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Course: Quantum Information Theory

Assignment - 1
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Problem 1

Let $\mathcal{X}$ be a finite set and p_X be a probability distribution or a probability mass function (PMF) on $\mathcal{X}$. The Shannon entropy of p_X is defined as

$$H(p_X) \triangleq -\sum_{x\in\mathcal{X}} p_X(x)\log p_X(x)$$

.

1. Prove $\log x \leq x-1$ and $\log\frac{1}{x} \geq 1-x$ for all $x>0$.
2. $\sum\limits_{x\in\mathcal{X}} p_X(x)\log\frac{1}{p_X(x)} \leq \log|\mathcal{X}|$
3. $H(X)+H(Y)\geq H(X,Y)$ where $H(X,Y)=H(p_{X,Y})$ is the entropy of a joint PMF, $H(X)=H(p_X)$ where p_X is marginal of $p_{X,Y}$

Solution:

1. We have $\log x = \displaystyle\int_1^x \frac{1}{t}\,dt$ and $x-1=\displaystyle\int_1^x dt$. Now for $x\geq 1$ for all $t\geq 1$ we have $1\geq\frac{1}{t}$. Hence

$$\int_1^x \frac{1}{t}\,dt \leq \int_1^x dt \iff \log x\leq x-1$$

For $0<x<1$ we have $t<1$ hence $\frac{1}{t}\geq 1$. Hence

$$\int_x^1 \frac{1}{t}\,dt \geq \int_x^1 dt \iff -\log x\geq 1-x \iff x-1\geq \log x$$

Therefore $\forall\, x>0$ we have $\log x\leq x-1$.

Now we have $\log x\leq x-1 \iff 1-x\leq -\log x \iff 1-x\leq \log\frac{1}{x}$.

2.

$$\begin{aligned}
\sum_{x\in\mathcal{X}} p_X(x)\log\frac{1}{p_X(x)}-\log|\mathcal{X}| &= \sum_{x\in X} p_X(x)\log\frac{1}{p_X(x)} - \sum_{x\in\mathcal{X}} p_X(x)\log|\mathcal{X}| \\
&= \sum_{x\in\mathcal{X}} p_X(x)\log\frac{1}{|\mathcal{X}|p_X(x)} \\
&\leq \sum_{x\in\mathcal{X}} p_X(x)\left[\frac{1}{|\mathcal{X}|p_X(x)}-1\right] && \text{[Using Part (1)]} \\
&= \sum_{x\in\mathcal{X}}\left[\frac{1}{|\mathcal{X}|}-p_X(x)\right] = 1-1=0
\end{aligned}$$

Hence we get

$$\sum_{x\in\mathcal{X}} p_X(x)\log\frac{1}{p_X(x)}-\log|\mathcal{X}| \iff \sum_{x\in\mathcal{X}} p_X(x)\log\frac{1}{p_X(x)}\leq\log|\mathcal{X}|$$

3.

$$
\begin{aligned}
H(X)+H(Y)-H(X,Y) &= -\sum_{x\in\mathcal{X}} p_X(x)\log p_X(x) - \sum_{y\in\mathcal{Y}} p_Y(y)\log p_Y(y) \\
&\qquad + \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(x,y)\log p_{XY}(x,y) \\
&= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{XY}(x,y)\log p_X(x) - \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p_{XY}(x,y)\log p_Y(y) \\
&\qquad + \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(x,y)\log p_{XY}(x,y) \\
&= -\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(x,y)\log \frac{p_X(x)p_Y(y)}{p_{XY}(x,y)} \\
&= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(x,y)\log \frac{p_{XY}(x,y)}{p_X(x)p_Y(y)} \\
&\geq \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(x,y)\left[1-\frac{p_X(x)p_Y(y)}{p_{XY}(x,y)}\right] && \text{[Using Part (1)]} \\
&= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(x,y) - \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(x,y)\frac{p_X(x)p_Y(y)}{p_{XY}(x,y)} \\
&= 1-\sum_{x\in\mathcal{X}} p_X(x)\left[\sum_{y\in\mathcal{Y}} p_Y(y)\right] \\
&= 1-\sum_{x\in\mathcal{X}} p_X(x) = 1-1 = 0
\end{aligned}
$$

Hence we got $H(X)+H(Y)\geq H(X,Y)$.

