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Course: Probability Theory

Assignment - 2
Dept: STCS
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Problem 1 [H] Problem 1.3: Ordering of three random variables

Suppose X, Y and U are mutually independent, such that X and Y are each exponentially distributed with some common parameter $\lambda > 0$ and U is uniformly distributed on the interval $[0, 1]$. Express $\mathbb{P}\{X < U < Y\}$ in terms of λ. Simplify your answer.

Solution: X and Y are exponentially distributed with some common parameter $\lambda > 0$. Hence $F_X(x) = 1 - e^{-\lambda x}$ and $F_Y(y) = 1 - e^{-\lambda y}$ for some $x, y \geq 0$.and U is uniform on $[0, 1]$. So

$$
\begin{aligned}
\mathbb{P}[X < U < Y] &= \int_0^1 \mathbb{P}[X < u, Y > u]du \\
&= \int_0^1 \mathbb{P}[X < u] \cdot \mathbb{P}[Y > u]du \\
&= \int_0^1 F_X(u)(1 - F_Y(u))du \\
&= \int_0^1 \left(1 - e^{-\lambda u}\right)\left(1 - \left(1 - e^{\lambda u}\right)\right) du \\
&= \int_0^1 \left(1 - e^{-\lambda u}\right) e^{-\lambda u} du \\
&= \int_0^1 \left[e^{-\lambda u} - e^{-2\lambda u}\right] du \\
&= \left[\frac{e^{-\lambda u}}{-\lambda} - \frac{e^{-2\lambda u}}{-2\lambda}\right]_0^1 \\
&= \left[\frac{e^{-2\lambda}}{2\lambda} - \frac{e^{-\lambda}}{\lambda}\right] - \left[\frac{1}{2\lambda} - \frac{1}{\lambda}\right] = \frac{\left(e^{-\lambda}\right)^2 - 2e^{-\lambda} + 1}{2\lambda} = \frac{\left(e^{-\lambda} - 1\right)^2}{2\lambda}
\end{aligned}
$$

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Problem 2 [H] Problem 1.5: Congestion at output ports

Consider a packet switch with some number of input ports and eight output ports. Suppose four packets simultaneously arrive on different input ports, and each is routed toward an output port. Assume the choices of output ports are mutually independent, and for each packet, each output port has equal probability.

(a) Specify a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to describe this situation.

(b) Let X_i denote the number of packets routed to output port i for $1 \leq i \leq 8$. Describe the joint pmf of $X_1, \ldots, X_8$.

(c) Find $\mathrm{Cov}(X_1, X_2)$

(d) Find $\mathbb{P}[X_i \leq 1 \text{ for all } i]$

(e) Find $\mathbb{P}[X_i \leq 2 \text{ for all } i]$

Solution:

(a) Since each packet can be routed toward any of the 8 output ports we keep all possible 4−tuples in the sample space i.e.

$$\Omega = \{(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \mid \sigma_i \in [8] \;\forall\, i \in [4]\}$$

Then we take $\mathcal{F}$ to be the power set of Ω, $\mathcal{P}(\Omega)$ and the probability measure $\mathbb{P}$ is uniform i.e. for any $k_1, \ldots, k_4 \in [8]$ we have

$$\mathbb{P}[(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (k_1, k_2, k_3, k_4)] = \frac{1}{8^4}$$

Hence we have $\Omega = \{1, \ldots, 8\}^4$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}[\omega] = \frac{1}{8^4}$ for each $\omega \in \Omega$.

(b) The random variables X_i for $i \in [8]$ satisfy the following property

$$X_1 + X_2 + \cdots + X_8 = 4$$

Hence

$$\mathbb{P}[X_1 = x_1, \ldots, X_8 = x_8] = \frac{1}{8^4} \prod_{i=1}^{8} \binom{4 - \sum_{j=1}^{i-1} x_j}{x_i}$$

(c) Now

$$\mathbb{P}[X_k = i] = \frac{\binom{4}{i} 7^{4-i}}{8^4} \text{ and } \mathbb{P}[X_k = i, X_l = j] = \frac{\binom{4}{i}\binom{4-i}{j} 6^{4-(i+j)}}{8^4}$$

