CSS.201.1 Algorithms

Instructor: Umang Bhaskar

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Scribe: Soham Chatterjee

soham.chatterjee@tifr.res.in
Website: sohamch08.github.io

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Chapter 1

Approximation Algorithms using LP

In this chapter we will study some approximation algorithms using linear programming to get better approximation ratios of the optimal solution.

1.1 Set Cover

Set Cover
Input: $\mathcal{U}$: Universe of all elements $e_1, \ldots, e_n$
$\mathcal{S} = \{S_1, \ldots, S_m\}$, $S_i \subseteq \mathcal{U}$ forall $i \in [m]$
Function $c : \mathcal{S} \to \mathbb{Z}_+$
Question: Given $\mathcal{U}, \mathcal{S}$ and the function c find $T \subseteq [m]$ such that $\bigcup\limits_{i \in T} S_i = \mathcal{U}$ to minimize the total cost $\sum\limits_{i \in T} c(S_i)$

Since the special case of Set Cover is basically the Vertex Cover problem we discussed earlier, we know that Set Cover is NP-hard.

Theorem 1.1.1
Set Cover is NP-hard.

Since we are going to find approximate solutions using LP let's first write the linear program for Set Cover:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{S \in \mathcal{S}} c(S) x_S \\
\text{subject to} \quad & \sum_{S : e \in S} x_S \geq 1 \quad \forall\, e \in \mathcal{U}, \\
& x_S \geq 0 \quad \forall\, S \in \mathcal{S}
\end{aligned}
$$

1.1.1 Frequency f-Approximation Algorithm

Let for any element $e \in \mathcal{U}$, f_e is the frequency of the element e in $\mathcal{S}$ i.e. $f_e = |\{S \in \mathcal{S} : e \in S\}|$. Then let $f = \max\{f_e : e \in \mathcal{U}\}$. Then we want to find a f-approximation algorithm for set cover.

Question 1.1: F

r vertex cover what is f?

For all $e \in E$ we have $f_e = 2$ since the elements of universe corresponds to the edges and sets corresponds to vertices and each edge constained in exactly 2 sets. So $f = 2$.

Algorithm 1: f-Approximate Algorithm

Input: $\mathcal{U}, \mathcal{S}, c$
Output: $T \subseteq [m]$ such that $\bigcup_{i \in T} S_i = \mathcal{U}$ and $\sum_{i \in T} c(S_i)$ is minimized

begin
  $T \longleftarrow \emptyset$
  $\hat{x} \longleftarrow 0^{|\mathcal{S}|}$
  Let x^* is the optimal solution of the LP for Set Cover problem
  for $S_i \in \mathcal{S}$ **do**
    if $x^*_{S_i} \geq \frac{1}{f}$ **then**
      $T \longleftarrow T \cup \{i\}$
      $\hat{x}_{S_i} \longleftarrow 1$
  return T

Lemma 1.1.2

$\hat{x}$ is a feasible solution.

Proof: For all $e \in \mathcal{U}$ there are at most f sets containing e. Thus at most f terms in the summation in *LHS* of the first constraint for each $e \in \mathcal{U}$ Thus in x^* at least one such term is $\geq \frac{1}{f}$. ■

Lemma 1.1.3

$\sum_{S \in \mathcal{S}} c(S)\hat{x}_S \leq f \cdot \sum_{S \in \mathcal{S}} c(S) x^*_S$

Proof: In $\hat{x}$ if $\hat{x}_S = 1$ that means $x^*_S \geq \frac{1}{f}$. Therefore we have the lemma. ■

Hence with this algorithm we can get a f-approximation for Set Cover problem. But this is not good enough since one element can be in too many sets and then it doesn't give a good approximation. In the next sections we will show how to get a better approximation ratio.

1.1.2 $O(n \log n)$-Approximation Algorithm through Randomized Rounding

Algorithm 2: $n \log n$-Approximate Algorithm

Input: $\mathcal{U}, \mathcal{S}, c$
Output: $T \subseteq [m]$ such that $\bigcup_{i \in T} S_i = \mathcal{U}$ and $\sum_{i \in T} c(S_i)$ is minimized

begin
  $\hat{x} \longleftarrow 0^{|\mathcal{S}|}$
  Let x^* is the optimal solution of the LP for Set Cover problem
  for $S \in \mathcal{S}$ **do**
    Set $\hat{x}_S \longleftarrow 1$ with probability x^*_S.
  return $\hat{x}$

1.1.3 Frequency f-Approximation Algorithm through Dual Fitting

1.2 Makespan Minimization

Chapter 2

Bibliography