□

Problem 2

Let $p_X(x)$ be a PMF on $\mathcal{X}$. For $n \in \mathbb{N}$, $\delta > 0$, let

$$T_\delta^n(p_X) \triangleq \left\{ x^n \in \mathcal{X}^n \mid \left| \frac{N(a|x^n)}{n} - p_X(a) \right| \leq \frac{\delta p_X(a)}{\log|\mathcal{X}|} \; \forall\, a \in \mathcal{X} \right\}$$

where $N(a|x^n) = \sum_{i=1}^{n} \mathbb{1}_{\{x_i = a\}}$ denotes the number of occurrences of a in the sequences $x_1 x_2 \cdots x_n$.

1. Prove that

$$\sum_{x^n \notin T_\delta^n(p_X)} \prod_{i=1}^{n} p_X(x_i) \leq \exp\left[-\frac{2n\delta^2 \eta_{p_X}^2}{(\log|\mathcal{X}|)^2} \right]$$

   where $\eta_{p_X} = \min_{a \in \mathcal{X}} \{ p_X(a) \mid 0 < p_X(a) < 1 \}$

2. Prove that

$$\left[1 - \exp\left(\frac{2n\delta^2 \eta_{p_X}^2}{(\log|\mathcal{X}|)^2} \right) \right] \exp\left[n(H(p_X) - \delta) \right] \leq |T_\delta^n(p_X)| \leq \exp\left[n(H(p_X) + \delta) \right]$$

3. Prove that

$$x^n \in T_\delta^n(p_X) \implies \exp[-n(H(p_X) + \delta)] \leq \prod_{i=1}^{n} p_X(x_i) \leq \exp[-n(H(p_X) - \delta)]$$

Solution:

1. $\sum_{x^n \notin T_\delta^n(p_X)} \prod_{i=1}^{n} p_X(x_i) = \sum_{x^n \notin T_\delta^n(p_X)} p_X^n(x^n) = Pr[x^n \notin T_\delta^n(p_X)]$. If $x^n \notin T_\delta^n(p_X)$ then there exists $a \in \mathcal{X}$ such that $\left| \frac{N(a|x^n)}{n} - p_X(a) \right| > \frac{\delta p_X(a)}{\log|\mathcal{X}|}$. Now $N(a|x^n) = \sum_{i=1}^{n} \mathbb{1}_{x_i = a}$. Hence take the indicator random variables $\mathbb{1}_{x_i = a}$ for a, $i \in [n]$ then $\mathbb{E}\left[\mathbb{1}_{x_i = a} \right] = p_X(a)$. Then by Hoeffding Inequality we get

$$Pr\left[\left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{x_i = 1} - p_X(a) \right| > \frac{\delta p_X(a)}{\log|\mathcal{X}|} \right] \leq 2\exp\left[-2n \left(\frac{\delta p_X(a)}{\log|\mathcal{X}|} \right)^2 \right] \leq 2\exp\left[-2n \left(\frac{\delta \eta_{p_X}}{\log|\mathcal{X}|} \right)^2 \right]$$

   So

$$\begin{aligned} Pr[x^n \notin T_\delta^n(p_X)] &\leq \sum_{a \in \mathcal{X}} Pr\left[\left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{x_i = 1} - p_X(a) \right| > \frac{\delta p_X(a)}{\log|\mathcal{X}|} \right] \\ &\leq \sum_{a \in \mathcal{X}} 2\exp\left[-2n \left(\frac{\delta p_X(a)}{\log|\mathcal{X}|} \right)^2 \right] \leq 2\exp\left[-2n \left(\frac{\delta \eta_{p_X}}{\log|\mathcal{X}|} \right)^2 \right] \\ &= 2|\mathcal{X}| \exp\left[-\frac{2n\delta^2 \eta_{p_X}^2}{\log^2|\mathcal{X}|} \right] \end{aligned}$$

2. Using part (3) of we have

$$1 \geq \sum_{x^n \in T_\delta^n(p_X)} p_X^n(x^n) \geq \sum_{x^n \in T_\delta^n(p_X)} \exp[-n(H(p_X) + \delta)] \geq |T_\delta^n(p_X)| \exp[-n(H(p_X) + \delta)]$$

