Part - I

Mathematics

CHAPTER 1

Set Theory

1.0 OBJECTIVES

After studying the concept of a set and its fundamental operations you can explain the following :

- Concept of a set
- Complement of a set
- Union and intersection of sets.

Students can verify :

- De Morgan's Laws.
- Distributive laws of union and intersection.

Students can solve problems involving numerical data.

1.1 INTRODUCTION

The theory of sets forms the basis of the modern mathematics. The beginning of the theory can be traced from the German mathematician Cantor at the end of the 18th century. At present it covers a very extensive part of mathematics. However, we restrict ourselves to a very elementary part of it.

1.2 SET - CONCEPT

In our everyday life we often make use of collective phrases. Like,

(i) Students wearing spectacles,

(ii) Wagons attached to a train,

(iii) Flowering trees in a garden,

(iv) Students in a class,

(v) A pack of cards.

Such collections are called 'sets' in mathematics.

To describe a set we need the following considerations.

(a) set or collection

(b) elements in a set

(c) rule of property which enables us to say whether a given object is in a set or not. e.g. In a set "students wearing spectacles" we note that an element is "every student who wears spectacles" and the rule is "only those students who wear spectacles."

1.2.1 Sets and Elements

Consider a set consisting of numbers 1, 2, 3, 4, 5. Each of these numbers is an element of the set.

We denote this set by a letter A.

$$A = \{1, 2, 3, 4, 5\}$$

The fact that a number 3 is in a set A is written as "3 belongs to the set A". This concept of belonging to the set is symbolically denoted as

$3 \in A$

and read as " 3 belongs to A".

While the fact that the element 6 is not in the set A; i.e. 6 does not belong to the set A is symbolically written as $6 \notin A$ and read as "6 does not belong to A".

There are various ways of representing a set. One such method is indicated above to describe a set. Another method is to state all the elements of a set. There is one more method, which we shall study now. The set is described by using the property of the elements.

e.g. The fact that the element x of the set A having a property P is described as P(x) and the set is written as

$$A = \{x \mid P(x)\}$$

The above set A = {1, 2, 3, 4, 5} can now be written as in this notation as

$$A = \{x \mid x \text{ is a positive integer between 0 and 6}\}$$

An element of the set is counted once only,

i.e. {1, 2, 3, 3} is the same as {1, 2, 3}.

Also set is regarded as the same even if its elements are written in different order.

e.g. {p, q, r, s} = {r, p, s, q} = {s, r, p, q}.

Definition : Two sets are said to be **equal** if they contain the same elements.

e.g. $A = \{2,4\}$,

$B = \{ x | x^2 – 6x + 8 = 0 \}$

The elements of the set B are the roots of the equation $x^2 – 6x + 8 = 0$. We know from Elementary Algebra that the roots of this equation are 2 and 4.

Thus by the definition two sets A and B are equal; and we write

$$A = B.$$

Definition : A set having only one element is called a singleton set.

e.g. {prime minister of India}, {a}, {100}, {b} are. all singleton sets of : prime minister of India, a, 100, b.

Next, consider a set $\{ x \mid x = 1, x = – 1 \}$.

We know that there is no number which is equal to –1 and 1 at the same time. Thus this set has no element. Such a set having no element is called an empty set.

Definition : A set having no element is called an empty (or null or void) set and is denoted by ϕ.

Thus $\phi = \{x \mid x \neq x\}$

All empty sets are equal.

1.2 Check your progress.

1. Use appropriate symbols in the blank.

   (a) 3 ………….. {1,2,3,4}

   (b) 5 ………….. {1,6,7,8}

2. Are the following sets equal ?

   $A = \{x \mid x \text{ is a positive integer, } 1 \leq \times \leq 5\}$

   $B = \{1,2, 3, 4, 5\}$

   $C = \{x \mid x \text{ is a root of the equation } x^2 - 3x + 2 = 0\}$

3. State whether true or false.

   $A = \{x \mid x \text{ is a positive integer, } 2 \leq \times \leq 4\}$, $B = \phi$ and $A = B$.

   $C = (x \mid x \text{ is a root of the equation } 4x + 3 = 0\}$, and C is a singleton set.

1.3 SUBSET

Consider two sets.

$A = \{x \mid x \text{ is a student in } 1^{st} \text{ year class}\}$

$B = \{y \mid y \text{ is a student wearing spectacles in } 1^{st} \text{ year class}\}$

All students who wear spectacles are in a 1^{st} year class. In other words every y is x. Obviously every student in the class may not wear spectacles i.e. every x is not y. i.e. Every element of the set B is an element of A, but every element of A is not an element of B.

In this case, we say that B is a subset of A

We introduce the following notation.

If P and Q are two statements such that, if P is true then Q must be true; we say that P implies Q. We express this in a symbolic form as

$$P \Rightarrow Q. \text{ (read as P implies Q)}$$

$$\text{e.g. } (x = 3) \Rightarrow (x^2 - 6x + 9 = 0)$$

If in addition to $P \Rightarrow Q$, we have $Q \Rightarrow P$, we have both sided implications, and we write it as $P \Leftrightarrow Q$.

read as "P implies and is implied by Q."

e.g. $x = 2 \Leftrightarrow x^3 = 8$, when x is real.

For the two sets C, D defined as

$C = \{x \mid x \text{ is a triangle}\}$

$D = \{x \mid x \text{ is an equilateral triangle}\}$

We see that $x \in D \Rightarrow x \in C$.

Definition : B is called a **subset** of A if

$x \in B \Rightarrow x \in A$.

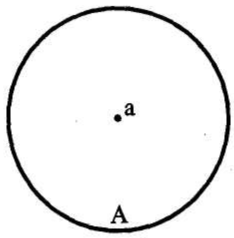
The statement that B is a subset of A is symbolically written as $B \subset A$.

If $x \in A \Rightarrow x \in B$, then A is a subset of B.

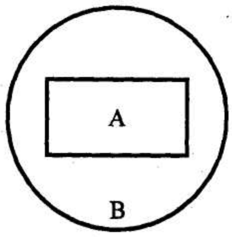
i.e. $A \subset B$

and is read as "A is contained in B" or "B contains A" or "A is a subset of B".

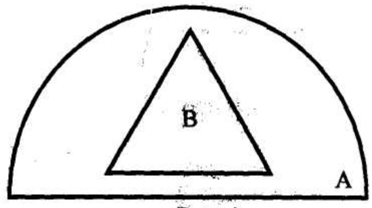
To illustrate the relationship between gets, we use different diagrams called **Venn diagrams.**



$a \in A$
Fig. 1.1



$A \subset B$
Fig. 1.2



$B \subset A$
Fig. 1.3

1.3.1 Proper and Improper Subsets

We shall now consider two extreme cases of subsets.

In the illustration of sets

A = {x | x is a student in 1st year class}

B = {y | y is a student wearing spectacle in 1st year class}

(i) It may happen that no student in 1st year class wears spectacles.

i.e. $B = \phi$

Hence ϕ can be regarded as a subset of A.

In fact $\phi \subset A$ for every set A.

(ii) Another possibility is, every student in 1st year class may wear spectacles.

i.e. B = A.

In this case $B\subset A \Rightarrow A\subset A$.

In fact every set A can be regarded as a subset of itself.

Both ϕ and A are called **improper** subsets of A.

Except ϕ and A all other subsets of A are called **proper** subsets of A.

e.g. Let A = {a, b, c}

The possible subsets of A are

ϕ , A_1 = {a}, A_2 = {b}, A_3 = {c}, A_4 = {a, b}, A_5 = {b. c},

A_6 = {c, a), A = a, b, c}

Thus A_1, A_2, A_1, A_4, A_5, A_6 are proper subsets of A.

The set of all subsets of A is called a **power set** of A and is denoted by P(A).

The total number of elements of P(A) is given by a formula :

$$n(P(A)) = 2^{n(A)}$$

where n (A) denote the number of elements in the set A.

Here n (A) = 3 and n (P (A)) = 2^3 = 8.

We easily count the number subsets of A as 8.

Next consider,

S = {1,2,3,5,7}, T = {3,5,7}

Here T is a proper subset of S. $T \subset S$.

Sometimes S is called superset of T.

1.3.2 Equality of Sets

The equality of two sets can be defined as :

Definition : Two sets A and B are said to be **equal** if each of A and B is a subset of other.

$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A.$$

This can be proved as follows :

Let $A \subset B$ and $B \subset A$.

$\therefore \quad x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \Rightarrow A$.

As all these relations hold at the same time, it is obvious that A and B have the same elements.

i.e. $A = B$.

Conversely, let us suppose that $A = B$.

Hence $x \in A \Rightarrow x \in B$ i.e. $A \subset B$

and $x \in B \Rightarrow x \in A$ i.e. $B \subset A$.

This provides us with a very important tool to prove the equality of two sets (especially when they contain infinite number of elements)

1.3.3 Transitivity of set inclusion

Now consider,

$A = \{p, q, r, s\}$

$B = \{o, p, q, r, s, t\}$

$C = \{o, p, q, r, s, t, u, v\}$

We clearly see the following relations.

$A \subset B$, $B \subset C$. Also $A \subset C$.

i.e. $(A \subset B, B \subset C) \Rightarrow A \subset C$.

This property known as **set inclusion** is said to be **transitive.**

The general proof of the above property is given below.

$A \subset B$ means $x \in A \Rightarrow x \in B$

$B \subset C$ means $x \in B \Rightarrow x \in C$

Hence $A \subset B$ and $B \subset C$ means

$x \in A \Rightarrow x \in B \Rightarrow x \in C$

i.e. $x \in A \Rightarrow x \in C$

i.e. $A \subset C$.

We have the following venn diagram to illustrate the above two properties.

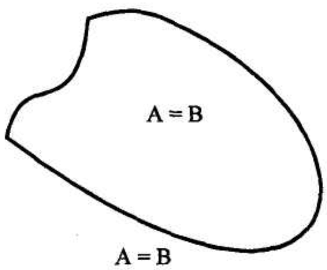
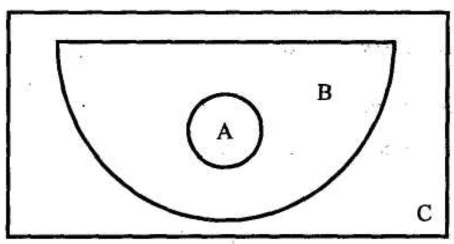


Fig. 1.4



$A \subset B, B \subset C \Rightarrow A \subset C$

Fig. 1.5

1.3 Check your progress.

1. State whether true or false.

   (a) $2 \subset \{1,2,3\}$

   (b) $3 \in \{2, 4, 6\}$

(c) $\{1, 2\} \subset \{x \mid x \text{ is a positive integer}, 1 \leq x \leq 3\}$

(d) ϕ and A are improper subsets of A.

(e) The proper subsets of $A = \{2, 5, 6\}$ are $A_1 = \{2\}$, $A_2 = \{5\}$, $A_3 = \{6\}$, $A_4 = \{2, 5\}$, $A_5 = \{2, 6\}$, $A_6 = \{5, 6\}$.

2. Write down all possible subsets of the set
   $A = \{10, 20, 30\}$.

3. (a) Is $A = \{a, b, c\}$ a super set of $B = \{a, b, c, d\}$?

   (b) Write down all possible subsets of $B = \{p, q, r, s\}$ each containing two - elements only.

1.4 UNIVERSAL SET

Consider the following sets.

$A = \{x \mid x \text{ is a student in } 1^{st} \text{ year class}\}$

$B = \{x \mid x \text{ is a student in a college in India}\}$

$C = \{x \mid x \text{ is a student in a college in the world}\}$

$U = \{x \mid x \text{ is a student}\}$

We see that $A \subset B, B \subset C, C \subset U$.

In fact A, B, C are all subsets of a fixed set U. We may find number of sets which are subsets of a fixed set like U. This fixed set is called the **universal set.**

Let P, Q, R, S, T,... be the set of books written in Marathi, Gujarathi, Bengali, Hindi French, German, languages. All these are subsets of "a set of books' say X.

In this case X is the **universal set**.

Many other illustrations of universal sets can be constructed.

1.5 COMPLEMENT OF A SET

Consider a universal set

$X = \{x \mid x \text{ is a tree}\}$

and a set $A = \{x \mid x \text{ is a tree having height more than 3 meters}\}$.

We can form another set $B = \{x \mid x \text{ is a tree having height less than 3 meters}\}$.

We observe that sets A and B together form a universal set X.

In this case, B is called complement of the set A. The elements of B are the elements of X but are not elements of A. The complement of a set A is denoted by A'.

Definition: The **complement** of a set A has elements in X which are not in A.

i.e. $A' = \{x \mid x \in X, x \notin A\}$

e.g. Let $X = \{0, 1, 2, 3, 4, 5\}$

and $A = \{0, 2, 4\}$

Then $A' = \{1, 3, 5\}$

The complement of A' is $\{0, 2, 4\}$, which is A again.

$\therefore \quad (A')' = A$

This can be proved as follows :

$x \in (A')' \Leftrightarrow x \notin A' \Leftrightarrow x \in A$

i.e. $(A')' = A$

We shall represent X, A, A' by the following venn diagram.

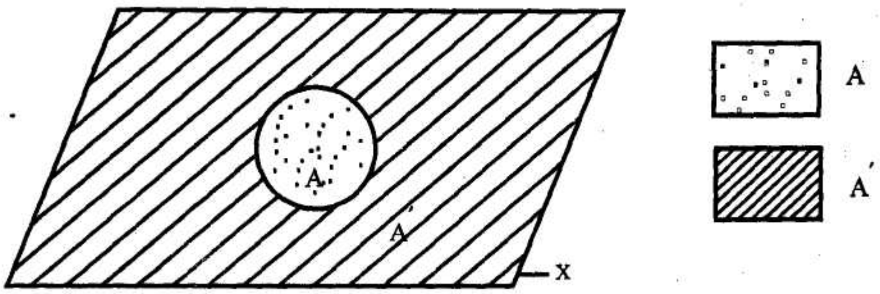


Fig. 1.6

We note the following points regarding complement of a set.

(i) $\phi' = X$ (ii) $X' = \phi$

(iii) If $A \subset B$ then $B' \subset A'$

We shall prove it as:

Let $x \in B' \Rightarrow x \notin B$, But $A \subset B$

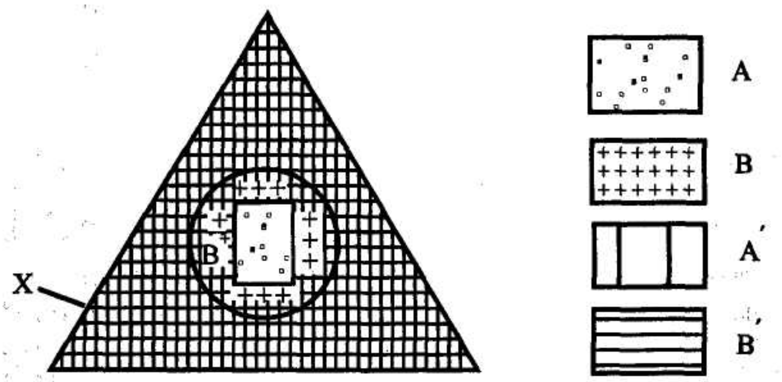
$\Rightarrow x \notin A$

$\Rightarrow x \in A'$

Thus $x \in B' \Rightarrow x \in A'$

$\therefore$ $B' \subset A'$.

The representation by a venn diagram is as :



$A \subset B \Rightarrow B' \subset A'$

Fig. 1.7

1.4 & 1.5 Check your progress:

Solve the Following

Given a universal set X = {1, 2, 3, 4, 5, 6, 7, 8, 9}

Obtain the complement of the following sets using universal X (above)

(a) A = {1, 3, 5, 7, 9}

(b) B = {2, 4, 6, 8}

(c) C = {1, 3, 9}

(d) D = {1,2,3,4, 5,6,7,8,9}

(e) ϕ

1.6 UNION OF SETS

We shall now begin with operations on sets or what is usually called as **Algebra of sets.**

Let A = {a, b, c, d, e}, B = {a, c, f, g, h} and
C = {a, b, c, d, e, f, g, h}

We find here that every element of A is an element of C and every element of B is an element of C. In other words, if an element belongs to A, it belongs to C and similarly if an element belongs to B, it belongs to C. i.e. An element of C is either an element of A or an element of B or it is a common element of both A and B. Such a set

C is called **union** of two sets A and B.

This set is denoted by $A \cup B$, and is read as "A union B".

Definition: The **union** of two sets A and B is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in \text{ both A and B } \}$$

or $$A \cup B = \{x \mid x \in A \text{ and} \mid \text{or } x \in B\}$$

Thus $A \cup B$ is the set of all elements of A and B, the common elements, if any, being taken once only.

The following Venn-diagram will illustrate the statement.

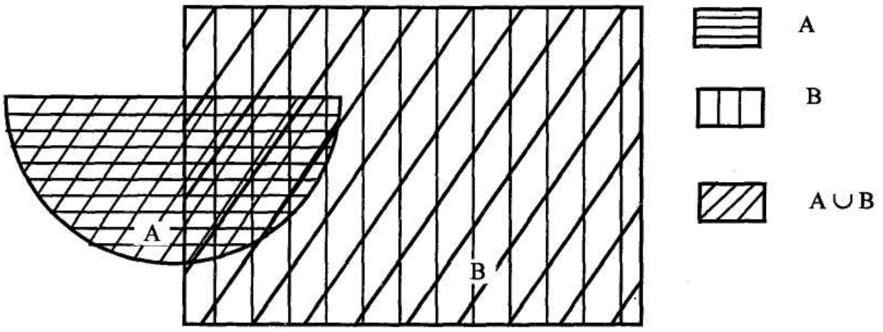


Fig. 1.8 : A ∪ B

1.6.1 Properties of Union Operation

We shall state some properties of the union operation

(i) **$A \cup B = B \cup A$ (commutativity)**

Proof.: $x \in A \cup B \Leftrightarrow x \in A$ and | or $x \in B$

$\Leftrightarrow x \in B$ and | or $x \in A$

$\Leftrightarrow x \in B \cup A$

i.e. we get the same set, in whatever order the sets are taken for union. In this case we say that union is commutative.

(ii) **$A \cup \phi = A$**

(iii) **$A \cup X = X$**

(iv) **$A \cup A' = X$**

Proof.: $x \in A \cup A' \Leftrightarrow x \in A$ and | or $x \in A'$

$\Leftrightarrow x \in A$ and $x \in A'$

$\Leftrightarrow x \in X$

(v) **$A \cup A = A$**

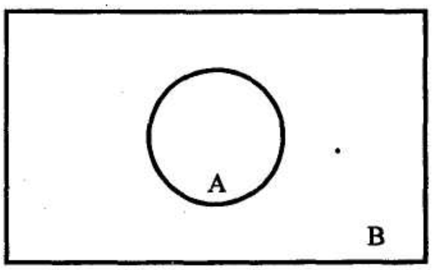
Proof.: $x \in A \cup A \Leftrightarrow x \in A$ and | or $x \in A$

$\Leftrightarrow x \in A$

(vi) **$A \subset A \cup B$ and $B \subset A \cup B$**

Both these follow from the Venn diagram of $A \cup B$. (Fig. 1.8)

(vii) **If $A \subset B$, then $A \cup B = B$**



$A \subset B \Leftrightarrow A \cup B = B$

Fig. 1.9

Proof.: Since $A \subset B$, $x \in A \Rightarrow x \in B$

$X \in A \cup B \Rightarrow x \in A$ and | or $x \in B$

$\Rightarrow x \in B$ (in either case $x \in B$)

$\therefore \quad A \cup B \subset B$

Similarly, we can show that $B \subset A \cup B$

$\therefore \quad A \cup B = B.$

(viii) $A \cup (B \cup C) = (A \cup B) \cup C$ (associativity)

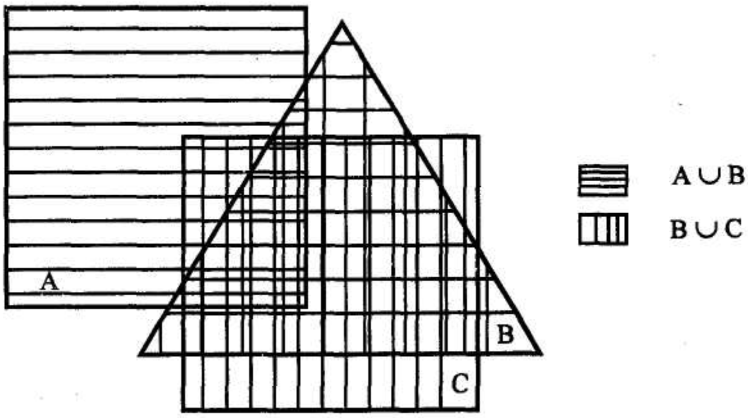


Fig. 1.10 : A ∪ B ∪ C

Proof. : $x \in A \cup (B \cup C)$

$\Leftrightarrow$ $x \in A$ and |or $x \in (B \cup C)$

$\Leftrightarrow$ $x \in A$ and | or $x \in B$ and | or $x \in C$

$\Leftrightarrow$ $x \in (A \cup B)$ and | or $x \in C$

$\Leftrightarrow$ $x \in (A \cup B) \cup C$

In this case we say that **union** operation is **associative.** i.e. if we remove the brackets on either side we get the same result,

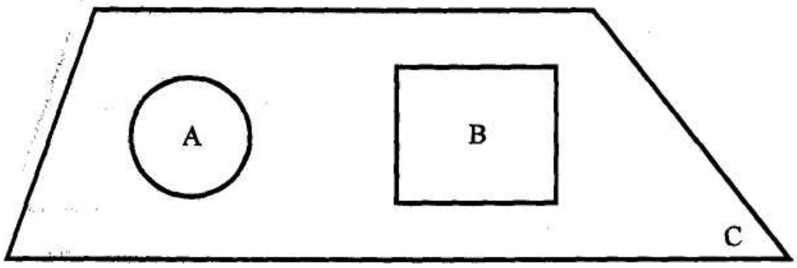
viz. $A \cup B \cup C.$

$A \cup B \cup C$

$= \{x \mid x \in A \text{ and } \mid \text{or } x \in B \text{ and } \mid \text{or } x \in C\}$

$= \{x \mid x \in \text{at least one of } A, B, C\}$

(ix) If $A \subset C$ and $B \subset C$ then $(A \cup B)$ C C



A ⊂ C, B ⊂ C ⇒ (A ∪ B) ⊂ C

Fig. 1.11

Proof.: $A \subset C$ means $x \in A \Rightarrow x \in C$

$B \subset C$ means $x \in B \Rightarrow x \in C$

$x \in (A \cup B) \Rightarrow x \in A$ and | or $x \in B$

$\Rightarrow \; x \in C$

$\therefore \quad (A \cup B) \subset C.$

Here we say that **union** operation is **transitive.**

(x) Principle of inclusion and exclusion.

Let A and B be finite subsets of a universal set.

If A and B are disjoint sets (i.e. $A \cap B = \phi$)

$$n(A \cup B) = n(A) + n(B)$$

In particular $n(A) + n(A') = n(X)$

1.6 Check your progress:

Given a universal set $X = \{5, 10, 15, 20, 25, 30, 35\}$

$A = \{10, 20, 30\}$, $B = \{5, 10, 25, 35\}$, $C = \{30\}$

1. Write down the following sets

   (a) $A \cup B$

   ____

   (b) $B \cup C$

   ____

   (c) $A \cup C$

   ____

   (d) $A' \cup B'$

   ____

   (e) $A' \cup C'$

   ____

2. State whether true or false.

   (a) $C \subset B$ ______________

   (b) $A \cup C = A$ ______________

   (c) $B \cup C = B$ ______________

   (d) $A \cup B \supset A \cup C$ ______________

3. Verify the associative law

   $A \cup (B \cup C) = (A \cup B) \cup C$

   ____

1.7 INTERSECTION OF SETS

This is another operation with sets.

Let $A = \{100,200,300,400,600\}$

$B = \{400, 600, 800, 1000\}$

$C = \{400, 600\}$

We find here that every element of C is an element of A as well as an element of B. In other words, if an element belongs to C; it belongs to both A and B. i.e. an element of C is common to both A and B. Such a set C is called **intersection** of two sets A and B. This set is denoted by $A \cap B$; read as "A intersection B"**.**

Definition: The **intersection** of two sets A and B is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

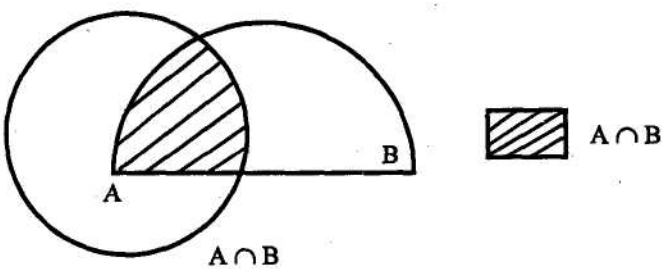


Fig. 1.12

Thus $A \cap B$ is a set of all common elements of A and B. This has been shown above in Fig. 1.12.

1.7.1 Disjoint Sets

Now consider the two sets,

$A = \{x \mid x \text{ is an even integer}\}$

$B = \{x \mid x \text{ is an odd integer})$

We know that there is no integer which is both even and odd at the same time. Thus these two sets A and B have no common element.

Their intersection is an empty set.

$A \cap B = \phi$

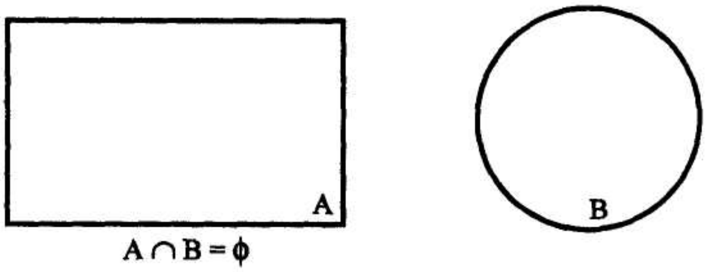


Fig. 1.13

Two such sets are called **disjoint** sets.

Definition : Two sets are **disjoint,** if they have no element in common.

e.g. A and A' are always disjoint.

$A \cap A' = \phi$ for any set A.

1.7.2 Properties of Intersection Operation

We shall list some properties of intersection.

(i) $\mathbf{A \cap B = B \cap A}$ **(commutativity)**

Proof.: $x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B$

$\Leftrightarrow x \in B \text{ and } x \in A$

$\Leftrightarrow x \in (B \cap A)$

i.e. $A \cap B = B \cap A$

We say that intersection operation is **commutative.**

i.e. We get the same set in whatever order the sets are taken for intersection.

(ii) $\mathbf{A \cap \phi = \phi}$

(iii) $\mathbf{A \cap X = A}$

(iv) $\mathbf{A \cap A' = \phi}$

(v) $\mathbf{A \cap A = A}$

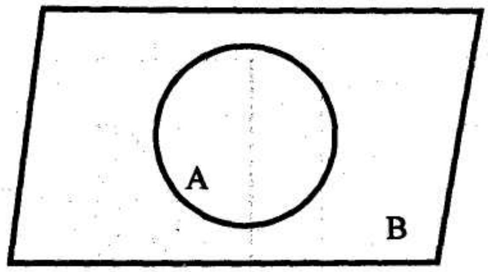
Proof.: $x \in A \cap A \Leftrightarrow x \in A \text{ and } x \in A$

$\Leftrightarrow x \in A$

$\therefore A \cap A = A$

(vi) We can easily see from Venn diagram of

Venn diagram representation is as follows:



$A \subset B \Rightarrow A \cap B = A$

Fig. 1.14

$A \cap B$ that (a) $(A \cap B) \subset A$

(b) $(A \cap B) \subset B$

(vii) If $A \subset B$ then $A \cap B = A$

Proof.: Since $A \subset B$, $x \in A \Rightarrow x \in B$

$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A$ (in either case)

i.e. $A \cap B = A$.

(viii) $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity)

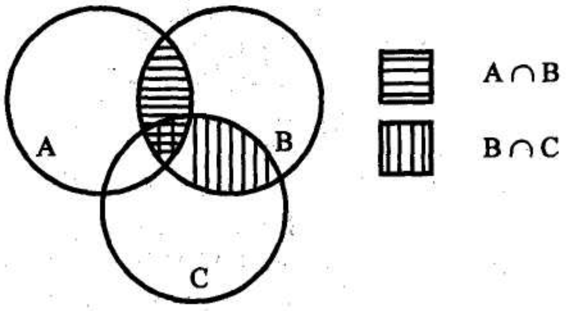


Fig. 1.15

Proof.: $x \in A \cap (B \cap C)$

$\Leftrightarrow x \in A$ and $x \in (B \cap C)$

$\Leftrightarrow x \in A$ and $x \in B$ and $x \in C$

$\Leftrightarrow (x \in A$ and $x \in B)$ and $x \in C$

$\Leftrightarrow x \in (A \cap B)$ and $x \in C$

$\Leftrightarrow x \in (A \cap B) \cap C$

We say that intersection operation is **associative.**

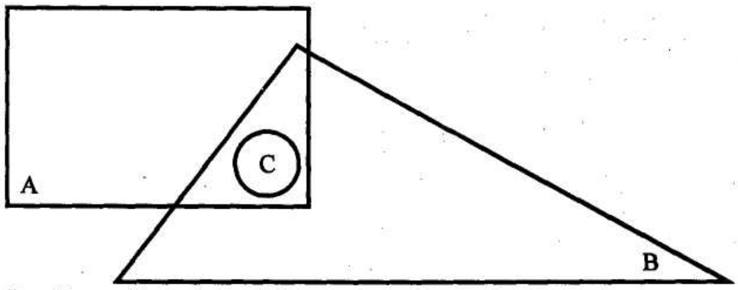
i.e. if we remove the brackets on either side we get the same result.

viz. $A \cap B \cap C$

$= \{x \mid x \in A$ and $x \in B$ and $x \in C\}$

$= \{x \mid x \in$ all the sets A, B, C$\}$.

(ix) If $C \subset A$ and $C \subset B$ then $C \subset (A \cap B)$



$C \subset A, C \subset B \Rightarrow C \subset (A \cap B)$

Fig. 1.16

Proof.: $C \subset A$ means $x \in C \Rightarrow x \in A$

$C \subset B$ means $x \in C \Rightarrow x \in B$

$x \in C \Rightarrow x \in A$ and $x \in B$

$\Leftrightarrow x \in (A \cap B)$ $\therefore$. $C \subset (A \cap B)$.

1.7.3 Relative Complement of a Set

We have seen the concept of a complement of a set in relation to a universal set. Now we shall consider **complement of** a set A **relative** to another set **B.**

Definition: Relative complement of A with respect to **B** is defined as $A \cap B'$ and is usually denoted by $A - B$.

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$

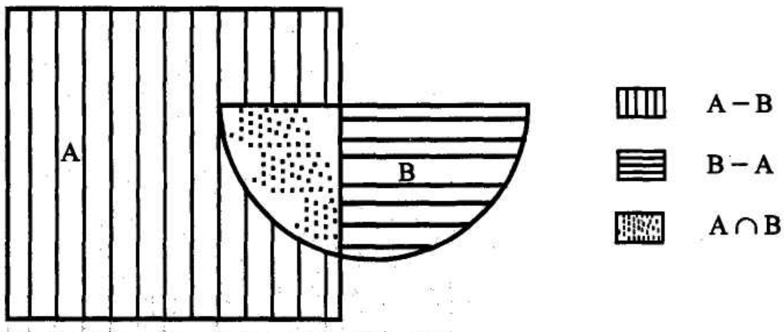
i.e. $A - B$ is a set of all elements of A except the common elements of A and B. $A - B$ is also called the **difference** of two sets A and B.

Similarly, relative complement of B with respect to A is $B \cap A' = B - A$.

$B - A = \{x \mid x \in B \text{ and } x \notin A\}$

i.e. $B - A$ is a set of all elements of B except the common elements of A and B.

The Venn-diagram representation in as :



i.e. $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

Fig. 1.17

(ix) Incidentally we observe the following from the above Venn diagram Fig. 1.17.

(a) sets $A - B$, $B - A$ and $A \cap B$ are pair wise disjoints sets. i.e. every pair of sets $A - B$, $B - A$ and $A \cap B$ have no element is common.

(b) $A \cup B$ is union of 3 disjoint sets viz

$A - B$, $B - A$, $A \cap B$.

i.e. $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

e.g. Let $A = \{2, 4, 6, 8, 10, 12\}$

$B = \{4, 8, 12, 16\}$

We have $A - B = \{2, 6, 10\}$

$B - A = \{16\}$

$A \cap B = \{8, 4, 12\}$

$A \cup B = \{2,4,6,8, 10, 12, 16\}$

$= \{2,6, 10\} \cup \{16\} \cup \{4, 8, 12\}$.

If X is the universal set and A is any of its subset then $A' = X - A$.

(x) Generalized Principle of Inclusion and Exclusion

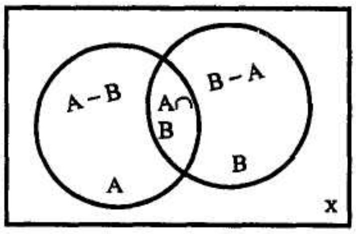
Se shall state various results without proof. The utility of them will be seen in illustrative examples.

Let A, B, C be any finite subsets of a universal set. We have the following rules.

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

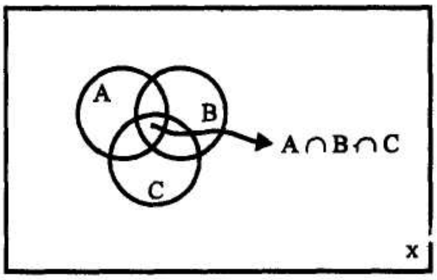
(ii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$

$- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$



A ∪ B

Fig. 1.18



A ∪ B ∪ C

Fig. 1.19

From the Venn diagram of $A \cup B$, we have the following:

(iii) $n(A-B) = n(A)-n(A \cap B) = n(A \cap B')$

(iv) $n(B-A) = n(B) - n(A \cap B) = n(A' \cap B)$

Similarly from the Venn diagram of $A \cup B \cup C$. it follows that

(v) Number of elements of A only

$= n(A)-n(A \cap B)-n(A \cap C) + n(A \cap B \cap C)$.

1.7.4 Illustrative Example

We shall illustrate the various operations that we have studied so far by means of an example.

Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A = \{0, 2, 4, 6, 8\}$

$B = \{1,3,7,8,9\}$

$C = \{2,5,7,9\}$

We have the following :

(i) $A \cup B = \{0, 1, 2, 3, 4, 6, 7, 8, 9\} = X - \{5\}$

(ii) $B \cup C = \{1, 2, 3, 5, 7, 8, 9\}$

(iii) $A \cup C = \{0,2,4, 5, 6,7,8,9\}$

(iv) $A \cap B = \{8\}$

(v) $B \cap C = \{7,9\}$

(vi) $A \cap C = \{2\}$

(vii) $A' = X - A = \{1, 3, 5, 7, 9\}$

(viii) $B' = X - B = \{0, 2, 4, 5, 6\}$

(ix) $C' = X - C = \{0, 1, 3, 4, 6, 8\}$

(x) $A - B = \{0, 2, 4, 6\} = A - (A \cap B)$

(xi) $B - C = \{1, 3, 8\} = B - (B \cap C)$

(xii) $C - A = \{5, 7, 9\} = C\ (C \cap A)$

(xiii) $A' - B = \{5\}$

(xiv) $A \cup B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} = X$

(xv) $A \cap B \cap C = \phi$

(xvi) $B - A = \{1,3,7, 9\}$

(xvii) Symmetric difference $A \oplus B = (A - B) \cup (B - A) = \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$

The Venn-diagram representation of some of these operations is given below.

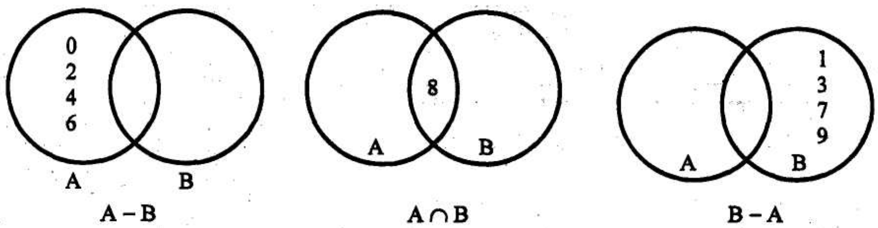


Fig. 1.20

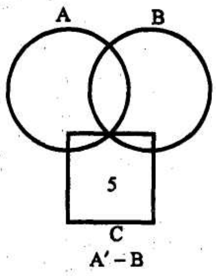


Fig. 1.21

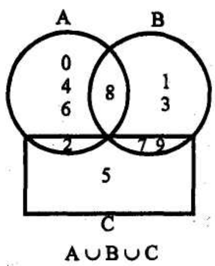


Fig. 1.22

1.7 Check your progress:

Solve the following

Given $X = \{a, b, c, d, e, f, g, h, i\}$

$A = \{a, c, d, f, g\}$, $B = \{c, d, e, f, h\}$, $C = \{a, i\}$

1. Write down the following sets.

   (i) $A \cap B$ ____________________ (ii) $B \cap C$ ____________________

   (iii) $A \cap C$ ____________________ (iv) $A' \cup B'$ ____________________

   (v) $A' - B$ ____________________ (vi) $B' - A$ ____________________

   (vii) Symmetric difference $A \oplus B$

   ____

2. Draw Venn-diagram to represent the following sets.

   (i) $A \cup B$ ____________________ (ii) $A - B$ ____________________

   (iii) $A - C$ ____________________ (iv) $A' \cap C$ ____________________

   (v) $A \oplus B$

   ____

1.8 DE MORGAN'S LAWS

We shall study two results known as De Morgan's laws. We shall offer theoretical proofs and their verifications for certain sets.

(I) $\mathbf{(A \cup B)' = A' \cap B'}$

Proof.:

$$\begin{aligned} x \in (A\cup B)' &\Leftrightarrow x \notin (A\cup B) \\ &\Leftrightarrow x \notin A \text{ and/or } x \notin \mathbf{B} \\ &\Leftrightarrow x \in A' \text{ and } x \notin B' \\ &\Leftrightarrow x \in (A'\cap B') \\ \therefore\ (A\cup B)' &= A'\cap B' \end{aligned}$$

(II) $\mathbf{(A \cap B)' = A' \cup B'}$

Proof. :

$$\begin{aligned} x \in (A'\cup \mathbf{B'}) &\Leftrightarrow x \in A' \text{ and/ o r } x \in \mathbf{B'} \\ &\Leftrightarrow x \notin A \text{ and } x \notin B \\ &\Leftrightarrow x \notin (A\cap B) \\ &\Leftrightarrow x \in (A\cap B)' \\ \therefore (A\cap B)' &= A'\cup B' \end{aligned}$$

We shall verify these laws for the following sets.

$$\begin{aligned} X &= \{x \mid x \text{ is a positive integer and } 1 \le x \le 10\} \\ A &= \{1, 2, 7\},\ B = \{2, 4, 6, 8, 10\} \end{aligned}$$

We have

$$\begin{aligned} A\cap B &= \{2\} \\ A\cup B &= \{1, 2, 4, 6, 7, 8, 10\} \\ A' &= \{3, 4, 5, 6, 8, 9, 10\} \\ B' &= \{1,3,5,7,9\} \\ (A\cap B)' &= \{1,3,4,5,6,7,8,9,10\} \\ (A\cup B)' &= \{3,5,9\} \\ A'\cap B' &= \{3,5,9\} \\ A'\cup B' &= \{1, 3, 4, 5, 6, 7, 8, 9, 10\} \\ \therefore\ (A\cup B)' &= \{3, 5, 9\} = A'\cap B' \\ \text{And } (A\cap B)' &= \{1, 3, 4, 5, 6, 7, 8, 9, 10\} = A'\cup B' \end{aligned}$$

This completes the verification.

Check your progress - 1.8

$X = \{a, b, c, d, p, q, r, s\}$

$A = \{a, c, d, p, s\}$

$B = \{b, c, d, p, q\}$

verify: (i) $(A\cup B)' = A\cap B'$

____

(ii) $(A \cap B)' = A'\cup B'$

____

1.9 DISTRIBUTIVE LAWS OF UNION AND INTERSECTION

We are familiar with the following simple operations.

$$2\times(4 + 6) = (2\times4) + (2 \times6)$$

The two operations used are multiplication and addition. The above relation is expressed as "multiplication is distributive over addition."

In the above illustration, we can not interchange multiplication and addition.

$$2+ (4\times6) \neq (2 + 4) \times (2 + 6)$$

The laws for union and intersection are as:

(I) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

By interchanging $\cup$ and $\cap$ in (I), we get another distributive law:

(II) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

We shall verify these laws by the following Venn diagram:

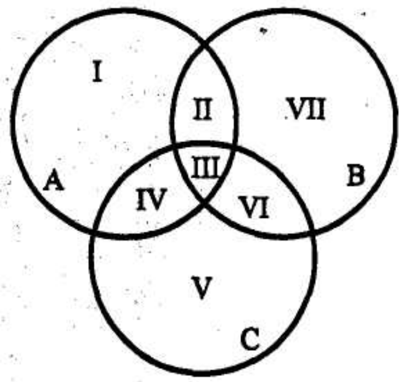


Fig. 1.23

Let $A = \{I, II, III, IV\}$

$B = \{II, III, VI, VII\}$

$C = \{III, IV, V, VI\}$

$\therefore A \cup B = \{I, II, III, IV, VI, VII\}$

$A \cup C = \{I, II, III, IV, V, VI\}$

$B \cup C = \{II, III, IV, V, VI, VII\}$

$A \cap B = \{II, III\}$

$A \cap C = \{III, IV\}$

$B \cap C = \{III, VI\}$

Hence $A \cup (B \cap C) = \{I, II, III, IV, VI\}$

$= (A \cup B) \cap (A \cup C)$

And $A \cap (B \cup C) = \{II, III, IV\}$

$= (A \cap B) \cup (A \cap C)$.

Check your progress – 1.9

For the sets $A = \{a, b, c, d\}$, $B = \{c, d, e, f\}$, $C = \{a, d, f, g\}$

Verify: (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

____

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

____

1.10 ILLUSTRATIVE EXAMPLES

(1) Show that each of the following condition is equivalent to $A \subset B$.

(i)$A \cap B = A$ **(ii)$A \cup B = B$**

(iii)$B' \subset A'$ **(iv)$B \cup A' = X$** **(v)$A \cap B' = \phi$**

Solution:

(i) $A \cap B = A$ means all the elements that are common to A and B are in A.

i.e. all the elements in A are elements in B while all elements in B are not in A.

i.e. $A \subset B$.

(ii) We know that $A \subset (A \cup B)$ for any two sets A and B.

$\therefore A \subset B$.

(iii) $B \subset A'$

i.e. $x \in B' \Rightarrow x \in A'$

i.e. $x \notin B \Rightarrow x \notin A$

i.e. $x \in A \Rightarrow x \in B$

i.e. $A \subset B$

(iv) We know that for any set B,

$B \cup B' = X$

We are given that $B \cup A' = X$

$\therefore B' \subset A'$

and hence from (iii), it follows that $A \subset B$.

(v) We know that for any set A,

$A \cap A' = \phi$

We are given that $A \cap B' = \phi$

$\therefore B' \subset A'$

and the result follows from (iii).

(2) Prove that for any two set A and B

$(A \cap B) \subset A \subset (A \cup B)$.

Solution : $x \in (A \cap B) \Rightarrow x \in A$ and $x \in B$

In particular let $x \in A$

$\therefore x \in (A \cap B) \Rightarrow x \in A$.

i.e. $(A \cap B) \subset A$ ...(i)

Further, if $x \in A$, then $x \in A$ and/or $x \in B$.

i.e. $x \in (A \cup B)$...(ii)

From (i) and (ii), it follows that

$(A \cap B) \subset A \subset (A \cup B)$.

(3) Given A = {a, b, c, d, e, f}, B = {d, e, f, g, h, i}

C = {b, e, h, i}, D = {d, e}, E = {b, d}, F = {b}.

Let x be an unknown set. Determine which sets A, B, C, D, E or F can be equal to X, if we are given the following information.

(i) $X \subset A$ and $X \subset B$

(ii) $X \not\subset B$ and $X \subset C$

(iii) $X \not\subset A$ and $X \not\subset C$

(iv) $X \subset B$ and $X \not\subset C$.

Solution :

(i) The set which is subset of A and B both, is $A \cap B$. Here this set is D.

$\therefore X = D$.

(ii) The set is not a subset of B, but it is a subset of C. The only element of C which is not contained in B is b.

Hence X = F.

(iii) The set is not a subset of A and C. The only set which satisfies the condition is B.

$\therefore X = B$.

(iv) The set is a subset of B, but it is not a subset of C. Hence the set may be B or D.

$\therefore X = B, D$.

(4) Prove the following :

(i) If $A \cup B = X$, then $A' \subset B$

(ii) $A \cap (A' \cup B) = A \cap B$.

Solution :

(i) We know that $X \cap A' = A'$ and we are given that $A \cup B = X$.

$\therefore (A \cup B) \cap A' = A'$

By distributive law, we get

$$(A \cup A') \cup (B \cap A') = A'$$
$$\text{i.e. } \phi \cup (B \cap A') = A'$$
$$\text{i.e. } B \cap A' = A'$$
$$\text{i.e. } A' \subset B$$

(ii) By distributive law, we have

$$\begin{aligned} A \cap (A' \cup B) &= (A \cap A') \cup (A \cap B) \\ &= \phi \cup (A \cap B) \\ &= A \cap B \\ \therefore A \cap (A' \cup B) &= A \cap B. \end{aligned}$$

(5) The students in a hostel were asked whether they had a TV set or a computer in their rooms. The result showed that 650 students had a TV set, 150 did not have a TV set, 175 had a computer and 50 had neither a TV set nor a computer. Find the number of students who,

(a) live in the hostel.

(b) have both a TV set and a computer.

(c) have only a computer.

Solution : We shall draw a Venn-diagram,. Let C = set of students having a computer**,** T = set of students having a TV set, X = set of students who live in the hostel.

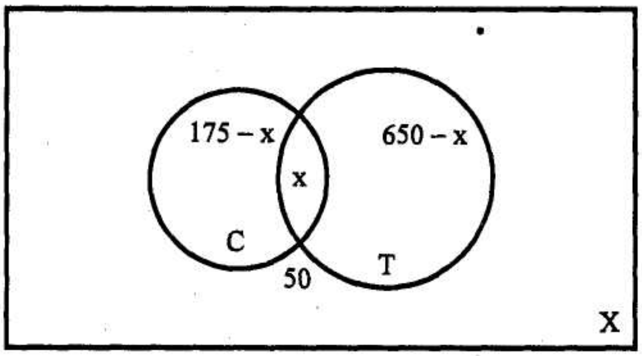


Fig. 1.24

Let x be the number of students having TV and computer both.

By data

$n(C \cap T) = x$

$n(T) = 650$

$n(C) = 175$

$n(C \cup T)' = 50$

Also there are 50 students who do not have a TV set.

$\therefore 175 - x + 50 = 150$

i.e. $225 - 150 = x$

i.e. $x = 75$.

i.e. number of students having both TV & computer is 75.

(b) Thus the number of students having both TV set and a computer is 75.

(a) Total number of students staying in the hostel

$$\begin{aligned} &= n(X) = n(C \cup T) + n(C \cup T)' \\ &= n(C) + n(T) - n(C \cap T) + 50 \end{aligned}$$

$= 650 + 175 - 75 + 50$

$= 800.$

(c) Number of students having only a computer

$= n(C) - n(C \cap T)$

$= 175 - 75$

$= 100.$

(6) Among 100 students, 32 study Mathematics, 20 study Physics, 45 study Biology. 15 study Mathematics and Biology. 7 study Mathematics and Physics. 10 study Physics and Biology. 30 do not study any of the three subjects. Then find,

(a) number of students who study all the three subjects.

(b) number of students who study Mathematics only.

Solution : Here also we can draw a Venn-diagram to represent the data.

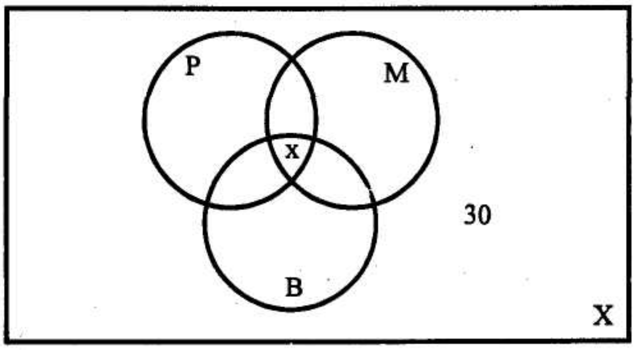


Fig. 1.25

Let X = set of 100 students

P = set of students who study Physics

M = set of students who study Mathematics

B = set of students who study Biology

x = number of students who study all the three subjects.

By data

$n(P) = 20$

$n(M) = 32$

$n(B) = 45$

$n(P \cap M) = 7$

$n(M \cap B) = 15$

$n(P \cap B) = 10$

$n(P \cup M \cup B)' = 30$

$n(X) = 100$ and $n(P \cap M \cap B) = x$.

Number of students who are studying at least one of the subjects $= n(P \cup M \cup B)$.

$$n(P \cup M \cup B) = n(P) + n(M) + n(B) - n(P \cap M) - n(M \cap B) - n(P \cap B) + n(P \cap M \cap B)$$

i.e. $n(X) - n(P \cup M \cup B)' = 20 + 32 + 45 - 7 - 15 - 10 + x$

i.e. $100 - 30 = 97 - 32 + x$

i.e. $x = 5.$

(a) Number of students who study all the three subjects is 5.

(b) Number of students who study Mathematics only

$= n(M) - n(M \cap P) - n(M \cap B) + n(P \cap M \cap B)$

$= 32 - 7 - 15 + 5$

$= 15.$

1.11SUMMARY

- A set is a well defined collection of objects.
- There are 10 types of sets.
- For any two sets A and B we can write

a) $A \cup B$

b) $A \cap B$

c) $A^{'}$, B'

d) (AB)' = B' A'

e) A^{-1} exits

f) $(A \cup B)' = A' \cap B'$

g) $(A \cap B)' = A' \cup B'$

h) $n(A \cup B) = n(A) + n(B)$

i) $n(A \cap B) = n(A) + n(B) - n(A \cap B)$

j) $n(A \cap B) = n(A) + n(B)$ When $A \cap B = \phi$

1.12 CHECK YOUR PROGRESS–ANSWERS

1.2 1. (a) $3 \in \{1, 2, 3, 4\}$ (b) $5 \notin \{1, 6, ,7\ 8\}$

2. $A = B$, $A \neq C$, $B \neq C$

3. (a) False (b) True

1.3 1. (a) False (b) False (c) True (d) True (e) True

2. $\Phi\{10\}, \{20\}, \{30\}, \{10, 20\}, \{10, 30\}, \{20, 30\}$ A.

