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Course: Quantum Information Theory

Assignment - 2.2 : Quantum Foundations
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For all the questions

- $[k] := \{1, 2, \dots, k\}$ where $k \in \mathbb{N}$.
- $\mathscr{L}(\mathcal{H}) :=$ Linear operators on $\mathcal{H}$
- $\mathscr{R}(\mathcal{H}) :=$ Self-adjoint or hermitian operators on $\mathcal{H}$
- $\mathscr{P}(\mathcal{H}) :=$ Positive semi-definite operators on $\mathcal{H}$
- $\mathscr{D}(\mathcal{H}) :=$ Density operators on $\mathcal{H}$

Problem 1

For $T : \mathcal{H} \to \mathcal{H}$, prove that

$$\sum_{i=1}^{d} \langle e_i \, | T e_i \rangle = \sum_{i=1}^{d} \langle f_i \, | T f_i \rangle$$

if $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ and $\{|f_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ are ONB.

Solution: Let $S : \mathcal{H} \to \mathcal{H}$ where it maps the basis vectors from $|e_i\rangle \to |f_i\rangle$. Then $S\,|e_i\rangle = |f_i\rangle$. Hence S is an unitary matrix since

$$\langle e_j |\, S^\dagger S \,|e_i\rangle = \langle f_j |\, f_i\rangle = \delta_{ji} \quad \text{and} \quad \langle f_j |\, SS^\dagger \,|f_i\rangle = \langle e_j |\, e_i\rangle = \delta_{ji}$$

Hence

$$\sum_{i=1}^{d} \langle f_i | T f_i \rangle = \sum_{i=1}^{d} \langle e_i | S^\dagger T S \,|e_i\rangle = tr(S^\dagger T S) = tr(SS^\dagger T) = tr(T) = \sum_{i=1}^{d} \langle e_i \, | T e_i \rangle$$

Therefore we have

$$\sum_{i=1}^{d} \langle e_i \, | T e_i \rangle = \sum_{i=1}^{d} \langle f_i \, | T f_i \rangle$$

□

Problem 2

If $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \le i \le d\}$ and $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \le i \le d\}$ are ONB, then $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \le i, j \le d\} \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ is ONB

Solution: Let $|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then $|\psi\rangle = \sum\limits_{i=1}^{d} \alpha_i \,|e_i\rangle$ where $\alpha_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|e_i\rangle \in \mathcal{H}_1 \mid 1 \le i \le d\}$ is ONB for $\mathcal{H}_1$. Hence

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i \,|e_i\rangle \otimes |\phi\rangle$$

Now $|\phi\rangle = \sum\limits_{i=1}^{d} \beta_i \,|f_i\rangle$ where $\beta_i \in \mathbb{C}$ for all $i \in [d]$ since $\{|f_i\rangle \in \mathcal{H}_2 \mid 1 \le i \le d\}$ is ONB for $\mathcal{H}_2$. Hence

$$\forall\, i \in [d] \;\; |e_i\rangle \otimes |phi\rangle = \sum_{j=1}^{d} \beta_j \,|e_i\rangle \otimes |f_j\rangle$$

Thereofore we get

$$|\psi\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle \otimes |\phi\rangle = \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{d} \beta_j |e_i\rangle \otimes |f_j\rangle = \sum_{1\le i,j\le d} \alpha_i\beta_j |e_i\rangle \otimes |f_j\rangle$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \le i,j \le d\}$ is a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Now for any $i1, i2, j1, j2 \in [d]$

$$(\langle e_{i1}| \otimes \langle f_{j1}|)(|e_{i2}\rangle \otimes |f_{j2}\rangle) = \langle e_{i1}|e_{i2}\rangle \langle f_{j1}|f_{j2}\rangle = \delta_{i1,i2}\,\delta_{j1,j2}$$

Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \le i,j \le d\}$ is orthonormal. Therefore $\{|e_i\rangle \otimes |f_j\rangle \mid 1 \le i,j \le d\}$ is a ONB for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

□

Problem 3

Let $\{|g_k\rangle \mid 1 \le k \le d_2\} \subseteq \mathcal{H}_2$ be ONB. For $T \in \mathscr{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, let $tr_2(T) \in \mathscr{L}(\mathcal{H}_1)$ denote the operator satisfying

$$\langle u| tr_2(T) |v\rangle = \sum_k \langle u \otimes g_k| T |v \otimes g_k\rangle$$

for any choice $|u\rangle, |v\rangle \in \mathcal{H}_1$. Prove that $\sum_k \langle u \otimes g_k| T |v \otimes g_k\rangle$ is invariant.