   Therefore we obtain

$$|T_\delta^n(p_X)| \leq \exp[n(H(p_X) + \delta)]$$

Now

$$Pr[x^n \notin T^n_\delta(p_X)] \leq 2|\mathcal{X}| \exp\left[-\frac{2n\delta^2\eta^2_{p_X}}{\log^2|\mathcal{X}|}\right] \implies Pr[x^n \in T_\delta] \geq 1 - 2|\mathcal{X}| \exp\left[-\frac{2n\delta^2\eta^2_{p_X}}{\log^2|\mathcal{X}|}\right]$$

And again using part (3)

$$Pr[x^n \in T_\delta] = \sum_{x^n \in T^n_\delta(p_X)} p^n_X(x^n) \leq \sum_{x^n \in T^n_\delta(p_X)} \exp[-n(H(p_X)-\delta)] \leq T^n_\delta(p_X)| \exp[-n(H(p_X)-\delta)]$$

Therefore we have

$$|T^n_\delta(p_X)| \geq \left[1 - 2|\mathcal{X}| \exp\left(-\frac{2n\delta^2\eta^2_{p_X}}{\log^2|\mathcal{X}|}\right)\right] \exp[n(H(p_X)-\delta)]$$

Hence we finally obtain

$$\left[1 - 2|\mathcal{X}| \exp\left(\frac{2n\delta^2\eta^2_{p_X}}{(\log|\mathcal{X}|)^2}\right)\right] \exp\left[n(H(p_X)-\delta)\right] \leq |T^n_\delta(p_X)| \leq \exp\left[n(H(p_X)+\delta)\right]$$

3. $p_X(x^n) = \prod_{i=1}^{n} p_X(x_i) = \prod_{a\in\mathcal{X}} p_X(a)^{N(a|x^n)}$. Now from the definition we get for all $a \in \mathcal{X}$ if $x^n \in T^n_\delta(p_X)$

$$-\frac{\delta p_X(a)}{\log|\mathcal{X}|} \leq \frac{N(a|x^n)}{n} - p_X(a) \leq \frac{\delta p_X(a)}{\log|\mathcal{X}|} \implies np_X(a)\left[1 - \frac{\delta}{\log|\mathcal{X}|}\right] \leq N(a|x^n) \leq np_X(a)\left[1 + \frac{\delta}{\log|\mathcal{X}|}\right]$$

Now we get

$$\begin{aligned}
\prod_{a\in\mathcal{X}} p_X(a)^{N(a|x^n)} &\leq \prod_{a\in\mathcal{X}} p_X(a)^{np_X(a)\left[1-\frac{\delta}{\log|\mathcal{X}|}\right]} \\
&= \prod_{x\in\mathcal{X}} \exp[np_X(a)\left[1 - \frac{\delta}{\log|\mathcal{X}|}\right] \log p_X(a)] \\
&= \exp\left[\sum_{x\in\mathcal{X}} np_X(a)\left(1 - \frac{\delta}{\log|\mathcal{X}|}\right) \log p_X(a)\right] \\
&= \exp\left[n\left(1 - \frac{\delta}{\log|\mathcal{X}|}\right) \sum_{x\in\mathcal{X}} p_X(a) \log p_X(a)\right] \\
&= \exp\left[-n\left(1 - \frac{\delta}{\log|\mathcal{X}|}\right) H(p_X)\right]
\end{aligned}$$

Similarly we get

$$\prod_{a\in\mathcal{X}} p_X(a)^{N(a|x^n)} \geq \exp\left[-n\left(1 + \frac{\delta}{\log|\mathcal{X}|}\right) H(p_X)\right]$$

By Problem 1.(2) we have $H(p_X) \leq \log|\mathcal{X}|$. Hence

$$-n\left(H(p_X) + \frac{\delta H(p_X)}{\log|\mathcal{X}|}\right) \geq -n(H(p_X)+\delta)$$

$$-n\left(H(p_X) - \frac{\delta H(p_X)}{\log|\mathcal{X}|}\right) \leq -n(H(p_X)-\delta)$$

Therefore we get

$$\exp[-n(H(p_X)+\delta)] \leq \prod_{i=1}^{n} p_X(x_i) \leq \exp[-n(H(p_X)-\delta)]$$

□

Definitions: Let $p_{X,Y}$ be a joint PMF on $\mathcal{X} \times \mathcal{Y}$ where $\mathcal{X}$, $\mathcal{Y}$ are finite sets. (Essentially $p_{XY}(x,y) \geq 0$ and $\sum\limits_{x \in \mathcal{X}} \sum\limits_{y \in mcY} p_{XY}(x,y) = 1$). We define the marginal of p_{XY} on X as $p_X(x) \triangleq \sum\limits_{y \in \mathcal{Y}} p_{XY}(x,y)$ for $x \in \mathcal{X}$ and marginal of p_{XY} on Y as $p_Y(y) \triangleq \sum\limits_{x \in \mathcal{X}} p_{XY}(x,y)$ for $y \in \mathcal{Y}$.

