CSS.317.1 Algorithmic Game Theory

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5.1 Best Response Dynamics

The existence of a Nash equilibrium is clearly a desirable property of a strategic game. In this chapter and the next we discuss some natural classes of games that do have a Nash equilibrium. The *Best-Response-Dynamics* is a straightforward procedure by which players search for a pure Nash equilibrium (PNE) of a game.

Algorithm 1: BEST-RESPONSE-DYNAMICS (BRD)

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1 **begin**
2   **for** $t = 1, \ldots, T$ **do**
3   **if** $t = 1$ **then**
4   Each player plays an arbitrary pure strategy
5   **else**
6   Pick a player $i \in [n]$
7        $s_i^t \longleftarrow \arg\min_{s_i \in S_i} c_i(s_i, s_{-i}^{t-1})$
8        $s_j^t \longleftarrow s_j^{t-1} \ \forall \ j \in [n], j \neq i$

> **Note:-**
> Best-response dynamics can only halt at a PNE and it cycles in any game without one. It can also cycle in games that have a PNE. For example consider the following 2 player.

5.2 Network (Atomic) Congestion Games

Definition 5.1: Network (Atomic) Congestion Games

A network (atomic) congestion game or in short NCG consists of the following:

- A directed graph $G = (V, E)$.
- N players where each player $i \in [n]$ has some source-sink pair $(s_i, t_i) \in V \times V$ associated with it.
- Edge cost functions $c_e : [n] \to \mathbb{R}$ for each edge $e \in E$.
- Player $i \in [N]$ has strategy set S_i = Set of all $s_i \rightsquigarrow t_i$ paths in G. $S = \underset{i=1}{\overset{N}{\times}} S_i$.
- For a strategy profile $f \in S$ (often called *flow*), let $n_e(f) = |\{i : e \in f_i\}|$. Then the cost to player i of strategy profile f is $C_i(f) = \sum_{e \in S_i} c_e(n_e(f))$.

So we can define (atomic) NCG by the tuple

$$\left(G = (V, E), N, \{(s_i, t_i) \mid i \in [N]\}, \{c_e : [N] \to \mathbb{R}_{\geq 0} \mid e \in E\}\right)$$

Note that unlike the last few lectures where we've been talking about utility-maximization games, this is a cost-minimization game. But of course we could just let a player's utility be the negative of its cost and everything would work as you expect.

> **Lemma 5.2.1**
> Every NCG has a PNE.

Proof: Given a strategy profile $f \in S$, we will define a potential function $\Phi : S \to \mathbb{R}_{\geq 0}$ with the property that if f is not an equilibrium then $\exists f' \in S$ such that $\Phi(f) > \Phi(f')$. Thus if $f^* \in S$ minimizes Φ then f^* must be a PNE.

Consider the potential function $\Phi : S \to \mathbb{R}_{\geq 0}$:

$$\Phi(s) = \sum_{e \in E} \sum_{i=1}^{n_e(f)} c_e(i)$$

Now it is enough to calculate the change in potential when a player deviates to any other strategy since for $f, f' \in S$

$$\Phi(f) - \Phi(f') = \sum_{i=0}^{N-1} \Phi(f^{(i)}) - \Phi(f^{(i+1)})$$

where $f^{(i)} = (f_1', f_2', \ldots, f_i', f_{i+1}, \ldots, f_N)$ and for $f^{(0)} = f$. Now for any strategy profile $f \in S$ if the player i deviates to the strategy $f_i' \in S_i$ then

$$C_i(f) - C_i(f_i', f_{-i}) = \left[\sum_{e \in f_i \cap f_i'} c_e(n_e(f)) + \sum_{e \in f_i \setminus f_i'} c_e(n_e(f)) \right] - \left[\sum_{e \in f_i \cap f_i'} c_e(n_e(f_i', f_{-i})) + \sum_{e \in f_i' \setminus f_i} c_e(n_e(f_i', f_{-i})) \right]$$

$$= \underbrace{\sum_{e \in f_i \cap f_i'} c_e(n_e(f)) - c_e(n_e(f_i', f_{-i}))}_{=0} + \sum_{e \in f_i \setminus f_i'} c_e(n_e(f)) - \sum_{e \in f_i' \setminus f_i} c_e(n_e(f_i', f_{-i}))$$

$$= \sum_{e \in f_i \setminus f_i'} c_e(n_e(f)) - \sum_{e \in f_i' \setminus f_i} c_e(n_e(f) + 1)$$

