CSS.201.1 Algorithms

Instructor: Umang Bhaskar

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Scribe: Soham Chatterjee

soham.chatterjee@tifr.res.in
Website: sohamch08.github.io

Contents

Chapter 1

Finding Closest Pair of Points

Find Closest
Input: Set $S = \{(x_i, y_i) \mid x_i, y_i \in \mathbb{R}, \forall\, i \in [n]\}$. We denote $P_i = (x_i, y_i)$.
Question: Given a set of points find the closest pair of points in $\mathbb{R}^2$ find P_i, P_j that are at minimum l_2 distance i.e. minimize $\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$.

1.1 Naive Algorithm

Now the naive algorithm for this will be checking all pairs of points and take their distance and output the minimum one. There are total $\binom{n}{2}$ possible choices of pairs of points. And calculating the distance of each pair takes $O(1)$ time. So it will take $O(n^2)$ times to find the closest pair of points.

Idea: $\forall\, P_i, P_j \in S$ find distance $d(P_i, P_j)$ and return the minimum. Time taken is $O(n^2)$.

1.2 Divide and Conquer Algorithm

Below we will show a Divide and Conquer algorithm which gives a much faster algorithm.

Definition 1.2.1: Divide and Conquer

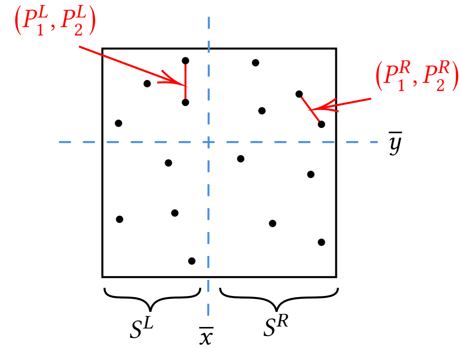
- Divide: Divide the problem into two parts (roughly equal)
- Conquer: Solve each part individually recursively. If the subproblem sizes are small enough, however, just solve the subproblems in a straightforward manner.
- Combine: Combine the solutions to the subproblems into the solution.

1.2.1 Divide

So to divide the problem into two roughly equal parts we need to divide the points into two equal sets. That we can do by sorting the points by their $x-$coordinate. Suppose S^x denote we get the new sorted array or points. And similarly we obtain S^y which denotes the array of points after sorting S by their $y-$coordinate.

Algorithm 1: Step 1 (Divide)

Function *Divide*:
  Sort S by $x-$coordinate and $y-$coordinate
  $S^x \longleftarrow S$ sorted by $x-$coordinate
  $S^y \longleftarrow S$ sorted by $y-$coordinate
  $\bar{x} \longleftarrow \lfloor \frac{n}{2} \rfloor$ highest $x-$coordinate
  $\bar{y} \longleftarrow \lfloor \frac{n}{2} \rfloor$ highest $y-$coordinate
  $S^L \longleftarrow \{P_i \mid x_i < \bar{x}, \ \forall\ i \in [n]\}$
  $S^R \longleftarrow \{P_i \mid x_i \geq \bar{x}, \ \forall\ i \in [n]\}$



1.2.2 Conquer

Now we will recursively get pair of closest points in S_L and S_R. Suppose the (P_1^L, P_2^L) are the closest pair of points in S^L and (P_1^R, P_2^R) are the closest pair of points in S^R.

Algorithm 2: Step 1 (Solve Subproblems)

Function *Conquer*:
  Solve for S_L, S^R.
  (P_1^L, P_2^L) are closest pair of points in S_L.
  (P_1^R, P_2^R) are closest pair of points in S_R.
  $\delta^L = d(P_1^L, P_2^L)$, $\delta^R = d(P_1^R, P_2^R)$
  $\delta_{min} \longleftarrow \min\{\delta^L, \delta^R\}$

1.2.3 Combine

Now we want to combine these two solutions.

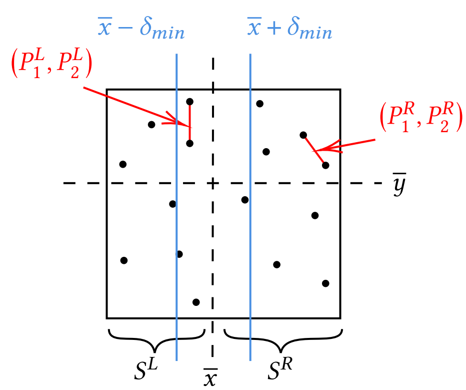
Question 1.1: We are not done

Is there a pair of points $P_i, P_j \in S$ such that $d(P_i, P_j) < \delta_{min}$

If Yes:

- One of them must be in S_L and the other is in S_R.
- $x-$coordinate $\in [\overline{x} - \delta_{min}, \overline{x} + \delta_{min}]$.
- $|y_i - y_j| \leq \delta_{min}$

So we take the strip of radius δ_{min} around $\overline{x}$. Define $T = \{P_i \in S \mid |x_i - \overline{x}| \leq \delta_{min}\}$

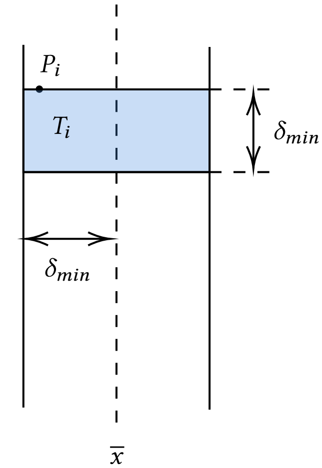


We now sort all the points in the T by their decreasing $y-$coordinate. Let T_y be the array of points. For each $P_i \in T_y$ define the region

$$T_i = \{P_j \in T_y \mid 0 \leq y_j - y_i \leq \delta_{min}, j > i\}$$

Lemma 1.2.1

Number of points (other than P_i) that lie inside the box is at most 8



Proof: Suppose there are more than 8 points that lie inside the box apart from P_i. The box has a left square part and a right square part. So one of the squares contains at least 5 points. WLOG suppose the left square has at least 5 points. Divide each square into 4 parts by a middle vertical and a middle horizontal line. Now since there are 5 points there is one part which contains 2 points but that is not possible as those two points are in S_L and their distance will be less than δ_{min} which is not possible. Hence contradiction. Therefore there are at most 8 points inside the box. ■

Hence by the above lemma for each $P_i \in T_y$ there are at most 8 points in T_i. So for each $P_j \in T_i$ we find the $d(P_i, P_j)$ and if it is less than δ_{min} we update the points and the distance

1.2.4 Pseudocode and Time Complexity

Assumption. *We will assume for now that for all $P_i.P_j \in S$ we have $x_i \neq x_j$ and $y_i \neq y_j$. Later we will modify the pseudocode to remove this assumption*

Algorithm 3: FIND-CLOSEST(S)

Input: Set of n points, $S = \{(x_i, y_i) \mid x_i, y_i \in \mathbb{R},\ \forall\, i \in [n]\}$. We denote $P_i = (x_i, y_i)$.
Output: Closest pair of ponts, (P_i, P_j, δ) where $\delta = d(P_i, P_j)$

1 **begin**
2   **if** $|S| \leq 10$ **then**
3   Solve by Brute Force (Consider every pair of points)
4   $S^x \longleftarrow S$ sorted by $x-$coordinate, $\quad S^y \longleftarrow S$ sorted by $y-$coordinate
5   $\overline{x} \longleftarrow \lfloor \frac{n}{2} \rfloor$ highest $x-$coordinate, $\quad \overline{y} \longleftarrow \lfloor \frac{n}{2} \rfloor$ highest $y-$coordinate
6   $S^L \longleftarrow \{P_i \mid x_i < \bar{x},\ \forall\, i \in [n]\}, \quad S^R \longleftarrow \{P_i \mid x_i \geq \bar{x},\ \forall\, i \in [n]\}$
7   $(P_1^L, P_2^L, \delta^L) \longleftarrow$ FIND-CLOSEST(S^L), $\quad (P_1^R, P_2^R, \delta^R) \longleftarrow$ FIND-CLOSEST(S^R)
8   $\delta_{min} \longleftarrow \min\{\delta^L, \delta^R\}$
9   **if** $\delta_{min} < \delta^L$ **then**
10     $P_1 \longleftarrow P_1^R,\ P_2 \longleftarrow P_2^R$
11   **else**
12     $P_1 \longleftarrow P_1^L,\ P_2 \longleftarrow P_2^L$
13   $T \longleftarrow \{P_i \mid |x_i - \overline{x}| \leq \delta_{min}\}$
14   $T_y \longleftarrow T$ sorted by decreasing $y-$coordinate
15   **for** $P \in T_y$ **do**
16     $U \longleftarrow$ Next 8 points
17   **for** $\hat{P} \in U$ **do**
18   **if** $d(P, \hat{P}) < \delta_{min}$ **then**
19         $\delta_{min} \longleftarrow d(P, \hat{P})$
20         $(P_1, P_2) \longleftarrow (P, \hat{P})$
21   **return** (P_1, P_2, δ_{min})

Notice we used the assumption in the line 5 for finding the medians. So the line 4 takes $O(n \log n)$ times. Lines 5,6 takes $O(n)$ time. Since $\overline{x}$ is the median, we have $|S^L| = \lfloor \frac{n}{2} \rfloor$ and $|S^R| = \lceil \frac{n}{2} \rceil$. Hence FIND-CLOSEST($S^L$) and FIND-CLOSEST($S^R$) takes $T\left(\frac{n}{2}\right)$ time. Now lines $8 - 12$ takes constant time. Line 13 takes $O(n)$ time. And line 14 takes $O(n \log n)$ time. Since U has 8 points i.e. constant number of points the lines $16 - 20$ takes constant time for each $P \in T_y$. Hence the for loop at line 15 takes $O(n)$ time. Hence total time taken

$$T(n) = O(n) + O(n \log n) + 2T\left(\frac{n}{2}\right) \implies T(n) = O(n \log^2 n)$$

1.3 Improved Algorithm for $O(n \log n)$ Runtime

Notice once we sort the points by x−coordinate and y−coordinate we don't need to sort the points anymore. We can just pass the sorted array of points into the arguments for solving the smaller problems. Their is another time where we need to sort which is in line 14 of the above algorithm. This we can get actually from S^y without sorting just checking one by one backwards direction if the x−coordinate of the points satisfy $|x_i - \overline{x}| \leq \delta_{min}$. So

$$T_y = \textsc{Reverse}(\{P_i \in S^y \mid |x_i - \overline{x}| \leq \delta_{min}\})$$

So we form a new algorithm which takes the input S^x and S^y and then finds the closest pair of points. Then we will use that subroutine to find closest pair of points in any given set of points.

Algorithm 4: $\textsc{Find-Closest-Sorted}(S^x, S^y)$

Input: Set of n points, $S = \{(x_i, y_i) \mid x_i, y_i \in \mathbb{R}, \ \forall\, i \in [n]\}$. S^x and S^y are the sorted array of points with respect to x−coordinate and y−coordinate respectively
Output: Closest pair of ponts, (P_i, P_j, δ) where $\delta = d(P_i, P_j)$

- **begin**
  - **if** $|S| \leq 10$ **then**
    - Solve by Brute Force
  - $\overline{x} \longleftarrow \lfloor \frac{n}{2} \rfloor$ highest x−coordinate
  - $\overline{y} \longleftarrow \lfloor \frac{n}{2} \rfloor$ highest y−coordinate
  - $S^L \longleftarrow \{P_i \in S^x \mid x_i < \bar{x}, \ \forall\, i \in [n]\}$
  - $S^L_y \longleftarrow \{P_i \in S^y \mid x_i < \overline{x}\}$
  - $S^R \longleftarrow \{P_i \in S^x \mid x_i \geq \bar{x}, \ \forall\, i \in [n]\}$
  - $S^R_y \longleftarrow \{P_i \in S^y \mid x_i \geq \overline{x}\}$
  - $(P^L_1, P^L_2, \delta^L) \longleftarrow \textsc{Find-Closest-Sorted}(S^L, S^L_y)$
  - $(P^R_1, P^R_2, \delta^R) \longleftarrow \textsc{Find-Closest-Sorted}(S^R, S^R_y)$
  - $\delta_{min} \longleftarrow \min\{\delta^L, \delta^R\}$
  - **if** $\delta_{min} < \delta^L$ **then**
    - $P_1 \longleftarrow P^R_1$, $P_2 \longleftarrow P^R_2$
  - **else**
    - $P_1 \longleftarrow P^L_1$, $P_2 \longleftarrow P^L_2$
  - $T \longleftarrow \{P_i \mid |x_i - \overline{x}| \leq \delta_{min}\}$
  - $T_y \longleftarrow \textsc{Reverse}(\{P_i \in S^y \mid |x_i - \overline{x}| \leq \delta_{min}\})$
  - **for** $P \in T_y$ **do**
    - $U \longleftarrow$ Next 8 points
    - **for** $\hat{P} \in U$ **do**
      - **if** $d(P, \hat{P}) < \delta_{min}$ **then**
        - $\delta_{min} \longleftarrow d(P, \hat{P})$
        - $(P_1, P_2) \longleftarrow (P, \hat{P})$
  - **return** (P_1, P_2, δ_{min})

Algorithm 5: $\textsc{Find-Closest}(S)$

Input: Set of n points, $S = \{(x_i, y_i) \mid x_i, y_i \in \mathbb{R}, \ \forall\, i \in [n]\}$. We denote $P_i = (x_i, y_i)$.
Output: Closest pair of ponts, (P_i, P_j, δ) where $\delta = d(P_i, P_j)$

- **begin**
  - **if** $|S| \leq 10$ **then**
    - Solve by Brute Force
  - $S^x \longleftarrow S$ sorted by x−coordinate
  - $S^y \longleftarrow S$ sorted by y−coordinate
  - **return** *Find-Closest-Sorted*(S^x, S^y)

This algorithm only sorts one time. So time complexity for $\textsc{Find-Closest-Sorted}(S^x, S^y)$ is

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n) \implies T(n) = O(n \log n)$$

and therefore times complexity for $\textsc{Find-Closest}(S)$ is $O(n \log n)$.

1.4 Removing the Assumption

For this there nothing much to do. For finding the median $\overline{x}$ if we have more than one points with same $x-$coordinate which appears as the $\left\lfloor \frac{n}{2} \right\rfloor$ highest $x-$coordinate we sort only those points with respect to their $y-$coordinate update the S^x like that and then take $\left\lfloor \frac{n}{2} \right\rfloor$ highest point in S^x. We do the same for S^y and update accordingly. All this we do so that S^L and S^R has the size $\frac{n}{2}$.

Chapter 2

Median Finding in Linear Time

Median Find
Input: Set S of n distinct integers
Question: Find the $\lfloor \frac{n}{2} \rfloor^{th}$ smallest integer in S

2.1 Naive Algorithm

The naive algorithm for this will be to sort the array in $O(n \log n)$ time then return the $\lfloor \frac{n}{2} \rfloor^{th}$ element. This will take $O(n \log n)$ time. But in the next section we will show a linear time algorithm.

2.2 Linear Time Algorithm

In this section we will show an algorithm to find the median of a given set of distinct integers in $O(n)$ time complexity. Consider the following two problems:

Rank-Find (S, k)
Input: Set S of n distinct integers and an integer $k \leq n$
Question: Find the k^{th} smallest integer in S

Approximate-Split(S)
Input: Set S of n distinct integers
Question: Given S, return an integer $z \in S$ such that z where $rank(z) \in \left[\frac{n}{4}, \frac{3n}{4}\right]$

2.2.1 Solve Rank-Find using Approximate-Split

Algorithm 6: Rank-Find(S,k)

```
Input: Set S of n distinct integer and k ∈ [n]
Output: k^th smallest integer in S
begin
    if |S| ≤ 100 then
        Sort S, return k^th smallest element in S
    z ⟵ Approximate-Split(S)        (z is the r^th smallest element for some r ∈ [n/4, 3n/4])
    S_L ⟵ {x ∈ S | x ≤ z}, S_R ⟵ {x ∈ S | x > z}
    if k ≤ |S_L| then
        return Rank-Find(S_L, k)
    return Rank-Find(S_R, k − |S_L|)
```

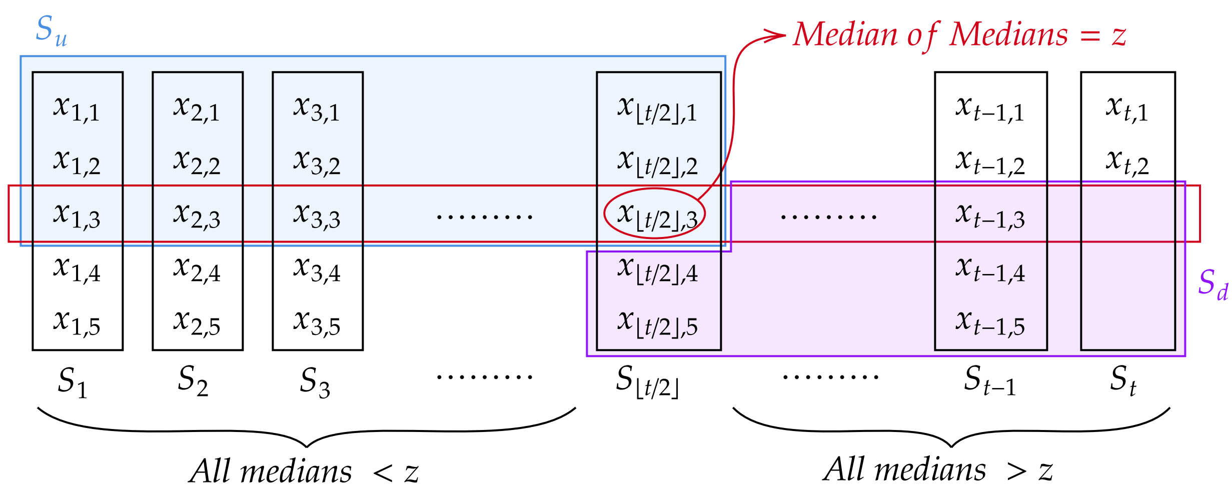
Certainly if we can solve RANK-FIND(S, k) for all $k \in [n]$ we can also solve MEDIAN-FIND. We will try to use both the problems and recurse to solve RANK-FIND in linear time.

In the above algorithm $rank(z) \in \left[\frac{n}{4}, \frac{3n}{4}\right]$. So $\frac{n}{4} \leq |S_L|, |S_R| \leq \frac{3n}{4}$. For now suppose RANK-FIND(S, k) takes $T_{RF}(n)$ time and APPROXIMATE-SPLIT(S) takes $T_{AS}(n)$ time. Then the time taken by the algorithm is

$$T_{RF}(n) \leq O(n) + T_{AS}(n) + T_{RF}\left(\frac{3n}{4}\right)$$

2.2.2 Solve APPROXIMATE-SPLIT using RANK-FIND

We first divide S into groups of 5 elements. So take $t = \left\lceil \frac{n}{5} \right\rceil$. Now we sort each group. Since each group have constant size this can be done in $O(n)$ time. So now consider the scenario:



After sorting each of the groups we takes the medians of each group. Let z be the median of the medians. We claim that $rank(z) \in \left[\frac{n}{4}, \frac{3n}{4}\right]$.

Algorithm 7: APPROXIMATE-SPLIT(S)

```
Input: Set S of n distinct integers
Output: An integer z ∈ S such that z where rank(z) ∈ [n/4, 3n/4]
begin
    if |S| ≤ 100 then
        Sort, return Exact median
    t ⟵ ⌈n/5⌉
    S_i ⟵ i^th block of 5 elements in S for i ∈ [t − 1]
    S_t ⟵ Whatever is left in S
    for i ∈ [t] do
        Sort S_i, Let h_i be the median of S_i
    T ⟵ {h_i | i ∈ [t]}
    return RANK-FIND(T, ⌊t/2⌋)
```

So in the picture among elements in upper left the highest element is z and among the elements in lower right the lowest element is z. We will show that the number of elements smaller than z is between $\frac{n}{4}$ and $\frac{3n}{4}$. Lets call the set of elements in upper left box is S_u and the set of elements in lower right box is S_d.

Lemma 2.2.1

$|S_u|, |S_d| \geq \frac{n}{4}$

Proof: $|S_u| \geq 3 \times \left\lfloor \frac{t}{2} \right\rfloor$. For $n \geq 100$, $3\left\lfloor \frac{t}{2} \right\rfloor > \frac{n}{4}$. Hence $|S_u| \geq \frac{n}{4}$. Now similarly $|S_d| \geq 3\left\lfloor \frac{t}{2} - 1 \right\rfloor \geq \frac{n}{4}$. ■

Lemma 2.2.2

Number of elements in S smaller than z lies between $\frac{n}{4}$ and $\frac{3n}{4}$.

Proof: Now number of elements in S smaller than $z \geq |S_u| \geq \frac{n}{4}$. The number of elements greater than $z \geq |S_d| \geq \frac{n}{4}$. So number of elements in S smaller than $z \leq n-$ number of elements greater than $z \leq n - \frac{n}{4} = \frac{3n}{4}$. ■

Hence the Approximate-Split(S) takes time

$$T_{AS}(n) = O(n) + T_{RF}\left(\frac{n}{5}\right)$$

2.2.3 Pseudocode and Time Complexity

Hence using Approximate-Split the final algorithm for Rank-Find is the following:

Algorithm 8: Rank-Find(S,k)

Input: Set S of n distinct integer and $k \in [n]$
Output: k^{th} smallest integer in S
begin
  if $|S| \leq 100$ **then**
    Sort S, **return** k^{th} *smallest element in* S
  $t \longleftarrow \left\lceil \frac{n}{5} \right\rceil$
  $S_i \longleftarrow i^{th}$ block of 5 elements in S for $i \in [t-1]$
  $S_t \longleftarrow$ Whatever is left in S
  for $i \in [t]$ **do**
    Sort S_i, Let h_i be the median of S_i
  $T \longleftarrow \{h_i \mid i \in [t]\}$
  $z \longleftarrow$ Rank-Find $\left(T, \left\lfloor \frac{t}{2} \right\rfloor\right)$
  $S_L \longleftarrow \{x \in S \mid x \leq z\}$, $S_R \longleftarrow \{x \in S \mid x > z\}$
  if $k \leq |S_L|$ **then**
    return *Rank-Find*(S_L, k)
  return *Rank-Find*$(S_R, k - |S_L|)$

Replacing $T_{AS}(n)$ in the time complexity equation of $T_{RF}(n)$ we get the equation:

$$T_{RF}(n) \leq O(n) + T_{RF}\left(\frac{n}{5}\right) + T_{RF}\left(\frac{3n}{4}\right)$$

Let $T_{RF}(n) \leq kn + +T_{RF}\left(\frac{n}{5}\right) + T_{RF}\left(\frac{3n}{4}\right)$. We claim that $T_{RF}(n) \leq cn$ for some $c \in \mathbb{N}$ for all $n \geq n_0$ where $n_0 \in \mathbb{N}$. By induction we have

$$T_{RF}(n) \leq kn + \frac{cn}{5} + \frac{3cn}{4} = \left(k + \frac{19c}{20}\right)n$$

To have $k + \frac{19c}{20} \leq c$ we have to have $k + \frac{19c}{20} \leq c \iff c \geq 20k$. So take $c \geq 20k$ and our claim follows. Hence $T_{RF}(n) = O(n)$. Since we can find any k^{th} smallest number in a given set of distinct integers in linear time we can also find the median in linear time.

Chapter 3

Polynomial Multiplication

Polynomial Multiplication

Input: Given 2 univariate polynomials of degree $n-1$ by 2 arrays of their coefficients $(a_0, \dots, a_{n-1})$ and $(b_0, \dots, b_{n-1})$ such that $A(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ and $B(x) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}$ respectively

Question: Given 2 polynomials of degree $n-1$ find their product polynomial $C(x) = A(x)B(x)$ of degree $2n-2$ by returning the array of their coefficients.

3.1 Naive Algorithm

We can do this naively by calculating each coefficient of C in $O(n)$ time since for any $i \in \{0, \dots, 2n-2\}$

$$c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

Since there are $2n-1 = O(n)$ total coefficient of C it takes total $O(n^2)$ time. In the following section we will do this in $O(n \log n)$ time.

3.2 Strassen-Schönhage Algorithm

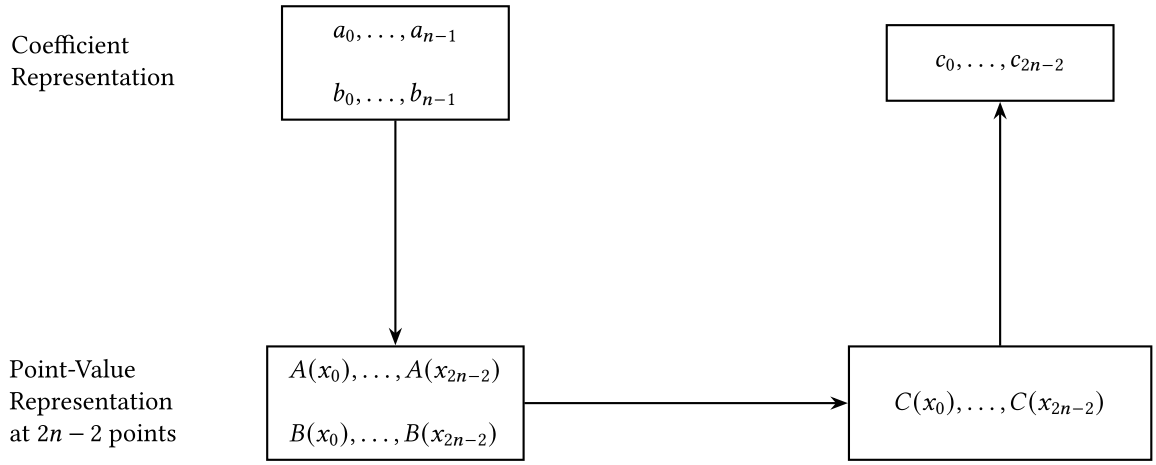
Before diving into the algorithm first let's consider how many ways we can represent a polynomial. Often changing the representation helps solving the problem in less time.

- **Coefficients**: We can represent a polynomial by giving the array of all its coefficient.
- **Point-Value Pairs**: We can evaluate the polynomial in distinct n points and give all the point-value pairs. This also uniquely represents a polynomial since there is exactly one polynomial of degree $n-1$ which passes through all these points.

Theorem 3.2.1

Given n distinct points $(x_0, y_0), \dots, (x_{n-1}, y_{n-1})$ in $\mathbb{R}^2$ there is an unique $(n-1)$–degree polynomial $P(x)$ such that $P(x_i) = y_i$ for all $i \in [\![n-1]\!]$

Since we want to find the polynomial $C(x) = A(x)B(x)$ and $C(x)$ has degree $2n-2$, we will evaluate the polynomials $A(x)$ and $B(x)$ in $2n-1$ distinct points. So we will have the algorithm like this:



3.2.1 Finding Evaluations of Multiplied Polynomial

Suppose we were given $A(x)$ and $B(x)$ evaluated at $2n-1$ distinct points $x_0, \ldots, x_{2n-2}$. Then we can get $C(x)$ evaluated at $x_0, \ldots, x_{2n-2}$ by

$$C(x_i) = A(x_i)B(x_i)\ \forall\ i \in [\![2n-2]\!]$$

Since there are $O(n)$ many points and for each point it takes constant time to multiply we can find evaluations of C at $x_0, \ldots, x_{2n-2}$ in $O(n)$ time.

3.2.2 Evaluation of a Polynomial at Points

Question 3.1

Suppose there is only one point, x_0. Can we evaluate a $n-1$ degree polynomial $A(x) = \sum\limits_{i=0}^{n-1} a_i x^i$ at x_0 efficiently?

We can rewrite $A(x)$ as

$$A(x) = a_0 + x(a_1 + x(a_2 + x(a_3 + \cdots (a_{n-1} + x(a_n))\cdots)))$$

In this represent it is clear that we have to do n additions and n multiplications to find $A(x_0)$. Hence we can evaluate a $n-1$ degree polynomial at a point in $O(n)$ time

But we have $O(n)$ points. And if each point takes $O(n)$ time to find the evaluation of the polynomial then again it will take total $O(n^2)$ time. We are back to square one. So instead we will evaluate the polynomial in some special points and we will evaluate in all of them in $O(n \log n)$ time. So now the problem we will discuss now is to find some special n points where we can evaluate a $n-1$-degree polynomial in $O(n \log n)$ time.

Idea: Evaluate at roots of unity and use Fast Fourier Transform

Assume n is a power of 2. NWe have the polynomial $A(x) = \sum\limits_{i=0}^{n-1} a_i x^i$. So now consider the following two polynomials

$$A^0(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{\frac{n}{2}-1} \qquad A^1(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{\frac{n}{2}-1}$$

Certainly we have

$$A(x) = A^0(x^2) + xA^1(x^2)$$

Hence we can get $A(1)$ and $A(-1)$ by

$$A(1) = A^0(1) + A^1(1) \qquad A(-1) = A^0(1) - A^1(1)$$

Hence like this by evaluating two $\frac{n}{2}-1$ degree polynomials at one point we get evaluation of A at two points. More generally for any $y \geq 0$ we have

$$A(\sqrt{y}) = A^0(y) + \sqrt{y}A^1(y) \qquad A(-\sqrt{y}) = A^0(y) - \sqrt{y}A^1(y)$$

So by recursing like this evaluating at $1, -1$ we can get evaluations of A at n^{th} roots of unity.

Let

$$\omega_n^k = n^{th} \text{ root of unity for } k \in [\![n-1]\!] = e^{i\frac{k}{n}2\pi} = \cos\left(\frac{k}{n}2\pi\right) + i \sin s\left(\frac{k}{n}2\pi\right)$$

Hence we have

$$A\left(\omega_n^k\right) = A^0\left(\omega_n^{2k}\right) + \omega_n^k A^1\left(\omega_n^{2k}\right) = A^0\left(\omega_{\frac{n}{2}}^k\right) + \omega_n^k A^1\left(\omega_{\frac{n}{2}}^k\right)$$
$$A\left(-\omega_n^k\right) = A\left(\omega_n^{\frac{n}{2}+k}\right) = A^0\left(\omega_n^{2k}\right) - \omega_n^k A^1\left(\omega_n^{2k}\right) = A^0\left(\omega_{\frac{n}{2}}^k\right) - \omega_n^k A^1\left(\omega_{\frac{n}{2}}^k\right)$$

Hence now we will solve the following problem:

RECURSIVE-DFT

Input: $(a_0, \dots, a_{n-1})$ representing $(n-1)-$degree polynomial $A(x) = \sum_{i=0}^{n-1} a_i x^i$

Question: Find the evaluations of the polynomial $A(x)$ in all n^{th} roots of unity

Since A^0 and A^1 have degree $\frac{n}{2} - 1$ we can use recursion. Hence the algorithm is

Algorithm 9: RECURSIVE-DFT(A)

Input: $A = (a_0, \dots, a_{n-1})$ such that $A(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$
Output: $A(x)$ evaluated at n^{th} roots of unity ω_n^k for all $k \in [\![n-1]\!]$

begin
  if $n == 1$ **then**
    return $A[0]$
  $A^0 \longleftarrow (A[0], A[2], \dots, A[n-2])$
  $A^1 \longleftarrow (A[1], A[3], \dots, A[n-1])$
  $Y^0 \longleftarrow$ RECURSIVE-DFT(A^0)
  $Y^1 \longleftarrow$ RECURSIVE-DFT(A^1)
  for $k = 0$ *to* $\frac{n}{2} - 1$ **do**
    $Y[k] \longleftarrow Y^0[k] + \omega_n^k Y^1[k]$ // $A\left(\omega_n^k\right) = A^0\left(\omega_{\frac{n}{2}}^k\right) + \omega_n^k A^1\left(\omega_{\frac{n}{2}}^k\right)$
    $Y\left[k + \frac{n}{2}\right] \longleftarrow Y^0[k] - \omega_n^{\frac{n}{2}+k} Y^1[k]$ // $A\left(-\omega_n^k\right) = A^0\left(\omega_{\frac{n}{2}}^k\right) - \omega_n^k A^1\left(\omega_{\frac{n}{2}}^k\right)$
  return Y

Time Complexity: $T(n) = 2T\left(\frac{n}{2}\right) + O(n) = O(n \log n)$.