3. (a) no, $A \subset B$

(b) {p, q}, {p, r}, {p, s}, {q, r}, {q, s}, {r, s}

1.4 and 1.5

(a) A' ={2, 4, 6, 8}

(b) B' = {1, 3, 5, 7, 9}

(c) C = {2, 4, 5, 6, 7, 8}

(d) Φ

(f) X

1.6 1. (a) $A \cup B = \{5, 10, 20, 25, 30, 35\}$

(b) $B \cup C = \{5, 10, 25, 30, 35\}$

(c) $A \cup C = \{10, 20, 30\}$

(d) $A' \cup B' = \{5, 15, 20, 25, 30, 35\}$

(e) $A' \cup C' = \{5, 15, 10, 20, 25, 35\}$

2. (a) False (b) True (c) False (d) True

3. Common set {5, 10, 15, 20, 25, 30 35}

1.7 1. (i) $A \cap B = \{c, d, e, f\}$ (ii) $B \cap B = \Phi$

(iii) $A \cap C = \{a\}$ (iv) $A' \cup B' = \{a, b, g, h, i\}$

(v) $A' - B = \{b, i\}$ (vi) $B' - A = \{b, i\}$

(vii) $A \oplus B = \{a, e, g, h\}$

2.

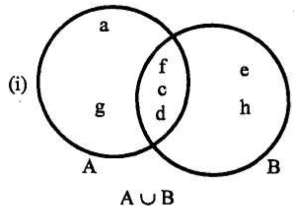


Fig. 1.26

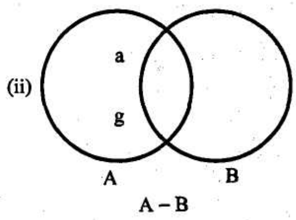


Fig. 1.27

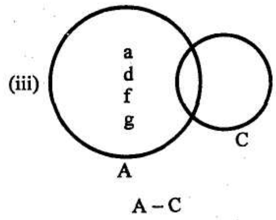


Fig. 1.28

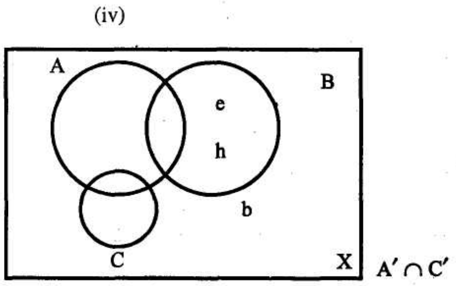


Fig. 1.29

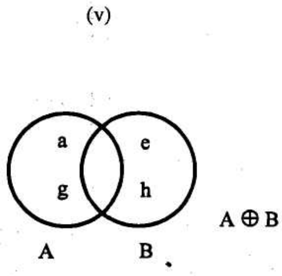


Fig. 1.30

1.13 QUESTIONS FOR SELF – STUDY

Problems For Practice

(1) 9000 students appeared for two papers in Mathematics at the first year examination. Exactly 7400 and 6600 students passed in papers I and II respectively. 6400 students passed in both the papers. Draw a Venn-diagram to indicate these results and hence or otherwise find the number of students who have failed in both papers.

(2) Among 600 families, 100 families have no children, 400 families have only boys and 200 have only girls. How many families have both boys and girls ?

(3) In a group of 200 students, 80 are taking a Mathematics class, 60 are taking a Chemistry class and 30 are taking both classes. (i) How many students are taking either a Mathematics class or a Chemistry class ? (ii) How many students are taking neither class ?

(4) Suppose that 100 students at a college take at least one of the languages French, German and Russian. 65 students study French, 45 study German and 42 study Russion. Also 20 students study French and German, 25 students study French and Russian, 15 students study German and Russian. Find the number of students who study (i) all the three languages, (ii) exactly one language.

(5) It is known that at the university 60% of professors play Tennis, 50% play Cricket, 70% play Hockey, 20% play Tennis and Cricket. 30% play Tennis and Hockey, 40% play Cricket and Hockey. Assuming that each professor play at least one of the games, determiner % of professors playing all the three games.

(6) A computer company must hire 25 programmers to handle systems programming tasks and 40 programmers for applications programming. Of those hired 15 will be expected to perform tasks of each type. How many programmers must be hired ?

1.14 SUGGESTED READINGS

1. Mathematics and Statistics by M. L. Vaidya, M. K. Kelkar
2. Pre- degree Mathematics by Vaze, Gosavi

◆◆◆

NOTES

NOTES

CHAPTER 2

Functions

2.0 OBJECTIVES

After studying the number system and certain preliminary concepts, students can explain the following –

* Functions in various notations
* Types of functions
* Functions of functions
* Graph of functions
* Formula of a function
* Function as a correspondence
* Students can solve problems involving the all above concepts.

2.1 INTRODUCTION

There is no permanent place in the world for ugly mathematics. It may be very hard to define mathematical beauty but that is just as true of beauty of any kind

- G.H. Hardy

The concept of term 'relation' in mathematics has been drawn from the meaning o relation in English language. Accordingly two objects or quantities are related if there is a connection or link between the two objects or quantities.

2.2 NUMBER SYSTEM

N = Set of all natural Numbers.
= { 1, 2, 3,…..}

W = Set of whole numbers.
= { 0, 1, 2, 3,…..}

I = Set of all Integers.
= { ………. -3, -2, -1, 0, 1, 2, 3, …………..}

Q = Set of rational Numbers

$= \left\{ \frac{p}{q} \middle| p, q, \in, I, q \neq 0 \right\}$

2.2.1 Basic operations in Mathematics

1) Addition –

i) a, b are two numbers then

a + b is called 'addition' of two numbers.

ii) $\frac{a}{b}$ and $\frac{c}{d}$ two fractions then

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

e.g. $\frac{2}{3}+\frac{5}{4}=\frac{2\times 4+5\times 3}{3\times 4}$

$$=\frac{8+15}{12}$$

$$=\frac{23}{12}$$

2) Subtraction

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

e.g. $\frac{2}{3}-\frac{4}{5}=\frac{2\times 5-4\times 3}{3\times 5}$

$$=\frac{10-12}{15}$$

$$=\frac{-2}{15}$$

3) Multiplication

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$$

e.g. $\frac{2}{3}\times\frac{4}{5}=\frac{2\times 4}{3\times 5}$

$$=\frac{8}{15}$$

4) Division

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$$

e.g. $=\frac{a\times d}{b\times c}$

where $\frac{d}{c}$ is called 'Multiplicative Inverse'

e.g. $\frac{2}{3}\div\frac{4}{5}=\frac{2}{3}\times\frac{5}{4}$ (Now use rule of Multiplication)

$$=\frac{2}{3}\times\frac{5}{4}$$

$$=\frac{10}{12}$$

5) Order

a and *b* are two given numbers then possible relation between these two are,

i) a < b	a is less than b
ii) a > b	a is greater than b
iii) a = b	a is equal to b

e.g. *a* and *b* are two students in TMV. then a and b must be admitted to any of the 15 coerces run by TMU.

If *a* take admission to B.C.A.

b take admission to M.C.A

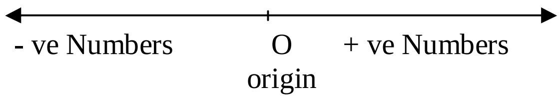
then by 'age'

a smaller than ***b***

$\therefore$ *a* not bigger than *b*

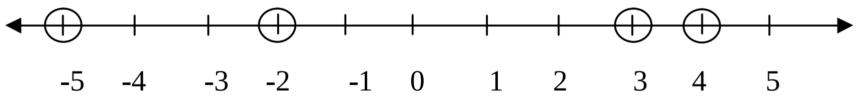
and *a* is not same age as *b*.

6) **Number line –** All numbers we can present on a line is called 'number line.'



e.g. Represent the numbers on line

-2, 3, 0, -5, 4



Means – All positive numbers are on right hand side of Number line whereas all –ve number on Left hand side.

2.2.2 Divisibility Test

Number a is called divisible by b when we divide no. a by no. b and remainder equal to 'zero'

If remain is not equal to zero, we say that 'a' is not divisible by 'b'

Test of Divisibility

1) **Test of 2** : A number is said to be divisible by 2 when its unit place digit is one of 0, 2, 4, 6, and 8.

T U

e.g. 1) 2 5 6 3 2

unit place digit = 2

$\therefore$ given number is divisible by 2

U

2) 1 3 4 5 6 7 8 9

unit place digit = 9

$\therefore$ given number is not divisible by 2

2) **Test of 3:** A number is said to be divisible by 3, when addition of digits of given number is divisible by 3.

e.g. 1) 1 2 3 4 5

$1 + 2 + 3 + 4 + 5 = 15$

$15 \div 3 = 5$ Remainder = 0

$\therefore$ given number is divisible by 3

2) 2 3 4 2

$2 + 3 + 4 + 2 = 11$

$11 \div 3 =$ is not Integer

$11 \div 3 = 3.333$

- given no is not divisible by 3

3) Test of 4 : A number is said to be divisible by 4, when last two digits T and U is divisible by 4.

e.g. 1) 4 5 1 2 3

last two digits 2 3

23 not divisible by 4

∴ given no. is not divisible by 4

2) 3 2 5 6 7 8 9 2 4

last two digits 2 4

24 is divisible by 4

∴ given no. is divisible by 4

4) Test of 5 : A number is said to be divisible by 5, when the last digit is 0 or 5.

e.g. 1) 4 3 1 5 6 2 5
last digit = 5
∴ given no. is divisible by 5

2) 2 5 6 2 0 1 5 2
last digit = 2
∴ given no. is not divisible by 5

5) Test of 10 : A number is said to be divisible by 10, when last digit is 0.

e.g. 1) 1 2 3 5 0 2 0 1
last digit = 1
∴ given no. is not divisible by 10

2) 2 4 0 1 2 3 4 0
last digit = 0
∴ given no. is not divisible by 10

2.3 Check your Progress
State True/False

1) 2 3 4 5 6 is divisible by 5

2) 1 0 0 0 is divisible by 10

3) 5 3 2 1 5 6 5 is divisible by 5

4) 1 2 3 4 5 6 is not divisible by 2

5) 5 6 2 3 1 2 5 2 0 is divisible by 2, 5, 10, and 4.

2.3 PRELIMINARY CONCEPTS

1, 2, 3, …… Natural Numbers.

1 takes always value one in any case, concept, example. so the numbers are called as constants. But when value is not constant e.g. Age of student in F.Y.

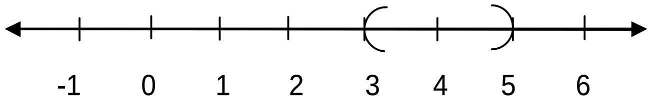
Probable year is after 12^{th} std. means above 17 years. But it may be 18, 19 also. So the age of student is a variable and denoted by small alphabets as, -, y, z, p, q, …

That means the ‘age of student’ lies between 17 and 19 and above. These values are called intervals of variables. There are four types of intervals.

1) Open – Open interval

e.g. (3,5) – means our variable takes values between 3 and 5. But not 3 and 5.

$\boxed{3 < x < 5}$

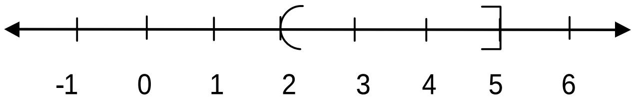


---

2) Open – Closed

$\boxed{2 < x \leq 5}$

e.g. (2 5)



means variable takes values between 2 and 5 and 5 also but not 2.

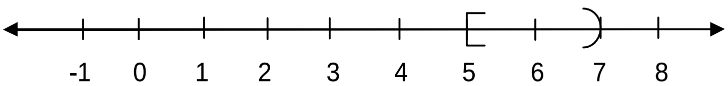
3) Closed – Closed

e.g. (3, 4)

means Variable takes values 3 and 4 between all values 3 and 4.

4) Closed Open – (5, 7)

means Variable takes values between 5 and 7 and 5 but not 7.



$\boxed{5 \leq x < 7}$

2.3.1 Absolute Value

x is a Variable . Then

$|x|$ = x if x > o

= x if x < o

$|x|$ can be read as 'modx'

Imp. Results –

1) $|x+y| \leq |x| + |y|$

2) $|x-y| \leq |x| - |y|$

3) $|xy| = |x||y|$

4) $\left|\frac{x}{y}\right| = \left|\frac{x}{\mathrm{y}}\right|$ if $y \neq 0$

2.4 CORRESPONDENCE

Definition – If A and B are two sets such that by some rule to an element a – A takes one or more elements in B, then the rule is called a correspondence

There are Four types of correspondence

1) One – One correspondence

e.g. Marks in examination of students.

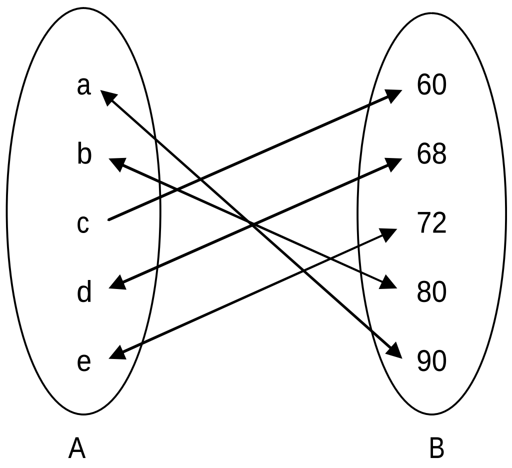
A = Set of Number of students.

B = Set of marks obtained by students.

A = { a, b, c, d, e }

B = { 90, 80, 60, 68, 72}

Venn – diagram



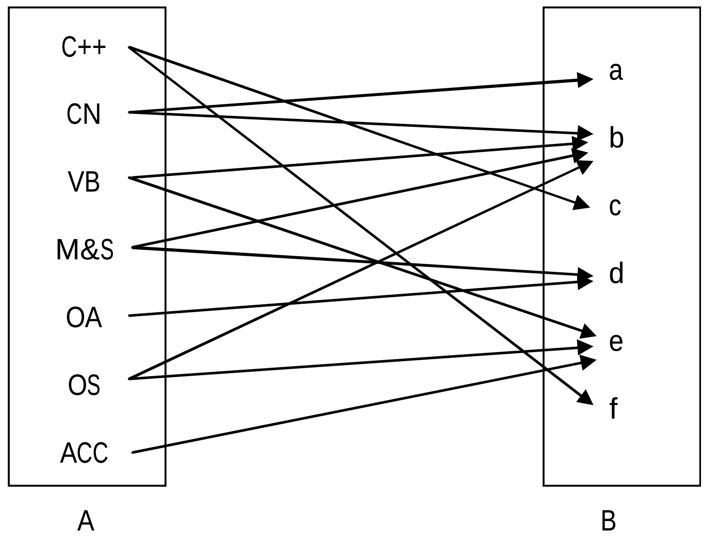
2) Many – One Correspondence

A = Number of subjects for F. Y.

B = Number of subjects in F. Y. in TMU

A = { Computer Network, VB Net, Math & stats office Automation, Operating system, C++. Accounting}

B = {a, b, c, d, e, f}

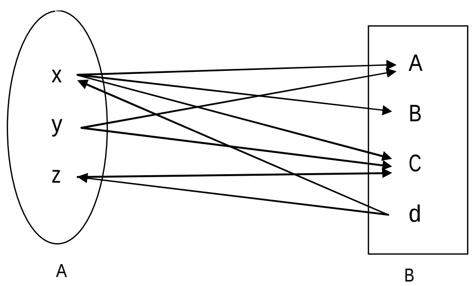


Sul

1) M & S
2) OA
3) C++
4) OS
5) VB 5 papers (subject) for each student.

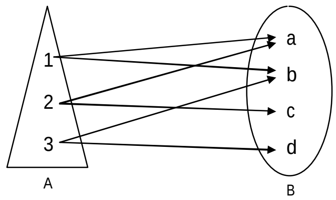
3) Many – Many

A = {x, y, z} B = {a, b, c, d}



4) One – Many

1) A = {1, 2, 3}
B = {a, b, c, d}



2) A = Set of states
B = Set of cities
R : A – B
R : called as Rule of correspondence from one set to another set.

Check your Progress – 2.4

1. State the correspondence. Draw venn diagrams.

i) A = {1, 2, 3}
B = {a, b, c, d}
R : A $\rightarrow$ B
R = { (1,a) (2,b), (3, c)}

ii) A = {2, 4, 6, 8}
B = {p, q, r, s}
R : A $\rightarrow$B
R = {(2,p), (2,q), (4,r) (6,s), (8,r)}

iii) A = {1, 3, 5, 7}
B = {p, q}
R : A $\rightarrow$ B
R = {(1,p), (3,p), (5,q), (7,q)}

iv) A = {2, 4}
B = {r, s}
R = A $\rightarrow$ B
R = {(2, r), (2,s), (4,r), (4,s)}

2.5 FUNCTIONS

Definition – Let A and B are two non empty sets A correspondence from the set A to set B is called function if it is either one-to-one or may-to-one.

Every function is correspondence but every correspondence is not to a function.

Function is a relation R : A $\rightarrow$ B (Relation from A to B) for every x $\in$A there is unique y $\in$B such that x R y (x related to y).

A function is denoted by letters f, g, h.
We write f : A - B
g : C – D
h : C – B
if f : A – B then
xRy – x is related to y
$\therefore$ y = f(x)

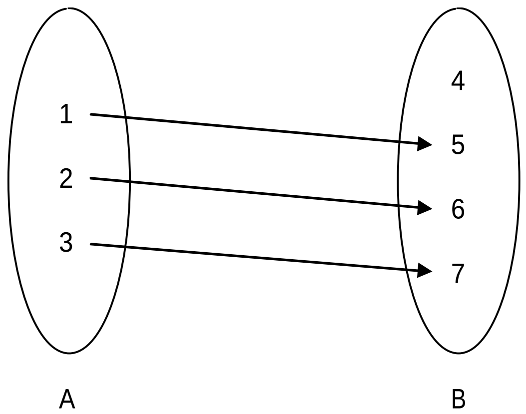
The set A is called as the domain of the function
The set B is called as the co-domain of the function
y = f(x) is called as Rule of correspondence
y $\in$ B is called as image of x under.
The set of images in B is called as the Range of the function.
$\therefore$ Range of r = {y | f(x) = y, where x $\in$ A, y $\in$ B}
The Range of a function is subset of its co domain

Note – If A – B , then,

1) Every element of set A is related to one and only one element of set B.
2) More than one element can be related to set A to one element of set B
3) Set B may contain elements which are not related to any element of set A.

e.g. Let $A = \{1, 2, 3,\}$
$B = \{4, 5, 6, 7,\}$
$f : A \rightarrow B$
$f = \{ (1, 5), (2,6), (3,7)\}$

then venn – diagram



Set A is domain
Set B is Co-domain
Range = {5, 6, 7}

Images $f(1) = 5$
$f(2) = 6$
$f(3) = 7$

Rule $f(x) = x + 4 = y \quad y \in B$
Set of order pair = {(1,5), (2,6), (3,7)}

Tabular form –

X	1	2	3
y = f(x)	5	6	7

2.5 Check your Progress

1. Find the image of following functions –
   a) $f(x) = 2x^2 - 3x + 4$
   find f(1), f(0), f(-1), f(-2), f(2)

   b) $f(x) = 3x^2 - 5$
   Find f (-1), f(-), f(3), f(-), f(0)

2.6 TYPES OF FUNCTIONS

1) Onto function
2) Into function
3) One-One function
4) Many – One function
5) Even function
6) Odd function
7) Composite function

1) Onto function – (subjective function)

Definition – A function $F : A \rightarrow B$ is said to be an Onto function if every element of set B is the image of some element of set A.

R = B (Range = codomain)

symbolically, we write,

onto

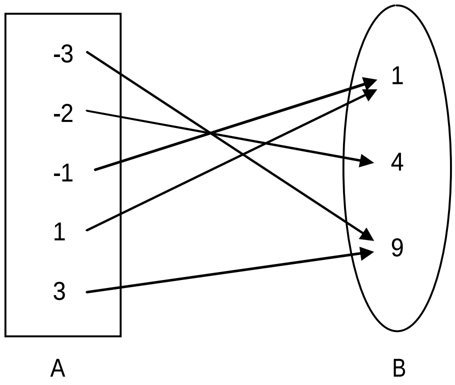
$f : A \to B$

e.g. $A = \{-3, -2, -1, 1, 3\}$

$B = \{1, 4, 9\}$

$f : A \to B$

$f(x) = x^2$



2) Into function (subjective function)

Definition – A function f : A – B is said to bean into function if there exits at least one element in B, which is not the image of any element of A.

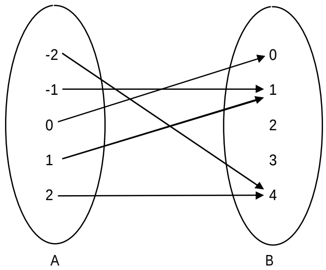
RCB (Range is proper subset of B)

e.g $A = \{-2, -1, 0, 1, 2\}$

$B = \{0, 1, 2, 3, 4\}$

$f : A \to B$

$f(x) = x^2 = y$



Range = $R = \{0, 1, 4\}$

$B = \{ 0, 1, 2, 3, 4\}$

$\therefore$ RCB

Note : The function is Onto or Into which depend on Range of that function.

3) One – One function (Injective function)

Definition – A function f : A – B is said to be One-to-one function if distinct elements of A have different images in B under f.

e.g. A = set of students.

B = set of Roll Numbers

$R : A \to B$

$\therefore$ Roll Number is fixed with student. Amit's Roll No. is 18. means No. 18 is Asine to Amit only and not other, any student.

$\therefore$ one-to-one correspondence.

$\therefore$ one-to one function.

4) Many-one function

Definition – A function $f : A \to B$ is said to be many-to-one function if two or more elements of A have the same image B i.e. there is at least element in B, which has more than one one pre-image in A.

5) Even function

Definition – A function f : A – B is said to that means f(x) does not change with x and –x replacement.

e.g. 1) $f : A \to B$

$f(x) = y = x^2 + 1$
Now put $x = -x$
$f(-x) = y = (-x)^2 + 1$
$= x^2 + 1 \qquad \therefore\ x^2 = (-x)^2$
$\therefore$ f(x) is even function.

2) $f(x) = 3x + 5$
put $x = -x$
$f(-x) = 3(-x) + 5$
$= -3x + 5$
$\therefore$ f(x) ‡ f(-x)
$\therefore$ Given function is not even function.

6) Odd function

Definition – A function f : A – B is said to be odd function if $f(x) = -f(x)$

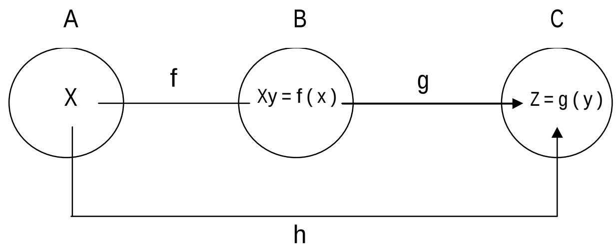
e.g. 1) A function $f(x) = x^3 + x$
$\therefore\ f(-x) = (-x)^3 + (-x)$
$= (-x)^3 - x$
$= [x^3 + x]$
$= - f(x)$
$\therefore$ Given function is An Odd function.

2) $f(x) = 3x^2 + 4x$
$f(-x) = 3(-x)^2 + 4(-x)$
$= 3x^2 - 4x$
$= - f(x)$
$\therefore$ Given function is not Odd function.

7) Composite function

Definition – A function $f : A \to B$ and g : B – C be two functions. x is any element of A. Then $y = f(x) \in B$. Since B is the domain of function g and C is its co-domain, $g(y) \in C$ so $z = g(y) \in C$

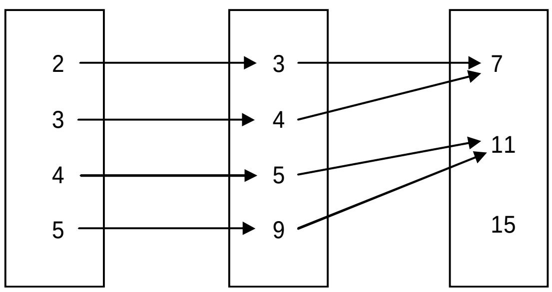
This function is called as composite function of f and g.
Let it denoted by h.
Thus $gof = h : A \to C$
such that $h(x) = g[f(x)]$



e.g. 1) Let f : {2, 3, 4, 5} – {3, 4, 5, 9} and g : {4, 3, 5, 9} – {7, 11, 15} functions defined as f(2) = 3, f(3) = 4, f (4) = = f(5) = 5 and g (3) = g(4) =7, g(5) = g(9) = 11 find g of.

Solution

1)



given

f(x) = 3

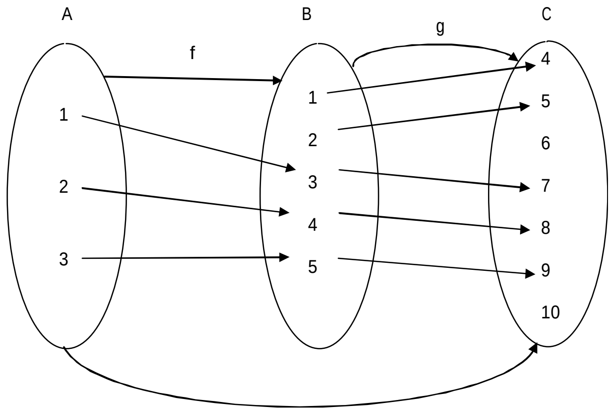
g of (2) = g[f(2)]
= g(3)
= 7

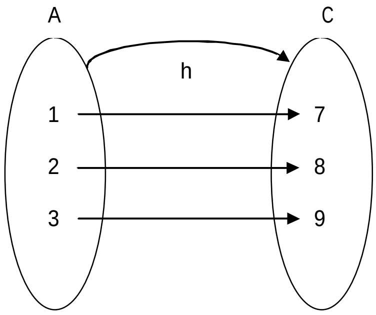
gof (3) = g[f(3)]
= g(4)
= 7

gof (4) = g[f(4)]
= 9(5)
= 11

gof (5) = g[f(5)]
= 9(5)
= 11

2)





g of (x) = g of (1) = g[f(x)]
= 9[f(1)]
= g(3)
= 7

g of (2) = g[f(2)] = 9(4)
= 8

gof(3) = g[f(3)] = g(5)
= 9

2.6 Check your progress

Fill in the blanks

i) The function is either onto or Into which depends on the given ………………

a) Range b) domain c) co-doman d) value

ii) A one-one function is also called ………………

a) Injective b) Bijective c) Onto d) Into

iii) One-one and Onto function is called ………………

a) Bijective b) Injective c) Into d) even.

iv) Many-one correspondence is called ………………function

a) even b) add c) Bijective d) Many-one.

v) A many-one function can be either ………………

a) into or onto b) even or odd c) even or bijective d) into or odd.

vi) Nature of function whether it is one-to-one or many-one depend, upon ………………of function.

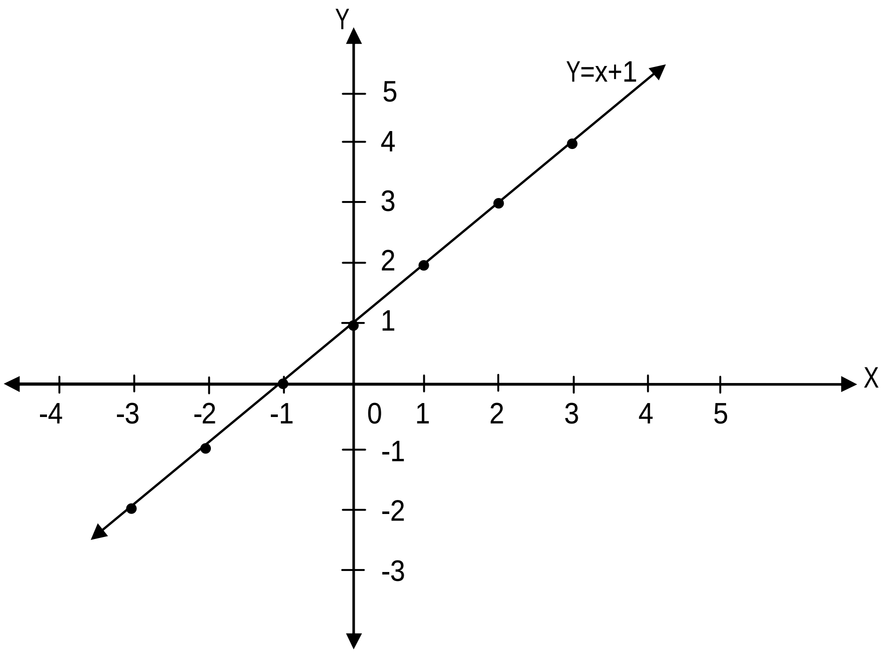
a) Domain b) Co-doman c) Range d) even.

2.7 GRAPH OF FUNCTION

Definition – A function f : A $\rightarrow$ B, x – A and y – B then (x,y) be an element of f. We can plot the point (x,y) in a plane by choosing a suitable co-ordinate system. On plotting all such order pair, we get geometrical representation (curve) of function f this is called graph of function f.

e.g. f(x) = x + 1

X	-3	-2	-1	0	1	2	3
F(x) = y	-2	-1	0	1	2	3	4



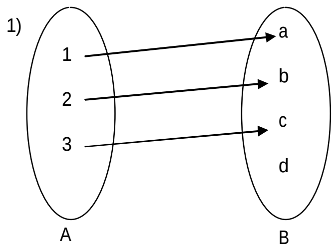
2.8 SUMMARY

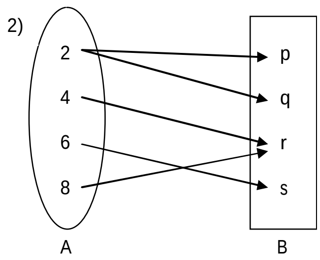
Functions means a action, the velation between variable and number. Correspondences are of four types. There are also types of functions. We can draw proper and net Venn diagram for each correspondences. The net graph is there for each function.

2.9 CHECK YOUR PROGRESS - ANSWERS

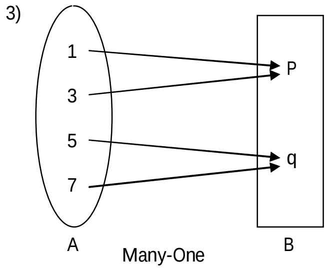
2.3 1) False 2) True 3) True 4) True 5) True

2.5

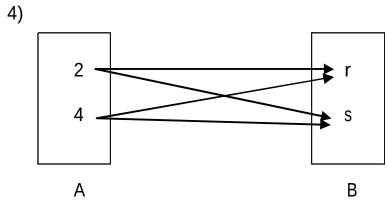




one-one one-many



Many-One



Many -Many

2.6 1) c) 2) a) 3) a) 4) d) 5) a) 6) a)

2.10 QUESTIONS FOR SELF – STUDY

Problems For Practice

1. If $f(x) = 3x^2 - 5$ $\quad -2 < x < 2$

   find f (-1), f ($\sqrt{3}$), f(3)

2. Find g of and fog if $f(x) = 2x + 1$

   and $g(x) = 3x^2 - x + 4$

3. If $f(x) = 2x + 6$ $\quad -3 \le x \le 0$

   $= 6$ $\quad 0 < x < 2$

   $= 2x - 6$ $\quad 2 \le x \le 4$

   Stat (a) Domain and Range

   (b) find f(1), f(-4), f(3)

4. $f : R \to R$, $g : R \to R$ are defined as,

   $f(x) = x^2$ $\quad x \le 0$

   $= -x$ $\quad x > 0$

   $g(x) = x + 2$ $\quad x \le 0$

   $= x^2$ $\quad x > 0$

   find g of (x), x – R

5. Test whether following functions are even or odd

   1) $f(x) = g - x^2$

   2) $f(x) = |x|$

   3) $f(x) = 2x + 1$

   4) $f(x) = \frac{1}{x}$ $\quad x \pm 0$

   5) $f(X) = 2x - 3$

6. Draw the graph of given functions –

   1) $f(x) = 3$

   2) $f(x) = x$ -x - R

   3) $f(x) = |x|$

   4) $f(x) = -2$ if $-2 \leq x < -1$
      $= -1$ if $-1 \leq x < 0$
      $= 0$ if $0 \leq x < 1$
      $= 1$ if $1 \leq x \leq 2$

find images of x = -1
= 0
= 2
= 3

7. Functions f and g are given by the following

   i) f = { (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7)}
      g = { (2, 4), (3, 6), (4, 8), (5, 10), (6, 12), (7, 14) }

   ii) If f = { (2, 3), (4, 5), (6, 7), (8, 9)}
       g = { (1, 2), (2, 4), (3, 6), (4, 8)}

   iii) If f = { (1, 1), (2, 3), (3, 5), (4, 7)), (5, 9)}
        g = { (1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}

   a) Express f and g by formula
   b) Show that f and g are one-one functions.
   c) find fog and gof.

Problems for practice

1) $f(-1) = -2$, $f(\sqrt{3}) = 4$, f(3) does not exits

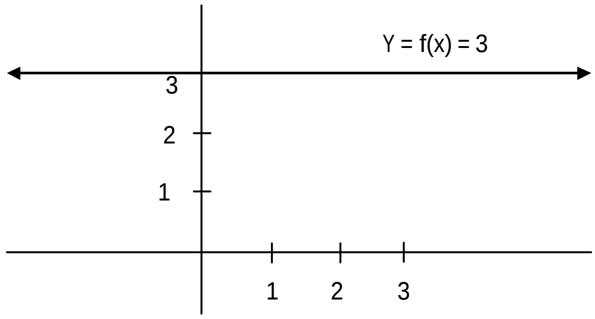
2) $gof(x) = 6x^2 - 2x + 9$
   $fog(x) = 12x^2 + 10x + 8$

3) Range = [-3,4]
   $f(1) = 6$, $f(-4)$ does not exist, $f(3) = 0$

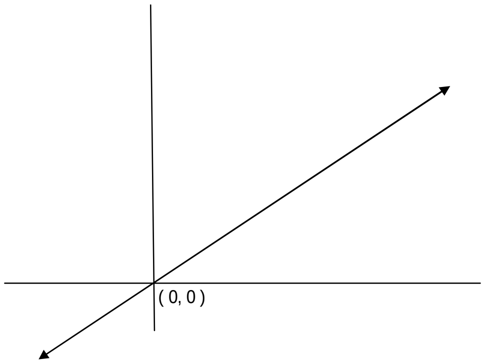
4) $gof(x) = 9[f(x)]$
   $x \leq 0$ $g[f(0)] = x^2 = 0$
   $g[f(-1) = x^2 = 1$
   $gof(x) = g[f(x)] = g[f(0)] = g(x^2) = (x + 2)^2 = 4$
   $gof(x) = 9[f(-1)] = g(1) = x^2 = 1$

5) 1) even 2) even 3) even 4) odd 5) odd.

6) 1)



2)



2.11 SUGGESTED READINGS

1. *Pre-degree Mathematics* by Vaze, Gosavi

2. *Discrete Mathematical Structures for Computer Science* by Bernard Kolman and Robert C Busby

3. *Statistical Analysis: A Computer - Oriented Approach Introduction to Mathematical Statistics by* S. P. Azen & A. A.Afifi

◈ ◈ ◈

NOTES

CHAPTER 3

Sequences, Progressions and Series

3.0 OBJECTIVES

After studying this chapter, you can explain and use various types of sequences, series given below :

- Sum of n terms of a sequence.
- Three types of sequences.
- Arithmetic progression.
- Geometric progression.
- Harmonic progression.
- The three means : A.M., G.M. and H.M.
- Summation of series of n terms.
- Summation of certain infinite series in G.P.

3.1 INTRODUCTION

In computer applications many concepts can be expressed as an ordered numbers using 0, 1. We have already seen one such case of expressing a set or a function formed by various combinations of 0, 1. Briefly the ordered set of numbers forms a sequence. We shall then -consider three types of sequences and summation of terms of sequences forming a series.

3.2 SEQUENCE

We know that the system of natural numbers is 1, 2, 3, ... n, n + 1, ... This is a collection of numbers satisfying the following properties.

(i) It is ordered i.e. each number of the collection has a definite position.

(ii) There exists definite law, according to which every number can be written down.

(iii) Every number is followed by next one.

Any other collection of numbers which satisfies the first two of the above properties is called a sequence. The sequence is further called infinite sequence, if the third property is also satisfied, otherwise it is called a finite sequence.

Definition : An ordered set of numbers formed according to a well defined law is called a **sequence.**

In terms function, it is a function

$f : N \to R$

If $n \in N$, then f(n) is called n^{th} term of the sequence. By giving different values to n, we get the corresponding term of a sequence.

e.g. (i) $1^2, 2^2, 3^2, \ldots\ldots, n^2, \ldots\ldots$

Here $f(n) = n^2$, i.e. square of a natural number. When n = 3, $f(3) = 3^2$ is the 3^{rd} term of a sequence.

(ii) $3, \frac{3}{2}, \frac{3}{3}, \frac{3}{4} \ldots\ldots, \frac{3}{n} \ldots$

Here $f(n) = \frac{3}{n}$

(iii) 2, 5, 8, 11, 14, ...

Here $f(n) = 3n - 1$ **or** $f(n) = 2 + 3(n - 1)$ **or** it is a sequence in which each term is obtained by adding 3 to the previous term.

(iv) 2, 4, 8, 16, 32, ...

Here each term is a power of 2.

$f(n) = 2^n$ is the n^{th} term.

(v) 1.2, 2.3, 3.4.......

Here $f(n) = n(n + 1)$ is the n^{th} term.

All these are illustrations of **infinite** sequences.

Now we shall consider certain **finite** sequences.

(vi) The sequence 1, 0, 0, 1, 0, 0, 1,0, 1, 1, 1 is a finite sequence with repeated terms. The digit 0, for, example occurs as the 2^{nd}, 3^{rd}, 5^{th}, 6^{th} and 8^{th} elements of the sequence.

The corresponding set is simply {0, 1}, in which the order of 0's and 1's is not specified.

(vii) An ordinary word in English, such as 'Physics' can be viewed as a finite sequence.

p, h, y, s, i, c, s

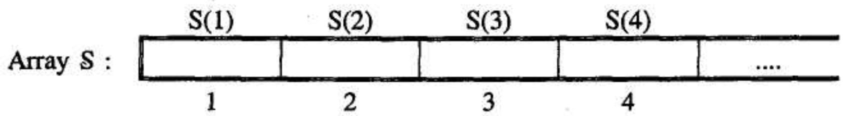
composed of letters from the ordinary alphabets.

If we omit commas, we get the word physics.

Such representation is referred to as **string.**

In computer science a sequence is sometimes called a **linear array** or **list.**

An array may be viewed as a "sequence of positions" which we represent below as boxes.



The position form a finite or infinite list, depending on the desired size of array. Elements from some set may be assigned to the positions of the array. The element assigned to position n will be denoted by S(n) [corresponding to f(n) in the definition] and the sequence S(l), S(2), S(3), S(4), will be called the **sequence of values** of the array S.

3.2.1 Summation of terms of a sequence:

Another problem related with sequences is to find out the sum of first n terms. It is tedious to add terms one after another in case of large number of terms. It is therefore necessary to find a law giving sum of the first n terms.

Let T_n and S_n denote the n^{th} term and sum of first n terms of a sequence respectively.

Thus $S_n = T_1 + T_2 + ... + T_n$

$$= \sum_{n=1}^{n} T_r$$

read as "sigma T_r, r varying from 1 to n"

With this notation, the sum of first (n – 1) terms is

$$S_{n-1} = \sum_{r=1}^{n-1} T_r = T_1 + T_2 + - + T_{n-1}$$

Hence $S_n - S_{n-1}$, = T_n , true for $n \geq 2$.

We shall illustrate it by simple examples.

Example (1) : If $T_n = 3n^2 + 4n + 7$, find T_4, T_9 and T_{13}

Solution : We have $T_n = 3n^2 + 4n + 7$

putting n = 4, 9 and 13 respectively,

$T_4 = 3(4)^2 + 4(4) + 7 = 71$

$T_9 = 3(9)^2 + 4(9) + 7 = 286$

$T_{13} = 3(13)^2 + 4(13) + 7 = 566.$

Example (2): If $S_n = \frac{n(n+1)}{2}$ Find T_5 and T_7.

Solution: We have $S_n - S_{n-1}$, = T_n

$\therefore$ putting n = 5, we get

$S_5 - S_4 = T_5$

$\therefore T_5 = \frac{5(5+1)}{2} - \frac{4(4+1)}{2}$

$= 15 - 10$

$= 5.$

Similarly $T_7 = S_7 - S_6$

$= \frac{7(7+1)}{2} - \frac{6(6+1)}{2}$

$= 28 - 21$

$= 7.$

Example (3): Find T_{25} given that $S_n = \frac{3}{2}(3^n - 1)$.

Solution : We have $S_n - S_{n-1}$, = T_n

$\therefore$ putting n = 25, we get,

$T_{25} = S_{25} - S_{24}$

$= \frac{3}{2}(3^{25} - 1) - \frac{3}{2}(3^{24} - 1)$

$= \frac{3}{2}(3^{25} - 3^{24})$

$= 3^{24}\left(\frac{3}{2}\right)(3-1)$

$= 3(3^{24})$

$= 3^{25}$

Check your progress – 3.2

(1) The array

2	4	6	8	…….

represents an infinite sequence

Fill in the blank.

(2) Choose correct figure from the bracket to fill in the blank.

(i) $T_n = n^2 – 2n + 1, T_3 = . .$ (4, 5, 6, 3)

(ii) $S_n = \frac{n(n+1)(2n+1)}{6}$, $T_2 = ...$(4, 6, 8, 10)

(iii) $S_n = n^2$, $T_n = ...$ ($2n^2$, n^2, $2n – 1$, $2n + 1$)

3.3 ARITHMETIC PROGRESSION

Amongst the three types of sequences which are very common, the arithmetic progression is one of them. Consider a sequence

2, 5, 8, 11, 14, ...

Here we see that 5 – 2 = 8 – 5 = 11 – 8 = ... = 3

i.e. the difference between a term and its preceding term is constant. Such a sequence is called arithmetic progression; abbreviated as A.P**.**

Definition : An arithmetic progression (A.P.) is a sequence in which the difference between any term and the immediately preceding term is constant**.**

This constant difference is called the **common difference** of the arithmetic progression and is usually denoted by d.

In the above example the common difference is 3 and first term is 2.

The general form of an arithmetic progression is

$$a, a + d, a + 2d, a + 3d, ...$$

where a = first term, and

d = common difference.

3.3.1 The n^{th} term of A.P. (T_n)

Consider a general A.P.

a, a + d, a + 2d, a + 3d, ...

We observe that

$T_1 = a \qquad = a + (0)d \qquad = a + (1–1)d$

$T_2 = a + d \qquad = a + (2–1)d$

$T_3 = a + 2d \qquad = a + (3 – 1)d$ etc.

We note that every term is obtained by adding to 'a' certain multiple of d; and this multiple is exactly one less than that of the suffix of T**.**

$$\therefore T_n = a + (n–1)d.$$

Example (1) : Find T_n for the following A.P. 1, 5, 9, 13, 17

Solution :

Here $T_1 = a = 1$ and common difference d = 4

$\therefore T_n = a + (n–1)d$

$= 1+(n–1)4$

$= 4n–3.$

Example (2): Find the number of terms in the A.P. 101, 104, 107, ..., 182.

Solution: Here $a = 101,\ d = 107 - 104 = 3$

Let 182 be the n^{th} term of A.P.

$$\therefore\ 182 = 101 + (n-1)\,3$$

$$\text{i.e. } 81 = 3(n-l)$$

$$\therefore\ n - 1 = 27 \text{ and } n = 28.$$

Thus there 28 terms in the given A.P.

Example (3) : If the n^{th} term of a sequence is 3n + 2, show that it is an A.P., what is its first term and the common difference ?

Solution : Here $T_n = 3n + 2$

Replacing n by (n – 1) we get,

$$T_{n-1} = 3(n-l) + 2$$

$$= 3n-l$$

$$\text{Consider, } T_n - T_{n-l} = 3n + 2 - (3n-1)$$

$$= 3$$

This is constant (independent of n)

Hence it is n^{th} term of A.P. with d = 3.

$$1^{st} \text{ term} = T_1 = 3(1) + 2 = 5$$

and the common difference is 3 ($= T_n - T_{n-1}$).

3.3.2 Sum of the first n terms of A.P. (S_n)

Let S_n denote the sum of 1^{st} n terms of the general A.P. with $T_1 = a$ and $T_n = a + (n-l)d$

$$\text{Let } T_n = a + (n-1)d = l$$

$$\text{Then we have } T_{n-1} = l-d,\ T_{n-2} ,= l-2d$$

$$\text{Now } S_n = T_1 + T_2 + T_3 + ... + T_{n-2} + T_{n-1} + T_n$$

$$= a + (a+d) + (a+2d) + ... + (l-2d) + (l-d)\ l$$

Reversing the order of terms in summation,

$$S_n = l + (l-d) + (l-2d) + ... + (a+2d) + (a+d) + a$$

Adding vertically, we get

$$2S_n = (a+l) + (a+l) + ... + (a+l) + (a+l) \text{ to n terms}$$

$$= n\,(a + l)$$

$$\therefore\ S_n = \frac{n}{2}(a + l)$$

Substituting the value of n, we get,

$$S_n = \frac{n}{2}\{2a + (n-l)d\}$$

We shall illustrate the use of these formula in certain simple examples.

Example (4) : Find T_n and S_n for the following A.P. – 29, – 25, – 21, –17,...

Solution : Here $a = -29,\ d = -25 - (-29) = 29 - 25 = 4$

$$\therefore\ T_n = a + (n-1)\,d$$

$$= -29 + (n-1)(4)$$

$$= 4n-33.$$

$$\text{and } S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$= \frac{n}{2}\{-58 + (n-l)4$$

$$= n(2n-3l).$$

Example (5): Find the sum of the first n odd natural number

Solution : The first n odd natural numbers are 1, 3, 5, ...

$$T_n = n^{th} \text{ odd natural number}$$

$$= l+(n-l)2 \qquad (a = 1, d = 2)$$

$$= 2n - 1 = l \text{ say}$$

$$\therefore S_n = \frac{n}{2}(a + l)$$

$$= \frac{n}{2}(1 + 2n-l)$$

$$= n^2.$$

Example (6) : If the sum of the first n terms of a sequence is $3n^2 + 4n$, show that it is an A.P. Find the first term and the common difference.

Solution : Here $s_n = 3n^2 + 4n$

Replace n by (n – 1) to obtain S_{n-1}.

$$\therefore S_{n-1} = 3(n-1)^2 + 4(n-1) = 3n^2 - 6n + 3 + 4n - 4$$

$$= 3n^2-2n-1$$

$$\therefore T_n = S_n - S_{n-1}$$

$$= 3n^2 + 4n-(3n^2-2n-l)$$

$$= 6n + l.$$

Replace n by (n – 1) to obtain T_{n-1}

$$\therefore T_{n-1} = 6(n-1) + 1$$

$$= 6n-5.$$

Consider, $T_n - T_{n-1} = 6n+1 - (6n-5)$

$= 6$, which is constant.

Hence $S_n = 3n^2+4n$ is sum of first n terms of an A.P. with d = 6.

The first term = $T_1 = 6(1) + 1 = 7$ or $S_1 = 3 + 4 = 7$

Check your progress – 3.3

(1) State whether true or false. If false write correct answer.

(i) Numbers a – d, a, a + d are in A.P.

(ii) For an A.P., $T_n - T_{n-1}$ is not constant.

(iii) For an A.P., $T_8 = 36$ then $S_{15} = 540$.

(iv) In an A.P., $-s_n = 2n^2 - n$ then $a = T_1 = 4$.

(v) In an A.P., $S_n = n^3$ then d = 6.

(vi) The sum of first n even natural numbers is n(n + 1).

(vii) For an A.P. a = 2, $T_7 = 20$, the common difference is 7.

(viii) For an A.P. $S_{31} = 186$, then $T_{16} = 6$.

(ix) For a sequence having n terms, $S_n = \frac{n(n+1)}{2}$ then the sequence is an A.P.

3.4 GEOMETRIC PROGRESSION

Consider a sequence 3, 9, 27, 81, ... We observe that the ratio of any term to its preceding term is $\frac{9}{3} = \frac{27}{9} = \frac{81}{27}$ = 3 and is constant. Such a sequence in which ratio of a term to its preceding term is constant is called geometric progression, abbreviated as G.P.

Definition: A geometric progression (G.P.) is a sequence in which every term bears a constant ratio to the one immediately preceding it.

This constant ratio is called the **common** ratio of the G.P. and is usually denoted by r.

In the above example the common ratio is 3. The first term is also 3 here.

The general form of a geometric progression is a, ar, ar^2, ...

where a is the first term, and

r is the common ratio.

3.4.1 The n^{th} term of G.P. (T_n)

Consider a G.P. a, ar, ar^2 ....

With the usual notation,

$$T_1 = a = ar^\circ = ar^{1-1}$$
$$T_2 = a\,r = a\,r^{2-1}$$
$$T_3 = ar^2 = ar^{3-1}$$

We note that each term is a product of two factors. The first factor is a , and is common to all the terms. The second factor is a certain power of r. This is exactly one less than the corresponding suffix of T.

$$\therefore\ T_n = ar^{n-1}$$

Example (1) : Find T_n for the following G.P. $1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}$ ....

Solution : Here a = 1, r = $\frac{1}{3}/1 = \frac{1}{3^2}/\frac{1}{3} = \frac{1}{3^3}/\frac{1}{3^2} = \frac{1}{3}$

$$\therefore T_n = a\,r^{n-1}$$
$$= 1\left(\frac{1}{3}\right)^{n-1}$$
$$= \frac{1}{3^{n-1}}$$

Example (2): Given $T_3 = 20$, $T_7 = 320$ of a certain G.P. Find T_8.

Solution : We have $T_3 = a\,r^2 = 20$ and $T_7 = a\,r^6 = 320$

where a is the first term and r is the common ratio of G.P.