Therefore we have $X_k \sim Bin\left(4, \frac{1}{8}\right)$ for all $k \in [8]$. Hence $\mathbb{E}[X_k] = \frac{4}{8} = \frac{1}{2}$ and $\mathrm{Var}[X_k] = 4\frac{1}{8}\frac{7}{8} = \frac{7}{16}$. Therefore $\mathbb{E}[X_k^2] = \mathrm{Var}[X_k] + \mathbb{E}[X_k]^2 = \frac{7}{16} + \frac{1}{4} = \frac{11}{16}$ And also we have

$$\mathbb{P}[X_l = j \mid X_k = i] = \frac{\binom{4-i}{j} 6^{4-(i+j)}}{7^4} = \binom{4-i}{j} \left(\frac{6}{7}\right)^{4-(i+j)} \frac{1}{7^j}$$

Therefore $(X_l \mid X_k = i) \sim Bin\left(4 - i, \frac{1}{7}\right)$. Hence $\mathbb{E}[X_l \mid X_k = i] = \frac{4-i}{7}$. Now we will calculate the covariance.

$$\begin{aligned}
\mathrm{Cov}(X_1, X_2) &= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] \\
&= \mathbb{E}\left[\mathbb{E}[X_1 X_2 \mid X_1]\right] - \frac{1}{4} \\
&= \mathbb{E}\left[X_1 \mathbb{E}[X_2 \mid X_1]\right] - \frac{1}{4} \\
&= \mathbb{E}\left[X_1 \frac{4 - X_1}{7}\right] - \frac{1}{4} \\
&= \mathbb{E}\left[\frac{4X_1 - X_1^2}{7}\right] - \frac{1}{4} \\
&= \frac{4}{7}\mathbb{E}[X_1] - \frac{1}{7}\mathbb{E}[X_1^2] - \frac{1}{4} \\
&= \frac{4}{7}\frac{1}{2} - \frac{1}{7}\frac{11}{16} - \frac{1}{4} = -\frac{1}{16}
\end{aligned}$$

[I discussed with Spandan]

(d) For all $i \in [8]$ we have

$$\begin{aligned}
\mathbb{P}[X_i \leq 1] &= \mathbb{P}[X_i = 0] + \mathbb{P}[X_i = 1] \\
&= \frac{\binom{4}{0} 7^{4-0}}{8^4} + \frac{\binom{4}{1} 7^{4-1}}{8^4} \\
&= \frac{7^4}{8^4} + \frac{4 \times 7^3}{8^4} = \frac{7^3 \times 11}{8^4}
\end{aligned}$$

(e) For all $i \in [8]$ we have

$$\begin{aligned}
\mathbb{P}[X_i \leq 2] &= 1 - \mathbb{P}[X_i = 3] - \mathbb{P}[X_i = 4] \\
&= 1 - \frac{\binom{4}{3} 7^{4-3}}{8^4} - \frac{\binom{4}{4} 7^{4-4}}{8^4} \\
&= 1 - \frac{4 \times 7}{8^4} - \frac{1}{8^4} = 1 - \frac{29}{8^4}
\end{aligned}$$

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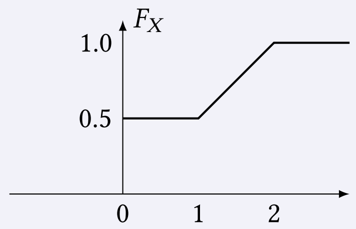
Problem 3 [H] Problem 1.13: A CDF of mixed type

Let X have the CDF shown.



(a) Find $\mathbb{P}[X \le 0.8]$

(b) Find $\mathbb{E}[X]$

(c) Find $\text{Var}[X]$

Solution:

(a) $\mathbb{P}[X \le 0.8] = \mathbb{F}_X[0.8] = 0.5$ since the value of F_X increases when $X \ge 1$.

(b) $F_X(x) = \begin{cases} 0 & \text{when } x \le 0 \\ 0.5 & \text{when} 0 \le x \le 1 \\ 0.5 + \frac{x-1}{2} & \text{when } 1 \le x \le 2 \\ 1 & \text{when } x \ge 2 \end{cases}$

Hence

$$\mathbb{E}[X] = \int_0^\infty [1 - F_X(x)]dx = \int_0^1 [1 - 0.5]dx + \int_1^2 \left[1 - 0.5 - \frac{x-1}{2}\right] dx + \int_2^\infty [1-1]dx$$
$$= \int_0^1 0.5dx + \int_1^2 \left[0.5 - \frac{x-1}{2}\right] dx$$
$$= 0.5 + \left[0.5x - \frac{(x-1)^2}{4}\right]_1^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

