l_e(f)]^2 &= \sum_{e\in\mathcal{E}} a_e \cdot \mathop{\mathbb{E}}_{f\sim\sigma}[l_e(f)]^2 \\
&\leq \mathop{\mathbb{E}}_{f\sim\sigma}\left[\sum_{e\in\mathcal{N}} a_e \cdot l_e^2(f)\right] && \text{[Jensen]} \\
&\leq \mathop{\mathbb{E}}_{f\sim\sigma}\left[\sum_{e\in\mathcal{N}} \mathrm{Cost}_j(f)\right] = \sum_{j\in\mathcal{N}} \mathop{\mathbb{E}}_{f\sim\sigma}[\mathrm{Cost}_j(f)]
\end{aligned}
$$

Bound on PoA : II

$$\begin{aligned}
\text{Primal-Sol} &\geq \sum_{j\in\mathcal{N}} \frac{1}{\delta} \cdot \mathop{\mathbb{E}}_{f\sim\sigma}[\mathrm{Cost}_j(f)] - \sum_{e\in\mathcal{E}} \frac{1}{\delta^2} \cdot \frac{1}{4} a_e \cdot \mathop{\mathbb{E}}_{f\sim\sigma}[l_e(f)]^2 \\
&\geq \frac{1}{\delta} \sum_{j\in\mathcal{N}} \mathop{\mathbb{E}}_{f\sim\sigma}[\mathrm{Cost}_j(f)] - \frac{1}{4\cdot\delta^2} \cdot \sum_{e\in\mathcal{E}} \mathop{\mathbb{E}}_{f\sim\sigma}[\mathrm{Cost}_j(f)] \\
&= \frac{4\delta - 1}{4\delta^2} \sum_{e\in\mathcal{E}} \mathop{\mathbb{E}}_{f\sim\sigma}[\mathrm{Cost}_j(f)]
\end{aligned}$$

Primal is $(1+\frac{1}{\delta})$-approximate solution to the optimal solution. So we get a bound of $(1+\frac{1}{\delta})\frac{4\delta^2}{4\delta-1}$ bound on PoA. Take $\delta = \frac{1+\sqrt{5}}{4}$ you will get a bound of $1+\phi$ where ϕ is the golden ratio.

Simultaneous Second-Price Auctions

Definition

- $\mathcal{M}$: Set of m items
- $\mathcal{N}$: Set of n players
- For each player $j \in \mathcal{N}$, $v_j : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$ is the valuation function of player j of $T \subseteq \mathcal{M}$. v_j is submodular.
- Each player j submits a bid $b_j \in \mathbb{R}^m_{\geq 0}$ which follows $\sum\limits_{i \in T} b_{ij} \leq v_j(T)$ for all $T \subseteq \mathcal{M}$.
- Let $W_j(b)$ denote the set of items won by player $j \in \mathcal{N}$ when the bids are b.
- Let $p(i, b)$ is the second highest bid for item i when the bids are b.
- Let $u_j(b)$ be the utility of player j when the bids are b. Then $u_j(b) = v_j(W_j(b)) - \sum\limits_{i \in W_j(b)} p(i, b)$.
- Auctions of each item follows Second-Price auctions rule.

GOAL: Maximize the social welfare of the players $V(b) = \sum\limits_{j \in \mathcal{N}} v_j(W_j(b))$

Property of Biddings

Theorem

$\forall j \in \mathcal{N}, \forall T \subseteq \mathcal{M}, \forall b \in \mathbb{R}^{m \times n}_{\geq 0}, \exists b_j(T) \in \mathbb{R}^m_{\geq 0}$ *such that*

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\}$$

Let $T = \{1, \ldots, i\}$. Take $b^*_{ij} = v_j(1, 2, \ldots, i) - v_j(1, 2, \ldots, i-1)$. Take $b_j(T) = b^*_j$

Observe: $\sum_{i \in T'} b^*_{i,j} \leq v_j(T')$ for all $T' \subseteq T$ by submodularity and for $T = T'$ its equality.

Proof of Theorem

$$
\begin{aligned}
u_j(b_j(T), b_{-j}) &= v_j(T^*) - \sum_{i \in T*} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \\
&\geq v_j(T^*) - \sum_{i \in T*} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} + \left[\sum_{i \in T \setminus T*} b^*_{i,j} - \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\} \right] \\
&\geq v_j(T) - \sum_{i \in T} \max_{j' \in \mathcal{N} \setminus \{j\}} \{b_{ij'}\}
\end{aligned}
$$

LP Formulation

- $x_{j,T} :=$ Variable for player j winning item T.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{T \subseteq \mathcal{M}} \sum_{j \in \mathcal{N}} x_{j,T} \cdot v_j(T) \\
\text{subject to} \quad & \boxed{\sum_{j \in \mathcal{N}} \sum_{i \in T} x_{j,T} \leq 1 \quad \forall i \in \mathcal{M},} \\
& \boxed{\sum_{T \subseteq \mathcal{M}} x_{j,T} \leq 1 \quad \forall j \in \mathcal{N},} \\
& x_{j,T} \geq 0 \quad \forall j \in \mathcal{N}, T \subseteq \mathcal{M}
\end{aligned}
$$

This constraint makes sure each agent receives exactly one set from $2^{\mathcal{M}}$.