Solution: Let $\{|f_k\rangle \mid 1 \le k \le d_2\} \subseteq \mathcal{H}_2$ be another ONB. Suppose $S : \mathcal{H}_2 \to \mathcal{H}$ be a map such that $S|g_k\rangle = |f_k\rangle$. As we previously showed in Problem 1, S is unitary. Then for all $k \in [d_2]$ we have

$$|f_k\rangle = \sum_{i=1}^{d_2} w_{i,k} |e_i\rangle$$

where $w_{i,k} \in \mathbb{C}$. Hence

$$\langle f_i| S^\dagger S |f_j\rangle = \sum_{k=1}^{d_2} w_{i,k}^* w_{j,k} = \delta_{i,j}$$

Now for any $|u\rangle, |v\rangle \in \mathcal{H}_1$ we have

$$\begin{aligned}
\langle u| tr_2(T) |v\rangle_{\{|f_k\rangle\}} &= \langle u| \left[\sum_{k=1}^{d_2} (I \otimes \langle f_k|) T (I \otimes |f_k\rangle)\right] |v\rangle \\
&= \langle u| \left[\sum_{k=1}^{d_2} \left(I \otimes \left(\sum_{i=1}^{d_2} w_{i,k}^* \langle g_i|\right)\right) T \left(I \otimes \left(\sum_{j=1}^{d_2} w_{j,k} |g_j\rangle\right)\right)\right] |v\rangle \\
&= \sum_{k=1}^{d_2}\sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \langle u| \left[w_{i,k}^* w_{j,k} (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \langle u| \left[\left(\sum_{k=1}^{d_2} w_{i,k}^* w_{j,k}\right) (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \langle u| \left[\delta_{i,j} (I \otimes \langle g_i|) T (I \otimes |g_j\rangle)\right] |v\rangle \\
&= \sum_{i=1}^{d_2} \langle u| \left[(I \otimes \langle g_i|) T (I \otimes |g_i\rangle)\right] |v\rangle \\
&= \langle u| tr_2(T) |v\rangle_{\{|g_k\rangle\}}
\end{aligned}$$

Hence $\sum_k \langle u \otimes g_k| T |v \otimes g_k\rangle$ is invariant.

□

Problem 4 Mark Wilde: Exercise 3.3.3

Show that the Pauli matrices are all Hermitian, unitary, they square to the identity, and their eigenvalues are ± 1

Solution: Pauli matrices are

$$X|0\rangle = |1\rangle, X|1\rangle = |0\rangle \qquad Y|0\rangle = -i|1\rangle, Y|1\rangle = i|0\rangle \qquad Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle$$

Therefore we have

$$X = |1\rangle\langle 0| + |0\rangle\langle 1| \qquad Y = i[|0\rangle\langle 1| - |1\rangle\langle 0|] \qquad Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Hence

$$\begin{aligned}
X^\dagger &= (|1\rangle\langle 0|)^\dagger + (|0\rangle\langle 1|)^\dagger = |0\rangle\langle 1| + |1\rangle\langle 0| = X \\
Y^\dagger &= (i|0\rangle\langle 1|)^\dagger + (-i|1\rangle\langle 0|)^\dagger = -i|1\rangle\langle 0| + i|0\rangle\langle 1| = Y \\
Z^\dagger &= (|0\rangle\langle 0|)^\dagger - (|1\rangle\langle 1|)^\dagger = |0\rangle\langle 0| - |1\rangle\langle 1| = Z
\end{aligned}$$

Therefore they are Hermitian.

Now

$$\begin{aligned}
X^\dagger X = XX^\dagger = X^2 &= [|1\rangle\langle 0| + |0\rangle\langle 1|][|1\rangle\langle 0| + |0\rangle\langle 1|] \\
&= |1\rangle\langle 0|1\rangle\langle 0| + |1\rangle\langle 0|0\rangle\langle 1| + |0\rangle\langle 1|1\rangle\langle 0| + |0\rangle\langle 1|0\rangle\langle 1| \\
&= |1\rangle\langle 1| + |0\rangle\langle 0| = I
\end{aligned}$$