(c) Take $Y = X^2$ and the distribution function for Y is F_Y. Now for any $y \ge 0$

$$F_Y(y) = \mathbb{P}[Y^2 \le y] = \mathbb{P}[X^2 \le y] = \mathbb{P}[X \le \sqrt{y}] = F_X(\sqrt{y})$$

Therefore

$$F_Y(y) = \begin{cases} 0 & \text{when } x \le 0 \\ 0.5 & \text{when} 0 \le x \le 1 \\ 0.5 + \frac{\sqrt{y}-1}{2} & \text{when } 1 \le y \le 4 \\ 1 & \text{when } y \ge 4 \end{cases}$$

Hence

$$\mathbb{E}[X^2] = \int_0^\infty [1 - F_Y(y)]dy = \int_0^1 [1 - 0.5]dy + \int_1^4 \left[1 - 0.5 - \frac{\sqrt{y}-1}{2}\right] dy + \int_4^\infty [1-1]dy$$
$$= \int_0^1 0.5dy + \int_1^4 \left[0.5 - \frac{\sqrt{y}-1}{2}\right] dy$$
$$= 0.5 + \left[0.5y - \frac{\frac{2}{3}y^{\frac{3}{2}} - y}{2}\right]_1^4 = 0.5 + 3 \cdot 0.5 - \left[\frac{\frac{2}{3}4^{\frac{3}{2}} - 4}{2} - \frac{\frac{2}{3}1^{\frac{3}{2}} - 1}{2}\right] = 2 - \frac{5}{6} = \frac{7}{6}$$

So $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X] = \frac{7}{6} - \frac{9}{16} = \frac{56-27}{48} = \frac{29}{48}$.

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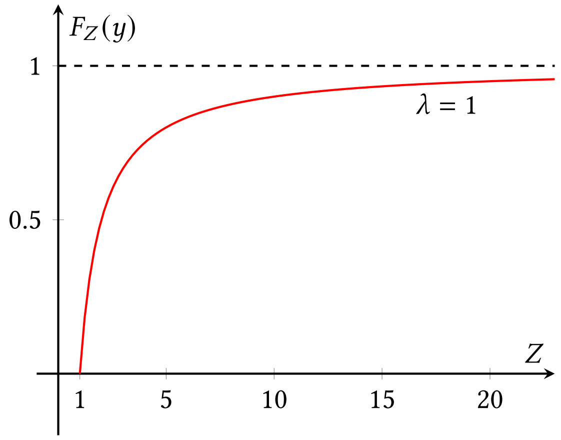
Problem 4 [H] Problem 1.17: Transformation of a random variable

Let X be exponentially distributed with mean λ^{-1}. Find and carefully sketch distribution functions for the random variables $Y = \exp(X)$ and $Z = \min(X, 3)$

Solution: X is exponentially distributed with mean λ^{-1}. So the density function of X for $x \geq 0$ is $f_X(x) = \lambda e^{-\lambda x}$. So for $y > 0$,

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[e^x \leq y] = \mathbb{P}[x \leq \ln y] = F_X[\ln y] = 1 - e^{-\lambda[\ln y]} = 1 - \left[e^{\ln y}\right]^{-\lambda} = 1 - y^{-\lambda}$$



And for $z \geq 0$ Z can be either equal to X or 3.

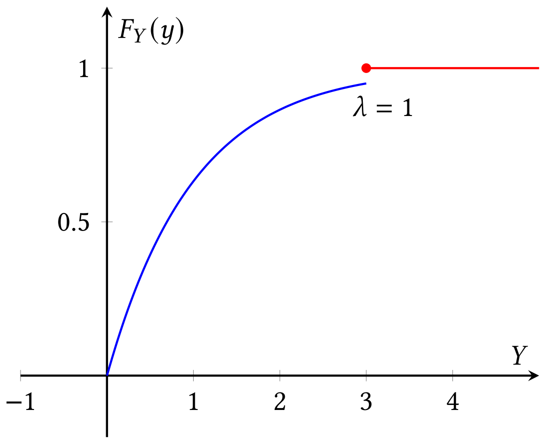
$$F_Z(z) = \mathbb{P}[Z \leq z] = \mathbb{P}[\min(X, 3) \leq z]$$

Now there will be two cases. $z < 3$ and $z \geq 3$. For $z < 3$

$$F_Z(z) = \mathbb{P}[Z \leq z] = \mathbb{P}[X \leq z] = F_X(z)$$

For $z \geq 3$

$$F_Z(z) = \mathbb{P}[Z \leq z] = \mathbb{P}[\{Z < 3\} \cup \{Z = 3\}] = \mathbb{P}[Z < 3] + \mathbb{P}[Z \geq 3] = \mathbb{P}[X < 3] + \mathbb{P}[X \geq 3] = F_X(3) + \left[1 - F_X(3)\right] = 1$$