Therefore the change in the potential is

$$\Phi(f) - \Phi(f_i', f_{-i}) = \sum_{e \in E} \sum_{i=1}^{n_e(f)} c_e(i) - \sum_{e \in E} \sum_{i=1}^{n_e(f_i', f_{-i})} c_e(i)$$

$$= \sum_{e \in E} \left[\sum_{i=1}^{n_e(f)} c_e(i) - \sum_{i=1}^{n_e(f_i', f_{-i})} c_e(i) \right]$$

$$= \sum_{e \in f_i \setminus f_i'} c_e(n_e(f)) - \sum_{e \in f_i' \setminus f_i} c_e(n_e(f) + 1)$$

$$= C_i(f) - C_i(f_i', f_{-i})$$

So the change in potential is exactly equal to the change in the cost of the player who deviates. Therefore if f is not a PNE then $\exists i \in [N]$ such that $\exists f_i' \in S_i$ such that $c_i(f) - c_i(f_i', f_{-i}) > 0$ and therefore $\Phi(f) - \Phi(f_i', f_{-i}) > 0$. Hence every NCG has a PNE.   ∎

5.3   Potential Games

> **Definition 5.2: Potential Game**
>
> A game Γ is a potential game if there exists a potential function $\Phi : S \to \mathbb{R}_{\geq 0}$ where S is the set of strategy profiles such that $\forall\, s \in S$ and $s_i' \in S_i$ $C_i(s) - C_i(s_i', s_{-i}) = \Phi(s) - \Phi(s_i', s_{-i})$

In the proof of Theorem 5.2.1 we showed that every NCG is a potential game. Now we will show that every potential game has a PNE.

> **Theorem 5.3.1**
>
> Every potential game has a Pure Nash Equilibrium

Proof: For a potential game Γ let Φ is the potential function for Γ. Then $C_i(s) - C_i(s_i', s_{-i}) = \Phi(s) - \Phi(s_i', s_{-i})$. Now consider the strategy profile $s = \arg\min\limits_{s \in S} \Phi(s)$. If any player had incentive to deviate there would be a strategy profile with smaller potential which is not possible by the definition of s. Therefore s also has the minimum cost. Therefore s is PNE.   ∎

> **Lemma 5.3.2**
>
> Best Response Dynamics cannot cycle in a potential game.

Proof: In each iteration of the BRD every time any player deviates to play a best response the potential must decrease. Hence BRD cannot cycle.   ∎

Suppose there exists a time T such that every player was chosen in the BRD to choose their best response in the Best response algorithm. Then:

> **Lemma 5.3.3**
>
> Let $s^* \in S$ be the strategy profile at time t. If s^* is the strategy profile after T further steps of BRD then s^* is a PNE.

Proof: Since in every T steps every player has the option to deviate to another strategy but chose not to. Therefore for each player $i \in [N]$, for all $s_i' \in S_i$, $C_i(s) \leq C_i(s_i', s_{-i})$. Therefore clearly s^* is a PNE.   ∎

> **Lemma 5.3.4**
>
> Let $s^{\in}S$ be the strategy profile after $T|S|$ steps of BRD. Then s^* is a PNE.

Proof: Since BRD cannot cycle, $\exists\, s \in S$ that must have persisted fro T time steps. Therefore by the previous lemma this must be a PNE.   ∎

> **Theorem 5.3.5**
>
> In a finite potential game from an arbitrary initial outcome the Best Response Dynamics converges to a PNE if $\exists\, T \in \mathbb{N}$ such that in every T steps of BRD every player is chosen at least once.

Since every (Atomic) NCG is a potential game we have the following corollary:

> **Corollary 5.3.6**
>
> In an (Atomic) NCG, BRD converges to a PNE if $\exists\, T \in \mathbb{N}$ such that in every T steps of BRD every player is chosen at least once. or "every player is chosen infinitely often".

5.3.1 General Congestion Games

General Congestion Games are generalized version of (atomic) NCG. We will show that they are also potential game.

