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Course: Quantum Information Theory

Assignment - 1
Roll: BMC202175
Date: January 31, 2024

Problem 1

Let $\mathcal{X}$ be a finite set and p_X be a probability distribution or a probability mass function (PMF) on $\mathcal{X}$. The Shannon entropy of p_X is defined as

$$H(p_X) \triangleq -\sum_{x\in\mathcal{X}} p_X(x)\log p_X(x)$$

.

1. Prove $\log x \leq x-1$ and $\log\frac{1}{x} \geq 1-x$ for all $x>0$.
2. $\sum\limits_{x\in\mathcal{X}} p_X(x)\log\frac{1}{p_X(x)} \leq \log|\mathcal{X}|$
3. $H(X)+H(Y) \geq H(X,Y)$ where $H(X,Y)=H(p_(X,Y))$ is the entropy of a joint PMF, $H(X)=H(p_X)$ where p_X is marginal of $p_{X,Y}$

Solution:

1. We have $\log x = \displaystyle\int_1^x \frac{1}{t}\,dt$ and $x-1=\displaystyle\int_1^x dt$. Now for $x\geq 1$ for all $t\geq 1$ we have $1\geq\frac{1}{t}$. Hence

$$\int_1^x \frac{1}{t}\,dt \leq \int_1^x dt \iff \log x\leq x-1$$

For $0<x<1$ we have $t<1$ hence $\frac{1}{t}\geq 1$. Hence

$$\int_x^1 \frac{1}{t}\,dt \geq \int_x^1 dt \iff -\log x\geq 1-x \iff x-1\geq\log x$$

Therefore $\forall\ x>0$ we have $\log x\leq x-1$.

Now we have $\log x\leq x-1 \iff 1-x\leq -\log x \iff 1-x\leq \log\frac{1}{x}$.

□

Problem 2

Let $p_X(x)$ be a PMF on $\mathcal{X}$. For $n\in bbN$, $\delta>0$, let

$$T_\delta^n(p_X) \triangleq \left\{x^n\in\mathcal{X}^n \mid \left|\frac{N(a|x^n)}{n}-p_X(a)\right| \leq \frac{\delta p_X(a)}{\log|\mathcal{X}|}\ \forall\, a\in\mathcal{X}\right\}$$

where $N(a|x^n)=\sum\limits_{i=1}^n \mathbb{1}_{\{x_i=a\}}$ denotes the number of occurrences of a in the sequences $x_1x_2\cdots x_n$.

1. Prove that

$$\sum_{x^n\notin T_\delta^n(p_X)}\prod_{i=1}^n p_X(x_i) \leq \exp\left[-\frac{2n\delta^2\eta_{p_X}^2}{(\log|\mathcal{X}|)^2}\right]$$

where $\eta_{p_X}=\min\limits_{a\in\mathcal{X}}\{p_X(a)\mid 0<p_X(a)<1\}$

2. Prove that

$$\left[1-\exp\left(\frac{2n\delta^2\eta_{p_X}^2}{(\log|\mathcal{X}|)^2}\right)\right]\exp\left[n(H(p_X)-\delta)\right] \leq |T_\delta^n(p_X)| \leq \exp\left[n(H(p_X)+\delta)\right]$$

3. Prove that

$$x^n \in T^n_\delta(p_X) \implies \exp[-n(H(p_X)+\delta)] \le \prod_{i=1}^{n} p_X(x_i) \le \exp[-n(H(p_X)-\delta)]$$

Solution:

$\square$

Definitions: Let $p_{X,Y}$ be a joint PMF on $\mathcal{X} \times \mathcal{Y}$ where $\mathcal{X}$, $\mathcal{Y}$ are finite sets. (Essentially $p_{XY}(x,y) \ge 0$ and $\sum_{x\in\mathcal{X}} \sum_{y \in mcY} p_{XY}(x,y) = 1$). We define the marginal of p_{XY} on X as $p_X(x) \triangleq \sum_{y\in\mathcal{Y}} p_{XY}(x,y)$ for $x \in \mathcal{X}$ and marginal of p_{XY} on Y as $p_Y(y) \triangleq \sum_{x\in\mathcal{X}} p_{XY}(x,y)$ for $y \in \mathcal{Y}$. or a pair $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ of sequences we define $N(a,b \mid x^n, y^n) = \sum_{i=1}^{n} \mathbb{1}_{\{(x_i,y_i)=(a,b)\}}$ as the number of occurances of (a,b) in (x^n, y^n).

Next the joint typical set wrt p_{XY} is defined as

$$T^n_\delta(p_{XY}) \triangleq \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n \mid \left| \frac{N(a,b \mid x^n, y^n)}{n} - p_{XY}(a,b) \right| \le \frac{\delta p_{XY}(a,b)}{\log |\mathcal{X}||mcY|} \ \forall\ (a,b) \in \mathcal{X} \times \mathcal{Y} \right\}$$

Problem 3

1. Prove that if $p_{XY}(a,b) = 0$ for some $(a,b) \in \mathcal{X} \times \mathcal{Y}$ and $(x^n, y^n) \in T^n_\delta(p_{XY})$ then $N(a,b|p_{XY}) = 0$. In other words, a pair that - probability does not occur in any typical pair of sequences.

