

**Problem 1**

Let  $\mathcal{X}$  be a finite set and  $p_X$  be a probability distribution or a probability mass function (PMF) on  $\mathcal{X}$ . The Shannon entropy of  $p_X$  is defined as

$$H(p_X) \triangleq - \sum_{x \in \mathcal{X}} p_X(x) \log p_X(x)$$

1. Prove  $\log x \leq x - 1$  and  $\log \frac{1}{x} \geq 1 - x$  for all  $x > 0$ .
2.  $\sum_{x \in \mathcal{X}} p_X(x) \log \frac{1}{p_X(x)} \leq \log |\mathcal{X}|$
3.  $H(X) + H(Y) \geq H(X, Y)$  where  $H(X, Y) = H(p_{X,Y})$  is the entropy of a joint PMF,  $H(X) = H(p_X)$  where  $p_X$  is marginal of  $p_{X,Y}$

**Solution:**

1. We have  $\log x = \int_1^x \frac{1}{t} dt$  and  $x - 1 = \int_1^x dt$ . Now for  $x \geq 1$  for all  $t \geq 1$  we have  $1 \geq \frac{1}{t}$ . Hence

$$\int_1^x \frac{1}{t} dt \leq \int_1^x dt \iff \log x \leq x - 1$$

For  $0 < x < 1$  we have  $t < 1$  hence  $\frac{1}{t} \geq 1$ . Hence

$$\int_x^1 \frac{1}{t} dt \geq \int_x^1 dt \iff -\log x \geq 1 - x \iff x - 1 \geq \log x$$

Therefore  $\forall x > 0$  we have  $\log x \leq x - 1$ .

Now we have  $\log x \leq x - 1 \iff 1 - x \leq -\log x \iff 1 - x \leq \log \frac{1}{x}$ .

2.

$$\begin{aligned} \sum_{x \in \mathcal{X}} p_X(x) \log \frac{1}{p_X(x)} - \log |\mathcal{X}| &= \sum_{x \in \mathcal{X}} p_X(x) \log \frac{1}{p_X(x)} - \sum_{x \in \mathcal{X}} p_X(x) \log |\mathcal{X}| \\ &= \sum_{x \in \mathcal{X}} p_X(x) \log \frac{1}{|\mathcal{X}| p_X(x)} \\ &\leq \sum_{x \in \mathcal{X}} p_X(x) \left[ \frac{1}{|\mathcal{X}| p_X(x)} - 1 \right] && \text{[Using Part (1)]} \\ &= \sum_{x \in \mathcal{X}} \left[ \frac{1}{|\mathcal{X}|} - p_X(x) \right] = 1 - 1 = 0 \end{aligned}$$

Hence we get

$$\sum_{x \in \mathcal{X}} p_X(x) \log \frac{1}{p_X(x)} - \log |\mathcal{X}| \iff \sum_{x \in \mathcal{X}} p_X(x) \log \frac{1}{p_X(x)} \leq \log |\mathcal{X}|$$

3.

$$\begin{aligned}
H(X) + H(Y) - H(X, Y) &= - \sum_{x \in \mathcal{X}} p_X(x) \log p_X(x) - \sum_{y \in \text{mc}Y} p_Y(y) \log p_Y(y) \\
&\quad + \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p_{XY}(x, y) \log p_{XY}(x, y) \\
&= - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{XY}(x, y) \log p_X(x) - \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} p_{XY}(x, y) \log p_Y(y) \\
&\quad + \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p_{XY}(x, y) \log p_{XY}(x, y) \\
&= - \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p_{XY}(x, y) \log \frac{p_X(x) p_Y(y)}{p_{XY}(x, y)} \\
&= \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p_{XY}(x, y) \log \frac{p_{XY}(x, y)}{p_X(x) p_Y(y)} \\
&\geq \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p_{XY}(x, y) \left[ 1 - \frac{p_X(x) p_Y(y)}{p_{XY}(x, y)} \right] \quad [\text{Using Part (1)}] \\
&= \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p_{XY}(x, y) - \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p_{XY}(x, y) \frac{p_X(x) p_Y(y)}{p_{XY}(x, y)} \\
&= 1 - \sum_{x \in \mathcal{X}} p_X(x) \left[ \sum_{y \in \mathcal{Y}} p_Y(y) \right] \\
&= 1 - \sum_{x \in \mathcal{X}} p_X(x) = 1 - 1 = 0
\end{aligned}$$

Hence we got  $H(X) + H(Y) \geq H(X, Y)$ .