Therefore we can evaluate a $n-1$ degree polynomial in all the n^{th} roots of unity in $O(n \log n)$ time. Hence with this algorithm we will get evaluations of the polynomial $C(x) = A(x)B(x)$ in all the $2n^{th}$ roots of unity. Now we need to interpolate the polynomial $C(x)$ from its evaluations. We will describe the process in the next subsection.

3.2.3 Interpolation from Evaluations at Roots of Unity

In this section we will show how to interpolate a $n-1$ degree polynomial from evaluations at all n^{th} roots of unity. Previously we had

$$\underbrace{\begin{bmatrix} C\left(\omega_n^0\right) \\ C\left(\omega_n^1\right) \\ C\left(\omega_n^2\right) \\ \vdots \\ C\left(\omega_n^{n-1}\right) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} 1 & \omega_n^0 & \omega^{0\cdot 2} & \cdots & \omega^{0\cdot(n-1)} \\ 1 & \omega_n^1 & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot(n-1)} \\ 1 & \omega_n^2 & \omega^{2\cdot 2} & \cdots & \omega^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega^{(n-1)\cdot 2} & \cdots & \omega^{(n-1)\cdot(n-1)} \end{bmatrix}}_{V = \text{Vandermonde Matrix}} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}}_{C}$$

Now vandermonde matrix is invertible since all the n^{th} roots are distinct. Therefore $C = V^{-1}Y$. But we can not do a matrix inversion to interpolate the polynomial because that will take $O(n^2)$ time. Instead we have this beautiful result:

Lemma 3.2.2
$\left(V^{-1}\right)_{j,k} = \frac{1}{n}\omega_n^{-jk}$ for all $0 \leq j, k \leq n-1$

Proof: Consider the matrix $n \times n$ matrix T such that $(T)_{j,k} = \frac{1}{n}\omega_n^{-jk}$. Now we will show $VT = I$ This will confirm that $V^{-1} = T$. Now

$$\sum_{k=0}^{n-1}(V)_{i,j}\,(T)_{j,k} = \sum_{k=0}^{n-1}\omega_n^{ij} \times \frac{1}{n}\omega_n^{-jk} = \frac{1}{n}\sum_{k=0}^{n-1}\left(\omega_n^{i-k}\right)^j = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} 1 = 1 & \text{when } i = k \\ \dfrac{1}{n}\dfrac{1-\omega_n^n}{1-\omega} = 0 & \text{when } i \neq k \end{cases}$$

Hence in VT there are $1'$s on the diagonal and rest of the locations are 0. Hence $VT = I$. So $V^{-1} = T$. ■

Hence we can see the inverse of the vandermonde matrix is also a vandermode matrix with a scaling factor. We will denote $y_i = C\left(\omega_n^i\right)$ for $i \in [\![n-1]\!]$ since these values are given to us some how and we have to find the corresponding polynomial. Therefore we have

$$\underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}}_{C} = \frac{1}{n}\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^{-1} & \omega^{-1\cdot 2} & \cdots & \omega^{-1\cdot(n-1)} \\ 1 & \omega_n^{-2} & \omega^{-2\cdot 2} & \cdots & \omega^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega^{-(n-1)\cdot 2} & \cdots & \omega^{-(n-1)\cdot(n-1)} \end{bmatrix}}_{V^{-1}} \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}}_{Y}$$

Observation. $nc_j = y_0 + y_1\omega_n^{-j} + y_2\omega_n^{-2j} + \cdots + y_{n-1}\omega_n^{-(n-1)j}$ *for all* $j \in [\![n-1]\!]$.

We can also see this situation as we have the polynomial $Y(x) = y_0 + y_1x + y_2x^2 + \cdots + y_{n-1}x^{n-1}$ and c_j is just $Y(x)$ evaluated as $\omega_n^{-j} = \omega_n^{n-j}$ multiplied by n. Hence we just reindex the n^{th} roots of unity and evaluate Y n^{th} roots of unity in $O(n \log n)$ time using the algorithm described in subsection 3.2.2

Chapter 4

Dynamic Programming

Definition 4.1: Dynamic Programming

Dynamic Programming has 3 components:

1. Optimal Substructure: Reduce problem to smaller independent problems
2. Recursion: Use recursion to solve the problems by solving smaller independent problems
3. Table Filling: Use a table to store the result to solved smaller independent problems.

4.1 Longest Increasing Subsequence

Longest Increasing Subsequence
Input: Sequence of distinct integers $A = (a_1, \dots, a_n)$
Question: Given an array of distinct integers find the longest increasing subsequence i.e. return maximum size set $S \subseteq [n]$ such that $\forall\, i, j \in S, i < j \implies a_i < a_j$

4.1.1 $O(n^2)$ Time Algorithm

Given $A = (a_1, \dots, a_n)$ first we will create a n-length array where i^{th} entry stores the length and longest increasing subsequence ending at a_i. Certainly we have the following recursion relation

$$\text{LIS}(k) = 1 + \max_{j<k,\ a_j<a_k} \{\text{LIS}(j)\}$$

since if a subsequence $S \subseteq [n]$ is the longest increasing subsequence ending at a_k then certainly $S - \{k\}$ is the longest increasing subsequence which ends at $a_j < a_k$ for some $j < k$. Hence in the table we start with 1st position and using the recursion relation we fill the table from left. And after the table is filled we look for which entry of the table has maximum length. So the algorithm will be following:

Algorithm 10: LIS(A)

Input: Sequence of distinct integers $A = (a_1, \dots, a_n)$
Output: Maximum size set $S \subseteq [n]$ such that $\forall\, i, j \in S, i < j \implies a_i < a_j$.

```
begin
    Create an array T of length n
    for i ∈ [n] do
        T[i][1] ⟵ 1 + max{T[j][1] : j < k, a_j < a_k}        // Finds LIS[i]
        T[i][2] ⟵ T[T[i][1] − 1][2]
    Index ⟵ max{T[j][1] : j ∈ [n]}
    return T[Index]
```

Time Complexity: For each iteration of the loop it takes $O(n)$ time to find LIS$[i]$. Hence the time complexity of this algorithm is $O(n^2)$.

4.1.2 $O(n \log n)$ Time Algorithm

In the following algorithm we update the longest increasing sequence every time we see a new element of the given sequence. At any time we keep the best available sequence.

Idea. *We can make an increasing subsequence longer by picking the smallest number for position k so that there is an increasing subsequence of length k. Doing this we can maximize the length of the subsequence.*

> **Theorem 4.1.1**
>
> Is $S \subseteq A$ is the longest increasing subsequence of length t then for any $k \in [t]$ the number $S(k)$ is the smallest number in subarray of A starting at first and ending at $S(k)$ such that there is an increasing subsequence of length k ending at $S(k)$.

Proof: Suppose $\exists\, k \in [t]$ such that k is the smallest number in $[t]$ such that $S(k)$ is not the smallest number to satisfy the condition. Now denote the subarray of A starting at first and ending at $S(k)$ by A_k. Now let $x \in A_k$ be the smallest number in A_k such that there is an increasing subsequence of length k ending at x. Certainly $x < S(k)$. Now since k is the smallest index which does not satisfy the given condition, $\forall\, j \in [k-1]$, $S(j)$ is the smallest number in A_j such that there is an increasing subsequence of length j ending at $S(j)$. Then consider the subsequence $\{S(1), \dots, S(k-1), x, S(k), S(k+1), \dots, S(t)\}$. This is an increasing subsequence of A and has length $t+1$. But this contradicts the minimality of S. Hence contraditon. Every element of S follows the given condition. ■

So we will construct an increasing subsequence by gradually where each step this property is followed, i.e. at each step we will ensure that the sequence built at some time have the above propert. So now we describe the algorithm.

Algorithm 11: QUICKLIS(A)

Input: Sequence of distinct integers $A = (a_1, \dots, a_n)$
Output: Maximum size set $S \subseteq [n]$ such that $\forall\, i, j \in S, i < j \implies a_i < a_j$.

1 **begin**
2 Create an array T of length n with all entries 0
3 Create an array M of length n
4 **for** $i = 1, \dots, n$ **do**
5 $M[i] \longleftarrow \infty$
6 **for** $i = 1, \dots, n$ **do**
7 $k \longleftarrow$Find smallest index i such that $M[k] > a_i$ using BINARY-SEARCH
8 $M[k] \longleftarrow i$
9 $T[i] \longleftarrow M[k-1]$ `// Pointer to the previous element of the sequence`
10 $l \longleftarrow$ Largest l such that $M[l]$ is finite
11 Create an array S of length l
12 **for** $i = l, \dots, 1$ **do**
13 **if** $i = l$ **then**
14 $S[l] \longleftarrow M[l]$
15 Continue
16 $S[i] \longleftarrow T\big[S[i+1]\big]$ `//` $T[S[i+1]]$ `is pointer to previous value of sequence`
17 **return** (l, S)

Time Complexity: To create the arrays and the first for loop takes $O(n)$ time. In each iteration of the for loop at line 6 it takes $O(\log n)$ time to find k and rest of the operations in the loop takes constant time. So the for loop takes $O(n \log n)$ time. Then To find l and creating S it takes $O(n)$ time. Then in the for loop at line 12 in each iteration it takes constant time. So the for loop at line 12 takes in total $O(n)$ time. Therefore the algorithm takes $O(n \log n)$ time.

We will do the proof of correctness of the algorithm now.

Lemma 4.1.2

For any index $M[k]$ is non increasing

Proof: Every time we change a value of $M[k]$ we replace by something smaller. So $M[k]$ is non increasing. ■

We denote the state of array M at i^{th} iteration by M^i. Then we have the following lemma:

Lemma 4.1.3

At any time i, $M^i[1] \leq M^i[2] \leq \cdots \leq M^i[n]$

Proof: We will prove this by induction on i. The base case follows naturally. Now for i^{th} iteration suppose $M^i[k]$ is replaced by x_i. Then we know $\forall\ j < k$ we have $M^i[j] \leq x_i$. By inductive hypothesis at time $t-1$ we have M as an increasing sequence. Now before replacing $M^i[k] \leq M^i[k+1] \leq \cdots M^i[n]$. Now by Lemma 4.1.2 $M^i[k]$ is nonincreasing. So So we still have $M^i[1] \leq \cdots M^i[k-1] \leq x_i \leq M^i[k+1] \leq \cdots \leq M^i[n]$. Hence bt mathematical induction it holds. ■

Now suppose at i^{th} iteration k_i is largest such that $M^i[k_i] < \infty$. Then S^i denote the set constructed like the way we constructed at line 12–16 in the algorithm i.e.

$$S^i[k_i] = M^i[k_i] \qquad \text{and} \qquad S^i[j] = T[S^i[j+1]] \quad \forall\ j \in [k_i - 1]$$

Lemma 4.1.4

After any i^{th} iteration, for $k \in [n]$ if $M^i[k] < \infty$ then $S^i[k]$ stores the smallest value in $x_1, \ldots, x_i$ such that there is an increasing subsequence of size k that ends in $S^i[k]$.

Proof: We will induction on i. Base case: This is true after first iteration since only $M^1[1] < \infty$. So this naturally follows.

Suppose this is true after i iterations. Now at $(i+1)^{th}$ iteration suppose t be the smallest index such that $M^i[t] > x_{i+1}$. Then we have

$$M^i[1], \ldots, M^i[t-1] < x_{i+1} < M^i[k], \ldots, M^i[n] \implies S^i[1], \ldots, S^i[t-1] < x_{i+1} < S^i[k], \ldots, S^i[k_i]$$

Now for $k \leq t-1$ it is true by the inductive hypothesis. For $k > t$ and if $M^{i+1}[k] < \infty$ then $S^{i+1}[k]$ is the smallest value in $x_1, \ldots, x_{i+1}$ such that there is an increasing subsequence of size k that ends in $S^{i+1}[k]$ since this was true for i^{th} iteration.

Now only the case when $k = t$ is remaining. If $S^{i+1}[k]$ is not the smallest value in $x_1, \ldots, x_{i+1}$ to have an increasing subsequence of size k ending at $S^{i+1}[k]$ then let x_j was the smallest value to satisfy this condition where $j < i+1$. Then naturally $x_j < x_{i+1}$. Then $M^i[t] \leq x_j < x_{i+1}$. But we t was the smallest number such that $M^i[t] > x_{i+1}$. Hence contradiction. Therefore $S^i[k]$ is the smallest value in $x_1, \ldots, x_{i+1}$ to have an increasing subsequence of size k ending at $S^{i+1}[k]$.

Therefore by mathematical induction this is true for all iterations. ■

Theorem 4.1.5

S is the longest increasing subsequence of A.

Proof: After the n^{th} iteration $S^n = S$ and $k_n = l$. Hence by Lemma 4.1.4 we can say for all $k \in [l]$, $S[k]$ is the smallest number such that there is an increasing sequence of length k ending at $S[k]$. Now we want to show that this increasing sequence is the longest increasing subsequence of A. Suppose S is not the longest increasing subsequence. Let T be the longest increasing subsequence of length t. Then suppose $j \leq l$ be the smallest index such that $S[j] \neq T[j]$. Now $S[j]$ is the smallest number in $x_1, \ldots, x_n$ such that there is an increasing subsequence of length j ending at $S[j]$. Hence we have $S[j] < T[j]$. Now for all $i < j$ we have $S[i] = T[i]$. Then we form this new subsequence $\hat{T} = \{T[1], T[2], \ldots, T[j-1], S[j], T[j], \ldots, T[t]\}$. Certainly $\hat{T}$ has length $t+1$ and it is also an increasing subsequence. But this contradicts the maximality condition of T. Hence S is indeed the longest increasing subsequence. ■

4.2 Optimal Binary Search Tree

Optimal BST

Input: A sorted array $A = (a_1, \dots, a_n)$ of search keys and an array of their probability distributions $P = (p(a_1), \dots, p(a_n))$

Question: Given array of keys A and their probabilities the probability of accessing a_i is $p(a_i)$ then return a binary tree with the minimum cost where for any binary tree T, $\textsc{Cost}(T) = \sum_{i=1}^{n} p(a_i) \cdot height_T(a_i)$.

So let T be the optimal binary search tree with a_k as its root for some $k \in [n]$. Let T_l and T_r denote the tree rooted at the left child and right child of a_k in T respectively. Then:

$$\textsc{Cost}(T) = p_k + \sum_{i<k} p_i \left(1 + height_{T_l}(a_i)\right) + \sum_{i>k} p_i \left(1 + height_{T_r}(a_i)\right) = \sum_{i=1}^{n} p_i + \underbrace{\sum_{i<k} p_i \cdot height_{T_l}(a_i)}_{\textsc{Cost}(T_l)} + \underbrace{\sum_{i>k} p_i \cdot height_{T_l}(a_i)}_{\textsc{Cost}(T_r)}$$

We will use the use of notion in general $\text{OPT}\textsc{Cost}(i,k) = \textsc{Cost}(T_i^k)$ where T_i^k is the optimal binary tree of the subarray $A[i \dots k]$ for any $i \leq k \leq n$. Therefore we arrive at the following recurrence relation

$$\text{OPT}\textsc{Cost}(i,k) = \begin{cases} 0 & \text{when } i > k \\ \sum_{j=i}^{k} p(a_j) + \min_{i \leq r \leq k} \{\text{OPT}\textsc{Cost}(i, r-1) + \text{OPT}\textsc{Cost}(r+1, k)\} & \text{otherwise} \end{cases}$$

So the algorithm for constructing the optimal binary search tree is following:

Algorithm 12: $\textsc{OptimalBST}(A, P)$

Input: A sorted array $A = (a_1, \dots, a_n)$ of search keys and an array of their probability distributions $P = (p(a_1), \dots, p(a_n))$

Output: Binary Tree T with the minimum search cost, $\textsc{Cost}(T) = \sum_{i=1}^{n} p(a_i) \cdot height_T(a_i)$

```
1  begin
2      for i = 1, ..., n do
3          OPTCost[i, i] ⟵ (p(a_i), a_i), OPTCost[0, i] ⟵ (0, None)
4      for d = 2, ..., n do
5          for i ∈ [n + 1 − d] do
6              minval ⟵ 0
7              for k = i + 1, ..., i + d − 2 do
8                  newval ⟵ OPTCost[i, k − 1][1] + OPTCost[k + 1, i + d − 1][1]
9                  if minval > newval then
10                     minval ⟵ newval
11                     Index ⟵ k
12             OPTCost[i, i + d − 1] ⟵ (minval + Σ_{k=1}^{i+d−1} p(a_k), k)
13             a_k.left ⟵ OPTCost[i, k − 1][2]
14             a_k.right ⟵ OPTCost[k + 1, i + d − 1][2]
15     return OPTCost[1, n]
```

Time Complexity: To two for loops ar line 4 and line 5 takes $O(n^2)$ many iterations. Now the inner most for loop at line 7 runs $O(n)$ iterations where in each iterations it takes constant runtime. So the total running time of the algorithm is $O(n^3)$.

CHAPTER 5

Greedy Algorithm

5.1 Maximal Matching

MAXIMAL MATCHING
Input: Graph $G = (V, E)$
Question: Find a maximal matching $M \subseteq E$ of G

Before diving into the algorithm to find a matching or maximal matching we first define what is a matching.

Definition 5.1.1: Matching

Given a graph $G = (V, E)$, $M \subseteq E$ is said to be a matching if M is an independent set of edges i.e. no two edges of M are incident on same vertex.

Definition 5.1.2: Maximal Matching

For a graph $G = (V, E)$ a matching $M \subseteq E$ is maximal if it cannot be extended and still by adding an edge.

There is also a maximum matching which can be easily understood from the name:

Definition 5.1.3: Maximum Matching

For a graph $G = (V, E)$ a matching $M \subseteq E$ is maximum if it is maximal and has the maximum size among all the maximal matchings.

Idea. *The idea is to create a maximal matching we will just go over each edge one by one and check if after adding them to the set M the matching property still holds.*

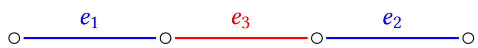
Algorithm 13: MAXIMAL-MATCHING

Input: Graph $G = (V, E)$
Output: Maximal Matching $M \subseteq E$ of G
begin
    $M \longleftarrow \emptyset$
    Order the edges $E = \{e_1, \ldots, e_k\}$ arbitrarily
    for $e \in E$ **do**
        if $M \cup \{u\}$ *is matching* **then**
            $M \longleftarrow M \cup \{e\}$
    return M

Question 5.1

Do we always get the largest possible matching?

Solution: Clearly algorithm output is not optimal always. We get a maximal matching sure. But we don't get a maximum matching always. For example the following graph



If we start from e_1 we get the matching $\{e_1.e_2\}$ which is maximum matching but if we start from e_3 then we get only the maximal matching $\{e_3\}$ which is not maximum. ■

Since the algorithm output may not be optimal always we can ask the following question

Question 5.2

How large is the matching obtained compared to the maximum matching?

This brings us to the following result:

Theorem 5.1.1

For any graph G let the greedy algorithm obtains the matching M and the maximum matching is $M^\star$. Then

$$|M| \geq \frac{1}{2}|M^\star|$$

Proof: Consider an edge $e \in M^\star$ but $e \notin M$. Since e wasn't picked in M, $\exists\, e' \in M \setminus M^\star$ such that e and e' are incident on same vertex. Thus define the function $f : M^\star \longrightarrow M$ where

$$f(e) = \begin{cases} e & \text{when } e \in M \\ e' & \text{when } e \in M^\star \setminus M \text{ where } e' \in M \setminus M^\star \text{ such that } e' \cap e \neq \emptyset \end{cases}$$

Now note that there are at most two edges in $M^\star$ that are adjacent to an edge $e' \in M$ which will be mapped to e'. Hence

$$|M \setminus M^\star| \geq \frac{1}{2}|M^\star \setminus M|$$

Therefore $|f^{-1}(e')| \leq 2\ \forall\, e' \in M$. Hence

$$|M^\star| = |M \cap M^\star| + |M^\star \setminus M| \leq |M \cap M^\star| + 2|M \setminus M^\star| \leq 2|M|$$

Therefore we have the result $|M| \geq \frac{1}{2}|M^\star|$. ■

Alternate Proof : Let M_1 and M_2 are two matchings. Consider the symmetric difference $M_1 \triangle M_2$. This consists of edges that are in exactly one of M_1 and M_2. Now in $M \triangle M^\star$ we have the following properties:

(a) Every vertex in $M \triangle M^\star$ has degree $\leq 2 \implies$ Each component is a path or an even cycle.

(b) The edges of M and $M^\star$ alternate.

Now we will prove the following property about the connected components of $M \triangle M^\star$.

Claim: No connected component is a single edge.
Proof: This is because let e be a connected component. So the two edges e_1, e_2 which are adjacent to e, they are either in both M and $M^\star$ or not in M and $M^\star$. The former case is not possible because then e_1, e_2, e are all in either M or $M^\star$ which is not possible as they do not satisfy the condition of matching. For the later case since $M^\star$ is maximal matching, $e \in M^\star$. Then $e \notin M$. That means $e, e_1, e_2 \notin M$ which is not possible since M is also a maximal matching. Therefore no connected component is a single edge. ■

Therefore every path has length ≥ 2. Therefore ratio of # edges of M to # edges of $M^\star$ in a path is ≤ 2. And for cycles we have # edges of M = # edges of $M^\star$. So in every connected component C of $M \triangle M^\star$ the ratio $\frac{|M^\star \cap C|}{|M \cap C|} \leq 2$. Therefore we have

$$\frac{|M^\star|}{|M|} = \frac{|M \cap M^\star| + \sum\limits_C |M^\star \cap C|}{|M \cap M^\star| + \sum\limits_C |M \cap C|} \leq 2$$

Hence we have $|M| \geq \frac{1}{2}|M^\star|$.

■

5.2 Huffman Encoding

HUFFMAN CODING

Input: n symbols $A = (a-1, \ldots, a_n)$ and their frequencies $P = (f_1, \ldots, f_n)$ of using symbols

Question: Create a binary encoding such that:

- Prefix Free: The code for one word can not be prefix for another code
- Minimality: Minimize $\textsc{Cost}(b) = \sum\limits_{i=1}^{n} f_i \cdot \textsc{Len}(b(a_i))$ where $b : A \to \{0,1\}^*$ is the binary encoding

Assignment of binary strings can also be scene as placing the symbols in a binary tree where at any node 0 means left child and 1 means right child. Then the first condition implies that there can not be two codes which lies in the same path from the root to a leaf. I.e. it means that all the codes have to be in the leaves. Then the length of the binary coding for a symbol is the height of the symbol in the binary tree.

We can think the frequencies as the probability of appearing for a letter. We denote the probability of appearing of the letter a_i by $p(a_i) := \frac{f_i}{\sum\limits_{i=1}^{n} f_i}$. So the we can see the updated cost function

$$\textsc{Cost}(b) = \sum_{i=1}^{n} p(a_i) \cdot \textsc{Len}(b(a_i))$$

And from now on we will see the frequencies as probabilities and cost function like this

5.2.1 Optimal Binary Encoding Tree Properties

Then our goal is to finding a binary tree with minimum cost where all the symbols are at the leaves. We have the following which establish the optimality of Huffman encoding over all prefix encodings where each symbol is assigned a unique string of bits.

Lemma 5.2.1

In the optimal encoding tree least frequent element has maximum height.

Proof: Suppose that is not the case. Let T be the optimal encoding tree and let the least frequent element x is at height h_1 and the element with the maximum height is y with height h_2 and we have $h_1 < h_2$. Then we construct a new encoding tree T' where we swap the positions of x and y. So in T' height of y is h_1 and height of x is h_2. Then

$$\textsc{Cost}(T) - \textsc{Cost}(T') = (p(x)h_1 + p(y)h_2) - (p(x)h_2 + p(y)h_1) = (p(x) - p(y))(h_1 - h_2)$$

Since $p(x) < p(y)$ and $h_1 < h_2$ we have $\textsc{Cost}(T) - \textsc{Cost}(T') > 0$. But that is not possible since T is the optimal encoding tree. So T should have the minimum cost. Hence contradiction. x has the maximum height. ■

Lemma 5.2.2

The optimal encoding binary tree must be complete binary tree. (i.e. every non-leaf node has exactly 2 children)

Proof: Suppose T be the optimal binary tree and there is a non-leaf node r which has only one child at height h. By Lemma 5.2.1 the least frequent element x has the maximum height, h_m.

Then consider the new tree $\hat{T}$ where we place the least frequent element at height h and make it the second child of the node r. Then

$$\textsc{Cost}(T) - \textsc{Cost}(\hat{T}) = p(x)h_m - p(x)h = p(x)(h_m - h) > 0$$

But this is not possible as T is the optimal binary tree and it has the minimal cost. Hence contradiction. Therefore the optimal encoding binary tree must be a complete binary tree. ∎

Lemma 5.2.3

There is an optimal binary encoding tree such that the least frequent element and the second least frequent element are siblings at the maximum height.

Proof: Let T be optimal binary encoding tree. Suppose x, y are the least frequent element and the second least frequent element. And suppose b, c be two siblings at the maximum height of the tree (There may be many such siblings, and if so pick any such pair.). If $\{x, y\} = \{b, c\}$ we are done. So suppose not. Let the frequencies of x, y, b, c are respectively $p(x), p(y), p(b), p(c)$ and heights of x, y, b are h_x, h_y and h respectively. WLOG assume $p(x) \le p(y)$ and $p(b) \le p(c)$.

Now since we know x, y have the smallest frequencies we have $p(x) \le p(b)$ and $p(y) \le p(c)$. And since b, c have the maximum height we have $h_x, hy \ge h$. So we switch the position of x with b to form the new tree T'. And from T' we swap the positions fo y and c to form a new tree T''.

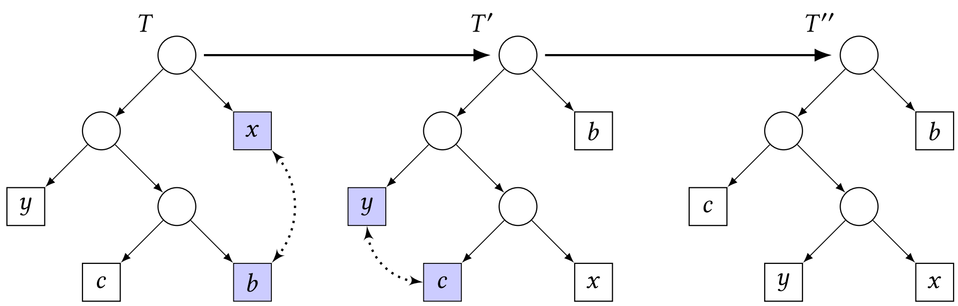


Figure 5.1: Showing that the lowest probability nodes are siblings at the tree's lowest level.

Now we will calculate how the cost changes as we go from T to T' and T' to T''. First check for $T \to T'$. Almost all the nodes contribute the same except x, b. So we have

$$\textsc{Cost}(T) - \textsc{Cost}(T') = (h_x \cdot p(x) + h \cdot p(b)) - (h_x \cdot p(b) + h \cdot p(x)) = (p(b) - p(x))(h - h_x) \ge 0$$

Therefore swapping x and b does not increase the cost and since T is the optimal binary encoding tree the cost doesn't decrease either. Therefore the costs are equal. Hence T' is also an optimal tree.

Similarly we calculate cost for going from T' to T'' we have

$$\textsc{Cost}(T') - \textsc{Cost}(T'') = (h_y \cdot p(y) + h \cdot p(c)) - (h_y \cdot p(c) + h \cdot p(y)) = (p(c) - p(y))(h - h_y) \ge 0$$

Therefore swapping y and c also does not increase the cost and since T' is the optimal binary encoding tree the cost doesn't decrease either. Therefore the costs are equal. Hence T'' is also an optimal tree. Hence T'' is the optimal tree where the least frequent element and second last frequent element are siblings. ∎

By the Lemma 5.2.2 and Lemma 5.2.3 we have that the least frequent element and the second least frequent element are siblings and they have the maximum height.

Observation. *The cost of the trees T_n and T_{n-1} differ only by the fixed term $p(z) = p(x) + p(y)$ which does not depend on the tree's structure. Therefore minimizing the cost for T_n is equivalent to minimizing the cost of T_{n-1}.*

Theorem 5.2.4

Given an instance with symbols $\mathcal{I}$:

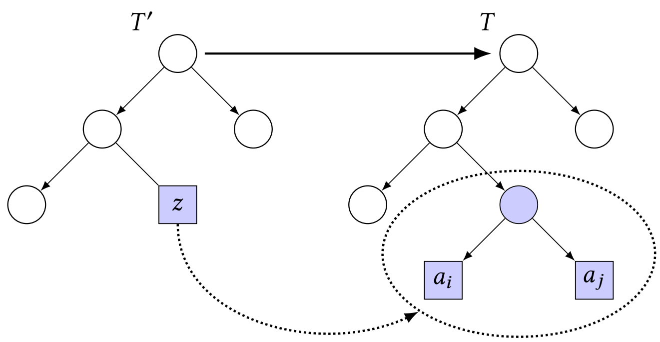
$$\begin{array}{ccccccc} a_1, & a_2, & \cdots, & a_i, & \cdots, & a_j, & \cdots, & a_n \\ p(a_1), & p(a_2), & \cdots, & p(a_i), & \cdots, & p(a_j), & \cdots, & p(a_n) \end{array} \quad \text{with probabilities}$$

such that a_i, a_j are the least frequent and second least frequent elements respectively. Consider the instance with $n-1$ symbols $\mathcal{I}'$:

$$\begin{array}{cccccccccc} a_1, & a_2, & \cdots, & a_{i-1}, & a_{i+1}, & \cdots, & a_{j-1}, & a_{j+1}, & \cdots, & a_n, & z \\ p(a_1), & p(a_2), & \cdots, & p(a_{i-1}), & p(a_{i+1}) & \cdots, & p(a_{j-1}), & p(a_{j+1}), & \cdots, & p(a_n), & p(a_i)+p(a_j) \end{array}$$

Let T' be the optimal tree for this instance $\mathcal{I}'$. Then there is an optimal tree for the original instance $\mathcal{I}$ obtained from T' by replacing the leaf of b by an internal node with children a_i and a_j.