2. Let $\eta_{p_{XY}} = \min_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \{p_{XY}(x,y) \mid 0 < p_{XY}(a,b) < 1\}$. Use the Hoeffding Inequality to prove that

$$\sum_{(x^n,y^n)\notin T^n_\delta(p_{XY})} p^n_{XY}(x^n,y^n) \le 2|\mathcal{X}||\mathcal{Y}| \exp\left[-\frac{2n\delta^2\eta^2_{p_{XY}}}{(\log|\mathcal{X}||\mathcal{Y}|)^2}\right]$$

Hoeffding Inequality: Let $Z_1, \ldots, Z_m$ are independent and identically distributed random variables for which $P[a \le Z_i \le b] = 1$ for ever $1 \le i \le m$ and $\mu = \mathbb{E}[Z_i]$. Then for every $\epsilon > 0$

$$p\left[\left|\frac{1}{m}\sum_{i=1}^{m} Z_i - \mu\right| > \epsilon\right] \le 2\exp\left[-2m\frac{\epsilon^2}{(b-a)^2}\right]$$

3. For any $(x^n, y^n) \in T^n_\delta(p_{XY})$ prove that

$$2^{-n[H(p_{XY})+\delta]} \le p^n_{XY}(x^n,y^n) = \prod_{i=1}^{n} p_{XY}(x_i,y_i) \le 2^{-n[H(p_{XY})-\delta]}$$

4. Prove that

$$(1-\tilde{\delta})2^{n[H(p_{XY})-\delta]} \le |T^n_\delta(p_{XY})| \le 2^{n[H(p_{XY})+\delta]}$$

where $\tilde{\delta} = 2|\mathcal{X}||\mathcal{Y}| \exp\left[\frac{2n\delta^2\eta^2_{p_{XY}}}{(\log|\mathcal{X}||\mathcal{Y}|)^2}\right]$

5. Prove that $(x^n, y^n) \in T^n_\delta(p_{XY})$ then $x^n \in T^n_\delta(p_X)$ and $y^n \in T^n_\delta(p_Y)$.

Solution:

$\square$

Definitions: Suppose p_{XY} is a probability distribution (probability mass function (PMF)) on $|mmcX \times \mathcal{Y}$. We recall the condition distribution $p_{Y|X}(y|x) = \frac{p_{XY}(x,y)}{p_X(x)}$ and for a pair $(x^n, y^n)\imath\mathcal{X}^n \times \mathcal{Y}^n$ of sequence $(x^n, y^n)\imath\mathcal{X}^n \times \mathcal{Y}^n$ of sequences $p^n_{Y|X}(y^n|x^n) = \prod_{i=1}^{n} p_{Y|X}(y_i|x_i)$

We define

$$H(Y|X = x) \triangleq H(p_{XY}|X = x) = -\sum_{y\in\mathcal{Y}} p_{Y|X}(y|x)\log p_{Y|X}(y|x)$$

and

$$H(Y|X) = H(p_{Y|X}|p_X) \triangleq \sum_{x\in\mathcal{X}} p_X(x)h9Y|X = x)$$

For any $x^n \in \mathcal{X}^n$ define the conditional typical set of x^n as

$$T^n_\delta(p_{Y|X}) = \{y^n \in \mathcal{Y}^n \mid (x^n, y^n) \in T^n_\delta(p_{XY})\}$$

Problem 4

1. Prove that $\sum_{y\in\mathcal{Y}} p_{Y|X)(y|x)=1}$
2. Prove that $H(Y|X) = H(X,Y) - H(X)$ and $H(Y|X) \geq 0$
3. Prove that Verify that if $x^n \notin T^n_\delta(p_X)$ then $T^n_\delta(p_{XY}|x^n) = \phi$
4. Suppose $x^n \in T^n_\delta(p_X)$ and $y^n \in T^n_\delta(p_{XY}|x^n)$ prove that
$$2^{-n[H(Y|X)+2\delta]} \leq p^n_{Y|X}(y^n|x^n) \leq 2^{-n[H(Y|X)-2\delta]}$$
5. Prove that if $x^n \in T^n_\delta(p_X)$ then
$$\sum_{y^n\in T^n_{2\delta}(p_{XY}|x^n)} p^n_{Y|X}(y^n|x^n) \geq 1 - 2|\mathcal{X}||\mathcal{Y}|\exp\left[-\frac{2n\delta^2}{(\log|\mathcal{X}||\mathcal{Y}|)^2}\eta_{p_{Y|X}}\right]$$
where $\eta_{p_{Y|X}} = \min_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \{p_{Y|X}(y|x) \mid 0 < p_{Y|X}(y|x) < 1\}$
6. Suppose $x^n \in T^n_\delta(p_X)$ then
$$(1-\tilde{\delta})2^{n[H(Y|X)-4\delta]} \leq |T^n_\delta(p_{XY}|x^n)| \leq 2^{n[H(Y|X)+4\delta]}$$
where $\tilde{\delta} = 2|\mathcal{X}||\mathcal{Y}|\exp\left[-\frac{2n\delta^2}{(\log|\mathcal{X}||\mathcal{Y}|)^2}\eta_{p_{Y|X}}\right]$

Solution:

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