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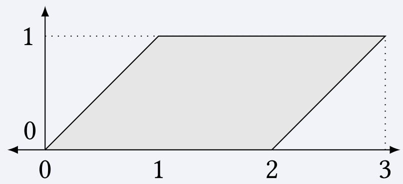
Problem 5 [H] Problem 1.27: Working with a two dimensional density

Let the random variables X and Y be jointly uniformly distributed over the region shown.



(a) Determine the value of $f_{X,Y}$ on the region shown

(b) Find f_X, the marginal pdf of X.

(c) Find the mean and variance of X.

(d) Find the conditional pdf of Y given that $X = x$, for $0 \le x \le 1$.

(e) Find the conditional pdf of Y given that $X = x$, for $1 \le x \le 2$.

(f) Find and sketch $\mathbb{E}[Y \mid X = x]$ as a function of x. Be sure to specify which range of x this conditional expectation is well defined for.

Solution:

(a) The parallelogram has base length 2 and height 1. Therefore area of the parallel is 2×1. Hence value of $f_{X,Y}(x, y) = \frac{1}{2}$ on the region shown.

(b) For $0 \le x \le 1$

$$f_X(x) = \int_0^x f_{X,Y}(x, y)dy = \int_0^x \frac{1}{2}dy = \frac{x}{2}$$

Now for $1 \le x \le 2$

$$f_X(x) = \int_0^1 f_{X,Y}(x, y)\ dy = \int_0^1 \frac{1}{2}\ dy = \frac{1}{2}$$

And $2 \le x \le 3$

$$f_X(x) = \int_{x-2}^1 f_{X,Y}(x, y)dy = \int_x^1 \frac{1}{2}dy = \frac{3-x}{2}$$

Therefore

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{when } 0 \le x \le 1 \\ \frac{1}{2} & \text{when } 1 \le x \le 2 \\ \frac{3-x}{2} & \text{when } 2 \le x \le 3 \\ 0 & \text{else} \end{cases}$$

(c)

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\ &= \int_0^1 \frac{x^2}{2}dx + \int_1^2 \frac{x}{2}dx + \int_2^3 \frac{x(3-x)}{2}dx \\ &= \left[\frac{x^3}{6}\right]_0^1 + \left[\frac{x^2}{4}\right]_1^2 + \left[\frac{3x^2}{4} - \frac{x^3}{6}\right]_2^3 \\ &= \frac{1}{6} + \frac{3}{4} + \frac{7}{12} = \frac{3}{2} \end{aligned}$$

And

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_0^1 x^2 f_X(x)dx \\
&= \int_0^1 \frac{x^3}{2}dx + \int_1^2 \frac{x^2}{2}dx + \int_2^3 \frac{x^2(1-x)}{2}dx \\
&= \left[\frac{x^4}{8}\right]_0^1 + \left[\frac{x^3}{6}\right]_1^2 + \left[\frac{x^3}{6} - \frac{x^4}{8}\right]_2^3 \\
&= \frac{1}{8} + \frac{7}{6} + \frac{11}{8} = \frac{8}{3}
\end{aligned}$$

Hence the variance will be

$$\mathrm{Var}[X] = \mathbb{E}[X]^2 - \mathbb{E}[X]^2 = \frac{8}{3} - \frac{9}{4} = \frac{5}{12}$$

(d) We have

$$f_{Y|X}(y \mid X = x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

For $0 \leq x \leq 1$ we have

$$f_{Y|X}(y \mid X = x) = \frac{\frac{1}{2}}{\frac{x}{2}} = \frac{1}{x}$$

(e) For $1 \leq x \leq 2$ we have

$$f_{Y|X}(y \mid X = x) = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

(f) For $2 \leq 3$ we have

$$f_{Y|X}(y \mid X = x) = \frac{\frac{3-x}{2}}{\frac{1}{2}} = \frac{1}{3-x}$$

Therefore we have

$$f_{Y|X=x}(y) = \begin{cases} \frac{1}{x} & \text{when } 0 < x \leq 1 \\ 1 & \text{when } 1 < x \leq 2 \\ \frac{1}{3-x} & \text{when } 2 < x \leq 3 \\ 0 & \text{else} \end{cases}$$