Proof: We will prove this by contradiction. Suppose $\hat{T}$ is optimal for $\mathcal{I}$. Then $\textsc{Cost}(\hat{T}) < \textsc{Cost}(T)$. In $\hat{T}$ we know a_i and a_j are siblings by Lemma 5.2.3. Now consider $\hat{T}'$ for instance $\mathcal{I}'$ where we merge a_i, a_j leaves and their parent into a leaf for symbol z.



Then

$$\textsc{Cost}(\hat{T}') = \textsc{Cost}(\hat{T}) - p(a_i) - p(a_j) < \textsc{Cost}(T) - p(a_i) - p(a_j) = \textsc{Cost}(T')$$

This contradicts the fact that T' is optimal binary encoding tree for $\mathcal{I}'$. Hence T is optimal. ■

5.2.2 Algorithm

Idea: We are going to build the tree up from the leaf level. We will take two characters x, y, and "merge" them into a single character, z, which then replaces x and y in the alphabet. The character z will have probability equal to the sum of x and y's probabilities. Then we continue recursively building the code on the new alphabet, which has one fewer character.

Since we always need the least frequent element and the second least frequent element we have to use the data structure called MIN-PRIORITY QUEUE. So the following algorithm uses a MIN-PRIORITY QUEUE Q keyed on the probabilities to identify the two least frequent objects.

Time Complexity: To create the priority queue it takes $O(n)$ time in line 4-5. Then for each iteration of the for loop in line 6 the EXTRACT-MIN operation takes $O(\log n)$ time and then to insert an element it also takes $O(\log n)$ time. Hence each iteration takes $O(\log n)$ time. Since the for loop has $n-1 = O(n)$ many iterations the running time for the algorithm is $O(n \log n)$.

Remark: We can reduce the running time to $O(n \log \log n)$ by replacing the binary min-heap with a van Emde Boas tree.

Algorithm 14: HUFFMAN-ENCODING(A, P)

Input: Set of n symbols $A = \{a_1, \dots, a_n\}$ and their probabilities $P = \{p_1, \dots, p_n\}$

Output: Optimal Binary Encoding $b : A \to \{0,1\}^*$ for A with minimum $\textsc{Cost}(b) = \sum_{i=1}^{n} p(a_i) \cdot \textsc{Len}(b(a_i))$.

```
begin
    n ⟵ |A|
    Q ⟵ MIN-PRIORITY QUEUE
    for x ∈ A do
        INSERT(Q, x)
    for i = 1, ... n − 1 do
        z ⟵ New internal tree node
        x ⟵ EXTRACT-MIN(Q), y ⟵ EXTRACT-MIN(Q)
        left[z] ⟵ x, right[z] ⟵ y
        p(z) ⟵ p(x) + p(y)
        INSERT(Q, z)
    return Last element left in Q as root
```

Theorem 5.2.5 Correctness of Huffman's Algorithm

The above Huffman's algorithm produces an optimal prefix code tree

Proof: We will prove this by induction on n, the number of symbols. For base case $n = 1$. There is only one tree possible. For $n = k$ we know that by Lemma 5.2.3 and Lemma 5.2.1 that the two symbols x and y of lowest probabilities are siblings and they have the maximum height. Huffman's algorithm replaces these nodes by a character z whose probability is the sum of their probabilities. Now we have 1 less symbols. So by inductive hypothesis Huffman's algorithm computes the optimal binary encoding tree for the $k-1$ symbols. Call it T_{n-1}. Then the algorithm replaces z with a parent node with children x and y which results in a tree T_n whose cost is higher by a fixed amount $p(z) = p(x) + p(y)$. Now since T_{n-1} is optimal by Theorem 5.2.4 we have T_n is also optimal. ■

5.3 Matroids

Definition 5.3.1: Matroid

A matroid $M = (E, \mathcal{I})$ has a ground set E and a collection I of subsets of E called the *Independent Sets* st

1. Downward Closure: If $Y \in \mathcal{I}$ then $\forall\, X \subseteq Y, X \in \mathcal{I}$.
2. Exchange Property: If $X, Y \in \mathcal{I}$, $|X| < |Y|$ then $\exists\, e \in Y - X$ such that $X \cup \{e\}$ also written as $X + e \in \mathcal{I}$

An element $x \in E$ extends $A \in \mathcal{I}$ if $A \cup \{x\} \in \mathcal{I}$. And A is maximal if no element can extend A.

Lemma 5.3.1

If A, B are maximal independent set, then $|A| = |B|$ i.e. all maximal independent sets are also maximum

Proof: Suppose $|A| \neq |B|$. WLOG assume $|A| > |B|$. Then by the exchange property $\exists\, e \in A - B$ such that $B \cup \{e\} \in \mathcal{I}$. But we assumed that B is maximal independent set. Hence contradiction. We have $|A| = |B|$. ■

Base: Maximal Independent sets are called bases.
Rank of $S \in I$: $\max\{|X| : X \subseteq S, X \in I\}$
Rank of a Matroid: Size of the base.
Span of $S \in I$: $\{e \in E : rank(S) = rank(S + e)\}$

5.3.1 Examples of Matroid

- **Uniform Matroid:** Given $E = \{e_1, \dots, e_n\}$, and $k \in \mathbb{Z}_0$ take $\mathcal{I} = \{S \subseteq E : |S| \le k\}$

Lemma 5.3.2

$M = (E, \mathcal{I})$ defined as above is a matroid

Proof:

① Downward Closure: $A \in \mathcal{I}, B \subseteq A \implies |B| \le k \implies B \in \mathcal{I}$

② Exchange Property: $A, B \in \mathcal{I},\ |B| < |A| \le k \implies |B| < k \implies \forall\, e \in A - B,\ |B \cup \{e\}| \le k \implies B \cup \{e\} \in \mathcal{I}$

Therefore M is a matroid ■

- **Partition Matroid:** Given E, $\{P_1, \dots, P_l\}$ such that $E = \bigsqcup_{i=1}^{l} P_i$ and $k_1, \dots, k_l \in \mathbb{Z}_0$ then take

$$\mathcal{I} = \{S \subseteq E : \forall\, k \in [l],\ |S \cap P_j| \le k_j\}$$

Lemma 5.3.3

$M = (E, \mathcal{I})$ defined as above is a matroid

Proof:

① Downward Closure: $A \in \mathcal{I}, B \subseteq A \implies \forall\, j \in [l]\ |B \cap P_j| \le |A \cap P_j| \le k_j \implies B \in \mathcal{I}$

② Exchange Property: $A, B \in \mathcal{I},\ |B| < |A| \implies \exists\, j \in [l],\ |B \cap P_j| < |A \cap P_j| \le k_j \implies e \in (A \cap P_j) - (B \cap P_j),\ |(B \cup \{e\}) \cap P_j| = |B \cap P_j| + 1 \le k \implies B \cup \{e\} \in \mathcal{I}$

Therefore M is a matroid ■

- **Laminar Matroid:** Given E, $\mathscr{L} = \{L_1, \dots, L_l\}$ such that $\forall\, i, j \in [l]$, either $L_i \subseteq L_j$ or $L_i \supseteq L_j$ or $L_i \cap L_j = \emptyset$ and also given $k_1, \dots, k_l \in \mathbb{Z}_0$. Then take

$$\mathcal{I} = \{S \subseteq E : \forall\, j \in [l],\ |S \cap L_j| \le k_j\}$$

For any $L \in \mathscr{L}$ we denote $k(L)$ be the given number corresponding to L.

Lemma 5.3.4

$M = (E, \mathcal{I})$ defined as above is a matroid

Proof:

① Downward Closure: $A \in \mathcal{I}, B \subseteq A \implies \forall\, j \in [l]\ |B \cap L_j| \le |A \cap L_j| \le k_j \implies B \in \mathcal{I}$

② Exchange Property: Let $A, B \in \mathcal{I}$ with $|B| < |A|$. If there exists $e \in A \backslash B$ such that $e \notin L$ for any $L \in \mathscr{L}$, then $|(B + e) \cap L| = |B \cap L| \le k(L)$ for any $L \in \mathscr{L}$.

Hence assume that for each $e \in A \backslash B$ there exists $L \in \mathscr{L}$ with $e \in L$. For each $e \in A \backslash B$, let $\mathscr{L}_e$ be the collection of $L \in \mathscr{L}$ with $e \in L$. For each $e \in A \backslash B$ and any $L \in \mathscr{L} \backslash \mathscr{L}_e$, we have $|(B + e) \cap L| = |B \cap L| \le k(L)$.

Hence it remains to show that there exists $e \in A \backslash B$ such that $|(B + e) \cap L| \le k(L)$ for any $L \in \mathscr{L}_e$. Note that $\mathscr{L}_e$ is a chain, as $\mathscr{L}$ is a laminar. Let $\mathscr{L}' = \{L_{e_1}, \dots, L_{e_l}\}$ be the collection of inclusion-wise maximal sets in $\mathscr{L}$ such that $|B \cap L_{e_i}| \le k(L_{e_i})$ with $e_i \in A \backslash B$. Then $L_{e_i} \cap L_{e_j} = \emptyset$. Moreover, $|A| > |B|$ and $|A \cap L_{e_i}| \le k(L_{e_i})$ imply that $|A \backslash (\cup L_{e_i})| > |B \backslash (\cup L_{e_i})|$. Hence there $\exists\, e_i$ such that $|A \cap L_{e_i}| > |B \cap L_{e_i}|$.

Now we take a look at the chain $\mathscr{L}_{e_i}$. For brevity we will use e instead of e_i. So in the chain $\mathscr{L}_e = \{L_1, \dots, L_n\}$ such that we have

$$L_n \supseteq L_{n-1} \supseteq \cdots \supseteq L_2 \supseteq L_1$$

Then take $i \in [n]$ to be the largest index such that $|A \cap L_i| \leq |B \cap L_i|$. There will be such index because otherwise we will have $|A| \leq |B|$ which is not possible. Then take $e^* \in (A \cap L_{i+1}) - (L_i \cup B)$. Such an e^* will exist because $|A \cap L_{i+1}| > |A \cap L_{i+1}| \implies A \cap (L_{i+1} - L_i \neq \emptyset$ and also $A \cap (L_{i+1} - L_i \not\subseteq B \cap (L_{i+1} - L_i)$ because otherwise we will have

$$|A \cap L_{i+1}| = |A \cap (L_{i+1} - L_i| + |A \cap L_{i+1}| \leq |B \cap (L_{i+1} - L_i)| + |B \cap L_i| = |B \cap L_{i+1}|$$

which is not possible. Hence there exists e^* such that $e^* \in (A \cap L_{i+1}) - (L_i \cup B)$. Therefore take $B^* = B \cup \{e^*\}$. Then for all $j < i$ we have $B^* \cap L_j = B \cap L_j$ so we don't have a problem there. Now for all $j \geq i$ we have $|A \cap L_j| > |B \cap L_j|$. Hence now $|B^* \cap L_j| \leq |B \cap L_j| + 1 \leq |A \cap L_j| \leq k(L_j)$. Therefore we have $B^* \in \mathcal{I}$. Hence the exchange property follows.

Therefore M is a matroid. ■

- **Graphic Matroid:** Given a graph $G = (V, E)$ E is the ground set and take

$$\mathcal{I} = \{E' \subseteq E : E' \text{ is acyclic}\}$$

Lemma 5.3.5

$M = (E, \mathcal{I})$ defined as above is a matroid

Proof:

1. Downward Closure: If a set of edges S is acyclic then naturally any subset of edges of S is also acyclic. Hence downward closure property follows.
2. Exchange Property: $A, B \in \mathcal{I}$, and $|B| < |A|$. Let $G_1, \dots, G_k$ are the connected components due to B. For each component G_i, we have $|G_i \cap A| \leq |G_i \cap B|$ since each component is a tree and B has maximum number of edges for that component. Then A contains an edge e connecting 2 components G_i and G_j. Then $B \cup \{e\} \in \mathcal{I}$.

Therefore M is a matroid ■

- **Linear Matroid:** Given a $m \times n$ matrix $M \in \mathbb{Z}^{m \times n}$, $E = [n]$ and take

$$\mathcal{I} = \{S \subseteq E : \text{Columns of } M \text{ corresponding to } S \text{ are linearly independent}\}$$

Lemma 5.3.6

$M = (E, \mathcal{I})$ defined as above is a matroid

Proof:

1. Downward Closure: $A \in \mathcal{I}$, $B \subseteq A$. Subset of linearly independent set is also linearly independent. Hence $B \in \mathcal{I}$.
2. Exchange Property: $A, B \in \mathcal{I}$, $|B| < |A|$. Then take span $\langle A \rangle$ over $\mathbb{Q}$. Now we know a set of integral vectors are linearly independent over integers if and only if they are linearly independent over rationals. Hence $|A| = \dim_{\mathbb{Q}} \langle A \rangle > \dim_{\mathbb{Q}} \langle B \rangle = |B|$. Hence we can extend B by an element $e \in A - B$ such that $\langle B \cup \{e\} \rangle = |B| + 1$. Hence $B \cup \{e\} \in \mathcal{I}$.

Therefore M is a matroid ■

This matroid is also called Metric Matroid.

5.3.2 Finding Max Weight Base

Max Weight Base
Input: A matroid $M = (E, I)$ is given as an input as an oracle and a weight function $W : E \longrightarrow \mathbb{R}$.
Question: Find the maximum weight base of the matroid.

We will solve this using greedy algorithm.

Algorithm 15: Max-Weight-Base(E, W)

Input: A matroid $M = (E, I)$ is given as an input as an oracle and a weight function $W : E \longrightarrow \mathbb{R}$.
Output: Find the maximum weight base of the matroid

```
begin
    Assume w(1) ≥ ··· ≥ w(n)
    S ← ∅
    I ← {S}
    for i = 1 to n do
        if S + i ∈ I then
            S ← S + i
    return S
```

Theorem 5.3.7

The above algorithm outputs a maximum weight base

Proof: Let M be a matroid. We will prove that this greedy algorithm works by inducting on i. At any iteration i we need to prove the following claim:

Claim 5.3.1

At any iteration i there is a max weight base B_i such that $S_i \subseteq B_i$ and $B_i \setminus S_i \subseteq \{i+1, \dots, n\}$.

Proof: Base case: $S = \emptyset$. So for base case the statement is true trivially. Assume that the statement is true up to $(i-1)$ iterations.

Now $S_{i-1} \subseteq B_{i-1}$ where B_{i-1} is a maximum weight base and $B_{i-1} - S_{i-1} \subseteq \{i, \dots, n\}$. Now three cases arise:

Case 1: If $i \in B_{i-1}$ then $S_{i-1} + i \subseteq B_{i-1}$. Therefore $S_{i-1} + i$ is independent. So now $B_i = B_{i-1}$ and $S_i = S_{i-1} + i$ and $B_i - S_i \subseteq \{i+1, \dots, n\}$.

Case 2: If $i \notin B_{i-1}$ and $S_{i-1} + i \notin \mathcal{I}$. Then $S_i = S_{i-1}$ and $B_i = B_{i-1}$. And $B_i - S_i \subseteq \{i+1, \dots, n\}$.

Case 3: If $i \notin B_{i-1}$ but $S_{i-1} + i \in \mathcal{I}$. Then $S_i = S_{i-1} + i$. Now S_i can be extended to a B' by adding all but one element of B_{i-1}. So $|B'| = |B_{i-1}|$. Let the element which is not added is $j \in B_{i-1}$. So $B' = B_{i-1} + i - j$.

$$wt(B') = Wt(B_{i-1}) - wt() + wt(i)$$

But we have $wt(i) \geq wt(j)$. So $wt(B') \geq wt(B_{i-1})$. Now since B_{i-1} has maximum weight we have $wt(B') = wt(B_{i-1})$. Then our $B_i = B'$. So $B_i - S_i \subseteq \{i+1, \dots, n\}$.

Hence the claim is true for the ith stage as well. Therefore the claim is true. ■

Claim 5.3.2

At any iteration, $T_i = \{t_1, \dots, t_k\}$, then T_i is a maximum weight independent set with at most i elements

Proof: We will prove by induction. Base Case: $i = 0$. Then $T_i = \emptyset$. So the statement follows naturally.
Assume T_{i-1} is maximum weight independent set with at most $i-1$ elements. Now for a contradiction, say $\hat{T}_i \in \mathcal{I}$ of size at most i with strictly larger weight than T_i. Then $\exists\, x \in \hat{T}_i - T_{i-1}$ such that $T_{i-1} \cup \{x\} \in \mathcal{I}$. Then we have

$$wt(\hat{T}_i - x) \leq wt(T_{i-1})$$

by inductive hypothesis. The only element that extend T_{i-1} are those t_{i-1}. Therefore $wt(x) \leq wt(t_i)$. Hence we have

$$wt(\hat{T}_i - x) + wt(x) \leq wt(T_{i-1}wt(t_i) \implies wt(\hat{T}_I) \leq wt(T_i)$$

But we assumes $wt(\hat{T}_i) > wt(T_i)$. Hence contradiction. ■

Therefore using the claims, after the algorithm finished we have no elements left to check, so the current set has the maximum weight which is also an independent set. So the algorithm successfully returns a maximum weight base. ■

5.3.3 Job Selection with Penalties

FIND FEASIBLE SCHEDULE

Input: Set J of n jobs with deadlines $d_1, \ldots, d_n$ and rewards $w_1, \ldots, w_n$

Question: Each jobs unit time and we have a single machine to process their jobs. Give a feasible schedule of jobs with maximum reward

First lets define what is a schedule and what is a feasible schedule:

Definition 5.3.2: Feasible Schedule

For a subset S of jobs:

1. A schedule is an ordering of S
2. A feasible schedule is one where one job in S gets finished by deadline.
3. A set $S \subseteq J$ is feasible if S has a feasible schedule.

Now for any $S \subseteq J$, and $t \in \mathbb{Z}_+$, define $N_t(S) = \{j \in S : d_j \leq t\}$. Then we have the following lemma:

Lemma 5.3.8

The following are equivalent:

1. S is feasible
2. $\forall\, t \in \mathbb{Z}_t$, $|N_t(S)| \leq t$
3. The schedule that orders jobs by deadline is feasible

Proof:

$3 \implies 1$: This follows naturally

$1 \implies 2$: Suppose not. Then $\exists\, t$ such that $|N_t(S)| > t$. Then by time t, greater than t many jobs have to be completed. But S is feasible so every job is finished by deadlines and each job takes unit take. Hence by time t, more than t jobs can not finished. Hence contradiction.

$2 \implies 3$: The schedule orders the jobs by deadline. We induction on t. For $t = 1$ we have $|N_1(S)| \leq 1$. Hence by $t = 1$ at most one job is completed. At $t = 1$ the jobs are completed within deadline. Suppose till time $t - 1$ the jobs are completed within deadlines. At time t we have $|N_t(S)| \leq t$. Therefore all the jobs with deadlines $\leq t$ in S. So they all can be completed within time t in any order. Therefore if we complete the jobs with deadline $< t$ first then also we can complete all the jobs with deadline t within time t. Hence at time t all the jobs are completed within their deadlines. Hence by mathematical induction at time $t = n$ all the jobs are completed within deadline. Therefore the schedule orders jobs by deadline then it is a feasible schedule.

■

Lemma 5.3.9

Consider $M = (J, \mathcal{I})$ where S is feasible $\implies S \in \mathcal{I}$. Then M is a matroid. (Assume that no two jobs have same deadline)

Proof: Suppose $D :=$ the maximum of all deadlines. Consider the set

$$\mathscr{L} = \{N_t(J) : t \in [D]\}$$

Then take $\mathcal{I}' = \{S \subseteq J : |N_t(S)| \leq t \ \forall t \in [D]\}$. By Lemma 5.3.4 $M = (J, \mathcal{I}')$ is a laminar matroid. And by Lemma 5.3.8 $\mathcal{I}'$ is the set of feasible schedules. Therefore $\mathcal{I}' = \mathcal{I}$. Hence M is a matroid. ■

Alternate Proof :

1. Downward Closure: If $S \in \mathcal{I}$ then S is feasible. Then for any subset T of S all the jobs are completed within deadlines since S is feasible. So $T \in \mathcal{I}$.

2. Exchanges Property: Given $S, T \in \mathcal{I}$ and $|T| < |S|$. Now order S and T by deadlines. Let j be the job with largest deadline that is not in S i.e. $j = \max\limits_{i \in S \setminus T} d_i$. Then we claim that $T \cup \{j\} \in \mathcal{I}$.

   Now define
   $$T^< = \{i \in T : d_i < d_j\} \qquad T^> = \{i \in T : d_i > d_j\}$$
   And also similarly define
   $$S^< = \{i \in S : d_i < d_j\} \qquad S^> = \{i \in S : d_i > d_j\}$$
   As we defined j we have $T^> = S^>$. Since we have $|S| > |T|$ we have $|S^<| \geq |T^<|$.

   Now if $T \cup \{j\}$ is not feasible then $\exists\, t$ such that $|N_t(T \cup \{j\})| > t$. Since T is feasible we have $|N_t(T)| \leq t$. Hence $t \geq d_j$ otherwise $N_t(T \cup \{j\}) = N_t(T)$. But then

   $$|N_t(T \cup \{j\})| = |T^<| + 1 + |\{i \in T \cup \{j\} : d_j < d_i \leq t\}| \leq |S^<| + 1 + |\{i \in S \cup \{j\} : d_j < d_i \leq t\}| = |N_t(S)| \leq t$$

   Therefore we obtain $|N_t(T \cup \{j\})| \leq t$. Hence contradiction. Therefore $T \cup \{j\}$ is feasible.

■

Chapter 6

Dijkstra Algorithm with Data Structures

Minimum Weight Path

Input: Directed Graph $G = (V, E)$, $s \in V$ is source and $W = \{w_e \in \mathbb{Z}_+ : e \in E\}$

Question: $\forall\, v \in V - \{s\}$ find minimum weight path $s \rightsquigarrow v$.

This is the problem we will discuss in this chapter. In this chapter we will often use the term 'shortest distance' to denote the minimum weight path distance. One of the most famous algorithm for finding out minimum weight paths to all vertices from a given source vertex is Dijkstra's Algorithm

6.1 Dijkstra Algorithm

We will assume that the graph is given as adjacency list. Dijkstra Algorithm is basically dynamic programming. Suppose $\delta(v)$ is the shortest path distance from $s \rightsquigarrow v$. Then we have the following relation:

$$\delta(v) = \min_{u:(u,v)\in E} \{\delta(u) + e(u, v)\}$$

And suppose for any vertex $v \in V - \{s\}$, $dist(v)$ be the distance from s estimated by the algorithm at any point. This is why Dijkstra's algorithm maintains a set S of vertices whose final shortest-path weights from the source s have already been determined. The algorithm repeatedly selects the vertex $u \in V - S$ with minimum shortest-path estimate and estimates the distances of neighbors of u. So here is the algorithm:

Algorithm 16: Dijkstra(G, s, W)

```
Input: Adjacency Matrix of digraph G = (V, E), source vertex s ∈ V and weight function W = {w_e ∈ Z_+ : e ∈ E}
Output: ∀ v ∈ V − {s} minimum weight path from s ⇝ v
begin
    S ⟵ ∅, U ⟵ V
    dist(s) ⟵ 0, ∀ v ∈ V − {s}, dist(v) ⟵ ∞
    while U ≠ ∅ do
        u ⟵ min_{u∈U} dist(u) and remove u from U
        S ⟵ S ∪ {u}
        for e = (u, v) ∈ E do
            dist(v) ⟵ min{dist(v), dist(u) + w(u, v)}
```

Here below we give an example of how the Dijkstra algorithm works:

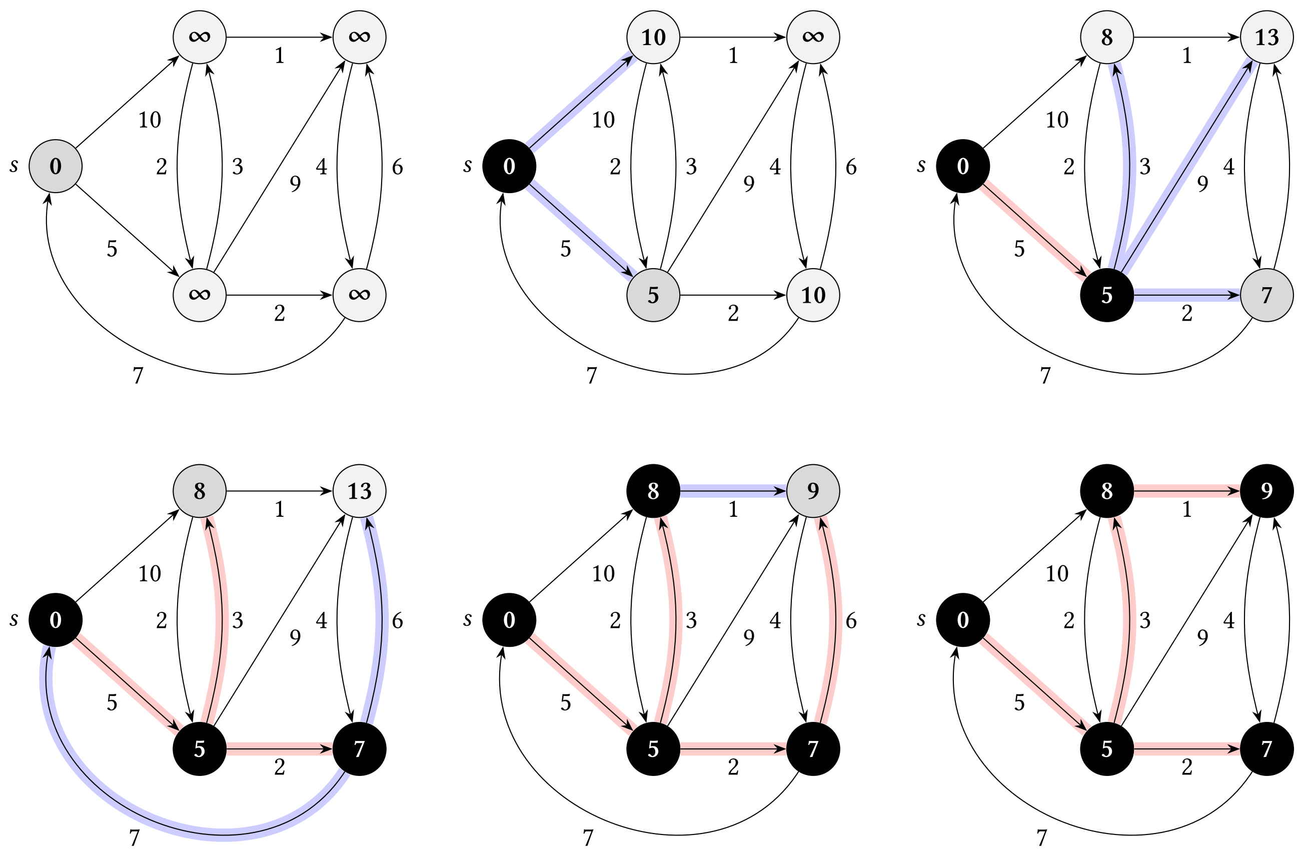


Figure 6.1: The execution of Dijkstra's algorithm. The source s is the leftmost vertex. The shortest-path estimates appear within the vertices, and shaded edges indicate predecessor values. Black vertices are in the set S and at any iteration of while loop the shaded vertex has the minimum value. At any iteration the red edges are the edges considered in minimum weight path from s using only vertices in S.

Suppose at any iteration t, let $dist_t(v)$ denotes the distance v from s calculated by algorithm for any $v \in V$ and $S^{(t)}$ denote the content of S at t^{th} iteration. In order to show that the algorithm correctly computes the distances we prove the following lemma:

Theorem 6.1.1

For each $v \in S^{(t)}$, $\delta(v) = dist_t(v)$ for any iteration t.

Proof: We will prove this induction. Base case is $|S^{(1)}| = 1$. S grows in size. Then only time $|S^{(1)}| = 1$ is when $S^{(1)} = \{s\}$ and $d(s) = 0 = \delta(s)$. Hence for base case this is correct.

Suppose this is also true for $t-1$. Let at t^{th} iteration the vertex $u \in V - S$ is picked. By induction hypothesis for all $v \in S^{(t)} - \{u\}$, $dist_t(v) = dist_{t-1}(v) = \delta(v)$. So we have to show that $dist_t(u) = \delta(u)$.

Suppose for contradiction the shortest path from $s \rightsquigarrow u$ is P and has total weight $= \delta(u) = w(P) < dist_t(u)$. Now P starts with vertices from $S^{(t)}$ by eventually leaves S. Let (x, y) be the first edge in P which leaves S i.e. $x \in S$ but $y \notin S$. By inductive hypothesis $dist_t(x) = \delta(x)$. Let P_y denote the path $s \rightsquigarrow y$ following P. Since y appears before u we have

$$w(P_y) = \delta(y) \leq \delta(u) = w(P)$$

Now

$$dist_t(y) \leq dist_t(x) + w(x, y)$$

since y is adjacent to x. Therefore

$$dist_t(y) \leq dist_t(x) + w(x, y) = \delta(y) \leq dist_t(y) \implies dist_t(y) = \delta(y)$$

Now since both $u, y \notin S^{(t)}$ and the algorithm picked up u we have $\delta(u) < dist_t(u) \leq dist_t(y) = \delta(y)$. But we can not have both $\delta(y) \leq \delta(u)$ and $\delta(u) < \delta(y)$. Hence contradiction. Therefore $\delta(u) = dist_t(u)$. Hence by mathematical induction for any iteration t, for all $v \in S^{(t)}$, $\delta(v) = dist_t(v)$. ■

Therefore by the lemma after all iterations S has all the vertices with their shortest distances from s and henceforth the algorithm runs correctly.

6.2 Data Structure 1: Linear Array

6.3 Data Structure 2: Min Heap

6.4 Amortized Analysis

6.5 Data Structure 3: Fibonacci Heap

Instead of keeping just one Heap we will no keep an array of Heaps. We will also discard the idea of binary trees. We will now use a data structure which will take the benefit of the faster time of both the data structure. I.e. The * is because in Fibonacci Heap the amortized time taken by Extract-Min is $O(\log n)$.

Since Fibonacci heap is an array of heaps there is a *rootlist* which is the list of all the roots of all the heaps in the Fibonacci heap. There is a *min-pointer* which points to the root with the minimum key. For each node in the Fibonacci heap we have a pointer to its parent and we keep 3 variables. The 3 variables are *degree*, *size* and *lost* where *lost* is a Boolean Variable. For any node x in the Fibonacci heap the $x.degree$ is the number of children x has, $x.size$ is the number of nodes in the tree rooted at x and $x.lost$ is 1 if and only if x has lost a child before. Why any node will lost a child that explanation we will give later. With this set up let's dive into the data structure.

