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Course: Quantum Information Theory

Assignment - 1
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Problem 1

Let $\mathcal{X}$ be a finite set and p_X be a probability distribution or a probability mass function (PMF) on $\mathcal{X}$. The Shannon entropy of p_X is defined as

$$H(p_X) \triangleq -\sum_{x\in\mathcal{X}} p_X(x)\log p_X(x)$$

.

1. Prove $\log x \leq x-1$ and $\log\frac{1}{x} \geq 1-x$ for all $x>0$.
2. $\sum\limits_{x\in\mathcal{X}} p_X(x)\log\frac{1}{p_X(x)} \leq \log|\mathcal{X}|$
3. $H(X)+H(Y) \geq H(X,Y)$ where $H(X,Y) = H(p_{X,Y})$ is the entropy of a joint PMF, $H(X) = H(p_X)$ where p_X is marginal of $p_{X,Y}$

Solution:

1. We have $\log x = \displaystyle\int_1^x \frac{1}{t}\,dt$ and $x-1 = \displaystyle\int_1^x dt$. Now for $x\geq 1$ for all $t\geq 1$ we have $1\geq \frac{1}{t}$. Hence

$$\int_1^x \frac{1}{t}\,dt \leq \int_1^x dt \iff \log x \leq x-1$$

For $0<x<1$ we have $t<1$ hence $\frac{1}{t}\geq 1$. Hence

$$\int_x^1 \frac{1}{t}\,dt \geq \int_x^1 dt \iff -\log x \geq 1-x \iff x-1\geq \log x$$

Therefore $\forall\, x>0$ we have $\log x \leq x-1$.

Now we have $\log x \leq x-1 \iff 1-x \leq -\log x \iff 1-x\leq \log\frac{1}{x}$.

2.

$$\begin{aligned}
\sum_{x\in\mathcal{X}} p_X(x)\log\frac{1}{p_X(x)} - \log|\mathcal{X}| &= \sum_{x\in X} p_X(x)\log\frac{1}{p_X(x)} - \sum_{x\in\mathcal{X}} p_X(x)\log|\mathcal{X}| \\
&= \sum_{x\in\mathcal{X}} p_X(x)\log\frac{1}{|\mathcal{X}|p_X(x)} \\
&\leq \sum_{x\in\mathcal{X}} p_X(x)\left[\frac{1}{|\mathcal{X}|p_X(x)}-1\right] && \text{[Using Part (1)]} \\
&= \sum_{x\in\mathcal{X}}\left[\frac{1}{|\mathcal{X}|}-p_X(x)\right] = 1-1=0
\end{aligned}$$

Hence we get

$$\sum_{x\in\mathcal{X}} p_X(x)\log\frac{1}{p_X(x)} - \log|\mathcal{X}| \iff \sum_{x\in\mathcal{X}} p_X(x)\log\frac{1}{p_X(x)} \leq \log|\mathcal{X}|$$

3. We have

$$1\geq p_{XY}(x,y) = p_Y(y)$$

□

Problem 2

Let $p_X(x)$ be a PMF on $\mathcal{X}$. For $n \in bbN$, $\delta > 0$, let

$$T_\delta^n(p_X) \triangleq \left\{ x^n \in \mathcal{X}^n \mid \left| \frac{N(a|x^n)}{n} - p_X(a) \right| \leq \frac{\delta p_X(a)}{\log|\mathcal{X}|} \; \forall \, a \in \mathcal{X} \right\}$$

where $N(a|x^n) = \sum_{i=1}^{n} \mathbb{1}_{\{x_i = a\}}$ denotes the number of occurrences of a in the sequences $x_1 x_2 \cdots x_n$.