Now if $0 < x \leq 1$

$$\begin{aligned}
\mathbb{E}[Y \mid X = x] &= \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)dy \\
&= \int_0^x y\frac{1}{x}dy = \left[\frac{y^2}{2x}\right]_0^x = \frac{x^2}{2x} = \frac{x}{2}
\end{aligned}$$

If $1 < x \leq 2$ then

$$\begin{aligned}
\mathbb{E}[Y \mid X = x] &= \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)dy \\
&= \int_0^1 y \cdot 1dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}
\end{aligned}$$

If $2 < x \leq 3$

$$\begin{aligned}
\mathbb{E}[Y \mid X = x] &= \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)dy \\
&= \int_{x-2}^1 \frac{y}{3-x}dy \\
&= \left[\frac{y^2}{2(3-x)}\right]_{2-x}^1 = \frac{1}{2(3-x)} - \frac{(2-x)^2}{2(3-x)} = \frac{4x-3-x^2}{2(3-x)} = \frac{(x-3)(x-1)}{2(x-3)} = \frac{x-1}{2}
\end{aligned}$$

Hence we have

$$\mathbb{E}[Y \mid X = x] = \begin{cases} \frac{x}{2} & \text{when } 0 < x \leq 1 \\ \frac{1}{2} & \text{when } 1 < x \leq 2 \\ \frac{x-1}{2} & \text{when } 2 < x \leq 3 \\ 0 & \text{else} \end{cases}$$

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Problem 6

Let X, Y, Z be three jointly distributed random variables. Suppose

(i) X and Y are independent

(ii) X and Z are independent conditioned on Y

Prove or disprove the following claims:

(i) X and Y are independent conditioned on Z

(ii) X and Z are independent

Solution: Given that X, Y are independent and X, Z are independent conditioned on Y. Now

$$\begin{aligned} &\mathbb{P}[X \cap Z \mid Y] = \mathbb{P}[X \mid Y]\mathbb{P}[Z \mid Y] \\ \implies &\frac{\mathbb{P}[X \cap Z \cap Y]}{\mathbb{P}[Y]} = \frac{\mathbb{P}[X \cap Y]}{\mathbb{P}[Y]}\frac{\mathbb{P}[Z \cap Y]}{\mathbb{P}[Y]} \\ \implies &\mathbb{P}[X \cap Y \cap Z] = \frac{\mathbb{P}[X]\mathbb{P}[Y]}{\mathbb{P}[Y]}\mathbb{P}[Y \cap Z] \\ \implies &\mathbb{P}[X \cap Y \cap Z] = \mathbb{P}[X]\mathbb{P}[Y \cap Z] \end{aligned}$$

Now since X, Y are independent

$$\mathbb{P}[X \cap Y^c] = \mathbb{P}[X] - \mathbb{P}[X \cap Y] = \mathbb{P}[X] - \mathbb{P}[X]\mathbb{P}[Y] = \mathbb{P}[X](1 - \mathbb{P}[Y])$$

That means X, Y^c is also independent. Therefore using the same process as above we get

$$\begin{aligned} &\mathbb{P}[X \cap Z \mid Y^c] = \mathbb{P}[X \mid Y^c]\mathbb{P}[Z \mid Y^c] \\ \implies &\frac{\mathbb{P}[X \cap Z \cap Y^c]}{\mathbb{P}[Y^c]} = \frac{\mathbb{P}[X \cap Y^c]}{\mathbb{P}[Y^c]}\frac{\mathbb{P}[Z \cap Y^c]}{\mathbb{P}[Y^c]} \\ \implies &\mathbb{P}[X \cap Y^c \cap Z] = \frac{\mathbb{P}[X]\mathbb{P}[Y^c]}{\mathbb{P}[Y^c]}\mathbb{P}[Y^c \cap Z] \\ \implies &\mathbb{P}[X \cap Y^c \cap Z] = \mathbb{P}[X]\mathbb{P}[Y^c \cap Z] \end{aligned}$$

Therefore we get

$$\begin{aligned} \mathbb{P}[X \cap Z] &= \mathbb{P}[X \cap Y \cap Z] + \mathbb{P}[X \cap Y^c \cap Z] \\ &= \mathbb{P}[X]\mathbb{P}[Y \cap Z] + \mathbb{P}[X]\mathbb{P}[Y^c \cap Z] \\ &= \mathbb{P}[X](\mathbb{P}[Y \cap Z] + \mathbb{P}[Y^c \cap Z]) \\ &= \mathbb{P}[X]\mathbb{P}[Z] \end{aligned}$$