On division, we get,

$$\frac{a\,r^6}{a\,r^2} = \frac{320}{20}$$

i.e. $r^4 = 16$ $\qquad\qquad \therefore\ r = 2$

Then $a\,r^2 = 20$ gives $a = \frac{20}{r^2} = \frac{20}{4} = 5$

$$\therefore\ T_8 = a\,r^7 = 5\,(2^7) = 640.$$

3.4.2 Sum of first n terms of a G.P. (S_n)

Let a be the first term and r be the common ratio of a general G.P.

Then $T_n = ar^{n-1}$ and

$$S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1} \quad ....(I)$$

Multiply by r. $rS_n = ar + ar^2 + + ar^{-n-2} + ar^{n-1} + ar^n$. (II)

Subtracting (II) from (I), we get

$$S_n - rS_n = a - ar^n$$

Divide by (1 – r).

i.e. $S_n (1 - r) = a (1 - r^n)$

$$\therefore S_n = \frac{a(1-r^n)}{(1-r)}, \text{ if } r < 1$$

$$= \frac{a(r^n - 1)}{r-1}, \text{ if } r > 1$$

However if r = 1, G.P. becomes a, a,....

Here S_n = a + a + ... to n terms

= n a

We shall illustrate the use of these formula by means of some simple examples.

Example (3) : Find T_n and S_n for the following G.P. 1, $\frac{-3}{2}, \frac{9}{4}, \frac{-27}{8}$,.....

Solution: Here a = 1, $r = \frac{-3/2}{1} = \frac{9/4}{-3/2} = \frac{-27/8}{9/4} = \frac{-3}{2}$

$$\therefore T_n = a\,r^{n-1} = 1\left(\frac{-3}{2}\right)^{n-1} \quad = \left(\frac{-3}{2}\right)^{n-1}$$

$$\text{and } S_n = \frac{a(1-r^n)}{1-r} = \frac{1\left(1-\left(\frac{-3}{2}\right)^n\right)}{1-\left(\frac{-3}{2}\right)} = \frac{2}{5}\left[1-\left(\frac{-3}{2}\right)^n\right]$$

Example (4): For a G.P. a = 5, r = 2, S_n = 635, find n.

Solution : Here a = 5, r = 2, $S_n = \frac{a(r^n - 1)}{r-1}$

$$\therefore 635 = \frac{5(2^n - 1)}{2-1}$$

$$\therefore 127 = 2^n - 1$$

$$\therefore 128 = 2^n$$

$$\text{i.e. } 2^7 = 2^n$$

$$\therefore n = 7$$

Check you progress – 3.4

(1) Fill in the blanks by choosing appropriate number given in the bracket

(i) In a G.P. with a = 2, r = 3, T_3 = ... (18,2,3,9)

(ii) a, b, $\frac{b^2}{a}$ .... are in G.P. then the common ratio is ...

$\left(\frac{a}{b}, b, a^2, \frac{b}{a}\right)$

(iii) If for a sequence, $S_n = 3\,(4^n - 1)$, then the sequence is ....
(A.P., G.P., none of these)

(iv) In a G.P. a = $\frac{2}{3}$ and T_6 = 162 then the common ratio r = ...
(2, 3, 4, 6)

(v) $1, \frac{-3}{2}, \frac{9}{4}, \frac{-27}{8}$ are in ....

(A.P., G.P., H.P., none of them)

(2) In a G.P. $T_3 = 200$ $T_7 = 3200$ find a and r.

(3) In a G.P., $a = \sqrt{5}$, $r = 2\sqrt{5}$ find T_5.

(4) State whether true or false and if false write the correct statement,

(i) In a G.P. $S_n = 3^n - 1$, then common ratio is 3.

(ii) In a sequence 2, 6, 18, ... the 4^{th} term is 36.

(iii) In a G.P. $1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}$ the n^{th} term is $\frac{1}{a^{1-n}}$.

(iv) In a G.P. $T_3 = \frac{1}{18}$ and $T_5 = \frac{1}{162}$ then $T_4 = \pm\frac{1}{54}$.

3.5 HARMONIC PROGRESSION (H.P.)

Consider a sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...(A)$

The reciprocal of the terms of the sequence, form another sequence viz. 1, 2, 3, 4,…..(B)

The terms of the sequence (B) are in A.P. Such a sequence (A), in which the reciprocals of the terms form A.P. is known as Harmonic progression, and abbreviated as H.P.

Definition : The terms $a_1, a_2, a_3, ..., a_n, ...$ are said to be in **harmonic progression (H.P.)** if..

$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ... \frac{1}{a_n}$ be in arithmetic progression.

To find n^{th} term of H.P., we have to find n^{th} term of the corresponding A.P.

Example: Find T_n and hence T_5 and T_7 in the following harmonic progression. $\mathbf{\frac{1}{7}, \frac{3}{22}, \frac{3}{23}, ...}$

Solution: The corresponding A.P. obtained by taking reciprocal of the terms in the given sequence is $7, \frac{22}{3}, \frac{23}{3}, ...$

$$\text{Here } a = 7,\ d = \frac{22}{3} - 7 = \frac{22-21}{3} = \frac{1}{3}$$

Hence T_n of A.P. is a + (n–1) d

$$= 7 + (n-1)\frac{1}{3} = \frac{20+n}{3}$$

$\therefore$ The n^{th} term of given H.P. is,

$$T_n = \frac{3}{20+n}$$

Hence putting n = 5 and 7, we get

$$T_5 = \frac{3}{20+5} = \frac{3}{25} \text{ and}$$

$$T_7 = \frac{3}{20+7} = \frac{3}{27} = \frac{1}{9}$$

There is no convenient formula that can be developed to obtain sum of first n terms of H.P.

Check you progress: 3.5

Answer the following.

(1) Given A.P. as 1, 3, 5, 7, 9, ... Write down the corresponding H.P.

(2) In a harmonic progression first two terms are $\frac{1}{a}$ and $\frac{1}{b}$, find the common difference of corresponding A.P.

3.6 THE THREE MEANS

If x, y, z, are consecutive terms of a sequence then y is called mean of x and z. We shall? consider the three means corresponding to three progressions that we have studied so far; and! also study some of its properties.

Definition : If a, A, b are consecutive terms of an A.P. then A is called **arithmetic mean** between and a and b and is abbreviated as A.M.

Since a, A, b are in A.P., the common difference of A.P. is = A – a = b – A

$$\text{i.e. } 2A = a + b$$

$$\text{i.e. } A = \frac{a+b}{2}$$

Thus the A.M. between two numbers a and b is. $\frac{a+b}{2} = A$

e.g. (i) The A.M. of 3 and 5 is $\frac{3}{10}, \frac{3}{8}, \frac{15}{32}, \frac{75}{128}, = 4$.

(ii) The A.M. of – 3 and $\frac{7}{2}$ is $\frac{1}{2}\left(-3+\frac{7}{2}\right) = \frac{1}{4}$.

Definition : If a, G, b are all positive and are three consecutive terms of G.P. then G is called **geometric mean** between a and b and is abbreviated as G,M.

Since a, G, b (a > 0, G > 0, b > 0) are in G.P. the common ratio = $\frac{G}{a} = \frac{b}{G}$

$$\text{i.e. } G^2 = ab$$

$$\text{i.e. } G = \sqrt{ab}$$

Thus the G.M. between two positive numbers a and b is $G = \sqrt{ab}$

e.g. (iii) The G.M. of 5 and 15 is $\sqrt{5\times 15} = 5\sqrt{3}$

(iv) The G.M. of $\frac{a}{r}$ and a r is $\sqrt{\frac{a}{r}(ar)} = a$.

Definition : If a, H, b are consecutive terms of H.P. then H is called the **harmonic mean** between a and b.

Since a, H, b are in H.P., the reciprocals of them viz $\frac{1}{a}$, $\frac{1}{H}$ and $\frac{1}{b}$ are in A.P.

The common difference of A.P. is

$$= \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

$$\text{i.e. } \frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

$$\text{i.e. } \frac{2}{H} = \frac{a+b}{ab}$$

$$\text{i.e } H = \frac{2ab}{a+b}$$

Thus the H.M. between and b is $H = \frac{2ab}{a+b}$

e.g. (v) The H.M. of – 2 and 3 is $\frac{2(-2)(3)}{-2+3} = -12$.

(vi) The H.M. of 4 and 7 is $\frac{2(4)(7)}{4+7} = \frac{56}{11}$

3.6.1 Properties of means

Let a, b be the given unequal positive real numbers.

We have A.M. is $A = \frac{a+b}{2}$

G.M. is $G = \sqrt{ab}$

H.M.is $H = \frac{2ab}{a+b}$

We have the following properties.

(1) A, G, H are in G.P. or $AH = G^2$

Proof :Consider, $AH = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab = \left(\sqrt{ab}\right)^2 = G^2$.

(2) A > G > H. (a > 0, b > 0)

Proof: Consider $A - G = \frac{a+b}{2} - \sqrt{ab}$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}$$

Since a, b are positive . $\sqrt{a}$, $\sqrt{b}$ are real.

$$\therefore \left(\sqrt{a}-\sqrt{b}\right)^2 > 0 \quad (\because a \neq b)$$

$$\therefore A - G > 0$$

$$\therefore A > G ....(I)$$

$$\text{or } A / G > 1$$

By property , (1) $\frac{A}{G} = \frac{G}{H}$

and by (I), $\frac{A}{G} > 1$

$$\therefore \frac{G}{H} > 1$$

$$\therefore G > H \qquad .... (II)$$

It now follows from (I) and (II) that A > G > H.

This- gives a very convenient way to assume 3, 4, 5 numbers in the respective progressions. We prepare the following table.

progression	3 nos.	4 nos.	5 nos.
A.P.	a – d, a, a + d	a – 3d, a – d, a + d, a + 3d	a – 2d, a – d, a, a + d, a + 2d
G.P.	$\frac{a}{r}$, a,	a/r^3 , a/r, ar, ar^3	a/r^2, a/r, a, ar, ar^2

We shall illustrate the use of the means in the following examples.

Example (1): Find three numbers in A.P. such that their sum is 15 and their product is 105.

Solution *:* Let the three numbers in A.P. be a – d, a, a + d.

Their sum is 15.

$$\therefore \quad (a-d)+a+(a+d) = 15$$
$$\text{i.e. } 3a = 15$$
$$a = 5$$

Hence numbers are 5 – d, 5, 5 + d.

Their product is 105.

$$(5-d)(5)(5+d) = 105$$
$$25-d^2 = 21$$
$$\therefore d = \pm 2$$

Hence the numbers are 5 – 2, 5, 5 + 2 or (d = 2)

5 + 2, 5, 5 – 2 (d = – 2)

$\therefore$ 3, 5, 7 or 7, 5, 3 are the numbers.

Example (2) : Find four numbers in G.P. such that their product is 1 and the sum of two middle is $\frac{5}{2}$.

Solution : Let $\frac{a}{r^3}, \frac{a}{r}(ar), ar, ar^3$ be four numbers in G.P.

Their product is 1.

$$\left(\frac{a}{r^3}\right)\left(\frac{a}{r}\right)(ar)(ar3) = 1$$
$$\therefore a^4 = 1$$
$$\therefore a = 1$$

Hence the numbers are $\frac{1}{r^3}, \frac{1}{r}, r, r^3$

The sum of the two middle numbers is $\frac{5}{2}$.

$$\therefore \frac{1}{r}+r = \frac{5}{2}$$
$$\text{i.e. } 2+2r^2-5r = 0$$
$$\text{i.e. } 2r^2-4r-r+2 = 0$$
$$\text{i.e. } (2r-l)(r-2) = 0$$
$$\therefore \text{ Either } r = \frac{1}{2} \text{ or } r = 2.$$

When r = 2, number are $\frac{1}{8}, \frac{1}{2}, 2, 8$

or when r = $\frac{1}{2}$, number are 8, 2, $\frac{1}{2}, \frac{1}{8}$..

Example (3) : If 6 is G.M. and $\frac{72}{13}$ is the H.M. between two numbers, find them.

Solution : Let a, b be the two numbers.

G.M. is 6. $\therefore \sqrt{ab} = 6$

Squaring we get ab = 36 ...(I)

Their H.M. is $\dfrac{72}{13}$

$$\therefore \frac{2ab}{a+b} = \frac{72}{13}$$

$$\text{i e } \frac{2(36)}{a+b} = \frac{72}{13}$$

$$\text{i.e. } a + b = 13$$

$$\text{consider, } (a-b)^2 = (a+b)^2 - 4ab$$

$$= 169 - 144$$

$$= 25$$

$$\therefore a - b = 5$$

$$\text{and } a + b = 13$$

$$\text{Adding, we get } 2a = 18$$

$$\therefore a = 9 \text{ and } b = 4$$

Thus 9 and 4 are the required numbers.

Example (4) : If A.M. and G.M. between two numbers be 17 $\dfrac{7}{9}$ and 10 $\dfrac{2}{3}$ respectively. Find their H.M.

Solution : Let a, b be the two numbers.

$$\text{Their A.M. is } 17\frac{7}{9} = \frac{160}{9}$$

$$\text{and G.M. is } 10\frac{2}{3} = \frac{32}{3}$$

$$\therefore \frac{a+b}{2} = \frac{160}{9} \text{ and } \sqrt{ab} = \frac{32}{3}$$

$$\text{i.e } a + b = \frac{320}{9} \text{ and } ab = \frac{(32)^2}{9}$$

$$\text{The H.M is given by } H = \frac{2ab}{a+b}$$

$$\text{i.e. H.M} = \frac{2(32)^2}{9} \times \frac{9}{320}$$

$$= \frac{32}{5}$$

$\therefore$ H.M. between the numbers is $\dfrac{32}{5}$

Check your progress - 3.6

Answer the following :

(1) Given two numbers 3 and 8, find A.M., G.M. and H.M.

(2) a and b are two numbers such that their A.M. is 2 and H.M. is $\dfrac{3}{2}$, show that a = 3 and b = 1.

(3) If a is A.M. between b and c, b is G.M. between c and a, show that H.M. between a and b is c.

(4) Find five numbers in G.P. such that their product is 32 and the product of the last two numbers is 108.

(5) If 4, H, 9 are in H.P. find H.

3.7 SERIES

We have seen that f : N → R is a sequence; with f(n) as the n^{th} term of a sequence. Now $\sum_{n=1}^{n} f(n)$ is called a series. If it contains sum of a finite number of terms, then it is a finite series. We shall consider mainly a finite series.

3.7.1 Standard series

We assume certain standard results without proof.

(1) $\sum_{n=1}^{n} r = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$.

(2) $\sum_{n=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + ... + n^2 \ \frac{n(n+1)(2n+1)}{6}$.

(3) $\sum_{n=1}^{n} r^3 = 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

(4) $\sum_{n=1}^{n} (ar^3 + br^2 + cr + d) = a\sum_{n=1}^{n} r^3 + b\sum_{n=1}^{n} r^2 + c\sum_{n=1}^{n} r + n(d)$

We shall consider examples in which these results are used.

Example (1): Evaluate $\sum_{n=1}^{n}$ r (r +1) (r + 3)

Solution :We have r (r + 1) (r + 3) = $r(r^2 + 4r + 3)$

$= r^3 + 4r^2 + 3r$

Hence the given sum = $\sum_{n=1}^{n} (r^3) + 4\sum_{n=1}^{n} (r^2) + 3\sum_{n=1}^{n} (r)$

$$= \frac{n^2(n+1)^2}{4} + \frac{4n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)}{12}[3n(n+1)+8(2n+1)+18]$$

$$= \frac{n(n+1)}{12}[3n^2+19n+26]$$

$$= \frac{n(n+1)}{12}[3n^2+6n+13n+26]$$

$$= \frac{n(n+1)}{12}[(3n+13)(n+2)]$$

$$= \frac{n(n+1)(n+2)(3n+13)}{12}$$

Example (2) : Find T_r and sum to n terms of the following 2.5 + 5.8 + 8.11 +....

Solution : The 1st factors in each term are 2, 5, 8,....

These are in A.P. with a = 2, d = 3

$$\therefore T_{r_1} = 2 + (r - 1)3 = 3r - 1$$

The 2nd factors are 5, 8, 11,...

These are also in A.P. with a = 5, d = 3

$$\therefore T_{r_2} = 5 + (r - 1)(3) = 3r + 2$$

$$\therefore T_r = T_{r_1} . T_{r_2} = (3r-1)(3r + 2)$$

$$= 9r^2 + 3r–2$$

$$\text{Now } S_n = \sum_{n=1}^{n} T_r = \sum_{n=1}^{n} (9r^2 + 3r–2)$$

$$= 9\sum_{n=1}^{n} r^2 + 3\sum_{n=1}^{n} r–2n$$

$$= \frac{9n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} - 2n$$

$$= \frac{n}{6}[9(n+l)(2n + l) + 9(n+l)–12]$$

$$= \frac{n}{6}[9(2n^2 + 3n + 1) + 9n – 3]$$

$$= \frac{n}{6}[18n^2 + 36n + 6]$$

$$= n(3n^2 + 6n + l).$$

Example (3): sum to n terms of 9 + 99 + 999 + 9999 + ...

Solution : Here we shall express each term as a difference of two terms, one of the which is a geometric series.

$$\text{Let } S_n = 9 + 99 + 999 + \text{... to n terms}$$

$$= (10–1) + (10^2–1)+(10^3–1) + \text{... to n terms}$$

$$= [10+10^2+10^3+ \text{... to n terms}]– [1+1+\text{... to n terms}]$$

$$= S_1–S_2 \text{ say}$$

$$\text{Now } S_1 = 10 + 10^2 + \text{.... to n terms.}$$

These terms are in G.P. with r = a = 10, n = n.

$$\therefore S_1 = \frac{10(10^n - 1)}{10-1} = \frac{10}{9}(10^n - 1)$$

$$\text{and } S_2 = 1 + 1 + \text{... + to n terms}$$

$$= n$$

$$\therefore S_n = S_1–S_2 = \frac{10}{9}(10^n - 1) – n.$$

Example (4) : Find the sum of the first n terms of the following arithmetico - geometric sequence. 1, 2×2, 3×4, 4×8, 5×16, 6×32, ...

Solution : The 1st factors are 1, 2, 3, 4, ...

$$\therefore T_{r_1} = r$$

The 2nd factors are 1, 2, 2^2, 2^3, 2^4, ...

These factors are in G.P. with a = 1, r = 2

$$\therefore T_{r_2} = l(2)^{r-1} = 2^{r-1}$$

$$\therefore \quad T_r = T_{r_1} \times T_{r_2} = r2^{r-1}$$

$$\text{Hence } S_n = \sum_{n=1}^{n} T_r$$

$$= 1+2\times 2 + 3\times 2^2 + 4\times 2^3 + ... + n\times 2^{n-1} \qquad ..(A)$$

The common ratio of G.P. is 2. Hence multiply S_n by 2

$$2S_n = 2+2\times 2^2+3\times 2^3+....+(n–l)\times 2^{n-1}+n\times 2^n \qquad ... (B)$$

Subtract (A) from (B)

$$\therefore 2 S_n– S_n = –(1+1\times 2 +1\times 2^2 + l\times 2^3 + ...+ 2^{n-1})+n\times 2^n$$

$$= n\times 2^n–(l + 2 + 2^2 + ... + 2^{n-1})$$

$$\therefore \quad S_n = n \times 2^n - S_1$$

where $S_1 = 1 + 2 + .. + 2^{n-1}$

are in G.P. with a = 1, r = 2, n = n.

$$\therefore S_1 = \frac{1(2^n - 1)}{2-1} = 2^n - 1$$

$$\text{Hence } S_n = n \times 2^n - (2^n - l)$$

$$= (n - l)\, 2^n + l.$$

Example (5): Express the following recurring decimals as rational numbers $2.3\overline{45}$

Solution : We have $2.3\overline{45} = 2.3 + 0.045 + 0.00045 + ...$

$$= \frac{23}{10} + \frac{45}{10^3} + \frac{45}{10^5} + ...$$

The terms after the first term form a G.P. whose first term is $\frac{45}{1000}$ = a and the common ratio $r = \frac{1}{100}$

Now $r = \frac{1}{100} < 1$ and when n becomes very large, r^n almost becomes zero.

$\therefore$ Sum of these terms in G.P is $\frac{a}{1-r} = \frac{45/1000}{1-(1/100)}$

$$= \frac{45}{99 \times 10} = \frac{1}{22}$$

Hence the given number $2.3\overline{45} = \frac{23}{10} + \frac{1}{22} = \frac{129}{55}$

3.7.2 Infinite Geometric Series

Consider a geometric progression with $T_n = \frac{1}{3^n}$.

The terms are $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3} ... \frac{1}{3^n} ...$

We observe that as n becomes larger and larger the terms of a G.P. go on becoming smaller and smaller. If we take n to be very large, the corresponding term approaches zero.

We say $\frac{1}{3^n}$ approaches zero**.**

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{\frac{1}{3}\left(1 - \frac{1}{3^n}\right)}{1 - \frac{1}{3}} \text{ becomes } \frac{\frac{1}{3}}{2/3} = \frac{1}{2}$$

This happens only when $|r| < 1$**.**

For values of r such that $|r| > 1$, such a sum does not exist**.**

Definition : If for a G.P., S_n approaches a certain number S as n becomes indefinitely large, the number S is called **sum to infinity** of the G.P**.**

Let a be the first term and r (< 1) be the common ratio**.**

$$S_n = \frac{a(1-r^n)}{1-r} \quad \therefore S = \frac{a}{1-r}$$

Thus $a + ar + ar^2 + ... + ar^{n-1} + ... = \frac{a}{1-r}$

We shall explain this concept by the following example.

Example (6): Find S_n and sum to infinity of the following G.P.
$\mathbf{\frac{-5}{3}, \frac{5}{9}, \frac{-5}{27}, \frac{5}{81}, \frac{-5}{243}, ...}$

Solution : Here $a = \frac{-5}{3}$ and $r = - -\frac{1}{3}$

$$\therefore S_n = \frac{a(1-r^n)}{1-r} = \frac{\left(\frac{-5}{3}\right)\left[1-\left(\frac{-5}{3}\right)^n\right]}{1-\left(-\frac{1}{3}\right)} = \frac{-5}{4} + \frac{5}{4}\left(-\frac{1}{3}\right)^n$$

Since $| r | = |-\frac{1}{3}| = \frac{1}{3} < 1$, the sum to infinity of the G. P. exists and

$$S = \frac{a}{1-r} = \frac{-5/3}{1-\left(-\frac{1}{3}\right)} = -\frac{5}{4}.$$

Example (7): The first term of a G.P. is 2 and the sum to infinity is 6. Find the common ratio.

Solution : Here a = 2, S = 6

We have to find r.

We know that $S = \frac{a}{1-r}$

$$\therefore 6 = \frac{2}{1-r}$$

$$\therefore 1-r = \frac{2}{6} = \frac{1}{3}$$

$$\therefore r = \frac{2}{3}$$

Thus the common ratio is $\frac{2}{3}$.

Check your progress – 3.7

(1) Determine whether the sum to infinity of the following G.P.'s exits. Determine it, when it ; exists.

(i) 1, 2, 4, 8. 16, ... (ii) $4, \frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \frac{4}{81}$, ...

(iii) $1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}$,....... (iv) $\frac{3}{10}, \frac{3}{8}, \frac{15}{32}, \frac{75}{128}$, ...

(v) $\frac{1}{5}, \frac{2}{15}, \frac{4}{45}, \frac{8}{135}$, ...

(2) Given that 9 + 99 + 999 + ... to n terms = S

$= \frac{10}{9}(10^n - 1) - n$, find the following sums in terms of S.

(i) 5 + 55 + 555 + ... to n terms

(ii) 1 + 11 + 111 + ... to n terms

(3) Show that the recurring decimal $2.3\overline{56}$ is $\frac{2333}{990}$.

(4) Prove that the sum

$3\times1+5\times8 + 7\times15 + \ldots\ldots$ to n terms = $\frac{n}{6}\ (28n^2 + 27n - 37)$

(5) If the sum of infinity of the sequence 3, 5r, $7r^2$, ... is $4\frac{8}{9}$, find r.

3.8 SUMMARY

Mathematics means numbers and sequence means arrangement of proper number in proper way. E.g. 1, 2, 3, 4, 5 given numbers are increasing and increased with one unit. So it is sequence. Series is concept of addition of sequence terms.

3.9 CHECK YOUR PROGRESS - ANSWERS

3.2 (1) 2, 4, 6, 8, ...

(2) (i) 4 (ii) 4 (iii) (2n – 1)

3.3 (1) (i) True

(ii) False; For an A.P., $T_n - T_{n-1}$ is constan**t.**

(iii) True, Hint : $S_{15} = \frac{15}{2}\ (2\ T_8)$

(iv) False; In an A.P., $S_n = 2n^2 - n$, then $a = T_1 = 1$

(v) True, Hint : $S_1 = a = T_1 = 1$, $S_2 = 2a + d = 8$

(vi) True

(vii) False; For an A.P. a = 2, $T_7 = 20$, the common difference is 3.

(viii) True, Hint : $S_{31} = \frac{31}{2}\ (2T_{16})$ (ix) True**.**

3.4 (1) (i) 18 (ii) $\frac{b}{a}$ (iii) G.P. (iv) 3 (v) G.P.

(2) a = 50, r = 2

(3) $400\sqrt{5}$

(4) (i) True

(ii) False; In a sequence 2, 6**,** 18, ...the 4^{th} term is 54

(iii) False; In a G.P., 1, $\frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}$, .. the n^{th} term is a^{1-n} (iv) True.

3.5 (1) $\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$, **....** (2) b–a

3.6 (1) $\frac{11}{2}, 2\sqrt{6}, \frac{48}{11}$

(4) $\frac{2}{9}, \frac{2}{3}, 2, 6, 18$

(5) $\frac{72}{13}$

3.7 (1) (i) No (ii) yes, S = 6 (iii) yes, S = $\frac{3}{5}$ (iv) No (v) S = $\frac{3}{5}$, yes

(2) (i) $\frac{5}{9}$ S (ii) $\frac{1}{9}$ S

(5) $\frac{1}{4}$.–Hint: $\frac{a}{1-r} + \frac{dr}{(1-r)^2} = 4\frac{8}{9}$ with a = 3**,** d = 2**.**

3.9 QUESTIONS FOR SELF - STUDY

Self Study Problems

(1) Find the 10^{th} and n^{th} term of the Sequence 2, 4,8,16 -----Also find the sum of First n terms.

(2) In a G. P. the third term is $\frac{1}{18}$ and fifth term is $\frac{1}{162}$ find n^{th} term of G.P.

(3) Find three numbers in a G. P. Such that their sum is 7/2 and sum of their squares is $21/4$

(4) The Sum of First n terms of a Sequence is $3(4^n-1)$ show that it is a G. P. Find its Common Ratio.

(5) How many terms of the A. P. 3, 7, 11.... are needed to yield the sum 1275?

3.10 SUGGESTED READINGS

1. *Pre-degree Mathematics* by Vaze, Gosavi

2. *Mathematics and Statistics* by M. L. Vaidya and M. K. Kelkar

◆◆◆

NOTES

CHAPTER 4

Permutations and Combinations

4.0 OBJECTIVES

After studying this chapter you will be able to use and explain the following ideas very freely and confidently**.**

- Permutations
- Permutations of things not all alike
- Combinations
- Complementary combinations

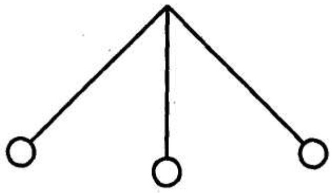
4.1 INTRODUCTION

We shall start with a simple but general result known as multiplication principle in combinatorics or fundamental principle in old language of mathematics. This has numerous applications in this chapter and else where. It will followed by permutations and combinations.

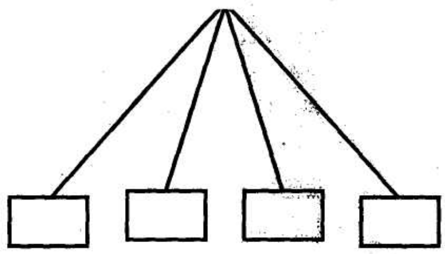
4.2 MULTIPLICATION PRINCIPLE

Let us suppose that two tasks T_1 and T_2, are to be performed in a sequence. A task T_1 can be performed in 3 ways and for each of these ways a task T_2 can be performed in 4 ways. We have to determine a sequence of performing tasks T_1 T_2.

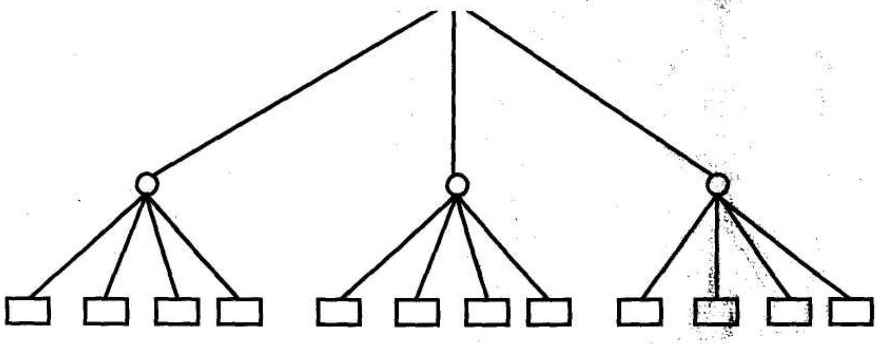
We shall exhibit this by means of following tree diagram.



Possible ways of performing task T_1
Fig. 4.1



Possible ways of performing task T_2
Fig. 4.2



Possible ways of performing task T_1 and task T_2 in sequence
Fig. 4.3

Each task T_1 can be performed in 3 ways and after that for each of these tasks (T_1), a task T_2 can be performed in 4 ways. Thus there are $3 \times 4 = 12$ ways of performing tasks T_1, T_2 in a sequence.

If a task T_1 can be performed in m ways and after performing the task T_1 in any of these m ways a second task T_2 can be performed in n different ways, then possible ways, of performing tasks T_1 and T_2 in a sequence is mn.

This can obviously be generalised as follows :

If a third task T_3 can be performed in p different ways, then possible ways of performing tasks T_1, T_2, T_3 in a sequence is mnp and so on.

We shall now illustrate this principle by the following examples.

Example (1) : A label identifier, for a computer program consists of one letter followed by three digits. How many distinct lable identifiers are possible (i) if no digit is repeated, (ii) if repeitition of digit is allowed ?

Solution : We consider an array consisting of four empty boxes.

(A)

26	10	9	8
Box 1	Box 2	Box 3	Box 4

(B)

26	10	10	10
Box 1	Box 3	Box 3	Box 4

(i) When none of the 10 digit is repeated refer to array (A). The successive boxes can be filled in 26, 10, 9, 8 ways. Hence by the extended multiplication principle, possible label identifiers are

$$26 \times 10 \times 9 \times 8 = 18720.$$

(ii) When any of the digit is repeated with reference to array (B), the successive boxes can be filled in 26, 10, 10, 10 ways. Hence by the extended multiplication principle, possible label identifiers are

$$26 \times 10 \times 10 \times 10 = 26000.$$

Example (2) : Show that the number of subsets of a set containing n elements is 2^n.

Solution : We use the concept of characteristic function of the set A having n elements. F_A consists of an array of n boxes, and each box can be filled in by 2 ways viz 0, s or l's. Thus by the extended principle of multiplication, there are

$2 \times 2 \times$ .... n factors = 2^n ways of filling the array, and therefore 2^n subsets of A.

Example (3) : Let set A contain n elements and $1 \leq r \leq n$. The number of sequences of length r (allowing repetitions) that can be formed from elements of A is n^r.

Solution : A sequence of length r can be formed by filling r boxes in an array.

n	n	n	....	n	n
Box 1	Box 2	Box 3		Box r – 1	Box r

Let T_1 be the task of filling box 1. We can choose any of n elements of A to fill it. Since repetition is allowed the task T_2 of filling second box can be done in n ways. This is true for all the boxes. T_r, the task of filling r^{th} box can also be done in n ways.

By the extended multiplication principle, the number of sequences that can be formed is

$n \times n \times$ ... to r factors = n^r.

Example (4) : In how many ways can the first and second prize in mathematics and first and second prize in physics be awarded in a class of 25 students.

Solution : The first prize in Mathematics can be awarded in 25 ways and having done this in any one way, a 2nd prize can be awarded in 24 ways.

Thus task of awarding prizes is mathematics can be done in $25 \times 24 = 600$ ways.

As awarding prize in Physics is irrespective of whether student has obtained a mathematics prize or not, the task of awarding prizes in Physics can be done in $25 \times 24 = 600$ ways.

By the multiplication principle both tasks can be performed in $600 \times 600 = 360000$ ways.

Check your progress – 4.2

(1) A coin is tossed four times and the result of each toss is recorded. How many different sequences of heads and tails are possible ?

(2) A six faced die is tossed four times and the numbers shown are arranged in a sequence. How many different sequences are there ?

(3) In how many ways can I write letters to three out of 9 friends, if I have exactly one post-card, one inland letter and one envelope ?

(4) Four persons enter a first class railway compartment in which there are 6 seats. In how many ways can they take their seats ?

4.3 FACTORIAL NOTATION

Let n be any positive integer or any natural number. The product of first n natural numbers

n(n–l)(n–2) 3.2.1

is denoted by n! or |_n and is read as "factorial n"

Thus n! = n (n – 1) (n – 2) 3.2.1

We define o! = 1

and 1! = 1

2! = $2 \times 1 = 2$

$$3! = 3\times2\times1 = 6$$
$$4! = 4 \times 3 \times 2 \times 1 = 24$$
$$5! = 5\times4\times3\times2\times 1 = 120 \text{ and so on.}$$
We can also write $n! = (n)\, \underline{|n-1}$

or $n! = (n)\,(n-1)\, \underline{|n-2}$.

4.4 PERMUTATION

Consider a set A containing n elements. If we select r elements out of them and arrange in an array each such arrangement is called **permutation.**

We can now formulate the problem in terms of subsets of A, each subset having r distinct elements.

Consider an array of r boxes.

n	n − 1	n − 2	....	n − (r − 2)	(n − r + 1)
Box 1	Box 2	Box 3		Box (r − 1)	Box r

The task T_1 of filling Box 1 can be done by choosing "any of 'n' elements of A. Hence task T_1 can be done in n ways. When this has been done task T_2 of filling Box 2 can be done in (n – 1) ways (element is not to be repeated). Continuing this way the task T_{r-1} , of filling a Box (r– 1) can be done inn– (.r–2) = n–r + 2 ways. The last task T_r of filling

^{rth} box can be done in n–r + 2 – l = n–r + l ways.

By the extended multiplication principle, the tasks $T_1, T_2, \dots, T_r$ can be done in

n(n–l)(n–2) .... (n – r + 1) ways.

This number is denoted by nP_r or P^n_r and is often called as "r – permutations of n things"

Thus the total number of subsets containing r elements out of a set A containing n elements is

$$nP_r = n(n-1)(n-2) \dots (n-r+1)$$

In terms of factorial notation, we get

$$nP_r = \frac{n(n-1)(n-2)\dots(n-r+1)[(n-r)(n-r-1)\dots.3.2.1]}{(n-r)(n-r-1)\dots3.2.1}$$

$$= n\,(n-1)\,(n-2)\dots.3.2.1\,/(n-r)(n-r-1)\dots.3.2.1$$

$$= \frac{n!}{(n-r)!}.$$

***Note :** Here the order of selecting elements is important. i.e. 1st element is selected, 2nd element is selected and so on.*

Alternatively, selecting r elements out of n given elements and arranging them in a row gives rise to nP_r permutations.

We shall illustrate it by means of the following examples.

Example (1) : Find the number of different arrangements that can be made by using all the letters of the word absurd.

Solution : The word absurd contains 6 different letters. The corresponding arrays has six boxes. Hence number of possible arrangements of letters is $6P_6 = 6! = 720$.

Example (2) : Find r, if rP_4 = 24024.

Solution : We know that $rP_4 = r(r-1)(r-2)(r-3)$.

We have to factories 24024 and arrange the factors in an array having 4 boxes.

$$24024 = 24\,(1001) = 24 \times 7 \times 143$$

$$= 24\times7\times13\times11$$

$$= 14\times13\times12\times11$$

$$\therefore r(r-1)(r-2)(r-3) = 14\times13\times12\times11$$

$$\therefore r = 14.$$

Example (3) : Show that the number of permutations of n different things taken all at a time such that two particular things are not together is $(n-2)\,\underline{|n-1}$.

Solution : The number of permutations of all n things without any restriction is $nP_n = n!$

When two particular things say T_1, T_2 are tied so that they are always together can be arranged in $(n-1)\,P_{(n-1)} = (n-1)!$ ways. But in each such arrangement T_1, T_2 can interchange their positions. Hence by multiplication principle, the total number of arrangements, when two particular things are always together are $2 \times (n-1)!$

Thus the number of permutations of n things taken all at a time when two particular things are not together is

$$n! - 2\times(n-1)! = n\times(n-1)! - 2\times(n-1)!$$

$$= (n-1)!\,[n-2]$$

$$= (n-2)\,\underline{|n-1}$$

Example (4): A number of five different digits is to be formed with the help of digits 1,2,3,4,5,6,7 in all possible ways, (a) How many such numbers can be formed ? (b) How many of these are greater than 34000 ?

Solution : We have seven digits 1,2,3,4,5,6,7. Out of these just five digits are to be chosen to form five digit number.

(a) We have to fill five boxes in the array. This can be done as follows.

7	6	5	4	3
Box 1	Box 2	Box 3	Box 4	Box 5

By using extended multiplication principle the number of digits so formed is $7P_5 = 7\times6\times5\times4\times3 = 2520$.

(b) A number greater than 34000 can be formed in two ways.

(i) Box 1 is filled by a number 3

(ii) Box 1 is filled by a number by choosing any of 4 digits viz. 4,5,6,7.

Case (i) : Box 1 can be filled by 3 in only one way. Then Box 2 can be filled by any of 4 digits viz. 4,5,6,7. Hence Box 2 can be filled in 4 ways.

The remaining three boxes can be filled by choosing any 3 of remaining 5 digits in $5\,P_3$ ways.

Hence by extended multiplication principle, the number of digits that can be formed is

$1\times4\times5\,P_3 = 4\times5\times4\times3 = 240.$

Case (ii) : Box 1 can be filled by any of 4 digits viz. 4,5,6,7 in 4 ways. After this remaining 4 boxes can be filled by any of the remaining six digits in $6P_4.=6\times5\times4\times3 = 360$ ways.

By extended multiplication principle, the number of digits thus formed are

$$4\times6\,P_4 = 4\times360 = 1440.$$

As the two cases are mutually exclusive the total number of numbers greater than 34000 is their sum.

Hence 240 + 1440 = 1680

numbers can be formed.,

Example (5) : In how many ways can letters of the word MOBILE be arranged ? In many of these, the consonants occupy the even places ?

Solution : There are 6 letters in the word mobile. These can be arranged in $6P_6 = 6! = 720$ ways.

Amongst these letters M, B, L are consonants, and O, E, I are vowels. The even places are Box 2, Box 4 and Box 6 in an array of 6 boxes.

These three boxes can be filled by the three consonant in $3\ P_3 = 3! = 6$ ways. Also the remaining three boxes can be filled by the three vowels in $3\ P_3 = 3! = 6$ ways.

By using the extended multiplication principle, the number of arrangements having consonants at even places is $6 \times 6 = 36$ ways.

Example (6) : Six boys and seven girls are to be seated for a photograph in a row. Find the number of ways in which they can be seated, if no two girls sit together.

Solution : Six boys can be seated in a row in $6\ P_6 = 6! = 720$ ways.

As no two girls are to be together, there are

$$G_1 \boxed{B_1} G_2 \boxed{B_2} G_3 \boxed{B_3} G_4 \boxed{B_4} G_5 \boxed{B_5} G_6 \boxed{B_6} G_7$$

seven places for the girls to be seated as shown in the above array. They can occupy the seats in $7\ P_7 = 5040$ ways.

By the extended multiplication principle, the required number of seating arrangements is $720 \times 5040 = 3628800$.

Check your progress – 4.3 – 4.4

(1) Compute $n\ P_{n-1}$

(2) In how many ways can twelve persons of which 6 are men and six are women be seated in a row if

(i) any person may sit next to any other person

(ii) men and women must occupy alternate seats ?

(3) Find the number of different permutations of the letters in the word "group" ?

(4) In how many ways can six people be seated in a circle ?

(5) Find n if

$nP_3 = (n–lP_3) + 3(7P_2)$

(6) Find r if

$(56\ P_{r+6}) : (54P_{r+3}) = 30800 : 1$

(7) Eight papers are to be set at an examination, two of which are on computer. In how many orders, can the papers be arranged, so that the two computer papers are (i) consecutive (ii) not consecutive ?

(8) How many numbers lying between 2000 and 6000 can be formed by using the digits 1,2,4,5,7,9 ?

4.4.1 Permutations of things not all different

We shall consider a case when all the n things are not all the same, some n_1, of them alike, another n_2 of them are of same, n_3 of them are similar.

The number of district permutations in this case are given by,

$$\frac{n!}{n_1!\,n_2!\,n_3!}$$

We omit the proof and see certain examples.

Example (7) : Find the total number of distinct permutations of the letters of the word constitution.

Solution : The word constitution contains 12 letters of which T occurs 3 times, each of N, O, I occurs 2 times and the remaining C, S, and U occurs once only. Hence the total number of distinct permutations of these 12 letters is

$$= \frac{12!}{(3!)(2!)(2!)(2!)} = \frac{12\times 11!}{6\times 2\times 2\times 2} = \frac{11!}{4}$$

Example (8) : In how many ways can 4 red, 3 yellow and 2 green discs be arranged in arrow if the discs of the same colour are indistinguishable?

Solution : The total number of given discs are

4 (red) + 3 (yellow) + 2 (green)

= 9 discs.

$\therefore$ the total number of arrangements

$$= \frac{9!}{4!3!2!}$$

= 1260

Example (9): Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

(i) Do the words start with P

(ii) Do all the vowels always occur together

(iii) Go the vowels never occur together

(iv) Do the words begin with I and end with P?

Solution **: Given word is**

INDEPENDENCE

$\therefore$ **total letters = 12**

Letter	**Appearance (times)**
N	**3**
E	**4**
D	**2**
I	**1**
P	**1**
C	**1**

	12

$\therefore$ **the required number of arrangements** $= \frac{12!}{3!4!2!1!1!1!}$

= 1663200

(i)

P											

Box 1 2 3 4 5 6 7 8 9 10 11 12

$\therefore$ P in Box 1

$\therefore$ 11 places remain

$\therefore$ The required of words starting with P

$$= \frac{11!}{3!2!4!1!1!}$$

= 138600

ii) In English language,

total letters = 26
total number of vowels = 5 (a, e, i, o, u)
Now in given word,
Number of vowels are,
E – 4 times and
I – 1 times

We want, all vowels always occur together.

$\therefore$ $(EEEEI)$ we treats them as single object; for this situation; but arrange

$= \frac{5!}{4!1!}$ ways.

$\therefore$ 12 – 5 = 7 remaining objects (letters)

$\therefore$ 7 + 1 = 8

Now but in 8 object N – 3 limits & D – 2 times

$\therefore$ The required arrangement

$$= \frac{8!}{1!3!2!} x \frac{5!}{4!}$$

$= 16800$

iii) The required number of arrangements

= total number of arrangements (without restriction)

- the number of arrangements where all the vowels occur together

= 16632200 – 16800

= 16464400.

iv)

I											P

Box 1 2 3 4 5 6 7 8 9 10 11 12

$\therefore$ Remaining places = 12-2 = 10

Remaining letters	times
N	3
E	4
D	2
C	1

	10

Hence, the required number of arrangements

$$= \frac{10!}{3!4!2!1!1!}$$

= 12600.

4.5 Combination

Consider a problem of counting the number of subsets of A, such that each subset contains r elements and the set A contains n elements.

We observe that each permutation of elements of A, taken r at a time can be produced by performing the following two tasks in a sequence.

Task 1 : Choose a subset B of A containing r elements of A.

Task 2 : Choose a particular permutation of B.

Let x be the number of ways of choosing B. Task 1 can be performed in x ways; and task 2 can be performed in r! ways. Then the total number of ways of performing both the tasks which is nP_r, and by the multiplication principle it is x r! Hence

$$x\,r! = nP_r = \frac{n!}{(n-r)!}$$

$$\therefore x = \frac{n!}{r!(n-r)!}$$

When r elements are chosen without reference to the order in which they are selected we get a **combination.** Total number of r– combinations out of n things in denoted by nC_r and has formula :

$$nC_r = \frac{nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

We have the following results regarding combinations.

(i) $nC_r = nC_{n-r}$ (usually called as complementary combinations)

verbally selecting r things out of n given things is equivalent to rejecting (n – r) things.

Proof : R.H.S. = $nC_{n-r} = \dfrac{n!}{(n-r)!(n-n+r)!}$ [by formula]

$$= \frac{n!}{(n-r)!r!}$$

$$= nC_r$$

(ii) nCn = 1

Selecting n things out of n given things can be done only in 1 way.

(iii) nCo = 1

Selecting nothing out of n given things can also be done in only one way.

(iv) $nC_r + nC_{r-1} = (n+l)C_r$

Proof : L.H.S. = $\dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)!}$

$$= \frac{n!}{r!(n-r+1)!}[n-r+1+r]$$

$$= \frac{n!(n+1)}{r!(n+1-r)!} = \frac{(n+1)!}{r!(n+1-r)!}$$

$$= (n+l)C_r$$

We shall illustrate the use of these results in the following examples :

Example (1) : Find r, if $21C_r = 21C_{3r-3}$

Solution : By complementary combinations,

$$21C_r = 21C_{21-r}$$

$$\therefore \text{ By data } 21C_{21-r} = 21C_{3r-3}$$

$$\therefore 21 - r = 3r - 3$$

$$\therefore 24 = 4r$$

$$\therefore r = 6$$

Note : $21C_r = 2IC_{3r-3}$ *may give r* = 3 r – 3

and $r = \frac{3}{2}$ which is absurd.

Example (2) : Find n, if nC_4, nC_5 and nC_6 are in A.P.

Solution : By data $nC_4 + nC_6 = 2\,(nC_5.)$

i.e. $\frac{n!}{4!(n-4)!} + \frac{n!}{6!(n-6)!} = \frac{2(n!)}{5!(n-5)!}$

i.e. $\frac{n!}{4!(n-4)(n-5)(n-6)!} + \frac{n!}{6 \times 5 \times 4!(n-6)!} = \frac{2(n!)}{5 \times 4!(n-5)(n-6)!}$

Canceling $\frac{n!}{(n-6)!4!}$ through out, we get

$$\frac{1}{(n-4)(n-5)} + \frac{1}{30} = \frac{2}{5(n-5)}$$

L.C.M. is 30 (n – 4) (n – 5)

$$\therefore 30 + (n-4)(n-5) = 12(n-4)$$

$$\text{i.e. } n^2 - 9n + 50 = 12n - 48$$

$$\text{i.e. } n^2 - 21n + 98 = 0$$

$$\text{i.e. } (n-14)(n-7) = 0$$

$$\therefore n = 7, 14.$$

Example (3) : Find r, if $14\,C_s + 14\,C_6 + 15\,C_7 + 16\,C_8 = 17\,C_r$.

Solution : We use the identity

$$nC_r + nC_{r-1} = (n+l)C_r$$

Consider, $14\,C_5 + 14\,C_6 = (14+1)\,C_6$ n = 14, r = 6

$= 15C_6$

Next, $15\,C_6 + 15\,C_7 = (15+1)\,C_7$ n = 15, r = 7

$= 16C_7$

Finally L.H.S. = $16\,C, + 16\,C_8 = (16+1)\,C_8$ n = 16, r = 8

$= 17C_8$

By data $17\,C_8 = 17\,C_r$

$\therefore r = 8$ ......(A)

or , by using complementary combination

$$17\,C_{17-8} = 17C_r$$

i.e. $17C_9 = 17C_r$

$\therefore r = 9$ ...(B)

Thus, from (A) and (B) we get r = 8 or 9

Example (4) : From a pack of 52 playing cards, a hand of 5 cards is drawn. Find in how many ways such a hand can be drawn.

Solution : Out of 52 playing cards a hand of 5 cards is drawn at random.

This can be done in 52 C_5. ways.

$$= \frac{52P5}{5!}$$

$$= \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 52 \times 51 \times 10 \times 49 \times 2$$

= 2598960 ways.

Example (5) : A valid computer password consists of four character, the first of which is chosen from the set A = {p, q, r, s, t} and the remaining three characters are chosen from the English alphabet or digits chosen from the set

T = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}.

How many different passwords are there ?

Solution : A password can be constructed by performing the following two tasks T_1 and T_2 in succession.

Task T_1 : Choose a starting letter from the set A.

Task T_2 : Choose a sequence, three characters of letters or digits (repetition is allowed)

Task T_1 can be performed in 5 C_1 =5 ways.

Since there 26 alphabets and 10 digits that can be chosen for each of the remaining characters, and since repetition is allowed T_2, can be performed in $(36)^3$ = 45656 ways. By the multiplication principle, number of different passwords is

5(45656) = 233280.

Example (6) : How many seven-person committees can be formed each containing three female members from an available set of 20 female and four male members from an available set of 30 males ?

Solution : A committee can be formed by performing the following two tasks in succession :

Task T_1 : Choose 3 female member from a set of 20 females.

Task T_2 : Choose 4 males from a set of 30 males.

Here order does not matter in individual choices, so we are merely counting the number of possible subsets. Thus task T_1 can be performed in 20 C_3 $= \frac{20 \times 19 \times 18}{3 \times 2 \times 1}$ = 1140 ways.

and task T_2 can be performed in

$$30C_4 = \frac{30 \times 29 \times 28 \times 27}{4 \times 3 \times 2 \times 1} = 27405 \text{ ways.}$$

By the multiplication principle, there are (1140) (27405)= 31, 241, 700 different committees.

Check your progress : 4.5

(1) Given that $15\ C_r = 15\ C_{r+1}$ find $r\ C_4$.

(2) Prove that $5\ C_1 + 5\ C_2 + 5\ C_3 + 5\ C_4 = 30$.

(3) Find n, if $n\ C_6 : (n-3)\ C_3 = 33 : 4$.

(4) A person has 12 friends of whom 8 are relatives. In how many ways can he invite 7 guests so that 5 of them are relatives ?

(5) An urn contains 15 balls, eight of which are red and 7 are black. In how many ways can five balls be chosen so that

(a) all five are red,
(b) all five are black,
(c) two are red and three are black,
(d) three are red and two are black.

(6) Find the total number of
(i) rectangles,
(ii) squares on a chessboard.

(7) A person has 4 English and 16 Marathi books. He wants 5 of these to be bound. In how many ways can he select the 5 books so as to include (i) exactly 3 English books (ii) at least 3 English books (iii) at most 3 English books ?