□

### Problem 2

Let  $p_X(x)$  be a PMF on  $\mathcal{X}$ . For  $n \in \mathbb{N}$ ,  $\delta > 0$ , let

$$T_\delta^n(p_X) \triangleq \left\{ x^n \in \mathcal{X}^n \mid \left| \frac{N(a|x^n)}{n} - p_X(a) \right| \leq \frac{\delta p_X(a)}{\log |\mathcal{X}|} \forall a \in \mathcal{X} \right\}$$

where  $N(a|x^n) = \sum_{i=1}^n \mathbb{1}_{\{x_i=a\}}$  denotes the number of occurrences of  $a$  in the sequences  $x_1 x_2 \cdots x_n$ .

1. Prove that

$$\sum_{x^n \notin T_\delta^n(p_X)} \prod_{i=1}^n p_X(x_i) \leq \exp \left[ - \frac{2n\delta^2 \eta_{p_X}^2}{(\log |\mathcal{X}|)^2} \right]$$

where  $\eta_{p_X} = \min_{a \in \mathcal{X}} \{p_X(a) \mid 0 < p_X(a) < 1\}$

2. Prove that

$$\left[ 1 - \exp \left( - \frac{2n\delta^2 \eta_{p_X}^2}{(\log |\mathcal{X}|)^2} \right) \right] \exp [n(H(p_X) - \delta)] \leq |T_\delta^n(p_X)| \leq \exp [n(H(p_X) + \delta)]$$

3. Prove that

$$x^n \in T_\delta^n(p_X) \implies \exp[-n(H(p_X) + \delta)] \leq \prod_{i=1}^n p_X(x_i) \leq \exp[-n(H(p_X) - \delta)]$$

**Solution:**

□

**Definitions:** Let  $p_{X,Y}$  be a joint PMF on  $\mathcal{X} \times \mathcal{Y}$  where  $\mathcal{X}, \mathcal{Y}$  are finite sets. (Essentially  $p_{X,Y}(x,y) \geq 0$  and  $\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{X,Y}(x,y) = 1$ ). We define the marginal of  $p_{X,Y}$  on  $X$  as  $p_X(x) \triangleq \sum_{y \in \mathcal{Y}} p_{X,Y}(x,y)$  for  $x \in \mathcal{X}$  and marginal of  $p_{X,Y}$  on  $Y$  as  $p_Y(y) \triangleq \sum_{x \in \mathcal{X}} p_{X,Y}(x,y)$  for  $y \in \mathcal{Y}$ . or a pair  $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$  of sequences we define

$N(a,b | x^n, y^n) = \sum_{i=1}^n \mathbb{1}_{\{(x_i, y_i) = (a,b)\}}$  as the number of occurrences of  $(a,b)$  in  $(x^n, y^n)$ .

Next the joint typical set wrt  $p_{X,Y}$  is defined as

$$T_\delta^n(p_{X,Y}) \triangleq \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n \mid \left| \frac{N(a,b | x^n, y^n)}{n} - p_{X,Y}(a,b) \right| \leq \frac{\delta p_{X,Y}(a,b)}{\log |\mathcal{X}| |\mathcal{Y}|} \forall (a,b) \in \mathcal{X} \times \mathcal{Y} \right\}$$

### Problem 3

1. Prove that if  $p_{X,Y}(a,b) = 0$  for some  $(a,b) \in \mathcal{X} \times \mathcal{Y}$  and  $(x^n, y^n) \in T_\delta^n(p_{X,Y})$  then  $N(a,b | p_{X,Y}) = 0$ . In other words, a pair that - probability does not occur in any typical pair of sequences.

2. Let  $\eta_{p_{X,Y}} = \min_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \{p_{X,Y}(x,y) \mid 0 < p_{X,Y}(x,y) < 1\}$ . Use the Hoeffding Inequality to prove that

$$\sum_{(x^n, y^n) \notin T_\delta^n(p_{X,Y})} p_{X,Y}^n(x^n, y^n) \leq 2|\mathcal{X}||\mathcal{Y}| \exp \left[ -\frac{2n\delta^2 \eta_{p_{X,Y}}^2}{(\log |\mathcal{X}| |\mathcal{Y}|)^2} \right]$$