For a pair $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ of sequences we define $N(a, b \mid x^n, y^n) = \sum\limits_{i=1}^{n} \mathbb{1}_{\{(x_i, y_i) = (a,b)\}}$ as the number of occurrences of (a, b) in (x^n, y^n).

Next the joint typical set wrt p_{XY} is defined as

$$T_\delta^n(p_{XY}) \triangleq \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n \mid \left| \frac{N(a, b \mid x^n, y^n)}{n} - p_{XY}(a,b) \right| \leq \frac{\delta p_{XY}(a,b)}{\log |\mathcal{X}||\mathcal{Y}|} \; \forall \, (a,b) \in \mathcal{X} \times \mathcal{Y} \right\}$$

Problem 3

1. Prove that if $p_{XY}(a,b) = 0$ for some $(a,b) \in \mathcal{X} \times \mathcal{Y}$ and $(x^n, y^n) \in T_\delta^n(p_{XY})$ then $N(a, b|x^n, y^n) = 0$. In other words, a pair that has 0 probability does not occur in any typical pair of sequences.

2. Let $\eta_{p_{XY}} = \min\limits_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \{p_{XY}(x,y) \mid 0 < p_{XY}(a,b) < 1\}$. Use the Hoeffding Inequality to prove that

$$\sum_{(x^n, y^n) \notin T_\delta^n(p_{XY})} p_{XY}^n(x^n, y^n) \leq 2|\mathcal{X}||\mathcal{Y}| \exp\left[- \frac{2n\delta^2 \eta_{p_{XY}}^2}{(\log |\mathcal{X}||\mathcal{Y}|)^2} \right]$$

   Hoeffding Inequality: Let $Z_1, \ldots, Z_m$ are independent and identically distributed random variables for which $P[a \leq Z_i \leq b] = 1$ for ever $1 \leq i \leq m$ and $\mu = \mathbb{E}[Z_i]$. Then for every $\epsilon > 0$

$$p\left[\left| \frac{1}{m} \sum_{i=1}^{m} Z_i - \mu \right| > \epsilon \right] \leq 2 \exp\left[-2m \frac{\epsilon^2}{(b-a)^2} \right]$$

3. For any $(x^n, y^n) \in T_\delta^n(p_{XY})$ prove that

$$\exp[-n[H(p_{XY}) + \delta]] \leq p_{XY}^n(x^n, y^n) = \prod_{i=1}^{n} p_{XY}(x_i, y_i) \leq \exp[-n[H(p_{XY}) - \delta]]$$

4. Prove that
$$(1 - \tilde{\delta}) 2^{n[H(p_{XY}) - \delta]} \leq |T_\delta^n(p_{XY})| \leq 2^{n[H(p_{XY}) + \delta]}$$
   where $\tilde{\delta} = 2|\mathcal{X}||\mathcal{Y}| \exp\left[\frac{2n\delta^2 \eta_{p_{XY}}^2}{(\log |\mathcal{X}||\mathcal{Y}|)^2} \right]$

5. Prove that $(x^n, y^n) \in T_\delta^n(p_{XY})$ then $x^n \in T_\delta^n(p_X)$ and $y^n \in T_\delta^n(p_Y)$.

Solution:

1. Given that $p_{XY}(a,b) = 0$. Now if $(x^n, y^n) \in T_\delta^n(p_{XY})$

$$\left| \frac{N(a, b \mid x^n, y^n)}{n} - p_{XY}(a,b) \right| \leq \frac{\delta p_{XY}(a,b)}{\log |\mathcal{X}||\mathcal{Y}|}$$

   putting the given value $p_{XY}(a,b) = 0$ we get

$$\left| \frac{N(a, b \mid x^n, y^n)}{n} \right| \leq 0$$

   Hence we get $\frac{N(a,b|x^n,y^n)}{n} = 0 \iff N(a, b \mid x^n, y^n) = 0$.