(8) Prove that $nC_r = \frac{n-r+1}{r}\ (nC_{r-1})$

4.6 SUMMARY

Permutations and Combinations means the proper arrangement of given thing. If there is some conditions then it is permutation otherwise combination. Formula for Permutation is in Factorial Notation. We can use these formula for solving examples.

4.7 CHECK YOUR PROGRESS - ANSWERS

4.2

(1) n{H, T} = 2, 2^4 sequences.

(2) $4^6 = 4096$ sequences.

(3) $9 \times 8 \times 7 = 504$ ways.

(4) $6 \times 5 \times 4 \times 3 = 360$ ways.

4.3 and 4.4

(1) n!

(2) (i) 12! = 479, 001, 600

(ii) $2(6!)^2$ = 1, 036, 800

(3) 5! = 120

(4) 1×5! = 120

(5) n = 8

(6) r = 41

(7) (i) 2×7! = 10080

(ii) 8 ! – 2 × 7 ! = 30240

(8) 180

4.5

(1) 35

(3) 11

(4) $8\,C_5 \times 4\,C_2 = 336$

(5) (a) $8\,C_5 = 56$

(b) $7\,C_5 = 21$

(c) $8\,C_2 \times 7\,C_3 = 980$

(d) $8C_3 \times 7C_2 = 1176$

(6) (i) $(6^2)^2 = (36)^2 = 1296$

(ii) 204

(7) (i) $4\,C_3 \times 16\,C_2 = 480$

(ii) $4\,C_3 \times 16\,C_2 + 4\,C_4 \times 16\,C_1 = 496$

(ii) $16C_5 + 16C_4 \times 4C_1 + 16C_3 \times 4C_2 + 16C_2 \times 4C_3 = 15840$

(8)

$$nCr = \frac{n!}{(\text{n - v!r!})}$$

$$nCr - 1 = \frac{n!}{(n-r+1)!(r-1)}$$

$$= \frac{\text{n! x (r)}}{\text{(n - r)!(n - r + 1(r - 1)}}$$

$$\therefore = \frac{n!}{(n-n!r!)} x \frac{(r)}{(n-r+1)}$$

$$= nCrx \frac{r}{(h-r+1)}$$

$$\therefore nCr = \left(\frac{n-r+1}{r}\right)(nCr-1)$$

Hence the proof.

4.8 QUESTIONS FOR SELF – STUDY

1) Evaluate 1) $\frac{30!}{27!}$ 2) $\frac{12!}{6!6!}$

2) Find the number of Permutations obtained by arranging all letters of the word “COMBINATION”

3) From a group of 10 students, how many committees can be formed consisting of either 2, 3 or 4 students?

4) In how many ways, 3 cards of same suit can be taken from a pack of playing cards?

5) Find the value of n if $nC_4 = 5.np_1$

4.9 SUGGESTED READINGS

1. *Pre-degree Mathematics* by Vaze, Gosavi

2. *Discrete Mathematical Structures for Computer Science* by Bernard Kolman and Robert C Busby

◆◆◆

NOTES

CHAPTER 5

Linear Equations

5.0 OBJECTIVES

After studying the basic concepts such as determinants, matrices, Cramer's rule you can very well use the following

- solving homogeneous equations
- using Cramer's rule
- solving non-homogeneous equations
- testing consistency of equations
- perform addition, subtraction and multiplication operations with matrices.

5.1 INTRODUCTION

Before we begin our discussion with linear equations, we shall consider certain mathematical tools like determinants and matrices in brief. These concepts are very useful in solving the general linear equations. At the outset we shall restrict ourselves to the case when the general linear equations have a unique solution.

5.2 DETERMINANT

The four elements a, b, c and d arranged in two arrays containing two elements and having a definite value is called a **determinant of second order.**

e.g. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is a determinant of 2^{nd} order,

It has two rows a b and c d. Also it has two columns $\begin{matrix} a \\ c \end{matrix}$ and $\begin{matrix} b \\ d \end{matrix}$.

We measure rows from top to bottom and measure columns from left to right.

(ad–bc) is called the value or expansion of a determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$..

$$\text{Thus}\ \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{array}{l} \rightarrow 1^{st}\text{ row} \\ \rightarrow 2^{nd}\text{ row} \end{array} = (ad - bc)$$

$$\begin{array}{cc} \downarrow & \downarrow \\ 1^{st}\text{ column} & 2^{nd}\text{ column} \end{array}$$

We consider some simple examples.

Examples : Evaluate the following :

(1) $\begin{vmatrix} 2 & 4 \\ -3 & 4 \end{vmatrix} = 2\,(4) - (-3)\,(4) = 8 + 12 = 20.$

(2) $\begin{vmatrix} a & -x \\ 2 & -3 \end{vmatrix} = a\,(-3) - (2)\,(-x) = 2\,x - 3a..$

(3) $\begin{vmatrix} 2 & 4 \\ 3 & 0 \end{vmatrix} = 2\,(0) - 3\,(4) = -12..$

5.2.1 Determinant of 3rd order

A determinant of 3rd order has 3 arrays each containing 3 elements. Thus it has 3 rows and 3 columns. The expansion of a determinant of 3rd order is given by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1\,(b_2\,c_3 - b_3\,c_2) - b_1\,(a_2\,c_3 - a_3\,c_2) + c_1\,(a_2\,b_3 - a_3\,b_2)$

Examples : Evaluate the following determinants

(4) $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1\,(1) - 1\,(-1) = 2$

(5) $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 1 \begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & x \end{vmatrix} + 1 \begin{vmatrix} 1 & x \\ 1 & 1 \end{vmatrix}$

$= x^2 - 1 - (x-1) + 1\,(1-x)$

$= x^2 - 2x + 1$

(6) $\begin{vmatrix} 1 & 2 & 3 \\ 8 & 4 & 6 \\ 4 & 4 & 3 \end{vmatrix} = 1 \begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 8 & 6 \\ 4 & 3 \end{vmatrix} + 3 \begin{vmatrix} 8 & 4 \\ 4 & 2 \end{vmatrix}$

$= (12 - 12) - 2\,(24 - 24) + 3\,(16 - 16) = 0$

5.2.2 Consistency of Equations :

So far we have seen the method to solve two simultaneous' equations in two unknowns and three simultaneous equations in three unknowns by using Cramer's rule.

Now consider a system of three equations in two unknowns.

$$a_1\,x + b_1\,y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$a_3\,x + b_3\,y + c_3 = 0$$

It is not always possible to solve these equations and get the values of x and y.

If these equations have a common solution we say that the **equations** are **consistent.**

The condition that these equations are consistent is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

When the condition is fulfilled, any two of the above equations can be solved to get values of x and y. i.e. solution of the system. Consider an example, which will give us the method.

Example (7) : **Test whether the following equations are consistent.**

$2x + 3y + 4 = 0$, $x + 2y + 3 = 0$, $3x + 4y + 5 = 0$

If they are consistent solve them.

Solution : Consider, the determinant for the consistency condition.

$$D = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} \quad \begin{matrix} \text{Perfor min g} \\ R_1 - 2R_2 \\ R_3 - 3R_2 \end{matrix} \qquad D = \begin{vmatrix} 0 & -1 & -2 \\ 1 & 2 & 3 \\ 0 & -2 & -4 \end{vmatrix}$$

$$= -1 \begin{vmatrix} -1 & -2 \\ -2 & -4 \end{vmatrix} = -4 + 4 = 0 \qquad \text{i.e. } D = 0$$

$\therefore$ The given equations are consistent.

To solve them, we rewrite the first two equations as

$$2x + 3y = -4$$
$$x + 2y = -3$$

By Cramer's rule, we have

$$x = \frac{\begin{vmatrix} -4 & 3 \\ -3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = \frac{-8+9}{4-3} = 1 \qquad y = \frac{\begin{vmatrix} 2 & -4 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = \frac{-6+4}{4-3} = -2$$

Hence the equations are consistent and has the solution $x = 1$, $y = -2$.

Example (8) : Solve the following equations by Cramer's rule $3x + 4y - 7 = 0$, $7x - y - 6 = 0$.

Solution : We shall rewrite the equations in the form

$$3x + 4y = 7$$
$$7x - y = 6$$

By Cramer's rule, we get

$$x = \frac{\begin{vmatrix} 7 & 4 \\ 6 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 7 & -1 \end{vmatrix}} = \frac{-7-24}{-3-28} = 1$$

$$y = \frac{\begin{vmatrix} 3 & 7 \\ 7 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 7 & -1 \end{vmatrix}} = \frac{18-49}{-3-28} = 1$$

Hence x = 1, y = 1 is the solution.

The same Cramer's rule can be extended to three linear equations in three unknowns.

Consider, a system of linear equations

$$a_1 x + b_1 y + C_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_3$$

$$a_3 x + b_3 y + c_3 z = d_3$$

The solution of the system is given by $x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$ if $D \neq 0$.

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\qquad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $\qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

The example solved below will illustrate the technique involved.

Example (9) : Solve the following system of linear equations

x + y + 4z – 4 = 0, 2x + 3y + 6z – 5 = 0, 2y + z – 3x + 4 = 0.

Solution : The given system of equation can be rewritten as

$$x + y + 4z = 4$$

$$2x + 3y + 6z = 5$$

$$-3x + 2y + z = -4$$

Using Cramer's rule we solve the given system of equation –

$$\therefore \quad D = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 3 & 6 \\ -3 & 2 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 3 & 6 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 6 \\ -3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix}$$

= 1 (3x1 – 2x6) -1 (2x1-(-3)x6) +4 (2x2-(-3)8)

= 1 (3-12) -1 (2+18) + 4 (4+9)

=1x-9 – 1x20 +4 x 13

= -9 – 20 + 52

= -29 + 52

= 23

$$Dx = \begin{vmatrix} 4 & 1 & 4 \\ 5 & 3 & 6 \\ -4 & 2 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 3 & 6 \\ 2 & 1 \end{vmatrix} -1\begin{vmatrix} 5 & 6 \\ -4 & 1 \end{vmatrix} +4\begin{vmatrix} 5 & 3 \\ -4 & 2 \end{vmatrix}$$

= 4 (3x1 – 2x6) -1 (5x1 – (-4) x 6) +4 (5x2 – (-4)x3)
= 4 (3-12) -1 (5+24) +4 (10-12)
= 4x – 9 -1 x 29 + 4x 22
=-36 – 29 + 38
= 23

$$\text{Weget, } Dy = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 5 & 6 \\ -3 & -4 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 5 & 6 \\ -4 & 1 \end{vmatrix} -4\begin{vmatrix} 2 & 6 \\ -3 & 1 \end{vmatrix} +4\begin{vmatrix} 2 & 5 \\ -3 & -4 \end{vmatrix}$$

= 1 (5 x1 – (-4) x 6) -4 (2 x 1 – (-3) x 6) +4 (2 x (-4) – (-3) x5)
= 1 (5 + 24) -4 (2 + 18) +4 (-8 + 15)
= 1 x 29 – 4 x 20 + 4 x 7
= 29 – 80 + 28
= -23

$$\text{Lastly } Dz = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 3 & 5 \\ -3 & 2 & -4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 5 \\ 2 & -4 \end{vmatrix} -1\begin{vmatrix} 2 & 5 \\ -4 & -4 \end{vmatrix} +4\begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix}$$

= 1 (3 x -4 – 5 x 2) -1 (2x – 4 – (-3) x 5) +4 (2x2- (-3) x 3)
= 1 (-12-10) -1(-8+15) + 4(4 + 9)
= 1 x -22 – 1 x 7 + 4 x 13
= -22 – 7 + 52
= 23

Now by Cramer's Rule, we have,

$$x = \frac{Dx}{D} = \frac{23}{23} = 1$$

$$y = \frac{Dy}{D} = \frac{-23}{23} = -1$$

$$z = \frac{Dz}{D} = \frac{23}{23} = 1$$

Hence solution in (x, y, z) = (1, -1, 1)

5.2.3 Cramer's rule

Consider the system of two simultaneous linear equations in two unknowns.

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

The solution of these equations is given by Cramer's rule as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \qquad y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

We shall illustrate this by means of a simple example.

Check your progress – 5.2

(1) Fill in the blank by choosing correct value from the bracket

(i) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$ = (2,3,4,–14)

(ii) $\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}$ = (3, 21, –21, 7)

(iii) $\begin{vmatrix} 2 & 5 & 7 \\ 3 & -1 & 1 \\ 5 & 4 & 8 \end{vmatrix}$ = ... (0,5,7,3)

(iv) If 2x + 5y = 5 (x = 1, y = 1; x = y = 0; x = –1,

and –x + 7y = 6 y = 1; x = 2, y=l)

then x =........., y =..........

(v) If $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$

and $\frac{1}{x}-\frac{1}{y}=\frac{1}{4}$ x = ..., y = ...

(x = 4, y = 4; x = $\frac{1}{4}$ –, y = $\frac{1}{4}$ –; x = 2, y = – 2; x = 1, y = 1)

(2) Test for consistency the following equations. If found consistent, obtain the values of x, y.

(i) x + y = 4

3x + 4y = 7

llx+10y = 7

(ii) 4x – 3y + 1 = 0

7x – 8y + 10 = 0

x + y–5 = 0

(3) Show that

(i) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

(ii) (b + c) x + ay +1 = 0

(c + a) x + by +1 = 0

(a + b) x + cy +1 = 0

are consistent.

5.3 MATRIX

We shall now consider another entity called a matrix, which has numerous applications in almost every field.

Definition : A **matrix** is a rectangular array of numbers arranged in m rows and n columns as

$$A=\begin{bmatrix} a_{11} & a_{12} & .... & a_{In} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{ml} & a_{m2} & .... & a_{mn} \end{bmatrix}$$

The i th row of A is [a_{i1}., a_{i2}.. .... a_{in}] $(1 \le i \le m)$

and j^{th} column of A is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ : \\ a_{mj} \end{bmatrix}$ $(1 \le j \le n)$

The **order of matrix A** is said to m × n, as it has m rows and n columns. The element common to i^{th} row and j^{th} column is a.ij.. It is (i, j)th entry of A.

We briefly write $A = (a_{ij})_{m \times n}$

e.g. (i) $A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \end{pmatrix}$ is a matrix of order 2×3.

(ii) $B = \begin{pmatrix} 2 & -1 \\ 3 & 2 \\ 4 & 3 \end{pmatrix}$ is a matrix of order 3×2.

(iii) $C = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ -1 & 3 & 7 \end{pmatrix}$ is a matrix of order 3×3.

The elements a_{11}, a_{22}, a_{33} .....a_{nn} form the **principal diagonal** of a matrix $(a_{.ij.})n \times n$

$a_{11} + a_{22},–+ a_{33} + ... + a_{nn}$ is called the **trace** of the matrix $A = (a_{.ij.})_{n \times n}$

5.3.1 Types of Matrices :

We shall now consider certain types of matrices.

(1) Row matrix : It is a matrix having a single row.

e.g. $[2\ 3\ 4\ –5]_{l \times 4}$

(2) Column Matrix : It has a single column.

e.g. $\begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$ is a column matrix of order 4×1

(3) Null Matrix : It is a matrix of any order, whose all the elements are zero.

(4) Square Matrix : It is matrix having number of rows and number of columns equal.

e.g (i) $\begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix}$ is a square matrix of order 2 × 2 or 2 rowed square matrix.

(ii) $\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 1 & 3 \end{pmatrix}$

are square matrices of order 3.

(5) **Diagonal Matrix :** It is a square matrix whose non-zero elements are along the diagonal $a_{ii} \neq 0$

e.g. $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ = diag(4,2), $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ = diag (2, –1, 3)

are diagonal matrices of orders 2, 3 respectively.

(6) **Scalar Matrix** : It is a diagonal matrix having all elements equal. $a_{ii} = k$ for $i = j$

e.g. diag (k k k) = $\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$

(7) **Unit Matrix** : It is a scalar matrix, having all diagonal elements equal to 1.

eg. $I_{2\times2}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(8) **Triangular Matrix** : It is a square matrix in which all the elements above or below the principal diagonal are zero.

e.g. $A=\begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 4 & 7 & 8 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & -2 \\ 0 & 0 & 8 \end{bmatrix}$

are both triangular matrices.

(9) **Singular Matrix** : It is a square matrix whose determinant is equal to zero.

e.g. (i) $A =\begin{pmatrix} 2 & 4 \\ 6 & 12 \end{pmatrix}$ Consider, $|A| = \begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix}$ =24–24 = 0.

(ii) $B=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 5 & 7 & 3 \end{bmatrix}$ Consider, $|B| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 5 & 7 & 3 \end{vmatrix}$

Performing $R_3 - R_2$, we get

$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 0$ because $R_1 = R_3$, $|B| = 0$

Thus both matrices A and B are singular.

(10) **Non-singular matrix** : It is a square matrix, whose determinant is non-zero.

e.g. (i) $C = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$, Consider, $|C| = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}$ =1+6 = 7≠0

$\therefore\ |C| \neq 0.$

(ii) $D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ -1 & -2 & 5 \end{bmatrix}$, Consider, $|D| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ -1 & -2 & 5 \end{vmatrix}$

Performing $R_2 - 4R_1$, $R_3 + R_1$ we get

$$|D| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -12 \\ 0 & 0 & 8 \end{vmatrix} = \begin{vmatrix} -3 & -12 \\ 0 & 8 \end{vmatrix} = -24 \neq 0$$

$\therefore \quad |D| \neq = 0.$

Thus both C and D are non-singular matrices.

5.3.2 Algebra of Matrices :

We shall briefly consider the following operations with matrices :

(I) Transpose of a Matrix : It is a matrix obtained by interchanging rows and columns.

The transpose of A is denoted by A or A^T.

Let $A = (a_{ij})_{m\times n}$ then $A' = A^T = (a_{ji})_{n\times m}$

e.g. $A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 0 & -5 \end{pmatrix}_{2\times 3}$, $\quad A' = A^T = \begin{pmatrix} 2 & 4 \\ 3 & 0 \\ 4 & -5 \end{pmatrix}_{3\times 2}$

are transposes of each other.

(II) Multiplication by a Scalar : Let $k \in R$ and $A = (a._{ij})_{m\times n}$- Then $kA = (ka_{ij})_{m\times n}$ i.e. every element of the matrix A is multiplied by the scalar k.

e.g. $A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 2 & 0 \end{pmatrix}$ then $3A = \begin{pmatrix} 1x3 & 2x3 & 5x3) \\ 3x3 & 2x3 & 0x3 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 15 \\ 9 & 6 & 0 \end{pmatrix}_{2\times 3}$

(III) Equality of Matrices : Two matrices A and B of the same order are said to be equal, if the corresponding elements are equal.

i.e. $A = (a._{ij}.)_{m\times n}$ $\quad B = (b_{ij}.)_{m \times n}$

$A = B$, if and only if $a_{ij}. = b_{ij}.$ for all i, j.

(IV) Addition of two matrices : Two matrices of **the same order** can be added.

Let $A^{:} = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m \times n}$

$A + B = (a_{ij} + b_{ij})_{m \text{ x } n}$

i.e. corresponding elements of the two matrices are added.

e.g. $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & -7 & 1 \end{pmatrix}_{2\times 3}$ $\quad B = \begin{pmatrix} 3 & 4 & -1 \\ 0 & 5 & 7 \end{pmatrix}_{2\times 3}$

$$A + B = \begin{pmatrix} 1+3 & 2+4 & 4-1) \\ 2-0 & -7+5 & 1-7 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 3 \\ 2 & -2 & 8 \end{pmatrix}_{2\times 3}$$

(V) Subtraction of Matrices : Two matrices of the **same order** can be subtracted.

$A–B = (a_{ij}..–b_{ij})_{m\times n}$

Refer to matrices A,B in.(IV).above.

$$A-B = \begin{pmatrix} 1-3 & 2-4 & 4-(-1) \\ 2-0 & -7-5 & 1-7 \end{pmatrix} = \begin{pmatrix} -2 & -2 & 5 \\ 2 & -12 & -6 \end{pmatrix}_{2\times 3}$$

(VI) Multiplication of Matrices :

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{jk})_{n\times p}$ be two matrices. The product AB will be defined if the **number of columns of A = number of rows of B.** In the above case AB is defined and is of order m×p.

Let $C = (c_{ik})_{m\times p} = AB$

$$\text{Then, } c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1}b_{1k} + a_{i2}\,b_{2k} + \dots + a_{in}b_{nk}$$

$$\text{e.g. } A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & -2 & 0 \end{pmatrix}_{2\times 3}, \quad B = \begin{pmatrix} -2 & 3 \\ -4 & 5 \\ -1 & 6 \end{pmatrix}_{3\times 2}$$

The product AB is defined and is of order 2×2. Let $AB = C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}_{2\times 2}$

$$c_{11} = (2)(-2)+3(-4)+4(-1)=-20$$

$$c_{12} = 2(3)+3(5)+4(6)=45$$

$$c_{21} = (-1)(-2)+(-2)(-4)+(0)(-1)=10$$

$$c_{22} = -1(3)+(-2)5+0(6)=-13$$

$$\therefore AB = \begin{pmatrix} -20 & 45 \\ 10 & -13 \end{pmatrix}_{2\times 2}$$

Here the product BA is also defined and is of order 3 ×3.

$$\text{Let } \quad BA = \begin{pmatrix} -2 & 3 \\ -4 & 5 \\ -1 & 6 \end{pmatrix}_{3\times 2} \begin{pmatrix} 2 & 3 & 4 \\ -1 & -2 & 0 \end{pmatrix}_{2\times 3} = D = (d_{ij})_{3\times 3}$$

$$d_{11} = -2(2)+3(-1)= -7$$

$$d_{12} = -2(3) + 3(-2)= -12$$

$$d_{13} = -2(4)+3(0)= -8$$

$$d_{21} = -4(2) + 5(-1)= -13$$

$$d_{22} = -4(3) + 5(-2)= -22$$

$$d_{23} = -4(4) + 5(0)= -16$$

$$d_{31} = -1(2) + 6(-1)= -8$$

$$d_{32} = -1(3) + 6(-2)= -15$$

$$d_{33} = -1(4) + 6(0)= -4$$

$$\therefore BA = \begin{pmatrix} -7 & -12 & -8 \\ -13 & -22 & -16 \\ -8 & -15 & -4 \end{pmatrix}$$

$$= -\begin{pmatrix} 7 & 12 & 8 \\ 13 & 22 & 16 \\ 8 & 15 & 4 \end{pmatrix}_{3\times 3}$$

Note : *Even though AB and BA are both defined, they are of different order.*

Hence $AB \neq BA$.

Check your progress : 5.3

(1) Fill in the blanks by choosing appropriate type of a matrix

(i) (2 3 4) is a _____ matrix.

(ii) $\begin{pmatrix} 7 \\ 3 \\ 4 \end{pmatrix}$ is a ________ matrix.

(iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a______matrix.

(iv) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a _________matrix.

(v) $\begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{pmatrix}$ is a______matrix.

(vi) $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ is a______matrix.

(2) State whether true or false.

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

(ii) $3\begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 4 & 3 \\ 9 & 0 & 12 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 4 & -1 \\ 3 & 7 & 11 \end{pmatrix}$ and $\begin{pmatrix} 1 & 4 \\ 4 & 7 \\ -1 & 11 \end{pmatrix}$

can be added / subtracted.

(iv) $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} a & 4 \\ b & 1 \end{pmatrix}$ then a = 2, b = 3

(v) $A=\begin{pmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \end{pmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -2 & 2 \\ 5 & -4 \end{bmatrix}$ then $AB = \begin{pmatrix} -20 & 20 \\ 14 & -4 \end{pmatrix}$

(vi) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}$

are compatible for multiplication.

5.4 LINEAR HOMOGENEOUS EQUATIONS

Consider a system of m equations in n unknowns viz. $x_r\ x_2, \ldots\ldots\ x_n$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{ln}x_n = 0$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = 0$

:

:

$a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\ldots+a_{mn}x_n = 0$

The system can be written in a matrix form as AX = 0

where A= $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{ml} & a_{m2} & & a_{mn} \end{pmatrix}_{m\times n}$

and X = $\begin{pmatrix} x_1 \\ x_2 \\ : \\ x_n \end{pmatrix}_{n\times 1}$ and O= $\begin{pmatrix} 0 \\ 0 \\ : \\ 0 \end{pmatrix}_{n\times 1}$

A is of order m×n, X and O are column matrices of order n×1 each.

In general such a system has a trivial solution. i.e. $x_1 = 0 = x_2 = \ldots = x_n$ is a solution.

Under certain conditions a system may have a non-trivial solutions.

Examples : (1) Solve the following system of equations and obtain a set of non-zero solutions; if possible

x + y + z = 0 2x + 3y–z = 0 5x + 4y + z = 0

Solution : The matrix equivalent of the system is AX = B

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 5 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

[***Note*** *: We may use elementary transformations to reduce the above matrix equation. Only row transformations are permissible.*]

(i) Interchange of i^{th} row / column with i^{th} row / column and is denoted by, $R._{ij}/ C_{ij}$..

(ii) Multiply elements of (i^{th}) row / column by a scalar k and is denoted by, $k(R_i)/k(C_i.)$

(iii) Multiply elements of i^{th} row / column by k and adding it to the corresponding elements of j^{th} row / column and is denoted by, $R_j. + k(R_i.)/C_j. + k(C_i)$]

Performing $R_2 + R_1$ and $R_3 – R_1$ we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 0 \\ 4 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Here non-zero solution does not exist.

The determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 0 \\ 4 & 3 & 0 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 9–16=–7 \neq 0$$

i.e. The coefficient matrix is non-singular.

Hence only solution is given by x+y+z = 0, 3x+4y = 0, 4x+3y = 0

$$\therefore x = 0 = y = z$$

Thus system has only a trivial solution.

Example (2) : Obtain a set of non-zero solutions.

x + 2y + 3z = 0

2x + y – z = 0

3x + 3y + 2z = 0

Solution : The matrix equivalent of the system is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Performing $R_3 - R_1 - R_2$ and $R_2 - 2R_1$, we get

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -7 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The system reduces to equations

$x + 2y + 3z = 0$ and $3y + 7z = 0$

$$\therefore\ y = -\frac{7}{3}z \text{ and } x = -2y - 3z = \frac{14}{3}z - 3z = \frac{5z}{3}.$$

Let $z = 3k$. Then $x = 5k$ and $y = -7k$.

Thus for any non-zero value of k, we get infinitely many solutions of the form

$x = 5k$, $y = -7k$ and $z = 3k$

Example (2) suggests another method to get infinitely many solutions in case of two equations in three unknowns.

Let $a_1 x + b_1 y + c_1 z = 0$

and $a_2x + b_2y + c_2z = 0$

Solving these for proportional values of x, y, z we get

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

i.e. $$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

Example (3) : Solve for proportional values (or obtain infinitely many solutions of)

2x + 3y + 4z = 0

–x + y + 6z = 0

Solution : Writing the solutions in the determinant form, we get

$$\frac{x}{\begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 4 & 2 \\ 6 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}}$$

i.e. $$\frac{x}{18-4} = \frac{y}{-4-12} = \frac{z}{2+3}$$

i.e. $$\frac{x}{14} = \frac{y}{-16} = \frac{z}{5} = k \text{ say}$$

Solution is $x = 14k$, $y = -16k$, $z = 5k$, $k \in R$.

Check your progress : 5.4

Solve the following system of equations by matrix method:

(1) $2x + 3y + 4z = 0$, $x + y + z=0$, $-x + 2y + 3z = 0$

(2) $x+2y-z=0$, $3x+4y+z=0$, $4x+6y=0$

(3) $2x+y+z=0$,

$3x+2y+3z=0$ (proportional values only)

5.5 LINEAR NON-HOMOGENEOUS EQUATIONS

Consider a system of m equations in n unknowns viz. $x_1, x_2, \ldots, x_n$

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots + a_{2n} x_n = b_2$$
$$\vdots$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \cdots + a_{mn} x_n = b_m$$

The system can be reduced to a matrix form AX = B

$$\text{where } A = (a_{ij})_{m \times n},\ X = \begin{pmatrix} x_1 \\ x_2 \\ : \\ x_n \end{pmatrix}_{n\times1},\ B = \begin{pmatrix} b_1 \\ b_2 \\ : \\ b_m \end{pmatrix}_{m\times1}$$

If the coefficient matrix is non-singular then the solution exists and is unique. We shall illustrate this concept by the following examples.

Example (1) : Solve the following system of equations by matrix method,

$$\mathbf{x + y + z = 6}$$
$$\mathbf{2x - y + z = 3}$$
$$\mathbf{-x - y + z = 0}$$

Solution : The matrix equivalent of the system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} \text{ i.e. } AX = B$$

$$\text{consider,} \quad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & -1 & 1 \end{vmatrix}$$

Performing $R_2 - 2R_1$, $R_3 + R_1$, we get

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -3 & -1 \\ 0 & 2 \end{vmatrix} = -6 \neq 0$$

Here $|A| \neq - 0$ $\therefore$ A is non-singular.

$\therefore$ System has a unique solution. By using elementary row transformations, we can reduce A to a triangular matrix.

$$\text{We have} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}$$

Performing $R_2 - 2R_1$, $R_3 + R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 6 \end{pmatrix}$$

i.e. $x + y + z = 6$, $-3y - z = -9$, $2z = 6$

$\therefore$ $z = 3$, $-3y = -9 + z = -6$ $\therefore y = 2$

and $x = 6 - y - z = 6 - 2 - 3 = 1$

Hence the system has a unique solution given by

$x = 1$, $y = 2$, $z = 3$.

Example (2) : Solve the following system by matrix method,

$x + y - z = -2$

$2x - y + z = 5$

$5x - 2y + 3z = 13$

Solution : The matrix equivalent of the system is

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 5 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 13 \end{pmatrix} \text{ i.e. } AX=B$$

The coefficient matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 5 & -2 & 3 \end{pmatrix}$

Consider, $|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 5 & -2 & 3 \end{vmatrix}$

Performing $R_2 - 2R_1$, $R_3 - 5R_1$, we get

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & -7 & 8 \end{vmatrix} = \begin{vmatrix} -3 & 3 \\ -7 & 8 \end{vmatrix} = -24+21 = -3 \neq 0$$

Hence coefficient matrix is non-singular. The system has a unique solution.

We have $\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 5 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 13 \end{pmatrix}$

Performing $R_2 - 2R_1$, $R_3 - 5R_1$, we get

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & -7 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \\ 23 \end{pmatrix}$$

Performing, $R_2\left(\frac{1}{3}\right)$ we get

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -7 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 23 \end{pmatrix}$$

Now $R_3 - 7R_2$ gives

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$$

i.e. $x + y - z = -2$, $-y + z = 3$, $z = 2$

$\therefore$ $z = 2$, $y = -1$, $x = 1$

The solution is x = .1, y = – 1, z = 2.

Example (3) : Solve the following system of equations

$$\mathbf{x - y + 2z = -1}$$

$$\mathbf{2x + y + z = 1}$$

$$\mathbf{3x + 3z = 0}$$

Solution : The matrix equivalent of the system is

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

i.e. AX = B; where $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix}$

Consider, $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 3 \end{vmatrix}$

Performing $R_2 - 2R_1$ $R_3 - 3R_1$, we get

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{vmatrix} \text{ Performing } R_3 - R_2 \text{ we get}$$

$$= \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\therefore$ The matrix A is singular. Hence the solution is not unique.

Consider, the system

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Performing $R_2 - 2R_1$ $R_3 - 3R_1$, we get

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$$

Performing $R_3 - R_2$ and $R_2\left(\frac{1}{3}\right)$, we get

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

i.e. $x - y + 2z = -1$ and $y - z = 1$

The system has infinitely many solutions.

Let $y = k$ $\therefore z = k - l$ and

$x = -1 + y - 2z = -l + k - 2k + 2 = 1 - k$

The solution is

$x = 1 - k, \quad y = k, \quad z = k - 1, \; k \in R.$

By giving different values to k, we get infinitely many solutions.

Check your progress – 5.5

Solve the following system of equations,

(1) $4x + 3y - 4z = 2$

$x - y + z = 1$

$2x - y = 1$

(2) $x + y + z = \frac{6}{5}$

$2y - z = \frac{1}{5}$

$6x + y - z = 1$

(3) $x + y + z = 2$

$2x - 3y + 2z = -6$

$3x - 2y + 3z = -4$

5.6 SUMMARY

Linear Equations are the mathematical relationship between variables. Degree of this equation is one so it is called as linear equation. There are one, two, three, any number of variables you can find values for these variables. To solve system of equation we have Cramers Rule, Matrix Method.

5.7 CHECK YOUR PROGRESS - ANSWERS

5.2

(1) (i) –14 (ii) –21, (iii) 0 (iv) $x = 1, y = 1$ (v) $x = 4, y = 4$

(2) (i) inconsistent (ii) consistent, $x = 2, y = 3$

5.3

(1) (i) row (ii) column (iii) diagonal (iv) unit

(v) singular (vi) scalar

(2) (i) true (ii) false (iii) false (iv) true (v) true (vi) false

5.4

(1) $x = 0, y = 0, z = 0$

(2) $x = -3k, y = 2k, z = k, k \in R - \{0\}$

(3) $x = k, y = -3k, z = k, k \in R - \{0\}$

5.5

(1) $x = 1, y = 1, z = 1$

(2) $x = \frac{1}{5}, y = \frac{2}{5}, z = \frac{3}{5}$

(3) $x = -k, y = 2, z = -k \; k \in R$

5.8 QUESTIONS FOR SELF - STUDY

1) Find the value of a, b, c, d from following equations

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

2) If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ then

Calculate :

AB, BA and check AB = BA.

3) Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Find a) A+B b) A–B c) 3A–C

4) Find Value of x.

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$$

5.9 SUGGESTED READINGS

1. *Pre-degree Mathematics* by Vaze, Gosavi

2. *Discrete Mathematical Structures for Computer Science* by Bernard Kolman and Robert C Busby

3. *Statistical Analysis: A Computer - Oriented Approach Introduction to Mathematical Statistics by* S. P. Azen & A. A.Afifi

◆◆◆

NOTES

NOTES

CHAPTER 6

Quadratic Equations

6.0 OBJECTIVES

After studying the concept and operations with complex numbers you can very well make use of the following :

- solving the quadratic equation
- discussing the nature of roots
- formation of a quadratic equation
- relation between roots and coefficients of the quadratic equation.

6.1 INTRODUCTION

We have seen a linear expression in one variable viz ax+b. Similarly a quadratic expression in one variable is $ax^2 + bx + c$ ($a \neq 0$). By equating to zero, a quadratic expression, we get a quadratic equation. viz $ax^2+bx+c = 0$. To solve such an equation, we shall express the quadratic expression as a difference of two squares and then factories it. Before we come to the solution of such an equation we shall define complex numbers and study its elementary operations.

6.2 COMPLEX NUMBERS

We define positive square root of –1 by i i.e. $i = +\sqrt{-1}$ so that $i^2 = -1$. i is called **imaginary unit.** If a, b are real numbers that a + ib is called a **complex number.**

a is called **real part** and b is called **imaginary part** of the complex number (a + ib).

The numbers (a + ib) and (a –ib) are called **complex conjugates** of each other.

The real number $\sqrt{a^2 + b^2}$ is called modulus of (a+ib) and is denoted by $|a + ib|$.

Also $|a + ib| = \sqrt{a^2 + b^2} = |a - ib|$.

For example,

$$2 + 3i, (-1) + \sqrt{3i}, 4 + \left(\frac{-1}{11}\right)i$$

Are complex numbers.

For the complex number Z = a+ ib, a is called the real part, denoted by Rez and b is called imaginary part denoted by Imz of the complex number z.

For example,

If $Z = 2 + 5i$

Then Rez + 2 and Imz = 5.

The two complex numbers

$Z_1 = a + ib$ and $Z_2 = a - ib$

Are called as complex conjugate of each other.

Where $Z_1 = a + ib$

And $Z_2 = a - ib = Z_1$

Let $Z_1 = a + ib$ and $Z_2 = C + id$ then these two complex numbers are said to be equal if a = c and b = d

6.2.1 Algebra of Complex Numbers

(I) Addition

Let $Z_1 = a + ib$, $Z_2 = C + id$ be any two complex numbers. Then sum (addition) is defined as follows :

$Z_1 + Z_2 = (a + ib) + (C + id)$

$= (a + c) + i(b + d)$

For example –

$Z_1 = 2 + 3i$ and

$Z_2 = 5 - 2i$

Then

$Z_1 + Z_2 = (2 + 3i) + (5 - 2i)$

$= (2 + 5) + (3 - 2) i$

$= 7 + 1i$

$= (7 + i)$

(II) Subtraction –

$Z_1 = a + ib$

$Z_2 = C + ib$

then $Z_1 - Z_2 = (a + ib) - (C + ib)$

$= (a - c) + (b - d)i$

e.g.

$Z_1 = 3 + 2i$, $Z_2 = 5 + i$

then $Z_1 - Z_2 = (3 + 2i) - (5 + i)$

$= (3 - 5) + (2 - 1)i$

$= (-2) + 1i$

$= (-2+i)$

(III) Multiplication –

$Z_1 = (a + ib)$, $Z_2 = (C + id($

Then $Z_1 \text{ x } Z_2 = Z_1Z_2$

$= (a + ib) (c + ib)$

$= (ac - bd) + (ad + bc)i$

For example,

$Z_1 = 3 + i5$, $Z_2 = 2 + 6i$

Then $Z_1Z_2 = (3 + 5i) (2 + 6i)$

$= 3 (2+6i) + 5i(2+6i)$

$= 3 \text{ x } 2 + 3 \text{ x } 6i + 5i \text{ x } 2 + 5i \text{ x } 6i$

$= 6 + 18i + 10i + 30i^2C$

$= 6 + 28i + 30x - 1$ $\quad\quad \because i^2 = -1$

$= 6 + 28i - 30$

$= -24 + 28i$

(IV) Division –

$Z_1 = a + ib$, $Z_2 = C + id$

then $\frac{Z_1}{Z_2} = \frac{a+ib}{c+id}$

but

We have multiply numerator and denominator by complex conjugate of Z_2 as $Z_2 = C - id$

$$\therefore \frac{Z_1}{Z_2} = \frac{a+ib}{c=id} x \frac{c-id}{c-id}$$

$$= \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

$$= \frac{a(c-id)+ib(c-id}{c(c-id)=id(c-id)}$$

$$= \frac{ac-iad+ibc-i^2bd}{c^2-cid+cid-i^2d^2}$$

$$= \frac{ac+i(bc-ad-bdx-1}{c^2-i^2-d^2}$$

$$= \frac{ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right)+i\left(\frac{bc-ad}{c^2+d^2}\right)$$

(V) Modulus Value of Complex Number

Let $Z_1 = a + ib$

then $Z_1 =$ a + ib = $\sqrt{a^2+b^2}$

and

Complex Conjugate of Z_1 is Z_1

And $Z_1 = a - ib$

Then 1z1 = 1a+ib1 = $\sqrt{a^2+(-b)^2}$

$\sqrt{a^2+b^2}$

$\therefore 1z1 = 1z1 = \sqrt{a^2+b^2}$

For example –

$Z_1 = a+2i$, $Z_2 = 3 + 5i$ then find the modules values of given complex numbers

Solution – Given

$Z_1 \quad = 9 + 2i$

$|z1| \quad = \sqrt{(9)^2+(2)^2}$

$= \sqrt{81+4}$

$= \sqrt{85}$

And

$Z_2 = 3 - 5i$

then $|Z_2| = \sqrt{3^2 + (-5)^2}$

$= \sqrt{9+25}$

$= \sqrt{34}$

Example (1) : Express the following in the form a + ib.

(i) $(2 + 3i) + (4-7i) = (2 + 4) + i(3-7)$

$= 6-3\,i$

(ii) $(2 + 3i) + (2-3i) = (2 + 2) + i(3-3)$

$= 4 + i(0)$

(iii) $(4 + 5i)-(7-2i) = (4-7) + i(5 + 2)$

$= -3+7i$

(iv) $(4 + 5i)-(4-5i) = (4-4) + i(5 + 5)$

$= 0 + 10\,i$

(v) $(l + i)(3-i) = (l)(3) + i^2(l)(-l) + 3i-i$

$= 4 + 2i$

(vi) $(l+i)(1-i) = l^2-i^2 = 2$

(vii) $\dfrac{3+i}{2-i} = \dfrac{(3+i)(2+i)}{(2+i)(2-i)} = \dfrac{6+i^2+3i+2i}{4-i^2}$

$= \dfrac{5}{4} + \dfrac{5i}{5} = 1+i$

Example (2) : Find the modulus of the following numbers.

(i) (2 + i)(3–5i) **(ii)** $\dfrac{7+i}{1-i}$

Solution :

(i) $(2 + i)(3-5i) = 6-5i^2 + 3i-10i$

$= 6 + 5-7i = 11 - 7\,i$

$|(2+i)(3-5i)| = |11-7i| = \sqrt{121+(-7)^2}$

$= \sqrt{121+49} = \sqrt{170}$.

(ii) $\dfrac{7+i}{1-i} = \dfrac{(7+i)(l+i)}{(1+i)(1-i)} = \dfrac{7+i^2+7i+i}{1-i^2}$

$= \dfrac{6+8i}{2} = 3+4i$

$\left|\dfrac{7+i}{1-i}\right| = |3+4i| = \sqrt{9+16} = \sqrt{25} = 5.$

Here 3 is real part and 4 is imaginary part of (3 + 4 i).

Example (3) : Factorise : $x^2 + 2x + 5$.

Solution : We have $x^2 + 2x + 5$

$= x^2 + 2x + l+4 = (x+l)^2-(-4)$

$= (x+l)-(-l)(2)^2$ $\qquad i^2 = -1$

$= (x+1)^2-(2i)^2$ $\quad\quad [a^2 - b^2 = (a+b)(a-b)]$

$= (x + 1+2i)(x + 1-2i)$.

Check your progress – 6.2

(1) Fill in the blanks.

(i) the real part of (3 – 7 i) is_______

(ii) the imaginary part of (3 – 2 i) is_______

(iii) The conjugate of $(7 - \sqrt{3}\,i)$ is_____

(iv) the sum of the complex numbers $(\sqrt{3} - 5\,i)$ and $(-\sqrt{3} + 5\,i)$ is______

(v) the difference (3 + 3i)–(2–i)is_____

(vi) the product of (2 + i) and (7 – i) is______

(vii) the ratio of $\frac{2-i}{2+i}$ is______

(viii) the modulus of $(2 - 3\,i)^2$ is_______

(2) Express as a complex number with real denominator and state its real and imaginary part of $\frac{1+i}{1-2i}$

(3) Factorise : $x^2 + 2x + 7$.

6.3 SOLUTION OF A QUADRATIC EQUATION

We know that a quadratic equation is of the form $ax^2+bx+c=0$ $(a \neq 0)$

Divide by a.

$$x^2 + \frac{b}{a} \times + \frac{c}{a} = 0$$

$$\text{i.e. } x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\text{i.e. } \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

$$\text{i e } \left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0$$

$$\text{i e } \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$$

$$\therefore \text{ roots are } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Let us denote these roots by α, β.

Then the sum of the roots is

$$\alpha + \beta = \frac{-b\sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{2b}{2a} = -\frac{b}{a} = \frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

and the product of the roots is

$$\alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - \left(\sqrt{b^2-4ac}\right)^2}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{c}{a} = \frac{Cons\tan tterm}{coefficient\, of\ x^2}$$

Thus $\alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

We shall illustrate the use of formulae derived, by the following examples.

Example (1) Solve the following equations

(i) $6x^2 - 13x - 63 = 0$ **(ii) $6x^2 - 5x + \frac{1}{3} = 0$**

(iii) $9x^2 + 60x + 100 = 0$ **(iv) $256x^2 - 192x + 85 = 0$**

Solution :

(i) On comparing $6x^2 - 13x - 63 = 0$ with $ax^2 + bx + c = 0$, we get

a = 6, b = – 13, c = – 63.

The solutions are $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Here $x = \frac{13 \pm \sqrt{169+1512}}{12} = \frac{13 \pm \sqrt{1681}}{12} = \frac{13 \pm 41}{12}$

$\therefore\ x = \frac{9}{2}$ and $x = \frac{-7}{3}$ are the roots.

(ii) we have a = 6, b = — 5, c = $\frac{1}{3}$.

Given $6x^2 - 5 + \frac{1}{3} = 0$

Comparing coefficients with

$ax^2 + bx + c = 0$

We have

$\therefore\ x = \frac{5 \pm \sqrt{25-8}}{12} = \frac{5 \pm \sqrt{17}}{12}$.

$\therefore\ x = \frac{5+\sqrt{17}}{12}$ and $x = \frac{5-\sqrt{17}}{12}$ are the roots

(iii) we have a = 9, b = 60, c = 100.

Given $ax^2 + 60x + 100 = 0$

Comparing with

$ax^2 + bx + c = 0$

$\therefore\ x = \frac{-60 \pm \sqrt{3600-3600}}{18} = \frac{-60}{18} = \frac{-10}{3}$

Here both the roots are equal to $\frac{-10}{3}$.

(iv) here a = 256, b = – 192, c = 85.

The solutions are

Given $256x^2 - 192x + 85 = 0$

Comparing with

$ax^2 - bx + c = 0$

We have

The solution are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As

z x = $\dfrac{192 \pm \sqrt{(192)^2 - 4(156)(85)}}{512}$

$= \dfrac{192 \pm \sqrt{-50176}}{512}$

$= \dfrac{192 \pm 224\sqrt{-1}}{512} = \dfrac{6 \pm 7\,i}{16}$

Here the roots are $\left(\dfrac{6+7i}{16}\right), \left(\dfrac{6-7i}{16}\right)$

i.e. a pair of complex conjugates.

Example (2) : If α and β are the roots of the equation $2x^2$–5x +7 = 0, evaluate the following :

(i) $\alpha^2+\beta^2$ (ii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ (iii) $\dfrac{\alpha}{\alpha}+\dfrac{\beta}{\alpha}$

Solution : We have α, β are roots of $2x^2 - 5x + 7 = 0$.
Comparing with $ax^2 + bx + c = 0$,

Here a = 2, b = –5, c = 7 $\alpha + \beta = -\dfrac{b}{a} = \dfrac{5}{2}$,

$\alpha\,\beta = \dfrac{c}{a} = \dfrac{7}{2}$.

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\,\beta$

$= \left(\dfrac{5}{2}\right)^2 - 2x\left(\dfrac{7}{2}\right)$

$= \left[\dfrac{25}{4}\right] - \left(\dfrac{14}{2}\right)$

$= \left(\dfrac{-3}{4}\right)$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha+\beta}{\alpha\beta} = \dfrac{5/2}{7/2} = \left(\dfrac{5}{7}\right)$

(iii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{-3/4}{7/2} = -\left(\dfrac{3}{14}\right)$

Check your progress – 6.3

(1) Solve the following equations

(i) $x^2 + 4x + 4 = 0$ (ii) $x^2 - 14x + 1 = 0$

(iii) $2x^2 + 15x + 18 = 0$ (iv) $x^2 + 5x - 2444 = 0$

(2) Given that $\alpha\ \beta$ are the roots of $3x^2 + 5x - 1 = 0$.

Fill in the blanks :

(i) $\alpha + \beta =$ .... (ii) $\alpha\ \beta =$ ....

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} =$ .... (iv) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} =$ ....

(v) $(\alpha + l)\ (\beta + l) =$ ....

6.4 NATURE OF ROOTS

In example (1) of the previous article, we have seen that the roots of the quadratic equations are of four types viz (i) real and unequal (ii) rational and unequal (iii) equal (iv) pair of complex conjugates.

All these roots depend on the value of $\sqrt{b^2 - 4ac}$. We call $b^2 - 4ac$ as a discriminant of $ax^2 + bx + c = 0$ and denote it by Δ. We will classify the roots as above and write down the necessary condition in terms of Δ.

(i) real and unequal roots : $\Delta>0$

(ii) rational and unequal roots : a, b, c, are all rational and Δ is a perfect square.

(iii) equal roots (each equal to $-\frac{b}{2a}$) : $\Delta = 0$

(iv) a pair of complex conjugate roots : $\Delta < 0$

Example (1) : Determine the nature of the roots of the following equations :

(i) $x^2 + 3x + \frac{9}{4} = 0$ **(ii)** $8x^2 - 19x + 8 = 0$

(iii) $9x^2 + 9x - 4 = 0$ **(iv)** $0.3x^2 - 1.4x + 2 = 0$

Solution :

Given $x^2 + 3x + \frac{a}{4} = 0$

(i) On comparing $x^2 + 3x + \frac{9}{4} = 0$ with $ax^2 + bx + c = 0$, we have a = 1, b = 3, c = $\frac{9}{4}$,

$\therefore\ \Delta = b^2 - 4ac = 9 - 4\ (1) \left(\frac{9}{4}\right) = 0$

Hence the roots are equal and each root is $= -\frac{b}{2a} = -\frac{3}{2}$.

(ii) Given $8x^2 = 19x + 9 = 0,$

Here a = 8, b = – 19, c = 8

$\therefore \Delta = b^2 - 4ac = (-19)^2 - 4(8)(8) = (19)^2 - (16)^2$

i.e. $\Delta = 3(35) > 0$

$\therefore$ The roots are real and unequal.

Given

(iii) Here a = 9, b = 9, c = —4

$\therefore \Delta = b^2 - 4ac = 81 - 4(-4)(9) = 81 + 144 = 225 = (15)^2$

Thus Δ is a perfect square, and a, b, c are rational. Hence roots are rational and unequal.

Given

(iv) Here a = 0.3, b = – 1.4 and c = 2

$\therefore \Delta = b^2 - 4ac = (-1.4)^2 - 4(0.3)(2) = 1.96 - 2.4$

$= -0.44 < 0$

Hence roots are complex conjugates of each other.

Example (2) : Find the value of k in each of the following, given $3x^2 + 6kx + k + 2 = 0$ is a quadratic equation.

(i) The sum of roots is zero.

(ii) The product of roots is zero.

(iii) One root is reciprocal of other.

(iv) Both roots are equal.

(v) The roots are complex conjugates of each other.

Solution : Let α, β be the roots. Here $a = 3$, b = 6k, c = k + 2

Then $\alpha + \beta = \frac{-b}{a} = \frac{-6k}{3} = -2k$

and $\alpha\beta = \frac{c}{a} = \frac{k+2}{3}$

(i) $\alpha + \beta = 0$ gives $-2k = 0$ $\therefore$ k = 0.

(ii) $\alpha\beta = 0$ gives $\frac{k+2}{3} = 0$ $\therefore$ k = – 2

(iii) If one root is reciprocal of the other root, product of the roots = 1.