Hence we obtain X, Z are independent. Now since X, Z are independent we will derive that X and Y are independent conditioned on Z.

$$\begin{aligned}
&\mathbb{P}[X \cap Y \cap Z] = \mathbb{P}[X]\mathbb{P}[Y \cap Z] \\
\implies &\frac{\mathbb{P}[X \cap Y \cap Z]}{\mathbb{P}[Z]} = \frac{\mathbb{P}[X]\mathbb{P}[Y \cap Z]}{\mathbb{P}[Z]} \\
\implies &\mathbb{P}[X \cap Y \mid Z] = \frac{\mathbb{P}[X]\mathbb{P}[Z]}{\mathbb{P}[Z]}\mathbb{P}[Y \cap Z] \\
\implies &\mathbb{P}[X \cap Y \mid Z] = \frac{\mathbb{P}[X \cap Z]}{\mathbb{P}[Z]}\mathbb{P}[Y \cap Z] \\
\implies &\mathbb{P}[X \cap Y \mid Z] = \mathbb{P}[X \mid Z]\mathbb{P}[Y \mid Z]
\end{aligned}$$

Therefore X, Y are independent conditioned on Z.

Hence we have both results: X, Y are independent conditioned on Z and X, Z are independent. ■

[I discussed the solution with Spandan]

Problem 7

Let $X_1, X_2, \dots, X_n$ be independent, identically distributed random variables. Suppose their (common) marginal distribution function is F_X. Let $Y = \max(X_1, X_2, \dots, X_n)$ and $Z = \min(X_1, X_2, \dots, X_n)$. Find the distribution functions of Y and Z. Assume F_X has a density function f_X. Find the density functions of Y, Z and $Y - Z$.

Solution: $Y = \max(X_1, X_2 \dots, X_n)$. Hence for any y

$$Y \leq y \iff \max(X_1, \dots, X_n) \leq y \iff X_i \leq y \;\forall\, i \in [n]$$

Suppose F_Y is the distribution function of Y. Then

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[\max(X_1, \dots, X_n) \leq y] = \mathbb{P}[X_1 \leq y, X_2 \leq y, \dots, X_n \leq y] = \prod_{i=1}^{n} \mathbb{P}[X_i \leq y] = F_X^n(y)$$

Therefore $F_Y(y) = F_X^n(y)$. Hence density function of Y is $f_Y(y) = nF_X^{n-1}(y)f_X(y)$

Now $Z = \min(X_1, X_2, \dots, X_n)$. Hence for any z

$$Z > z \iff \min(X_1, \dots, X_n) > z \iff X_i > z \;\forall\, i \in [n]$$

Suppose F_Z be the distribution function of Z. Then

$$1 - F_Z(z) = \mathbb{P}[Z > z] = \mathbb{P}[\min(X_1, \dots, X_n) > z] = \mathbb{P}[X_1 > z, \dots, X_n > z] = \prod_{i=1}^{n} \mathbb{P}[X_i > z] = \left(1 - F_X(z)\right)^n$$

Hence $F_Z(z) = 1 - \left(1 - F_X(z)\right)^n$. Hence density function of Z is $f_Z(z) = n\left(1 - F_X(z)\right)^{n-1} f_X(z)$.

Now we will compute the density functions of $Y - Z$. First we will compute density function of Y, Z. Now

$$F_{Y,Z}(y, z) = \mathbb{P}[Y \leq y, Z \leq z] = \mathbb{P}[Y \leq y] - \mathbb{P}[Y \leq y, Z > z] = F_X(y)^n - \mathbb{P}[Y \leq y, Z > z]$$

Hence if $y < z$ then $\mathbb{P}[Y \leq y, Z > z] = 0$ otherwise when $y \geq z$ we have

$$\mathbb{P}[Y \leq y, Z > z] = \left(F_X(y) - F_X(z)\right)^n$$

Therefore we have

$$F_{Y,Z}(y, z) = \begin{cases} F_X^n(y) - \left(F_X(y) - F_X(z)\right)^n & \text{when } z \leq y \\ F_X^n(y) & \text{else} \end{cases}$$

Therefore we get the density function

$$f_{Y,Z}(y,z) = \frac{\partial^2}{\partial y \partial z} F_{Y,Z}(y,z) = \begin{cases} n(n-1)\left(F_X(y) - F_X(z)\right)^{n-2} f_X(y) f_X(z) & \text{when } z \le y \\ 0 & \text{else} \end{cases}$$