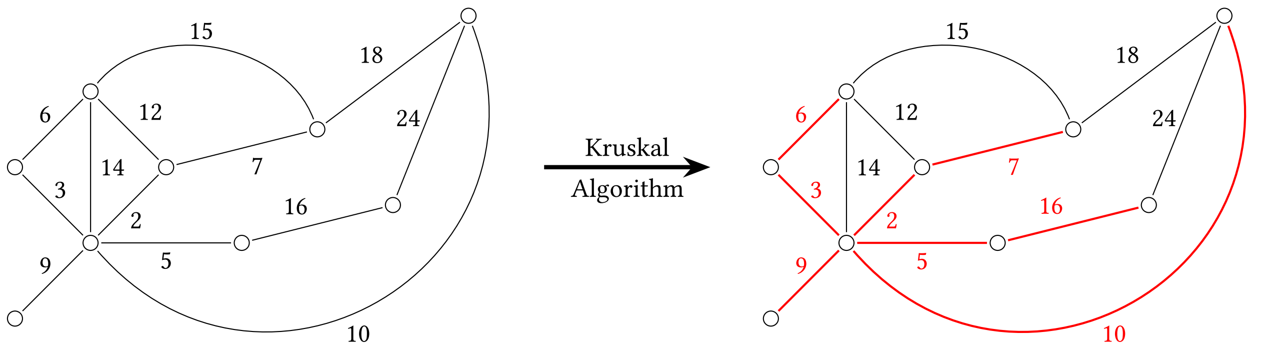
6.5.1 Inserting Node

To insert a node we call the Fib-Insert function and in the function the algorithm initiates the node with setting up all the pointers and variables then add the node to the *rootlist*.

Chapter 7

Kruskal Algorithm with Data Structure

7.1 Kruskal Algorithm



7.2 Data Structure 1: Array

7.3 Data Structure 2: Left Child Right Siblings Tree

7.4 Data Structure 3: Union Find

7.4.1 Analyzing the Union-Find Data-Structure

We call a node in the union-find data-structure a *leader* if it is the root of the (reversed) tree.

Lemma 7.4.1

Once a node stop being a leader (i.e. the node in top of a tree). it can never become a leader again.

Proof: A node x stops being a leader only because of the Union operation which made x child of a node y which is a leader of a tree. From this point on, the only operation that might change the parent pointer of x is the Find operation which traverses through x. Since path-compression only change the parent pointer of x to point to some other node y. Therefore the parent pointer of x will never become equal to itself i.e. x can never be a leader again. Hence once x stops being a leader it can never be a leader again. ∎

Lemma 7.4.2

Once a node stop being a leader then its rank is fixed.

Proof: The rank of a node changes only by an Union operation. But the Union operation only changes the rank of nodes that are leader after the operation is done. Therefore once a node stops being a leader it's rank will not being changed by an Union operation. Hence once a node stop being a leader then its rank is fixed. ■

Lemma 7.4.3

Ranks are monotonically increasing in the reversed trees, as we travel from a node to the root of the tree.

Proof: To show that the ranks are monotonically increasing it suffices to prove that for all edge $u \longrightarrow v$ in the data structure we have $\text{rank}(u) < \text{rank}(v)$. ■

Lemma 7.4.4

When a node gets rank k than there are at least $\geq 2^k$ elements in its subtree.

Corollary 7.4.5

For all vertices v, $v.rank \leq \lfloor \log n \rfloor$

Corollary 7.4.6

Height of any tree $\leq \lfloor \log_2 n \rfloor$

Lemma 7.4.7

The number of nodes that get assigned rank k throughout the execution of the Union-Find data-structure is at most $\frac{n}{2^k}$.

Define $N(r)$ = #vertices with rank at least k. Then by the above lemma we have $N(r) \leq \frac{n}{2^k}$.

Lemma 7.4.8

The time to perform a single find operation when we perform union by rank and path compression is $O(\log n)$ time.

We will show that we can do much better. In fact we will show that for m operations over n elements the overall running time is $O((n+m)\log^* n)$

Lemma 7.4.9

During a single $\textsc{Find}(x)$ operation, the number of jumps between blocks along the search path is $O(\log^* n)$.

Lemma 7.4.10

At most $|Block(i)| \leq Tower(i)$ many Find operations can pass through an element x which is in the i^{th} block (i.e. $\text{INDEX}_B(x) = i$) before $x.parent$ is no longer in the i^{th} block. That is $\text{INDEX}_B(x.parent) > i$.

Lemma 7.4.11

There are at most $\frac{n}{Tower(i)}$ nodes that have ranks in the i^{th} block throughout the algorithm execution.

Lemma 7.4.12

The number of internal jumps performed, inside the i^{th} block, during the lifetime of UNION-FIND data structure is $O(n)$.

Theorem 7.4.13

The number of internal jumps performed by the UNION-FIND data structure overall $O(n \log^* n)$.

Theorem 7.4.14

The overall time spent on m FIND operations, throughout the lifetime of a Union-Find data structure defined over n elements is $O((n+m) \log^* n)$.

Chapter 8

Red Black Tree

Definition 8.1: Perfect Binary Tree

df

Definition 8.2: Red Black Tree

df

Lemma 8.1

Every perfect binary tree with k leaves has $2k-1$ nodes (i.e. $k-1$ internal nodes).

Chapter 9

Maximum Flow

9.1 Flow

Suppose we are given a directed graph $G = (V, E)$ with a source vertex s and a target vertex t. And additionally for every edge $e \in E$ we are given a number $c_e \in \mathbb{Z}_0$ which is called the capacity of the edge.

Definition 9.1.1: Flow

An $s - t$ flow is a function $f : E \longrightarrow \mathbb{R}_0$ which satisfies the following:

1. $\forall\, e \in E,\ f(e) \le c_e$
2. $\forall\, v \in V \setminus \{s, t\},\ \sum\limits_{e \in in(v)} f(e) = \sum\limits_{e \in out(v)} f(e)$

Also the value of a flow f is denoted by $|f| := \sum\limits_{e \in out(s)} f(e)$.

Before proceeding into the setup and the problem first we will assume some things

Assumption.

- *$in(s) = \emptyset$ i.e. there is no edge into s.*
- *$out(t) = \emptyset$ i.e. there is no edge out of t.*
- *There are no parallel edges*

Lemma 9.1.1

For any flow f, $|f| = \sum\limits_{e \in in(t)} f(e)$

Proof: We have for every edge $e \in E$, $\exists\, v \in V$ such that $e \in in(v)$ and $\exists\, u \in V$ such that $e \in out(u)$. Hence we get

$$\sum_{e \in E} f(e) = \sum_{v \in V} \sum_{e \in in(v)} f(e) = \sum_{v \in V} \sum_{e \in out(v)} f(e) \implies \sum_{v \in V} \left[\sum_{e \in in(v)} f(e) - \sum_{e \in out(v)} f(e) \right] = 0$$

Now we know $\forall\, v \in V \setminus \{s, t\}$. $\sum\limits_{e \in in(v)} f(e) = \sum\limits_{e \in out(v)} f(e)$. Therefore we get

$$\sum_{v \in V} \left[\sum_{e \in in(v)} f(e) - \sum_{e \in out(v)} f(e) \right] = 0 \implies \sum_{v \in \{s,t\}} \left[\sum_{e \in in(v)} f(e) - \sum_{e \in out(v)} f(e) \right] = 0 \implies \sum_{e \in out(s)} f(e) - \sum_{e \in in(t)} f(e)$$

Hence we have $|f| = \sum\limits_{e \in in(t)} f(e)$. ∎

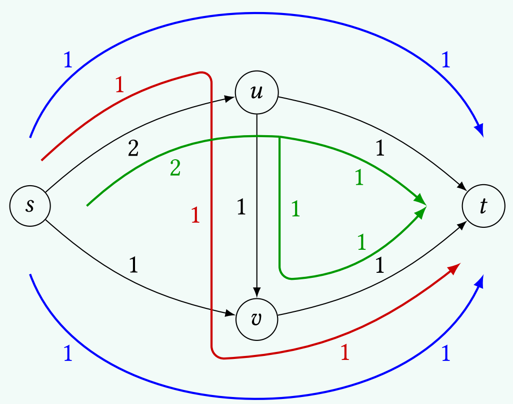
Max Flow

Input: A directed graph $G = (V, E)$ with source vertex s and target vertex t and for all edge $e \in E$ capacity of the edge $c_e \in \mathbb{Z}_+$

Question: Given such a graph and its capacities find an $s - t$ flow which has the maximum value

Example 9.1.1

Consider the following directed graph with capacities: $V = \{s, t, u, v\}$, $c_{s,u} = 2, c_{s,v} = c_{u,t} = c_{v,t} = c_{u,v} = 1$. Firstly the following function: $f' : f'(s,u) = 2 = f(u,t)$. It is not a flow since $f(u,t) = 2 > 1 = c_{u,t}$. Now we define three different flow functions:



- f: $f(s,u) = f(u,v) = f(v,t) = 1$ and otherwise 0. Therefore $|f| = 1$
- g: $g(s,u) = g(u,t) = 1$, $g(s,v) = g(v,t) = 1$ and otherwise 0. Therefore $|g| = 2$
- h: $h(s,u) = 2$, $h(u,t) = h(u,v) = h(v,t) = 1$ and otherwise 0. Therefore $|h| = 2$

Notice here g and h has the maximum flow value.

9.2 Ford-Fulkerson Algorithm

Definition 9.2.1: Residual Graph

Given a directed graph $G = (V, E)$ and capacities C_e for all $e \in E$ and an $s - t$ flow f the residual graph $G_f = (V, E_f)$ has the edges with the following properties:

1. If $(u,v) \in E$ and $f(u,v) > 0$ then $(v,u) \in E_f$ and $c^f_{v,u} = f(u,v)$. Such an edge is called a *backward* edge.
2. If $(u,v) \in E$ and $f(u,v) < c_{u,v}$ then $(u,v) \in E_f$ and $c^f_{u,v} = c_{u,v} - f(u,v)$. It is called *forward* edge.

Algorithm 17: Ford-Fulkerson

Input: Directed graph $G = (V, E)$, source s, target t and edge capacities C_e for all $e \in E$
Output: Flow f with maximum value

1 **begin**
2   **for** $e \in E$ **do**
3     $f(e) = 0$
4   **while** $\exists\ s \rightsquigarrow t$ *path* P *in* G_f **do**
5     $\delta \longleftarrow \min_{e \in P}\{c^f_e\}$ **for** $e = (u,v) \in P$ **do**
6   **if** *e is Forward Edge* **then**
7         $f(u,v) \longleftarrow f(u,v) + \delta$
8   **else**
9         $f(u,v) \longleftarrow f(v,u) - \delta$

We call one iteration of the While loop at line 4 *Flow Augmentation.*

Lemma 9.2.1

At any iteration the f' obtained after the flow augmentation of the flow f is a valid flow

Proof: At any iteration let P be the path from $s \rightsquigarrow t$ and $\delta = \min_{e\in P} c_f(e)$. Let f' be the new function such that for each $(u,v) \in P$ if (u,v) is forward edge in G_f then $f'(u,v) = f(u,v)+\delta$ and if (u,v) is backward edge in G_f then $f'(v,u) = f(v,u)-\delta$ and for other edges $e \in E\setminus P$, $f'(e) = f(e)$.

Now since $\delta = \min_{e\in P} c_f(e)$, $c_f(e) \geq \delta$ for all $e \in P$. Hence if (u,v) is backward edge then $(v,u) \in E$ and $c_f(u,v) = f(u,v)$. Hence $f'(v,u) = f(v,u)-\delta \geq 0$. Therefore for all $e \in E$, $f'(e) \geq 0$.

Now first we will show $f'(e) \leq c_e$ for all $e \in E$. If $(u,v) \in P$ is a forward edge then $(u,v) \in E$ and $c_f(u,v) = c_{u,v} f(u,v)$. Therefore $f'(u,v) = f(u,v)+\delta \leq f(u,v)+c_{u,v}-f(u,v) = c_{u,v}$. Now if $(u,v) \in P$ is a backward edge then $(v,u) \in E$ and $c_f(u,v) = f(u,v)$. Therefore $f'(u,v) = f(u,v)-\delta \leq f(u,v) \leq c_{u,v}$. For other edges $e \in E\setminus P$, $f'(e) = f(u) \leq c_e$. Therefore $f'(e) \leq c_e$ for all $e \in E$

Now we will prove for all $v \in V\setminus\{s,t\}$, $\sum\limits_{e\in in(v)} f'(e) = \sum\limits_{e\in out(v)} f'(e)$. If v is not in the path P in G_f then, $f'(e) = f(e)$ for all edges $e \in in(v)\cup out(v)$. Hence the condition is satisfied for such vertices. Suppose v is in the path P. Then there are two edges e_1 and e_2 in P which are incident on e. If both are forward edges or both are backward edges then one of them is in $in(v)$ and other one is in $out(v)$. WLOG suppose $e_1 \in in(v)$ and $e_2 \in out(v)$ we have

$$\sum_{e\in in(v)} f'(e) = \sum_{e\in in(v)\setminus\{e_1\}} f(e)+f(e_1)\pm\delta = \sum_{e\in out(v)\setminus\{e_2\}} f(e)+f(e_2)\pm\delta = \sum_{e\in out(v)} f'(e)$$

If one of e_1, e_2 forward edge and other one is backward edge then either $e_1,e_2 \in in(v)$ (when e_1 is forward and e_2 is backward) or $e_1,e_2 \in out(v)$ (when e_1 is backward and e_2 is forward). Now if $e_1,e_2 \in in(v)$, $f'(e_1)+f'(e_2) = f(e_1)+\delta+f(e_2)-\delta = f(e_1)+f(e_2)$ and if $e_1,e_2 \in out(v)$ then $f'(e_1)+f'(e_2) = f(e_1)-\delta+f'(e_2)+\delta = f(e_1)+f(e_2)$. Hence

$$\sum_{e\in in(v)} f'(e) = \sum_{e\in in(v)} f(e) = \sum_{e\in out(v)} f(e) = \sum_{e\in out(v)} f'(e)$$

Hence f' is a valid flow. ■

Lemma 9.2.2

At any iteration Given G_f if the flow, f' obtained after flow augmentation of f by δ then

$$|f'| = |f|+\delta$$

Proof: Since we augment flow along an $s \rightsquigarrow t$ path, the first edge of the path is always in $out(s)$. Let the first edge is $e = (s,u)$. Now e has to be a forward edge because otherwise $(u,s) \in E$ and then there is an incoming edge in G which is not possible. Hence

$$|f'| = \sum_{e\in out(s)} f'(e) = \sum_{e\in out(s)\setminus\{e\}} f(e)+f'(e) = \sum_{e\in out(s)\setminus\{e\}} f(e)+f(e)+\delta = \sum_{e\in out(s)} f(e)+\delta = |f|+\delta$$

Hence we have the lemma. ■

Lemma 9.2.3

At every iteration of the Ford-Fulkerson Algorithm the flow values and the residual capacities of the residual graph are non-negative integers.

Proof: Initial flow and the residual capacities are non-negative integers. Let till i^{th} iteration the flow values and the residual capacities were non-negative integers. Let the flow after i^{th} iteration was f. Hence $\forall\, e \in E$, $f(e) \in \mathbb{Z}_0$. Therefore in the G_f for all $e \in E_f$, $c_f(e) \in \mathbb{Z}_0$. Hence $\delta \in \mathbb{Z}_0$. Therefore $\forall\, e \in E$, $f'(e) \in \mathbb{Z}_0$. And therefore for all $e \in E_{f'}$ where $G_{f'}$ is the residual graph of the flow f', $c_{f'}(e) \in \mathbb{Z}_0$. Hence by mathematical induction the lemma follows. ■

At any iteration let P be the path from $s \rightsquigarrow t$. Then for all $e \in P$, $c_f(e) > 0$. Therefore $\delta = \min\limits_{e\in P} c_f(e) \geq 1$. Therefore the algorithm must stop in at most $\sum\limits_{e\in out(s)} c_e$ since we can have the value of a flow to be at max the value of the sum of capacities of edges in $out(s)$ and therefore we can increase the flow at max that many times.

Lemma 9.2.4

If f is a max flow then there is no $s \rightsquigarrow t$ path in G_f.

Proof: Suppose there is an $s \rightsquigarrow t$ path P in G_f. We will show that then f is not a max flow following the algorithm. Then $\forall\ e \in P$, $c_f(e) > 0$. Hence $\delta = \min\limits_{e\in P} c_f(e) \geq 1$. Now after the flow augmentation process of f by δ we get a new valid flow f' by Lemma 9.2.1 and by Lemma 9.2.2 we have $|f'| = |f| + \delta > ||f|$. Hence f is not a maximum flow. Hence contradiction. Therefore there is no $s \rightsquigarrow t$ path in G_f. ∎

9.2.1 Max Flow Min Cut

Definition 9.2.2: Cut Set

For a graph $G = (V, E)$ and a subset $A \subseteq V$, the cut $(A, V \setminus A)$ is a bipartition of V where the edges E_A of the graph $G_A = (A, V \setminus A, E_A)$ is the set $E_A = E \cap (A \times (V \setminus A))$.

Now if s, t are two vertices of G then an *$s - t$ Cut* $(A, V \setminus A)$ is a cut such that $s \in A$ and $t \in V \setminus A$.

Now we define for a cut $(A, V \setminus A)$ the *Capacity of the Cut* $(A, V \setminus A) = \sum\limits_{e\in E_A} c_e$. For an $s - t$ cut $(A, V \setminus A)$ we denote the capacity of the cut by $cap(A)$ A *Min $s - t$ Cut* is a $s - t$ cut of minimum capacity. Then we have the following relation between cut and flow.

Lemma 9.2.5

Given a graph $G = (V, E)$, s, t, $c_e \in \mathbb{Z}_0$ for all $e \in E$ for any flow f and a $s - t$ cut $(A, V \setminus A)$

$$|f| \leq cap(A)$$

Proof: Given f and the $s - t$ cut $(A, V \setminus A)$ we have

$$\begin{aligned}
|f| &= \sum_{e\in out(s)} f(e) \\
&= \sum_{v\in A}\left[\sum_{e\in out(v)} f(e) - \sum_{e\in in(v)} f(e)\right] \\
&= \sum_{\substack{e=(u,v),\\ u\in A, v\notin A}} f(e) - \sum_{\substack{e=(u,v),\\ u\notin A, v\in A}} f(e) && \text{[Edges for both endpoints in } A \text{ are canceled out]} \\
&= \sum_{e\in out(A)} f(e) - \sum_{e\in in(A)} f(e) \\
&\leq \sum_{e\in out(A)} f(e) \leq \sum_{e\in out(A)} c_e = cap(A)
\end{aligned}$$

Hence we have the lemma. ∎

Having this lemma we have have for any flow f and $s - t$ cut $(A, V \setminus A)$ we have

$$|f| \leq cap(A) \implies \max_f |f| \leq \min_{s-t\ cut\ (A,V\setminus A)} cap(A)$$

So we have the following theorem that the value of maximum flow is equal to the capacity of minimum cut.

Theorem 9.2.6 Max Flow Min Cut

Given a graph $G = (V, E)$, s, t, $c_e \in \mathbb{Z}_0$ for all $e \in E$. Then the following are equivalent:

(1) f is a maximum flow.

(2) There is no $s \rightsquigarrow t$ path in G_f

(3) There exists an $s - t$ cut of capacity $|f|$

Proof:

(1) $\implies$ (2): This is by Lemma 9.2.4.

(2) $\implies$ (3): We are given a flow f such that there is no $s \rightsquigarrow t$ path in G_f. We will construct a $s - t$ cut which has the capacity $|f|$. Now take A to be all the vertices reachable from s in G_f. This is a valid $s - t$ cut since $s \in A$ and as there is no $s \rightsquigarrow t$ path in G_f, $t \notin A$. Now

$$|f| = \sum_{e \in out(A)} f(e) - \sum_{e \in in(A)} f(e)$$

Now $\forall\, e = (u, v) \in E$ where $u \in A$ and $v \notin A$ we have $c_{u,v} = f(u, v) \implies c_{u,v} - f(u, v) = 0$ since otherwise $c_{u,v} - f(u, v) \neq 0 \implies c_{u,v} > f(u, v) \implies (u, v) \in E_f$ and therefore v is reachable from s but $v \notin A$, contradiction. Therefore (u, v) is a backward edge and hence . Now $\forall\, e = (u, v) \in E$ where $u \notin A$ and $v \in A$ we have $f(u, v) = 0$ since otherwise $f(u, v) > 0 \implies (v, u) \in E_f$ and therefore u is reachable from s but $u \notin A$, contradiction. Hence we have

$$|f| = \sum_{e \in out(A)} f(e) - \sum_{e \in in(A)} f(e) = \sum_{e \in out(A)} c_e = cap(A)$$

(3) $\implies$ (1): Now by Lemma 9.2.5 we have for any flow f and $s - t$ cut

$$|f| \leq cap(A) \implies \max_f |f| \leq \min_{s-t \text{ cut } (A, V \setminus A)} cap(A)$$

Now given f there exists an $s - t$ cut of capacity $|f|$. Hence f is a max flow.

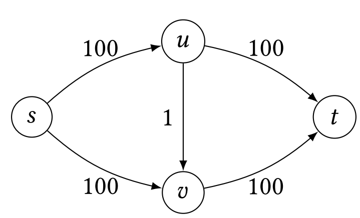
$\blacksquare$

We will get another proof of the Max Flow Min Cut Theorem in subsection 14.4.4.

Hence at the end of the Ford-Fulkerson Algorithm let the flow returned by the algorithm is f. The algorithm terminates when there is no $s \rightsquigarrow t$ path in G_f. Hence by Max Flow Min Cut Theorem we have f is a maximum flow. This completes the analysis of the Ford-Fulkerson Algorithm.

Since the capacities of the edges can be very large we want an algorithm return the maximum flow with running time $poly(n, m, \log c_e)$ where n is the number of vertices and m is number of edges and $\log c_e$ basically means number of bits at most needed to represent the capacities.

But Ford-Fulkerson algorithm takes does not run in $poly(n, m, \log c_e)$ instead $poly(n, m, c_e)$ as the while loop in the algorithm takes $poly(c_e)$ many iterations. For example in the following graph: it takes around 100 steps



and in general Ford-Fulkerson takes $O(|f_{\max}|)$ time. For this reason we will now discuss a modification of the Ford-Fulkerson Algorithm which takes $poly(n, m, \log c_e)$ time, Edmonds-Karp Algorithm.

9.2.2 Edmonds-Karp Algorithm

To get a $poly(n, m, \log c_e)$ time algorithm we will always pick the shortest $s \rightsquigarrow t$ path in the residual graph. This algorithm is known as the Edmonds-Karp Algorithm

Suppose f_i be the total flow after i^{th} iteration. And G_{f_i} be the residual graph with respect f_i. Then $f_0(e) = 0$ for all $e \in E$ and $G_{f_0} = G$. Also suppose $dist_i(v)$ = Shortest $s \rightsquigarrow v$ path distance in the residual graph G_{f_i}. Hence $dist_i(s) = 0$ for all i and $dist_i(t) = \infty$ at the end of the algorithm.

> **Note:-**
>
> In i^{th} iteration of the Ford-Fulkerson Algorithm or Edmonds-Karp Algorithm if P is the $s \rightsquigarrow t$ in the residual graph G_{f_i} where $e = (u, v) \in P$ and $c_{f_i}(u, v) = \delta = \min\limits_{e \in P} c_{f_i}(e)$ then the edge (u, v) is not present in the next residual graph $G_{f_{i+1}}$. Thus at least one edge disappears in each iteration of Ford-Fulkerson or Edmonds-Karp Algorithm.

Now we will prove following two lemmas which will help us to prove that the Edmond-Karp algorithm takes $O(mn)$ iterations.

Lemma 9.2.7

At any iteration i, $\forall\, v \in V$, $dist_i(v) \leq dist_{i+1}(v)$

Proof: Suppose this is not true. Then let i be the first iteration in which there exists a vertex $v \in V$ such that $dist_i(v) > dist_{i+1}(v)$. We pick such v which minimizes $dist_{i+1}(v)$. Consider the shortest path P from $s \rightsquigarrow v$ in $G_{f_{i+1}}$. Hence length of P, $|P| = dist_{i+1}(v)$. Let (u, v) be the last edge of P.

$$P: \quad s \rightsquigarrow u \longrightarrow v$$

Then

$$dist_{i+1}(v) = dist_{i+1}(u) + 1 \geq dist_i(u) + 1$$

Here the last inequality follows because v is the vertex which has the minimum $dist_{i+1}(v)$ among all the vertices $w \in V$ which follows $dist_i(w) > dist_{i+1}(w)$. Now we will analyze case wise.

- **Case 1:** $(u, v) \in E_{f_i}$. Then
$$dist_i(v) \leq dist_i(u) + 1 \leq dist_{i+1}(v)$$
But this is not possible since $dist_i(v) > dist_{i+1}(v)$.

- **Case 2:** $(u, v) \notin E_{f_i}$. Then $(v, u) \in E_{f_i}$. Since $(u, v) \in E_{f_{i+1}}$ then we must have sent flow along (v, u). Since we take the shortest $s \rightsquigarrow t$ path in G_{f_i} in the algorithm we have $dist_i(u) = dist_i(v) + 1$. But then
$$dist_i(u) \leq dist_{i+1}(v) - 1 \implies dist_{i+1}(v) \geq dist_i(v) + 2$$
But this is not possible.

Hence contradiction ↯ Therefore for all iterations i, for all vertices $v \in V$, $dist_i(v) \leq dist_{i+1}(v)$. ■

Lemma 9.2.8

For any edge $e = (u, v) \in E$ the number of iterations where either (u, v) appears or (v, u) appears is at most $O(n)$ i.e.

$$\left|\left\{i\colon (u, v) \notin G_{f_i}, (u, v) \in G_{f_{i+1}}\right\}\right| + \left|\left\{i\colon (v, u) \notin G_{f_i}, (v, u) \in G_{f_{i+1}}\right\}\right| = O(n)$$

Proof: Following the proof of Lemma 9.2.7 in the second case we showed if $(u, v) \notin G_{f_i}$ but $(u, v) \in G_{f_{i+1}}$ then $dist_{i+1}(v) \geq dist_i(v) + 2$. Hence the distance increases by at least 2. Now this can happen at most $O(n)$ many times since $\forall\, i$, $dist_i(v) \leq n - 1$. Hence the number of iterations where either (u, v) appears or (v, u) appears is at most $O(n)$. ■

With this this lemma we will prove that the Edmonds-Karp Algorithm takes $O(mn)$ iterations.

Theorem 9.2.9

Edmonds-Karp Algorithm terminates in $O(mn)$ many iterations.

Proof: For k iterations at least k edges must disappear. Since each edge can reappear $O(n)$ times by Lemma 9.2.8, it can disappear at most $O(n)$ many times. In each iteration at least one edge disappears. Now after $O(mn)$ iterations number of disappearances is at most $O(mn)$. But after $O(mn)$ many disappearances there are no edge remaining and therefore there is no $s \rightsquigarrow t$ path. Hence the algorithm terminates. Therefore the Algorithm terminates in $O(mn)$ iterations. ■

Hence Edmond-Karp Algorithm takes $O(m^2n)poly(\log c_e) = O\left(m^2n\log^{O(1)}(c_e)\right)$ time since it takes $O(mn)$ iterations and in each iteration it finds the shortest $s \rightsquigarrow t$ path in G_{f_i} in $O(m)$ time and in each iteration it does addition and subtraction and finds minimum of the capacities which takes polynomial of the bits needed to represent them time.

9.3 Preflow-Push/Push-Relabel Algorithm

In this algorithm we will maintain something called "Preflow" which is not a valid flow. Unlike Ford-Fulkerson, Edmonds-Karp it does not maintain a $s \rightsquigarrow t$ path in the residual graph and the algorithm stops when the preflow is actually a valid flow.

Definition 9.3.1: Preflow

Given a graph $G = (V, E)$ and the edge capacities c_e, a function $f : E \longrightarrow \mathbb{R}_0$ is a preflow if is satisfies:

1. $\forall\, e \in E,\ f(e) \le c_e$.
2. $\forall\, v \in V \setminus \{s\},\ \sum\limits_{e in(v)} f(e) \ge \sum\limits_{e\in out(v)} f(e)$

Notice here unlike the definition of Flow here in the second criteria we need $\sum\limits_{e in(v)} f(e) \ge \sum\limits_{e\in out(v)} f(e)$ instead of $\sum\limits_{e in(v)} f(e) = \sum\limits_{e\in out(v)} f(e)$.

Now define for all $v \in V$ and for all preflow f, $excess_f(v) = \sum\limits_{e\in in(v)} f(e) - \sum\limits_{e\in out(v)} f(e)$. If f is a preflow then $excess_f(s) \le 0$ and $\forall\, v \in V \setminus \{s\}$, $excess_f(v) \ge 0$

Lemma 9.3.1

For all preflow f

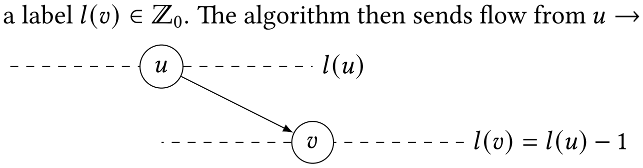
$$\sum_{v\in V} excess_f(v) = 0$$

Proof:

$$\begin{aligned}
\sum_{v\in V} excess_f(v) &= \sum_{v\in V}\left[\sum_{e\in in(v)} f(e) - \sum_{e\in out(v)} f(e)\right] \\
&= \sum_{v\in V}\sum_{e\in in(v)} f(e) - \sum_{v\in V}\sum_{e\in out(v)} f(e) \\
&= \sum_{e\in E} f(e) - \sum_{e\in E} f(e) = 0
\end{aligned}$$

■

Now for each $v \in V$ we assign a label $l(v) \in \mathbb{Z}_0$. The algorithm then sends flow from $u \longrightarrow v$ if $l(v) = l(u) - 1$.