$\therefore \quad \alpha\beta = 1 \qquad \therefore \frac{k+2}{3} = 1 \qquad \therefore k + 2 = 3$

$\therefore \quad k = 1.$

(iv) If both roots are equal, $\Delta = b^2 - 4ac$ and $\Delta = b^2 - 4ac = \mathbf{0}$

then $36k^2 - 4(3)(k+2) = 0$

i.e. $3k^2 - (k+2) = 0$

i.e. $3k^2 - 3k + 2k - 2 = 0$

i.e. $(3k+2)(k-1) = 0$

$\therefore k = 1$ or $k = -\frac{2}{3}$

(v) The roots are complex conjugates of each other.

$\therefore \quad \Delta = b^2 - 4ac < 0$

i.e. $\quad (3k+2)(k-l) < 0$

Case (i) $\quad 3k + 2 > 0, k - 1 < 0$

i.e. $k > -\frac{2}{3}$ and $k < 1$

i.e. $-\frac{2}{3} < k < 1$.

Hence roots will be complex conjugates,

if $-\frac{2}{3} < k\ 1$.

Case (ii) $3k + 2 < 0, k - 1 > 0$

i.e. $k < -\frac{2}{3}, k > 1$

There is no common value satisfying both inequalities. Hence it is not feasible.

Check your progress – 6.4

(1) Fill in the blanks by choosing the appropriate word from (real and distinct, rational, irrational, complex conjugate, real and equal)

(i) The roots of the equation

$x^2 + 5x + 1 = 0$ are ....

(ii) The roots of the equation

$9x^2 + 12x + 4 = 0$ are ....

(iii) The equation $x^2 + x + 1 = 0$ has .... roots.

(iv) The equation $x^2 - 8 = 0$ has .... roots.

(v) The roots of the equation $2x^2 + x - 3 = 0$ are ....

(2) State whether true or false, if false correct it.

(i) One root of the equation $10x^2 - 29x + k + 4 = 0$ is the reciprocal of the other, then $k = 6$.

(ii) The sum of the roots of the equation $x^2 - 6x + 3kx - 3k = 0$ is zero, then $k=1$.

(iii) The equation $2x^2 - 6x + k = 0$ has equal roots, then $k = 6$.

(iv) The equation $2x^2 + 3hx + 18 = 0$ has complex conjugate roots,

if $-4 < h < 4$.

6.5 QUADRATIC EQUATION WITH GIVEN ROOTS

If α, β are the roots of the equation $ax^2 + bx + c = 0$, we have

$\alpha+\beta = -\frac{b}{a}, \ \alpha\ \beta = \frac{c}{a}$

$a \neq O$ Since divide quadratic equation by a.

$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0$ i.e. $x^2 - \left(-\frac{b}{a}\right) \times + \left(\frac{c}{a}\right) = 0$

i.e. $x^2 - (\alpha + \beta)\ x + \alpha\beta = 0$

Hence x^2 – (sum of roots) x + product of the roots = 0

This gives us the formula to obtain a quadratic equation with given roots.

***Note** : (i) if (α + i β) is one root of the quadratic equation, then the other root is* (α–iβ).

(ii) If one root is($\alpha + \sqrt{\beta}$), then the other root is ($\alpha - \sqrt{\beta}$).

We shall illustrate the formulation of a quadratic equation by the following examples.

Example (1) : Form the quadratic equation with real coefficients, one of whose root is

(i) $(7 + i\sqrt{3})$ (ii) $(3 - 5\sqrt{3})$

Solution :

(i) By the above note, the other root is conjugate of $(7 + i\sqrt{3})$.

Hence the roots are $(7 + i\sqrt{3})$, $(7 - i\sqrt{3})$

$$\therefore \text{Sum of the roots} = (7 + i\sqrt{3}) + (7 - i\sqrt{3}) = 14$$

$$\text{and product of the roots} = (7 + i\sqrt{3})(7 - i\sqrt{3})$$

$$= 7^2 - (i\sqrt{3})^2 = 52.$$

Hence the quadratic equation is

$$x^2 - 14x + 52 = 0.$$

(ii) The other root is $(3 + 5\sqrt{3})$.

$$\therefore \text{The roots are } (3 + 5\sqrt{3}),\ (3 - 5\sqrt{3}).$$

$$\text{Sum of the roots} = (3 + 5\sqrt{3}) + (3 - 5\sqrt{3}) = 6$$

$$\text{and product of the roots} = (3 + 5\sqrt{3})(3 - 5\sqrt{3}) = 9 - (5\sqrt{3})^2$$

$$= 9 - 75 = -66$$

Hence the quadratic equation is

$$x^2 - 6x - 66 = 0.$$

Example (2) : Find the quadratic equation whose roots are $2\alpha + 5$, $2\beta + 5$, given that $\alpha\ \beta$ are the roots of the equation $2x^2 - 3x + 5 = 0$.

Solution : Since α, β are the roots of $2x^2 - 3x + 5 = 0$,

$$= 2x\frac{3}{3} + 10$$

$$= 3 + 10$$

$$= 13$$

$$\text{we have } \alpha+\beta = -\left(\frac{b}{a}\right) = -\left(\frac{-3}{2}\right) = \frac{3}{2}$$

$$\alpha\beta = \frac{c}{a} = \frac{5}{2}$$

$$\text{Sum of new roots} = (2\alpha + 5) + (2\beta + 5)$$

$$= 2(\alpha+\beta) + 10$$

$$= 2x\frac{3}{2} + 10$$

$$= 3 + 10$$

$$= 13$$

$$\text{Product of new roots} = (2\alpha + 5)(2\beta + 5)$$

$$= 4\alpha\beta + 10(\alpha + \beta) + 25$$

$$= 4\left(\frac{5}{2}\right) + 10\left(\frac{3}{2}\right) + 25$$

$= 50.$

Hence the required equation is

$x^2 - 13x + 50 = 0.$

Example (3) : If the sum of the roots of a quadratic equation is 5 and the sum of their squares is 27, find the equation.

Solution : Let α, β be the roots of the quadratic equation.

The sum of the roots is 5. $\therefore \alpha + \beta = 5$

The sum of their square is 27 $\therefore \alpha^2 + \beta^2 = 27.$

We have $2\alpha\beta = (\alpha + \beta)^2 - (\alpha^2 + \beta^2) = (5)^2 - 27$

$= -2$

$\therefore \alpha\beta = -1$

Hence the required quadratic equation is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\therefore x^2 - 5x - 1 = 0.$

Check your progress – 6.5

(1) State whether true or false. If false write correct answer.

α, β are the roots of the equation

$3x^2 + 5x - 4 = 0$

(i) The equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ is

$3x^2 + 5x + 4 = 0$

(ii) The equation with roots $-\alpha, -\beta$ is

$3x^2 - 5x - 4 = 0$

(iii) The equation having roots $\alpha + 1, \beta + 1$ is

$3x^2 - x - 2 = 0$

(2) Fill in the blanks.

(i) One root of the equation $x^2 + px + q = 0$ is $\alpha - 2\beta i$, then the other root is ______

(ii) One root of the equation $x^2 + 3x + p = 0$ is $2 + \sqrt{3}$, then the other root is ________ and p = ____

6.6 SUMMARY

Quadratic equation means a equation with degree two. There or three variables. We can calculate value of one or two unknown variables with given methods of solving examples of quadratic equations. There are two roots α and β. There is proper relation between α and β. If value of one root is given we can calculate the other root and also the respective quadratic equation.

6.7 CHECK YOUR PROGRESS - ANSWERS

6.2

(1) (i) 3 (ii) –2 (iii) $7 + \sqrt{3}i$ (iv) 0 (v) $1 + 4i$

(vi) $15 + 5i$ (vii) $\frac{5-4i}{5}$ (viii) 13

(2) $\frac{-1+3i}{5}, -\frac{1}{5}, \frac{3}{5}$

(3) $x^2 + 2x + 7 = (x+l+\sqrt{6}i)(x+l-\sqrt{6}\ i)$

6.3

(1) (i) –2, –2 (ii) $7 + 4\sqrt{3}$, $7 - 4\sqrt{3}$

(iii) $-6, -\frac{3}{2}$ (iv)47, –52

(2) (i) $-\frac{5}{3}$ (ii) $-\frac{1}{3}$ (iii) 5 (iv) $-\frac{19}{3}$.

(v) Hint :$(\alpha+l)(\beta+l) = \alpha\beta + (\alpha + \beta) + l = -1$

6.4

(1) (i) real and distinct
(ii) real and equal
(iii) complex conjugates
(iv) irrational
(v) rational

(2) (i) True (ii) False, correct : k = 2

(iii) False, correct : $k = \frac{9}{2}$ (iv) True.

6.5

(1) (i) False, correct : $4x^2 - 5x - 3 = 0$
(ii) True
(iii) False, correct : $x^2 - \frac{1}{3}x - 2 = 0$

(2) (i) $\alpha + 2\beta i$
(ii) $2-\sqrt{3}$, p=l.

6.8 QUESTIONS FOR SELF - STUDY

1) Find (a) $z_1 z_2$ (b) $z_1 \overline{z_2}$ (c) $\overline{z_1} z_2$ (d) $\overline{z_1 z_2}$ (e) $\frac{z_1}{z_2}$

Where $z_1 = (i+i)$, $z_2 = (2-3i)$

2) Express the in form of a + ib

$$\frac{2+i}{(3+i)(1+2i)}$$

3) If $\left(\frac{1+3i}{1-1}\right)^2 = x + iy$ find x, y.

4) Find Modulas value of z = 3+2i

5) Simplify $(2+5i)^2$

6.9 SUGGESTED READINGS

1. *Mathematics and Statistics* by M. L. Vaidya and M. K. Kelkar

2. *Discrete Mathematical Structures for Computer Science* by Bernard Kolman and Robert C Busby

3. *Statistical Analysis: A Computer - Oriented Approach Introduction to Mathematical Statistics by* S. P. Azen & A. A.Afifi

◆◆◆

NOTES

CHAPTER 7

Probability

7.0 OBJECTIVES

After studying the concept of probability you can easily solve and explain problems based on the following :

- Addition theorem
- Conditional probability
- Independent events
- Multiplication theorem
- Probability model

7.1 INTRODUCTION

The theory of probability has its origin in the study of experiments with uncertain outcomes. One such popular experiment is tossing a coin or tossing a six faced dice. In case of a coin, we can almost say that either tail or head will appear. In a six faced dice, we say that any one of the six numbers {1, 2, 3, 4, 5, 6} will appear on top, when it is rolled.

7.2 DEFINITIONS

(1) The set of possible outcomes of an experiment is called a **sample space,** usually denoted by S or X. or Ω.

Every outcome is called a **sample point.**

e.g. (i) when a coin is tossed, sample points are H and T and sample space is

$S1=\{H, T\}$

(ii) When a six faced dice bearing nos. 1 to 6 is rolled, sample space is

$S2=\{1,2,3,4,5,6\}$.

(2) Any subset of a sample space is called an **event.**

e.g. getting an odd number on top of a dice, the event is = $\{1, 3, 5\} \subset S_2$

(3) When the event is a singleton set, it is called a **simple event.**

(4) The entire sample space is a **certain event.**

(5) Probability function : $P : S \rightarrow [0, 1]$

If $S = \{x_1, x_2, \ldots.. x_n\}$then P has to satisfy two conditions.

(i) $P (x_i.) \geq 0$ and (ii) $\sum_{i=1}^{n} p(x_i) = 1$.

Thus P(x) is a **non-negative fraction** lying between 0 and 1.

e.g. If S is a sample space having 10 sample points and occurrence of all them are equally likely (equi-probable space) then probability of every sample point is $\frac{1}{10}$.

(6) **Probability of an event** is the sum of the probabilities of all sample points in the event. e.g. : A pack of 52 playing cards is shuffled and a card is drawn. The probability that it is a king is $\frac{4}{52} = \frac{1}{13}$. For, let A be the event that a king is drawn. Sample space consists of 52 sample points and there are 4 kings.

$\therefore$ n (S) = 52, n (A) = 4.

n (A) means number of elements in the set A.

The probability of every sample point is $\frac{1}{52}$.

$$\therefore P(A) = \frac{1}{52} + \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{4}{52} = \frac{1}{13}$$

$$\text{Alternatively } P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

Incidentally this gives us an **formula** for

$$\text{Probability of an event A} = \boxed{P(A) = \frac{n(A)}{n(S)}}$$

(7) The **complement of the event A** is denoted by $A^{|}$ or ac and $P(A^{|})$ = l-P(A).

(8) A and B be two events.

(i) **$A \cap B$** means occurrence of both events A and B.

(ii) **$A \cup B$** means one of the events A and B may happen **or** both A and B happen together or at least one of the two events happen.

(9) Events A and B are **mutually exclusive** means if A happens, B does not happen or if B happens then A does not happen.

e.g. when a dice is thrown,

Let A={1, 3, 5} and B = {2, 4, 6} then $A \cap B = \phi$ Thus A and B are mutually exclusive events.

Also events A, A′ are always mutually exclusive.

(10) The empty set ϕ is an **impossible event.**

(11) Events A, B, C are **exhaustive** if, $S = A \cup B \cup C$

Events A, A are exhaustive events.

(12) The probability of **certain event** S is P(S) = 1

(13) If A and B are mutually exclusive events then probability of events A or B is $P(A \cup B) = P(A)+P(B)$.

We shall illustrate these definitions by the following examples :

Example (1) : Three coins are tossed simultaneously. State the sample space and mention the events A, B and $A \cap B$ where

(i) A is said to have occurred when an outcome is 2 heads and 1 tail.

(ii) B is said to have occurred when an outcome of the experiment is 2 or more heads.

Solution : The sample space S contains $2^3 = 8$ sample points. n(S) = 8.

$\therefore$ S = { HHH, TTT, HTT, THT, HHT, THH, HTH, TTH }

(i) A = {HHT, THH, HTH } n (A) = 3

(ii) B = {HHT, THH, HTH, HHH } n (B) = 4

(iii) $A \cap B$ = {HHT, THH, HTH } $\therefore$ $n(A \cap B) = 3$

Example (2) : Three unbaised coins are tossed. Find the probability of (i) getting at least two heads up (ii) getting at least one head up.

Solution : The sample space has $2^3 = 8$ sample points.

S = { HHH, HHT, HTH, THH, TTH, TTT, THT, HTT }

$\therefore$ n (S) = 8

(i) Let A_1 be the event of getting two heads or three heads.

A_1 = {HHT, HTH, THH, HHH }

$\therefore n(A_1) = 4$

$$\therefore p(A_1) = \frac{n(A_1)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

(iii) Let A_2 be the event of getting at least one head up. i.e. A_2 consists of getting 1 head, 2 heads or 3 heads. This is complementary to the event of getting no head.

Alternative method getting one head

{HTT, THT, TTH}, $p = \frac{3}{8}$

Getting two heads

{HHT, HTH, THH}, $p = \frac{3}{8}$

Getting three heads

{HHH} $p = \frac{1}{8}$

$$\therefore P(A2) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

$$= \frac{7}{8}$$

$\therefore$ A'_2 = { TTT }

$n(A'_2) = 1$.

$$\therefore \quad P(A_2) = 1 - p(A'_2) = 1 - \frac{n(A'_2)}{n(S)}.$$

$$= 1 - \frac{1}{8} = \frac{7}{8}$$

(3) Two unbaised dice are thrown in air. Find the probability in each of the following events.

(i) score is a perfect square

(ii) score is a multiple of five

(iii) score is at most five

(iv) score is a prime number or a perfect square

(v) score on each dice is the same

(vi) score on the second dice is greater than the score on the first dice.

Solution : Let x, y be the scores on the 1st and 2nd dice respectively. The sample space S can be shown as

$$S = \{ (x, y) \mid x, y = 1, 2, 3, 4, 5, 6 \}$$

$$\therefore \ n(S) = 36$$

and probability of each sample point is $\frac{1}{36}$.

(i) Let A_1 be the event that the score is a perfect square.
$A_1, = \{ (x, y) \mid x + y = 4 \text{ or } 9 \}$.

Score	**Sample points**	**Number of sample points**
x + y = 4	(1, 3), (3, 1), (2, 2)	3
x + y = 9	(3, 6), (6, 3), (4, 5), (5, 4)	4
	Total	7

Table 8.1

$$\therefore \ n(A_1) = 7$$

$$\text{and } P(A_1) = \frac{n(A_1)}{n(S)} = \frac{7}{36}$$

(ii) Let A_2 be the event that the score is a multiple of 5.
$A_2 = \{ (x, y) \mid x + y = 5 \text{ or } 10 \}$

Score	**Sample points**	**Number of sample points**
x + y = 5	(1. 4), (4, 1), (2, 3), (3, 2)	4
x + y = 10	(4. 6). (6. 4). (5. 5}	3
	Total	7

Table No. 8.2

$$\therefore \ n(A_2) = 7$$

$$\text{and } P(A_2) = \frac{n(A_2)}{n(S)} = \frac{7}{36}$$

(iii) Let A_3be the event that the score is atmost five.
$\therefore \ A_3 = \{ (x, y) \mid x + y = 2, 3, 4, 5 \}$

Score	**Sample points**	**Number of sample points**
X +y = 2	(1.1)	1
x + y = 3	(1, 2), (2, 1)	2
x + y = 4	(1, 3), (3, 1), (2, 2)	3
x + y = 5	(1. 4), (4, 1), (2, 3), (3, 2)	4
	Total	10.

Table 7.3

$$\therefore \ n(A_3) = 10 \text{ and}$$

$$P(A_3) = \frac{n(A_3)}{n(S)} = \frac{10}{36} = \frac{5}{18}$$

iv) Let A_4. be the event that score is a prime number or a perfect square

$A_4 = \{ (x, y) \mid x + y = 2, 3,4,5,7, 9,11 \}$

Score	**Sample points**	**Number of sample points**
x + y = 2	(1,1)	1
x + y = 3	(1, 2), (2, 1)	2
x + y = 5	(1, 4), (2, 3), (4, 1), (3, 2)	4
x + y = 7	(1, 6), (6, 1), (2, 5), (5, 2), (3, 4) (4, 3)	6
x + y = 11	(5, 6), (6, 5)	2
x + y = 4	(1, 3), (2, 2), (3, 1)	3
x + Y = 9	(3, 6), (4, 5), (5, 4), (6, 3)	4
	Total	22

Table 7.4

$$P(A_4) = \frac{n(A_4)}{n(S)} = \frac{22}{36} = \frac{11}{18}$$

(iv) Let A_5 be the event that the score on each dice is the same

$A_5 = \{ (x, y) \mid x = y \}$

$A_5, = \{ (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) \}$

$\therefore\ n(A_5) = 6$

$$P(A_5) = \frac{n(A_5)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

(vi) Let A_6 be the event that the score on the second dice is greater than the score on the first dice.

$$A_6 = \{ (x, y) \mid x < y \}$$

We shall list the possible sample points of A_6, as

$$\begin{aligned} A_6 = \{ & (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\ & (2, 3), (2, 4), (2, 5), (2, 6), \\ & (3, 4), (3, 5), (3, 6), \\ & (4, 5), (4, 6), \\ & (5, 6) \ \} \end{aligned}$$

$\therefore\ n(A_6) = 15$ and

$$P(A_6) = \frac{n(A_6)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

(4) Four books on Physics, six books on Mathematics and three books on Biology are to be arranged on a shelf. Find the probability that books of the same subject are

(i) together (ii) never together.

Solution : The composition of books on various subjects is as follows :

Subject	Physics	Mathematics	Biology	Total
No. of books	4	6	3	13

Table 7.5

Thus total of 13 books can be arranged on a shelf in 13 P_{13}=13! ways.

There are 3 groups of books (books on one subject forms a group)

These groups can be arranged in 3! ways.

In each arrangement within a group,

4 books on Physics can be arranged in 4! ways,

6 books on Mathematics can be arranged in 6! Ways, 3 books on Biology can be arranged in 3! ways.

Thus all the books in of the same subject can be arranged in

$$3! \times 4! \text{ x} 6! \text{ x} 3! \text{ ways.}$$

(i) Let A be the event that the books of the same subject are together.

$$n(A) = 3! \times 4! \times 6! \times 3!$$

$$= 6 \times 24 \times 6 \times 6!$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{6 \times 24 \times 6 \times 6!}{13!}$$

$$= \frac{6x24x6x6!}{61x7x8x9x10x11x12x13}$$

$$= \frac{6 \times 24 \times 6}{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7}$$

$$= \frac{1}{10010}$$

(ii) The event that books of same subject are never together is the complement of event A.(ie A')

$$\therefore P(A^1) = 1\frac{P(A)}{n(S)}$$

$$\therefore P(A) = 1 - = 1 - \left[\frac{1}{10010}\right]$$

$$= \left[\frac{10009}{10010}\right].$$

(5) A box contains 8 tickets bearing numbers 1, 2, 3, 5, 7, 8, 10. A ticket is drawn at random from the box and kept aside, then the second ticket is drawn. Find the probability that both tickets show odd numbers.

Solution : Two tickets without replacement can be drawn in 8 C_2 ways. $\therefore$ n (S) = 8 C_2.

Two odd numbered tickets out 5 odd numbered tickets can be drawn in 5 C_2 ways.

$$\text{Probability} = \frac{5c_2}{8c_2} = \frac{5 \times 4}{8 \times 7} = \frac{5}{14}$$

(6) A purse contains 4 silver coins and 5 copper coins. Another purse contains 3 silver and 4 copper coins. A purse is selected at random and a coin is drawn at random. What is the probability that it is a copper coin ?

Solution :

Purse	Silver	Copper	Total
A	4	5	9
B	3	4	7

Table 7.6

Probability of selecting a purse at random is $\frac{1}{2}$. There are two cases as :

(i) Purse A is selected and a copper coin is selected from it. Its probability is

$$= \frac{1}{2} \times \frac{5}{9} = \frac{5}{18}.$$

(ii) Purse B is selected and copper coin is drawn from it. its probability is

$$= \frac{1}{2} \times \frac{4}{7} = \frac{2}{7}$$

Two cases are mutually exclusive. Hence the required probability is the sum of these probabilities.

$\therefore$ required probability $= \frac{5}{18} + \frac{2}{7} = \frac{35+36}{126} = \frac{71}{126}$

(7) A bag contains 5 white and 4 black balls and another contains 4 white and 6 black balls. One ball is transferred from 1st bag to the 2nd bag, and a ball is drawn from the 2nd bag. Find the probability that the ball drawn is black.

Solution :

Bag	White	Black	Total
A	5	4	9
B	4	6	10

Table 7.7

We shall consider two cases depending upon the colour of the ball being transferred. The composition of bag changes accordingly.

Case (i) : Let us suppose that a white ball is transferred from bag A to a bag B.

Probability of selecting a white ball from bag A is $\frac{5}{9}$.

Composition of bag B now becomes a white (5) and black (6) : Total (11)

Now probability of drawing a black ball from bag B is $\frac{6}{11}$

Hence the probability is $\frac{5}{9} \times \frac{6}{11} = \frac{10}{33}$.

Case (ii) : Let us suppose that a black ball is transferred from bag A to a bag B.

Probability of selecting a black ball from bag A is $\frac{4}{9}$.

Composition of bag B now becomes : White (4), Black (7) : Total (11)

Next, probability of drawing a black ball from bag B is $\frac{7}{11}$.

Hence the probability is $\frac{4}{9} \times \frac{7}{11} = \frac{28}{99}$.

As the two cases are mutually exclusive, the required probability is the sum of probabilities of two cases.

$\therefore$ The required probability

$$= \frac{10}{33} + \frac{28}{99} = \frac{58}{99}.$$

(8) A room has 4 sockets for lamps. From a collection of 15 bulbs of which 8 are defective, 4 are selected at random and put in the sockets. Find the probability that the room is (i) dark (ii) lighted.

Solution : The composition of bulbs is as

Good bulbs	Bad bulbs	Total
7	8	15

Table 7.8

Of these 4 bulbs can be selected in $15\,C_4$ ways

$\therefore$ n (S) = $15\,C_4$.

(i) Let A_1 be the event that the room is dark. This is possible only if all 4 bulbs selected are coming out of 8 bad bulbs.

This can be done in $8\,C_4$ ways.

$\therefore$ $n(A_1) = 8C_4$

$$\text{Hence } P(A_1,) = \frac{n(A_1)}{n(S)} = \frac{8C_4}{15C_4} = \frac{8\times7\times6\times5}{15\times14\times13\times12}$$

$$= \frac{2}{39}$$

(ii) Let A_2 be the event that the room is lighted. This is complement of the event A_1

$$\therefore P(A_2) = 1 - P(A_1) = 1 - \frac{2}{39} = \frac{37}{39}$$

(9) A committee of 4 boys and 3 girls is to be formed from a group of 8 boys and 5 girls selecting randomly. What is the probability that the committee contains a particular boy and a particular girl ?

Solution : A committee of 7 members out of 13 can be formed in $13\,C_7 = 13\,C_6$ ways.

$\therefore$ n(S) = $13\,C_6$.

A committee of 4 boys and 3 girls containing a particular boy and a particular girl can be formed as follows :

3 boys out of 7 boys in $7\,C_3$ ways and 2 girls out of 4 girls in $4\,C_2$ ways.

$\therefore$ $n(A) = 7C_3 \times 4C_2$

$$= \frac{7\times6\times5}{3\times2}\times\frac{4\times3}{2} = 35\times6 \text{. ways}$$

$$\therefore p(A) = \frac{n(A)}{n(s)} = \frac{35\times6}{13C_6} = \frac{35\times6\times6\times5\times4\times3\times2}{13\times12\times11\times10\times9\times8}$$

$$= \frac{35}{286}.$$

(10) A card is drawn from a pack of 52 playing cards. Find the probability that the card drawn is (i) red or bears a number between 5 and 10 both inclusive (ii) an ace or a king (iii) ace or spade (iv) diamond or a face card.

Solution : Since a card is drawn out of 52 playing cards. this can be done in $52\,C_1$ = 52 ways.

$\therefore$ n(S) = 52.

(i) Let A_1 be the event that a card drawn is red or bears a number between 5 and 10 both inclusive.

There are 26 red cards and number of cards bearing nos. 5, 6, 7, 8, 9 10 are is 24. Out of these 24 cards, 12 are red cards.

$$\therefore\ n(A_1) = 26 + 24 - 12 = 38$$

$$\text{Hence, } P.(A_1) = \frac{n(A)}{n(S)} = \frac{38}{52} = \frac{19}{26}.$$

(ii) Let A_2 be the event that card drawn is ace or king. There are 4 aces and 4 kings.

$$\therefore\ n(A_2) = 4 + 4 = 8$$

$$\text{Hence, } P(A_2) = \frac{n(A_2)}{n(S)} = \frac{8}{52} = \frac{2}{13}.$$

(iii) Let A_3 be the event that card drawn is ace or spade. There are 4 aces and 13 spade cards including an ace.

$$\therefore\ n(A_3) = 4 + 13 - 1 = 16$$

$$\therefore\ p(A_3) = \frac{n(A_3)}{n(S)} = \frac{16}{52} = \frac{4}{13}.$$

(iv) Let A_4 be the event that card drawn is a diamond or a face card. There are 13 diamond and 12 face cards. Out of them, there are 3 diamond face cards.

$$\therefore\ n(A_4) = 13 + 12 - 3 = 22$$

$$\text{Hence, } P(A_4) = \frac{n(A_4)}{n(S)} = \frac{22}{52} = \frac{11}{26}$$

(11) An urn contains 4 red, 3 white and 3 black balls. If 2 balls are drawn at random, find the probability that they are of different colour.

Solution : The composition of the bag is as :

Red (4)	White (3)	Black (3)	Total (10)

Table 7.9

Two balls can be drawn out of 10 balls in $10\ C_2$, $= \frac{10 \times 9}{2} = 45$ ways.

Let A be the event that the balls drawn are of different colours. The following are the possible cases of draw, and the respective number of ways in which this can be done.

Case	Red (4)	White (3)	Black (3)	Total (10)
(i)	1	1	_	$4C_1 \times 3C_1 = 12$
(ii)	1	_	1	$4C_1 \times 3C_1 = 12$
(iii)	–	1	1	$3C_1 \times 3C_1 = 9$

Table 7.10

All these case are mutually exclusive. Hence n(A) is the sum of these cases of selection.

$$n(A) = 12 + 12 + 9 = 33$$

$$\therefore\ P(A) = \frac{n(A)}{n(S)} = \frac{33}{45} = \frac{11}{15}.$$

Check your progress – 7.2

(1) Fill in the blanks by choosing appropriate number from the bracket.

(i) A class contains 10 boys and 6 girls. If a committee of 3 is chosen at random from this class, the probability that exactly 2 boys are selected is $\left(\frac{27}{56}, \frac{29}{56}, \frac{15}{56}, \frac{31}{56}\right)$

(ii) Two men and two women are seated in a row at random. The probability that women are neighbours is …….. $\left(\frac{3}{4}, \frac{1}{4}, \frac{1}{2}, 1\right)$

(iii) A cricket eleven is to be selected out of a group of 14 players. The probability that the team includes at least one of two specified players A and B is ….. $\left(\frac{3}{91}, \frac{14}{91}, \frac{7}{91}, \frac{55}{91}\right)$

(iv) Three cards are drawn from a pack of 52 playing cards. The probability that they contain exactly two hearts is……… $\left(\frac{117}{850}, \frac{733}{850}, \frac{201}{850}, \frac{1}{850}\right)$

(v) Two fair dice are rolled. The probability that the maximum of the two numbers is greater than four is……. $\left(\frac{4}{9}, \frac{3}{9}, \frac{7}{9}, \frac{5}{9}\right)$

(vi) An urn contains 5 red, 2 white and 3 blue balls. If 5 balls are drawn at random the probability of obtaining 3 red, 1 white and 1 blue ball is…… $\left(\frac{3}{27}, \frac{5}{21}, \frac{16}{21}, \frac{7}{27}\right)$

(vii) In a batch of 12 electric bulbs, 3 are defective. Two bulbs are selected at random and put into 3 sockets in a room. The probability that the room is illuminated is................. $\left(\frac{1}{22}, \frac{2}{11}, \frac{3}{7}, \frac{21}{22}\right)$

7.3 ADDITION THEOREM ON PROBABILITY

If A and B are any two events then $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

Proof :

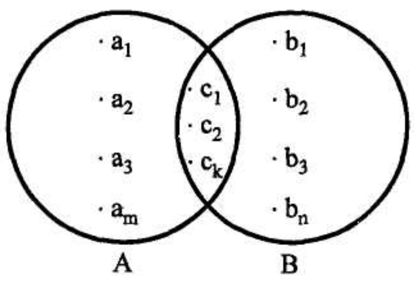


Fig. 7.1

Let $A = \{ a_1\ a_2, \ldots\ldots, a_m, c_1, c_2, \ldots\ldots c_k \}$

$B = \{ b_1\ b_2 \ldots\ldots, b_n, c_1\ c_2 \ldots\ldots c_k \}$

Hence $A\cap B = \{ c_1, c_2, \ldots\ldots c_k \}$

and $A\cup B = \{ a_1\ a_2, \ldots\ldots a_m, c_1, c_2 \ldots\ldots c_{fc}, b_1, , b_2 \ldots\ldots b_n \}$

$$\therefore P(A) = \sum_{i=1}^{m} P(a_i) + \sum_{l=1}^{k} p(c_l)$$

$$P(B) = \sum_{j=1}^{n} p(b_j) + \sum_{l=1}^{k} P(P_l)$$

$$P(A \cap B) = \sum_{l=1}^{k} P(c_l)$$

$$\text{Now } P(A \cup B) = \sum_{i=1}^{m} P(a_i) + \sum_{l=1}^{k} P(c_l) + \sum_{j=1}^{n} P(b_j)$$

$$= \left[\sum_{i=1}^{m} P(a_i) + \sum_{l=1}^{k} P(c_l)\right]$$

$$+ \left[\sum_{j=1}^{n} P(b_j) + \sum_{l=1}^{k} P(c_l)\right] - \left[\sum_{l=1}^{k} P(C_l)\right]$$

$$= P(A) + P(B) - P(A \cap B)$$

$$\text{i.e. } P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Cor. I.

If A and B are mutually exclusive events

$$A \cap B = \phi \text{ and } P(A \cap B) = P(\phi) = 0$$

$$\therefore P(A \cup B) = P(A)(B)$$

Cor. II

If $B = A'$ then $A \cap A' = \phi$ and $A \cup A' = S$

$$\therefore P(A \cup A') = P(A) + P(A') - P(A \cap A')$$

$$P(S) = P(A) + P(A') - P(\phi)$$

$$\therefore 1 = P(A) + P(A') - 0$$

$$\text{Thus } P(A') = l - P(A).$$

We shall illustrate the use of the theorem in the following examples :

Example (1) The probability that a person stopping at a petrol pump will ask for petrol is 0.80 and the probability that he will ask for water is 0.70 and the probability that he will ask for both is 0.65. Find the probability that a person stopping at this petrol pump and will ask neither petrol nor water.

Solution : Let A be the event that a person asks for petrol and B be the event that a person asks for water.

$$\text{we have } P(A) = 0.8, P(B) = 0.7$$

$$\text{and } P(A \cap B) = 0.65$$

$$\text{we have } P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.7 - 0.65 = 0.85$$

i.e. a person stopping at a petrol pump will ask for petrol and / or water has probability $P(A \cup B) = 0.85$.

The event that he will neither ask for petrol nor for water is $A' \cap B' = (A \cup B')$.

$$\therefore P(A \cup B)' = l - P(A \cup B)$$

$$= 1 - 0.85$$

$$= 0.15.$$

Example (2) If A and B are two events such that P (A) = 0.8, P (B) = 0.6 and P (A $\cap$ B) = 0.5, find P(A$\cup$B).

Solution : We have

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$
$$= 0.8 + 0.6 - 0.5 = 0.9.$$

Example (3) A and B are two events such that P (A $\cup$ B) = $\frac{5}{6}$,

P(A$\cap$B) = $\frac{1}{3}$ and P(B′) = $\frac{1}{3}$; find P(A).

Solution : We have P (B) = 1 – P (B′)

$$= 1-\frac{1}{3}=\frac{2}{3}$$

Next $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$$\frac{5}{6} = P(A) + \frac{2}{3}-\frac{1}{3}$$

$$\therefore P(A) = \frac{5}{6}-\frac{1}{3}=\frac{3}{6}=\frac{1}{2}$$

Check your progress – 7.3

1) State whether true or false, if false correct the same and write the correct answer.

(i) $P(A) = \frac{4}{5}, P(B') = \frac{2}{5}$ and $P(A \cap B) = \frac{1}{2}$

(α) $P(A\cup B)=\frac{9}{10}$ (β) $P(A\cup B')=\frac{1}{4}$

(γ) $P(A\cap B)=\frac{2}{3}$ (δ) $P(A\cap B')=\frac{2}{5}$

(ii) A bowl contains 100 slips numbered 1 to 100. A slip is drawn at random from the bowl. The probability that the slip bears a number which is

(α) divisible by 5 is $\frac{1}{5}$,

(β) divisible by 7 is $\frac{7}{50}$,

(γ) divisible by 5 and / or 7 is $\frac{8}{50}$,

(δ) divisible by 5 but not by 7 is $\frac{7}{50}$

(iii) An urn contains 3 white and 5 red balls and another urn contains 2 white and 4 red balls and a ball is drawn at random. The probability that the ball is red is $\frac{31}{48}$

7.4 CONDITIONAL PROBABILITY

Suppose that event A has already happened. Then probability that event B will happen is the **conditional probability** of **B** given A and is denoted by P (B / A) and is read as; probability of B given A. Similarly probability that event A will happen on the assumption that event B has already happened is conditional probability of A given B and is denoted by P (A / B) and is read as : Probability of A given B.

e.g. Let A be the event that one card drawn from a pack is spade. It is kept aside and then a second card is drawn. Let B be the event that 2^{nd} card drawn is also spade.

$$P(A) = \frac{13}{52} = \frac{1}{4}$$

$$P(B/A) = \frac{12}{51} = \frac{4}{17} \text{ (12 spade cards are in a pack of 51 cards)}$$

If the first card drawn is not, spade and still a second card drawn is spade,

$$P(B/A') = \frac{13}{51}$$

We shall use **Multiplication theorem** as :

$$P(A \cap B) = P(A/B) \cdot P(B)$$

or

$$P(A \cap B) = P(B/A) \cdot P(A)$$

We shall verify it for the above illustration. $A \cap B$ means both cards drawn are spade. $n(s) = 52\,C_2$ and $n(A \cap B) = 13\,C_2$

$$\therefore P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{n(A \cap B)}{n(S)} = \frac{13C_2}{52C_2} = \frac{13 \times 12}{52 \times 51} = \frac{1}{17}$$

$$\text{Hence } P(B/A) = \frac{P(A \cap B)}{n(A)} = \frac{1}{17} \div \frac{1}{4} = \frac{4}{17}$$

We shall illustrate this concept by following examples.

Example (1) : A and B are two, events in a sample space such that P(A) = 0.6, P(B) = 0.2 and P (A/B) = 0.5 Find P [A/(A ∪ B)]

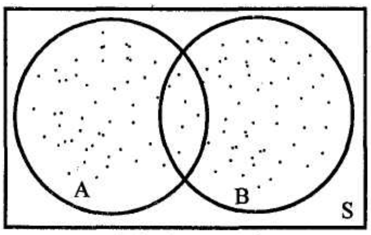
Solution : We have $P(A \cap B) = P(A/B) \times P(B)$

$$= (0.5)(0.2) = 0.1$$

Also by addition theorem,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.6 + 0.2 - 0.1 = 0.7$$



$\boxed{} = (A \cup B)$

Fig. 7.2

Next $A \cap (A \cup B) = A$ as shown in Fig. 7.2.

$$\therefore P[A/(A \cup B)] = \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$$

$$= \frac{P(A)}{P(A \cup B)} = \frac{0.6}{0.7} = \frac{6}{7}$$

Example (2) : In a certain examination out of 50 candidates 30 passed in Economics, 35 passed in psychology and 10 failed in both the subjects. A

candidate is selected at random. Find the probability that he has passed in Economics, if it is known that he has passed in Psychology.

Solution :

Let E = Set of students passed in Economics.

P = Set of students passed in Psychology.

Here n (S) = 50, n(E) = 30, n(P) = 35.

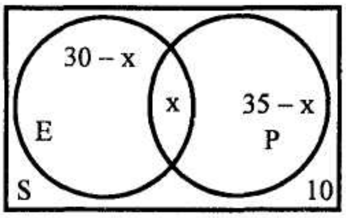


Fig. 7.3

Let x be the number of students passed in both subjects Economics and Psychology. Then number of students passed in Economics only = 30 –x, and number of students passed in Psychology only is 35 – x. (Refer to Fig. 7.3)

$$\therefore\ 30 - x + x + 35 - x + 10 = 50 \text{ (total number of students.)}$$

$$\text{i.e. } 75 - x = 50$$

$$\text{i.e. } x = 25 = n(E \cap P)$$

$$\therefore P(E \cap P) = \frac{n(E \cap p)}{n(S)} = \frac{25}{50} = \frac{1}{2}$$

$$\text{and } P(P) = \frac{n(P)}{n(s)} = \frac{35}{50} = \frac{7}{10}$$

By conditional probability, probability of student selected at random who has passed in Economics, given that he has passed in Psychology is

$$P(E/P) = \frac{P(E \cap P)}{P(P)} = \frac{1/2}{7/10} = \frac{1}{2} \times \frac{10}{7} = \frac{5}{7}$$

7.5 INDEPENDENT EVENTS

Two events A and B are said to independent if occurrence of one of them does not depend on the occurrence of another.

Thus events A and B are independent

$$\text{if } P(A).P(B) = P(A \cap B).$$

We shall verify it for one simple case :

Let a single unbiased dice be rolled. Let A be the event that score is even number; and B be the event that score is multiple of three.

$$\text{Here } n(S) = 6$$

$$\text{We have } A = \{2, 4, 6\},\ B = \{3, 6\},\ A \cap B = \{6\}$$

$$\therefore\ n(A) = 3,\ n(B) = 2,\ n(A \cap B) = 1$$

$$\text{Hence } P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{6}$$

Thus we verify that,

$$P(A).\ P(B) = \frac{1}{2}.\frac{1}{3} = \frac{1}{6}$$

$$= P(A \cap B).$$

Hence A and B are independent events.

7.5.1 Multiplication theorem :

When two events A and B are independent the probabilities of occurrence of both the events A and B simultaneously is the product of the probabilities of events A and B.

Thus when A and B are independent events.

$$P(A \cap B) = P(A).P(B)$$

In the above illustration $A \cap B$ is the event that the score is multiple of three and an even number. When the events are not independent, the theorem is

$$P(A \cap B) = P(A).P(B/A) = P(B) \bullet P(A/B).$$

We shall illustrate the above principles by the following examples.

Example (1) : A fair dice is tossed twice. If A = the event that sum of the two numbers is 7. B = the event that the number, in the second toss is 6. Show that A and B are independent.

Solution : n (S) = 36, as any of number 1 to 6 can appear on top at any toss.

Here A = {(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}

n (A) = 6

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

We have B = {(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)}

n (B) = 6

$$\therefore P(B) = \frac{1}{6}$$

Also, $A \cap B$ = {(1,6)}

$\therefore n(A \cap B) = 1$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{36}$$

$$\text{and } P(A).P(B) = \frac{1}{6}.\frac{1}{6} = \frac{1}{36} = P(A \cap B)$$

Thus $P(A \cap B) = P(A).P(B)$.

Hence A and B are independent events.

Example (2) : If A and B are two events such that P(A) = $\frac{3}{4}$, and P (B)= $\frac{2}{3}$ (A $\cap$ B) = $\frac{1}{2}$. Find (i) P (A $\cup$ B (ii) P (B/A) (iii) P (A/B).

Solution : We have addition theorem P $(A \cup B)$ =P(A) + p (B) – $(A \cap B)$

$$= \frac{3}{4} + \frac{2}{3} - \frac{1}{2}$$

$$= \frac{9+8-6}{12} = \frac{11}{12}.$$

(i) Thus $P(A \cup B) = \frac{11}{12}$

(ii) $P(B/A) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2} \div \frac{3}{4} = \frac{2}{3}.$

(iii) $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2} \div \frac{2}{3} = \frac{3}{4}$

Example (3) : If A and B arc two events such that P(A) = 0.7, P(B) = 0.4 and P ($A \cap B$) = 0.3 find (i)P($A \cap B'$) (ii) P (B/A) (iii) P($A' \cap B$)

Solution :

(i) By the adjoining Venn–diagram (Fig. 7.4)

$$\text{We have } A \cap E' = A - (A \cap B)$$

$$\therefore P(A \cap B') = P(A) - P(A \cap B)$$

$$= 0.7 - 0.3 = 0.4$$

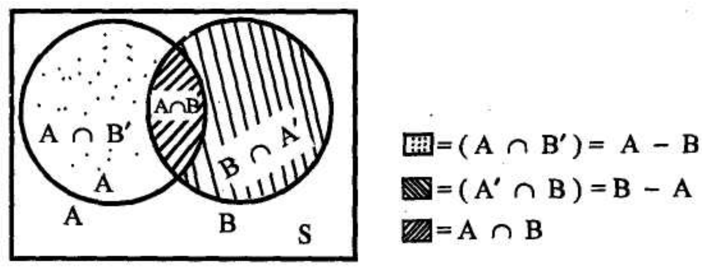


Fig. 7.4

(ii) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.7} = \frac{3}{7}.$

(iii) By the Venn-diagram $A' \cap B = B - (A \cap B)$

$$\therefore P(A' \cap B) = P(B) - P(A \cap B)$$

$$= 0.4 - 0.3 = 0.1.$$

Example (4) : Given that P(A) = $\frac{3}{5}$, P(B) = $\frac{1}{2}$, P (B/A) = $\frac{2}{3}$ find (i) P (A/B) (ii) P ($A \cup B$)

Solution : We shall first find out P($A \cap B$).

By conditional probability, we have $P(B/A) = \frac{P(A \cap B)}{P(A)}$

$$\therefore \quad \frac{2}{3} = \frac{P(A \cap B)}{3/5}$$

$$\text{i.e. } P(A \cap B) = \frac{2}{3} \times \frac{3}{5} = \frac{2}{5}.$$

(i) $P(A/B) = P(A \cap B)/P(B)$

$$= \frac{2}{5} / \frac{1}{2} = \frac{4}{5}.$$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{3}{5} + \frac{1}{2} - \frac{2}{5}$$

$$\therefore \text{ P } (A \cup B) = \frac{6+5-4}{10} = \frac{7}{10}$$

Example (5) : In a group of equal number of men and women, 8% men and 54% women are unemployed. Find the chance that a person selected at random from this group is employed.

Solution : The composition of employed and unemployed from the group and their respective probabilities are as.

1 Prob.	employed	unemployed
Men	0.92	0.8
Women	0.46	0.54

Table 7.11

The group contains equal number of men and women. Hence probability of selecting man/woman is $\frac{1}{2}$

The probability of selecting an employed man from this group is $\frac{1}{2}$ (0.92) = 0.46

The probability of selecting an employed women from this group is $\frac{1}{2}$ (0.46) = 0.23

As these two cases are mutually exclusive the required probability of selecting an employed person from the group is the sum 0.46 + 0.23 = 0.69.

Example (6) : A problem in Mathematics is given to three students Rama, Govinda and Seeta whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that the problem will be solved if they try independently.

Solution : We shall first obtain the probability that the problem is not solved. By putting the data in a tabular form we get,

Name	**pr. of solving p**	**pr. of not solving (1 – p)**
Rama	$\frac{1}{2}$	$\frac{1}{2}$
Govinda	$\frac{1}{3}$	$\frac{2}{3}$
Seeta	$\frac{1}{4}$	$\frac{3}{4}$

Table 7.12

As they try independently, the probability that the problem will not be solved is the product of the probabilities of not solving.

$\therefore$ Probability that the problem will not be solved

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

The required probability of the problem being solved is probability of the complementary event and is $1 - \frac{1}{4} = \frac{3}{4}$

Note : *Instead of considering various cases like Rama solves, Govinda and Seeta does not solve, it is convenient to consider complementary event.*

Example (7) : A husband and a wife appeared in an interview for two vacancies in the office. The pr. of the selection of the husband is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. Find the probability that (a) both of them are selected (b) one of them is selected (c) at least one of them is being selected.

Solution : The data can be put in a tabular form as

	pr. of selection	pr. of rejection
husband	$\frac{1}{7}$	$\frac{6}{7}$
wife	$\frac{1}{5}$	$\frac{4}{5}$

Table 7.13

(a) probability of both of them being selected is,

$$\frac{1}{5}\times\frac{1}{7}=\frac{1}{35}$$

(b) Here we shall consider two cases :

Case (i) : Husband is selected and wife is rejected. The probability is,

$$\frac{1}{7}\times\frac{4}{5}=\frac{4}{35}$$

Case (ii) : Husband is rejected while wife is selected. The probability is,

$$\frac{6}{7}\times\frac{1}{5}=\frac{6}{35}$$

The two cases are mutually exclusive. The required probability is sum of them.

$\therefore$ The required probability $=\frac{4}{35}+\frac{6}{35}=\frac{10}{35}=\frac{2}{7}$

(c) It is complement of both being rejected, probability that both are rejected is $\frac{6}{7}\times\frac{4}{5}=\frac{24}{35}$

The required Probability = $1-\frac{24}{35}=\frac{11}{35}$

or

It is sum of both (a) and (b) above viz. either husband is selected or wife is selected or both are selected.

$\therefore$ Probability is $\frac{1}{35}+\frac{10}{35}=\frac{11}{35}$

Example (8) : Three persons A, B and C fire a target simultaneously. The probabilities that A, B and C can hit the target are $\mathbf{\frac{1}{3},\frac{1}{4}}$ and $\mathbf{\frac{1}{5}}$ respectily. Find the probability that exactly two of them hit the target.

person	pr. of hitting	pr. of not hitting
A	$\frac{1}{3}$	$\frac{2}{3}$
B	$\frac{1}{4}$	$\frac{3}{4}$
C	$\frac{1}{5}$	$\frac{4}{5}$

Table 7.14

We shall consider three cases :

(i) A, B hit and C does not hit (A, B and C′)
(ii) B and C hit, while A does not hit (B, C and A′)
(iii) A and C hit, while B does not hit (A, C and B)
The probabilities in each case are as,

(i) $P(A).P(B).P(C') = \frac{1}{3} \times \frac{1}{4} \times \frac{4}{5} = \frac{1}{15}$

(ii) $P(B).P(C).P(A') = \frac{1}{4} \times \frac{1}{5} \times \frac{2}{3} = \frac{1}{30}$

(iii) $P('A) \bullet P(C) \bullet P(B') = \frac{1}{3} \times \frac{1}{5} \times \frac{3}{4} = \frac{1}{20}$

All these cases are mutually exclusive, the required probability of exactly two of them hitting the target is the sum of the probabilities.

$\therefore$ The required probability is $= \frac{1}{15} + \frac{1}{30} + \frac{1}{20} = \frac{4+2+3}{60} = \frac{3}{20}$

Check your progress – 7.4-7.5

(1) Fill in the blanks by choosing correct number from the bracket.

(i) Of 10 girls in a class, 3 have blue eyes. If two of the girls are chosen at random; the probability that (α) both have blue eyes is .... $\left(\frac{1}{15}, 1, \frac{2}{15}, \frac{4}{15}\right)$

(β) neither has blue eyes is $\left(\frac{2}{15}, \frac{7}{15}, \frac{8}{15}, 0\right)$

(γ) at least one has blue eyes is $\left(\frac{1}{15}, \frac{7}{15}, \frac{67}{91}, \frac{8}{15}\right)$

(ii) Three bolts and three nuts are put in a box. If two parts are chosen at random, the Probability that 1 is a nut and 1 is a bolt is ….. $\left(\frac{2}{5}, \frac{3}{5}, \frac{1}{3}, \frac{1}{2}\right)$

(2) Given that $P(A) = \frac{1}{2}$ $P(B) = \frac{1}{3}$, $P(A \cup B) = \frac{2}{3}$, Test for independence of A and B.

(3) The probability that a problem can be solved by A, B, C is $\frac{1}{3}, \frac{1}{4}, \frac{1}{2}$ respectively. Find the probability that the problem is not solved.