1. Prove that

$$\sum_{x^n \notin T_\delta^n(p_X)} \prod_{i=1}^{n} p_X(x_i) \leq \exp\left[-\frac{2n\delta^2 \eta_{p_X}^2}{(\log|\mathcal{X}|)^2} \right]$$

   where $\eta_{p_X} = \min_{a \in \mathcal{X}} \{ p_X(a) \mid 0 < p_X(a) < 1 \}$

2. Prove that

$$\left[1 - \exp\left(\frac{2n\delta^2 \eta_{p_X}^2}{(\log|\mathcal{X}|)^2} \right) \right] \exp\left[n(H(p_X) - \delta) \right] \leq |T_\delta^n(p_X)| \leq \exp\left[n(H(p_X) + \delta) \right]$$

3. Prove that

$$x^n \in T_\delta^n(p_X) \implies \exp[-n(H(p_X) + \delta)] \leq \prod_{i=1}^{n} p_X(x_i) \leq \exp[-n(H(p_X) - \delta)]$$

Solution:

$\square$

Definitions: Let $p_{X,Y}$ be a joint PMF on $\mathcal{X} \times \mathcal{Y}$ where $\mathcal{X}$, $\mathcal{Y}$ are finite sets. (Essentially $p_{XY}(x,y) \geq 0$ and $\sum_{x \in \mathcal{X}} \sum_{y \in mcY} p_{XY}(x,y) = 1$). We define the marginal of p_{XY} on X as $p_X(x) \triangleq \sum_{y \in \mathcal{Y}} p_{XY}(x,y)$ for $x \in \mathcal{X}$ and marginal of p_{XY} on Y as $p_Y(y) \triangleq \sum_{x \in \mathcal{X}} p_{XY}(x,y)$ for $y \in \mathcal{Y}$. or a pair $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ of sequences we define $N(a, b \mid x^n, y^n) = \sum_{i=1}^{n} \mathbb{1}_{\{(x_i, y_i) = (a,b)\}}$ as the number of occurances of (a,b) in (x^n, y^n).

Next the joint typical set wrt p_{XY} is defined as

$$T_\delta^n(p_{XY}) \triangleq \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n \mid \left| \frac{N(a, b \mid x^n, y^n)}{n} - p_{XY}(a,b) \right| \leq \frac{\delta p_{XY}(a,b)}{\log|\mathcal{X}||mcY|} \; \forall \, (a,b) \in \mathcal{X} \times \mathcal{Y} \right\}$$

Problem 3

1. Prove that if $p_{XY}(a,b) = 0$ for some $(a,b) \in \mathcal{X} \times \mathcal{Y}$ and $(x^n, y^n) \in T_\delta^n(p_{XY})$ then $N(a, b | p_{XY}) = 0$. In other words, a pair that - probability does not occur in any typical pair of sequences.

2. Let $\eta_{p_{XY}} = \min_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \{ p_{XY}(x,y) \mid 0 < p_{XY}(a,b) < 1 \}$. Use the Hoeffding Inequality to prove that

$$\sum_{(x^n, y^n) \notin T_\delta^n(p_{XY})} p_{XY}^n(x^n, y^n) \leq 2|\mathcal{X}||\mathcal{Y}| \exp\left[-\frac{2n\delta^2 \eta_{p_{XY}}^2}{(\log|\mathcal{X}||\mathcal{Y}|)^2} \right]$$

   Hoeffding Inequality: Let $Z_1, \ldots, Z_m$ are independent and identically distributed random variables for which $P[a \leq Z_i \leq b] = 1$ for ever $1 \leq i \leq m$ and $\mu = \mathbb{E}[Z_i]$. Then for every $\epsilon > 0$

$$p\left[\left| \frac{1}{m} \sum_{i=1}^{m} Z_i - \mu \right| > \epsilon \right] \leq 2 \exp\left[-2m \frac{\epsilon^2}{(b-a)^2} \right]$$

3. For any $(x^n, y^n) \in T_\delta^n(p_{XY})$ prove that

$$2^{-n[H(p_{XY})+\delta]} \leq p_{XY}^n(x^n, y^n) = \prod_{i=1}^{n} p_{XY}(x_i, y_i) \leq 2^{-n[H(p_{XY})-\delta]}$$