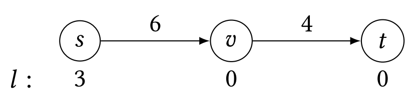
Algorithm 18: PREFLOW-PUSH

Input: Directed graph $G = (V, E)$, source s, target t and edge capacities C_e for all $e \in E$
Output: Flow f with maximum value

1 **begin**
2 Initially $\forall\, e = (s, u) \in E$, $f(e) = c_e$ and $f(e)$ for all other edges.
3 $l(s) \longleftarrow n$
4 **for** $v \in V \setminus \{s\}$ **do**
5 $l(v) \longleftarrow 0$
6 **while** $\exists\, v \neq t$, $excess_f(v) > 0$ **do**
7 **if** $\exists\, u$, *such that* $(v, u) \in E_f$ *and* $l(u) = l(v) - 1$ **then**
8 $\delta \longleftarrow \min\left\{excess_f(v), c_f(v, u)\right\}$
9 **if** (v, u) *is Forward Edge* **then**
10 $f(v, u) \longleftarrow f(v, u) + \delta$
11 **else**
12 $f(u, v) \longleftarrow f(u, v) - \delta$
13 **else**
14 $l(v) \longleftarrow l(v) + 1$ //Relabeling

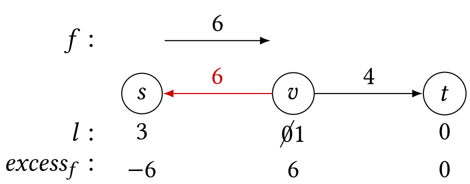
In the algorithm in line 8 if $\delta = c_f(v, u)$ then we call it *saturating push* and if $\delta = excess_f(v)$ then we call it *non-saturating push.*

Now we will show an example of how the algorithm on a graph. We will start the algorithm with the following graph:



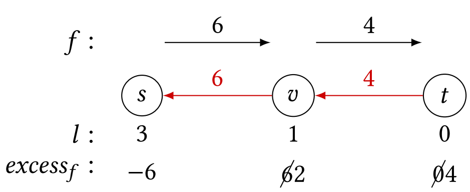
Below we will show change of the residual graph and preflow in each iteration of the WHILE loop:

- Step 1:



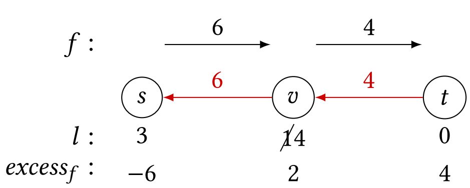
Since $excess_f(v) = 6 > 0$. So in first iteration v is taken. Since there is no edge (v, u) with $l(u) = l(v) - 1$, label of v got increased

- Step 2:



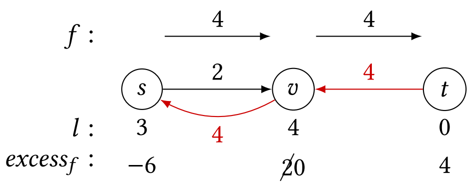
Since $excess_f f(v) = 2 > 0$, in second iteration again v is selected. There is an edge (v, t) with $l(t) = 0 = l(v) - 1 = 1 - 1$. Now $\delta = c_f(v, t) = 4$. Hence saturating push. The preflow gets updated, $f(s, v) = 6$, $f(v, t) = 4$.

- Step 5:



Since $excess_f f(v) = 2 > 0$, in next 3 iterations again v is selected. Since there is no edge (v, u) with $l(u) = l(v) - 1$, label of v gets increased every time. Which becomes 4 after 3 iterations.

- Step 6:



Since $excess_f f(v) = 2 > 0$, in this iteration again v is selected. There is an edge (v, s) with $l(t) = 3 = l(v) - 1 = 4 - 1$. Now $\delta = excess_f(v, s) = 2$. Hence it's non-saturating push. So the preflow gets updated $f(s, v) = 6 - 2 = 4$, $f(v, t) = 4$. Now it's a valid flow. Now there is no vertex with postive *excess*. Hence the algorithm stops.

Observation 9.1. *Labels are monotone non-decreasing.*

Observation 9.2. *For every iteration f is always a preflow. The proof is similar to Lemma 9.2.1 but use inequalities.*

Observation 9.3. $\sum\limits_{v\in V} excess_f(v) = 0$ *and* $\forall\, v \in V \setminus \{s\}$, $excess_f(v) \geq 0$. *Hence* $excess_f(s) \leq 0 \implies l(s)$ *is unchanged.*

Now suppose f^i denote the preflow after the i^{th} iteration of the algorithm. Then

$$f^0(e) = \begin{cases} c_e & \text{when } e = (s, u) \\ 0 & \text{otherwise} \end{cases}$$

Now we will show the correctness of the algorithm.

Lemma 9.3.2

$\forall\, v \in V, \forall\, i,\ excess_{f^i}(v) > 0 \implies \exists\, v \rightsquigarrow s$ in G_{f^i}

Proof: First we fix v and i such that $excess_{f^i} > 0$. Let X be the set of vertices reachable from v in G_{f^i}. Now

$$\sum_{u\in X} excess_{f^i}(u) = \sum_{u\in X}\left[\sum_{e\in in(v)} f^i(e) - \sum_{e\in out(v)} f^i(e)\right] = \sum_{e\in in(X)} f^i(e) - \sum_{e\in out(X)} f^i(e)$$

Now if $\sum\limits_{e\in in(X)} f^i(e) > 0$ then $\exists\, e = (u', u) \in E$ such that $u' \notin X$ and $u \in X$ and $f^i(e) > 0$. Then the backward edge $(u, u') \in E_{f^i}$. Then u' is reachable from v in G_{f^i}. But $u' \notin X$. Contradiction $\lightning$ Therefore $\sum\limits_{e\in in(X)} f^i(e) = 0$. Hence

$$\sum_{u\in X} excess_{f^i}(u) = \cancel{\sum_{e\in in(X)} f^i(e)} - \sum_{e\in out(X)} f^i(e) \leq 0$$

But from Observation 9.3 we have $\forall\, w \in V \setminus \{s\}$, $excess_{f^i}(w) \geq 0$. But at the same time $\sum\limits_{u\in X} excess_{f^i}(u) \leq 0$ and $excess_{f^i}(v) > 0$. Hence $\exists$ a vertex $u \in X$ such that $excess_{f^i}(u) < 0$. But we know only vertex with negative *excess* is s. Therefore $s \in X$. Hence s is reachable from v. ■

Lemma 9.3.3

$\forall\, i$, if $(u, v) \in G_{f^i}$ then $l(v) \geq l(u) - 1$.

Proof: We will prove this using induction on i. Initially $l(s) = n$ and $l(v) = 0$ for all $v \in V \setminus \{s\}$. Hence for all edges (u, v) where $u, v \neq s$ this is satisfied. All the other edges incident on s are in $in(s)$ in the residual graph. And $l(s) = n \geq l(u) = 0$. Therefore the base case is followed.

Now suppose the condition is true for f^{i-1}. Now in the i^{th} iteration suppose the selected vertex is $v \in V \setminus \{t\}$ with $excess_{f^{i-1}} > 0$. Now there are two possible cases.

- **Case 1:** If the step is relabeling then $f^{i-1} = f^i$, $G_{f^{i-1}} = G_{f^i}$ but v is relabeled by $l(v) + 1$. Now for any edge $e = (u, v) \in in(v)$ by Inductive Hypothesis $l(v) \geq l(u) - 1 \implies l(v) + 1 \geq l(u) - 1$. Now consider any edge $e = (v, w) \in out(v)$. By Inductive Hypothesis we have $l(w) \geq l(v) - 1$. Now if $l(w) = l(v) - 1$ then we would have pushed flow along the edge (v, w). Since that is not the case we have $l(w) > l(v) - 1$. Therefore $l(w) \geq (l(v) + 1) - 1$. Hence the condition is satisfied.

- **Case 2:** If the step is pushing flow then suppose we push flow along the edge $(v,w) \in E_{f^{i-1}}$ and $l(w) = l(v) - 1$. Now if we push flow along the edge (v,w) we might introduce the reverse edge (w,v) in G_{f^i}. In that case $l(v) = l(w) + 1 \geq l(w) - 1$. Hence the condition is satisfied.

Therefore by mathematical induction $\forall\ i, \forall\ (u,v) \in E_{f^i}, l(v) \geq l(u) - 1$. ■

Corollary 9.3.4

There is no $s \rightsquigarrow t$ path in G_{f^i} in any iteration i. Thus when the algorithm terminates f is a max flow.

Proof: Now $l(s) = n$ and $l(t) = 0$. We fix v and i. If there is a $s \rightsquigarrow v$ path in G_{f^i} then length of the path is at most $n-1$. For each edge in the path the label decreases by at most 1 by Lemma 9.3.3. Hence $l(v) \geq 1$. Therefore for every vertex $v \in V$, reachable from s we have $l(v) \geq 1$. But $l(t) = 0$. Hence t is not reachable from s. Hence if the algorithm terminates, band if f is a valid flow then y Max Flow Min Cut Theorem it is a max flow. ■

Corollary 9.3.5

$\forall\ v \in V, \forall\ i, l(v) \leq 2n$.

Proof: Suppose $\exists\ v, i$ such that $l(v) = 2n$ and $excess_{f^i}(v) > 0$. By Lemma 9.3.2 there exists an $v \rightsquigarrow s$ path in G_{f^i}. Now by Lemma 9.3.3 for each edge in the path the label decreases by at most 1 and the length of the path is at most $n-1$. Since $l(v) = 2n$, $l(s) \geq n+1$. But we know $l(s)$ for all i by Observation 9.3. Hence contradiction ↯ Therefore for all $v \in V$ and $\forall\ i, l(v) \leq 2n$. ■

Corollary 9.3.6

Total number relabeling operations is $\leq 2n^2$

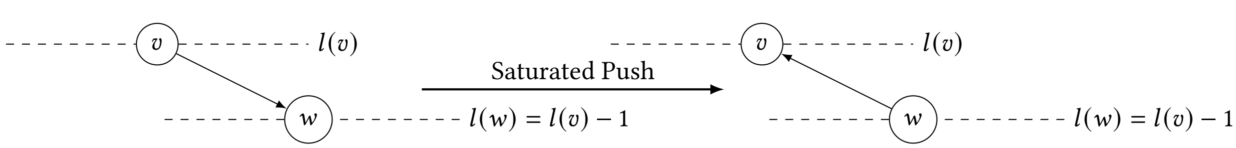
Proof: By Corollary 9.3.5 each vertex label can be at most $2n$. So total number of relabeling operations done in the algorithm is at most $2n^2$ ■

Now we need a bound on the number of push operations. We will count separately the number of Saturating Pushes and number of Non-Saturating Pushes.

Lemma 9.3.7

Total number of saturating pushes is $\leq 2mn$

Proof: We first fix an edge (v,w). Now we will count the number of saturating pushes along (v,w). Then $\delta = c_f(v,w)$. Now consider the scenario of two consecutive saturating pushes along (v,w). When the first saturating push along (v,w) occurred we have $l(w) = l(v) - 1$. Now if (v,w) is forward edge then $\delta = c_f(v,w) = c_{v,w} - f(v,w)$. Then new flow along (v,w) is $f(v,w) + c_f(v,w) = c_{v,w}$. Hence the edge (v,w) vanishes and the flow along (w,v) is $c_{v,w}$. If (v,w) is a backward edge then $\delta = c_f(w,v) = f(w,v)$. Hence then new flow along (w,v) is $f(w,v) - \delta = 0$. Hence again the (w,v) edge vanishes and the flow along (w,v) is $f(w,v)$.



Therefore after a saturated push along (v,w) the edge vanishes and the (w,v) edge is there. Hence in order for another push along (v,w) the algorithm must push flow along (w,v). And this happens when we have the new labels of

v, w follow the condition $l'(w) = l'(v) + 1$. Since by Observation 9.1 the labels never decreases in order for $l(w) = l(v) + 1$ the label of v must increase by at least 2.

Now starting from $l(v) = 0$ we have by Lemma 9.3.5 $l(v) \le 2n$ and for each saturating push along (v, w) the $l(v)$ increase by 2. Hence at most n many saturating pushes occurred along (v, w). Now in the original graph since there are m edges the total number of saturating pushes is $\le 2mn$. ∎

Now we will count the number of non-saturating pushes. For such pushes along any edge (v, u) the $excess_f(v)$ goes to 0. We define the potential function for a preflow f,

$$\Phi(f) = \sum_{v:\ excess_f(v)>0} l(v)$$

Now $\Phi(f) \ge 0$ for all preflow f and initially at the start of the algorithm $\Phi(f^0) = 0$.

Lemma 9.3.8

For each non-saturating push $\Phi(f)$ decreases by at least 1.

Proof: Suppose at any iteration i a non-saturating push occur along an edge (v, w). Therefore $l(w) = l(v) - 1$. We will show that $\Phi(f^i) \le \Phi(f^{i-1}) - 1$. We have $\delta = excess_{f^{i-1}}(v)$. Now if (v, w) is a forward edge then new flow along (v, w) is $f^i(v, w) = f^{i-1}(v, w) + excess_{f^{i-1}}(v)$. Since $(v, w) \in out(v)$

$$excess_{f^i}(v) = \sum_{e\in in(v)} f^i(e) - \sum_{e\in out(v)} f^i(e) = \sum_{e\in in(v)} f^{i-1}(e) - \sum_{e\in out(v)\setminus\{(v,w)\}} f^{i-1}(e) - f^i(v, w) = excess_{f^{i-1}}(v) - \delta = 0$$

Otherwise if (v, w) is a backward edge. Then ew flow along (w, v) is $f^i(w, v) = f^{i-1}(w, v) - excess_{f^{i-1}}(v)$. Since $(w, v) \in in(v)$

$$excess_{f^i}(v) = \sum_{e\in in(v)} f^i(e) - \sum_{e\in out(v)} f^i(e) = f^i(w, v) + \sum_{e\in in(v)\setminus\{(w,v)\}} f^{i-1}(e) - \sum_{e\in out(v)} f^{i-1}(e) = -\delta + excess_{f^{i-1}}(v) = 0$$

In both cases $excess_{f^i}(v) = 0$. Therefore v goes out of the summation. Now there are two cases depending on the value of $excess_{f^{i-1}}(w)$

- **Case 1:** If $excess_{f^{i-1}}(w) > 0$ i.e. w had excess flow before push operation then $\Phi(f^{i-1})$ decreases by $l(v)$ i.e. $\Phi(f^i) = \Phi(f^{i-1}) - l(v)$. Since $l(w) = l(v) - 1$ and by Observation 9.1 $l(v) \ge 1$. Therefore $\Phi(f^i) = \Phi(f^{i-1}) - l(v) \le \Phi(f^{i-1}) - 1$.
- **Case 2:** If $excess_{f^{i-1}}(w) = 0$, then $excess_{f^i}(w) = excess_{f^{i-1}}(w) + \delta > 0$ since $\delta = excess_{f^{i-1}}(v) > 0$ and therefore $\Phi(f^i) = \Phi(f^{i-1}) - l(v) + l(w) = \Phi(f^{i-1}) - 1$

Hence for both the cases $\Phi(f^i) \le \Phi(f^{i-1}) - 1$. Therefore $\Phi(f^{i-1})$ decreases by at least 1. ∎

Observation 9.4. *For relabeling operation $\Phi(f)$ increases by* 1.

Since there are at most $2n^2$ relabeling operations by Corollary 9.3.6, $\Phi(f)$ increases by at most $2n^2$ with relabeling operations.

Observation 9.5. *For each saturating push $excess_f(v, w)$ might not go to* 0 *and therefore Φ might increase.*

Now by Lemma 9.3.7 total number of saturated pushes is at most $2mn$. And by Corollary 9.3.5 each vertex has label at most $2n$. Hence in total $\Phi(f)$ can increase at most $2mn \times 2n = 4mn^2$ by saturated pushes. Hence $\Phi(f)$ increases at most $2n^2 + 2mn \times 2m = O(mn^2)$.

Now

$$\#\text{Non-saturating Pushes} \le \text{Total decrease in } \Phi \le \text{Total increase in } \Phi \le 2n^2 + 4mn^2 = O(mn^2)$$

Therefore total number of iterations of the WHILE loop is #Relabeling+#Saturated Push+#Non-saturated Push= $2n^2 + 2mn + O(mn^2) = O(mn^2)$. There fore the algorithm takes $O(mn^2)$ iteration. In each iteration it takes $O(m + n)$ time. Therefore the runtime of the algorithm is $O(mn^2)O(n + m) = O(m^2n^2)$.

Chapter 10

Randomized Algorithm

Here we will study randomized algorithm for tow basic problems. Later we will discuss other randomized algorithms too in the next chapters. We will also try to derandomize an algorithm in the next chapter.

10.1 Estimated Binary Search Tree Height

In this section we will calculate the expected height of a tree obtained by constructing a binary tree by picking elements uniformly at random from a given array. For this we have the following simple Intersection Algorithm

Algorithm 19: Simple Intersection Algorithm

Input: Array A of n elements of $[n]$ in any order.
Output: Construct a binary tree from A

begin
  $S \longleftarrow A$
  $T \longleftarrow \emptyset$
  while $S \neq \emptyset$ **do**
    $u \longleftarrow$ Extract(S)
    Insert each element at the appropriate leaf of T
  return T

Question 10.1

What is the expected height of the tree obtained by this Simple Intersection Algorithm assuming sequence of keys is uniformly random permutation of $[n]$.

Suppose X_n be the random variable for the height of the tree obtained by the algorithm running on any permutation of $[n]$. Let R_n be the random variable for the root of the tree obtained by the algorithm. Now consider the random variable Y_n defined as $Y_n = 2^{X_n}$. Then if we know $R_n = i$ we have

$$X_n = 1 + \max\{\text{Height of left subtree, Height of right subtree}\} = 1 + \max\{X_{i-1} + X_{n-i}\} \implies Y_n = 2\max\{Y_{n-1}, Y_{n-i}\}$$

Now for the case of $n = 1$ $Y_1 = 1$ since there is only one element and for the convenience we define $Y_0 = 0$. Now consider the following indicator random variable $Z_{n,i}$ where

$$Z_{n,i} = \begin{cases} 1 & \text{if } i \text{ is first element} \\ 0 & \text{otherwise} \end{cases}$$

So basically $Z_{n,i} = \mathbb{1}\{R_n = i\}$. Now if i is the first element then i the root of the tree obtained by the algorithm. Therefore we have

$$
\begin{aligned}
Y_n &= \sum_{i=1}^{n} Z_{n,i}\left(1+\max\{Y_{i-1},Y_{n-i}\}\right) \\
&\le 2\sum_{i=1}^{n} Z_{n,i}(Y_{i-1}+Y_{n-i}) && \text{[Using Lemma 10.1.1]}
\end{aligned}
$$

Lemma 10.1.1 Soft Max

For any $a, b \in \mathbb{R}$,

$$\max\{a,b\} \le \log(2^a + 2^b)$$

Therefore we have

$$
\begin{aligned}
\mathbb{E}[Y_n] &\le 2\sum_{i=1}^{n} \mathbb{E}\left[Z_{n,i}(Y_{i-1}+Y_{n-i})\right] \\
&= 2\sum_{i=1}^{n} \mathbb{E}[Z_{n,i}]\mathbb{E}[Y_{i-1}+Y_{n-i}] \\
&= \frac{2}{n}\sum_{i=1}^{n}(\mathbb{E}[Y_{i-1}]+\mathbb{E}[Y_{n-i}]) = \frac{4}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i]
\end{aligned}
$$

Now to compute $\mathbb{E}[Y_n]$ we use the following lemma

Lemma 10.1.2

$\mathbb{E}[Y_n] \le \frac{1}{4}\binom{n+3}{3}$

Proof: We will prove this using induction on n. The base case is true for $n = 0$. Suppose this is true for $0, \ldots, n-1$.

$$\mathbb{E}[Y_n] \le \frac{4}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i] \le \frac{1}{n}\sum_{i=0}^{n-1}\binom{i+3}{3} = \frac{1}{n}\binom{n+3}{4} = \frac{1}{n}\frac{(n+3)!}{4!(n-1)!} = \frac{1}{4}\binom{n+3}{3}$$

Hence by mathematical induction this is true for all n. ■

Hence by the lemma we have $\mathbb{E}[Y_n] \le \frac{1}{4}\binom{n+3}{3} = O(n^3)$. Now by Jensen Inequality we have

$$\mathbb{E}[Y_n] = \mathbb{E}[2^{X_n}] \ge 2^{\mathbb{E}[X_n]}$$

Therefore $\mathbb{E}[X_n] \le O(\log n)$. Therefore the expected height of a binary search tree is $O(\log n)$.

10.2 Solving 2-SAT

In this section we will discuss a randomized algorithm for deciding if a n-variate 2-SAT boolean formula is satisfiable or not.

2-SAT
Input: 2-SAT formula φ consisting of n variables.
Question: Given n-variate 2-SAT boolean formula determine if φ is satisfiable.

Here we give a simple randomized algorithm for solving the 2-SAT problem:

Algorithm 20: 2-SAT Randomized Algorithm

```
Input: n variate 2-SAT formula φ
Output: Decide if φ is satisfiable or not
begin
    ∀ i ∈ [n], Set x_i = 0
    while ∃ clause C that is not satisfied do
        Let x_i and x_j be variables in C
        Pick from {x_i, x_j} with equal probability and flip the assignment for that variable.
    return x
```

Now if the algorithm terminates it terminates with a satisfying assignment. For now assume that φ is satisfiable. We will deal with the case that φ is not satisfiable later.

Now since there are n variables there can be at most $O(n^2)$ many clauses can be in the formula. Therefore for each step of the while loop to occur it can at most take $O(n^2)$ time to find a clause which is not satisfied.

Let S represents the set of satisfying assignments for φ. Let at j^{th} iteration let A_j denote the current assignment of the variables. Let X_j be the random variable which denotes maximum number of variables of A_j that matches with some satisfying assignment of S i.e.

$$X_j = \max\{n - |x - A_j| : x \in S\}$$

At any step if $X_j = n$ then the algorithm terminates since the algorithm has found a satisfying assignment. Now starting with $X_j < n$ we consider how X_j evolves over time and how long it takes before X_j reaches n.

Now at each step we pick a clause which is unsatisfied. So we know A_j and all assignments of S disagree on the value of at least one variable of this clause. If all the assignments in S disagree with A_j on both variables changing either one will increase X_j. If there are assignments in S which disagree on the value of one of the two variables then with probability $\frac{1}{2}$ we choose that variable and increase X_j by 1 and with probability $\frac{1}{2}$ we choose the other variable and decrease X_j by 1.

Therefore X_j behaves like a random walk on a line starting from 0 which denotes the worst possible case and ends once it reaches at n where at any nonzero point it goes up or down by 1 with probability $\frac{1}{2}$. This is a Markov Chain. We want to calculate how many steps does it take on average for X_j to stumble all the way up to n. Before that we first properly define our Markov Chain.

The Markov Chain consists states from 0 to n. Where from 0 it goes to 1 with probability 1 and from n it always stays at n. And for any other state i it goes to $i+1$ with probability $\frac{1}{2}$ and goes to $i-1$ with probability $\frac{1}{2}$. Now let

$$T(k) = \text{Expected time to walk from } k \text{ to } n$$

Then we have

$$T(n) = 0, \qquad T(0) = T(1) + 1, \qquad \forall\, i \in [n-1],\ T(i) = \frac{T(i-1)}{2} + \frac{T(i+1)}{2} + 1$$

Then we have n unknowns and n equations in the above system. Therefore on average at most $O(n^2)$ steps needed to find a solution.

Now at first we said we are assuming we are dealing with the case of there exists a solution.

Question 10.2

How to deal with the issue of no solution?

In this case we will run for more number of iterations before we give up since when we give up we me might just not have found the solution. So we will run the algorithm for $100n^2$ steps. And if no solution was found then we will give up.

We first of all divide the execution of the algorithm into segments of $2n^2$ steps each. We will calculate the failure case of each segment. If the 2-SAT formula has no solution then the algorithm gives correct output. Suppose it has a solution. Then by Markov's Inequality the probability of number of steps needed to find the solution is greater than the expected number of steps needed to find a solution is at most $\frac{1}{2}$. Now after total $100n^2$ steps the probability none of the segments found a solution is 2^{-50}.

Chapter 11

Derandomization

In this section we will see a derandomization technique called Conditional Expectation. With this technique we will show derandomization of some randomized algorithms in the following sections.

11.1 Conditional Expectation

Let $\mathscr{A}$ be a randomized algorithm which is successful with probability at least $\frac{2}{3}$. Suppose $\mathscr{A}$ uses m random bits and suppose the random bits are $R_1, \ldots, R_m$. Then we have

$$\mathbb{P}_{R_1,\ldots,R_m}[\mathscr{A}(x, R_1, \ldots, R_m) = \text{Correct}] \geq \frac{2}{3}$$

We want to derandomize $\mathscr{A}$.

Now think of $\mathscr{A}$ as a binary tree which, given x, branches on the sampled value of each random bit R_i where it goes to left child if the random bit takes value 0 and goes to right child if the random bit takes value 1. Every path in this tree from root to leaf corresponds to different possible random strings and the leaf nodes corresponds to the output of the algorithm with the corresponding random string. Since $\mathscr{A}$ succeeds with probability at least $\frac{2}{3}$ means that at least $\frac{2}{3}$ of the leaves are good outputs for the input x.

Idea. *To derandomize $\mathscr{A}$ we need to find a deterministic algorithm that traverses from the root to a leaf which at any branch at level i chooses a direction which leads to a good output.*

Now suppose $r_1, \ldots, r_m \in \{0, 1\}$ denote the values taken by the random variables $R_1, \ldots, R_m$. Now let $P(r_1, \ldots, r_i)$ denote the fraction of the leaves of the subtree below the node obtained by following the path $r_1, \ldots, r_i$. Formally,

$$P(r_1, \ldots, r_i) = \mathbb{P}[\mathscr{A}(x, R_1, \ldots, R_m) \mid R_1 = r_1, \ldots, R_i = r_i] = \frac{1}{2}P(r_1, \ldots, r_i, 0) + \frac{1}{2}P(r_1, \ldots, r_i, 1)$$

From the last equality it is clear that there is a choice r_{i+1} such that $P(r_1, \ldots, r_{i+1}) \geq P(r_1, \ldots, r_i)$. Therefore to find a good path in the tree it suffices at each branch to pick such an $r \in \{0, 1\}$. Then we would have

$$P(r_1, \ldots, r_m) \geq P(r_1, \ldots, r_{m-1}) \geq \cdots \geq P(r_1) \geq \mathbb{P}[\mathscr{A}(x, R_1, \ldots, R_m) = \text{Correct}] \geq \frac{2}{3}$$

Since $P(r_1, \ldots, r_m)$ is either 0 or 1 it must be 1.

11.2 Max-SAT

Max-SAT
Input: SAT formula φ with n variables and m clauses and non negative weights w_c on clauses.
Question: Given a SAT formula φ with n variables and m clauses and non negative weights w_c on clauses find an assignment that maximizes weight of satisfied clauses.

We will first show a randomized algorithm for this problem. Then we will use conditional expectation to derandomize the algorithm.

11.2.1 Randomized Algorithm

First lets see what is the expected weight of satisfied clauses. Let Y_c be the indicator random variable if clause C is satisfied. Suppose there are k variables in C. Then we have $\mathbb{E}[Y_c] = 1 - \frac{1}{2^k} \geq \frac{1}{2}$. Therefore expected weight of satisfied clauses is

$$\mathbb{E}\left[\sum_C w_c Y_c\right] = \sum_C w_c \mathbb{E}[Y_c] \geq \frac{1}{2} \sum_C w_c$$

Let OPT be the optimal Max-SAT solution for the given formula. Then we have $\sum_C w_c \geq \text{OPT}$. Therefore

$$\mathbb{E}\left[\sum_C w_c Y_c\right] \geq \frac{1}{2}\text{OPT}$$

Hence we have the following randomized algorithm:

Algorithm 21: 2-Approximate Max-SAT

Input: SAT formula φ with n variables and m clauses and non negative weights w_c on clauses.
Output: Find an assignment that maximizes weight of satisfied clauses.

```
begin
    for i ∈ [n] do
        x_i ⟵ Pick a value from {0, 1} uniformly at random
    return x
```

By the above discussion we have an assignment with an expected weight of satisfied clauses at least half the maximum.

11.2.2 Derandomization

Now we want to derandomize the algorithm using conditional expectation. Let $X_1, \dots, X_n$ denote the random variable for each variables and $x_1, \dots, x_n \in \{0,1\}$ denote the value the random variables took. A key step will be evaluate the conditional probabilities:

$$\mathbb{E}\left[\sum_C w_c Y_c \mid X_1 = x_1, \dots, X_i = x_i\right] = \sum_C w_c \mathbb{P}[Y_c = 1 \mid X_1 = x_1, \dots, X_i = x_i] \quad \forall\, i \in [n]$$

Hence we have to find the value of $\mathbb{P}[Y_c = 1 \mid X_1 = x_1, \dots, X_i = x_i], \forall\, i \in [n]$. Now if the clause C is already satisfied by the setting $x_1, \dots, x_i$ then $Y_C = 1$. Else if C has r variables from $x_{i+1}, \dots, x_n$ then

$$\mathbb{P}[Y_c = 1 \mid X_1 = x_1, \dots, X_i = x_i] = 1 - \frac{1}{2^r}$$

. Now if at height i, we find $\mathbb{E}\left[\sum_C w_c Y_c \mid X_1 = x_1, \dots, X_i = 0\right]$ and $\mathbb{E}\left[\sum_C w_c Y_c \mid X_1 = x_1, \dots, X_i = 1\right]$ and which ever gives the higher value we will set the assignment for X_i to be that one. Thus we can derandomize the algorithm.

11.3 Set Balancing

Set-Balance
Input: $A \in \{0,1\}^{n\times n}$ matrix with A_i is the i^{th} row of A and $A_{i,j}$ is the $(i,j)^{th}$ entry
Question: Given $n \times n$, 0-1 matrix A find $b \in \{1,-1\}^n$ to minimize $\|Ab\|_\infty = \max_{i\in[n]} |A_i b|$.

In the following sections we will not optimize on $\|Ab\|_\infty$. Instead we will give bound on how large $\min \|Ab\|_\infty$ can be for any A.

Algorithm 22: SET-BALANCING

Input: $A \in \{0,1\}^{n\times n}$ matrix
Output: Find an $b \in \{1,-1\}^n$ to minimize $\|Ab\|_\infty$

```
begin
    for i ∈ [n] do
        x_i ⟵ Pick a value from {1, −1} uniformly at random
    return x
```

11.3.1 Randomized Algorithm

Clearly for each row $i \in [n]$ we have

$$\mathbb{E}[A_i b] = \mathbb{E}\left[\sum_j A_{i,j} b_j\right] = \sum_j \mathbb{E}[A_{i,j} b_j] = 0$$

But that does not mean $\mathbb{E}[|A_i b|] = 0$. To get a bound on $\mathbb{E}[|A_i b|]$ we will use Hoeffding's Inequality

Theorem 11.3.1 Hoeffding's Inequality

Let $Y_1, \dots, Y_n$ be independent random variables with bounded supposer $[l_i, u_i]$ for Y_i and let $Y = \sum_{i=1}^{n} Y_i$. Then for any $\delta > 0$

$$\mathbb{P}[|Y - \mathbb{E}[Y]| > \delta] \le 2e^{-\frac{2\delta^2}{\sum_i (u_i - l_i)^2}}$$

In our case we have $Y_{i,j} = A_{i,j} b_j$ and $Y_i = \sum_j A_{i,j} b_j$. Then each $Y_{i,j} \in \{-1, 0, 1\}$, $\mathbb{E}[Y_{i,j}] = 0$ and $\mathbb{E}[Y_i] = 0$. Therefore

$$\mathbb{P}[|Y_i| > \delta] \le 2e^{-\frac{2\delta^2}{4n}}$$

Now we choose $\delta = 2\sqrt{n \ln n}$

$$\mathbb{P}[|A_i b| \ge 2\sqrt{n \ln n}] \le \frac{2}{n^2}$$

Therefore $\mathbb{P}[\|Ab\|_\infty \ge 2\sqrt{n \ln n}] \le \frac{2}{n}$ by union bound. Hence choosing each entry b uniformly at random from ± 1 we can obtain $\|Ab\|_\infty \le 2\sqrt{n \ln n}$ with high probability.

