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Course: Probability Theory

Assignment - 2
Dept: STCS
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Problem 1 [H] Problem 1.3: Ordering of three random variables

Suppose X, Y and U are mutually independent, such that X and Y are each exponentially distributed with some common parameter $\lambda > 0$ and U is uniformly distributed on the interval $[0, 1]$. Express $\mathbb{P}\{X < U < Y\}$ in terms of λ. Simplify your answer.

Solution: X and Y are exponentially distributed with some common parameter $\lambda > 0$. Hence $F_X(x) = 1 - e^{-\lambda x}$ and $F_Y(y) = 1 - e^{-\lambda y}$ for some $x, y \geq 0$.and U is uniform on $[0, 1]$. So

$$\begin{aligned}
\mathbb{P}[X < U < Y] &= \int_0^1 \mathbb{P}[X < u, Y > u] du \\
&= \int_0^1 \mathbb{P}[X < u] \cdot \mathbb{P}[Y > u] du \\
&= \int_0^1 F_X(u)(1 - F_Y(u)) du \\
&= \int_0^1 \left(1 - e^{-\lambda u}\right)\left(1 - \left(1 - e^{\lambda u}\right)\right) du \\
&= \int_0^1 \left(1 - e^{-\lambda u}\right) e^{-\lambda u} du \\
&= \int_0^1 \left[e^{-\lambda u} - e^{-2\lambda u}\right] du \\
&= \left[\frac{e^{-\lambda u}}{-\lambda} - \frac{e^{-2\lambda u}}{-2\lambda}\right]_0^1 \\
&= \left[\frac{e^{-2\lambda}}{2\lambda} - \frac{e^{-\lambda}}{\lambda}\right] - \left[\frac{1}{2\lambda} - \frac{1}{\lambda}\right] = \frac{\left(e^{-\lambda}\right)^2 - 2e^{-\lambda} + 1}{2\lambda} = \frac{\left(e^{-\lambda} - 1\right)^2}{2\lambda}
\end{aligned}$$

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Problem 2 [H] Problem 1.5: Congestion at output ports

Consider a packet switch with some number of input ports and eight output ports. Suppose four packets simultaneously arrive on different input ports, and each is routed toward an output port. Assume the choices of output ports are mutually independent, and for each packet, each output port has equal probability.

(a) Specify a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to describe this situation.

(b) Let X_i denote the number of packets routed to output port i for $1 \leq i \leq 8$. Describe the joint pmf of $X_1, \ldots, X_8$.

(c) Find $\mathrm{Cov}(X_1, X_2)$

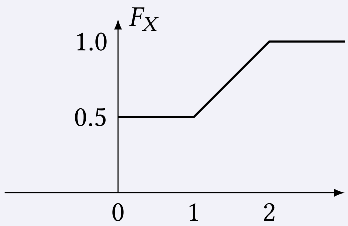
(d) Find $\mathbb{P}[X_i \leq 1 \text{ for all } i]$

(e) Find $\mathbb{P}[X_i \leq 2 \text{ for all } i]$

Solution: ■

Problem 3 [H] Problem 1.13: A CDF of mixed type

Let X have the CDF shown.



(a) Find $\mathbb{P}[X \leq 0.8]$

(b) Find $\mathbb{E}[X]$

(c) Find $\mathrm{Var}[X]$

Solution:

(a) $\mathbb{P}[X \leq 0.8] = \mathbb{F}_X[0.8] = 0.5$ since the value of F_X increases when $X \geq 1$.

(b) $F_X(x) = \begin{cases} 0 & \text{when } x \leq 0 \\ 0.5 & \text{when} 0 \leq x \leq 1 \\ 0.5 + \frac{x-1}{2} & \text{when } 1 \leq x \leq 2 \\ 1 & \text{when } x \geq 2 \end{cases}$

Hence

$$\begin{aligned} \mathbb{E}[X] &= \int_0^\infty [1 - F_X(x)]dx = \int_0^1 [1 - 0.5]dx + \int_1^2 \left[1 - 0.5 - \frac{x-1}{2}\right] dx + \int_2^\infty [1 - 1]dx \\ &= \int_0^1 0.5dx + \int_1^2 \left[0.5 - \frac{x-1}{2}\right] dx \\ &= 0.5 + \left[0.5x - \frac{(x-1)^2}{4}\right]_1^2 = 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

(c) Take $Y = X^2$ and the distribution function for Y is F_Y. Now for any $y \geq 0$

$$F_Y(y) = \mathbb{P}[Y^2 \leq y] = \mathbb{P}[X^2 \leq y] = \mathbb{P}[X \leq \sqrt{y}] = F_X(\sqrt{y})$$

Therefore

$$F_Y(y) = \begin{cases} 0 & \text{when } x \leq 0 \\ 0.5 & \text{when} 0 \leq x \leq 1 \\ 0.5 + \frac{\sqrt{y}-1}{2} & \text{when } 1 \leq y \leq 4 \\ 1 & \text{when } y \geq 4 \end{cases}$$

Hence

$$\begin{aligned} \mathbb{E}[X^2] &= \int_0^\infty [1 - F_Y(y)]dy = \int_0^1 [1 - 0.5]dy + \int_1^4 \left[1 - 0.5 - \frac{\sqrt{y}-1}{2}\right] dy + \int_4^\infty [1 - 1]dy \\ &= \int_0^1 0.5dy + \int_1^4 \left[0.5 - \frac{\sqrt{y}-1}{2}\right] dy \\ &= 0.5 + \left[0.5y - \frac{\frac{2}{3}y^{\frac{3}{2}} - y}{2}\right]_1^4 = 0.5 + 3 \cdot 0.5 - \left[\frac{\frac{2}{3}4^{\frac{3}{2}} - 4}{2} - \frac{\frac{2}{3}1^{\frac{3}{2}} - 1}{2}\right] = 2 - \frac{5}{6} = \frac{7}{6} \end{aligned}$$

So $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X] = \frac{7}{6} - \frac{9}{16} = \frac{56-27}{48} = \frac{29}{48}$.

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Problem 4 [H] Problem 1.17: Transformation of a random variable

Let X be exponentially distributed with mean λ^{-1}. Find and carefully sketch distribution functions for the random variables $Y = \exp(X)$ and $Z = \min(X, 3)$

Solution: X is exponentially distributed with mean λ^{-1}. So the density function of X for $x \geq 0$ is $f_X(x) = \lambda e^{-\lambda x}$. So for $y > 0$,

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[e^x \leq y] = \mathbb{P}[x \leq \ln y] = F_X[\ln y] = 1 - e^{-\lambda[\ln y]} = 1 - \left[e^{\ln y}\right]^{-\lambda} = y^{-\lambda}$$

And for $z \geq 0$

$$F_Z(z) = \mathbb{P}[Z \leq z] = \mathbb{P}[\min(X, 3) \leq z]$$

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