$$\begin{aligned}
Y^\dagger Y = Y^\dagger = Y^2 &= [i(|0\rangle\langle 1| - |1\rangle\langle 0|)][i(|0\rangle\langle 1| - |1\rangle\langle 0|)] \\
&= -[|0\rangle\langle 1|0\rangle\langle 1| - |0\rangle\langle 1|1\rangle\langle 0| - |1\rangle\langle 0|0\rangle\langle 1| + |1\rangle\langle 0|1\rangle\langle 0|] \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = I
\end{aligned}$$

$$\begin{aligned}
Z^\dagger Z = Z^\dagger = Z^2 &= [|0\rangle\langle 0| - |1\rangle\langle 1|][|0\rangle\langle 0| - |1\rangle\langle 1|] \\
&= |0\rangle\langle 0|0\rangle\langle 0| - |0\rangle\langle 0|1\rangle\langle 1| - |1\rangle\langle 1|0\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 1| \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = I
\end{aligned}$$

Therefore X, Y, Z are unitary and they square to the identity.

Since $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$ we have

$$X\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \qquad X\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

So the for the eigenvalue 1 the corresponding eignevector is $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and for the eigenvalue -1 the corresponding eigenvalue is $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Since $Y|0\rangle = -i|1\rangle$ and $Y|1\rangle = i|0\rangle$ we have

$$\begin{aligned}
Y\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) &= \frac{1}{\sqrt{2}}\left(-i|1\rangle + i^2|0\rangle\right) = -\frac{1}{\sqrt{2}}(i|1\rangle + |0\rangle) \\
Y\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) &= \frac{1}{\sqrt{2}}\left(-i|1\rangle - i^2|0\rangle\right) = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)
\end{aligned}$$

So the for the eigenvalue 1 the corresponding eigenvector is $|0\rangle - i|1\rangle$ and for the eigenvalue -1 the corresponding eigenvalue is $|0\rangle + i|1\rangle$.

Since $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$. So the for the eigenvalue 1 the corresponding eigenvector is $|0\rangle$ and for the eigenvalue -1 the corresponding eigenvalue is $|1\rangle$.

$\square$

Problem 5

For $S, T \in \mathscr{L}(\mathcal{H})$, show that

$$tr(T) = tr(T^\dagger)^*, \qquad tr(ST) = tr(TS)$$

[Recall T^+ denotes adjoint of T]. For $|x\rangle, |y\rangle \in \mathcal{H}$ show

$$tr(|x\rangle\langle y| T) = tr(T|x\rangle\langle y|) = \langle y|Tx\rangle$$

Solution:

- $tr(T)$ is the summation of the diagonal entries of T. Now $T^\dagger = (T^t)^*$. Now the diagonal elements of T remains in the same same position even after transpose. Hence the diagonal elements of $T^\dagger$ are the complex conjugate of the diagonal elements of T. Hence sum of the diagonal entries of $T^\dagger$ will also be the complex conjugate of the sum of the diagonal entries of T. Therefore we get

$$tr(T) = tr(T^\dagger)^*$$

- Let $\dim \mathcal{H} - d$. Suppose $\{|e_k\rangle \mid k \in [d]\} \subseteq \mathcal{H}$ be an ONB of $\mathcal{H}$

$$\begin{aligned}
tr(ST) &= \sum_{k=1}^{d} \langle e_k| ST |e_k\rangle = \sum_{k=1}^{d} \langle e_k| SIT |e_k\rangle \\
&= \sum_{k=1}^{d} \langle e_k| S \left[\sum_{i=1}^{d} |e_i\rangle\langle e_i|\right] |e_k\rangle \\
&= \sum_{k-1}^{d}\sum_{i=1}^{d} \langle e_k| S |e_i\rangle\langle e_i| T |e_k\rangle \\
&= \sum_{i=1}^{d}\sum_{k=1}^{d} \langle e_i| T |e_k\rangle\langle e_k| S |e_i\rangle \\
&= \sum_{i=1}^{d} \langle e_i| T \left[\sum_{k=1}^{d} |e_k\rangle\langle e_k|\right] S |e_i\rangle \\
&= \sum_{i=1}^{d} \langle e_i| TIS |e_i\rangle = \sum_{i=1}^{d} \langle e_i| TS |e_i\rangle = tr(TS)
\end{aligned}$$

- $tr(|x\rangle\langle y| T) = tr([|x\rangle\langle y|]T) = tr(T[|x\rangle\langle y|]) = tr(T|x\rangle\langle y|)$

□

Problem 6

Suppose $\mathcal{H}$ is finite dimensional complex inner product spacewith $\dim(\mathcal{H}) = d$. Show complex dimensionality of $\mathscr{L}(\mathcal{H})$ is d^2, real dimensionality of $\mathscr{R}(\mathcal{H})$ is d^2.