2. $\sum\limits_{(x^n,y^n)\notin T_\delta^n(p_{XY})} p_{XY}^n(x^n,y^n) = Pr[(x^n,y^n) \notin T_\delta^n(p_{XY})]$. If $(x^n,y^n) \notin inT_\delta^n(p_{XY})$ then there exists $(a,b) \in \mathcal{X} \times \mathcal{Y}$ such that

$$\left|\frac{N(a,b \mid x^n,y^n)}{n} - p_{XY}(a,b)\right| > \frac{\delta p_{XY}(a,b)}{\log|\mathcal{X}||\mathcal{Y}|}$$

Now we have $N(a,b \mid x^n,y^n) = \sum\limits_{i=1}^{n} \mathbb{1}_{\{(x_i,y_i)=(a,b)\}}$. Take the indicator random variables $\mathbb{1}_{(x_i,y_i)=(a,b)}$ for $(a,b)7in\mathcal{X} \times \mathcal{Y}$ for each $i \in [n]$. Then $\mathbb{E}\left[\mathbb{1}_{(x_i,y_i)=(a,b)}\right] = p_{XY}(a,b)$. Hence by Hoeffding Inequality

$$\begin{aligned}
Pr\left[\left|\frac{1}{n}\sum_{(a,b)\in\mathcal{X}\times\mathcal{Y}} \mathbb{1}_{(x_i,y_i)=(a,b)} - p_{XY}(a,b)\right| > \frac{\delta p_{XY}(a,b)}{\log|\mathcal{X}||\mathcal{Y}|}\right] &\leq 2\exp\left[-2n\left(\frac{\delta p_{XY}(a,b)}{\log|\mathcal{X}||\mathcal{Y}|}\right)^2\right] \\
&\leq 2\exp\left[-\frac{2n\delta^2\eta_{XY}^2}{\log^2|\mathcal{X}||\mathcal{Y}|}\right]
\end{aligned}$$

So by union bound we get

$$\begin{aligned}
Pr[(x^n,y^n) \notin T_\delta^n(p_{XY})] &\leq \sum_{(a,b)\in\mathcal{X}\times\mathcal{Y}} Pr\left[\left|\frac{1}{n}\sum_{(a,b)\in\mathcal{X}\times\mathcal{Y}} \mathbb{1}_{(x_i,y_i)=(a,b)} - p_{XY}(a,b)\right| > \frac{\delta p_{XY}(a,b)}{\log|\mathcal{X}||\mathcal{Y}|}\right] \\
&\leq \sum_{(a,b)\in\mathcal{X}\times\mathcal{Y}} 2\exp\left[-\frac{2n\delta^2\eta_{XY}^2}{\log^2|\mathcal{X}||\mathcal{Y}|}\right] = 2|\mathcal{X}|\mathcal{Y}|\exp\left[-\frac{2n\delta^2\eta_{XY}^2}{\log^2|\mathcal{X}||\mathcal{Y}|}\right]
\end{aligned}$$

Therefore we get

$$\sum_{(x^n,y^n)\notin T_\delta^n(p_{XY})} p_{XY}^n(x^n,y^n) \leq 2|\mathcal{X}|\mathcal{Y}|\exp\left[-\frac{2n\delta^2\eta_{XY}^2}{\log^2|\mathcal{X}||\mathcal{Y}|}\right]$$

3. $p_{XY}^n(x^n,y^n) = \prod\limits_{i=1}^{n}(x^n,y^n) = \prod\limits_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(a,b)^{N(a,b|x^n,y^n)}$. Now from the definition of $T_\delta^n(p_{XY})$ we get

$$np_{XY}(a,b)\left[1-\frac{\delta}{\log|\mathcal{X}||\mathcal{Y}|}\right] \leq N(a,b|x^n,y^n) \leq np_{XY}(a,b)\left[1+\frac{\delta}{\log|\mathcal{X}||\mathcal{Y}|}\right]$$