4. Prove that

$$(1-\tilde{\delta})2^{n[H(p_{XY})-\delta]} \leq |T_\delta^n(p_{XY})| \leq 2^{n[H(p_{XY})+\delta]}$$

where $\tilde{\delta} = 2|\mathcal{X}||\mathcal{Y}| \exp\left[\frac{2n\delta^2\eta_{p_{XY}}^2}{(\log|\mathcal{X}||\mathcal{Y}|)^2}\right]$

5. Prove that $(x^n, y^n) \in T_\delta^n(p_{XY})$ then $x^n \in T_\delta^n(p_X)$ and $y^n \in T_\delta^n(p_Y)$.

Solution:

□

Definitions: Suppose p_{XY} is a probability distribution (probability mass function (PMF)) on $|mmcX \times \mathcal{Y}$. We recall the condition distribution $p_{Y|X}(y|x) = \frac{p_{XY}(x,y)}{p_X(x)}$ and for a pair $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ of sequence $(x^n, y^n)\imath\mathcal{X}^n \times \mathcal{Y}^n$ of sequences $p_{Y|X}^n(y^n|x^n) = \prod_{i=1}^{n} p_{Y|X}(y_i|x_i)$

We define

$$H(Y|X=x) \triangleq H(p_{XY}|X=x) = -\sum_{y\in\mathcal{Y}} p_{Y|X}(y|x) \log p_{Y|X}(y|x)$$

and

$$H(Y|X) = H(p_{Y|X}|p_X) \triangleq \sum_{x\in\mathcal{X}} p_X(x) h9Y|X=x)$$

For any $x^n \in \mathcal{X}^n$ define the conditional typical set of x^n as

$$T_\delta^n(p_{Y|X}) = \{y^n \in \mathcal{Y}^n \mid (x^n, y^n) \in T_\delta^n(p_{XY})\}$$

Problem 4

1. Prove that $\sum_{y\in\mathcal{Y}} p_{Y|X)(y|x)=1}$

2. Prove that $H(Y|X) = H(X,Y) - H(X)$ and $H(Y|X) \geq 0$

3. Prove that Verify that if $x^n \notin T_\delta^n(p_X)$ then $T_\delta^n(p_{XY}|x^n) = \phi$

4. Suppose $x^n \in T_\delta^n(p_X)$ and $y^n \in T_\delta^n(p_{XY}|x^n)$ prove that

$$2^{-n[H(Y|X)+2\delta]} \leq p_{Y|X}^n(y^n|x^n) \leq 2^{-n[H(Y|X)-2\delta]}$$

5. Prove that if $x^n \in T_\delta^n(p_X)$ then

$$\sum_{y^n \in T_{2\delta}^n(p_{XY}|x^n)} p_{Y|X}^n(y^n|x^n) \geq 1 - 2|\mathcal{X}||\mathcal{Y}| \exp\left[-\frac{2n\delta^2}{(\log|\mathcal{X}||\mathcal{Y}|)^2}\eta_{p_{Y|X}}\right]$$

where $\eta_{p_{Y|X}} = \min_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \{p_{Y|X}(y|x) \mid 0 < p_{Y|X}(y|x) < 1\}$

6. Suppose $x^n \in T_\delta^n(p_X)$ then

$$(1-\tilde{\delta})2^{n[H(Y|X)-4\delta]} \leq |T_\delta^n(p_{XY}|x^n)| \leq 2^{n[H(Y|X)+4\delta]}$$

where $\tilde{\delta} = 2|\mathcal{X}||\mathcal{Y}| \exp\left[-\frac{2n\delta^2}{(\log|\mathcal{X}||\mathcal{Y}|)^2}\eta_{p_{Y|X}}\right]$

Solution:

$\square$