Suppose $\mathcal{H}$ is a real inner product space of dim d, show $\mathscr{L}(\mathcal{H})$ has dimension d^2 and the space of all symmetric operators is a real vector space of dimension $\frac{d(d+1)}{2}$.

Solution:

- Suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is an ONB of $\mathcal{H}$. Let $T \in \mathscr{L}(\mathcal{H})$. Then for all $i \in [d]$

$$T|e_i\rangle = \sum_{j=1}^{d} \alpha_{i,j} |e_j\rangle$$

where $\alpha_{i,j} \in \mathbb{C}$. Hence, the map T is uniquely decided by the numbers $\alpha_{i,j} \in \mathbb{C}$ for all $i, j \in [d]$. Hence, there are d^2 many numbers which uniquely decides T. Therefore $\dim(\mathscr{L}(\mathcal{H})) = d^2$.

- Now let $T \in \mathscr{R}(\mathcal{H})$. Then $T^\dagger = T$. Again suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is an ONB of $\mathcal{H}$. Let (i,j)th element of T is denoted by $t_{i,j}$. Then for all $i \in [d]$, $T_{i,i} \in \mathbb{R}$ since $T^\dagger = T$. Now for all off diagonal entries $t_{j,i} = t^*_{i,j}$. So there are $\frac{n^2-n}{2}$ many complex numbers which decides T uniquely apart from the n real entries in the diagonal. Now for each $i,j \in [d]$ let $t_{i,j} = x_{i,j} + iy_{i,j}$ where $x_{i,j}, y_{i,j} \in \mathbb{R}$. Therefore,

$$t_{j,i} = t^*_{i,j} = x_{i,j} - iy_{i,j}$$

So for each off-diagonal entries there are corresponding 2 real numbers. And there are total $\frac{d^2-d}{2}$ many off-diagonal entries which participates in uniquely deciding T. Hence there are total

$$2 \times \frac{d^2-d}{2} + d = d^2$$

real numbers which uniquely decides T. Hence $\dim(\mathscr{R}(\mathcal{H})) = d^2$.

- Suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is a basis of $\mathcal{H}$. Let $T \in \mathscr{L}(\mathcal{H})$. Then for all $i \in [d]$

$$T\,|e_i\rangle = \sum_{j=1}^{d} \alpha_{i,j}\,|e_j\rangle$$

where $\alpha_{i,j} \in \mathbb{R}$. Hence, the map T is uniquely decided by the numbers $\alpha_{i,j} \in \mathbb{C}$ for all $i,j \in [d]$. Since there are d^2 many numbers which uniquely decides T, $\dim(\mathscr{L}(\mathcal{H})) = d^2$.

- Let $T \in \mathscr{R}(\mathcal{H})$. Then $T^t = T$. Again suppose $\{|e_i\rangle \in \mathcal{H} \mid 1 \le i \le d\}$ is an basis of $\mathcal{H}$. Let (i,j)th element of T is denoted by $T_{i,j}$. Now for all off diagonal entries $T_{j,i} = T_{i,j}$. So there are $\frac{d^2-d}{2}$ many real numbers which decides T uniquely apart from the d entries in the diagonal. Therefore, there are total $\frac{d^2-d}{2}$ many off-diagonal entries which participates in uniquely deciding T. Hence there are total

$$\frac{d^2-d}{2} + d = \frac{d^2+d}{2} = \frac{d(d+1)}{2}$$

real numbers which uniquely decides T. Hence $\dim(\mathscr{R}(\mathcal{H})) = d^2$.

□

Problem 7

Show that $\mathscr{D}(\mathcal{H})$ is a convex subset of the real vector space of all Hermitian operators on $\mathcal{H}$. Show that the extreme points of $\mathscr{D}(\mathcal{H})$ are pure states, i.e. rank 1 projection operators.