Now take the random variables $U = Y - Z$ and $V = Z$. Then we replace variables by $y = u + v$ and $z = v$ when $z \le y$ or $u \ge 0$. Hence

$$\mathcal{J}(u,v) = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Hence when $u \ge 0$

$$f_{U,V}(u,v) = f_{Y,Z}(y(u,v), z(u,v))\,|\mathcal{J}| = f_{Y,Z}(y(u,v), z(u,v)) = n(n-1)\left(F_X(u+v) - F_X(v)\right)^{n-2} f_X(u+v) f_X(v)$$

Therefore

$$f_U(u) = \begin{cases} \int_{-\infty}^{\infty} n(n-1)\left(F_X(u+v) - F_X(v)\right)^{n-2} f_X(u+v) f_X(v)\, dv & \text{when } u \ge 0 \\ 0 & \text{else} \end{cases}$$

■

[I discussed with Spandan]

Problem 8 [H] Problem 1.31: Transformation of densities

Let U and V have the joint pdf:

$$f_{UV}(u,v) = \begin{cases} c(u-v)^2 & 0 \le u, v \le 1 \\ 0 & \text{else} \end{cases}$$

for some constant c.

(a) Find the constant c

(b) Suppose $X = U^2$ and $Y = U^2V^2$. Describe the joint pdf $f_{X,Y}(x,y)$ of X and Y. Be sure to indicate where the joint pdf is zero.

Solution:

(a) We know $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f_{UV}(u,v)\, dudv = 1$. Therefore we have

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{UV}(u,v)\, dudv &= \int_0^1 \int_0^1 c(u-v)^2\, dudv \\ &= \int_0^1 c\left[\int_0^1 (x^2 - 2xy + y^2)\, dx\right] dy \\ &= \int_0^1 c\left[\frac{x^3}{3} - x^2 y + y^2 x\right]_0^1 \\ &= c\int_0^1 \left[\frac{1}{3} - y + y^2\right] dy \\ &= c\left[\frac{y}{3} - \frac{y^2}{2} + \frac{y^3}{3}\right]_0^1 = c\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{3}\right] = \frac{c}{6} \end{aligned}$$

Therefore we have $\frac{c}{6} = 1 \iff c = 6$.

(b) $X = U^2 \implies U = \sqrt{X}$. Now $Y = U^2V^2 \implies \sqrt{Y} = UV \implies V = \sqrt{\frac{Y}{X}}$. Since $0 \le u, v \le 1$ take $0 \le y \le x \le 1$. So take $u = \sqrt{x}$ and $v = \sqrt{\frac{y}{x}}$. Therefore for $0 < y \le x \le 1$

$$\mathcal{J}(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{x}} & -\frac{1}{2}x^{-\frac{3}{2}}\sqrt{y} \\ 0 & \frac{1}{2\sqrt{xy}} \end{vmatrix} = \frac{1}{4x\sqrt{y}}$$

Hence we have

$$f_{X,Y}(x,y) = f_{U,V}(u(x,y), v(x,y))\,|\mathcal{J}| = f_{U,V}\left(\sqrt{x}, \sqrt{\frac{y}{x}}\right)\frac{1}{4x\sqrt{y}}$$

Therefore for $0 < y \le x \le 1$ we have

$$f_{X,Y}(x,y) = \left(\sqrt{x} - \sqrt{\frac{y}{x}}\right)^2 \frac{1}{4x\sqrt{y}} = \left(x - 2\sqrt{y} + \frac{y}{x}\right)\frac{1}{4x\sqrt{y}} = \frac{1}{4\sqrt{y}} - \frac{1}{2x} + \frac{\sqrt{y}}{4x^2}$$

Hence

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4\sqrt{y}} - \frac{1}{2x} + \frac{\sqrt{y}}{4x^2} & 0 < y \le x \le 1 \\ 0 & \text{else} \end{cases}$$

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Problem 9 [H] Problem 1.33: Transformation of joint densities

Assume X and Y are independent, each with the exponential pdf with parameter $\lambda > 0$. Let $W = X - Y$ and $Z = X^2 + X - Y$. Find the joint pdf of (W, Z). Be sure to specify its support (i.e. where it is not zero).