Dual Program

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in \mathcal{N}} y_j + \sum_{i \in \mathcal{M}} z_i \\
\text{subject to} \quad & y_j + \sum_{i \in T} z_i \geq v_j(T) \quad && \forall j \in \mathcal{N},\ T \subseteq \mathcal{M}, \\
& z_i \geq 0 && \forall i \in \mathcal{M}, \\
& y_j \geq 0 && \forall j \in \mathcal{N}
\end{aligned}
$$

Setting the Dual Variables

Given a CCE σ of the game, we set the dual variables as follows:

- $y_j = \mathop{\mathbb{E}}_{b\sim\sigma}[u_j(b)]$ for all $j \in \mathcal{N}$.
- $z_i = \mathop{\mathbb{E}}_{b\sim\sigma}\left[\max_{j\in\mathcal{N}} b_{ij}\right]$ for all $i \in \mathcal{M}$.

Since σ is an CCE

$$\mathop{\mathbb{E}}_{b\sim\sigma}[u_j(b)] \geq \mathop{\mathbb{E}}_{b\sim\sigma}\left[u_j(b_j(T), b_{-j})\right] \qquad \forall\, T \subseteq \mathcal{M}$$

By the theorem

$$u_j(b_j(T), b_{-j}) \geq v_j(T) - \sum_{i\in T} \max_{j'\in\mathcal{N}\setminus\{j\}}\{b_{ij'}\} \geq v_j(T) - \sum_{i\in T} \max_{j'\in\mathcal{N}}\{b_{ij'}\}$$

So $\mathop{\mathbb{E}}_{b\sim\sigma}[u_j(b)] \geq v_j(T) - \sum_{i\in T} \mathop{\mathbb{E}}_{b\sim\sigma}\left[\max_{j'\in\mathcal{N}}\{b_{ij'}\}\right]$. So it is feasible solution to the dual program.

Bound on PoA

$$
\begin{aligned}
\text{Primal-Sol} &\leq \sum_{j \in \mathcal{N}} \mathop{\mathbb{E}}_{b \sim \sigma} [u_j(b)] + \sum_{i \in \mathcal{M}} \mathop{\mathbb{E}}_{b \sim \sigma} \left[\max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\
&= \mathop{\mathbb{E}}_{b \sim \sigma} \left[\sum_{j \in \mathcal{N}} u_j(b) \right] + \mathop{\mathbb{E}}_{b \sim \sigma} \left[\sum_{i \in \mathcal{M}} \max_{j \in \mathcal{N}} \{b_{ij}\} \right] \\
&\leq 2 \cdot \mathop{\mathbb{E}}_{b \sim \sigma} [V(b)]
\end{aligned}
$$

So we get a bound of 2.

Facility Location Games

Definition

- $\mathcal{M}$: Set of m clients (Indexed by i)
- $\mathcal{N}$: Set of n service providers (Indexed by j)
- $\mathcal{L}$: Set of locations (Indexed by l)
- Each player $j \in \mathcal{N}$ has its strategy set of locations $S_j \subseteq \mathcal{L}$. $S = \bigtimes_{j \in \mathcal{N}} S_j$
- Each client $i \in \mathcal{M}$ has some value $\pi_j \geq 0$ for the service money he is wiling to pay.
- There is a cost $c(l, i)$ for serving the client $i \in \mathcal{M}$ from the location $l \in \mathcal{L}$

More Definitions

Each supplier chooses a single location $l \in S_j$ to set up a facility and offers prices to the clients.
Let $s \in S$ be any strategy profile.

- $\mathcal{K}(s)$: Set of locations chosen by the suppliers in s i.e. $\mathcal{K}(s) = \bigcup_{j \in \mathcal{N}} \{s_j\}$
- $p_s(i,j)$: Price charged from client i by supplier j in strategy profile s.
- $P_j(i,l,s_{-j})$: Profit of supplier j from client i when it is served from location l and the other suppliers are playing s_{-j}.
- $D_i(s)$: Savings of client i in strategy profile s which is $\pi_i - p_s(i, SP(i))$.
- Total utility of the supplier $j \in \mathcal{N}$ is $u_j(s) = \sum_{i:SP(i)=j} P_j(i, s_j, s_{-j})$
- $V(s)$: Social welfare of the strategy profile s, $W(s) = \sum_{j \in \mathcal{N}} u_j(s) + \sum_{i \in \mathcal{M}} D_i(s)$

Choosing Prices

Theorem

For any strategy profile s, for any client i and supplier j, $SP(i) = j$

(i) $c(s_j, i) = \min_{j' \in \mathcal{N}} c(s_{j'}, i)$

(ii) $p_s(i,j) = \max \left\{ c(s_j, i), \min_{l \in \mathcal{K}(s) \setminus \{s_j\}} c(l, i) \right\}$

Since prices charged by suppliers doesn't depend on which supplier charges we can as well take all the locations distinct.