So we have

$$\begin{aligned}
\prod_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(a,b)^{N(a,b|x^n,y^n)} &\leq \prod_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(a,b)^{np_{XY}(a,b)\left[1-\frac{\delta}{\log|\mathcal{X}||\mathcal{Y}|}\right]} \\
&= \prod_{(a,b)\in\mathcal{X}\times\mathcal{Y}} \exp\left[np_{XY}(a,b)\left(1-\frac{\delta}{\log|\mathcal{X}||\mathcal{Y}|}\right)\log p_{XY}(a,b)\right] \\
&= \exp\left[\sum_{(a,b)\in\mathcal{X}\times\mathcal{Y}} np_{XY}(a,b)\left(1-\frac{\delta}{\log|\mathcal{X}||\mathcal{Y}|}\right)\log p_{XY}(a,b)\right] \\
&= \exp\left[n\left(1-\frac{\delta}{\log|\mathcal{X}||\mathcal{Y}|}\right)\sum_{(a,b)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(a,b)\log p_{XY}(a,b)\right] \\
&= \exp\left[-n\left(1-\frac{\delta}{\log|\mathcal{X}||\mathcal{Y}|}\right)H(p_{XY})\right]
\end{aligned}$$

Similarly we obtain

$$\prod_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{XY}(a,b)^{N(a,b|x^n,y^n)} \geq \exp\left[-n\left(1+\frac{\delta}{\log|\mathcal{X}||\mathcal{Y}|}\right)H(p_{XY})\right]$$

Now we will prove a claim

Claim: $H(p_{XY}) \leq \log|\mathcal{X}||\mathcal{Y}|$

Proof:

$$
\begin{aligned}
&\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{XY}(x,y)\log\frac{1}{p_{XY}(x,y)} - \log|\mathcal{X}||\mathcal{Y}| \\
&= \sum_{x\in X}\sum_{y\in\mathcal{Y}} p_{XY}(x,y)\log\frac{1}{p_X(x)} - \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{XY}(x,y)\log|\mathcal{X}| \\
&= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{XY}(x,y)\log\frac{1}{(|\mathcal{X}||\mathcal{Y}|)p_{XY}(x,y)} \\
&\leq \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{XY}(x,y)\left[\frac{1}{(|\mathcal{X}||\mathcal{Y}|)p_{XY}(x,y)} - 1\right] && \text{[Using 1.(1)]} \\
&= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}}\left[\frac{1}{|\mathcal{X}||\mathcal{Y}|} - p_{XY}(x,y)\right] = 1-1=0
\end{aligned}
$$

□

Now using the claim we get

$$
\exp\left[-n\left(H(p_{XY}) - \frac{\delta H(p_{XY})}{\log|\mathcal{X}||\mathcal{Y}|}\right)\right] \leq \exp[-n(H(p_{XY})-\delta)]
$$
$$
\exp\left[-n\left(H(p_{XY}) + \frac{\delta H(p_{XY})}{\log|\mathcal{X}||\mathcal{Y}|}\right)\right] \geq \exp[-n(H(p_{XY})+\delta)]
$$

Hence we get if $(x^n, y^n) \in T_\delta^n(p_{XY})$ then

$$
\exp[-n(H(p_{XY})+\delta)] \leq p_{XY}^n(x^n,y^n) \leq \exp[-n(H(p_{XY})-\delta)]
$$

4. Using part (2) we have

$$
1 \geq \sum_{(x^n,y^n)\in T_\delta^n(p_{XY})} p_{XY}^n(x^n,y^n) \geq \sum_{(x^n,y^n)\in T_\delta^n(p_{XY})} \exp[-n(H(p_{XY})+\delta)] \geq |T_\delta^n(p_{XY})\exp[-n(H(p_{XY})+\delta)]
$$

Hence we get

$$
|T_\delta^n(p_{XY})| \leq \exp[n(H(p_{XY})+\delta)]
$$

In part (1) we proved $Pr[(x^n,y^n)\notin T_\delta^n(p_{XY})] \leq 2|\mathcal{X}|\mathcal{Y}|\exp\left[-\frac{2n\delta^2\eta_{XY}^2}{\log^2|\mathcal{X}||\mathcal{Y}|}\right]$. Hence

$$
Pr[(x^n,y^n)\in T_\delta(p_{XY})] \geq 1 - 2|\mathcal{X}|\mathcal{Y}|\exp\left[-\frac{2n\delta^2\eta_{XY}^2}{\log^2|\mathcal{X}||\mathcal{Y}|}\right]
$$