Solution: $W = X - Y$. $Z = X^2 + X - Y = X^2 + W \iff X^2 = Z - W \implies X = \sqrt{Z - W}$. Therefore $Y = X - W = \sqrt{Z - W} - W$. So we take $x = \sqrt{z - w}$ and $y = \sqrt{z - w} - w$. Then for $z \ge w$ we have

$$\mathcal{J}(w,z) = \begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2\sqrt{z-w}} & -\frac{1}{2\sqrt{z-w}} - 1 \\ \frac{1}{2\sqrt{z-w}} & \frac{1}{2\sqrt{z-w}} \end{vmatrix} = -\frac{1}{2\sqrt{z-w}}\frac{1}{2\sqrt{z-w}} + \left[\frac{1}{2\sqrt{z-w}} + 1\right]\frac{1}{2\sqrt{z-w}} = \frac{1}{2\sqrt{z-w}}$$

Hence

$$fW,Z(w,z) = f_{X,Y}(x(w,z), y(w,z))\,|\mathcal{J}| = f_X(x(w,z))\,f_Y(y(w,z))\,|\mathcal{J}|$$

Now X, Y are independent and both of them are exponential random variables with parameter $\lambda > 0$. Hence $f_X(x) = \lambda e^{-\lambda x}$ and $f_Y(y) = \lambda e^{-\lambda y}$. Therefore

$$f_X(x(w,z)) = \lambda e^{-\lambda\sqrt{z-w}} \qquad f_Y(y(z,w)) = \lambda e^{-\lambda(\sqrt{z-w}-w)}$$

Therefore

$$f_{W,Z}(w,z) = \lambda e^{-\lambda\sqrt{z-w}}\,\lambda e^{-\lambda(\sqrt{z-w}-w)}\,\frac{1}{2\sqrt{z-w}} = \frac{\lambda^2 e^{-\lambda(2\sqrt{z-w}-w)}}{2\sqrt{z-w}}$$

when $z \ge w$. Hence we have

$$f_{W,Z}(w,z) = \begin{cases} \frac{\lambda^2 e^{-\lambda(2\sqrt{z-w}-w)}}{2\sqrt{z-w}} & z \ge w \\ 0 & \text{else} \end{cases}$$

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Problem 10 [H] Problem 1.35: Conditional densities and expectations

Suppose the random variables X and Y have the joint pdf:

$$f_{XY}(u,v) = \begin{cases} 4u^2 & 0 < v < u < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find $\mathbb{E}[XY]$

(b) Find $f_Y(v)$. Be sure to specify it for all values of v.

(c) Find $f_{X|Y}(u \mid v)$. Be sure to specify where it is undefined and where it is zero.

(d) Find $\mathbb{E}[X^2 \mid Y = v]$ for $0 < v < 1$.

Solution:

(a)

$$\begin{aligned}
\mathbb{E}[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uv\, f_{X,Y}(u,v)\ dudv \\
&= \int_0^1 \int_0^u uv \cdot 4u^2\ dudv \\
&= \int_0^1 \int_0^u 4u^3 v\ dudv \\
&= \int_0^1 \left[2u^3v^2\right]_0^u\ dv \\
&= \int_0^1 2u^5\ du \\
&= \left[\frac{2u^6}{6}\right]_0^1 = \frac{1}{3}
\end{aligned}$$

(b) Now for $y > 0$ we have

$$f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\ du = \int_v^1 4u^2\ du = \left[\frac{4u^3}{3}\right]_v^1 = \frac{4(1-v^3)}{3}$$

Hence we have

$$f_Y(v) = \begin{cases} \frac{4(1-v^3)}{3} & \text{when } v > 0 \\ 0 & \text{else} \end{cases}$$

(c) When $0 < v < u < 1$ we have

$$f_{X|Y}(u \mid v) = \frac{f_{X,Y}(u,v)}{f_Y(v)} = \frac{4u^2}{\frac{4(1-v^3)}{3}} = \frac{3u^2}{1-v^3}$$

Therefore

$$f_{X|Y}(u \mid v) = \begin{cases} \frac{3u^2}{1-v^3} & \text{when } 0 < v < u < 1 \\ 0 & \text{else} \end{cases}$$

(d)

$$
\begin{aligned}
\mathbb{E}[X^2 \mid Y = v] &= \int_{-\infty}^{\infty} u^2 f_{X|Y}(u \mid v)\, du \\
&= \int_{v}^{1} \frac{3u^4}{1 - v^3} du \\
&= \left[\frac{3u^5}{5(1 - v^3)} \right]_v^1 \\
&= \frac{3(1 - v^5)}{5(1 - v^3)}
\end{aligned}
$$

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