Algebra and Computation

Instructor: Amit Kumar Sinhababu and Sumanta Ghosh

Scribe: Soham Chatterjee

sohamchatterjee999@gmail.com
Website: sohamch08.github.io

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CHAPTER 1

Introduction

CHAPTER 2

Integer and Polynomial Addition

CHAPTER 3

Integer and Polynomial Multiplication

Definition 3.1: Multiplication Time Function: $M(n)$

The function $M : \mathbb{N} \to \mathbb{R}_+$ for any commutative ring $R[x]$ is called multiplication time function for if polynomials in $R[x]$ of degree less than n can be multiplied using at most $M(n)$ operations in R.

Similarly we can define the function M as above for multiplication time for $\mathbb{Z}$ if two integers of length n bits can be multiplied using at most $M(n)$ operations

Assumption 3.0.1. *content...*

Proof of Claim c: ontent... ■

CHAPTER 4

Polynomial Evaluation

4.1 Introduction

We will consider the following situation: R is a commutative ring as always and $f \in R[x]$ where $\deg(f) = d$. We also have k points $u_0, \ldots, u_{k-1} \in R$. Now we want to discuss here the fast algorithms of finding out $(f(u_0), \ldots, f(u_{k-1}))$. So we basically want the evaluation map

$$\begin{aligned} \varphi : R[x]/\langle m \rangle &\to R^n \\ f &\to (f(u_0), \ldots, f(u_{k-1})) \end{aligned}$$

which is a ring homomorphism. If R is a field then $R[x]$ is a vector space over R and the ϕ is an isomorphism. Formally we want to solve the following two problems with fast algorithms:

Problem 4.1: Single Point evaluation

Given $f \in R[x]$ with $\deg(f) = d$ and $\alpha \in R$ compute $f(\alpha)$

Problem 4.2: Multi-Point evaluation

Given $f \in R[x]$ with $\deg(f) = d$ and $u_o, \ldots, u_{n-1} \in R$ compute $f(u_0), \ldots, f(u_{n-1})$

4.2 Single Point Evaluation

4.2.1 Horner's Method

Theorem 4.2.1 Horner's Method

Given a polynomial $f(x) = \sum_{i=0}^{d} a_i x^i$ where $a_i \in R$ for all $i \in [n]$ and a point $\alpha \in R$ using only $O(d)$ many additions and multiplications.

Proof: Consider the following algorithm:

Algorithm 1: Horner's Method

begin
$p(x) = a_0 + x(a_1 + x(a_2 + (x(\cdots + x(a_n)\cdots))))$

Clearly we are using only d many additions and d many multiplications. So overall we need $2d = O(d)$ ring operations to evaluate the polynomial. The following lower bound results we obtain. ■

This is the minimal number of additions and multiplications for any algorithm to evaluate a polynomial.

Theorem 4.2.2 [OST13]

Any algorithm to evaluate an arbitrary degree d polynomial $f \in R[x]$ at any point $\alpha \in R$ must use at least n additions

Theorem 4.2.3 [Pan66]

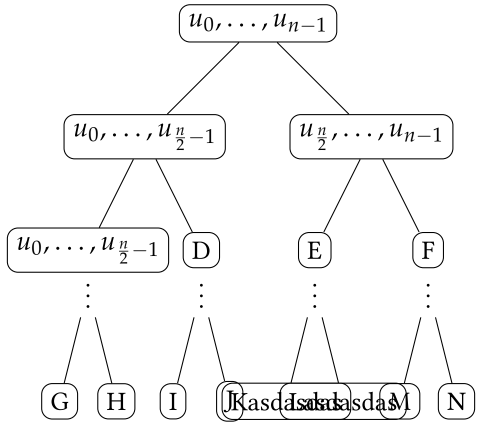
Any algorithm to evaluate an arbitrary degree d polynomial $f \in R[x]$ at any point $\alpha \in R$ without initial conditioning of coefficients has at least n multiplications and at least n additions.

Theorem 4.2.4 [Pan66],[Mot55]

Any degree d real polynomial can be evaluated using $\left\lfloor \frac{d}{2} \right\rfloor + 2$ multiplications and d additions.

4.3 Fast Multi-point Evaluation

A trivial algorithm for using $O(d^2)$ ring operations is to apply Horner's Method for each point and since it takes $O(d)$ operations for each point we can find the evaluations at all d points in $O(d^2)$ many ring operations. But we want to get close to linear operations. Since Horner's rules uses lowest number of ring operations doesn't mean for d points $O(d^2)$ is lowest. There is an fast algorithm to evaluate the polynomial at all d points using $O(M(d) \log d)$ operations.


$u_0, \ldots, u_{n-1}$
$u_0, \ldots, u_{\frac{n}{2}-1}$
$u_{\frac{n}{2}}, \ldots, u_{n-1}$
$u_0, \ldots, u_{\frac{n}{2}-1}$
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CHAPTER 5

Polynomial Interpolation

CHAPTER 6

Bibliography

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