7.6 THE PROBABILITY MODEL

So far we have considered cases in which the probability of every sample point is the same. In other words, we have considered equiprobable space. Now we shall consider cases where the probability of each sample point is different. But the basic conditions : P (w_i.) $\geq$ 0 and $\sum P(w_i) = 1$, must hold here also. If any of these conditions are not fulfilled, it will not be a probability model.

e.g. (1) Let S = { x_1 , x_2 , x_3, x_4 }

$$\text{and } P(x_1) = \frac{1}{4},\ P(x_2) = 0,\ P(x_3) = \frac{1}{2},\ P(x_4) = \frac{1}{4}.$$

Here both the above conditions are fulfilled and hence it is a probability model.

(2) Let S = {a, b, c, d, e} and

$$P(a) = \frac{1}{5},\ p(b) = -\frac{2}{5},\ P(c)=\frac{2}{5},\ P(d)=\frac{2}{5},\ P(e)=\frac{3}{5}$$

Here, $P(b) = -\frac{2}{5}$ <0 and sun of the probabilities = $\frac{1}{5}+\frac{2}{5}-\frac{2}{5}+\frac{2}{5}+\frac{3}{5}=\frac{6}{5}>1$

Hence it is not a probability model.

Now we shall consider examples where we have to find out probabilities of various; sample points from the data :

Example (1) : Dice is loaded such that the occurrence of even number on top is twice as likely as the occurrence of odd number. Find the probability that the number shown on uppermost face is greater than 4.

Solution : let x be the probability of occurrence of odd numbers. Then 2x is the probability of occurrence of even number.

The probability model is,

number	1	2	3	4	5	6
pr. model	x	2x	x	2x	x	2x
actual pr.	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$

Table 7.15

Sum of the probabilities is 9x. Since it is a probability model, 9x = 1 $\therefore$. $x = \frac{1}{9}$.

Now we can complete the above table 7.16.

Let A be the event that number is greater than 4. $\therefore$ A = {5, 6}

$$P(A) = P(5) + P(6) = \frac{1}{9}+\frac{2}{9}=\frac{1}{3}$$

Example (2) : Three horses H_1, H_2 and H_3 are in a race. H_1 is twice as likely to win as H_2 and H_1 is twice as likely to win as H_3 . Find their respective probabilities of winning a race.

Solution : Let the probability that horse H_3 wins a race be x.

$$\therefore P(H_3) = x$$

Horse H_2 is, twice as likely to win as H_3

$$\therefore P(H_2) = 2P(H_3) = 2x$$

Further, H_1 is twice as likely to win as H_2.

$$\therefore P(H_1) = 2P(H_2) = 4x$$

As only one of them will win the race, the sum of the probabilities is 1. Also these events are mutually exclusive and exhaustive.

$$\therefore 1 = P(H_1)+P(H_2) + P(H_3)$$

$$\text{i.e. } 1 = x + 2x + 4x = 7x$$

i.e. $x = \frac{1}{7}$

$\therefore$. $P(H_3) = \frac{1}{7}$, $P(H_2) = 2P(H_3) = \frac{2}{7}$ and $P(H_1) = 2P(H_2) = \frac{4}{7}$.

Example (3) : There are three groups of children. Group I contains 2 boys and 1 girl. Group II contains 1 boy and 2 girls and Group III contains 2 boys and 2 girls; one child is selected at random from each group. Find the probability that three selected children comprise of 2 girls and 1 boy.

Solution : The composition of each group, possible modes of selection have been tabulated. The corresponding probabilities are expressed in brackets.

Group	Boys	Girls	Total	Modes of selection (probabilities)		
I	2	1	3	$B\left(\frac{2}{3}\right)$	$G\left(\frac{1}{3}\right)$	$G\left(\frac{1}{3}\right)$
II	1	2	3	$G\left(\frac{2}{3}\right)$	$B\left(\frac{1}{3}\right)$	$G\left(\frac{2}{3}\right)$
III	2	2	4	$G\left(\frac{1}{2}\right)$	$G\left(\frac{1}{2}\right)$	$B\left(\frac{1}{2}\right)$
			Probability	$\frac{2}{3}\times\frac{2}{3}\times\frac{1}{2}=\frac{2}{9}$	$\frac{1}{3}\times\frac{1}{3}\times\frac{1}{2}=\frac{1}{18}$	$\frac{1}{3}\times\frac{2}{3}\times\frac{1}{2}=\frac{1}{9}$

Table 7.16

As the 3 cases are mutually exclusive,

the required probability = $\frac{2}{9}+\frac{1}{18}+\frac{1}{9}=\frac{4+1+2}{18}=\frac{7}{18}$

7.6 Check your progress

(1) Fill in the blanks by choosing appropriate number from the bracket.

(i) coin is weighted so that head is four times as likely to appear as tail.

$P(H) = \ldots\ldots\ldots\ldots\ldots \left(\frac{1}{5},\frac{2}{5},\frac{4}{5},1\right)$

(ii) A dice is loaded such that appearing 4 on a top is certain.

$P(4) = \ldots\ldots\ldots \left(0,1,\frac{1}{2},\frac{1}{6}\right)$

(2) State whether each of the following is a probability model on $S = \{S_1, S_2, S_3, S_4, S_5\}$

(i) $P(S_1) = 0, P(S_2) = \frac{1}{5}, P(S_3) = \frac{2}{5}, P(S_4) = \frac{1}{5}$ $P(S_5) = \frac{1}{5}$

(ii) $P(S_1) = \frac{1}{6}. P(S_2) = \frac{1}{3}, P(S_3) = \frac{1}{4}, P(S_4) = \frac{1}{5}, P(S5)$ $\frac{1}{2}$

(iii) $P(S_1) = -\frac{1}{2}$ $P(S_2) = \frac{1}{2}$, $P(S_3) = \frac{1}{3}$, $P(S_4) = \frac{2}{3}$, $P(S_5) = 0$

7.7 SUMMARY

Probability is a theory of chance. In each and every subject there is probability of pass and fail we want to be half and half. When we have to take some decision the probability of positive activity should be more than half. We learn this concept with games like tossing a coin, coins dice, dices, playing cards drawn randomly. But important is equal to one and not more than one.

7.8 CHECK YOUR PROGRESS - ANSWERS

7.2

(1) (i) $\frac{27}{56}$ (ii) $\frac{1}{4}$ (iii) $\frac{55}{91}$ (iv) $\frac{117}{850}$

(v) $\frac{5}{9}$ (vi) $\frac{5}{21}$ (vii) $\frac{21}{22}$

7.3

(1) (i) (α) true (β) False, correct is $\frac{1}{2}$

(γ) true (δ) false, correct is $\frac{3}{10}$

(ii) (α) true (β) true

(γ) false, correct is $\frac{8}{25}$ (δ) false, correct is $\frac{9}{50}$

(iii) true

7.4 and 7.5

(1) (i) (α) $\frac{1}{15}$ (β) $\frac{7}{15}$ (γ) $\frac{8}{15}$ (ii) $\frac{3}{5}$

(2) independent (3) $\frac{1}{4}$

7.6

(1) (i) $\frac{4}{5}$ (ii) 1

(2) (i) yes (ii) no (iii) no

7.9 QUESTIONS FOR SELF - STUDY

1) Two dice are rolled. Write down the sample space for the experiment. Hence write down the following event sets –

   a) The numbers on both dice are identical.

   b) The sum of numbers appearing on them is divisible by 4.

2) Three coins are tossed. Find the probability that at least two heads appear.

3) A room has 3 electric lamps. From a collection of 15 electric bulbs of which only 10 are good 3 are selected at random and put in the lamps. Find the probability that the room is lighted by at least one of the bulbs.

4) The probability that A can shoot at a target is 5/7 and the probability that B can shoot at the same target is 3/5 A and B shot independents. Find the probability that

a) the target is not shot at all

b) the target is shot at least one of them

7.10 SUGGESTED READINGS

1. *Pre-degree Mathematics* by Vaze, Gosavi

2. *Discrete Mathematical Structures for Computer Science* by Bernard Kolman and Robert C Busby

3. *Statistical Analysis: A Computer - Oriented Approach Introduction to Mathematical Statistics by* S. P. Azen & A. A.Afifi

◆◆◆

NOTES

CHAPTER 8

BINARY SYSTEM

8.0 0BJECTIVES

After studying the concept of a BINARY SYSTEM you will be able to use and solve problems related to the followings:

1) BINARY NUMBERS
Conversion of BINARY to a) DECIMAL
b) OCTAL
c) HEXADECIMAL
2) BINARY ARITHMATIC : a) ADDITION
b) SUBTRACTION

8.1 INDRODUCTION

There are different number systems we learnt. 1.2.3.4.5.6.7.8.9, are ten natural numbers, o is added to set of these numbers .so we get total ten numbers to study mathematics. There are four basic operations ADDITION, SUBTRACTION, MULTIFICATION & DIVISION .In DECIMAL SYSTEM ten digits are used as 0,1,2,3,4,5,6,7,8,9. But computer do not understand these numbers. Every computer stores numbers, letters & other special characters in coded form.

8.2 DECIMAL SYSTEM

$1 = 10^0$

$10 = 10^1$

$100 = 10^2$

$1000 = 10^3$

$10000 = 10^4$

8.2.1 Conversion of a number into DECIMAL SYSTEM:

PLACE in NUMBERS:

TL LAKHS TT THOU HUNDERS TENS UNIT

PLACE VALUES:

PLACE NAME	PLACE DIGIT	PLACE VALUE	VALUE
UNIT	4	$4 * 10^0$	4
TENS	9	$9 * 10^0$	90
HANDERD	8	$8 * 10^2$	800
THOUSAND	7	$7 * 10^3$	7000
TENTHOUSAND	6	$6 * 10^4$	60000
LAKH	5	$5 * 10^5$	500000
TENLAKHS	3	$3 * 10^6$	3000000
CRORES	2	$2 * 10^7$	20000000

$23567894 = 2*10^7 + 3*10^6 + 5*10^5 + 6*10^4 + 7*10^3 + 8*10^2 + 9*10_1 + 4*10^0$

8.2 Check your progress:

Express the given numbers in DECIMAL NUMBER system:

1) 2567
2) 93256
3) 3212
4) 5632
5) 9320

8.3 BINARY SYSTEM

Computer understands only two digits 0, 1 that means only two numbers. BI means two so this number system is called BINARY NUMBER SYSTEM .The BASE of Binary system is 2.

8.3.1 CONVERTIN number to BINARY SYSTEM:

EX: 235

2	235	REMENDER
2	117	1
2	58	1
2	29	0
2	14	1
2	7	0
2	3	1
2	1	1
	0	1

$235_{10} = 11101011_2$

EX: 21

2	21	REMENDER
2	10	1
2	5	0
2	2	1
2	1	0
	0	1

$21_{10} = 10101_2$

8.4 OCTAL NUMBER SYSTEM

OCTAL means 8.Base of this number system is 8 ,that is it use only eight digits from the ten numbers of DECIMAL SYSTEM 0.1.2.3.4.5.6.7 .So number of digits are 8 but the last number is 7 .

8.4.1: Conversion of number into OCTAL number system:

EX: 56

8	56	REMENDER
8	7	0
	0	

$56_{10} = 70_{8}$

8.5 HEXADECIMAL NUMBER SYSTEM

HEXA + decimal means it is a number system using 16 digits to write a number. Digits 0.1.2.3.4.5.6.7.8.9 the ten digits and more six digits from capital letters from English language A,B,C,D,E,F .

Where

A is 11^{th}

B is 12^{th}

C is 13^{th}

D is 14^{th}

E is 15^{th}

F is 16^{th}

So largest digit is F or 15 which is one less than the base 16. Each position in HEXADECIMAL system represents a power of base 16.

8.5.1 Conversion of a number into the HEXADECIMAL system:

EX: 431

16	431	Reminders in HEXADECIMAL system
16	26	15 =F
16	1	10 =A
	0	1 = 1

$431_{10} = 1AF_{16}$

8.6 CONVERSION OF DECIMAL TO BINARY

EX: 230

2	230	REMENDER
2	115	0
2	57	1
2	28	1
2	14	0
2	7	0
2	3	1
2	1	1
	0	1

$230_{10} = 11100110_{2}$

8.6.1 COVERSION OF BINARY to DECIMAL

EX: 11001

FIVE digested number is given

So maximum power of 2 is four

$11001 = 1 * 2^4 + 1 * 2^3 + 0 * 2^2 + 0 * 2^1 + 1 * 2^0$

$= 16 + 8 + 0 + 0 + 1$

$= 25_{10}$

CONVERSION OF DECIMAL TO OCTAL

EX: 952

8	952	Remainder
8	119	0
8	14	7
8	1	6
	0	1

$952_{10} = 1670_8$

8.7 BINARY ARITHMATIC

Basic arithmetic operations are ADDITION (+) .SUBTRACTON (-) .MULTIPLICATION (*) .DIVISION (/).

These operations are interrelated .we use that in previous standers. But in BINARY system only two digits are used so there are some RULES to calculations

8.7.1 RULES for ADDITION:

1) 0+0 = 0
2) 0+1=1
3) 1+0=1
4) 1+1=0 PLUS CARRT OF 1to next higher column

1 is the largest digit in binary number system any sum greater than 1 requires a digit to be carried over.

EX: 1) 10

+10

20

This is the addition DECIMAL SYSTEM .But in BINARY system

10

\+ 10

100

8.7.2 BINARY SUBTRACTION:

IN subtraction we have to borrow a number from left digit this number is depending on the base in which we are subtracting.

RULES:

1) 0-0=0
2) 0-1=1 WITH BORROW from the next column
3) 1 -0=1
4) 1 - 1 = 0

EX: 5-2

DECIMALSYSTEM	BINARY SYSTEM
5	101
- 2	- 10
3	11

$11 = 1 * 2^1 + 1 * 2^0$

8.8 SUMMARY

Binary system is very important for the computer students. Computer can understand only concept of 0 and 1 means yes / on. For any mathematical number there is proper arrangement of 0 and 1. There are also other systems e.g. Ocatal, Hexadecimal. The base for each system is different.

8.9 CHECK YOUR PROGRESS - ANSWERS

8.2

1. $2567 = 2 \times 10^3 + 5 \times 10^2 + 6 \times 10^1 + 7 \times 10^0$
2. $93256 = 9 \times 10^4 + 3 \times 10^3 + 2 \times 10^2 + 5 \times 10^1 + 6 \times 10^0$
3. $3212 = 3 \times 10^3 + 2 \times 10^2 + 1 \times 10^1 + 2 \times 10^0$
4. $5632 = 5 \times 10^3 + 6 \times 10^2 + 3 \times 10^1 + 2 \times 10^0$
5. $9320 = 9 \times 10^3 + 3 \times 10^2 + 2 \times 10^1 + 0 \times 10^0$

8.10 QUESTIONS FOR SELF - STUDY

Add binary numbers in both binary and DECIMAL forms.

1.	1011, 101	2.	1010110, 1011010
3.	10111, 1011	3.	101, 10110001
5.	101101, 001101001110		

SUBTRACT the following numbers in BINARY & DECIMAL both the system:

1) 50 – 25
2) 588 -134
3) 216 -172
4) 100- 23
5) 12 - 9
6) 10000 - 01010
7) 101110 -110111
8) 110111 - 011011
9) 1100 - 1111
10) 1011 - 101

8.11 SUGGESTED READINGS

1. *Pre-degree Mathematics* by Vaze, Gosavi

2. *Discrete Mathematical Structures for Computer Science* by Bernard Kolman and Robert C Busby

3. *Mathematics and Statistics* by M. L. Vaidya and M. K. Kelkar

◈◈◈

NOTES

CHAPTER 9

MATHEMATICAL LOGIC AND TRUTH TABLE

9.0 OBJECTIVES

Dear Friends this is new mathematical addition to your language.
After studying this chapter you will be able to –

- Think Logically
- Differentiate the logic and mathematical logic
- Explain Bits
- Explain Yes or No

9.1 INTRODUCTION

Mathematics is a way of thinking and reasoning. Logic is the discipline that deals with art of reasoning. Systematic reasoning is a base of Mathematics. Study of correct and systematic reasoning .Everybody thinks but everybody cannot distinguish between good and bad thinks .Logic gives idea to show how one should think if one has to think clearly.

Logic was first given by George Boole, so Mathematics logic is called as BOOLEAN LOGIC. Symbolic logic is very important in Computers.

9.2 STATEMENT OR PROPOSITION

9.2.1 Definition

A statement or proposition is a declarative sentence which is either TRUE or FALSE but not both.

EX: 1 1) 2 is a even prime number.
2) Three plus two is equal to six.
3) Mumbai is capital of India.

In Logic the above sentences are true or false, but not both (true and false) . This is called “Law of excluded middle”

EX: 2 1) Work hard
2) College life is very good.
3) Open the classroom.
We denote statement by small letters, p, q, r, s…….etc.

EX: 3 1) p= Square of a number is odd number.
2) q= This is Mathematics classroom
3) r= study hard for your exam.

9.2.2 Truth Value of a Statement

Definition: Truth value – If a statement is true then its truth value is defined as to be T (or 1) and if statement is false its truth value is F (or 0).

EX: 4 1) p= Square of a number is odd.
ANS: Statement is false.
Truth value = F

Ex: 5 q: This is Mathematic s classroom.
Ans: Statement is true.
Truth value =T

Ex:6 3) r = study hard for your exam.
Statement is true.
Truth value = T

Check your Progress – 9.2

Write truth values of following statements –

1) $3+5=0$

2) Pune is not a big city.

3) Zero is a natural number.

4) $\sqrt{2}$ is a rational number.

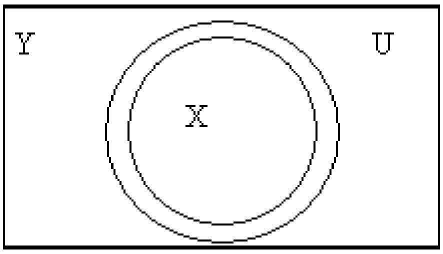
5) Empty set is a subset of every set.

9.3 USE OF VENN DIAGRAM

9.3.1 –All Xs are Ys-

The statement is true with true value T. X is proper subset of Y.

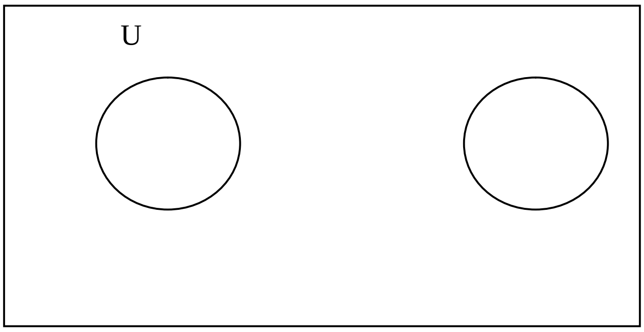
$X \subset Y$



Ex: 7 1) All Natural Number are whole numbers.

2) All even numbers are natural numbers.

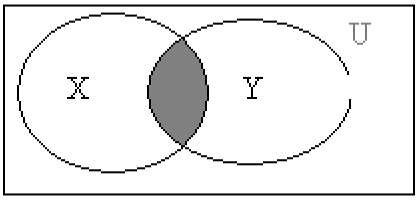
9.3.2 –Type 3)-NO Xs are Ys –



Ex: 8 1) X is set of even number s.
2) Y is set of odd numbers.
3) U is set of natural numbers.

9.3.2 – Type 4) Some Xs are Ys

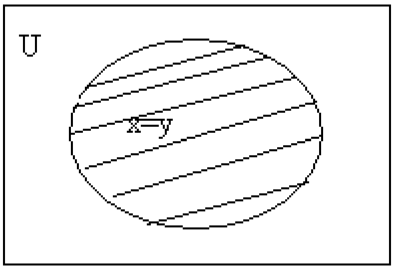
$X \subset Y$



9.3.3: Type 4-ALL Ys are Ys –

All Xs are Xs -

X =Y



9.4 LOGICAL CONNECTIVES

1) NOT
2) AND
3) OR
4) IF …….THEN
5) IF AND ONLY IF

9.4.1 Simple statement -

A combination which doesn't include any logical connective is called a simple statement.

EX: 1) Today is Sunday.

2) Newton was my friend.

9.4.2 - Compound statement –

A combination of simple statement formed by using logical connectives is called compound statement.

EX:9 1) Seeta and Geeta dancing

2) x= 0 or y=0

9.4.3 - Negation of a statement (NOT) –

Definition - If P is any statement then NOT P is called the negation of the statement P and is denoted by ~ p.

EX: 10 1) 2 is a prime number.

2) Three plus two is five.

P: 2 is a prime number.

~ p : 2 is not a prime number.

2) q: Three pulse two is five.

~ q: Three plus two is not five .

9.4.4 Conjunction (AND) (p Λ q)

If p and q are two simple statement .then the compound statement p and q is called there conjunction.

It is denoted by pΛq

Read as p and q

1) p: 25 is a odd number .

q: 25 is a perfect square.

P Λ q: 25 is an odd number and perfect square.

2) p: It is soft .

q: It is good .

p Λ q : It is soft and good .

9.4.5 Disjunction (OR Alternation) (pVq)

Definition: If p, q are two simple statements then compound statement.

It is denoted by p V q

Read as p or q

EX: 11 p: I will study at home.

q: I will go to my friend s house for study.

p V q: I will study at home or go to my friend's house.

EX: 12 p: x∈ A

q: x∈ B

p V q: x ∈ A or x ∈ B

Here p or q means - either p or q

Or both p and q

X ∈ A means x ∈ A or x ∈ B OR

X ∈ A x ∈ A and x ∈ B

9.4.6 Implication or Conditional statement ($p \rightarrow q$)

Definition: If p and q are two simple statements then compound statement if p and q is called a conditional or an implication.

It is denoted by $p \rightarrow q$

Read as p implies q

EX: 13 1) p: 12 is even number.

2) q: 12 is divisible by 2.

P q; If 12 is even number then it is divisible by 2.

2) p: Lines L_1 and L_2 are parallel lines .

q: Lines L1andL2 have no intersection point.

P q: If lines L_1 and L2 are parallel lines then they have no intersection point.

9.4.7: Double implication or Biconditional ($p \leftrightarrow q$)

Definition: If p and q are two statements, then the compound statement p if and only if q (or p if q) is called biconditional or double implication or equivalence .

It is denoted by $p \Rightarrow q$

Read as p if and only if q

EX; 13 1) p: Triangle ABC has three sides.

q: Triangle ABC is a triangle if and only if it have three sides

2) p: Pratik studies an Engineering .

q: Pratik passed 12th science examination with 90%marks.

P q Pratik studies an Engineering if and only if he passed 12th science with 90% marks.

The various connectives TABLE:

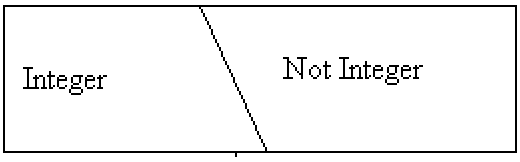
Connectives	Symbol	Compound Statement
NOT	~	Negation
OR	V	Disjunction
AND	Λ	Conjunction
IF…………….THEN	$\longrightarrow$	Implication
IF AND ONLY IF	$\leftrightarrow$	Double Implication

9.5 DIAGRAMMATIC REPRESENTATION OF LOGICAL CONNECTIVES

9.5.1: NEGETION (NOT):

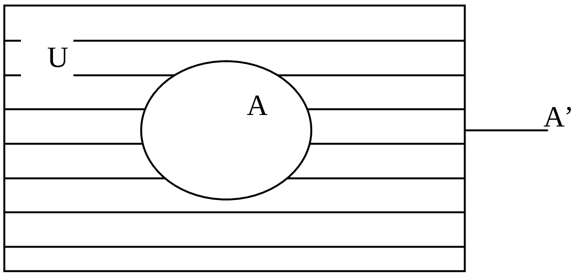
EX: 15 1) p: x is an integer.

~ P: x is not an integer.



2) P: A is set of numbers from 1 to 10.

~ P: A is set of numbers other than 1 to 10.



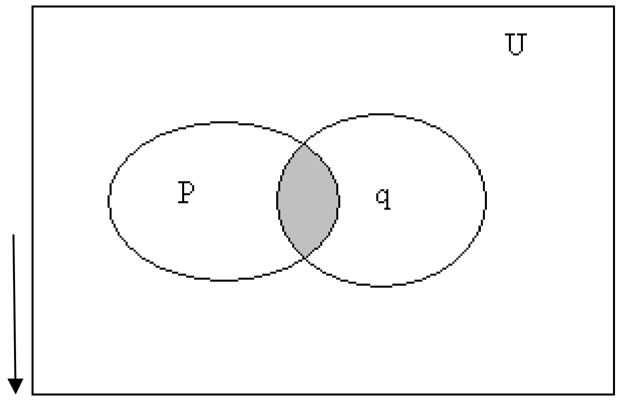
Note: The Venn diagram for the negation is similar to that of complemented of a set.

9.5.2: CONJUNCTION (AND)

EX: 16 p: Ram is intelligent.

q : Ram always studs first in the class.

$p \wedge q$: Ram is intelligent and always studs first in the class.



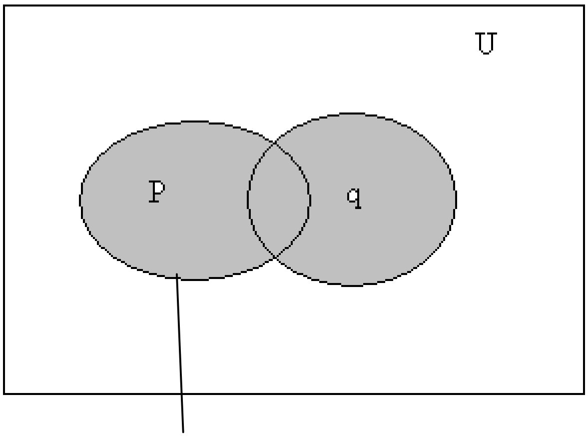
$p \wedge q$

9.5.3: DISJUNCTOIN: (OR)

EX: 17 p: 2+ 3 =5

q: 9- 3=5

PVq: 2 + 3 = 5 or 9 -3 = 5



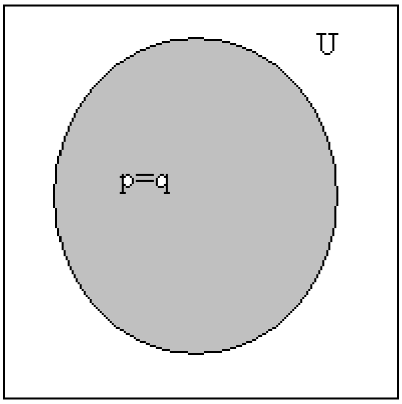
PVq

9.5.4: Implication: (IF ….THEN):

P: I go mad.

q: I bite you.

→ p q : If I go mad then I bite you.

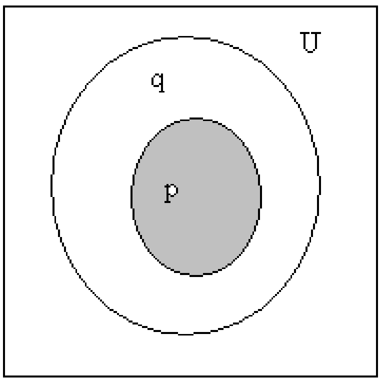


9.5.5: Double Implication – (IF AND ONLY IF):

P: Triangle ABC is isosceles triangle.

q : Base angles of triangle ABC are congruent.

P ↔ q; Triangle ABC are isosceles if and only if its base angles are congruent.



9.6 TRUTH TABLES

9.6.1: Truth table for NEGATION (~P):

Rule: If p has truth value T

Then ~p has truth value F

And if p has truth value F

Then ~p has truth value T

Thus statement and its negation have always opposite truth value s.

P	~p
T	F
F	T

EX: 18 p: 41 is a prime number.

~ P: 41 is not a prime number.

P	~P
T	F
F	T

9.6.2: TRUTH TABLE for DISJUNTION (OR)

RULE: If p and q true pVq is true.

P and q false pVq is false.

Otherwise pVq is false.

P	Q	PVq
T	T	T
T	F	T
F	T	T
F	F	F

9.6.3 TUTH TABLE for CONJUCTION (AND):

Rule: Both p and q TRUE then pΛq TRUE.

Otherwise pΛq FALSE.

P	q	PΛq
T	T	T
T	F	F
F	T	F
F	F	F

9.6.4: If p THEN q CONDITIONAL

Rule: Both p and q TRUE Then p-q TRUE.

If p is TRUE , q is FALSE then p-q FALSE.

If p is false, q is true then p-q TRUE.

If p is false. q is false then p-q TRUE .

P	q	→ P q
T	T	T
T	F	F
F	T	T
F	F	T

EX: 19 p: Today Is Sunday.

q: It is holiday.

$P \longrightarrow q$: If today is Sunday then it is holiday.

P	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	F

9.6.5 TRUTH TALE for IF AND ONLY IF (BICONDITONAL):

Rule: p and q TRUE , then p $\longrightarrow$ *q TRUE.*

P and q FALSE, then p $\longrightarrow$ *q FALSE .*

P or q FALSE p-q FALSE.

P	q	$P \longrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	F

Check your progress – 9.6

Write TRUTH TABLE of the following:

1) $pV \sim p$ 2) $\sim p \; \Lambda q$ 3) $(p\, Vq)\, V \sim p$ 4) $\sim (p \,\Lambda \sim q)$

5) $(p \,\Lambda\, q) - \sim(\sim p\, V \sim q)\;\;)$

9.7 TAUTOLOGY

Definition: A statement pattern which is true (always takes value only TRUE)

EX: 20 p V ~ p

TRUTH table

P	$\sim p$	$P \; V \sim p$
T	F	T
F	T	T

ANS: It is tautology.

Since the TRUTH table values of a tautology is always TRUE ,the last column of the TRUTH TABLE of tautology has all Ts only 002

EX: $[p\Lambda \; (p \rightarrow q\;)] \rightarrow q$

P	q	$p \rightarrow q$	$P \;\Lambda\, (p \quad q)$	$[p\Lambda \; (p \rightarrow q\;)] \rightarrow q$
T	T	T	T	T
T	F	F	F	F
F	T	F	F	F
F	F	T	F	T

ANS: It is Tautology.

9.8 CONTRADICTION (Fallacy)

Definition: Statement which is always false for all TRUTH values is called a CONTRADICTION.

EX: 21 $p \,\Lambda \sim p$

P	$\sim p$	$P \Lambda \sim p$
T	F	F
F	T	F

Last column of TRUTH table of (p Λ ~p)is F.
It is CONTRADICTION (Fallacy).

Check your progress – 9.8

Using TRUTH tables check which of the following statement is a TAUTOLOGY or CONTRADICTION or neither

1) $p \, V \, q$

2) $(p \, V \, q) \; - \; q$

3) $(p \, \Lambda \, q) \Lambda \; \sim q$

4) $(pV \, q) \; \Lambda \sim q$

5) $(p\Lambda \, q) \; \Lambda \sim q$

9.8 SUMMARY

Mathematical Logic means we take a proper meaning of Mathematics. If proper condition and situation is given we can transfer it in mathematical relation as variable x and double of it means 2x. Tautology means all sentences are true with given proper condition. Contradiction is, we contradict the situation, the condition given in the sentence.

e.g. 2 is prime number. Sentence is true. But 2 is divisible by 3, this contradict the condition of prime number.

9.9 CHECK YOUR PROGRESS – ANSWERS

9.2

1. False 2. False
3. False 4. False
5. True

9.5

1. $p \Lambda \sim q$

p	~p	$\sim p \Lambda \sim q$
T	F	T
F	T	T

2. $\sim p \, \Lambda \sim q$

p	~p	q	~ p Λ q
T	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F

3. **(***p V q***)** *V~* p

p	q	p Λ q	~p	(p V q) V~ p
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

4. ~ **(***p* Λ ~ *q***)**

p	q	~ q	(p Λ ~ q)	~ (p Λ ~ q)
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

5. **(**~p V ~ q)

p	q	p Λ q	~ p	~ q	**(**~p V ~ q)
T	T	T	F	F	F
T	F	F	F	T	F
F	T	F	T	F	F
F	F	F	T	T	T

9.8

1.

p	~p	*p* V *q*
T	T	T
T	F	F
F	T	F
F	F	F

Ans. : Not Tautology

Not Contradiction

2.

p	q	p V q	~ q	(p V q) V ~q
T	T	T	F	F
T	F	F	T	T
F	F	F	T	F
F	T	F	F	F
T	F	F	T	F

Ans. : Not Tautology

Not Contradiction

3. Contradiction
4. Tautology
5. Not contradiction, Not Tautology

9.10 QUESTIONS FOR SELF - STUDY

Solve the following problems.

1. Given statements –

   p : Pratik is good student

   q : Pratik is honest student

   Write i) ~ *p*

   ii) ~ q

   iii) p Λ q

   iv) p V q

   v) Truth table of p Λ q and p V q

2. Draw Venn diagram of –

   p : is positive number

   q : is even number

   i) p

   ii) q

   iii) ~ *p*

   iv) ~ *q*

   v) *p Λ q*

   vi) *p V q*

3. Check for Tautology

   p : 3 is odd number

   q : 5 is divisible by one and itself

   Write –

   ii) ~ *p*

   iii) ~ *q*

   iv) *p Λ ~q*

   v) ~ *p V~ q*

9.11 SUGGESTED READINGS

1. *Pre-degree Mathematics* by Vaze, Gosavi
2. *Mathematics and Statistics by* M. L. Vaidya and M. K. Kelkar
3. *Statistical Analysis: A Computer - Oriented Approach Introduction to Mathematical Statistics by* S. P. Azen & A. A.Afifi

◈◈◈

NOTES

NOTES

Part -II

Statistics

CHAPTER 1

Introduction to Statistics

1.0 OBJECTIVES

In our everyday life we make use of numbers or figures. These numbers is an information expressed in numerical form and is generally refered as data in statistics. It may also be in the form of tables. After studying this chapter you will be able to

- explain terminologies such as statistics, statistical methods.
- discuss and explain the need of and scope of statistics in different fields.

1.1 INTRODUCTION

The word statistics seems to be derived from the word statist, the known use of which dates back to 1602, when it was used in 'Hamlet' by Shakespeare. The numerical information used by the statists for the purpose of administration of state was termed as statistics. At present the word statistics is used to mean numerical data pertaining to some department of inquiry and it also means the science of Statistics which includes a number of statistical methods such as collection, classification, analysis and interpretation of numerical data.

1.2 DEFINITIONS OF STATISTICS

Different persons have defaced statistics in different ways. Some of them have defined statistics as numerical data and the others have defined it as a science. Most of them have described statistics as it appealed to them. Therefore none of the definitions has defined statistics quite comprehensively. Some of these definitions are given below.

a) **Webster** defined statistics as 'The classified facts representing the condition of people in a state, especially those facts which can be stated in numbers or in tables or in any tubular or classified arrangement.' This definition is too narrow as it confines the scope of statistics only to such facts and figures that represent the condition of the people in a state. Since statistics may represent various other facts such as biological, physical, commercial and others, this definition is quite inadequate.

b) **Prof. Horace Secrist** has given a more comprehensive definition which read as 'statistics are the aggregates of facts affected to a marked extent by multiplicity of causes, numerically expressed, enumerated estimated according to reasonable standards of accuracy, collected in a systematic manner for a predetermined purpose and placed in relation to each other.' This definition clearly points out the characteristics which numerical data must possess in order that they can be called statistics.

Check Your Progress – 1.1-1.2

1. What is statistics?

____

____

2. How statistics id defined by Webster?

____

____

1.3 IMPORTANCE OF STATISTICS

Importance of Statistics as a science lies in the service it has rendered to the mankind. In recent years, the growth of statistics has made itself felt in almost every phase of human activity. It no longer consists merely of collection of data and presenting it in tables or diagrams. It is now considered to encompass the science of making decisions in the face of uncertainty. This covers considerable ground since uncertainties are met when we roll a die, a doctor treats a patient, an actuary determines life insurance premiums, meteorologists make weather forecasts, a broker makes predictions about prices of shares, a newspaper predicts an election, so on and so forth.

Statistics, in its present state of development, can handle most of the situations involving uncertainties. It at least provides the models that are needed to study situations involving uncertainties. Statements, in any form become stronger, precise and more appealing when they are supported by relevant statistics. Statistical methods provide tools to summaries the complex numerical data and to present them in a manner which is easily intelligible. Statistics has provided techniques like statistical quality control which have made revolutionary changes in the field of industrial production. Statistical methods are widely used in the field of agriculture in estimating yield of a crop, in testing effectiveness of fertilizers, methods of irrigation and water management, in developing new varieties of seeds etc.

To sum up one can say that it is hardly possible to single out a department of human activity where statistics has not creeped in. It has rather become indispensible in all phases of human endeavour.

1.4 SCOPE OF STATISTICS

The field of application of statistics is expanding very fast in modern times. It is of immense value not only to the administrators of a state but also economists, businessmen, scientists and research workers in sociology and psychology as well.

In industry, statistical methods are used in estimating demand for a production in future and estimating the need for a raw material, labour, finances etc. In large scale production the statistical quality control techniques are used to reduce rejection of the product and wastage of raw materials, which results in increasing profits by reducing cost of production.

Statistical techniques are widely used in Economics also. Economics is mainly concerned with the production and distribution of wealth and with the consumption, saving and investment of income. Statistics is also used in formulating taxation policies. The economists have to depend on statistics to a great extent in solving problems confronted in production and distribution of essential commodities. The policies of reducing unemployment, poverty, rising prices etc. also depend on Statistics to a fairly good extent.

The management techniques are developing very fast in the twentieth century due to enormous growth of industry and business. Decision making is the prime function of any management and the statistical information and statistical techniques provide a sound basis for all sorts of decision. Since the complexities of business environment make the process of decision making difficult, the decision maker cannot rely entirely upon his observation, experience or evaluation to make a decision. Decisions have to be based on data which show relationship, indicate trends and show rates of change in various relevant variables. Statistics provides methods for collecting, presenting, analysing and interpreting meaningfully such data which is helpful in better decision making. The various statistical tools guide a manager in selecting the best course of action under given circumstances. The decisions relating to production, pricing purchasing and controlling various activities are rendered easier with the help of statistics.

There is a wide scope for application of statistical methods in the field of basic sciences like Biology, Astronomy, Meteorology, Physics, Chemistry etc. Research work carried out in different branches of science proves that it is impossible to conduct any research without the help of statistics. It is used as a scientific method in development of different branches of science. Statistical methods are used in establishing laws and principles in science and also in validating the same.

In the field of medical science almost all the conclusions are based on observations and experimentation. The statements like 'smoking is injurious to health, 'chewing tobacco causes cancer' are the statistical conclusions based on systematically collected data. Statistical methods are used in planning the experiments and analysing the result for testing the effectiveness of different medicines and their hazards. There can be no research in the medical sciences in modern times, without being supported by statistics.

The usefulness of statistics as a scientific method of studies in sociology and psychology has been widely recognised in modern age. The sociological studies are based on properly designed sample inquiries which involve planning of inquiry, collection of data, analysis and interpretation of these data. Since these sciences are not exact sciences, the observed facts can be handled more purposefully only by using statistical techniques. These studies are useful for planning and execution of social welfare activities to be undertaken by government or private agencies.

Check Your Progress. - 1.3 & 1.4

1. List the use of statistical methods in agriculture field.

   ____

   ____

2. List the use of statistical methods in medical science field.

   ____

   ____

3. List the branches of basic science in which statistical methods are used.

   ____

   ____

1.5 SUMMARY

This chapter explains in detail the importance of statistics to mankind and scope of statistics in different fields. Thus we see that statistics is very important subject and is useful in almost all areas.

1.6 CHECK YOUR PROGRESS - ANSWERS

1.1 &1.2

1. The numerical information used by the statists for the purpose of administration of state was termed as statistics.

2. **Webster** defined statistics as 'The classified facts representing the condition of people in a state, especially those facts which can be stated in numbers or in tables or in any tubular or classified arrangement.'

1.3 & 1.4

1. Statistical methods are widely used in the field of agriculture in estimating yield of a crop, in testing effectiveness of fertilizers, methods of irrigation and water management, in developing new varieties of seeds etc.
2. In medical science field Statistical methods are used in planning the experiments and analysing the result for testing the effectiveness of different medicines and their hazards. Also for research.
3. Application of statistical methods in the field of basic sciences like Biology, Astronomy, Meteorology, Physics, Chemistry etc.

1.7 QUESTIONS FOR SELF - STUDY

1. Explain the different meaning of the word Statistics
2. Give the definitions of statistics by Webster, Secrist
3. What is the role of statistics in economic planning?
4. Describe the scope of statistics in business and industry.
5. Explain the role of statistics in social sciences.
6. Describe the importance of statistics in management science.

1.8 SUGGESTED READINGS

1. Mathematics and Statistics by M. L. Vaidya, M. K. Kelkar
2. Statistical Analysis by S. P. Azen and A. A. Afifi

◆◆◆

NOTES

NOTES

Chapter 2

Statistical Data

2.0 OBJECTIVES

After studying this chapter you will be able to :-

- explain terms - attributes, variables, raw data, classification of data, population sample
- draw graphical representation of data (Histogram Frequency polygon and Ogive curves)
- describe Diagramatic representation of data. (Simple bar diagram subdivided Bar diagram, Pie diagram)

2.1 INTROUDCTION

Statistics in concerned with scientific methods for collecting organising, summerising, presenting and analysing data as well as drawing conclusions and making reasonable decisions on the basis of such analysis.

2.2 NATURE OF SUBJECT

Suppose we want to compare the performance in Mathematics of two divisions in the examination. The first thing we have to do is to collect the marks of students. These marks are collected is called, “Data” Hence first step of statistics is to collect the data. But, merely looking at the mark lists, we will not get any idea about performance of students in two divisions under consideration.

We have to find,

i) number of fail students

i.e. students getting less than 40 marks.

ii) number of pass students,

i.e. students getting equal to and more than 40 marks.

Here again we have consider the class of students means

I class – Marks 60 & up to 74 above

II class – 40 to 59

Distinction – 75 and above

This all means hat we must make “Classification”, of the collected data.

Hence, broadly speaking we can say that,

1) Collection of data

2) Classification of data

3) Diagrammatic representation of data

4) Analysis

5) Inference

are the different aspects of this subject (statistics).

Sampling : When a population is very large or infinite practically it is not possible to collect desired information on all the units of population. This may happen even in case of small or finite population when measurements of a variable is costly or in some cases destructive in nature. In such cases we select a small group of units drawn from the population to carry out investigation. This small group of units drawn from the population is called a sample. e.g. to find average height of student studying in a college instead of taking measurement of height on all students of college (population) we select a small group of no. of students (sample) and take measurements on such selected students. Similarly to estimate the average life of electric bulbs produced by a company a sample of bulbs is taken from the large population of no. of bulbs produced and the life time of each bulb from a sample is determined by actually burning out the bulbs. The no. of units in the sample is called the size of that sample. In real life there are many situations where we use sample from population for making judgement about the population.

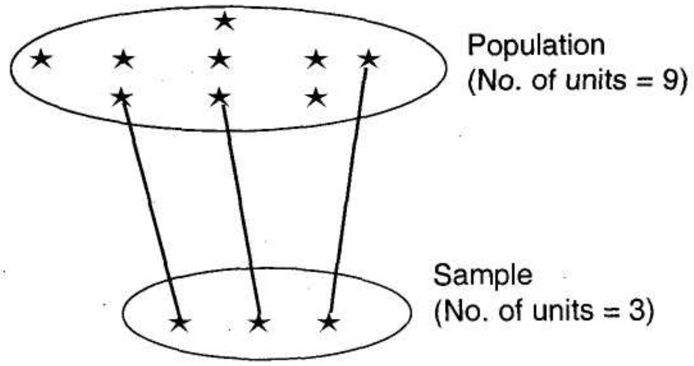
Following are few examples

(1) For judging the quality of rice in a bag we pick up handful of rice and judge the quality of rice in bag.

(2) The average yield of crop can be estimated by selecting a sample of farms and finding the mean yield per hectare for these forms.

(3) A housewife confirms whether the food is properly cooked or not with the help of few particles taken out of the container. Clearly the food in container is population and food taken out of container for inspection is sample.

(4) For testing quality of milk a small quantity of milk is tested instead of entire bulk. Concept of population and sample can be easily understand from following diagram



2.3 LANGUATE OF STATISTICS

Every subject has got its own special terminology; the special words are used for special purpose. So the terms used in, are

2.3.1 Population

In common language used word 'Population' as no of people live in that particular area. But in statistics, we use this word, 'Population' for any 'Collection of articles (items) under consideration of our study purpose.'

e.g.- 1) Students in class

2) Workers in Industry

3) Radio sets

4) T. V. sets

5) Variety of Mobiles available now a days.

6) Agricultural field yields.

So each member, object or observations of the population is called, 'an individual' or 'member' or 'element' of that population.

The population is also called "an Universe".

2.3.2 Variables

Each individual in the population is studied for a certain character or characteristic.

e.g. Height
Weight
Marks in a subject
Rain fall in a region
Yield of production of variety of crop
Production in factory.

Variables are of two types

I) Quantitative – can be counted as a number.
e. g. Height and weight of a student.
Temperature recorded in Month of May

II) Quantitative – When the character is qualitative in nature and hence not expressible in numerical forms, it is called qualitative (an attribute)
e. g. Sex, Religion. Mother tongue. Faculty Nationality.

None of these can be expressed numerically, but each will divide a population is two or more groups; as

Sex – gives you no. of males & no, of females.

Mother tongue – Marathi, Hindi, Tamil, Guajarati.

2.3.3 Size of Population

The number of units constituting the population is called size of that population.

e. g. B. C. A. = 100 Students.
Size of Population = 100
So this population is called, 'Finite Population'.
The other is 'In finite Population.'

e.g. N = Set of 'Natural Numbers'
= {1, 2, 3 ……….}
Number of elements are infinite means uncountable.

2.3.4 Discrete and Continuous Variables

I) Discrete variable – A variable is said to be discrete if it a takes distinct and isolated values.

e. g. Number of daily accidents in city.
Number of family members.
Number of decayed teeth of a child.

II) Continuous Variable – A Variable is said to be continuous when it takes all possible values in an interval.
e.g. weights of persons in a group
Temperature of Certain place

2.4 CLASSIFICATION OF DATA

The raw data are very difficult to understand and we cannot draw any conclusions from them unless we process it. The data so obtained after processing is called as secondary data.

Suppose a collection of on a certain characteristics. Such a set of numbers does not help in drawing any conclusion about the data. The data can be made more meaningful by an ordered arrangement or by dividing it into different groups or classes. This process is called, 'Classification of Data.'

2.4.1 Classification by attributes

When the characteristics under consideration is qualitative types or an attribute; the simplest way of classification is to put all the items or units possessing that attribute in one class and remaining items in other class. Such classification is called simple classification by attribute or dichotomy. e.g. we may classify group of persons into two classes males and females according to attribute sex. Similarly a group of individuals may be classified into smokers and non-smokers with attribute smoking habit.

If we classify group of items or units or individuals into more than tow classes then such classification is called manifold classification. e.g. group of persons may be classified according to their mother tongue into different classes such as persons having mother tongue Marathi, Tamil, Telgu, Punjabi etc.

In any type of classification by attributes either dichotomy or manifold the important thing is that the classes should be defined unambiguously. The classes should be mutually exclusive and exhaustive. An item should belong to one and only one class and no item should miss the classification.

2.4.2 Classification of variables

When the character under study is quantitative type or variable the classification is done according to values of variables.

In case of discrete variables like chest size of banians in the stock held by hosiery shop, the variable assumes only a few values like 60, 65, 70, 75, 80 -- 100 cms. Here each possible value of variable forms a class. These classes are said to form discrete series of observations on that variable. The no. of children in family, no. of accidents in a day in a city, size of footwear etc. are some examples of variable which can be classified in this way.

There are generally two types of variables we want to study for general consideration.

Suppose we have the following information about the number of accidents that in a month in a certain city.

1	0	3	1	3	4	3	4	0	2
2	2	3	2	3	4	3	5	0	5
4	2	1	1	4	3	2	3	4	5

We observed that the data we recorded is a discrete type means there was number of minimum '0' accident and maximum '5' accidents. But there is no such case that accident is 1.5 or -1. This means variable under consideration is positive and Integer.

So it is called discrete type and the 'Distribution' is called as,

'Ungrouped Frequency Distribution'

Using tally marks we write this information in a tabular form as shown in table-

How to prepare frequency distribution –

i) We find minimum number of accidents is zero and maximum is 5.

So first column as 'Number of accidents and values as 0, 1, 2, 3, 4, 5

ii) Next read the given observation in the data and make '1' called a tally mark in the next column. Read all the data & make such marks.

iii) Third column as 'frequency' count tally marks and accordingly write numbers as frequency.

iv) Check whether we are correct or not as the total of frequency column should be equal to total number of observations give in the data.

Frequency distribution for number accidents

Number of accident	**Tally-marks**	**Number of days Frequency**
0	ǀǀǀ	3
1	ǀǀǀǀ	4
2	卌 ǀ	6
3	卌 ǀǀǀ	8
4	卌 ǀ	6
5	ǀǀǀ	3
	Total	**30**

Grouped Frequency Distribution –

(I) Inclusive Method

(II) Exclusive Method

This type of classification is most popular in practice.

The weights in kg of 50 students in a class are give in the following data-

49	57	59	57	50	45	52	58	56	48
54	50	51	64	49	58	47	53	63	64
49	62	62	54	53	51	53	61	49	47
48	54	48	53	49	46	53	47	51	52
49	56	45	49	51	55	52	46	48	54

To solve the above example. We observed the data Minimum number as 46 and Maximum as 64. Now we will classify this data according to class-intervals.

We shall divide the numbers, in groups as 45-47, 48-50…….

1) Class 'limits (class boundaries) – There are two class limits lower and upper class limit

   Class : 45-47

   Where 45 – lower class limit

   47 – upper class limit.

2) Class intervals (classes)

   45-47, 48-50, 51-53 ……63-65 are called class intervals or classes.

3) Frequency – number of observations included in that class is called 'frequency' of that class.

4) Class width – The difference between lower limit (lower boundary) and upper limit (upper boundary) of that class is called class width.

∟ Class width = upper limit – lower limit

e. g. class : 63-65

class width = upper limit – lower limit

= 65 – 63

= 2

5) Class marks (mid value) – The arithmetic mean or average of the upper and lower limits of a class is called class marks or mid value of that class (class interval)

   ∟ the class mark (class mid value) = $\dfrac{\text{lowerlimit} + \text{upperlimit}}{2}$

   e.g. class – 45-47

   class mark = $\dfrac{45+47}{2}$

   = 46

   (I) Inclusive Method – The upper limit (boundary) is included in that class is called, 'Inclusive method of classification'.
   e.g. class : 51-53
   then values 51, 52, 53 are included.