> **Definition 5.3: General Congestion Games**
>
> A basic definition general Congestion Games or CG consists of the following:
>
> $$\left(E, N, \{S_i \mid i \in [N]\}, \{c_e : [N] \to \mathbb{R}_{\geq 0} \mid e \in E\}\right)$$
>
> - A base set E of congestible elements.
> - N players.
> - For each player $i \in [N]$ a finite set of strategies S_i where $S_i \subseteq 2^E$. $S = \underset{i=1}{\overset{N}{\times}} S_i$.
> - Cost functions $c_e : [N] \to \mathbb{R}$ for each element $e \in E$.
> - For a strategy profile $s \in S$ (often called *flow*), let $n_e(s) = |\{i : e \in s_i\}|$. Then the cost to player i of strategy profile s is $C_i(s) = \sum_{e \in S_i} c_e(n_e(s))$.

Consider the function $\Phi : S \to \mathbb{R}_{\geq 0}$ where for any strategy profile $s \in S$,

$$\Phi(s) = \sum_{e \in E} \sum_{i=1}^{n_e(s)} c_e(i)$$

that is the same function as the potential function in the case of NCG. This is also a potential function for general CG's which makes general CG's are also potential game.

5.3.2 Max Cut Game

> **Definition 5.4: Max Cut Game**
>
> A max cut game consists of the following:
>
> 1. An undirected weighted graph, $G = (V, E)$ and $w : E \to \mathbb{R}$.
> 2. N players.
> 3. For each player $i \in [N]$, has 2 strategies: $S_i = \{L, R\}$. $S = \underset{i=1}{\overset{N}{\times}} S_i$.
> 4. Utility functions $u_i : S \to \mathbb{R}_{\geq 0}$ for each player $i \in [N]$. For any strategy profile $s \in S$, $u_i(s) = \sum_{\substack{e=\{i,j\} \\ s_i \neq s_j}} w_e$

The max cut game is also a potential game. Consider the potential function $\Phi : S \to \mathbb{R}_{\geq 0}$ where for any strategy profile $s \in S$,

$$\Phi(s) = \sum_{\substack{e=\{i,j\} \\ s)i \neq s_j}} w_e$$

With this function we can prove that the Max Cut game is indeed a potential game and henceforth there exists a PNE.

5.4 Class: PLS

> **Definition 5.5: PLS (Polynomial Local Search)**
>
> A local search problem L has a set of problem instances $D_L \subseteq \Sigma^*$ where any $I \in D_L$ is a particular problem instance. For each instance $I \in D_L$ there exists a finite solution set $F_L(I) \subseteq \Sigma^*$. Let R_L be the relation that models L i.e.
>
> $$R_L := \{(I, s) \mid I \in D_L, s \in F_L(I)\}$$
>
> Then R_L is in PLS if:
>
> (i) The size of every solution $s \in F_L(I)$ for any $I \in D_L$ is polynomially bounded in the size of I.
>
> (ii) The problem instances $I \in D_L$ and the solutions $s \in F_L(I)$ are polynomial time verifiable.
>
> (iii) There is a polynomial time computable function $C_L : X \to \mathbb{R}_{\geq 0}$ that returns for each $I \in D_L$ and each $s \in F_L(I)$ the cost where $X := \bigcup_{I \in D_L} \{I\} \times F_L(I)$.
>
> (iv) There is a polynomial time computable function $N : (I, s) \mapsto S$ where $S \subseteq F_L(I)$ i.e. returns the set of neighbors for each $I \in D_L$ and each $s \in F_L(I)$.

Note that for each $I \in D_L$ and each $s \in F_L(I)$ using (iii) and (iv) we can find a neighboring solutions of lower cost of s or determine s is locally minimal. The problem we want to focus is to find a locally minimal cost solution given an instance I of L.

> **Definition 5.6: PLS-Reductions**
>
> fgsd

> **Theorem 5.4.1**
>
> The Max Cut Game is PLS-complete

> **Theorem 5.4.2**
>
> General Congestion Games are PLS-complete

Efficiency of Equilibria

Here we are going to leave aside for now the question of how a game arrived at an equilibrium and instead we will study '*quality of equilibria*'. We want to study how close to optimal the equilibria of a game are. But for that we have to define this 'closeness' and 'optimal' by introducing cost to every strategy and we basically want to find a equilibria which very close to the minimum cost strategy profile.