11.3.2 Derandomization

Again we will use conditional expectation to derandomize the algorithm. Let a node at height j corresponds to a setting of $b_1, \dots, b_j$ and we will calculate $\mathbb{P}[\|Ab\|_\infty > 2\sqrt{n \ln n} \mid b_1, \dots, b_j]$. Now consider a leaf corresponding to some choice of $b_1, \dots, b_n$ such that the value of the leaf is < 1. But there is no randomness at the leaf. Then $\mathbb{P}[\|Ab\|_\infty > 2\sqrt{n \ln n} \mid b_1, \dots, b_n] = 0$. Hence for this choice of $b_1, \dots, b_n$ it must have $\|Ab\|_\infty \le 2\sqrt{n \ln n}$. Now

$$\mathbb{P}[\|Ab\|_\infty > 2\sqrt{n \ln n} \mid b_1, \dots, b_j] = \mathbb{P}[\|Ab\|_\infty > 2\sqrt{n \ln n} \mid b_1, \dots, b_j, 0] + \mathbb{P}[\|Ab\|_\infty > 2\sqrt{n \ln n} \mid b_1, \dots, b_j, 1]$$

One of them have

$$\mathbb{P}[\|Ab\|_\infty > 2\sqrt{n \ln n} \mid b_1, \dots, b_j, b_{j+1}] \le \mathbb{P}[\|Ab\|_\infty > 2\sqrt{n \ln n} \mid b_1, \dots, b_j]$$

So we choose that one. Also note that at the root $\mathbb{P}[\|Ab\|_\infty > 2\sqrt{n \ln n}] < \frac{2}{n}$. Then for choosing such a path for the corresponding choice of b we will have $\|Ab\|_\infty \le M = 2\sqrt{n \ln n}$. But this depends on being able to calculate $\mathbb{P}[\|Ab\|_\infty > M \mid b_1, \dots, b_j]$ which we don't know how to do in polynomial time. Instead we will use pessimistic estimator which.

11.3.3 Using Pessimistic Estimator to Derandomize

Instead of $\mathbb{P}[\|Ab\|_\infty > M \mid b_1, \ldots, b_j]$ we will use $\sum_{i\in[n]} \mathbb{P}[|A_i b| > M \mid b_1, \ldots, b_j]$. Naturally we have

$$\sum_{i\in[n]} \mathbb{P}[|A_i b| > M \mid b_1, \ldots, b_j] \geq \mathbb{P}[\|Ab\|_\infty > M \mid b_1, \ldots, b_j]$$

Now we know how to calculate $\mathbb{P}[|A_i b| > M \mid b_1, \ldots, b_j]$. For any $i \in [n]$ we have

$$\mathbb{P}[|A_i b| > M \mid b_1, \ldots, b_j] = \sum_{k=M+1}^{n} \mathbb{P}[A_i b = k \mid b_1, \ldots, b_j] + \mathbb{P}[A_i b = -k \mid b_1, \ldots, b_j]$$

Let $S_i = \{j' > j : A_{i,j'} = 1\}$ and $l = \sum_{j'\leq j} A_{i,j'}$. Then

$$\mathbb{P}[A_i b = k \mid b_1, \ldots, b_j] = \mathbb{P}\left[\sum_{j'\in S_i} b_{j'} = k - l\right]$$

Let in S_i n_i coordinates of b are 1 and rest of the coordinates of b in S_i are -1. Then

$$\sum_{j'\in S_i} b_{j'} = 2n_i - |S_i| = k - l \implies n_i = \frac{1}{2}(k - l + |S_i|)$$

Therefore we have

$$\mathbb{P}[A_i b = k \mid b_1, \ldots, b_j] = \frac{1}{2^{|S_i|}} \binom{|S_i|}{\frac{1}{2}(k - l + |S_i|)}$$

Thus we can calculate $\mathbb{P}[A_i b = k \mid b_1, \ldots, b_j]$ for all $n \geq |k| > M$. Therefore we can calculate $\mathbb{P}[|A_i b| > M \mid b_1, \ldots, b_j]$ and henceforth $\sum_{i\in[n]} \mathbb{P}[|A_i b| > M \mid b_1, \ldots, b_j]$. With this pessimistic estimator we calculate at height j both $\sum_{i\in[n]} \mathbb{P}[|A_i b| > M \mid b_1, \ldots, b_j, b_{j+1} = 0]$ and $\sum_{i\in[n]} \mathbb{P}[|A_i b| > M \mid b_1, \ldots, b_j, b_{j+1} = 1]$ and the one which have value less than 1 we will follow that path and eventually we will get an assignment of b for which $\|Ab\|_\infty \leq 2\sqrt{n \ln n}$.

Chapter 12

Global Min Cut

Global Min Cut
Input: Undirected graph $G = (V, E)$
Question: Find cut $(S, V \setminus S)$ that minimizes $|\delta(S)|$ where $\delta(S) = \{e = (u, v) \mid u \in S, v \notin S\}$.

12.1 Naive Algorithm

In previous chapter we have seen the algorithm to find $s - t$ min cut given any $s, t \in V$ in $O(n^2\sqrt{m})$ time. So naively we can run over all possible vertex pairs (s, t) and output the global min cut in $O(n^4\sqrt{m})$ time.

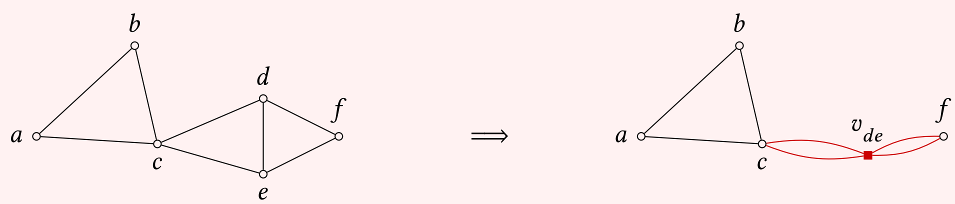
Or we can fix a vertex $s \in V$ and then for all $t \in V$ we can find the $s - t$ min cut and output the minimum. This takes $O(n^3\sqrt{m})$ time.

12.2 Karger's GMC Algorithm

Instead of naively solving the problem like above we will use randomization and will construct an algorithm which will output a global min-cut with high probability using edge contraction.

Definition 12.2.1: Edge Constraction

Given a graph $G = (V, E)$, $e = (u, v)$ edge contraction gives a multigraph (graph with multiple edges between two vertices but no self-loops) $G \setminus e = (V', E')$ where $V' = V \setminus \{u, v\} \cup \{v_e\}$ and for all $e' \in E$ if $e \cap e' = \emptyset$ then $e' \in E$ and otherwise $e' = (w, u)$ then $(w, v_e) \in E'$. The vertex v_e is called the supernode.



Observation. *For any edge $e \in G$:*

- *Any cut in $G \setminus e$ is also a cut in G of same size.*
- *Size of min cut in $G \setminus e$ is at least the size of min cut in G.*
- *Any cut in G that does not separate vertices of e is also cut in $G \setminus e$.*

Then we have the following lemma:

Lemma 12.2.1

Say k is the size of global min cut in $G' = (V', E')$ [G possible a multigraph] i.e. $\exists\, S \subseteq V'$ such that $|\delta(S)| = k$. Then $\min\{\deg(v) \mid v \in V'\} \geq k$ and $|E'| \geq \frac{k}{2}|V'|$.

Proof: If any vertex $v \in V'$ has degree less than k then we can take the cut $(\{v\}, V' \setminus \{v\})$ then $|\delta(v)| < k$, but that contradicts the fact that size of global min cut is k. Hence, contradiction ↯ Therefore $\forall\, v \in V'$, $\deg(v) \geq k$. Therefore, $|E'| = \frac{1}{2}\sum\limits_{v \in V'} \deg(v) \geq \frac{k}{2} \cdot |V'|$. ■

So we at each round we will pick an edge from the graph uniformly at random and then contract that edge and in the next round we will pick an edge from the contracted graph. We will do $n-2$ such iterations since after that we are left with 2 supernodes $(X, V \setminus X)$.

Algorithm 23: Karger's GMC Algorithm

Input: Undirected graph $G = (V, E)$
Output: Find a cut $(S, V \setminus S)$ such that $|\delta(S)|$ is minimum

```
begin
    H ⟵ G;
    for i = 1, ..., n − 2 do
        e ⟵ Picked uniformly at random from E;
        H ⟵ H \ e;
    return E(H)
```

Question 12.1

What is the probability that the above algorithm returns a global min cut?

Let $(S, V \setminus S)$ is the global min cut with $|\delta(S)| = k$. Now probability that the algorithm returns $(S, V \setminus S)$ is equal to the probability that none of the edges in $\delta(S)$ is picked. So let $e_1, \dots, e_{n-2}$ are the edges that are picked in the $n-2$ iterations of the algorithm. We need to calculate $\mathbb{P}[e_i \notin \delta(S),\ \forall\, i \in [n-2]]$

Lemma 12.2.2

$\mathbb{P}[e_1 \notin \delta(S)] \geq 1 - \frac{2}{n}$

Proof: We have $|\delta(S)| = k$. Hence, we have $|E| \geq \frac{n \cdot k}{2}$. Since e_1 is picked uniformly at random we have

$$\mathbb{P}[e_1 \notin \delta(S)] \geq 1 - \frac{k}{\frac{nk}{2}} = 1 - \frac{2}{n}$$

Hence we have the lemma. ■

Lemma 12.2.3

$\mathbb{P}[e_i \notin \delta(S) \mid e_1, \dots, e_{i-1} \notin \delta(S)] \geq 1 - \frac{2}{n-i+1}$

Proof: Let $e_1, \dots, e_{i-1} \notin \delta(S)$. Hence S is still a min cut in $G \setminus \{e_1, \dots, e_{i-1}\}$. Then number of edges after contracting $e_1, \dots, e_{i-1}$ is at least $\frac{k(n-i+1)}{2}$. Therefore

$$\mathbb{P}[e_i \notin \delta(S) \mid e_1, \dots, e_{i-1} \notin \delta(S)] 1 - \frac{k}{\frac{k(n-i+1)}{2}} = 1 - \frac{2}{n-i+1}$$

Therefore we have the lemma. ■

Hence we have

$$\begin{aligned}\mathbb{P}[\text{Success}] &\geq \mathbb{P}[e_i \notin \delta(S),\ \forall\, i \in [n-2]] \\ &= \prod_{i=1}^{n-2}\left(1 - \frac{2}{n-i+1}\right) \\ &= \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}} = O\left(\frac{1}{n^2}\right)\end{aligned}$$

So we run the above algorithm $2n^2 \log n$ times then take the cut which gives minimum size. Then we have

$$\begin{aligned}\mathbb{P}[\text{Succeeds}] &= 1 - \mathbb{P}[\text{All } 4n^2 \log n \text{ runs fails}] \\ &\geq 1 - \left(1 - \frac{2}{n^2}\right)^{4n^2 \log n} \\ &\geq 1 - \exp\left[-\frac{2}{n^2} 2n^2 \log n\right] \\ &= 1 - \frac{1}{n^4}\end{aligned}$$

Hence, this gives a much higher probability of success. So our final algorithm is

Algorithm 24: Multiple run of Karger's GMC Algorithm

Input: Undirected graph $G = (V, E)$
Output: Find a cut $(S, V \setminus S)$ such that $|\delta(S)|$ is minimum

```
begin
    S ⟵ ∅;
    cutEdgeSize ⟵ |E|;
    for i ∈ [2n² log n] do
        H ⟵ G;
        for j = 1, ..., n − 2 do
            e ⟵Picked uniformly at random from E;
            H ⟵ H \ e;
        if |E(H)| < cutEdgeSize then
            Let H = (X, V \ X);
            S ⟵ X;
            cutEdgeSize ⟵ |E(H)|;
    return S
```

12.3 Karger-Stein Algorithm

In Karger's algorithm the probability of getting a min cut is low because in later stages the probability of picking an edge from a min-cut is high because

$$\mathbb{P}[e_i \in \delta(S) \mid e_1, \ldots, e_{i-1} \notin \delta(S)] \leq \frac{2}{n-i+1} \implies \mathbb{P}[e_1, \ldots, e_i \notin \delta(S)] \geq \frac{\binom{n-i}{2}}{\binom{n}{2}}$$

If the above probability is at least $\frac{1}{2}$ then $2(n-i)^2 \geq n^2 \implies n - i \geq \frac{n}{\sqrt{2}}$. Hence, i can't be too high.

So instead of running the entire algorithm $\tilde{O}(n^2)$ times we can just run the later stages multiple times. So after $i \leq n - \frac{n}{\sqrt{2}} - 1$ iterations of Karger's GMC algorithm we have

$$\mathbb{P}[e_1, \ldots, e_i \notin \delta(S)] \geq \frac{(n-i)(n-i-1)}{n(n-1)} \geq \frac{n^2}{2n(n-1)} \geq \frac{1}{2}$$

from Lemma 12.2.3. We also have the following lemma:

Lemma 12.3.1

For any $1 \leq i < j \leq n-2$ we have

$$\mathbb{P}[e_i, e_{i+1}, \dots, e_j \notin \delta(S) \mid e_1, \dots, e_{i-1} \notin \delta(S)] \geq \frac{(n-j)(n-j-1)}{(n-i+1)(n-i)}$$

Now fix an $i \leq n-2$. Let $l = n-i+1$. Then For $j \leq n - \frac{l}{\sqrt{2}} - 1$ we have

$$\mathbb{P}[e_i, \dots, e_{i+j-1} \notin \delta(S) \mid e_1, \dots, e_{i-1} \notin \delta(S)] \geq \frac{l^2}{2l(l-1)} \geq \frac{1}{2}$$

So we have the following algorithm:

Algorithm 25: KS-Algorithm

Input: Undirected graph $G = (V, E)$
Output: Find a cut $(S, V \setminus S)$ such that $|\delta(S)|$ is minimum
begin
  if $|V| = 2$ **then**
    return *Any vertex of V*
  Run Karger's GMC Algorithm on H for $n - \frac{n}{\sqrt{2}} - 1$ iterations.;
  Let H be the resulting multigraph.;
  $S_1 \longleftarrow$ KS-Algorithm(H);
  $S_2 \longleftarrow$ KS-Algorithm(H);
  return $\arg\min\{|S_i| : i \in [2]\}$

Let $p(n)$ the probability of success for KS-Algorithm for a graph with n vertices. Then probability of not picking an edge until $\frac{n}{\sqrt{2}} + 1$ nodes remain is $\geq \frac{1}{2}$ as we have calculated above. Now the resulting graph has $\frac{n}{\sqrt{2}} + 1$ nodes. Hence, probability that KS-Algorithm(H) returns the min-cut is at least $\frac{1}{2} p\left(\frac{n}{\sqrt{2}} + 1\right)$. Therefore,

$$\mathbb{P}[\text{At least one of the run KS-Algorithm(H) returns the min cut}] \geq 1 - \left(1 - \frac{1}{2} p\left(\frac{n}{\sqrt{2}} + 1\right)\right)^2$$

Therefore we have

$$p(n) \geq 1 - \left(1 - \frac{1}{2} p\left(\frac{n}{\sqrt{2}} + 1\right)\right)^2$$

Solving this recursion relation we have $p(n) \geq \frac{1}{\log n}$. Hence, to succeed with high probability we need to run $2\log^2 n$ times.

Now For each run of the KS-Algorithm we have the recursion relation

$$T(n) \geq 2T\left(\frac{n}{\sqrt{2}} + 1\right) + O(n^2)$$

Solving the recursion relation we have $T(n) = O(n^2 \log n)$. Therefore, the time complexity of the total running time is $O(n^2 \log^3 n)$.

Chapter 13

Matching

In section 5.1 we saw how to find a maximal matching in a graph using matroids. Here we will try to find maximum matching.

Maximum Matching
Input: Graph $G = (V, E)$
Question: Find a maximum matching $M \subseteq E$ of G

First we will solve finding maximum matching in bipartite graphs first. Then we will extend the algorithm to general graphs.

13.1 Bipartite Matching

So in this section we will study the following problem:

Bipartite Maximum Matching
Input: Graph $G = (L \cup R, E)$
Question: Find a maximum matching $M \subseteq E$ of G

13.1.1 Using Max Flow

One approach to find a maximum matching is by using max-flow algorithm. For this we introduce 2 new vertices s and t where there is an edge from s to every vertex in L and there is an edge from every vertex in R to t and all edges have capacity 1. Then the max-flow for this directed graph is the maximum matching of the bipartite graph. So we have the algorithm:

Lemma 13.1.1

There exists a max-flow of value k in the modified graph $G' = (V, E')$ if and only there is a matching of size k

Proof: Suppose G' has a matching M of size k. Let $M = \{(u_i, v_i) \colon i \in [k]\}$ where $u_i \in L$ and $v_i \in R$ for all $i \in [k]$. Then we have the flow f, $f(s, u_i) = f(u_i, v_i) = f(v_i, t) = 1$ for all $i \in [k]$. This flow has value k.

Now suppose there is a flow f of value k. Since each edge has capacity 1 then either an edge has flow 1 or it has 0 flow. Since value of flow is k there are exactly k edges outgoing from s with positive flow. Let the edges are (s, u_i) for $i \in [k]$. Now from each u_i there is exactly one edge going out which has positive flow. Now if $\exists\ i \neq j \in [k]$ such that $\exists\ v \in R$, $f(u_i, v) = (u_j, v) = 1$ then $f(v, t) = 2$ but $c_{v,t} = 1$. So this is not possible. Therefore the edges going out from each u_i goes to distinct vertices. These edges now form a matching of size k. ■

Therefore the algorithm successfully returns a maximum matching of the bipertite graph. But we don't know any algorithm for finding maximum matching in general graphs using max-flow. In the next algorithm we will use something called Augmenting paths to find a maximum matching which we will extend to general graphs.

Algorithm 26: BP-MAX-MATCHING-FLOW

Input: $G = (L \cup R, E)$ bipartite graph
Output: Find a maximum matching

begin
  $V \longleftarrow A \cup B \cup \{s, t\}$
  $E' \longleftarrow E$
  for $v \in L$ **do**
    $E' \longleftarrow E' \cup \{(s, v)\}$
  for $v \in R$ **do**
    $E' \longleftarrow E' \cup \{(v, t)\}$
  for $e \in E'$ **do**
    $c_e \longleftarrow 1$
  $f \longleftarrow$ EDMONDS-KARP$(G' = (V, E'), \{c_e : e \in E'\})$
  return $\{e : f(e) > 0, e \in E\}$

13.1.2 Using Augmenting Paths

Definition 13.1.1: *M*-Alternating Path and Augmenting Path

In a graph $G = (V, E)$ and M be a matching in G. Then an M-alternating path is where the edges from M and $E \setminus M$ appear alternatively.

An M-alternating path between two unmatched (also called exposed) vertices is called an augmenting path.

Given a matching M and if there exists an augmenting path p then we can obtain a larger matching M' just by taking the edges in p not in M. Now suppose we are given a bipartite graph $G = (L \cup R, E)$. Let M is a matching in G. Suppse M is a maximum matching. If there exists an augmenting path p then we can obtain a larger matching just by taking the edges in p not in M. This contradicts with M is maximum matching. Hence there are no augmenting paths.

Now we will show that given G and M which is not maximum then we can find an augmenting path with an algorithm. Since M is not maximum there is a vertex v which is not matched

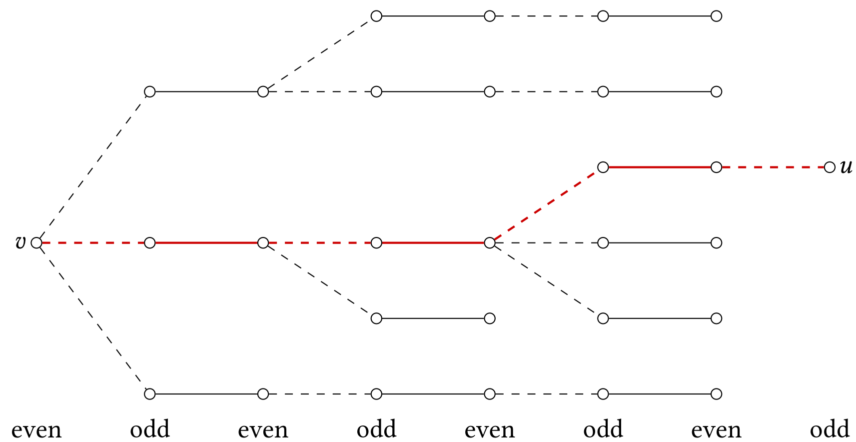
Algorithm 27: FIND-AUGMENTING-PATH(G, v)

Input: $G = (L \cup R, E)$ bipartite graph, matching M (not maximum) and an exposed vertex v
Output: Find an augmenting path starting from v

begin
  $v.mark \longleftarrow even$
  for $u \in L \cup R \setminus \{v\}$ **do**
    $u.mark \longleftarrow$ NULL
  QUEUE Q // For BFS
  ENQUEUE(Q, v)
  while Q *not empty* **do**
    AUTREE(Q)
  return *FAIL*

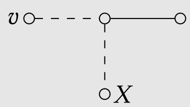
Algorithm 28: AUTREE(Q)

$u \longleftarrow$ DEQUEUE(Q)
if $u.mark == even$ **then**
  for $(u, w) \in E \setminus M$ **do**
    if $w.mark ==$ *NULL* **then**
      ENQUEUE(Q, w)
      $w.mark \longleftarrow odd$
      $w.p \longleftarrow u$
if $u.mark == odd$ **then**
  if $\exists\,(u, w) \in M$ *and* $w.mark ==$ *NULL* **then**
    $w.mark \longleftarrow even$
    $w.p \longleftarrow u$
    ENQUEUE(Q, w)
  else
    Print "$v \rightsquigarrow u$ augmenting path found"



The above algorithm in each iteration checks if the new vertex has mark NULL before adding to the queue. Because of this we are not adding same vertex more than one into the queue and if we follow the parent and child pointers, this forms a tree. We call this tree to be a M-alternating tree. Denote the tree by T.

Note:-



The algorithm may not visit all the vertices in $L \cup R$ in the tree. For example in case of the graph at left the algorithm will not find the vertex

Since the algorithm runs a BFS if there was an edge between two vertices at levels separated by 2 we would have explored that vertex earlier. So our first observation is:

Observation 13.1. *In the tree T there are no edges between vertices at levels separated by 2.*

Observation 13.2. *All even vertices except v are matched in T.*

Observation 13.3. *There are no edges between two odd levels or even levels.*

Lemma 13.1.2

If leaf u is odd there is a $v \rightsquigarrow u$ augmenting path.

Proof: If the odd vertex u is unmatched then clearly there is a $v \rightsquigarrow u$ augmenting path. So let's assume u is matched. Say $(u, w) \in M$. If w is not in T then u can not be a leaf as the algorithm will take the edge $(u, w) \in M$ for next iteration.

So suppose w is in T. Then $w.mark = even$ since otherwise we would have taken then (w, u) edge in T before. But by Observation 13.2 all the even vertices except v are matched in the tree already. So u can not be matched with w ■

Now from the tree T we partition the vertices of T into the even marked vertices and odd marked vertices. So let $L_T = L \cap T$ and $R_T = R \cap T$. Therefore L_T is the set of even marked vertices and R_T is the set of odd marked vertices.

Lemma 13.1.3

$N(L_T) = R_T$

Proof: Vertices in L_T are even vertices from which we explore all the edges not in M. Also all the even vertices except v are matched. So except v for all the vertices in L_T their parent is the matched vertex. Hence for all even vertices except v all the neighbors are in R_T. Since v is exposed v has no matched neighbor. So all the neighbors of v are also in R_T. Therefore $N(L_T) = R_T$. ■

Lemma 13.1.4

Suppose we start the algorithm from an exposed vertex v. Suppose there is no augmenting path from v and let the tree formed by the algorithm is T. Then $|L_T| = |R_T| + 1$.

Proof: Since there is no augmenting path the graph all the leaves of T are even vertices. Otherwise the leaves are odd vertices and then all of them have to be matched. If not then there will exists an augmenting path. Therefore all the leaves of T are even vertices. Now since the vertices in L_T are even vertices and all even vertices except v are matched to unique odd vertex in R_T we have $|L_T| = |R_T| + 1$. ■

Now suppose M is a matching. Let $L' = \{v_1, \dots, v_k\} \subseteq L$ are unmatched vertices. Therefore $|M| = |L| - k$. Then consider the following algorithm:

- Let T_1 be M-alternating tree from v_1 by FIND-AUGMENTING-PATH(G, v_1). L_{T_1}, R_{T_1} are vertices of T_1.
- Let T_2 be M-alternating tree from v_2 by FIND-AUGMENTING-PATH$(G \setminus T_1, v_2)$. L_{T_2}, R_{T_2} are vertices of T_2.
- Let T_3 be M-alternating tree from v_3 by FIND-AUGMENTING-PATH$(G \setminus (T_1 \cup T_2), v_3)$. L_{T_3}, R_{T_3} are vertices of T_3. $\cdots$
- Let T_k be M-alternating tree from v_k by FIND-AUGMENTING-PATH$\left(G \setminus \left(\bigcup_{i=1}^{k-1} T_i\right), v_k\right)$. L_{T_k}, R_{T_k} are vertices of T_k.

Observation 13.4. *v_i is not in T_j for any $j < i$ because otherwise we would have found an augmenting path in T_j.*

Now L_{T_i} for all $i \in [k]$ are disjoint and R_{T_i} for all $i \in [k]$ are disjoint. If G had no augmenting path from v_i for all $i \in [k]$ then there are no augmenting paths in $G \setminus \left(\bigcup_{i=1}^{j} T_i\right)$ for all $j \in [k-1]$ from v_{j+1}. Therefore by Lemma 13.1.4 we have $|L_{T_i}| = |R_{T_i}| + 1 \ \forall\, i \in [k]$. Hence we have

$$\sum_{i=1}^{k} |L_{T_i}| = \sum_{i=1}^{K} (|R_{T_i}| + 1) \implies \left|\bigcup_{i=1}^{k} L_{T_i}\right| = \left|\bigcup_{i=1}^{k} R_{T_i}\right| + k$$

Now by Lemma 13.1.3, $N(L_{T_{j+1}}) = R_{T_{j+1}}$ for all $j \in [k-1]$ in $G \setminus \left(\bigcup_{i=1}^{j} T_i\right)$. Hence

$$N(L_{T_j}) \subseteq \bigcup_{i=1}^{j} R_{T_i} \implies N\left(\bigcup_{i=1}^{k} L_{T_i}\right) = \bigcup_{i=1}^{k} R_{T_i}$$

But $\left|\bigcup_{i=1}^{k} L_{T_i}\right| = \left|\bigcup_{i=1}^{k} R_{T_i}\right| + k$. Therefore any matching of $\bigcup_{i=1}^{k} L_{T_i}$ must leave at least k vertices unmatched. Now all the vertices in $L \setminus \left(\bigcup_{i=1}^{k} L_{T_i}\right)$ with $R \setminus \left(\bigcup_{i=1}^{k} R_{T_i}\right)$ and vice versa. Therefore any matching of L must leave at least k vertices unmatched. Since M is a matching such that exactly k vertices are unmatched. M is a maximum matching. Therefore if there is no augmenting path in G then M is a maximum matching.

We also showed before that if M is a maximum matching then there is no augmenting path in G. Therefore we have the following theorem:

Theorem 13.1.5 Berge's Theorem

A matching M is maximum if and only if there are no augmenting path in G.

Therefore if we start with any matching and each time we find a augmenting path we update the matching by taking the odd edges in the augmenting path and obtain a larger matching. After continuously doing this once when there is no augmenting path we can conclude that we obtained a maximum matching.

Since every time the size of the maximal matching is increased by at least 1. The total number of iterations the algorithm takes to output the maximal matching is $O(n)$ where n is the number of vertices in G. In each iteration it calls the FIND-AUGMENTING-PATH algorithm which takes the time same as time taken in BFS. Hence FIND-AUGMENTING-PATH takes $O(m+n)$ time. Therefore the BP-MAXIMUM-MATCHING algorithm takes $O(n(n+m))$ time.

Algorithm 29: BP-MAXIMUM-MATCHING(G)

```
Input: G = (L ∪ R, E) bipartite graph
Output: Find a maximum matching
begin
    M ⟵ ∅
    while True do
        v ⟵ unmatched vertex
        p ⟵ FIND-AUGMENTING-PATH
        if p == FAIL then
            return M
        for e ∈ p do
            if e ∈ M then
                M ⟵ M \ {e}
            else
                M ⟵ M ∪ {e}
```

13.1.3 Using Matrix Scaling

Here we will show a new algorithm for deciding if a bipartite graph has a perfect matching using matrix scaling. The paper which we will follow is [LSW98]

BIPARTITE PERFECT MATCHING
Input: Graph $G = (L \cup R, E)$
Question: Decide if G has a perfect matching or not.

Suppose $G = (L \cup R, E)$ a bipartite graph. If bipartite adjacency matrix of the graph G is A then the permanent of the matrix A,

$$per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i,\sigma(i)}$$

counts the number of perfect matchings in G. So we want to check if for a given bipartite graph $(L \cup R, E)$, $per(A) > 0$ or not where A is the bipartite adjacency matrix. Now there is a necessary and sufficient condition for existence of perfect matching in a bipartite graph which is called Hall's condition.

Theorem 13.1.6 Hall's Condition

A bipartite graph $G = (L \cup R, E)$ has an L-perfect matching if and only if $\forall\, S \subseteq L$, $|S| \leq |N(S)|$ where $N(S) = \{v \in R \colon \exists\, u \in L,\ (u, v) \in E\}$

Proof: Now if G has a L-perfect matching then for every $S \subseteq L$, S is matched with some $T \subseteq R$ such that $|S| = |T|$. Therefore $T \subseteq N(S) \implies |S| = |T| \leq |N(S)|$.

Now we will prove the opposite direction. Suppose for all $S \subseteq L$ we have $|S| \leq |N(S)|$. Assume there is no L-perfect matching in G. Let M be a maximum L-matching in G. Let $u \in L$ is unmatched. Now consider the following sets:

$$X = \{x \in L \colon \exists\, M\text{-alternating path from } u \text{ to } x\}, \qquad Y = \{y \in R \colon \exists\, M\text{-alternating path from } u \text{ to } y\}$$

Now notice that $N(X) \subseteq Y$. Since in a M-alternating path from u whenever the odd edges are not matching edges and the even edges are matching edges. So in the odd edges we can pick any neighbor except the one it is matched with and the immediate even edge before that connects that vertex with the vertex in R it is matched with. Hence we have $N(X) \subseteq Y$.

Now it suffices to prove that $|X| > |Y|$. Now let $y \in Y$. Suppose $u \rightsquigarrow x' \longrightarrow y$ be the M-alternating path. If y is not matched then we could increase the matching by taking the odd edges of the path and thus obtain a matching with larger size than M. But M is maximum matching. Hence y is matched. Therefore we can extend the path by taking the matching edge incident on y and go the the vertex $x'' \in L$ i.e. the new M-alternating path becomes $u \rightsquigarrow x' \longrightarrow y \longrightarrow x''$ to have a M-alternating path $u \rightsquigarrow x''$. So $|X| > |Y|$.