   (II) Exclusive Method – The upper limit is excluded in that class is called, 'exclusive method of classification'
   e.g. class : 54 – 56 then weight of students upto 56 kg is added in this class but exact 56 kg are not consider it will consider in the next class as in 56-58.
   Class boundaries – In our example
   We considered weight of students which is in kg. first class interval is 45-47 & second is 48-50. But if a student has 47.5 kg weight. Then we have to make classes as 44.5-47.5, 47.5-50.5 …… thus all values between were considered, so this is called as class boundary.
   To get these class boundaries of a class, we add 0.5 to upper limit and subtract 0.5 from the lower limit.

Example : Find class boundaries of the following classes.

(1) 100–104, 104 –109, 110–114,115–119, 120–124

(2) 4–6,6–8,8–12,12–16,16–20

Solution :

(1) Here inclusive method of classification is used.
$\therefore$ class boundaries are
99.5–104.5, 104.5–109.5, 109.5–114.5, 114.5–119.5, 119.5–124.5

(2) Here exclusive method of classification is used

Hence class boundaries are same as class limits.

4–6,6–8,8–12,12–16,16–20

Frequency Distribution of weights

Class interval	**Tally marks**	**Frequency**	**Class boundaries**	**Class marks**
45-45	卌 II	7	44.5-47.5	46
48-50	卌 卌 III	13	47.5-50.5	49
51-53	卌 卌 II	12	50.5-53.5	52
54-56	卌 II	7	53.5-56.5	55
57-59	卌	5	56.5-59.5	58
60-62	III	3	59.5-62.5	61
63-65	III	3	62.5-65.5	64
	Total	50		

e.g. Following is a frequency distribution of no. of students according to their pocket money (in Rs.)

Pocket money	No. of students
50–55	7
55–60	20
60–70	30
70–100	5

In the above frequency distribution variable under study is pocket money (in Rs.) of a student and method of classification is exclusive method. No. of students belonging to respective classes are the class frequencies of those classes.

Cumulative frequencies : Class frequency is the no. of observations in that class. But many times we may be interested in knowing how many items have their values less than (or more than) the given value. e.g. We may be interested in finding no. of students having marks less than 60 or no. of students having marks more than 70. These numbers are also frequencies and are called cumulative frequencies.

The frequencies which give the no. of observations less than given value are called "less than" cumulative frequencies and those giving the no. of items having values more than the given value are called "more than" cumulative frequencies.

The computations of less than and more than frequencies is given below

Height (in cms)	No. of persons	Less than cumulative frequencies	More than cumulative frequencies
140–145	3	3	67
145–150	13	16	64
150–155	30	46	51
155–160	16	62	21
160–165	5	67	5

The less than cumulative frequency of a given class is the no. of observations having their values less than the upper boundary of the given class. In above example less than cumulative frequency of class 150 –155 is 46 means there are 46 persons having height less than 155 cms.

Similarly more than cumulative frequency of a given class is the no. of observations having their values more than the lower boundary of the given class.

In above example the more than cumulative frequency of class 150 –155 is 51 means there are 51 persons having height more than 150 cms.

The table showing classes together with their cumulative frequencies is called cumulative frequency distribution.

Example:

Following is a frequency distribute of no. of screws according to the length (in cms)

Find less than and more than cumulative frequency distribution.

Length (in cms)	No. of screws
1.0 – 1.5	12
1.5 – 2.0	32
2.0 – 2.5	27
2.5 – 3.0	10
3.0 – 3.5	9

Solution :

length (in cms)	No. of screws (frequency)	less than cumulative	more than cumulative
1.0–1.5	12	12	90
1.5–2.0	32	44	78
2.0–2.5	27	71	46
2.5–3.0	10	81	19
3.0–3.5	9	90	9

Check your Progress – 2.4

Fill in the blanks

1. Colour of eyes is a ……………………. Variable.
2. Marks of students is ………………….. Variable
3. We select item randomly is ………………… sampling.
4. The number of observation belong to a particular class is called …………………..
5. Midpoint of a class interval is called. ……………………….

2.5 GRAPHICAL REPRESENTATION OF DATA

The frequency distribution itself brings out some important features of raw data. However these features can be studied more conveniently it we represent it in the diagramatic or graphical form. Many questions about the data can be answered by means of these graphs. The various types of graphs used for presenting frequency distribution are (i) Histogram (ii) Frequency polygon (iii) Ogive curves.

2.5.1 Histogram

This is simple method of representing frequency distribution graphically. In this graph classes are represented by a series of adjacent rectangles. The base of each rectangle is the class interval of that class. The area of each rectangle is proportional to the frequency of that class. Hence when the class intervals are uniform throughout the distribution, the height of each rectangle is proportional to the frequency of corresponding class.

But when the class intervals are not uniform the height of rectangle is proportional to the frequency density of that class. Frequency density of class is ratio of class frequency to its width.

The histogram can distinguish more clearly, class with maximum concentration of frequency, This will be identified by the rectangle with maximum height irrespective of the fact that the class intervals are equal or not. It can be used to determine mode of the distribution.

In case of discrete frequency distribution the rectangles are reduced to vertical lines as the class interval are reduced to zero width. If class intervals are of type 5–9, 10–14, 15–19 etc. they are converted into continuous intervals by finding class boundaries as 4.5–9.5, 9.5–14.5, 14.5–19.5 respectively in order to have the rectangles adjacent to each other.

Example :

Following is a frequency distribution of no. of students according to their marks in a test. Draw histogram for it.

Marks	No. of students
10–20	5
20–30	13
30–40	37
40–50	14
50–60	6

Histogram :

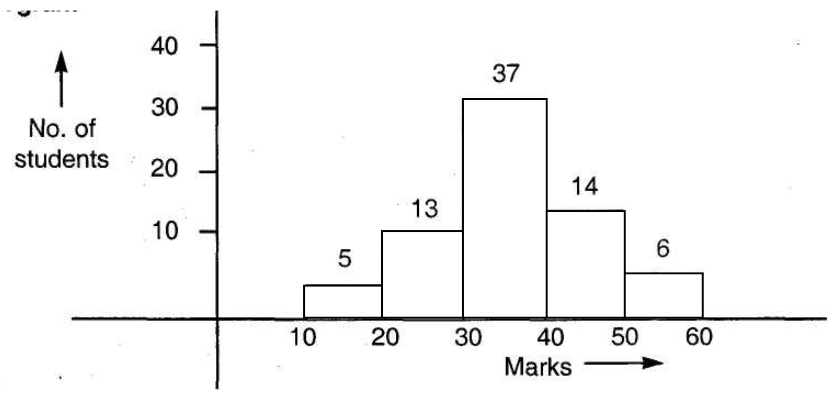


Fig.2.2

2.5.2 Frequency Polygon

Frequency polygon is plotted representing every class by a point on a graph paper. The class mark or mid value of class interval is taken as X - co-ordinate and the frequency of the class as Y-co-ordinate of the point representing the class. Consider two imaginary classes one at each end of the given distribution with frequency zero. These are represented by two points on X axis one at each end. The consecutive points are then connected by segments of straight lines. The figure enclosed by these lines and the X-axis is in the form of polygon and is called frequency polygon.

If the points representing different classes are joined by a smooth curve the curve is called as frequency curve.

From both of these graphs we can answer the queries about symmetry of distribution the points of maximum concentration of the frequency and the nature of frequency distribution.

Example :

Represent following frequency distribution by means of frequency polygon.

Class	Frequency
20–30	3
30–40	10
40–50	23
50–60	17
60–70	6
70–80	11

Solution :

First find the class marks of different classes.

Class	Class Mark	Frequency
20–30	25	3
30–40	35	10
40–50	45	23
50–60	55	17
60–70	65	6
70–80	75	11

Frequency Polygon

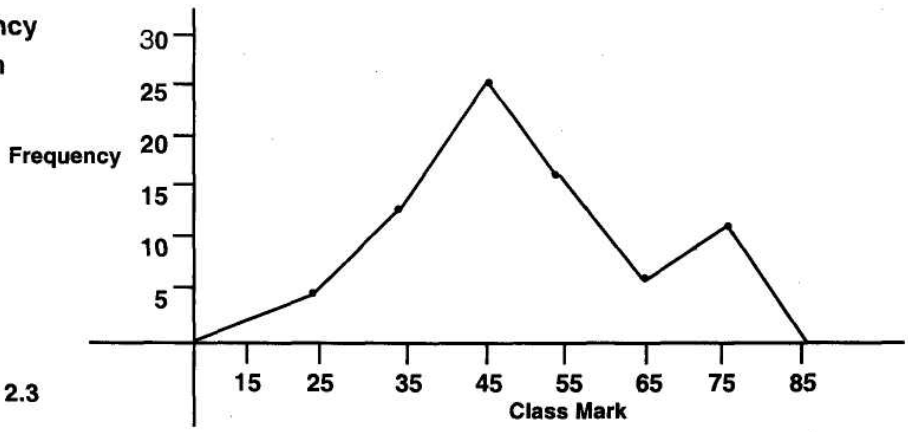


Fig. 2.3

2.5.3 Ogive curve

Ogive curve is also called as cumulative frequency curve. It is a smooth free hand curve passing through the points which have upper & lower class boundary as X-co-ordinate and less than (more than) cumulative frequency as Y-co-ordinate.

Accordingly curve is called less than or more than Ogive curve.

The less than Ogive curve goes on rising from left to right and on the other hand move than Ogive curve goes on declining from left to right. The Ogive curves are very useful as we can determine partition values like median, quantities etc. from them. We can also find the number and percentage of observations which lie between two given values of the variable.

Example 1 : Draw less than Ogive curve for the following distribution of daily wages (in Rs.) of workers in a small scale industry.

Wages	No. of workers	less than cumulative frequency
10–15	8	8
15–20	17	25
20–25	27	42
25–30	13	55
30–35	3	58
35–40	2	60

Less than Ogive curve

Scale

x axis : 1 unit = 5

y axis : 1 unit = 10

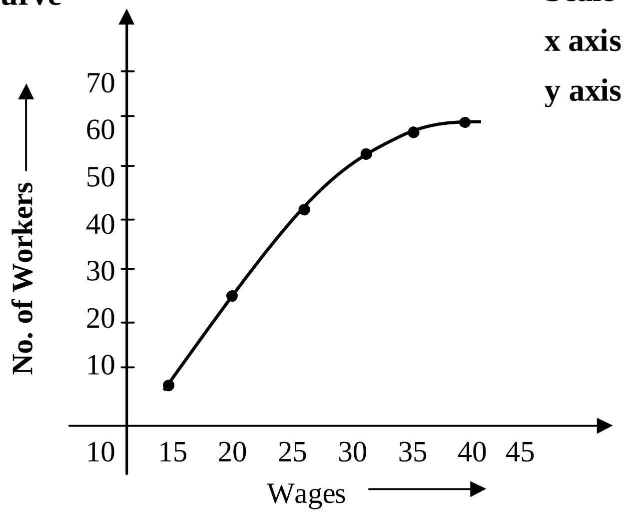


Fig. 2.4

Example 2 : Draw more than Ogive curve for the following data.

Class:	0–10	10–20	20–30	30–40	40–50
Frequency:	5	13	17	3	2

Solution :

Class	Frequency	less than cumulative frequency
0–10	5	40
10–20	13	35
20–30	17	22
30–40	3	5
40–50	2	2

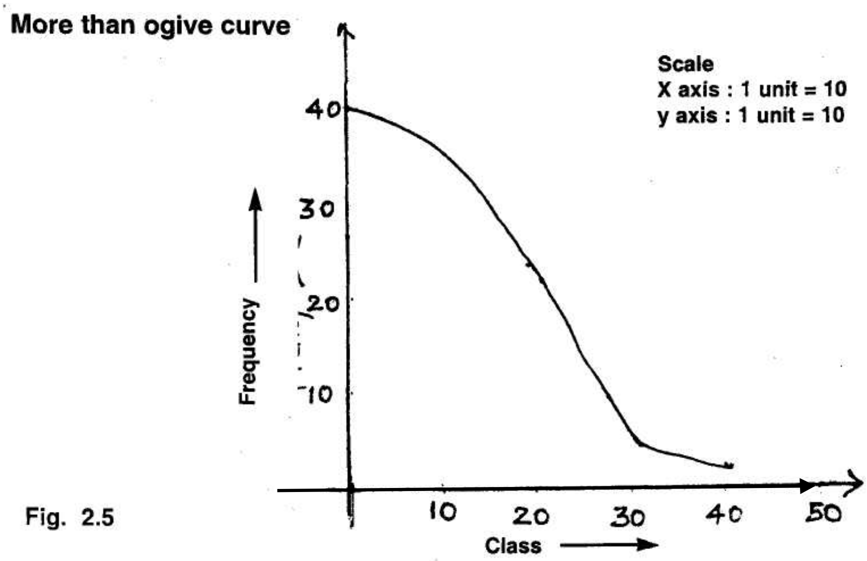


Fig. 2.5

Check Your Progress - 2.5

Write True or False

1. Histogram is a simple method to represent a frequency distribution.
2. Frequency polygon is three dimensional graph.
3. Ogive curve is also called as cumulative frequency curve.
4. There are four types of Ogive curves
5. Frequency polygon is a line graph.

2.6 DIAGRAMATIC REPRESENTATION OF DATA

Frequency distribution can be represented by a graph. But the categorical data cannot be represented by graphs. e.g. distribution of population of country according to religion cannot be represented by graph. Such data can be represented by means of a diagram very attractively. The diagrams are easy to remember as they create longer lasting impression on mind. Statistical data are made easily intelligence by means of diagrams. Following are some commonly used diagrams to represent statistical data.

(1) Simple bar diagram

(2) Sub divided bar diagram

(3) Pie diagram

2.6.1 Simple Bar Diagram

This is the simplest way of presenting the statistical data classified according to a single characteristic. It can be used to present data of population of different cities, exports of different countries etc. It can be used to represent any single series but generally it is used to show categorical series.

In drawing simple bar diagram quantities are represented by rectangular vertical bars separated from each other by uniform distance. The height of bar is proportional to the magnitude it represents. The width of bars must be the same for all bars as it does not have any significance. It is more convenient to use graph for drawing a bar diagram. Usually values of variable are marked along Y-axis and the factor of classification or category are marked on X-axis. The scale of Y-axis must have zero as starting point.

This diagram is also known as one dimentional diagram as it represents only one characteristic. Unless the order of bars has any significance it is suggested that the bars should be arranged in increasing or decreasing order of magnitudes represented by them. This makes the diagram more attractive as well as it facilitates the comparison.

Example :

Present the following information by bar diagram

Country	Birth rate (per 1000)
Iran	30
Libya	20
Malaysia	40
Mexico	30
Sweden	15

Solution :

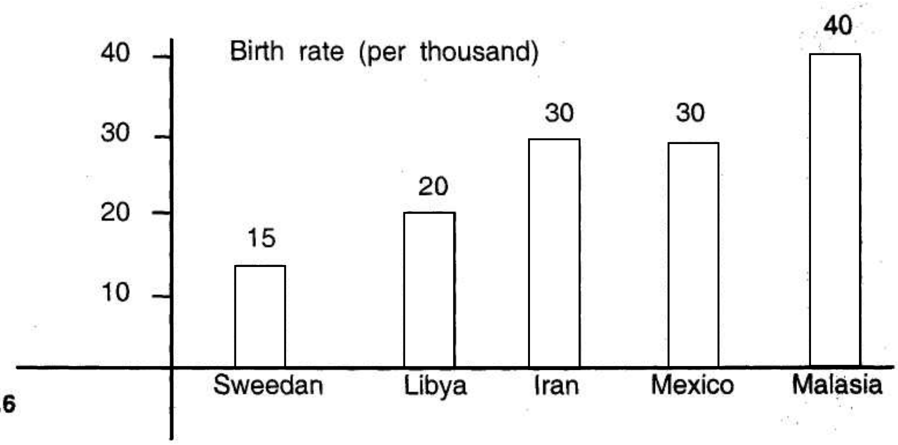


Fig. 2.6

2.6.2 Subdivided bar diagram

In many cases we have to represent a whole quantity and its sub divisions in the same diagram. In that case we can use bar diagram to represent whole quantities and the sub division can be represented proportionally by dividing each bar into number of parts. This type of bar diagram is called subdivided bar diagram.

The subdivided bar diagram is drawn using following steps.

(1) Draw one bar for each of the whole quantity with its height proportional to the magnitude it represents.

(2) Using same scale divide each bar into different parts proportionally. The order of subdivisions from bottom to top should be the same in each bar.

(3) Use different notations like horizontal, vertical or slanting lines or dots or columns for showing subdivisions.

(4) Give the title, scale and explanation of notation used at a suitable place in the diagram.

Subdivided bar diagram can be used (1) to represent data of population of different states in country with its subdivisions according to religion (2) to represent data of students studying in college for no. of years with subdivisions according to classes.

Example:

Represent following data by sub divided bar diagram

Year	No. of Students			
	FY	SY	TY	Total
1995–96	230	115	55	400
1996–97	210	120	60	390
1997–98	190	110	50	350

Solution :
Sub divided
bar diagram

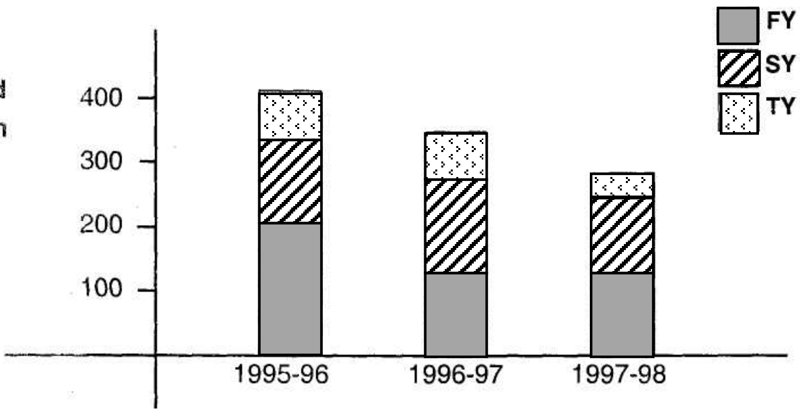


Fig. 2.7

2.6.3 Pie diagram

Pie diagram is special type of diagram used to represent whole quantity by a circle and subdivision of whole quantity are shown by sectors of that circle. The whole circle is divided into different sectors, areas of which are proportional to the magnitudes they represent. It is very easy to divide circle into sectors as the area of each sector is proportional to angle it subtends at the centre. Hence to divide the circle into sectors reduces to divide angle of 360° into proportional parts. The angle θ for a particular sector is given by the relation

$$\theta = \frac{\text{partial quantity}}{\text{Total quantity}} \times 360$$

This diagram is a two dimensional diagram because in this case area of sector represents the quantity. Pie diagram can be used to represent the subdivisions of total budget or total expenditure or total income etc. The name pie diagram is derived from the word pie which means a cake or slice of it with layer of custard on it. For drawing a pie diagram the first step is to convert all the sub quantities into angle θ using above formula. Then draw a circle of suitable size. Start measuring angles from some reference line from centre to circumstances. These angles divide the circle into different sectors. We may mark the sectors by different signs such as dots, crosses, parallel line or different colors, quantities represented by sectors may be written inside the sectors of the circle.

Example :

Draw Pie diagram for the following data on percentage of expenditure on different items in an average family budget.

Items :	Food	Rent	Clothing	Fuel	Others
% expenditure:	40	20	15	10	15

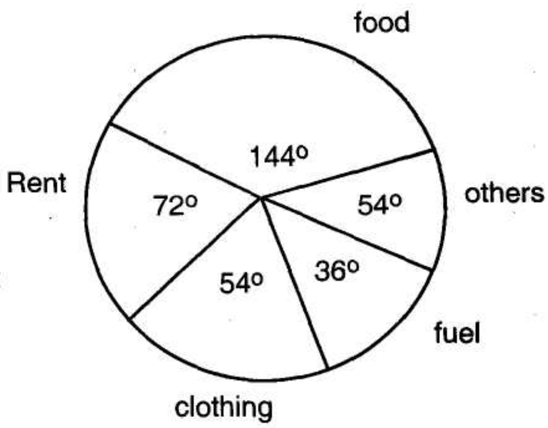
Solution : The first step is to convert the quantities into proportional angles. e.g. angle θ for the item food is

$$\theta = \frac{40}{100} \times 360 = 144°$$

The angles for remaining items are obtained as follows.

Item	**% expenditure**	**angle**
food	40	144°
Rent	20	72°
Clothing	15	54°
Fuel	10	36°
Others	15	54°

The Pie diagram representing these data is given below



Expenditure on different items in family budget

Fig. 2.8

Illustrative examples 2.0

Example 1 Frequency distribution of scores of 80 candidates is given below.

Score	No. of candidates
60–69	3
70–79	7
80–89	16
90–99	20
100–109	14
110–119	11
120–129	7
130–139	2

(1) Find class boundaries of all the classes.

(2) What is the lower class boundary of 4th class?

(3) What is width of 3rd class?

(4) What is the class mark of 6th class?

(5) Find less than cumulative frequencies.

Solution :

Class	Class Frequency	Frequency	Less than cumulative Frequency
60 – 69	59.5 – 69.5	3	3
70 – 79	69.5 – 79.5	7	10
80 – 89	79.5 – 89.5	16	26
90 – 99	89.5 – 99.5	20	46
100 – 109	99.5 – 109.5	14	60
110 – 119	109.5 – 119.5	11	71
120 – 129	119.5 – 129.5	7	78
130 – 139	129.5 – 139.5	2	80

(1) Class boundaries are obtained in above table

(2) Lower boundary of 4th class = 89.5

(3) Width of 3rd class = 89.5–79.5 = 10

(4) Class mark of 6th class = $\frac{110+119}{2}$ = 114.5

(5) Less than cumulative frequencies are obtained in above table

Example 2 Draw histogram and frequency polygon for the following data.

Size of farm (in hectors)	1–20	21–40	41–60	61–80	81–100
No. of farms	13	38	16	5	3

Solution : Here first step is to find class boundaries

Size of farms	**No. of farms**	**Class boundaries**	**Class mark**
1 – 20	13	0.5 – 20.5	10 – 5
21 – 40	38	20.5 – 40.5	30 – 5
41 – 60	16	40.5 – 60.5	50–5
61 – 80	5	60.5 – 80.5	70–5
81 – 100	3	80.5 – 100.5	90–5

Histograms

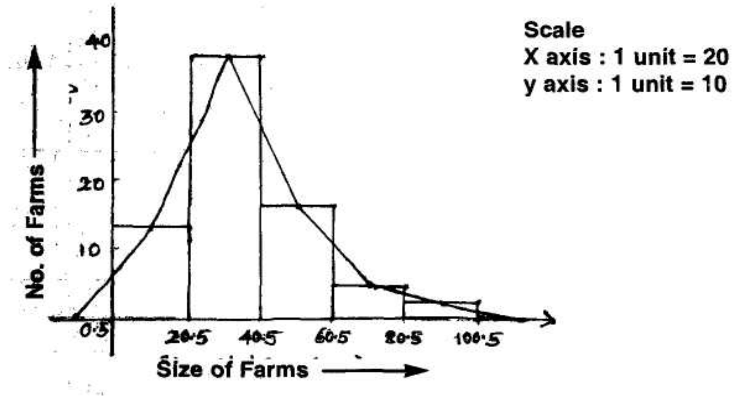


Fig. 2.9

Example 3 Draw less than and more than Ogive curves for the following data

class	**Frequency**	**Class**	**Frequency**
20–25	4	25–30	9
30–35	13	35–40	18
40–45	6	45–50	3
50 – 55	2		

Solution : First we shall find less than and more than cumulative frequencies

Class	**Frequency**	**Less than C.F.**	**Less than C.F**
20–25	4	4	55
25–30	9	13	51
30–35	13	26	42
35–40	18	44	29
40–45	6	50	11
45–50	3	53	5
50–55	2	55	2

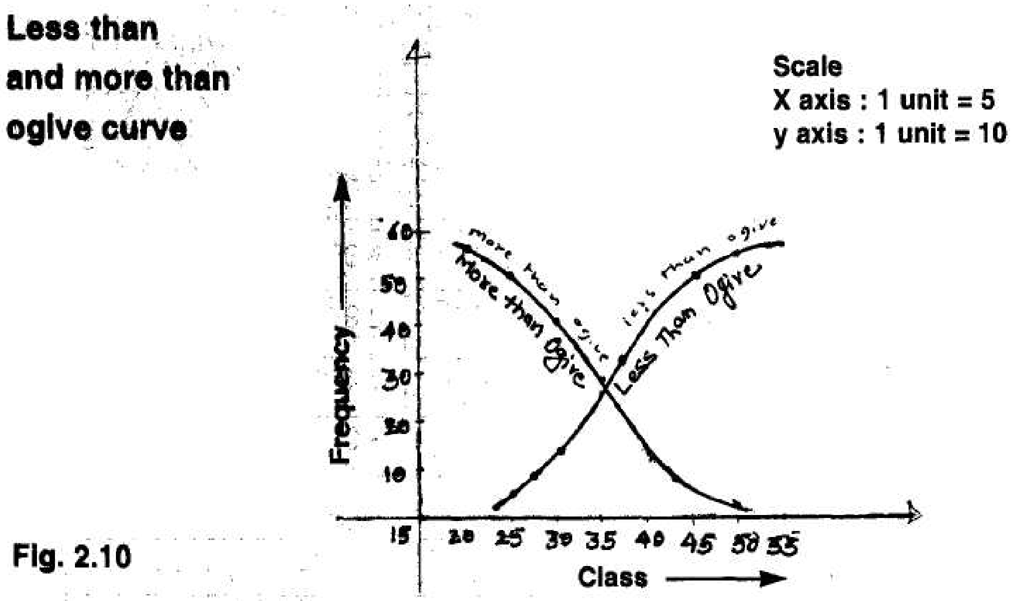


Fig. 2.10

Example 4 : Draw simple bar diagram for the following data on no. of students enrolls or certain course for different years

Year	No. of students
1994	140
1995	210
1996	170
1997	200
1998	180

Solution:

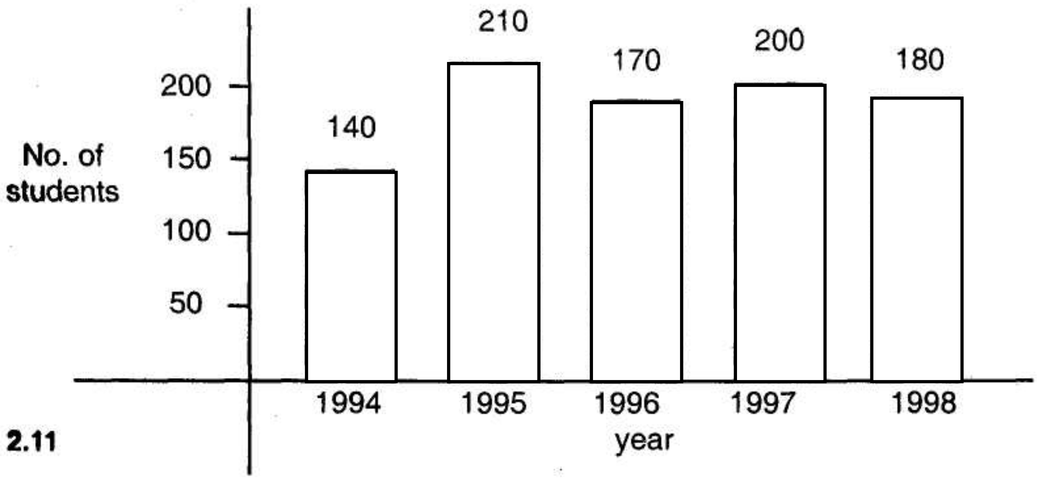


Fig. 2.11

Examples 5 : Present the following data using suitable diagram

Class	F.Y.	S.Y	T.Y
Pass	250	200	100
Fail	100	150	80
Total	350	350	180

Solution :

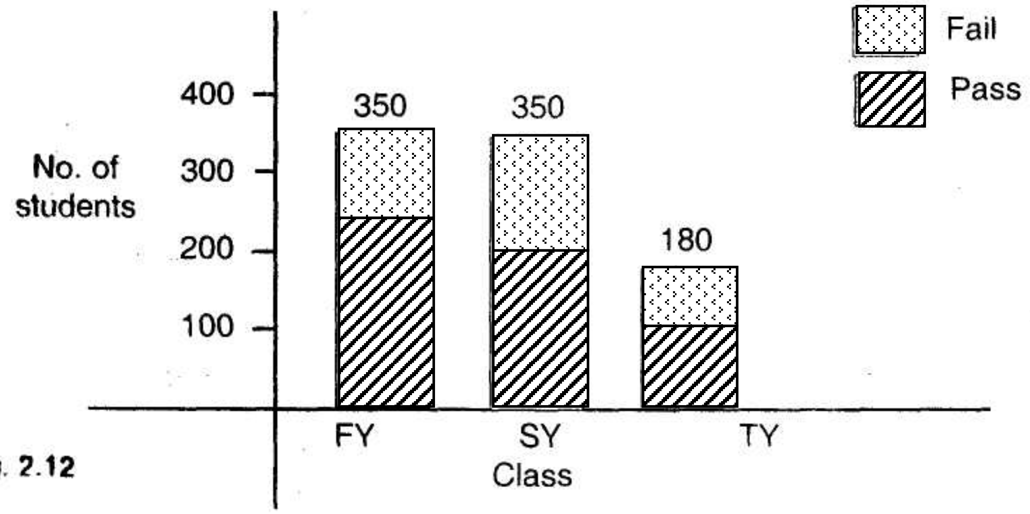


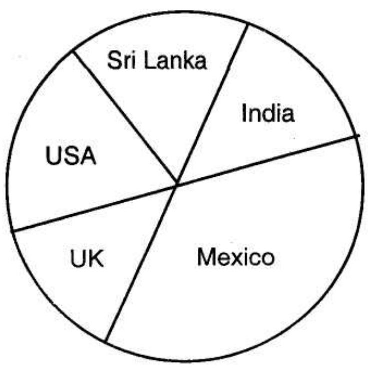
Fig. 2.12

Example 6 : Draw Pie diagram for the following data

Country	India	Sri Lanka	USA	U.K.	Mexico
Population growth rate (%)	2.2	1.8	2.0	1.8	3.2

Solution : Here first step is to find the angles for different countries

Country	**Population growth rate (%)**	**angle**
India	2.2	$\frac{2.2}{11.0}\times 360=72$
Sri Lanka	1.8	$\frac{1.8}{11.0}\times 360=58.91$
USA	2.0	$\frac{2.0}{11.0}\times 360=65.45$
UK	1.8	$\frac{1.8}{11.0}\times 360=58.91$
Mexico	3.2	$\frac{3.2}{11.0}\times 360=104.73$
Total	**11.0**	**360.0**



2.7 SUMMARY

Graphs is very strong statistical tools for presenting a given frequency data. We collect a data with different methods. From the data we draw sample with random sampling then process the raw data. Calculate measures of central tendency an then with the help of given type of decided variable we draw particular types of graphs to get values of variables.

2.8 CHECK YOUR PROGRESS – ANSWERS

2.4

1. Qualitative
2. Quantitative
3. Simple random sampling
4. Class frequency
5. Class marks.

2.5

1. True
2. False
3. True
4. False
5. False

2.9 QUESTIONS FOR SELF - STUDY

1. Explain different methods of classification briefly. Give suitable examples.

2. Explain the following terms with illustrations.

(i) Attribute (ii) Variable (iii) Class limits (iv) Class width (v) Class frequency (vi) Class mark (vii) less than and more than cumulative frequency

3. Following is a frequency distribution of heights in cm.

Height	No. of persons
150–154	2
155–159	17
160–164	29
165–169	21
170–174	1

(i) Find class boundaries of each class.

(ii) Determine class width of each class

(iii) Find less than and more than cumulative frequencies

4. Write a short notes on

(i) Histogram
(ii) frequency polygon
(iii) less than Ogive curve
(iv) more than Ogive curve

5. Draw histogram and frequency polygon for the following data

Class	Frequency
5–10	8
10–15	16
15–20	20
20–25	26
25–30	10
30–35	5

6. **Draw less than and more than Ogive curves for the following frequency distribution.**

Marks	No. of students
0–20	2
20–40	18
40–60	42
60–80	28
80–100	5

7. **Draw histogram for the following data**

Weight (in Kg)	No. of students
30–35	3
35–40	7
40–45	23
45–50	17
50–55	8
55–60	2

8. **Frequency distribution of screws according to their length in cms is given below**

length in cm.	No. of screws
4.0–4.1	13
4.1–4.2	23
4.2–4.3	42
4.3–4.4	67
4.4–4.5	30
4.5–4.6	13
4.6–4.7	12

(i) Determine class boundaries of all the classes
(ii) What is the width of 4th class?
(iii) Find less than cumulative frequencies
(iv) Draw more than Ogive curve.
(v) Draw histogram

9. **Draw suitable diagrams in each of the following cases**

(i) Following are the result of survey regarding viewership of different histograms telecast by Doordarshan

Mahabharata	96 %
Hindi film	65 %
Chitrahar	55 %
Rangoli	36 %
Hindi Serials	50 %
Hindi News	35 %
English News	20 %

(ii) The following table shows the cost of goods produces in a factory for different year

Year	Cost of goods (in Rs.)
1995	20000
1996	14000
1997	26000
1998	21000
1999	34000
2000	37000

(iii)

Category	Revenue in % to total
Corporate tax	43.5
Income tax	35.0
Excise duty	9.5
custom	12.0

(iv) Year No. of student in course

MCM	MCA	BCA	Total	MCM
2008-09	120	125	105	350
2009-10	90	75	90	225
2010-11	100	90	120	310
2011-12	130	120	150	350

(v)

Item	Food	clothing	Recreation	house rent
expenditure	500	325	150	400

10. Write short notes on

(i) Simple bar diagram

(ii) Subdivided bar diagram

(iii) Pie- diagram

11. Construct subdivided bar diagram for the following data

Year	Import	Export
1990	25	23
1995	35	37
2000	31	30
2005	28	32
2010	32	30

12. By the Economic budget of Maharashtra state of 2013-14, 'One Rupee comes from and one Rupee goes to' is given below-

Rupee comes from

No.	Tax & Revenue	Amount (Rs.)
1	Internal debt of the state	14.42%
2	State's own tax revenue	55.23%
3	Loans and advance by state government	0.43%
4	Grants-in-aid from central government	9.59%
5	Share of central taxes	9.31%
6	State's own nontax revenue	6.17%
7	Public account	3.43%
8	Loans from central government	1.42%

Rupee goes to –

No.	Tax and Grants	Amount (Rs.)
1	Social Service	37.08%
2	Grants-in-aid to Local bodies	0.81%
3	Loans and advances given by state government	0.64
4	Repayment of public debt	6.77%
5	Interest payment and debt services	11.69%
6	Capital expenditure	12.31%
7	Economic Services	13.69%
8	General Services	17.00%

2.10 SUGGESTED READINGS

1. Mathematics and Statistics by M. L. Vaidya, M. K. Kelkar
2. Statistical Analysis by S. P. Azen and A. A. Afifi

◆◆◆

NOTES

NOTES

Chapter 3

Measures of Central Tendency

3.0 OBJECTIVES

After studying this chapter you will be able to –

- Explain what is Mean
- Discuss how mode is calculated
- Calculate Median
- Discuss about central Tendency
- Discuss about value of Central item.
- Explain values coming again and again

3.1 INTRODUCTION

We have studied in the previous chapters that the first step towards condensation of raw and large data into compact form is to classify it and prepare frequency distribution. In the form of frequency distribution of data it becomes easy to understand many features of data such as pattern of variation of values, portion of concentration of values, symmetry of distribution etc. It is a descriptive measure as it depicts the pattern of behaviour of the variable. However for further statistical analysis we need the data to be condensed or summarized into a single number which may be taken as the representative number of the whole group. Such a number is called as an average or central value.

In most of the data we note a property of observations or values to concentrate in a central part of data. In other words large proportion of observations are gathered near central value. This property of observations in a data is called as property of central tendency. Naturally we select a representative observation from the central part and such observation in central part around which large no. of observations in a data are concentrated is called measure of central tendency. In most of data average is a centre of concentration of values. In that sense average is called measure of central tendency. The average locates centre of data and in many cases the whole distribution is identified by the average. The average is therefore called measure of location. Average is a descriptive measure and it can focus attention more sharply on various properties of data.

There are many types of averages each having particular properties and each being typical or represented in some unique way. The most frequently used averages are the arithmetic mean, the median and the mode.

3.2 ARITHMETIC MEAN OR MEAN

This is most commonly used and widely applicable average. Mean is a familiar average to a comman man.

Definition : Mean is defined as the ratio of sum of observations in the data to the number of observations.

Computational Formula

In statistics while computing different measures for the data on variable we require to consider two types of data.

(i) Ungrouped data (discrete variables)

(ii) Grouped data or frequency distribution (continuous variable)

Accordingly the computational formula for these two types of data are different.

(I) Ungrouped data

Suppose that x_1 x_2-------, x_n are the n given observations. Then mean of these n observations is denoted by $\bar{x}$ (read as x bar) and is given by

$$\bar{X} = \frac{\text{sum of n observations}}{\text{no. of observations}}$$

$$\bar{X} = \frac{x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n}{n}$$

$$\bar{X} = \frac{\sum x_i}{n}$$

Example 1 :

Annual sales (in ,000 Rs) of a company for 10 months are given below.

23, 47, 29, 32, 25, 30, 34, 32, 25, 35. Find mean annual sales.

Solution :

Here there are n = 10 observations

$$\bar{X} = \frac{x_1 + x_2 + \ldots\ldots\ldots\ldots x_n}{n} = \frac{23 + 47 + \ldots.. + 35}{10}$$

$$= \frac{312}{10} = 31.2$$

$\therefore$ Mean annual sales = 31200 Rs.

(II) Grouped data (frequency distribution)

In case of frequency distribution

suppose

k = no. of classes

X_i = class mark of i^{th} class i = 1,2 .....k

f_i = frequency of i^{th} class

Then mean of frequency distribution is denoted by x and is given by

$$\bar{X} = \frac{x_1 f_1 + x_2 f_2 + \ldots\ldots + x_k f_k}{f_1 + f_2 + \ldots\ldots + f_k}$$

$$\bar{X} = \frac{\sum f_i x_i}{\sum f_i}$$

Steps for finding mean

(1) Find class marks x_i - values of all the classes. i = 1, 2.......k

(2) Find values of $X_i f_i$ i =1,2......k

(3) Using formula find $\bar{x}$

Example 2: Frequency distribution of marks obtained by 100 students is given below

Marks	No. of students
10–20	2
20–30	17
30–40	23
40–50	31
50–60	15
60–70	6
70–80	4
80–90	2

Find mean marks

Solution : First it is required to find class marks of all classes

Marks	**No. of students (f)**	**class mark xi**	**fi-xi**
10–20	2	$\frac{10+20}{2}$ =15	2×15 = 30
20–30	17	$\frac{20+30}{2}$ =25	17×25 = 425
30–40	23	35	805
40–50	31	45	1395
50–60	15	55	825
60–70	6	65	390
70–80	4	75	300
80–90	2	85	170
	Total	100	4340

$$\text{Mean } \bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{4340}{100} = 43.4$$

Mean marks is 43.4

3.2.1 Properties of arithmetic mean

1) If we know the number of values n in the data and mean x then we can find sum of values in the data.

   sum of values = n x $\bar{x}$

2) The sum of deviations of values in the data from its mean is equal to zero.

   If $x_1, x_2, \ldots\ldots, x_n$ are n observations and $\bar{x}$ is their mean. Then $x_1 - \bar{x}$, $x_2 - \bar{x}$, ……$x_n - \bar{x}$ are deviations of these observations from their mean.

   $$\sum\left(x_i - \bar{x}\right) = \sum x_i - n\bar{x} = n\bar{x} - n\bar{x} = 0$$

3) Effect of change of origin and scale

   let x_1 , $x_2, \ldots\ldots\ldots x_n$ be n given observations and $\bar{x}$ is their mean.

   If we transform x_i to u_i

   i = 1, 2 – n using change of origin and scale as

$u_i = \frac{x_i - a}{4}$ i = 1,2....n a, n are constants

then the mean $\overline{u}$ of u_1 , u_2,……,u_n is given by

$\overline{u} = \frac{\overline{x} - a}{n}$ so that

$\overline{x} = a + n\overline{u}$

This result is useful for simplifying the computations of mean particularly when observations are large and in case of frequency distribution.

Example 3 : Following are data on no. of students in different colleges in first year.

105, 110, 98, 103, 105, 101,112, 106

calculate mean no of students

Solution : Here observations are large so we transform them to new observation by subtracting from each observation a suitable constant 100

The new observations are

$u_i = \frac{x_i - a}{n}$ i = 1,2 8

n = 1, a = 100,

$u_1 = 5$, $u_2 = 10$, $u_3 = -2$, $u_4 = 3$, $u_5 = 5$ $u_6 = 1$ $u_7 = 12$, $u_8 = 6$

$$\overline{u} = \frac{\sum u_i}{n} = \frac{40}{8} = 5.00$$

$\therefore$ mean $\overline{x} = a + n\overline{u} = 100 + 5.00 = 105.00$

Example 4 :

For the following frequency distribution on heights of students compute mean height.

Height (in cms)	**No. of students**
120–130	13
130–140	22
140–150	10
150–160	8
160–170	7

Solution : We shall solve this problem by transforming class marks (x_i) to u_i using

Height	No. of students (fi)	Class mark (X_i)	$u_i = \frac{x_i - 145}{10}$	f_iu_i
120–130	13	125	–2	–26
130–140	22	135	–1	–22
140–150	10	145	0	0
150–160	8	155	1	8
160–170	7	165	2	14
	60	Total	–26	

change of origin and scale

$$u = \frac{\sum f_i u_i}{\sum f} = \frac{-26}{60} = 0.4333$$

x=a+nu = 145.4.333=140.6667

3.2.2 Merits and Demerits of mean

The concept of mean is familiar and long usage and hence it seems to be best average or best measure of central tendency. Moreover there are certain limitations in using it. Following are merits and demerits of mean as a measure of central tendency.

Merits : (1) It is rigidly defined and uniquely determined.

(2) It is familiar to common man and easy to compute.

(3) !t is based on all values in the data and therefore is more stable.

(4) It is capable of further algebraic treatment.

(5) It is least affected by sampling fluctuations.

(6) It is widely used in practice and is most commonly used average in many fields.

(7) Observations need not required to be arranged in order for computations of mean.

Demerits : (1) The mean can be used only when characteristic under study is a variable. For attribute type character mean can not be determine.

(2) It is much affected by extreme observations specially when no. of observations in the data is small.

(3) If frequency distribution is having open end classes mean cannot be determined. Because we cannot find class mark value for such open end classes.

(4) It cannot be determined graphically.

3.3 THE MEDIAN

The median of data is the value of central item or observations when the observations are arranged in ascending order of magnitude. For most of the data the median can serve as an average as it will be always located at centre of the data. It is a positional average. There are equal no. of observations above and below the median in the data. It divides the data into two equal parts. It is the most suitable measure of central tendency for distribution's like income distribution or age distribution which are mostly non-symmetric.

Computational formula

(I) Ungrouped data : In case of ungrouped data when n observations are given

median = value of $\left(\frac{n+1}{2}\right)^{th}$ observation when observations are arranged in increasing order (n is odd)

median = mean of $\frac{n^{th}}{2}$ and $\left(\frac{n}{2}+1\right)^{th}$ observation when observations are arranged in ascending order. (n is even)

Example 5 :- Compute median of following observations in each set

(i)10,12,23,17,13,9,17

(ii)37, 31,42, 35, 27, 38, 18, 26

Solution

(1) Here no. of observations n = 7 which is odd no.

First we arrange observations in ascending order.

9,10,12,13,17,17,23

median = value of $(\frac{7+1}{2})^{th}$ = 4^{th} observation =13

(ii) Here no. of observations n = 8 which is even no.

first we arrange observations in ascending order.

18,26,27,31,35,38,42

Median = mean of $(\frac{n}{2})^{th} = 4^{th}$

and $(\frac{n}{2}+1)^{th} = 5^{th}$ observations = $\frac{31+35}{2}$ = 33

(II) **Grouped data :** (Frequency distribution)

In case of grouped data median is given by the following formula.

Median = $L + [\frac{N}{2} - cf]\,\frac{h}{f}$

L = lower class boundary of median class

N = Total frequency = $\sum f_i$

c. f = less than cumulative frequency of the median class.

h = width of median class.

f = frequency of median class.

Median class : It is the class having less than cumulative frequency just greater than or equal to N/2

Steps (1) first find less than cumulative frequencies of all the classes.

(2) Determine median class.

(3) Determine the values of

L, c. f., h, f,

(4) Using the formula compute median

Ratable No. of Value (Rs)	dwelling
0–1000	27
1000–2000	56
2000–3000	85
3000–4000	40
4000–5000	10

Example 6 : Frequency distribution of ratable value of dwelling in locality is given below.

Solutions :

Ratable	No. of value (Rs.)	less than cumulative dwellings frequency
0–1000	27	27
1000–2000	58	85
2000–3000	85	170
3000–4000	40	210
4000–5000	10	220
Total	220	

$\frac{N}{2} = \frac{220}{2} = 110$

for the class 2000–3000 less than cumulative frequency is just greater than $\frac{N}{2} = 110$

(Note that for classes 2000–3000 onwards less than cumulative frequency is greater than 110)

$\therefore$ Median class is 2000–3000

$\therefore$ L = 2000, c.f. = 85, h = 1000, f = 85

$$\text{Median} = L + [\frac{N}{2} - cf]\frac{h}{f}$$

$$= 2000 + [10 - 85]\frac{1000}{85}$$

$$= 2000 + \frac{1000 \times 5}{17}$$

$$= 2000 + 294.11764$$

$$= 2294.11764 \text{ Rs}$$

$$= 2294.12 \text{ Rs.}$$

3.3.1 Merits and Demerits of median : Whenever the mean fails to be a good measure of central tendency the median in general is found to be useful and the appropriate average. It has several advantages and limitations also.

Following are merits and demerits of median

Merits

(1) It is applicable to all kinds of data on variable or attributes. In case of qualitative data the items can be arranged in particular order according to a qualitative character and the quality of central item gives the median or average quality.

(2) For non-symmetric distributions like age distribution or income distribution the median is most appropriate average.

(3) It is not affected much by extreme observations in the data.

(4) Concept of median is easy to understand and is appealing.

(5) It can be determined even if there are open end classes in case of frequency distribution

(6) It is least affected by choice of class intervals.

(7) It is useful when the mean is either indeterminate or unsuitable.

(8) It can be determined graphically.

Demerits :

(1) It is not based on all the observation in the data.

(2) It is not as rigidly defined as the mean.

(3) It is not suitable average from small group of item.

(4) It is less capable of further mathematical treatment.

(5) It needs to arrange observations in ascending order.

3.4 THE MODE

The word mode means fashion. We say that wearing narrow bottom trousers is the current fashion among youngsters. It means that majority of youngsters wear that type of trousers.

The mode Mo is thus defined as the value of the variable occurring more or maximum no. of times in the data than any other value. It is the most frequently occurring value in the data.

Computation of mode: (I) Ungrouped data: In case of ungrouped data of n observations x_1, x_2,...... x_n mode is that observation which occurs maximum number of times.

Example7 : Calculate mode of the following observations. 11,13,17,20,17,15,17,13,17,19.

Solution : In the above observations observation 17 occurs more no. of times as compared to other observations. Hence mode is 17.

(II) Grouped data (Frequency distribution) : In case of frequency distribution mode is given by the following formula.

$$\text{Mode} = L + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right] xh$$

L = lower class boundary of modal class.

f_0 = frequency of pre-modal class

f_1 = frequency of modal class

f_2 = frequency of post modal class

h = width of modal class.

Modal class = It is the class having maximum frequency

Steps (1) Determine modal class

(2) Determine the values of f_0, f_1; f_2, h, L

(3) Using the formula determine the value of mode.

Example 8 : The marks obtained by 40 students in a certain test is given below. Find model marks.

Solution	No. of students
0–10	3
10–20	11
20–30	16
30–40	8
40–50	2

Marks	**No. of students**
0–10	3
10–20	11
20–30	**16**
30–40	8
40–50	2

Here maximum frequency 16 corresponding to class 20–30

modal class is 20–30

L = 20, f_0 = 11, f_1 =16, f_2 = 8h = 10

$$\text{Mode} = L + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right] xh$$

$$= 20 + \frac{16-11}{32-11-8} \times 10$$

$$= 20 + \frac{5\times 10}{13} = 23.8461$$

$$= 23.85$$

3.4.1 Merits and Demerits of mode

As compared to mean and median the mode has very limited utility. Following are merits and demerits of mode.

Demerits :

(1) It is not based on all observations in the data and hence it is not sensitive to the changes in extreme values in the data.

(2) It is not suitable average when the number of items in the data is very small.

(3) It is not suitable average for extremely non-symmetric distributions.

(4) It cannot be determined when maximum frequency is at one end of distribution.

(5) It is affected to a great extent by the choice of class intervals.

Merits :

(1) It is applicable to both qualitative and quantitative type data.

(2) It is useful in some special type of situations only.

(3) It is not influenced by extreme values in the data.

(4) It can be determined graphically

Check Your Progress - 3.2 to 3.4

1. What is mean?

____

____

2. What is median?

____

____

3. What is mode?

____

____

4. Choose the correct alternative.

i. For a set of 101 distinct values, The median value happened to be 55. Later it was observed that a value 74 was wrongly written as 64. With this correction now

a) The median will undergo a change and gets increased.

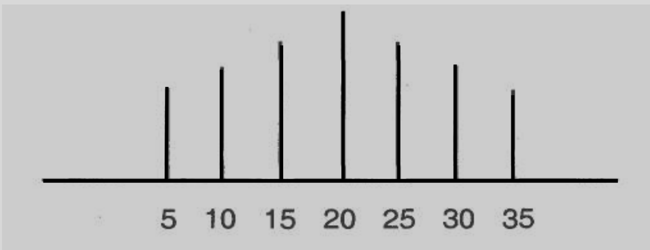
b) The median will undergo a change and gets decreased.

c) The median will be unchanged.

d) The given information is insufficient for recalculation of median.