Solution:

□

Problem 8

Show that if $\dim(\mathcal{H}) = d$, then $\mathscr{D}(\mathcal{H})$ can be embedded into a real vector space of dimension $n = d^2 - 1$

Solution: In any operator of $\mathscr{D}(\mathcal{H})$ there are ^{2} entries in the matrix of the operator. But density operator also has one extra condition which is its trace equal's to 1. Hence sum of the diagonal entries is 1. Hence for the diagonal entries it is enough to know about the $d-1$ entries instead of the all d entries because the last entry will be decided by the other $d-1$ entries as their sum is 1. Except the diagonal there are total $d^2 - d$ many off diagonal entries. Hence to uniquely characterize a operator in $\mathscr{D}(\mathcal{H})$ at most $(d^2-d)+(d-1) = d^2-1$ many numbers are needed. Therefore the set of operators, $\mathscr{D}(\mathcal{H})$ can be embedded into a real vector space of dimension $n = d^2 - 1$.

□

Problem 9

Prove the Singular value decomposition theorem stated in class.

Solution: We will first state the singular value decomposition theorem then we will prove it.
Singular Value Decomposition Theorem: Suppose $T : \mathcal{H} \to \mathcal{H}$ and $\dim(\mathcal{H}) = s$ then $\exists\, U, V \in \mathscr{L}(H)$ which are unitary and diagonal $D \in \mathscr{L}(\mathcal{H})$ with non-negative entries so that

$$T = UDV$$

Proof: Suppose we have an ONB of $\mathcal{H}$, $\{|e_i\rangle \mid i \in [d]\}$ of $\mathcal{H}$. Let's denote $S = T^\dagger T$. Now S is hermitian. Hence by spectral theorem there are unitary matrix W and a diagonal matrix Λ such that

$$S = W\Lambda W^\dagger$$

and also we get an orthonormal eigen basis $\{|v_i\rangle \mid i \in [d]\} \subseteq \mathcal{H}$ of $\mathcal{H}$ with corresponding eigenvalues $\{\lambda_i \mid i \in [d]\}$ of S which are the diagonal entries of Λ.

Now if λ is an eigenvalue of S with corresponding eigenvector $|v\rangle$ then

$$S\,|v\rangle = \lambda\,|v\rangle \implies \lambda\,\langle v|v\rangle = \langle v|\,Sv = (Sv)^\dagger\,|v\rangle = \lambda^\dagger\,\langle v|v\rangle$$

Therefore the eigenvalues are real. Also since

$$\lambda\,\langle v|v\rangle = \langle v|\,S\,|v\rangle = \langle v|\,T^\dagger T\,|v\rangle = (T\,|v\rangle)^\dagger (T\,|v\rangle) \geq 0$$

we have $\lambda \geq 0$ since $\langle v|v\rangle \geq 0$. Therefore eigenvalues of S are real and non-negative. Therefore entries of Λ are non-negative.

Let us denote the ith eigenvalue of Λ as λ_i. So now take $\sigma_i \triangleq \sqrt{\lambda_i}$. Now create the diagonal matrix Σ with ith eigenvalue of Σ is σ_i. Define the vectors $|u_i\rangle = \frac{1}{\sigma_i} T\,|v_i\rangle$ for all $i \in [d]$. Then

$$\langle u_i|u_j\rangle = \frac{1}{\sigma_i\sigma_j}\,\langle v_i|\,T^\dagger T\,|v_j\rangle = \frac{1}{\sigma_i\sigma_j}\,\langle v_i|\,S\,|v_j\rangle = \frac{1}{\sigma_i\sigma_j}\,\langle v_i|\,\lambda_j\,|v_j\rangle = \frac{\lambda_j}{\sigma_i\sigma_j}\,\langle v_i|v_j\rangle = \delta_{i,j}$$

Hence $\{|u_i\rangle \mid i \in [d]\}$ also forms an orthonormal basis.

Now we have two maps $V : \mathcal{H} \to \mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ which send the orthonormal basis $\{|e_i\rangle \mid i \in [d]\}$ to $\{|v_i\rangle \mid i \in [d]\}$ by $V\,|e_i\rangle = |v_i\rangle$ and $\{|e_i\rangle \mid i \in [d]\}$ to $\{|u_i\rangle \mid i \in [d]\}$ by $U\,|e_i\rangle = |u_i\rangle$. Therefore we have the matrix of U, V are orthonormal. By definition of u_i we have $Tv_i = \sigma_i u_i$ for all $i \in [d]$. Hence we have