Therefore we obtained a set of vertices $X \subseteq Y$ such that $|X| > |Y| \geq N(X)|$. This contradicts the assumption. Hence contradiction. Therefore G has a L-perfect matching. ■

We will use hall's condition on the adjacency matrix to check if $per(A)$ is positive or not. Now multiplying a row or a column of a matrix by some constant c also multiplies the permanent of the matrix by c as well. In fact if $d_1, d_2 \in \mathbb{R}_+^n$ and $D_1 = diag(d_1)$ and $D_2 = diag(D_2)$ then $per(D_1AD_2) = \left(\prod_{i=1}^n d_{1_i}\right)\left(\prod_{i=1}^n d_{2_i}\right) per(A)$. So we can scale our original matrix A to obtain a different matrix B and from B we can approximate $per(A)$ by approximating $per(B)$. A natural strategy is to seek an efficient algorithm for scaling A to a doubly stochastic B.

Definition 13.1.2: Doubly Stochastic Matrix

A matrix $M \in \mathbb{R}^{m\times m}$ is doubly stochastic if entries are non-negative and each row and column sum to 1.

First we will show that Hall's Condition holds for doubly stochastic matrix. First let's see what it means for a matrix to satisfy hall's condition. A matrix with all entries non-negative holds Hall's Condition if for all $S \subseteq [n]$ if $T = \{i \in [n] : \exists\, j \in S,\ A(i,j) \neq 0\}$ then $|T| \geq |S|$. This also corresponds to the bipartite adjacency matrix satisfying the hall's condition since for any set of rows S the number of columns for which in the S rows at least one entry is non zero should be greater than or equal to $|S|$.

Lemma 13.1.7

Hall's Condition holds for doubly stochastic matrix.

Proof: Let M be the doubly stochastic matrix. Let $S \subseteq [n]$. So consider the $|S| \times n$ matrix which only consists of the rows in S. Call this matrix M_S^r. Now suppose T be the set of columns in M_S^r which has nonzero entries. Now consider the $n \times |T|$ matrix which only consists of the columns in T. Call this matrix M_T^c. Now since M is doubly stochastic we know sum of entries of M_S^r is $|S|$ and sum of entries of M_T^c is $|T|$. Our goal is to show $|S| \leq |T|$. Now since T is the only set of columns which have nonzero columns in M_S^r the elements which contributes to the sum of entries in M_S^r are in the T columns in M_S^r. Since these elements are also present in M_T^c we have $|T| \geq |S|$. ■

Hence for doubly stochastic matrices the permanent is positive. Now not all matrices are doubly stochastic. And in fact matrices with permanent zero will not be doubly stochastic so no amount of scaling will make it doubly stochastic. So we will settle for approximately doubly stochastic matrix. In order to make a matrix doubly stochastic first for each row we will divide the row with their row some. Now it becomes row stochastic. Then if its not approximately doubly stochastic for each column we will divide the column entries with their column sum. But first what ϵ-approximate doubly stochastic matrix means.

Definition 13.1.3: $\epsilon-$Approximate Doubly Stochastic Matrix

A matrix is $\epsilon-$approximate doubly stochastic if for each column, the column sum is in $(1-\epsilon, 1+\epsilon)$ and for each row, the row sum is in $(1-\epsilon, 1+\epsilon)$

Now we will show that even for ϵ-approximate doubly stochastic matrix the hall's condition holds.

Lemma 13.1.8

Halls's Condition holds for ϵ-approximate doubly stochastic matrix for $\epsilon < \frac{1}{10n}$

Proof: Let M is $\epsilon-$approximate doubly stochastic matrix. Let $S \subseteq [n]$. So consider the $|S| \times n$ matrix which only consists of the rows in S. Call this matrix M_S^r. Now suppose T be the set of columns in M_S^r which has nonzero entries. Now consider the $n \times |T|$ matrix which only consists of the columns in T. Call this matrix M_T^c. Now the sum of entries in M_S^r is $\geq |S|(1-\epsilon)$ and sum of entries in M_T^c is $\leq |T|(1-\epsilon)$. Now since T is the only set of columns which have nonzero columns in M_S^r the elements which contributes to the sum of entries in M_S^r are in the T columns in M_S^r. Since these elements are also present in M_T^c we have $||T|(1+\epsilon) \geq |S|(1-\epsilon)$. Therefore we have

$$|T| \geq |S|\frac{1-\epsilon}{1+\epsilon} = |S|\left(1 - \frac{2\epsilon}{1+\epsilon}\right) \geq |S|(1-2\epsilon) > |S|\left(1-\frac{1}{5n}\right) \geq |S|\left(1-\frac{1}{|S|}\right) > |S| - 1$$

Since T is an integer we have $|T| \geq |S|$. Hence the Hall's condition holds. ■

Therefore permanent of ϵ-approximate doubly stochastic matrix is also positive. Hence our algorithm for bipartite perfect matching is:

Algorithm 30: BP-MATRIX-SCALING

```
Input: Bipartite adjacency matrix A of G = (L ∪ R, E)
Output: Decide if G has a perfect matching.
begin
    while True do
        A ⟵ Scale every rows of A to make it row stochastic.
        if All column-sums are in (1 − ϵ, 1 + ϵ) then
            return Yes
        A ⟵ Scale every column of A to make it column stochastic.
        if All row-sums are in (1 − ϵ, 1 + ϵ) then
            return Yes
```

In both if conditions we are checking if the matrix is $\epsilon-$approximate doubly stochastic matrix. The moment it becomes a $\epsilon-$approximate doubly stochastic matrix we are done.

Now if G doesn't have a perfect matching then we will never reach a ϵ-approximate doubly stochastic matrix since otherwise Hall's condition will hold and then we will have that the permanent is positive. So if G doesn't have a perfect matching the algorithm will run in an infinite loop. We only need to check if G has a perfect matching the algorithm returns Yes.

We will now define a potential function $\Phi\colon \mathbb{Z}_0 \longrightarrow \mathbb{R}_+$. Let $\sigma \in S_n$ such that $a_{i,\sigma(i)} \neq 0$ for all $i \in [n]$.Now if an entry of the matrix is nonzero then it is always nonzero since all the entries are non-negative. Now since the scalings are symmetric we will define the potential function for i^{th} scaling (row/column) is $\Phi(i) = \prod\limits_{i=1}^{n} a_{i,\sigma(i)}$. So we have $\Phi(0) = 1$ since at first all the entries of the matrix are from $\{0,1\}$. Also we know $\Phi(t) \leq 1$ for all t since every time we are scaling the matrix. Now $\Phi(1) \geq \frac{1}{n^n}$ since every row-sum can be at most n so it will be divided by n and therefore $a_{i,\sigma(i)} \geq \frac{1}{n}$ for all $i \in [n]$. Now to show the while loop stops if G has a perfect matching it suffices to show that $\Phi(t)$ increases by a multiplicative factor. So we have the following lemma.

Lemma 13.1.9

For all t, $\Phi(t+1) \geq \Phi(t)(1+\alpha)$ for some $\alpha \in (0,1)$.

Proof: Let A' denote the matrix at the t^{th} scaling where the $(t-1)^{th}$ scaling was column-scaling. Let A'' denote the matrix after row-scaling. Now since we went to the next iteration not all column-sums are in $(1-\epsilon, 1+\epsilon)$ after scaling the rows. Now the row sums of A'' are 1. Therefore we have

$$\frac{\Phi(t)}{\Phi(t+1)} = \prod_{i=1}^{n} \textit{Col-sum}_i(A'') \leq \left(\frac{\sum\limits_{i=1}^{n} \textit{Col-sum}_i(A'')}{n}\right)^n = \left(\frac{\sum\limits_{i=1}^{n} \textit{Row-sum}_i(A'')}{n}\right)^n = 1 \implies \Phi(t) \leq \Phi(t+1)$$

Similarly we can say the same if $(t-1)^{th}$ scaling was row-scaling. Since not all column-sums are in $(1-\epsilon, 1+\epsilon)$ we have $\sum\limits_{i=1}^{n} (\textit{Col-sum}_i(A'') - 1)^2 \geq \epsilon^2$. Therefore using Lemma 13.1.10 we have

$$\frac{\Phi(t)}{\Phi(t+1)} \leq 1 - \frac{\epsilon^2}{2} \implies \Phi(t+1) \geq \left(1 + \frac{\epsilon^2}{2}\right)\Phi(t)$$

Therefore we have the lemma. ■

We have $\epsilon < \frac{1}{10n}$. Therefore if $t \geq 200n^4$ then we have

$$1 \geq \Phi(t) \geq \frac{1}{n^n}\left(1 + \frac{1}{200n^2}\right)^t \geq \frac{1}{n^n} e^{n^2} > 1$$

Hence the while loop will iterate for at most $200n^4$ iterations. Hence this algorithm takes $O(n^4)$ time. Hence if G has a perfect matching the algorithm runs for at most $O(n^4)$ iterations. And if G doesn't have a perfect matching then the loop never stops. So we have the new modified algorithm to prevent infinite looping:

Algorithm 31: BP-Matrix-Scaling

Input: Bipartite adjacency matrix A of $G = (L \cup R, E)$
Output: Decide if G has a perfect matching.

begin
  $\epsilon \longleftarrow \frac{1}{20n}$
  for $i \in [200n^4]$ **do**
    $A \longleftarrow$ Scale every rows of A to make it row stochastic.
    if *All column-sums are in* $(1-\epsilon, 1+\epsilon)$ **then**
      return *Yes*
    $A \longleftarrow$ Scale every column of A to make it column stochastic.
    if *All row-sums are in* $(1-\epsilon, 1+\epsilon)$ **then**
      return *Yes*

We will prove the helping lemma needed to prove Lemma 13.1.9.

Lemma 13.1.10

Suppose $x_1, \dots, x_n \geq 0$ and $\sum\limits_{i=1}^{n} x_i = n$ and $\sum\limits_{i=1}^{n} (1-x_i)^2 \geq \delta$. Then $\prod\limits_{i=1}^{n} x_i \leq 1 - \frac{\delta}{2} + o(\delta)$.

Proof: Denote $\rho_i = x_i - 1$. So $\sum\limits_{i=1}^{n} \rho_i = 0$ and $\sum\limits_{i=1}^{n} \rho^2 \geq \delta$. Now

$$\log(1+\rho_i) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{\rho_i^j}{j} \implies \log(1+\rho_i) \leq \rho_i - \frac{\rho_i^2}{3} + \frac{\rho_i^3}{3} \implies 1 + \rho_i \leq e^{\rho_i - \frac{\rho_i^2}{3} + \frac{\rho_i^3}{3}}$$

Therefore we have

$$\prod_{i=1}^{n} x_i \leq \exp\left[\sum_{i=1}^{n} \rho_i - \sum_{i=1}^{n} \frac{\rho_i^2}{3} + \sum_{i=1}^{n} \frac{\rho_i^3}{3}\right] \leq \exp\left[0 - \frac{\delta}{2} + \frac{\left(\sum\limits_{i=1}^{n} \rho_i^2\right)^{\frac{3}{2}}}{3}\right] = \exp\left[-\frac{\delta}{2} + \frac{\delta^{\frac{3}{2}}}{3}\right] \leq 1 - \frac{\delta}{2} + o(\delta)$$

Therefore we have the lemma. ■

There is also a survey, [Ide16] on use of matrix scaling in different results.

13.2 Matching in General Graphs

Here we give a similar algorithm for finding maximum matching in general graph as in the case of bipartite graphs in subsection 13.1.2. We will give a similar characterization for the maximum matching in general graphs. First we will show a extension of berge's lemma to general graphs.

Theorem 13.2.1

For any graph $G = (V, E)$, $M \subseteq E$ is a maximal matching if and only if there is no augmenting paths in G.

Proof: Suppose M is a maximal matching. Then if G has an augmenting path p. Then we can just take the odd edges in p and then replace the edges in $M \cap p$ with those edges i.e. $M \triangle p$ and this is a larger matching than M. But this contradicts the maximum property of M. Hence G has no augmenting paths.

Now we will show that if M is not a maximum matching then G has an augmenting path. So suppose M is not a maximum matching. Let M' is a maximum matching. Then consider the graph $G' = (V, E')$ where $E' = M \triangle M'$. Now every vertex in V has degree $\in \{0, 1, 2\}$ in G'. Hence the connected components of G' are isolated vertices, paths and cycles. In a path or cycle the edges of M and M' not in both appear alternatively. Therefore the cycles are even cycles. Since $|M'| > |M|$ there exists a path p such that number $|p \cap M'| > |p \cap M|$. Therefore the starting and ending edge of p are in M'. Hence p is an augmenting path in G. ■

Therefore like in the case of bipartite graphs we will search for augmenting paths in G for matching M and if we can find an augmenting path p we will update the matching by taking $M' = M \triangle p$ and obtain a larger matching. But unlike bipartite graphs we can not run the same algorithm for finding augmenting paths as there can be edges between two odd layers or two even layers. So in the M-alternating tree there can be odd cycles but these odd cycles have all vertices except one vertex are matched using edges of the cycle. So we look for these special structures in the M-alternating tree called *blossom*.

Definition 13.2.1: Blossom

A blossom is an odd cycle in which only one vertex is unmatched and the remaining vertices are matched using edges of the cycle. The exposed vertex is called the *base* of the blossom.

Note that given a matching M and blossom B we can modify M so that any $v \in B$ is the base, since no vertex B has a matching outside.

The algorithm for finding augmenting path in non-bipartite graphs works by detecting blossoms in M-alternating tree starting from some exposed vertex. The idea is to then contract the blossoms into single vertex and then this process is repeated in the modified graph.

Let B be a blossom in G. Then the new graph is $G' = (V', E')$ where

$$V' = (V \setminus B) \cup \{v_B\}, \qquad E' = \Big(E \setminus \{(u,v) \colon u \in B \text{ or } v \in B\}\Big) \cup \{(u, v_B) \colon u \notin B, v \in B, (u,v) \in E\}$$

Observation 13.5. *v_B is an exposed vertex in G'*

For this first time we have to show that finding augmenting path in the contracted graph gives an augmenting path in original graph and vice versa.

Lemma 13.2.2

$G = (V, E)$ is a graph with matching $M \subseteq E$ and a blossom B. Let $G' = (V', E')$ be the contacted graph after contracting B into a single vertex v_B. Then G with matching M has an augmenting path if and only G' with matching $M' = M \setminus \{(u,v) \colon u, v \in B\}$ has an augmenting path

Proof: Let p be an augmenting path in G. Let $p = (v_0, \dots, v_k)$ where both v_0 and v_k are exposed vertices. Hence at least one of v_0 and v_k are not in B. WLOG $v_0 \notin B$. If none of the vertices in p are in B then p also exists in G' as well. Therefore G' has an augmenting path. So suppose $p \cap B \neq \emptyset$. Suppose v_r be the first vertex in p that is in B. Then $p' = (v_0, \dots, v_{r-1}, v_B)$ is an augmenting path since v_B is an exposed vertex in G'.

Now let p' is an augmenting path in G'. If v_B is not in p' then p' also exists in G. Therefore p' is an augmenting path in G. Suppose v_B is in p'. ■

Chapter 14

Linear Programming

14.1 Introduction

Definition 14.1.1: Linear Program

A linear programming problem asks for a vector $x \in \mathbb{R}^d$ that maximizes or minimizes a given linear function, among all vectors x that satisfy given set of linear inequalities.

The general form of a maximization linear programming problem is the following: given $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $a_i \in \mathbb{R}^n$ for each $i \in [m]$ then

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & a_i^T x \le b_i \quad \forall\, i \in [p], \\ & a_i^T x = b_i \quad \forall\, i \in \{p+1, \dots, p+q\}, \\ & a_i^T x \ge b_i \quad \forall\, i \in \{p+q+1, \dots, m\}, \\ & x_j \ge 0 \quad \forall j \in [k], \\ & x_j \le 0 \quad \forall j \in [\{k+1, \dots, k+l\} \quad \text{(Some } x_j\text{'s are free)} \end{aligned}$$

The similar goes for minimization linear programming problem. For maximization problem we can always write the LP in the form

$$\begin{aligned} \text{maximize} \quad & c^T \hat{x} \\ \text{subject to} \quad & \hat{a}_i^T x \le b_i' \quad \forall\, i \in [m], \\ & x_j' \ge 0 \quad \forall j \in [n] \end{aligned}$$

And then the LP is said to be in the ***canonical form***. What we can do is the following:

- For $i \in \{p+q+1, \dots, m\}$, we can replace $a_i^T x \le b_i$ with $-a_i^T x \ge -b_i$
- For $i \in \{p+1, \dots, p+q\}$, we can replace with two constraints $a_i^T x \ge b_i$ and $a_i^T x \le b_i$
- For $j \in \{k+1 \dots, k+l\}$, we can replace $x_j \le 0$ with $-x_j \ge 0$
- For $j \in \{k+l+1 \dots, n\}$, we can replace the free x_j's with $x_j^+ - x_j^-$ all the equations where $x_j^+, x_j^- \ge 0$

This way we can always get a LP of that form. Now we can replace the $\hat{a}_i$ for $i \in [m]$ with a matrix $A \in \mathbb{R}^{m \times n}$ and replace the constraint $\hat{a}_i^T x \le b_i', \forall\, i \in [m]$ with $Ax \le b$

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \le b, \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \ge b, \\ & x \ge 0 \end{aligned}$$

14.2 Geometry of LP

Definition 14.2.1: Feasible Point and Region

A point $x \in \mathbb{R}^n$ is *feasible* with respect to some LP if it satisfies all the linear constraints. The set of all feasible points is called the *feasible region* for that LP.

Feasible region of a LP has a particularly nice geometric structure. Before that we will first introduce some geometric terminologies used in the linear programming context:

Definition 14.2.2: Hyperplane, Polyhedron, Polytope

- **Line**: The set $\{x + \lambda d, \lambda \in \mathbb{R}\}$ is line for any $x, d \in \mathbb{R}^n$.
- **Hyperplane**: The set $\{x \in \mathbb{R}^n : a^x = b\}$ is a hyperplane for any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- **Hyperspace**: The set $\{x \in \mathbb{R}^n : a^x \leq b\}$ is a hyperspace or half-space for any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- **Polyhedron**: A polyhedron is the intersection of a finite set of half-spaces i.e. the set $\{x \in \mathbb{R}^n : Ax \leq b\}$ for any $A \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^m$.
- **Polytope**: A bounded polyhedron is called a polytope.

Now it is not hard to verify that any polyhedron is a convex set i.e. if a polyhedron contains two points then it contains the entire line segment joining those two points.

Lemma 14.2.1

Polyhedron is a convex set

Hence the feasible region of a LP creates a polyhedron in $\mathbb{R}^n$. And $c^T x$ is the hyperplane normal to the vector c and the objective of the LP is by moving the plane normal to the vector c for which point in the polyhedron the hyperplane $c^T x$ has the highest value. Since polyhedron can be unbounded there may not exists any point x where $c^T x$ is maximum.

Suppose we have a LP

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \end{aligned}$$

Let P be the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. Then given $x^* \in P$ if any constraint $a_i^T x^* = b_i$ then this constrain is said to be *tight* or *binding* or *active* at x^*. Now two constraints $a_i^T x \leq b_i$ and $a_j^T x \leq b_j$ are said to be linearly independent if a_i and a_j are linearly independent.

Definition 14.2.3: Basic Solution and Basic Feasible Solution

$x^* \in \mathbb{R}^n$ is a basic solution if n linearly independent constraints are active at x^* (Doesn't need to be feasible).

$x^* \in \mathbb{R}^n$ is a basic feasible solution if x^* is a basic solution and $x^* \in P$. The basic feasible solutions are also called *corners* of a polyhedron.

Theorem 14.2.2

Given a LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0 \end{aligned}$$

Let P is the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$. Suppose P is non-empty and has at least one basic feasible solution then either the optimal value is $-\infty$ or there is an optimal basic feasible solution.

Theorem 14.2.3

If polyhedron P does not contain a line it contains at least one basic feasible solution (Hence if P is bounded it contains at least one basic feasible solution).

With this geometry in hand, we can easily picture two pathological cases where a given linear programming problem has no solution. The first possibility is that there are no feasible points; in this case the problem is called *infeasible*. The second possibility is that there are feasible points at which the objective function is arbitrarily large; in this case, we call the problem *unbounded*. The same polyhedron could be unbounded for some objective functions but not others, or it could be unbounded for every objective function.

Example 14.2.1

- **Maximum Matchings:** Given undirected graph $G = (V, E)$. Say variable x_e for each $e \in E$, $x_e = 1 \implies e$ in matching and $x_e = 0$ otherwise.

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e \text{ incident on } v} x_e \leq 1 \quad && \forall\, v \in V, \\ & x_e \geq 0 && \forall\, e \in E, \\ & x_e \in \{0, 1\} && \forall\, e \in E \end{aligned}$$

Observation. *M is a matching iff $\{x \colon x_e = 1$ if $e \in M, = 0$ otherwise$\}$ is a feasible solution*

- **Maximum $s-t$ Flow:** Given directed graph $G = (V, E)$ with vertices s, t and capacity c_e on edges. Say variable x_e for each edge and equal to flow on that edge. Then the LP of this problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in out(s)} x_e \\ \text{subject to} \quad & \sum_{e \in in(v)} x_e - \sum_{c \in out(v)} x_e = 0 \quad && \forall\, v \in V, v \neq s, t, \\ & c_e \geq x_e \geq 0 && \forall\, e \in E \end{aligned}$$

We will now introduce a theorem without proof that for any LP with a polytope we can find a solution in polynomial time.

Theorem 14.2.4

Let $P = \{x \in \mathbb{R}^n \colon Ax \geq b\}$ be a polytope. Then we can find an optimal basic feasible solution for the LP $\min c^T x$ where $x \in P$ in polynomial time.

14.3 LP Integrality

For the LP for matchings in bipartite graphs $G = (L \cup R, E)$ we have:

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e \text{ incident on } v} x_e \leq 1 \quad && \forall\, v \in V, \\ & x_e \geq 0 && \forall\, e \in E \end{aligned}$$

We want $x_e \in \{0,1\}$ i.e. we want to have integral solution for this LP

Question 14.1

LP's can give fractional solutions. When is solution integral?

Sufficient Condition: Every basic feasible solution of the feasible polytope is integral i.e. x^* is basic feasible solution $\implies x^* \in \mathbb{Z}^n$. If all basic feasible solution are integral then for all $I \subseteq [m]$ with $|I| = n$, $A_I^{-1}b_I$ is integral. Let $x = A_I^{-1}b_I$ Then j^{th} component $x_j = \frac{|A_I^j|}{|A|}$ (Cramer's Rule).

14.3.1 Totally Unimodular Matrix

Definition 14.3.1: Totally Unimodular Matrix (TUM)

A matrix $A \in \{0,1,-1\}^{m\times n}$ is totally unimodular (TU) if every square submatrix of A has determinant $-1, 0, 1$.

Hence in the above LP is A is TU and b is integral then all basic feasible solutions are integral.

Lemma 14.3.1

Let A be TUM and $b \in \mathbb{Z}^n$ then $P = \{x : Ax \geq b\}$ is integral i.e. every basic feasible solution is integral.

Hence using Theorem 14.2.4 if the polytope is integral we can find optimal integral solution in polynomial time. We will now discuss properties of Totally Unimodular Matrix.

Lemma 14.3.2

$A \in \{0,1,-1\}^{m\times n}$ is TU iff the following are TU:

(i) $-A$

(ii) A^T

(iii) $\begin{bmatrix} A & e_i \end{bmatrix}$, $\begin{bmatrix} A & -e_i \end{bmatrix}$

(iv) $\begin{bmatrix} A & I \end{bmatrix}$, $\begin{bmatrix} A & -I \end{bmatrix}$

(v) $\begin{bmatrix} A & A_i \end{bmatrix}$, $\begin{bmatrix} A & -A_i \end{bmatrix}$ where A_i is the i^{th} column of A.

Corollary 14.3.3

If A is TUM and $a, b, c, d \in \mathbb{Z}^n$ are integer vectors then the polytope $Q = \{x \in \mathbb{R}^n : a \leq Ax \leq b, c \leq x \leq d\}$ is integral.

Proof: We can combine the four inequalities in one inequality. Consider the matrix $\begin{bmatrix} A & -A & I & -I \end{bmatrix}^T$. Then the given polytope is

$$Q = \left\{ x \in \mathbb{Z}^n : \begin{bmatrix} A \\ -A \\ I \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -a \\ d \\ -c \end{bmatrix} \right\}$$

By Lemma 14.3.2, $\begin{bmatrix} A & -A & I & -I \end{bmatrix}^T$ is a TUM since A is TUM. Therefore the polytope Q is integral. ■

The following theorem lets us to give a necessary and sufficient condition to check if a given matrix is TUM. Again we will accept the following theorem without the proof since the proof is a little nontrivial.

Theorem 14.3.4

Let $A \in \{-1,0,1\}^{m\times n}$. Then A is TU iff every set $S \subseteq [n]$ can be partitioned into S_1, S_2 such that

$$\sum_{i\in S_1} A_i - \sum_{i\in S_2} A_i \in \{-1,0,1\}^m$$

where A_i is the i^{th} column of A. C

14.3.2 Integrality of Some Well-Known Polytopes

Now using this theorem we will show that the polytope for bipartite maximum matching is integral. The LP for bipartite maximum matching is given by:

$$\begin{aligned} \text{maximize} \quad & \sum_{e\in E} x_e \\ \text{subject to} \quad & \sum_{e \text{ incident on } v} x_e \leq 1 && \forall\, v \in V, \\ & x_e \geq 0 && \forall\, e \in E \end{aligned}$$

Lemma 14.3.5

The polytope for bipartite maximum matching is integral.

Proof: Let A be the matrix for the polytope. Now clearly from the construction of the polytope we have $A \in \{0,1\}^{n\times m}$ where $n = |V|$ and $m = |E|$. Now we will show that A^T is TUM. Let L and R are the two sets of vertices in the bipartite graph. Now suppose $S \subseteq L \cup R$. Then take $S_1 = S \cap L$ and $S_2 = S \cap R$. Then for any row $e \in E$, we have

$$\sum_{i\in S_1} A_i - \sum_{i\in S_2} A_i \in \{-1,0,1\}$$

Hence A^T is TUM and therefore by Lemma 14.3.2 A is TUM. Hence the polytope for bipartite maximum matching is integral. ■

Note:-

For general graphs this polytope is not integral. Consider the triangle graph K_3. Then the point $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ is a feasible solution but not in the convex hull of the integral solutions $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Lemma 14.3.6

The LP for $s-t$ max flow is

$$\begin{aligned} \text{maximize} \quad & \sum_{e\in out(s)} x_e \\ \text{subject to} \quad & \sum_{e\in in(v)} x_e - \sum_{c\in out(v)} x_e = 0 && \forall\, v \in V, v \neq s,t, \\ & c_e \geq x_e \geq 0 && \forall\, e \in E \end{aligned}$$

Then the max flow polytope is integral.

Proof: Let A be the matrix for the polytope. We will show that A is TUM. Given $S \subseteq V \setminus \{s,t\}$ take $S_1 = S$ and $S_2 = \emptyset$. By the first condition of the polytope for all vertices we already have satisfied the condition

$$\sum_{i\in S_1} A_i - \sum_{i\in S_2} A_i = 0 \in \{-1,0,1\}^m$$

Therefore the polytope is TUM and hence integral. ■

14.4 Duality

Suppose we have the following LP:

$$\begin{aligned} \text{minimize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 - x_2 \geq 3, \\ & 2x_1 + x_2 \geq 1, \\ & x_1, x_2 \geq 0 \end{aligned}$$

Suppose we want to have a lower bount on the optimal solution of the LP. Then we will try to find a linear combination of the constriants such that in the LHS we obtain some thing which is at most the objective function and on the RHS we get the lower bound. So let we multiply the first constraint with y_1, second with y_2. For now y_1, y_2 are unknowns. Then we have the following:

$$\begin{aligned} x_1 + 2x_2 &\geq (y_1 + 2y_2)x_1 + (-y_1 + y_2)x_2 \\ &= y_1(x_1 - x_2) + y_2(2x_1 + x_2) \geq 3y_1 + y_2 \end{aligned}$$

But we also have the conditions that the coefficients of x_1 and x_2 can not be more than the coefficients of x_1 and x_2 in the objective function respectively. So we have the following conditions:

$$\begin{aligned} y_1 + 2y_2 &\leq 1 \\ -y_1 + y_2 &\leq 2 \end{aligned}$$

So now we have found a maximization LP which gives us a lower bound on the optimal solution of the original LP:

$$\begin{aligned} \text{maximize} \quad & 3y_1 + y_2 \\ \text{subject to} \quad & y_1 + 2y_2 \leq 1, \\ & -y_1 + y_2 \leq 2, \\ & y_1, y_2 \geq 0 \end{aligned}$$

This is called the *dual* of the original LP. The original LP is called the *primal* of the dual. The primal and dual are related in a very nice way. The following theorem gives us the relation between primal and dual.

For every minimization LP there is a dual LP that provides a lower bound on the optimal value of the primal LP.

Note:-
If the Primal LP is unbounded then the dual LP is infeasible.

Lemma 14.4.1
Dual of Dual is the primal LP

14.4.1 Dualization of LP

If the primal LP is in canonical form then we have the following:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \text{minimize} \quad & b^T y \\ \text{subject to} \quad & A^T y \leq c, \\ & x \geq 0 \end{aligned}$$

$\boxed{\text{Primal}}$ $\boxed{\text{Dual}}$

But if the primal LP is not in the canonical form then we have two options: either we can convert the primal to the canonical form and the dualize it or we can directly dualize the primal LP. The following method gives us a way to dualize the primal LP without converting it tot the canonical form.

$$
\begin{array}{llll}
\text{maximize} & c^T x & \\
\text{subject to} & A_j x \geq b_j & \forall\ j \in [d], \\
& A_j x = b_j & \forall\ j \in \{d+1, \dots, m\}, \\
& x_i \geq 0 & \forall\ i \in [k], \\
& x_i \text{ is free} & \forall\ i \in \{k+1, \dots, n\}
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{llll}
\text{minimize} & b^T y & \\
\text{subject to} & \sum_{j=1}^{m} A_{ji} y_j \leq c_i & \forall\ i \in [k], \\
& \sum_{j=1}^{m} A_{ji} y_j = c_j & \forall\ i \in \{k+1, \dots, n\}, \\
& y_j \geq 0 & \forall\ j \in [d], \\
& y_j \text{ is free} & \forall\ i \in \{d+1, \dots, m\}
\end{array}
$$

$\boxed{\text{Primal}}$ $\boxed{\text{Dual}}$

So we have the following observations:

Observation. *In dualization of a LP which is not in canonical form*

Primal		*Dual*
Non-negative variables	$\Longleftrightarrow$	*Inequality constraints*
Free variables	$\Longleftrightarrow$	*Equality constraints*

14.4.2 Weak and Strong Duality

Now as the motivation for constructing the dual LP. We have the following theorem which proves the any feasible solution of the dual LP indeed gives a lower bound on the optimal solution of the primal LP.

Theorem 14.4.2 Weak Duality Theorem

If x, y are feasible solutions for the primal and dual LPs respectively and then $c^T x \geq b^T y$.