and

$$
\begin{aligned}
Pr[(x^n,y^n)\in T_\delta^n(p_{XY})] &= \sum_{(x^n,y^n)\in T_\delta^n(p_{XY})} p_{XY}^n(x^n,y^n) \\
&\leq \sum_{(x^n,y^n)\in T_\delta^n(p_{XY})} \exp[-n(H(p_{XY})-\delta)] \leq |T_\delta^n(p_{XY})|\exp[-n(H(p_{XY})-\delta)]
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&|T_\delta^n(p_{XY})|\exp[-n(H(p_{XY})-\delta)] \geq 1 - 2|\mathcal{X}|\mathcal{Y}|\exp\left[-\frac{2n\delta^2\eta_{XY}^2}{\log^2|\mathcal{X}||\mathcal{Y}|}\right] \\
\implies &|T_\delta^n(p_{XY})| \geq \left[1 - 2|\mathcal{X}|\mathcal{Y}|\exp\left(-\frac{2n\delta^2\eta_{XY}^2}{\log^2|\mathcal{X}||\mathcal{Y}|}\right)\right]\exp[n(H(p_{XY})-\delta)]
\end{aligned}
$$

Therefore finally we get

$$\left[1-2|\mathcal{X}||\mathcal{Y}|\exp\left(-\frac{2n\delta^2\eta_{XY}^2}{\log^2|\mathcal{X}||\mathcal{Y}|}\right)\right]\exp[n(H(p_{XY})-\delta)] \;\leq\; |T_\delta^n(p_{XY})| \;\leq\; \exp[n(H(p_{XY})+\delta)]$$

5.

$\square$

Definitions: Suppose p_{XY} is a probability distribution (probability mass function (PMF)) on $\mathcal{X} \times \mathcal{Y}$. We recall the condition distribution $p_{Y|X}(y|x) = \frac{p_{XY}(x,y)}{p_X(x)}$ and for a pair $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ of sequence $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ of sequences $p^n_{Y|X}(y^n|x^n) = \prod_{i=1}^{n} p_{Y|X}(y_i|x_i)$

We define

$$H(Y|X = x) \triangleq H(p_{XY}|X = x) = -\sum_{y \in \mathcal{Y}} p_{Y|X}(y|x) \log p_{Y|X}(y|x)$$

and

$$H(Y|X) = H(p_{Y|X}|p_X) \triangleq \sum_{x \in \mathcal{X}} p_X(x) h(Y|X = x)$$

For any $x^n \in \mathcal{X}^n$ define the conditional typical set of x^n as

$$T^n_\delta(p_{Y|X}) = \{y^n \in \mathcal{Y}^n \mid (x^n, y^n) \in T^n_\delta(p_{XY})\}$$

Problem 4

1. Prove that $\sum_{y \in \mathcal{Y}} p_{Y|X}(y|x) = 1$
2. Prove that $H(Y|X) = H(X, Y) - H(X)$ and $H(Y|X) \geq 0$
3. Prove that Verify that if $x^n \notin T^n_\delta(p_X)$ then $T^n_\delta(p_{XY}|x^n) = \phi$
4. Suppose $x^n \in T^n_\delta(p_X)$ and $y^n \in T^n_\delta(p_{XY}|x^n)$ prove that

$$2^{-n[H(Y|X)+2\delta]} \leq p^n_{Y|X}(y^n|x^n) \leq 2^{-n[H(Y|X)-2\delta]}$$

5. Prove that if $x^n \in T^n_\delta(p_X)$ then

$$\sum_{y^n \in T^n_{2\delta}(p_{XY}|x^n)} p^n_{Y|X}(y^n|x^n) \geq 1 - 2|\mathcal{X}||\mathcal{Y}| \exp\left[-\frac{2n\delta^2}{(\log|\mathcal{X}||\mathcal{Y}|)^2} \eta_{p_{Y|X}}\right]$$

where $\eta_{p_{Y|X}} = \min_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \{p_{Y|X}(y|x) \mid 0 < p_{Y|X}(y|x) < 1\}$

6. Suppose $x^n \in T^n_\delta(p_X)$ then

$$(1 - \tilde{\delta}) 2^{n[H(Y|X)-4\delta]} \leq |T^n_\delta(p_{XY}|x^n)| \leq 2^{n[H(Y|X)+4\delta]}$$

where $\tilde{\delta} = 2|\mathcal{X}||\mathcal{Y}| \exp\left[-\frac{2n\delta^2}{(\log|\mathcal{X}||\mathcal{Y}|)^2} \eta_{p_{Y|X}}\right]$

Solution:

□