**Hoeffding Inequality:** Let  $Z_1, \dots, Z_m$  are independent and identically distributed random variables for which  $P[a \leq Z_i \leq b] = 1$  for ever  $1 \leq i \leq m$  and  $\mu = \mathbb{E}[Z_i]$ . Then for every  $\epsilon > 0$

$$P \left[ \left| \frac{1}{m} \sum_{i=1}^m Z_i - \mu \right| > \epsilon \right] \leq 2 \exp \left[ -2m \frac{\epsilon^2}{(b-a)^2} \right]$$

3. For any  $(x^n, y^n) \in T_\delta^n(p_{X,Y})$  prove that

$$2^{-n[H(p_{X,Y})+\delta]} \leq p_{X,Y}^n(x^n, y^n) = \prod_{i=1}^n p_{X,Y}(x_i, y_i) \leq 2^{-n[H(p_{X,Y})-\delta]}$$

4. Prove that

$$(1 - \tilde{\delta}) 2^{n[H(p_{X,Y})-\delta]} \leq |T_\delta^n(p_{X,Y})| \leq 2^{n[H(p_{X,Y})+\delta]}$$

$$\text{where } \tilde{\delta} = 2|\mathcal{X}||\mathcal{Y}| \exp \left[ -\frac{2n\delta^2 \eta_{p_{X,Y}}^2}{(\log |\mathcal{X}| |\mathcal{Y}|)^2} \right]$$

5. Prove that  $(x^n, y^n) \in T_\delta^n(p_{X,Y})$  then  $x^n \in T_\delta^n(p_X)$  and  $y^n \in T_\delta^n(p_Y)$ .

**Solution:**

□

**Definitions:** Suppose  $p_{X,Y}$  is a probability distribution (probability mass function (PMF)) on  $\mathcal{X} \times \mathcal{Y}$ . We recall the condition distribution  $p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$  and for a pair  $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$  of sequence  $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$  of sequences  $p_{Y|X}^n(y^n|x^n) = \prod_{i=1}^n p_{Y|X}(y_i|x_i)$

We define

$$H(Y|X = x) \triangleq H(p_{Y|X}|X = x) = - \sum_{y \in \mathcal{Y}} p_{Y|X}(y|x) \log p_{Y|X}(y|x)$$

and

$$H(Y|X) = H(p_{Y|X}|p_X) \triangleq \sum_{x \in \mathcal{X}} p_X(x) H(p_{Y|X=x})$$

For any  $x^n \in \mathcal{X}^n$  define the conditional typical set of  $x^n$  as

$$T_\delta^n(p_{Y|X}) = \{y^n \in \mathcal{Y}^n \mid (x^n, y^n) \in T_\delta^n(p_{XY})\}$$

#### Problem 4

1. Prove that  $\sum_{y \in \mathcal{Y}} p_{Y|X}(y|x) = 1$
2. Prove that  $H(Y|X) = H(X, Y) - H(X)$  and  $H(Y|X) \geq 0$
3. Prove that Verify that if  $x^n \notin T_\delta^n(p_X)$  then  $T_\delta^n(p_{XY}|x^n) = \emptyset$
4. Suppose  $x^n \in T_\delta^n(p_X)$  and  $y^n \in T_\delta^n(p_{XY}|x^n)$  prove that

$$2^{-n[H(Y|X)+2\delta]} \leq p_{Y|X}^n(y^n|x^n) \leq 2^{-n[H(Y|X)-2\delta]}$$

5. Prove that if  $x^n \in T_\delta^n(p_X)$  then

$$\sum_{y^n \in T_{2\delta}^n(p_{XY}|x^n)} p_{Y|X}^n(y^n|x^n) \geq 1 - 2|\mathcal{X}||\mathcal{Y}| \exp \left[ -\frac{2n\delta^2}{(\log |\mathcal{X}||\mathcal{Y}|)^2} \eta_{p_{Y|X}} \right]$$

where  $\eta_{p_{Y|X}} = \min_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \{p_{Y|X}(y|x) \mid 0 < p_{Y|X}(y|x) < 1\}$

6. Suppose  $x^n \in T_\delta^n(p_X)$  then

$$(1 - \tilde{\delta})2^{n[H(Y|X)-4\delta]} \leq |T_\delta^n(p_{XY}|x^n)| \leq 2^{n[H(Y|X)+4\delta]}$$

where  $\tilde{\delta} = 2|\mathcal{X}||\mathcal{Y}| \exp \left[ -\frac{2n\delta^2}{(\log |\mathcal{X}||\mathcal{Y}|)^2} \eta_{p_{Y|X}} \right]$

**Solution:**

□