Proof: We have

$$b^T \leq \sum_{j=1}^{d} y_j(A_j x) + \sum_{j=d+1}^{m} y_j(A_j x) = \sum_{j=1}^{d} y_j A_j x = \sum_{j=1}^{m}\sum_{i=1}^{n} y_j A_{ji} x_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} A_{ji} y_j \leq \sum_{i=1}^{m} x_i c_i = c^x$$

Hence we have the theorem. ∎

We also have a much stronger theorem which tells us that the optimal solutions of the primal and dual LPs are equal.

Theorem 14.4.3 Strong Duality Theorem

Let the primal and dual LP are feasible and x^*, y^* are the optimal solutions of the primal and dual LPs respectively. Then $c^T x^* = b^T y^*$.

Notice that if for any feasible solution y of the dual LP is $c^T x^* = b^T y$ then y must be the optimal solution of the dual LP.

14.4.3 Complementary Slackness

Question 14.2

Suppose we have optimal solutions x^*, y^* of the primal and dual LPs respectively. What can be said about which constraints are tight in the primal and dual?

Theorem 14.4.4 Complementary Slackness

Let x^*, y^* be the optimal solutions of the primal and dual LPs respectively iff:

(i) If $A_j x^* > b_j$ then $y_j^* = 0$.

(ii) If ${A^i}^T y^* < c_i$ then $x_i^* = 0$.

Proof: Suppose x^*, y^* are the optimal solutions of the primal and dual LPs respectively. Then by Strong Duality Theorem we have

$$\sum_{i=1}^{k} x_i \sum_{j=1}^{m} A_{ji} y_j + \sum_{i=k+1}^{n} x_i \sum_{j=1}^{m} A_{ji} y_j = \sum_{i=1}^{k} x_i c_i + \sum_{i=k+1}^{n} x_i c_i$$

So we have

$$\sum_{i=1}^{k} x_i \sum_{j=1}^{m} A_{ji} y_j = \sum_{i=1}^{k} x_i c_i$$

Hence either $x_i = 0$ or $\sum_{j=1}^{m} A_{ji} y_j = c_i$ for all $i \in [k]$. So ${A^i}^T y^* < c_i$ implies $x_i^* = 0$. Similarly we have $A_j x^* > b_j$ then $y_j^* = 0$. ■

There is also a relaxed version of the complementary slackness theorem, Theorem 15.1.4 which is useful in practice. It is explained in the next chapter.

14.4.4 Max-Flow Min-Cut Theorem

So here using LP-duality we give another proof of Max-Flow Min-Cut Theorem. The LP for maximum flow is given by:

$$\begin{aligned}
\text{maximize} \quad & \sum_{e \in out(s)} x_e \\
\text{subject to} \quad & \sum_{e \in in(v)} x_e - \sum_{c \in out(v)} x_e = 0 \qquad \forall\, v \in V, v \neq s, t, \\
& c_e \geq x_e \qquad \forall\, e \in E, \\
& x_e \geq 0 \qquad \forall\, e \in E
\end{aligned}$$

We can convert this LP by adding edges of $in(s)$ and giving them capacity 0. So we have the modified LP:

$$\begin{aligned}
\text{maximize} \quad & \sum_{e \in out(s)} x_e - \sum_{e \in in(s)} x_e \\
\text{subject to} \quad & \sum_{e \in in(v)} x_e - \sum_{c \in out(v)} x_e = 0 \qquad \forall\, v \in V, v \neq s, t, \\
& c_e \geq x_e \qquad \forall\, e \in E, \\
& x_e \geq 0 \qquad \forall\, e \in E
\end{aligned}$$

For the first constraint we have the variables α_v and for the second constrain we have the variables β_e. So the dual of this LP is given by:

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} c_e\beta_e \\ \text{subject to} \quad & -\alpha_u+\alpha_v+\beta_e \geq 0 \quad \forall\, e=(u,v)\in E, u,v\notin\{s,t\}, \\ & \alpha_v \geq 0 \quad \forall\, v\in V, v\neq s,t, \\ & \beta_e \geq 0 \quad \forall\, e\in E \end{aligned}$$

Now we can add $\alpha_s = 1$ and $\alpha_t = 0$ to the dual LP and obtain the modified dual LP:

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} c_e\beta_e \\ \text{subject to} \quad & \beta_e \geq \alpha_u-\alpha_v+ \quad \forall\, e=(u,v)\in E, u,v\notin\{s,t\}, \\ & \alpha_v \geq 0 \quad \forall\, v\in V, v\neq s,t, \\ & \beta_e \geq 0 \quad \forall\, e\in E, \\ & \alpha_s = 1, \\ & \alpha_t = 0 \end{aligned}$$

Now for the max-flow LP we already proved in Lemma 14.3.6 that the polytope is integral. By Lemma 14.3.2 the polytope for the dual is also integral. Let $x^*, (\alpha^*, \beta^*)$ be the optimal solution of the primal and dual LPs respectively. Now by Complementary Slackness we have the following:

$$x_e^* > 0 \implies \beta_e^* = \alpha_u^* - \alpha_v^* \qquad \text{and} \qquad \beta_e^* > 0 \implies x_e^* = c_e$$

Now $\alpha_s^* = 1$. Let $X = \{v\colon \alpha_v^* \geq 1\}$. Then $s\in X$ and $t\notin X$. Hence X is a $s-t$ cut. Now consider an edge (u,v) out of X. Then

$$\alpha_u^* \geq 1 \text{ and } \alpha_v^* < 1 \implies \beta_e^* > 0 \implies x_e^* = c_e$$

And for an edge $e=(u,v)$ in to X

$$x_e^* > 0, \alpha_u^* < 1, \alpha_v^* \geq 1 \implies \beta_e^* < 0$$

Hence for an edge e into X, $x_e^* = 0$. Hence maximum flow is equal to the $\sum\limits_{e\in out(X)} c_e$ and this is the minimum cut.

14.4.5 Maximum Bipartite Matching minimum Vertex Cover

The maximum bipartite matching problem is given by the following LP:

$$\begin{aligned} \text{maximize} \quad & \sum_{e\in E} x_e \\ \text{subject to} \quad & \sum_{e \text{ incident on } v} x_e \leq 1 \quad \forall\, v\in V, \\ & x_e \geq 0 \quad \forall\, e\in E \end{aligned}$$

The dual of the LP si given by

$$\begin{aligned} \text{minimize} \quad & \sum_{v\in V} y_v \\ \text{subject to} \quad & y_u+y_v \geq 1 \quad \forall\, (u,v)\in E, \\ & y_v \geq 0 \quad \forall\, v\in V \end{aligned}$$

Since in Lemma 14.3.5 we have proved the polytope for bipartite maximum matching is integral the polytope for the dual is also integral.

Definition 14.4.1: Vertex Cover

Given $G=(V,E)$ a vertex cover is a subset $C\subseteq V$ such that $\forall\, e\in E$ at least one of the endpoints of e is in C.

Then we have the following lemma:

Lemma 14.4.5

Let C be a vertex cover. Then there exists a dual feasible solution y such that $\sum_v y_v = |C|$.

Proof: Consider the vector $y \in \{0,1\}^{|V|}$ such that $y_v = 1$ if $v \in C$ and $y_v = 0$ otherwise. Then we have the lemma. ■

Lemma 14.4.6

Let y be an integral dual solution. Then $C = \{v : y_v \geq 1\}$ is a vertex cover.

Proof: For every edge $e = (u, v)$ we have $y_u + y_v \geq 1$. So either $y_u \geq 1$ or $y_v \geq 1$ as y is integral. Hence either $u \in C$ or $v \in C$. Hence every edge is covered by C and hence C is a vertex cover. ■

Note:-

In general graphs computing a minimum sized vertex cover in NP-hard. But since for bipartite graph the polytope is integral we can compute minimum weight vertex cover in polynomial time.

Chapter 15

Approximation Algorithms using LP

In this chapter we will study some approximation algorithms using linear programming to get better approximation ratios of the optimal solution.

15.1 Set Cover

Set Cover

Input: $\mathcal{U}$: Universe of all elements $u_1, \dots, u_n$

$\mathcal{S} = \{S_1, \dots, S_m\}$, $S_i \subseteq \mathcal{U}$ forall $i \in [m]$

Function $c : \mathcal{S} \to \mathbb{Z}_+$

Question: Given $\mathcal{U}, \mathcal{S}$ and the function c find $T \subseteq [m]$ such that $\bigcup\limits_{i \in T} S_i = \mathcal{U}$ to minimize the total cost $c(T) = \sum\limits_{i \in T} c(S_i)$

Since the special case of Set Cover is basically the Vertex Cover problem we discussed earlier, we know that Set Cover is NP-hard.

Theorem 15.1.1

Set Cover is NP-hard.

Since we are going to find approximate solutions using LP let's first write the linear program for Set Cover:

$$\begin{aligned}
\text{minimize} \quad & \sum_{S \in \mathcal{S}} c(S) x_S \\
\text{subject to} \quad & \sum_{S : u \in S} x_S \geq 1 \quad \forall\, u \in \mathcal{U}, \\
& x_S \geq 0 \quad \forall\, S \in \mathcal{S}
\end{aligned}$$

15.1.1 Frequency f-Approximation Algorithm

Let for any element $u \in \mathcal{U}$, f_u is the frequency of the element u in $\mathcal{S}$ i.e. $f_u = |\{S \in \mathcal{S} : u \in S\}|$. Then let $f = \max\{f_u : u \in \mathcal{U}\}$. Then we want to find a f-approximation algorithm for set cover.

Question 15.1: F

r vertex cover what is f?

For all $e \in E$ we have $f_e = 2$ since the elements of universe corresponds to the edges and sets corresponds to vertices and each edge is contained in exactly 2 sets. So $f = 2$.

Algorithm 32: f-Approximate Algorithm

Input: $\mathcal{U}, \mathcal{S}, c$
Output: $T \subseteq [m]$ such that $\bigcup_{i \in T} S_i = \mathcal{U}$ and $\sum_{i \in T} c(S_i)$ is minimized

begin
  $T \longleftarrow \emptyset$
  $\hat{x} \longleftarrow 0^{|\mathcal{S}|}$
  Let x^* is the optimal solution of the LP for Set Cover problem
  for $S_i \in \mathcal{S}$ **do**
    if $x^*_{S_i} \geq \frac{1}{f}$ **then**
      $T \longleftarrow T \cup \{i\}$
      $\hat{x}_{S_i} \longleftarrow 1$
  return T

Lemma 15.1.2

$\hat{x}$ is a feasible solution.

Proof: For all $e \in \mathcal{U}$ there are at most f sets containing e. Thus at most f terms in the summation in *LHS* of the first constraint for each $e \in \mathcal{U}$ Thus in x^* at least one such term is $\geq \frac{1}{f}$. ■

Lemma 15.1.3

$\sum_{S \in \mathcal{S}} c(S)\hat{x}_S \leq f \cdot \sum_{S \in \mathcal{S}} c(S)x^*_S$

Proof: In $\hat{x}$ if $\hat{x}_S = 1$ that means $x^*_S \geq \frac{1}{f}$. Therefore we have the lemma. ■

Hence with this algorithm we can get a f-approximation for Set Cover problem. But this is not good enough since one element can be in too many sets and then it doesn't give a good approximation. In the next section we will see a new way of getting the same approximation ratio.

15.1.2 Frequency f-Approximation Algorithm through Dual Fitting

First let's wrote the dual of the LP for Set Cover problem:

$$\begin{array}{llll} \text{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c(S)x_S & & \\ \text{subject to} & \displaystyle\sum_{S: u \in S} x_S \geq 1 & \forall\, u \in \mathcal{U}, & \\ & x_S \geq 0 & \forall\, S \in \mathcal{S} & \end{array} \quad \Longleftrightarrow \quad \begin{array}{llll} \text{maximize} & \displaystyle\sum_{u \in \mathcal{U}} y_u & \\ \text{subject to} & \displaystyle\sum_{u \in S} y_u \leq c(S) & \forall\, S \in \mathcal{S}, \\ & y_u \geq 0 & \forall\, u \in \mathcal{U} \end{array}$$

$\boxed{\text{Primal}}$ "Covering Problem" $\boxed{\text{Dual}}$ "Packing Problem"

Both the primal and dual are feasible. Let x, y are feasible solutions of the primal and dual respectively. Then by Weak Duality we have

$$\sum_{S \in \mathcal{S}} c(S)x_S \geq \sum_{u \in \mathcal{U}} y_u$$

Let x^*, y^* are the optimal solutions of primal and dual respectively. Then by [Complementary Slackness]

$$x_S^* > 0 \implies \sum_{u \in S} y_u^* = c(S) \qquad y_u^* > 0 \implies \sum_{S: u \in S} x_S^* = 1$$

Theorem 15.1.4 Relaxed Complementary Slackness

Suppose x, y are feasible solutions of the primal and dual respectively and the satisfy the following conditions:

1. If $x_j > 0$ then $\frac{1}{\alpha} \cdot c_j \leq {A^j}^T y \leq c_j$ where $\alpha \geq 1$.
2. If $y_i > 0$ then $b_i \leq A_i^T x \leq \beta \cdot b_i$ where $\beta \geq 1$.

Then

$$c^T x \leq \alpha\beta \cdot b^T y \leq \alpha\beta \cdot c^T x^* = \alpha\beta \cdot \mathrm{OPT}$$

Proof: x, y are the feasible solutions of the primal and dual respectively. Suppose first d constraints of the primal are equalities and rest are inequalities and similarly first l constraints of the dual are equalities and rest are inequalities. Then we have

$$c^T x = \sum_{j=1}^{m} c_j x_j \leq \sum_{j=1}^{m} \left(\alpha {A^j}^T y\right) x_j = \alpha \sum_{j=1}^{m} \sum_{i=1}^{n} A_{ij} y_i x_j = \alpha \sum_{i=1}^{n} \left(\sum_{i=1}^{m} A_{ij} x_j \right) y_i \leq \alpha \sum_{i=1}^{m} \beta \cdot b_i y_i = \alpha\beta \cdot b^T y$$

Hence we have $c^T x \leq \alpha\beta \cdot b^T y \leq \alpha\beta \cdot c^T x^* = \alpha\beta \cdot \mathrm{OPT}$. ■

To show a f-approximation algorithm for set cover we will first find x, y feasible primal, dual solution which satisfies:

1. x is integral.
2. x satisfies the first condition of Relaxed Complementary Slackness with $\alpha = f$.

Algorithm 33: Dual Fitting Algorithm for Set Cover

Input: $\mathcal{U}, \mathcal{S}, c$

Output: $T \subseteq [m]$ such that $\bigcup_{i \in T} S_i = \mathcal{U}$ and $\sum_{i \in T} c(S_i)$ is minimized

begin
- Initialize $\mathcal{U}' \longleftarrow \mathcal{U}, C \longleftarrow \emptyset$
- **while** $\exists\, u \in \mathcal{U}'$ **do**
  - Increase y_u until for some $S \in \mathcal{S}$ such that $u \in S$ we have $\sum_{u' \in S} y_{u'} = c(S)$
  - $C \longleftarrow C \cup \left\{ S \in \mathcal{S} : \sum_{u \in S} y_u = c(S) \right\}$
  - **for** $S \in C$ **do**
    - $\mathcal{U}' \longleftarrow \mathcal{U}' \setminus S$
- **return** C

From C we con construct x by $x_S = 1$ if $S \in C$ and otherwise $x_=0$ for all $S \in \mathcal{S}$. Now we have the observations:

Observation. *After the algorithm terminates we have:*

1. *At the beginning of the loop if $u \in \mathcal{U}$, $y_u = 0$.*
2. *If $x_S = 1$ and $u \in S$ then y_u is not increased.*
3. *$x_S \in \{0, 1\}^{|\mathcal{S}|}$ is integral.*

Lemma 15.1.5

1. x is feasible at the end of the algorithm.
2. y is feasible at every iteration of the while loop

Proof: The algorithm terminates when $\mathcal{U}' = \emptyset$. That means all the elements of the universe are covered. Hence the set C output after the algorithm terminates is indeed a set cover. Hence x is a feasible solution.

At the start of the algorithm $y = 0^{|\mathcal{U}|}$. Hence y is feasible. Now suppose at any iteration y is feasible. If the algorithm goes through another iteration then there exists an element in $\mathcal{U}'$ which is not covered. Let $u \in \mathcal{U}'$ which is not covered. Hence $y_u = 0$. Since in the previous iteration y was feasible we have $\sum\limits_{S:u\in S} y_u \le c(S)$. Now we increase y_u to the point we achieve the equality $\sum\limits_{u'\in S} y_{u'} = c(S)$ for all $S \in \mathcal{S}$. Therefore even after updating y_u all the constraints of dual are satisfied. Hence y is a feasible solution after another iteration of the while loop. Therefore y is feasible at every iteration of the while loop. ■

Lemma 15.1.6

x, y satisfy the Relaxed Complementary Slackness conditions.

Proof: If for any $S \in \mathcal{S}$, $x_S > 0$ then we have $\sum\limits_{u\in S} y_u = c(S)$ by the construction of C in the algorithm. Therefore

$$x_S > 0 \implies \sum_{u\in S} y_u = c(S)$$

Hence $\alpha = 1$.

Now let for some $u \in \mathcal{U}$, $y_u > 0$. Since f is the maximum frequency of any element of the universe we have $f \ge \sum\limits_{S:u\in S} x_S \ge 1$. Therefore

$$y_u > 0 \implies f \ge \sum_{S:u\in S} x_S \ge 1$$

Hence $\beta = f$. ■

Therefore by Relaxed Complementary Slackness C is an f-approximate solution for the set cover problem. In the next sections we will show how to get a better approximation ratio.

15.1.3 $O(n \log n)$-Approximation Algorithm through Randomized Rounding

Algorithm 34: $n \log n$-Approximate Algorithm

Input: $\mathcal{U}, \mathcal{S}, c$

Output: $T \subseteq [m]$ such that $\bigcup\limits_{i\in T} S_i = \mathcal{U}$ and $\sum\limits_{i\in T} c(S_i)$ is minimized

```
begin
    x̂ ⟵ 0^{|S|}
    Let x* is the optimal solution of the LP for Set Cover problem
    for S ∈ S do
        Set x̂_S ⟵ 1 with probability x*_S.
    return x̂
```

From the construction of $\hat{x}$ we have $\mathbb{E}\left[\sum_{S\in\mathcal{S}} c(S)\hat{x}_S\right] = \sum_{S\in\mathcal{S}} c(S)x_S^*$. Now suppose we fixed an element $u \in \mathcal{U}$. Then

$$\mathbb{P}[u \text{ is not covered}] = \prod_{S:u\in S} \mathbb{P}[S \text{ is not selected}] = \prod_{S:u\in S} (1 - x_S^*) \leq \prod_{S:u\in S} e^{-x_S^*} = \exp\left[-\sum_{S:u\in S} x_S^*\right] \leq e^{-1}$$

Hence to reduce the probability of not covering an element of $\mathcal{U}$ we repeat the algorithm multiple times. Hence we have the updated algorithm:

Algorithm 35: $n\log n$-Approximate Algorithm

Input: $\mathcal{U}, \mathcal{S}, c$
Output: $T \subseteq [m]$ such that $\bigcup_{i\in T} S_i = \mathcal{U}$ and $\sum_{i\in T} c(S_i)$ is minimized

```
begin
    Let x* is the optimal solution of the LP for Set Cover problem
    for i ∈ [2 log n] do
        C_i ⟵ ∅
        for S ∈ 𝒮 do
            Put S in C_i with probability x*_S.
    C ⟵ ⋃_{i=1}^{2 log n} C_i
    return C
```

Again now we fix an element $u \in \mathcal{U}$. Now we will calculate the probability that u is not covered in the union of all C_i's.

$$\mathbb{P}[u \text{ is not covered by } C] = \mathbb{P}[u \text{ is not covered by } C_i \text{ for all } i \in [2\log n]] \leq e^{-2\log n} = \frac{1}{n^2}$$

Hence the probability that e is covered is at least $1 - \frac{1}{n^2}$. Therefore

$$\mathbb{P}[\exists\ e \in \mathcal{U} \text{ is not covered by } C] \leq \sum_{u\in\mathcal{U}} \frac{1}{n^2} = \frac{1}{n}$$

Hence $\mathbb{P}[C \text{ is a set cover}] \geq 1 - \frac{1}{n}$. Now we have to bound the cost of C. By Markov's inequality we have

$$\mathbb{P}\left[c(C) \geq 6\log n \sum_{S\in\mathcal{S}} c(S)x_S^*\right] \leq \frac{1}{3}$$

$$\mathbb{P}\left[C \text{ is not a set cover OR cost of C} \geq 6\log n \sum_{S\in\mathcal{S}} c(S)x_S^*\right] \leq \frac{1}{n} + \frac{1}{3} \leq \frac{1}{2}$$

Therefore

$$\mathbb{P}\left[C \text{ is set cover AND } c(S) \leq 6\log n \sum_{S\in\mathcal{S}} c(S)x_S^*\right] \geq 12$$

Hence with probability at least $\frac{1}{2}$ we have a set cover C such that $c(C) \leq 6\log n \sum_{S\in\mathcal{S}} c(S)x_S^*$ which gives us an $O(\log n)$-approximation algorithm for Set Cover problem.

Note:-

$O(\log n)$-approximatiobn is also the best we can do for set cover. Doing better than that is NP-hard.

15.2 Makespan Minimization

MAKESPAN
Input: $\mathcal{M}$: Set of m machines
$\mathcal{J}$: Set of n jobs
$P \in \mathbb{N}^{m\times n}$: Matrix where P_{ij} is the time taken by machine i to complete job j.
Question: Given set of machines M, set of jobs $\mathcal{J}$ and the matrix of time taken by i^{th} machine to complete j^{th} job find an assignment $\sigma : \mathcal{J} \to \mathcal{M}$ of jobs to machines to minimize the makespan $S_\sigma = \max\{l_i : i \in \mathcal{M}\}$ where $l_i = \sum\limits_{j:\sigma(j)=i} P_{ij}$ i.e. time taken by machine i to complete all jobs assigned by σ

Theorem 15.2.1

Makespan problem is weakly NP-hard by reduction from subset-sum.

Note:-

Weakly NP-hard means there exists a pseudo polynomial time algorithm i.e. if all parameters are polynomially large the algorithm can solve the problem in polynomial time.

Theorem 15.2.2

It is NP-hard to approximate within a factor of 1.5

Here we will show a 2-approximate solution of makespan optimization. First let's construct the LP for makespan optimization.

15.2.1 LP Construction

We'll use the variable x_{ij} as an indicator for j^{th} job assigned to i^{th} machine. Then here is the LP:

$$\begin{aligned}
\text{minimize} \quad & T \\
\text{subject to} \quad & \sum_{i\in\mathcal{M}} x_{ij} \geq 1 && \forall\, j \in \mathcal{J}, \\
& \sum_{j\in\mathcal{J}} P_{ij}x_j \leq T && \forall\, i \in \mathcal{M}, \\
& x_{ij} \geq 0 && \forall\, i \in \mathcal{M}, j \in \mathcal{J}
\end{aligned}$$

So here the first constrain basically says that every job assigned to some machine. The second constraint says that for every machine the total time taken by the machine to complete the jobs should be at most the makespan where T denotes the makespan. But this LP is not good enough. Consider the following example where there is only one job and $P_{i1} = m$ then $\text{OPT}_{LP} = 1$ by setting $x_{i1} = \frac{1}{m}$ where as actually the optimal makespan is m. Hence this LP will not work. We have to strengthen the LP.

So now assume we already know the optimal makespan T. Then if any $P_{ij} > T$ then we know that we can't assign the j^{th} job to i^{th} machine. So now we have the new updated LP:

$$\begin{aligned}
\text{minimize} \quad & 0 \\
\text{subject to} \quad & \sum_{i\in\mathcal{M}} x_{ij} \geq 1 && \forall\, j \in \mathcal{J}, \\
& \sum_{j\in\mathcal{J}} P_{ij}x_j \leq T && \forall\, i \in \mathcal{M}, \\
& x_{ij} \geq 0 && \forall\, i \in \mathcal{M}, j \in \mathcal{J}, \\
& x_{ij} = 0 && \text{If } P_{ij} > T \forall\, i \in \mathcal{M}\ j \in \mathcal{J}
\end{aligned}$$

This basically checks the feasibility for a specific T. Hence now we can do a binary search over T's to find the smallest feasible T.

Theorem 15.2.3

By binary search $O(\log n)$ round we can find the smallest T such that $LP(T)$ is feasible.

Now suppose we have the smallest feasible time. Let's call this $\hat{T}$. Then $\hat{T} \leq \text{OPT}_I$. Let $\tilde{x}$ is the basic feasible solution for $\hat{T}$. We will now show a polynomial time algorithm to obtain an integral assignment with makespan $= 2\hat{T}$.

15.2.2 Rounding to Get 2-Approximate Solution

Now we have the smallest feasible time $\hat{T}$ and the basic feasible solution for that $\tilde{x}$. Now we can thing $\tilde{x}$ as a weighted bipartite graph between $\mathcal{J}$ and $\mathcal{M}$ with fractional weights i.e. one job assigned to multiple machines fractionally. Let the graph is $G = (L \sqcup R, E)$ where $e = (i, j) \in E$, if $\tilde{x}_{ij} > 0$ with $w(i, j) = \tilde{x}_{ij}$. Hence we also have for all $(i, j) \in E$, $\tilde{x}_{ij} \leq \hat{T}$.

Lemma 15.2.4

In $\tilde{x}$ at least $n - m$ jobs are assigned integrally.

Proof: There are total $n + m + nm$ constraints in the LP. But the LP is nm dimensional. Therefore at $\tilde{x}$, nm constraints are tight. So at most $m + n$ constraints of the type $x_{ij} \geq 0$ are not tight i.e. at most $m + n$ many $\tilde{x}_{ij}$ are not zero. Suppose α jobs are set integrally and β fractionally. So for each of the β jobs it is assigned to at least 2 machines. Now each of the $\tilde{x}_{ij}$ corresponds to an edge of the graph. Therefore we have the following two equations:

$$\alpha + \beta = n, \quad \alpha + 2\beta \leq m + n \implies \beta \leq m \implies \alpha \geq n - m$$

Therefore there at least $n - m$ jobs which are assigned integrally. ■

Lemma 15.2.5

In every connected component of G, #edges ≤ #vertices.

Proof: In the graph G, as we showed earlier at most $m + n$ constraints of the type $x_{ij} \geq 0$ are not tight i.e. at most $m + n$ many $\tilde{x}_{ij}$ are not zero. Hence

$$\#\text{edges} = |\{\tilde{x}_{ij} \mid \tilde{x}_{ij} > 0\}| \leq m + n = \#\text{vertices}$$

Suppose C is a connected component. Let $\mathcal{J}_C$, $\mathcal{M}_C$ be the jobs and machines of C and $\tilde{x}|_C$ is $\tilde{x}$ restricted to C. Then $\tilde{x}|_C$ is a basic feasible solution for the instance restricted to $\mathcal{M}_C$, $\mathcal{J}_C$ with $\hat{T}$ being a feasible time. If $\tilde{x}|_C$ was not feasible for $\mathcal{M}_C$ and $\mathcal{J}_C$ then there exists y_C and z_C with $y_C \neq z_C$ such that $wtdx|_C = \lambda y_C + (1 - \lambda) z_C$ where $\lambda \in (0, 1)$. Then

$$\tilde{x} = \lambda \left(y_C, \tilde{x}|_{\overline{C}} \right) + (1 - \lambda) \left(z_C, \tilde{x}|_{\overline{C}} \right)$$

Then $\tilde{x}$ can not be a basic feasible solution. And therefore by the same logic as above we have in the connected component #edges ≤ #vertices. Since C is arbitrary connected component this is true for every connected component. ■

Now we create a feasible solution $\hat{x}$ for $2\hat{T}$. We first initiate $\hat{x}$ setting all 0's. We fix a connected component C in G. We call a vertex in $\mathcal{J}_C \cup \mathcal{M}_C$ leaf if it has degree 1. If for any job $j \in \mathcal{J}_C$ it is assigned integrally in $\tilde{x}|_C$ then j is a leaf. So we remove the node j and assign the job to the machine $i \in \mathcal{M}_C$, j is connected to. This also removes the edge incident on j.

After doing this we still have #edges ≤ #vertices because we basically removed same number of jobs and edges from the graph. But now every job is connected to at least two machines.

If a machine $i \in \mathcal{M}_C$ is a leaf, let the edge incident on i is (i, j) then we remove both i, j from te graph and assign the job j to machine i i.e. basically we set $\hat{x}_{ij} = 1$. So the load added to i^{th} machine is at most $\hat{T}$. We do this for every leaf machine.

Now the graph has no leaves remaining. Since the graph is bipartite it is an even cycle. So find a matching of jobs to machines in the cycle and assign the jobs accordingly i.e. if M is a matching and $e = (i, j) \in M$ then set $\hat{x}_{ij} = 1$.

So we have the following final algorithm:

Algorithm 36: Makespan 2-Approximate Algorithm

Input: $\mathcal{M}, \mathcal{J}, P$ where $|\mathcal{M}| = m$, $|\mathcal{J}| = n$ and $P \in \mathbb{Z}_0^{m\times n}$
Output: $\sigma : \mathcal{J} \to \mathcal{M}$ assignment of jobs to machines to minimize $\max\{l_i : i \in \mathcal{M}\}$ where $l_i = \sum\limits_{j:\sigma(j)=i} P_{ij}$ i.e. time taken by machine i to complete all jobs assigned by σ

begin
    Do binary search to find the minimum feasible T for the LP.
    Let $\hat{T}$ is the minimum feasible time and $\tilde{x}$ is the basic feasible solution.
    Construct the weighted graph $G = (\mathcal{M} \sqcup \mathcal{J}, E)$ where $(i, j) \in E$ if $\tilde{x}_{ij} > 0$ and $w(i, j) = \tilde{x}_{i,j}$.
    $\mathcal{C} \longleftarrow$ Connected Components of G.
    for $C \in \mathcal{C}$ **do**
        while $\exists\, j \in \mathcal{J}_C$ *such that* $\deg(j) = 1$ **do**
            Let $(i, j) \in E$
            $\sigma(j) \longleftarrow i$
            $\mathcal{J} \longleftarrow \mathcal{J} \setminus \{j\}$
        while $\exists\, i \in \mathcal{M}_C$ *such that* $\deg(i) = 1$ **do**
            Let $(i, j) \in E$
            $\sigma(j) \longleftarrow i$
            $\mathcal{M} \longleftarrow \mathcal{M} \setminus \{i\}$
            $\mathcal{J} \longleftarrow \mathcal{J} \setminus \{j\}$
        $M \longleftarrow$BP-MATCHING-AUGMENTING-PATH. M will be a perfect matching.
        for $e = (i, j) \in M$ **do**
            $\sigma(j) \longleftarrow i$
    return σ

This algorithm works in polynomial time since solving the LP, constructing the weighted graph and finding the connected components can be done in polynomial time and then for every component the while loops and finding matching can also be done in polynomial time. So the algorithm is polynomial time.

This algorithm gives a 2-approximate solution because each machine i is assigned the jobs it is set integrally and another job j if $\tilde{x}_{ij} > 0$.

Chapter 16

P, NP and Reductions

Chapter 17

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