6.1 Cost Minimization Games

> **Definition 6.1: Cost Minimization Games**
>
> It is a game with n players $[n]$, with their strategy sets $S_1, \dots, S_n$ where $S = \bigtimes_{i=1}^{n} S_i$ and a cost function $C_i : S \to \mathbb{R}$ for each $i \in [n]$.

There is an objective function $f : S \to \mathbb{R}$ with which the different strategy profiles are compared. There are many common choices for f. Conventionally the concepts PNE, MNE, CE, CCE are defined for utility-maximization games with all of its inequalities reversed. But the two definitions are completely equivalent.

- **Pure Nash Equilibria**: A strategy profile $s \in S$ of a cost-minimization game Γ is a *Pure Nash Equilibrium* if for every player $i \in [n]$ and for all $s_i' \in S_i$, $C_i(s) \leq C_i(s_i', s_{-i})$.

- **Mixed Nash Equilibria**: A mixed strategy profile $\sigma \in \Sigma$ of a cost-minimization game Γ is a *Mixed Nash Equilibria* if for every player $i \in [n]$ and for all $s_i' \in S_i$, $\mathbb{E}_{s \sim \sigma}[C_i(s)] \leq \mathbb{E}_{s \sim \sigma}[C_i(s_i', s_{-i})]$

- **Correlated Equilibria**: A distribution μ over S of a cost-minimization game Γ is a *Correlated Equilibria* if for every player $i \in [n]$ and for all $s_i' \in S_i$, $\mathbb{E}_{s \sim \mu}[C_i(s) \mid s_i] \leq \mathbb{E}_{s \sim \mu}[C_i(s_i', s_{-i}) \mid s_i]$

- **Coarse Correlated Equilibria**: A distribution μ over S of a cost-minimization game Γ is a *Coarse Correlated Equilibria* if for every player $i \in [n]$ and for all $s_i' \in S_i$, $\mathbb{E}_{s \sim \mu}[C_i(s)] \leq \mathbb{E}_{s \sim \mu}[C_i(s_i', s_{-i})]$

6.2 Pareto Optimality

> **Definition 6.2: Pareto Optimal Strategy**
>
> Given a game Γ, a strategy profile $s \in S$ is pareto optimal also denoted by PO if $\nexists s' \in S$ such that
>
> $$\forall i \in [n], \ c_i(s') \leq c_i(s) \qquad \exists i \in [n] \ c_i(s') < c_i(s)$$
>
> or equivalently for all $s' \in S$, either $\forall i \in [n], c_i(s) = c_i(s')$ or $\exists i \in [n], c_i(s') > c_i(s)$.

Economists call Pareto Optimality "efficiency". PO induces a partial order over the set of all strategy profiles. Let $s, s' \in S$. We say that $s >_p s'$ if $\forall i \in [n]$, $c_i(s') \leq c_i(s)$ and $\exists i \in [n]$ $c_i(s') < c_i(s)$.

To introduce a total order we can think of social welfare function for example:

(1) Utilitarian Social Welfare: For any $s \in S$, $C(s) = \sum\limits_{i=1}^{n} c_i(s)$

(2) Nash Social Welfare: For any $s \in S$, $C(s) = \sum\limits_{i=1}^{n} c_i(s)$

(3) Egalitarian Social Welfare: For any $s \in S$, $C(s) = \max\limits_{i=1}^{n} c_i(s)$

This allows us to quantitatively see how good or bad a equilibrium is by comparing two strategy profiles. Typically by "social welfare" we mean utilitarian social welfare. We will focus on calculating utilitarian social welfare from now on.

6.3 Price of Anarchy

For a game Γ we also want to know how bad is the social welfare at equilibrium compared to the best possible social welfare. This ratio is know as Price of Anarchy.

Definition 6.3: Price of Anarchy

We denote it by PoA. For a game Γ:

$$\text{PoA}(\Gamma) = \frac{\text{Social welfare of "worst equilibrium"}}{\text{Optimal social welfare}}$$

$$= \frac{\max\left\{\sum\limits_{i=1}^{n} c_i(s) : s \in S \text{ is an MNE}\right\}}{\min\left\{\sum\limits_{i=1}^{n} c_i(s) : s \in S\right\}}$$

6.3.1 PoA of Network (Atomic) Congestion Games

Theorem 6.3.1

The PoA in network congestion games with affine cost functions is $\frac{5}{2}$.