$$TV = U\Sigma \implies T = U\Sigma V^\dagger$$

□

Problem 10

Suppose $|\psi\rangle_{AR_1} \in \mathcal{H}_A \otimes \mathcal{H}_{R_1}$, $|\psi\rangle_{AR_2} \in \mathcal{H}_A \otimes \mathcal{H}_{R_2}$ are purifications of $\rho_A \in \mathscr{D}(\mathcal{H}_A)$ and $\dim(\mathcal{H}_{R_2}) \geq \dim(\mathcal{H}_{R_1})$, then show that there exists an isometry $V : \mathcal{H}_{R_1} \to \mathcal{H}_{R_2}$ such that

$$|\psi\rangle_{AR_2} = (V \otimes I)\,|\psi\rangle_{AR_1}$$

Problem 11 Mark Wilde: Exercise 3.6.5

Show that the Bell states form an orthonormal basis:

$$\langle \Phi^{z_1 x_1}|\Phi^{z_2 x_2}\rangle = \delta_{z_1,z_2}\,\delta_{x_1,x_2}$$

Solution: By definition we have

$$|\Phi^{z,x}\rangle = (Z^z \otimes I)(X^x \otimes I)\,|\Phi^+\rangle = (Z^z X^x \otimes I)\,|\Phi^+\rangle$$

Therefore

$$\begin{aligned}\langle\Phi^{z_1x_1}|\Phi^{z_2x_2}\rangle &= \langle\Phi^+|\,(X^{x_1}Z^{z_1}\otimes I)(Z^{z_2}X^{x_2}\otimes I)\,|\Phi^+\rangle \\ &= \langle\Phi^+|\,(X^{x_1}Z^{z_1}Z^{z_2}X^{x_2})\otimes I\,|\Phi^+\rangle \\ &= \langle\Phi^+|\,(X^{x_1}Z^{z_1\oplus z_2}X^{x_2})\otimes I\,|\Phi^+\rangle\end{aligned}$$

□

Problem 12 Mark Wilde: Exercise 3.7.11

Show that the set of states $\{|\Phi^{x,z}\rangle_{AB}\}_{x,z=0}^{d-1}$ forms a complete, orthonormal basis:

$$\langle\Phi^{x_1,z_1}|\Phi^{x_2,z_2}\rangle = \delta_{x_1,x_2}\,\delta_{z_1,z_2} \qquad \sum_{x,z=0}^{d}|\Phi^{x,z}\rangle\langle\Phi^{x,z}| = I_{AB}$$

Problem 13 Mark Wilde: Exercise 4.1.5

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2},|0\rangle\},\{\frac{1}{2},|1\rangle\}\}$ and $\{\{\frac{1}{2},|+\rangle\},\{\frac{1}{2},|-\rangle\}\}$

Problem 14

Show that the set of states $\{|\Phi^{x,z}\rangle_{AB}\}_{x,z=0}^{d-1}$ forms a complete, orthonormal basis:

$$\langle\Phi^{x_1,z_1}|\Phi^{x_2,z_2}\rangle = \delta_{x_1,x_2}\,\delta_{z_1,z_2} \qquad \sum_{x,z=0}^{d}|\Phi^{x,z}\rangle\langle\Phi^{x,z}| = I_{AB}$$

Problem 15 Mark Wilde: Exercise 4.1.3

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2},|0\rangle\},\{\frac{1}{2},|1\rangle\}\}$ and $\{\{\frac{1}{2},|+\rangle\},\{\frac{1}{2},|-\rangle\}\}$

Problem 16 Mark Wilde: Exercise 3.7.12

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2},|0\rangle\},\{\frac{1}{2},|1\rangle\}\}$ and $\{\{\frac{1}{2},|+\rangle\},\{\frac{1}{2},|-\rangle\}\}$

Problem 17

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2},|0\rangle\},\{\frac{1}{2},|1\rangle\}\}$ and $\{\{\frac{1}{2},|+\rangle\},\{\frac{1}{2},|-\rangle\}\}$

Problem 18

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2},|0\rangle\},\{\frac{1}{2},|1\rangle\}\}$ and $\{\{\frac{1}{2},|+\rangle\},\{\frac{1}{2},|-\rangle\}\}$

Problem 19

Show that the following ensembles have the same density operator: $\{\{\frac{1}{2},|0\rangle\},\{\frac{1}{2},|1\rangle\}\}$ and $\{\{\frac{1}{2},|+\rangle\},\{\frac{1}{2},|-\rangle\}\}$