$$P_j(i, l, s_{-j}) = \begin{cases} \min_{l' \in \mathcal{K}(s) \setminus \{s_j\}} c(l', i) - c(l, i) & \text{If } c(l, i) \leq c(l', i) \\ 0 & \text{Otherwise} \end{cases}$$

$$W(s) = \sum_{j \in \mathcal{N}} u_j(s) + \sum_{i \in \mathcal{M}} D_i(s) = \sum_{i \in \mathcal{M}} \pi_i - c(s_{SP(i)}, i)$$

LP Formulation

- x_{ijl} := Variable indicating if the supplier j serves the client i from location l.
- x_{jl} := Variable indicating if the supplier j opens a facility at location l.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j \in \mathcal{N}} \sum_{l \in S_j} \sum_{i \in \mathcal{M}} (\pi_i - c(l,i)) \cdot x_{ijl} \\
\text{subject to} \quad & \sum_{j \in \mathcal{N}} \sum_{l \in S_j} x_{ijl} \leq 1 \quad && \forall\, i \in \mathcal{M}, \\
& \sum_{j \in \mathcal{N}} x_{jl} \leq 1 && \forall\, l \in \mathcal{L}, \\
& \sum_{k \in S_j} x_{jl} \leq 1 && \forall\, j \in \mathcal{N}, \\
& x_{ijl} \leq x_{jl} && \forall\, i \in \mathcal{M},\, j \in \mathcal{N},\, i \in \mathcal{M},\, l \in S_j, \\
& x_{ijl} \geq 0 && \forall\, i \in \mathcal{M},\, j \in \mathcal{N},\, l \in S_j
\end{aligned}
$$

Dual Program

We denote the dual variables by $\{\alpha_j\}_{j\in\mathcal{N}}$, $\{\beta_i\}_{i\in\mathcal{M}}$, $\{\gamma_l\}_{l\in\mathcal{L}}$ and $\{z_{ijl}\}_{i\in\mathcal{M},j\in\mathcal{N},l\in S_j}$.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j\in\mathcal{N}} \alpha_j + \sum_{i\in\mathcal{M}} \beta_i + \sum_{l\in\mathcal{L}} \gamma_l \\
\text{subject to} \quad & \beta_i + z_{ijl} \geq \pi_i - c_{il} && \forall\, i \in \mathcal{M},\, j \in \mathcal{N},\, l \in S_j, \\
& \gamma_l + \alpha_j \geq \sum_{i\in\mathcal{M}} z_{ijl} && \forall\, j \in \mathcal{N},\, l \in S_j, \\
& \alpha_j \geq 0 && \forall\, j \in \mathcal{N}, \\
& \beta_i \geq 0 && \forall\, i \in \mathcal{M}
\end{aligned}
$$

Setting the Dual Variables

We set the dual variables as follows:

- $\alpha_j = \mathop{\mathbb{E}}_{s\sim\sigma}[u_j(s)]$ for all $j \in \mathcal{N}$.
- $\beta_i = \mathop{\mathbb{E}}_{s\sim\sigma}[D_i(s)]$ for all $i \in \mathcal{M}$.
- $z_{ijl} = \mathop{\mathbb{E}}_{s\sim\sigma}[P_j(i, l, s_{-j})]$ for all $i \in \mathcal{M}, j \in \mathcal{N}$ and $l \in S_j$.
- Define $W_l(s) = u_j(s)$ if $l \in \mathcal{K}(s)$ and $s_j = l$ for some $j \in \mathcal{N}$ and otherwise 0. Then $\gamma_l = \mathop{\mathbb{E}}_{s\sim\sigma}[W_l(s)]$ for all $l \in \mathcal{L}$.

Feasibility Checking

- $\pi_i - p_s(i, SP(i)) \geq \pi_i - c(l, i)$ for any $l \in \mathcal{L}$. Now $P_j(i, l, s_{-j}) \neq 0$ when $l = SP(i)$. Then clearly $\pi_i - p_s(i, SP(i)) + P_j(i, SP(i), s_{-j}) = \pi_i - c(SP(i), i)$ and for other locations $P_j(i, l, s_{-j}) = 0$. So the first constraint is satisfied
- If $l \in \mathcal{K}(s)$ then $W_l(s) = \sum_{i \in \mathcal{M}} P_j(i, l, \theta_{-j})$ for some $j \in \mathcal{N}$ such that $s_j = l$. So it satisfies the second constraint. If $l \notin \mathcal{K}(s)$. $u_j(s) \geq P_j(i, l, s_{-j})$ since σ is a CCE. So the second constraint is satisfied.

Bound on PoA

$\sum_{j \in \mathcal{N}} \alpha_j + \sum_{i \in \mathcal{M}} \beta_i$ is the expected social welfare under the distribution σ.

$\sum_{l \in \mathcal{L}} W_l(s)$ is at most the social welfare since σ is a CCE.

So by Weak Duality

$$\text{Primal-Sol} \leq \sum_{j \in \mathcal{N}} \alpha_j + \sum_{i \in \mathcal{M}} \beta_i + \sum_{l \in \mathcal{L}} \gamma_l \leq 2 \cdot \mathop{\mathbb{E}}_{s \sim \sigma}[V(s)]$$