→

ii. For the following distribution, how would the mean compare with the median?



a) The Mean would be less than the Median

b) The Mean would be equal to the Median

c) The Mean would be greater than the Median

d) None of the above

iii. If a constant value 50 is subtracted from each observation of a set, the mean of the set is

a) increased by 50 b) decreased by 50

c) not affected d) 50 times the original value

iv. A distribution of 6 scores has a median of 21. If the highest score increases by 3 points, the median will become

a) 21 b)21.5

c) 24 d) cannot be determined without additional information

v. The value of $\Sigma(x_i\text{-}x)/n$, is

a) zero if $x = \frac{1}{n}\sum_{i=1}^{n} x_i$ b) always zero

c) n –1 d) none of the above

3.5 SUMMARY

– Mean is nothing but average which is the ratio of sum of observations in the data to the number of observations. It is denoted as x (read as x bar)

– Mean is calculated on two types of data.
  - Ungrouped
  - Grouped

for ungrouped data is formula is

$$\bar{x} = \frac{\sum x_i}{n}$$

Where $\Sigma x_i = x_1 + x_2 + \ldots\ldots x_n$

n = No. of observation

for Grouped data the formula is

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Where $\Sigma f_i\, x_i = x_1 f_1 + x_2 f_2 + \ldots\ldots + x_k\, f_k$

$\Sigma f_i = f_1 + f_2 + \ldots\ldots f_k$

Where k = no. of classes

X_i = class mark of i^{th} class i = 1, 2…..k

f_i = frequency of i^{th} class

– The Median of data is the value of central item of observations when the observations are arranged in ascending order of magnitede.

– Formula of Medion for

(I) Ungrouped data : In case of ungrouped data when n observations are given

median = value of $(\frac{n+1}{2})^{th}$ observation when observations are arranged in increasing order (n is odd)

median = mean of $\frac{n}{2}^{th}$ and $(\frac{n}{2}+1)^{th}$ observation when observations are arranged in ascending order. (n is even)

(II) Grouped data : (Frequency distribution)

In case of grouped data median is given by the following formula.

$$\text{Median} = L + [\frac{N}{2} - cf]\ \frac{h}{f}$$

L = lower class boundary of median class

N = Total frequency $=\Sigma f_i$

c. f = less than cumulative frequency of the median class.

h = width of median class.

f = frequency of median class.

Median class : It is the class having less than cumulative frequency just greater than or equal to N/2

Steps (1) first find less than cumulative frequencies of all the classes.

(2) Determine median class.

(3) Determine the values of L, c. f., h, f,

(4) Using the formula compute median

- The Mode is the value of the variable occurring more or maximum no. of times in the data

than any other value.

Grouped data (Frequency distribution): In case of frequency distribution mode is given by the following formula.

$$\text{Mode} = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$$

L = lower class boundary of modal class.

f_0 = frequency of pre-modal class

f_1 = frequency of modal class

f_2 = frequency of post modal class

h = width of modal class.

Modal class = It is the class having maximum frequency

Steps (1) Determine modal class

(2) Determine the values of f_0, f_1, f_2, h, L

(3) Using the formula determine the value of mode.

Ungrouped data: In case of ungrouped data of n observation $x_1\ x_2 \ldots x_n$ mode is that observation which occurs maximum number of times.

3.6 CHECK YOUR PROGRESS - ANSWERS

3.2 to 3.5

1. Mean is nothing but average which is the ratio of sum of observations in the data to the number of observations. It is denoted as x (read as x bar)
2. The Median of data is the value of central item of observations when the observations are arranged in ascending order of magnitude.
3. The Mode is the value of the variable occurring more or maximum no. of times in the data than any other value.
4. (i) – c, (ii) – b, (iii) – b, (iv) – a (v) - a

3.7 ILLUSTRATIVE EXAMPLES

Example 1 : The starting monthly salaries of 10 employees recruited in a firm are Rs. 1500, 1750, 1680, 1820, 1850, 1750, 2000, 1725, 1575 and 1750

Find the mean, median and the mode

Solution : Let x_1 , $x_{2.....}$ x_{10} be 10 given observations

Mean **:**

Mean $\bar{x} = \frac{\sum x_i}{n} = \frac{17400}{10} = 1740\text{Rs}$

Median : For finding median we arrange observations in ascending order.

1500, 1575, 1680, 1725, 1750, 1750, 1750, 1820, 1850, 2000

Here no. of observations n = 10

Median = mean of 5^{th} and 6^{th} observation = 1750 Rs.

Mode :

Observation 1750 is repeated maximum no. of times.

$\therefore$ mode = 1750

Example 2 The distribution of life time in hrs. of 200 radio tubes is given below.

Calculate the mean, median and the mode

Life	Tubes
600–800	40
800–1000	55
1000–1200	60
1200–1400	25
1400–1600	20

Solution :

Mean : $u = \frac{\sum fu}{} = -\frac{-70}{} = -0.350$

Life (in hrs)	No. of tubes (f_i)	Class mark x_i	less than cf	$u_i = \frac{x_i - 11}{200}$	$f_i u_i$
600–800	40	700	40	–2	–80
800–1000	55	900	95	–1	–55
1000–1200	60	1100	155	0	0
1200–1400	25	1300	180	1	25
1400–1600	20	1500	200	2	40
Total	N =200	–	–	–	–70

Σf 200

x = a + hu a = 1100 h = 200

x = 1100 – 200 x 0.350 = 1030 hrs.

Median

From less than cumulative frequency observe that for the class 1000 –1200 the less than cumulative frequency is just greater than $\frac{N}{2}$ =100

$\therefore$ Median class is 1000 –1200

L = 1000 c. f =95 h = 200 f = 60

$$\text{Median} = L + \left[\frac{N}{2} - cf\right]\frac{h}{f}$$

$$= 1000 + [100–95]\ \frac{200}{60}$$

= 1016.6667 hrs.

Mode : Maximum frequency corresponds to class 1000–1200

$\therefore$ Modal class is 1000–1200

$\therefore$ L = 1000 f_0 =55 f_1 = 60 f_2 = 25 h = 200

$$\text{Median} = L + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right] h$$

$$= 1000 + \left[\frac{60 - 55}{120 - 55 - 25}\right] 200$$

= 1025 hrs

3.8 QUESTIONS FOR SELF - STUDY

1. Define mean and state its important properties.
2. Define median. State merits and demerits of median.
3. Define mode. State merits and demerits of mode.
4. What do you mean by central tendency of data? What is measure of central tendency?
5. State merits and demerits of mean.
6. The lifetime in days of 8 small insects is given below.

   15, 14,28, 19,26, 17, 15,23
7. Find mean and median life time.
8. An incomplete frequency distribution is given below. The total frequency is 230 and the median is 46. Find the missing frequencies.

Marks :	10–20	20–30	30–40	40–50	50–60	60–70
Students	12	30	?	65	?	43

9. The frequency distribution of no. of tablets required to cure fever is given below.

   Find the mean, the median and the mode

Tablets	No. of persons
4–8	11
8–12	13
12–16	16
16–20	14
20–24	8
24–28	5
28–32	3

10. The following is the age distribution of life insurance policy holders whose mean age is 23.6 years. Find the missing frequencies.

Age:	0–10	10–20	20–30	30–40	40–50
Persons:	7	12	?	13	3

11. The monthly expenditure (in Rs.) of 10 families is given below. Find mean, median and mode.

   700, 750, 700, 800, 750, 775, 800, 750, 720, 750

3.9 SUGGESTED READINGS

1. Mathematics and Statistics by M. L. Vaidya, M. K. Kelkar
2. Statistical Analysis by S. P. Azen and A. A. Afifi

◆◆◆

NOTES

NOTES

CHAPTER 4

Measures of Dispersion

4.0 OBJECTIVES

Friends, Dispersion means the expanse of the given sample data. After studying this chapter you will be able to –

- Explain Range
- Discuss Mean
- Calculate Variance

4.1 INTRODUCTION

The average or measure of Central tendency is a good descriptive measure of a distribution of a variable. However it cannot describe the distribution completely. It gives us idea about the location of Centre of the distribution. For complete knowledge of the distribution some additional information is required. One such information is that about nature and extent of variation of the values in the data. The average scoress of two batsmen for a season may be equal or nearly equal but their consistency can be judged by studying the variability of their scores e.g. If the scores of one batsman are 40, 45, 50, 56, 59 and those of other are 20, 35, 50, 60, 85 then they do not differ in average but they differ in variation. Hence only average is not sufficient for comparing the performance of two batsmen. This nature and extent of variation of values in the data is known as dispersion.

The knowledge of dispersion helps in judging the reliability of the average. The average of the data will be more reliable or representative of data when the data has less variability. This analysis of variation in values in the data has no of practical applications in various fields like agriculture, industry medical etc.

For the study of dispersion present in the data we need some measure of the degree of dispersion and it is called measure of dispersion. In the remaining chapter we are going to study some measure of dispersion.

4.2 RANGE

Range is simplest measure among several measures of dispersion. Range is defined as the difference between maximum and minimum observations in the data. In case of frequency distribution range may be defined as the difference between smallest and largest class boundaries.

Since range is the simplest measure to Compute, it is the crude measure of dispersion. The range is used in limited applications and also it has certain defects. It is greatly affected by an unusal value of the extremity. We can not interpret the value of range properly without knowing the no. of observation.

The range is useful is situations where one desires to know only the extent of extreme dispersion. The stock market reports are frequently stated in terms of their range by quoting the high and low price of stock over a period. In weather reports also we find maximum and minimum temperatures stated. The daily mean temperature can be obtained by averaging these two temperatures. In quality contral range is used as a measure of variation within the sample.

The range being easy to compute and is a common way of describing dispersion is often used in engineering and medical reports.

Illustrations (1) : Following are the prices of stock market shares of a certain company for last 10 days. Find the range

123, 98, 96, 120, 115, 121, 117, 151, 101, 99

Answer : Here minimum observation is 96 and maximum observation is 131.

Hence range is 131 – 96 = 35.

4.3 MEAN DEVIATION

Range as a measure of dispersion does not takes into consideration all observations in the data. So it is Comparatively unstable and insensitive measure of dispersion. Hence it is not further useful for analysis of data. Mean deviation is a measure of dispersion based on all observations in the data. By deviation we mean subtracting same constant from given observation and is called deviation of that observation from that constant e.g. deviation of x_1 from arbitrary constant a is $x_1 - a$.

In mean deviation we consider the deviation of each observation from some constant a. The mean of absolute deviations of observations from a is called mean deviation about 'a'.

Definition. (I) In case of n observation $x_1, x_2, \ldots\ldots x_n$ the mean deviation about a is g

$$\text{MD about a} = \frac{\sum^{n} |x_i - a|}{n}$$

(II) In case of frequency distribution

x_i = class mark of i^{th} class i = 1,2, ……… k

f_i = frequency of i^{th}class

mean deviation about a is given by

$$\text{MD about a} = \frac{\Sigma f_i |x_i - a|}{\Sigma f_i}$$

Usually we obtain mean deviation about some central value such as mean or median or mode accordingly we get mean deviation about mean or mean deviation about median or mean deviation about mode.

Illustration : (1) Calculate mean deviation about mean for the following observation.

15, 17, 22, 18, 26, 13, 14, 20, 15, 10

Answer :

	x_i	$\lvert x_i - \bar{x} \rvert$
	15	2
	17	0
	22	5
	18	1
	26	9
	13	4
	14	3
	20	3
	15	2
	10	7
Total	170	36

$$\bar{x} = mean = \frac{\Sigma xi}{n}$$

$$= \frac{170}{10} = 17$$

$$\text{MD about mean} = \frac{\Sigma |x_i - \bar{x}|}{n}$$

$$= \frac{36}{10} = 3.6$$

(2) Frequency distribution of number of days of medical leaves enjoyed by 30 employees is given below.

No. of Days	No. of Employees
0–10	5
10–20	6
20–30	10
30–40	5
40–50	4

Calculates mean deviation about mean.

Solution :

No. of Days	No. of Employees f_i	x_i	f_ix_i	$\lvert x_i - \bar{x}\rvert$	$f_i\lvert x_i - \bar{x}\rvert$
0–10	5	5	25	19	95
10–20	6	15	90	9	54
20–30	10	25	250	1	10
30–40	5	35	175	11	55
40 – 50	4	45	180	21	84
Total	30	–	720		298

$$\bar{x} = \frac{720}{30}\ 24.0$$

$$\text{MD about mean} = \frac{\sum f_i\lvert x_i - \bar{x}\rvert}{\sum f_i} = \frac{298}{30} = 9.93 \text{ days}$$

4.4 VARIANCE

Variance is the mean square deviation about mean. Thus variance is defined as the mean of square of deviations taken from arithmetic mean. Variance is good measure of dispersion and it has many desirable properties. It is denoted by σ^2 (sigma squared)

(I) In case of n observations x_1 x_2……x_n the variance is defined as –

$$\sigma^2 = \frac{\Sigma\left(x_i - \bar{x}\right)^2}{n} \text{ where } \bar{x} = \text{mean} = \frac{\Sigma x_i}{n}$$

Computational formula

$$\sigma^2 = \frac{\Sigma\left(x_i - \bar{x}\right)^2}{n} = \frac{\Sigma x_i^{\ 2}}{n} - \frac{-\bar{2}x\,\Sigma x_i}{n} + \bar{x}^{\ 2}$$

$$= \frac{\Sigma x_i^{\ 2}}{n} - 2\bar{x}^{\ 2} + \bar{x}^{\ 2}$$

$$= \frac{\Sigma x_i^{\ 2}}{n} - \bar{x}^{\ 2}$$

$$\sigma^2 = \left[\frac{\Sigma x_1^{\ 2}}{n} - \bar{x}^{\ 2}\right]$$

(II) In case of frequency distribution.

x_1 = Class mark of i^{th} class I = 1,2,……k

f_i = frequency of i^{th} class

Variance is defined as

$$\sigma^2 = \frac{\Sigma f_I (x_i - \bar{x})^2}{\Sigma f_i} \text{ where } \bar{x} = \text{mean} = \frac{\Sigma f_i x_i}{\Sigma f_i}$$

Computational Formula

$$\sigma^2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{\Sigma f_i} = \frac{\Sigma f_i \left(x_i^2 - 2x_i.\bar{x} + \bar{x}^2\right)}{\Sigma f_i}$$

$$= \frac{\Sigma f_i x_i^2}{\Sigma f_i} - 2\bar{x}\frac{\Sigma f_i x_i}{\Sigma f_i} + \bar{x}^2 = \frac{\Sigma f_i x_i^2}{\Sigma f_i} - \bar{x}^2$$

$$\sigma^2 = \left[\frac{\Sigma f_i x_i^2}{\Sigma f_i} - \bar{x}^2\right]$$

In case of frequency distribution as well as individual observations calculations of variance can be simplifies by making use of charge of origin and scale.

Change of origin and scale :

Let $u_i = \left(\frac{x_i - A}{h}\right)$ be the new observations obtained from x_i by using charge of origin and scale so that -

$X_i = A + hu_i$

$\therefore \bar{x} = A + h\bar{u}$ then variance of u is given by

$\sigma_u^2 = \left[\frac{\Sigma u_i^2}{n} - \bar{u}^2\right]$ (in case of individual observation)

when $\bar{u} = \frac{\Sigma f_i u_i}{zf}$ and $\sigma_u^2 = \left[\frac{\Sigma f_i u_i^2}{\Sigma f_i} - \bar{u}^2\right]$ when $\bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i}$

(in case of frequency distribution)

Then variance of original observation is $\sigma_x^2 = h^2 \sigma_u^2$

Illustrations

(I) Followings are monthly savings (in Rs.) of 10 families.

1150, 750, 700, 1000, 800, 900, 720, 840, 980.

Find variance.

Solution :

x	**x^2**
1150	1322500
750	562500
700	490000
1000	1000000
800	40000
900	810000
720	578400
840	705600
980	960400
650	422500
Total 8490	7431900

$\bar{x} = \frac{8490}{10} = 849$

$\sigma = \frac{\Sigma x^2}{n} - \bar{x}^2$

$=7431900 - 720801$

$=22398$

(2) Find the variance of the following distribution of percentage dividend paid by 50 company.

Dividend	No. of Companies
0–6	8
6–12	10
12–18	15
18 –24	12
24 –30	5

Solution :

Dividend	**No. of Companies (f)**	**Class mark X**	**$u = \frac{x-15}{6}$**	**Fu**	**fu^2**
0–6	8	3	–2	–16	32
6–12	10	9	–1	–10	10
12–18	15	15	0	0	0
18–24	12	21	1	12	12
24–30	5	27	2	10	20
Total	50			–4	74

$u_i = \frac{x_i - 15}{6}$ $\qquad x_i = 15 + 6u_i = A + hu$

gives A = 15 $\qquad$ h = 6

$$\therefore \bar{u} = \frac{\Sigma fu}{\Sigma f} = \frac{-4}{50} = -0.08$$

$$\bar{x} = A + h\bar{u}$$

$= 15 + 6(-0.08) = 14.52$

$= -0.08$

$\sigma_x^2 = h^2 \sigma_u^2$

$$\sigma_u^2 = \frac{\Sigma fu^2}{\Sigma f} - \bar{u}^2 = \frac{74}{50} - (-0.08)^2$$

$$= 1.48 - 0.0064$$

$$= 1.6336$$

$\sigma_x^2 = 58.8096$

4.5 STANDARD DEVIATION

Standard deviation is most Commonly used measure of dispersion. It has been devised to remove the drawback of the variance that it is rather an awkward value to interpret. The units attached to variance are squares of the units in practice. e.g. cm^2, Rs^2 etc. But we define standard deviation as the positive square root of variance or the root mean square deviation from the arithmetic mean. Due to this standard deviation can be expressed in the same units as that of the original data. It also has all advantages of the variance as a measure of dispersion. However from magnitude of standard deviation we cannot immediately say whether distribution has small or high degree of variability.

Standard deviation is denoted by σ or SD.

Formula of Computing SD are as follows :

$$\sigma = \sqrt{\frac{\Sigma x_1^2}{n} - \bar{x}^2} \text{ where } \bar{x} = \frac{\Sigma x_i}{n}$$

(in case of observations $x_1, x_2, \dots\dots\dots x_n$)

$$\sigma = \sqrt{\frac{\Sigma f_i x_i^2}{\Sigma f_1} - \bar{x}^2} \text{ where } \bar{x} = \frac{\Sigma f_i x_1}{\Sigma f_i}$$

(in case of frequency distribution)

4.6 ABSOLUTE AND RELATIVE MEASURES OF DISPERSION

The measure of dispersion like range, mean deviation, variance, standard deviation measures the magnitude of dispersion and they are called measure of absolute dispersion. These are to be expressed with appropriate units. They are useful for comparison of variability of two sets of data only when both are in the same units and their central values ie. averages are nearly equal. But in many problems situations one or both of these conditions are not fulfilled. So we need measures of dispersion which are independent of units. Such a measure can be obtained by taking ratio of the absolute measure of dispersion to same central value of the data. It is called measure of relative dispersion. Most commonly used measure of relative dispersion is coefficient of variation.

4.7 COEFFICIENT OF VARIATION

Coefficient of variation (cv) is widely and commonly used measure of dispersion. Whenever we require to compare the variability of sets of values we use cv. It is defined as the ratio of standard devidation of the series to its arithmetic mean. It is always expressed in percentage.

$$CV = \frac{SD}{mean} \times 100$$

The series which has less CV is said to be more consistent or stable.

Check Your Progress - 4.1 to 4.6

1. Define the following terms.

(a) Range

____

____

(b) Mean deviation

____

____

(c) Variance

____

____

(d) Standard deviation

____

____

2. Choose the correct answer from given.

i) Let A = {–2, –1,0,1,2} and B = {–4, –2, 0, 2, 4}

Let m(.) and v(.) denote the mean and variance respectively of the set mentioned.

Then indentify the correct statement.

a) m(A) > m(B), v(A) < v(B)

b) m(A) > m(B), v(A) < v(B)

c) m(A) = m(B), v(B) = 4v(A)

d) m(A) = m(B), v(B) = 2v(A)

ii) Which of the following measures of dispersion does not depend on the units of measurement?

a) S. D. b) Mean Deviation c) Range d) C. V.

iii) Mean deviation is minimum when the observations are taken from

a) Mean b) Median

c) Mode d) Q4

iv) If you are told that a population has a mean of 25 and variance of zero what must you conclude?

a) Someone has made a mistake

b) There is only one element in the population

c) There are no elements in the population

d) All the elements in the population are 25

v) The following set of scores is obtained on a test

X : 4, 6, 8, 9, 11, 13, 16, 24, 24, 24, 26. The teacher computes all of the descriptive indices of central tendency and variability on these data, then he discovered that an error was made and one of the 24's is actually a 17. Which of the following will be changed from the original computation?

a) median b) range

c) S. D. d) None of the above

vi) If each observation of a series is multiplied by a constant C, the coefficient of variation as compared to the original value

a) is increased by C

b) is decreased by C

c) remains unchanged

d) is C times the original value

4.8 SUMMARY

– Variations of values in the data is known as dispersion.

– Measure of dispersions are Range, Mean deviation, Variance, Standard deviation.

– Range is defined as difference between maximum and minimum observations.

– Mean deviation is a measure of dispersion based on all observations in the data which is calculated by subtracting same constant from given observation and is called as deviation of that observation from that constant.

– Variance is the mean square deviation about mean. Variance is defined as the mean of square of deviations taken from arithmetic mean.

– Standard deviation is the positive square root of variance or the root mean square deviation from the arithmetic mean.

– Range, mean deviation, variance and standard deviation are called as absolute dispersion.

4.9 CHECK YOUR PROGRESS- ANSWERS

4.2 to 4.7

1. (a) Range is defined as difference between maximum and minimum observations.

(b) Mean deviation is a measure of dispersion based on all observations in the data which is calculated by subtracting same constant from given observation and is called as deviation of that observation from that constant.

(c) Variance is the mean square deviation about mean. Variance is defined as the mean of square of deviations taken from arithmetic mean.

(d) standard deviation is the positive square root of variance or the root mean square deviation from the arithmetic mean.

2 (i)–c, (ii)–d, (iii)–b, (iv)–d (v)–c, (vi) – d

4.10 ILLUSTRATIVE EXAMPLES

Example I: The scores of batsmen in a certain test are as given below :

35, 47, 52, 45, 61, 37, 40, 58

Find (i) variance (ii) coefficient of variation

Solution : Here data given are 8 observations

Say $x_1, x_2, \ldots\ldots\ldots\ldots x_8$

We use change of origin for simplifying calculation as

$u_i = x_1 - 45 \quad i = 1, 2\ldots\ldots 8$

So that new observations and their squares are as follows :

X_i	$u_i = X_i - 45$	u^2_i
35	–10	100
47	2	4
52	7	49
45	0	0
61	16	256
37	– 8	64
40	– 5	25
58	13	169
Total	15	667

$$\bar{u} = \frac{\Sigma u_i}{n} = \frac{15}{8} = 1.875$$

$$\therefore \bar{x} = \text{mean} = 45 + \bar{u} = 46.875$$

σ_u^2 = variance for u

$$= \frac{\Sigma u^2{}_i}{n} - \bar{u}^2$$

$$= \frac{667}{8} - (1.875)2$$

$$= 79.8594$$

$$\therefore \sigma_x^2 = \sigma_u^2 = 79.859,\ \sigma_x = +\sqrt{\sigma_x{}^2}$$

$$= 8.9365$$

Example 2 : The number of items of an industrial product sold by two salesman A and B in ten months in an year are given below. From these date determine which salesman is more consistent.

Number of Items Sold

A	128	132	143	140	152	145	135	129	130	145
B	142	150	160	130	120	125	135	145	140	142

Solution : For judging which salesman is more consistent we have to compare the variability of their performance. This can be done more appropriately by comparison of their coefficients of variation. So let us find the mean and the SD for each of the two series, here the given values are large in size. So we may use the method of change of origin.

Number of items sold		u= x - 140	u^2	v = y –140	v2
Salesman A x	Salesman B y				
128	142	–12	144	2	4
132	150	– 8	64	10	100

143	160	3	9	20	400
140	130	0	0	–10	100
152	120	12	144	–20	400
145	125	5	25	–15	225
135	135	– 5	25	– 5	25
129	145	–11	121	5	25
130	150	–10	100	10	100
145	142	5	25	2	4
		–21	657	–1	1383

The computations of standard deviations and means of the series are as follows :

$$\bar{u} = \frac{\Sigma u}{n} = \frac{-21}{10} = -2.1$$

$$\bar{v} = \frac{\Sigma v}{n} = \frac{-1}{10} = -0.1$$

$$\text{SD of u} = \sigma_u = \sqrt{\frac{\Sigma u^2}{n} - (\bar{u})^2}$$

$$= \sqrt{\frac{657}{10} - (-2.1)^2} = \sqrt{65.7 - 4.41}$$

$$= \sqrt{61.29}$$

$$= 7.83$$

$$\text{SD of v} = \sigma_v = \sqrt{\frac{\Sigma v^2}{n} - (\bar{v})^2}$$

$$= \sqrt{\frac{1383}{10} - (-0.1)^2}$$

$$= \sqrt{138.3 - 0.01}$$

$$= \sqrt{138.29}$$

$$= 11.76$$

Since SD is invariant to change of origin

$\sigma_x = \sigma_u = 7.67$ and $\sigma_y = \sigma_v = 11.76$

$$\bar{x} = 140 + \bar{u} = 140 - 2.1 = 137.9$$

$$\bar{y} = 140 + \bar{v} = 140 - 0.1 = 139.9$$

$$\therefore \text{CV of x} = \frac{\sigma_x}{\bar{x}} \times 100 = \frac{7.83}{137.9} \times 100$$

$$= 5.68\ \%$$

$$\therefore \text{CV of y} = \frac{\sigma_y}{\bar{y}} \times 100 = \frac{11.76}{139.9} \times 100$$

$$= 8.4\ \%$$

Since the salesman A has smaller CV, he is more consistent.

4.11 QUESTIONS FOR SELF - STUDY

1. What is dispersion? Why is it necessary to take into consideration the dispersion of . the data?
2. Define range as the measure of dispersion. Discuss its advantages and limitations. Mention some uses of range.
3. Define standard deviation. Establish its importance as a measure of variability?
4. What are the measures of absolute dispersion? Can they be used for comparison of variability?
5. What are the relative measures of dispersion? In what respect are they superior to the absolute measures?
6. Define coefficient of variation. In what situations is it useful?
7. A set of 10 observations has the sum of squares of diviation from the mean equal to 120. Find its variance. If two more values, each equal to mean, are added, what will the variance of the new set?
8. If all the observations in the data are of same value, what will be its SD?
9. If $x_i(i = 1,2 ...n)$ are observations on X, show that

$$\Sigma\, x_i^2 > \frac{(\Sigma\, x_i)^2}{n}$$

10. For 20 observations on Y, $\Sigma\, y^2 = 500$. Show that the mean of the data cannot exceed 5.
11. Are the data n = 10, $\Sigma x^2 = 500$. x = 8, consistent? Give reasons for your answer.
12. A variable takes values 1, 2, 4 ... n. Find the mean and variance.
13. From the following distribution of milk co–operative societies according to procurement of milk per day (in liters), compute standard deviation

Quantity of Milk	100–150	150–200	200–250	250–300	300–350
Societies :	10	20	35	25	10

14. A survey conducted to determine the distance travelled (in Km) per litre of petrol by newly introduced moped yielded the following distribution.

Distance	40–45	45–50	50–55	55–60	60–65
No. of Moped	13	12	25	35	50

Find the standard deviation.

15. The polythene bags were supplied by two suppliers A and B. These bags were tested for bursting pressure and the following data were obtained.
Pressure in Kg. : 30–40 40–50 50–60 60–70 70–80 80–90

Bags A :	4	6	15	25	20	10
Bags B :	6	14	20	25	10	5

Which supplier's bags have more consistency in bursting pressure?

16. Marks obtained by two students in the ten different papers at an examinations are given below. Find who is more consistent.

Marks A **:**	50	60	35	40	70	90	65	70	75	75
Marks B**:**	60	65	78	72	80	55	45	65	75	80

17. The mean of 5 observations is 4.4 and the variance is 8.24. If three of the five observations are 1, 2 and 6, find the other two.

18. The statistics of runs scored by the batsmen A and B in 1o innings are given below.

	Player A	Player B
Mean	53	45
Standard deviation	40	16

Which of the two players is more consistent?

19. Find the standard deviation of the following frequency distribution.

x :	1	2	3	4	5
f :	k	2k	3k	4k	5k

20. For a group of 10 observations on X

$\Sigma x = 452$ and $\Sigma x^2 = 24270$. Find the standard deviation.

4.12 SUGGESTED READINGS

1. Mathematics and Statistics by M. L. Vaidya, M. K. Kelkar
2. Statistical Analysis by S. P. Azen and A. A. Afifi
3. Pre-degree Mathematics by Vaze, Gosavi

◆◆◆

NOTES

Chapter 5

Correlation

5.0 OBJECTIVES

After studying this chapter you will be able to explain following -

- Two variables
- Relations between two variables
- Positive relations
- Negative relation
- Reduce the negative relation

5.1 INTRODUCTION

Bivariate Data :

The data we have studied upto this stage were consisting of observations on a single variable and are called the univariate data. But there are many situations in which we are interested in observations on two variables for every unit in a sample or a group of units. If we observe consumption of coal X and production of electricity Y for 30 days in a month. We get pairs of values (x_i, y_i) for i = 1, 2,…..30. These data on two variables constitute bivariate data. In short the set of pairs of observations on two variables are called bivariate data. For example, the observations on:

(i) Age of husband and age of wife in several married couples.

(ii) Intelligence quotient and score in aptitude test of students in a class.

(iii) Supply and price of a commodity in a market on several days, are some examples of the bivariate data.

5.2 CORRELATION

The major interest in collection and study of bivariate data is in finding whether there is any mutual relation between the two variables under consideration or not. This mutual or joint relation between the two variables is called correlation which can be ascertained by studying the joint variation of the two variables in the data. For example, if we observe the data on consumption of coal and production of electricity at a thermal power plant, we find that there is relation between these variables because more consumption of coal is bound to produce more electricity and shortage of coal is bound to result in shortage of electricity produced. In fact consumption of coal is the cause of production of electricity. Unless there exists such a logical relationship between the two variables the study of correlation will be meaningless. There is no point in studying correlation between height and intelligence quotient of a group of adults.

Thus two variables are said to be correlated when change in value of one variable causes corresponding change in the value of the other variable. Population of a town and number of vehicles in the town are correlated because towns with larger population are bound to have larger number of vehicles.

5.2.1 Positive and Negative Correlation :

The Variables showing corresponding changes in their values are said to be correlated. But these changes in different pairs of variables are not of the same kind. In some cases the changes in values of both the variables are in the *same direction.* Increase in value of one variable causes increase in value of the other variable. Correlation between these variables is said to be **positive.** The consumption of coal and the amount of electricity produced are positively correlated.

In some other cases the changes in the values of the two variables may be in *opposite direction.* Increase in value of one variable may cause decrease in value of the other variable. These variables are said to be negatively correlated. Since ample supply of a commodity results in fall of price and scarcity results in rise of price the variables supply of a commodity and its price have negative correlation between them.

5.3 COVARIANCE

As stated above, the study of correlation between two variables is in some sense a study of joint variation of the two variables which may be termed as covariation. In order to ascertain the degree of correlation we need a measure of this degree of covariation. Such a measure is provided by covariance which is defined as the mean of products of deviations of the observed values of X and Y from their respective means.

Let us have a sample of n pairs of observations $(x_i\ y_i)$ on the variables X and Y. Then the means of X and Y for the given sample are

$$\bar{x} = \frac{\Sigma\ x_i}{n} \text{ and } \bar{y} = \frac{\Sigma y_i}{n}$$

Then the covariance of X and Y for the given sample is

$$\text{Cov. } (x, y) = \frac{\Sigma\ (x_i - \bar{x})(y_i - \bar{y})}{n}$$

$$= \frac{\Sigma\ x_i\ y_i - n\bar{x}\,\bar{y}}{n}$$

The interesting feature of this measure is that it may be negative, zero or positive according to the nature of correlation between the variables. In case of data on positively correlated variables the covariance is also positive.

Let us now study the effect of change of origin and scale. Let us change the variables X and Y into u and v by the transformation.

$$u = \frac{X - a}{h} \text{ and } v = \frac{Y - a}{k}$$

Then $\quad x_i = a + hu_i$ and $y_i, = b + kv_i,$

$\therefore \quad \bar{x} = a + h\bar{u}$ and $\bar{y} = b + k\bar{v}$

$\therefore \qquad x_i - \bar{x} = h(u_i - \bar{u})$ and $y_i - \bar{y} = k(v_i - \bar{v})$

Hence Cov. (X, Y) = $\dfrac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{n}$

$$= hk \frac{\Sigma (u_i - \bar{u})(v_i - \bar{v})}{n}$$

$$= hk \text{ Cov. } (u, v)$$

Thus covariance is invariant to the change of origin but not to the change of scale.

5.4 COEFFICIENT OF CORRELATION

For further study of correlation and comparison of correlation it is necessary to measure the degree of correlation between the two variables under consideration. Professor Karl Pearson has suggested a measure of a degree of correlation called *coefficient of correlation.*

Karl Pearson's Coefficient:

It is defined as the ratio of covariance of two variables to the product of standard deviations of these variables. It is also known as *product moment correlation coefficient.* The coefficient of correlation computed for a sample from a bivariate population is denoted by r.

Let us have a sample of n pairs of observations (x_i , y_i) on variables X and Y. Then the sample correlation coefficient is given by

$$r = \frac{\text{Cov.}(x, y)}{\sigma x \, \sigma_y}$$

Since Cov. *(X, Y)* = $\dfrac{\Sigma\, x_i\, y_i - n\,\bar{x}\,\bar{y}}{n}$

and $\sigma_x = \sqrt{\dfrac{\Sigma\, x^2{}_i}{n} - \bar{x}^2}$; $\sigma_y = \sqrt{\dfrac{\Sigma y^2{}_i}{n} - \bar{y}^2}$

On simplification we get

$$r = \frac{\Sigma\, x_i\, y_i - n\,\bar{x}\,\bar{y}}{\sqrt{(\Sigma x_i^2 - n\bar{x}^2)(\Sigma y^2{}_i - n\bar{y}^2)}}$$

This form is more suitable for computation of correlation coefficient.

The Magnitude or numerical value of r measures the degree of correlation and the algebraic sign of r indicates the type of correlation – positive or negative. Thus the value of correlation coefficient gives us the complete idea about the correlation between two variables.

5.4.1 Properties of Correlation Coefficient:

Let *r* be the coefficient of correlation between the variables X and Y computed from the sample of n pairs $(x_i\, y_i)$

(i) The magnitude of the coefficient of correlation i.e|r| is invariant to the change of origin and scale.

Let us denote the coefficient of correlation between X and Y by r_{xy} and that between u and v by r_{uv} Let us change the variables X and Y into u and v by the transformation.

$$u = \frac{x - a}{h} \text{ and } v = \frac{y - b}{k}$$

Then. $X = a + hu$ and $Y = b + kv$ and $|r_{xy}| = |r_{uv}|$

This shows that the numerical value of the correlation coefficient is invariant to the change of origin and scale.

Further it can be concluded that (i) when h and k have same algebraic sign i.e. when both are positive or both are negative, $r_{xy} = r_{uv}$ and (ii) when h and k have different algebraic signs $r_{xy} = -r_{uv}$.

For example, if coefficient of correlation between X and Y is 0.8 that between 2X and 3Y is 0.8. Also the coefficient of correlation between –2X and –3Y is 0.8. But the correlation between 2X and –3Y or that between –2X and 3Y is –0.8.

Further the coefficient of correlation between (2X + 5) and (3Y –10) is the same as r_{xy} but that between (2X + 5) and (–3y +10) is $-r_{xy}$

ii) Karl Pearson's coefficient of correlation between two variables is numerically less than or equal to unity.

5.4.2 Interpretation of the value of Correlation Coefficient:

The numerical value of the correlation coefficient measures the degree of correlation between the two variables. The larger value of $|r|$ indicates closer relationship between the variables. When $|r| > 0.8$, it indicates correlation of very high degree. When $|r|$ lies between 0.3 and 0.7, one can say that there is significant or considerable, correlation between the two variables. Correlation is said to be very poor or insignificant when $|r| < 0.3$. The algebraic sign of r indicates whether the correlation is positive or negative.

The value r = 0 means that the variables are uncorrelated. When r = 1 or r = –1, there is perfect positive correlation or perfect negative correlation respectively, between the two variables. But these values of r are very uncommon. In real life situations, chance of occurrence of these values of r are very rare.

In all these interpretations it is assumed that the sample from which r is computed is sufficiently large.

5.4.3 Computing Correlation Coefficient For Ungrouped Data :

The data specifying all the pairs of observation (x_i, y_i), i = 1, 2 ... n; on two variables X and Y are called ungrouped data. The steps in computing correlation coefficient for these data are given below :

i) Compute means of X and Y

$$\bar{x} = \frac{\Sigma x_i}{n} \text{ and } \bar{y} = \frac{\Sigma y_i}{n}$$

ii) Compute standard deviations of x and y

$$\sigma_x = \sqrt{\frac{\Sigma x^2{}_i}{n} - (\bar{x})^2}, \ \sigma_y = \sqrt{\frac{\Sigma y^2{}_i}{n} - (\bar{y})^2}$$

This needs computation of sums of squares $\sum x^2{}_i$ and $\sum y^2{}_i$

iii) Compute the sum of products $\sum x_i y_i$.

Then Cov. (x, y) = $\left[\frac{\Sigma x_i y_i}{n} - \bar{x}\bar{y}.\right]$

iv) Compute the coefficient of correlation between X and Y.

$$r = \left[\frac{Cov.(x, y)}{\sigma_x \sigma_y}\right]$$

Note : If the values of X and Y in the data are inconveniently large so as to make computation of sums of squares and sum of products difficult, we may employ the techniques of change of origin and/or change of scale that would simplify the computations : Usually the scaling factors are both positive. So the value of r remains unaltered. For example, if we have the transformation

$$u = \left(\frac{X-a}{h}\right) \text{ and } v = \left(\frac{Y-a}{k}\right) \text{ where } h > 0, k > 0, r_{xy} = r_{uv}$$

Check Your Progress. - 5.2 to 5.4

1. What is correlation?

____

____

2. What is positive correlation & negative correlation?

____

____

3. What is Covariance?

____

____

4. What is Karl Pearson's coefficient of correlation?

____

____

5. Choose the correct answer from given.

(i) If X and Y are any two random variables then the covariance between aX + b, cY + d is given by
a) cov(X, Y)
b) abcd cov(X, Y)
c) ac cov(X, Y)
d) ac cov(X,Y) + ab

(ii) The correlation coefficient between college entrance exam grades and the final gardes was computed to be – 1.08. On the basis of this you would recommend that:

a) the entrance exam is a good predictor of success
b) students who do worst in this exam will do best in final
c) Students at this school are not scholars
d) Recomputed the correlation coefficient

(iii) The correlation coefficient between X and Y is known to be zero, We then conclude that

a) X and Y have standard distributions.
b) the variances of X and Y are equal.
c) there exists no relationship between X and Y
d) there exists no linear relationship between X and Y

(iv) Suppose the correlation coefficient between height as measured in feet and weight as measured in pounds is 0.40. What is the correlation coefficient of height measured in inches versus weight measured in ounces (12 inches = one feet, 16 ounces = one pound)

a) 0.40 b) 0.30

c) 0.533 d) cannot be determined from the information given

(v) Consider the following data.

x	10	11	12	13	14
y	43	40	37	34	31

which one of the following would be true?

(a) Correlation coefficient between X and Y is negative but not equal to –1.
(b) Correlation coefficient between X and Y is –1.
(c) Correlation coefficient between X and Y is 0.
(d) None of the above.

5.5 SUMMARY

- The mutual or joint relation between the two variables is call correlation.
- The two variables are said to be correlated when change in value of one variable causes corresponding change in the value of the other variable In some cases the changes in values of both the variables are in the same direction. Increase in value of one variable causes increase in value of the other variable such a variables are called as positively correlated variables. In some other cases the changes in the values of the two variables may be in opposite direction. Increase in value of one variable may cause decrease in value of the other variable. These variables are said to be negatively correlated.
- The study of correlation between two variables termed as covariation.
- A measure of the degree of covariation is called as covariance.
- According to professor Karl Pearson a measure of a degree of correlation called coefficient of correlation.
- It is defined as the ratio of covariance of two variables to the product of standard deviations of these variables. It is also known as product moment correlation coefficient.

5.6 CHECK YOUR PROGRESS– ANSWERS

5.2 to 5.4

1. The mutual or joint relation between the two variables is called correlation.

2. Increase in value of one variable causes increase in value of the other variable such a variables are called as positively correlated variables. Increase in value of one variable may cause decrease in value of the other variable. These variables are said to be negatively correlated.

3. The study of correlation between two variables termed as covariation. A measure of the degree of covariation is called as covariance.

4. According to Professor Karl Pearson a measure of a degree of correlation called coefficient of correlation. It is defined as the ratio of covariance of two variables to the product of standard deviations of these variables. It is also known as *product moment correlation coefficient.*

5. (i) – c, (ii) – d, (iii) – d (iv) – a (v) – a

5.7 ILLUSTRATIVE EXAMPLE

Example : The following are the values of exports of raw cotton (X) and the values of imports of manufactured cotton goods (Y) in Crores of Rs. Compute the coefficient between X and Y.

Table 5.1 : Computation of Coefficient of Correlation

X	Y	y = u – 70	v = Y – 60	uv	u2	v2
42	56	–28	–4	112	784	16
44	49	–26	–11	286	676	121
58	53	–12	–7	84	144	49
55	58	–15	–2	30	225	4
89	65	19	5	95	361	25
98	76	28	16	448	784	256
66	58	– 4	– 2	8	16	4
		–38	–5	1063	2990	475

Here the given values of X and Y are large. So we convert X and Y into u and v by change of origin.

Take u = X – 70 and v= Y – 60

Then $\bar{u} = \frac{\sum u_i}{n} = \frac{-38}{7}$ = -5.4285

$\bar{v} = \frac{\sum v_1}{n} = \frac{-5}{7}$ = - 0.7143

The standard deviations are

$$\sigma_u = \sqrt{\frac{\sum u^2 i}{n} - (\bar{u})} = \sqrt{\frac{2990}{7} - (-5.4285)^2} = 19.94$$

$$\sigma_v = \sqrt{\frac{\sum v^2 i}{n} - (\bar{v})^2} = \sqrt{\frac{475}{7} - (-0.7143)^2} = 8.2065$$

$$\text{Cov}(u,v) = \frac{\sum u_i v_i}{n} - \bar{u}\,\bar{v}$$

$$= \frac{1063}{7} - (-5.4285)\,(-0.7142) = 151.857 - 3.8776$$

$$= 147.9794$$

$$\text{Hence } r_{uv} = \frac{\text{Cov}\cdot(u,v)}{\sigma_u\,\sigma_v} = \frac{147.9794}{19.94x8.2065} = 0.9043$$

Since the coefficient of correlation is invariant to change of origin we have $r_{xy} = r_v$ = 0.9043

This shows that there is correlation of high degree between the variables X and Y.

Note : Transformation of variables need not be used unless it significantly facilities computations. Mere it is used only as an illustration.

5.8 QUESTIONS FOR SELF - STUDY

1. Explain with an example the concept of bivariate data.

2. When are two variables said to be correlated? What do you mean by (i) positive correlation and (ii) negative correlation? Give two examples of each type.

3. Define Karl Pearsion correlation coefficient and state its properties.

4. Show that the correlation coefficient is numerically invariant to the change of scale and origin.

5. Ten pairs of values of X and Y give the following result: $\sum x = 40$, $\sum y = 50$, $\sum x^2 = 200$, $\sum y^2 = 500$ and $\sum xy = 160$. Find the correlation coefficient between X and Y.

6. Twenty pairs of values of X and Y given x = 5, y = 3, $Ex^2 = 680$, $\sum y^2 = 500$ and xi (yi – y) = 120. Find the coefficient of correlation

7. From the data of 25 pairs of observations of X and Y a student got $\sum x = 100$, $\sum y = 1250$, $\sum y^2 = 1300$, $\sum xy = 50$. Are these result consistent?

8. Two series of X and Y with 50 observations have standard deviations 4.5 and 3.5 respectively. The sum of products of deviations of X and Y from their respective means is 420.0. Find the coefficient of correlation between X and Y.

9. From the following data of supply in quintals (X) and price in Rs. per quintal (Y) of a certain commodity compute the correlation coefficient between price and supply.

X :	80	82	86	91	83	85	89	96	93	90
Y:	145	140	130	124	133	127	120	110	116	130

10. From the following data of height X in cm and weight Y in kg. of 10 adults find the correlation coefficient between X and Y.

X:	155	185	175	145	180	178	158	150	180	165
Y :	50	65	63	50	60	61	55	54	65	54

11. The mean soil temperature (X) and number of days (Y) required for germination for winter wheat at 10 places are given below :

X	57	42	38	42	45	42	44	40	46	44
Y:	10	26	41	29	27	27	19	18	19	31

Compute the correlation coefficient between X and Y.

12. From the following data on water X (in ft) and yield of Alfalfa Y (in tons per acre) calculate the correlation coefficient between X and Y^1

X:	1.0	1.5	2.0	2.5	3.0	2.5	4.0
Y:	5.3	5.7	6.3	7.3	8.3	8.7	8.4

13. From the data of n pairs of observations of X and Y following result are obtained x = 199, y = 94, $\Sigma(X - x)^2 = 1298$, $\Sigma (y - y)^2 = 600$ and $\Sigma (X - y) = -262$. Find the coefficient of correlation.

14. Find n, if r = 0.5, $\sigma y = 8$, $\Sigma (X_j - x)^2 = 90$ and $(x_i - x)(y_i - y) = 120$.

15. Given $n = 20$, $\Sigma = 80$, $\Sigma y = 40$, $\Sigma x^2 = 1680$, $\Sigma y^2 = 320$, $\Sigma xy = 480$, find the correlation coefficient between x and y.

16. Compute the correlation coefficient between x and y from the following :

n =1,0, $\Sigma x = 100$, $\Sigma y = 150$, $\Sigma (x - 10)2 = 180$, $\Sigma (y - 15)2 = 150$, $\Sigma(x-10)(y-15) = 60$

17. Given $r_{xy} = 0.75$, find the correlation coefficient between

a) (x–10) and (y–15)

b) (2x–4)and(2–y)

c) $\frac{x}{2}$ and $\frac{y}{5}$

5.9 SUGGESTED READINGS

1. Mathematics and Statistics by M. L. Vaidya, M. K. Kelkar
2. Statistical Analysis by S. P. Azen and A. A. Afifi
3. Pre-degree Mathematics by Vaze, Gosavi

□□□

NOTES

NOTES

Chapter 6

Linear Regression

6.0 OBJECTIVES

After studying this chapter you will be able to discuss following –

- Regression
- Two variable
- Three variables
- Quantitative evidence
- Sophisticated Results
- Proper equation
- Interpretation of results
- Use of results (decisions)

6.1 INTRODUCTION

In the preceding chapter we have studied methods of measuring the, degree of correlation between the two variables by obtaining bivariate data on these variables. If the bivariate data provide a quantitative evidence of existence of correlation or association between the variables, our attempt would be to establish this association in some functional form mathematically, that would enable us to estimate quite accurately, on an average, the value of one variable on the basis of the value of other variable. Such a mathematical relationship between two variables is called regression equation or simply regression.

This estimation by association is quite sophisticated and very useful. This procedure is actually that of prediction and prediction is the central function of sciences. The main task of any scientific study is to discover the general relationships between the observed variables and to state the nature of such relationships in mathematical terms, so that the value of one variable can be predicted on the basis of that of another. This is what we are going to attempt in this chapter. Generally, the relationships between the variables understudy such as i) height and weight of adult men ii) number of infant deaths and number of births etc. are very blurred, vague and imprecise. Ordinary mathematical methods are not useful in this case but statistical methods are. The special contribution of statistics in this field is that of handling such vague, blurred, and imprecise relationships.'

As stated above the mathematical relationship between the two variables under study is called *regression equation* which is essentially a prediction equation. But the term regression is well established in statistics and no attempt has been made to replace it.

6.2 LINE OF REGRESSION

The simplest equation for expressing the relationship between the two variables is linear equation. In the case the regression is known as liner and the equation is called the **line of regression.** Among the two variables under consideration the regression equation expresses one variable in terms of the other. If the equation expresses Y in terms of X, Y is called 'dependent' or 'explained' variable and X the 'independent' or 'explanatory' variable. [Note that the term 'independent' is not used in statistical sense.]

Thus the equation $Y = a + bX$ is called line of regression of Y on X and is used for prediction of Y for given X. Here a and *b* are constants for the given line. The coefficient *b* of X, is called the **regression coefficient** of Y on X.

Likewise the equation $X = a' + b'Y$ gives the line of regression of X on Y and is used for prediction of X for given Y. The coefficient *b'* is the regression coefficient of X on Y.

There is only one measure of degree of correlation between the two variables X and Y.it is the correlation coefficient *r*. But for the same pair of variables we have two lines of regression because we have two choices for dependent and independent variables. The coefficient of correlation r_{xy} is not different from r_{yx}. Hence there is only one coefficient of correlation for the given pair of variables.

The constants in the regression equation are determined that fits the data is obtained by the principle of least squares.

6.3 EQUATION OF LINE OF REGRESSION BY THE METHOD OF LEAST SQUARES

Let us have a sample of n pairs of observations (x_i, y_i) on the variables X and Y. Let the equation of line of regression of Y on X be

$$Y = a + bX \qquad \text{……..(1)}$$

For the i^th^ observation, y_i is the observed of Y. The value of Y obtained from the equation

(1) for $X = X_i$, is called the linear regression estimate of Y denoted. $\hat{y}_i$

Thus $\hat{y}_i = a + bx_i$ ....(2)

Now the constants a and b in the equation (1) are evaluated so that the sum of squares of deviations of observed y_i from their regression estimates $\hat{y}_i$ is the least. This is known as the method of least squares.

Let the sample of n pairs $(x_i\ y_i)$ have the means $\bar{x}$ and $\bar{y}$ and the variances σ_x^2 and σ_y^2 for X and Y respectively and let $\text{Cov}(X, Y) = m_{11}$ be covariance between X and Y for the sample.

Let $D = \Sigma(y_i - \hat{y}_i)^2$, which is the sum of squares of deviations of observed y_i form the linear regression estimate $\hat{y}_i$

The constants a and *b* are found in such a way that D is minimum. These values of a and *b* will be given by the equations.

$$na + b\,\Sigma x_j = \Sigma y_i \qquad \text{.... (3)}$$

$$a\Sigma x_i + b\Sigma x^2_i = \Sigma x_i y_i \qquad \text{.... (4)}$$

The sums ΣX_i, Σy_i, Σx^2_i and $\Sigma X_i y_i$ are known from the data. Thus we have two equations 3and 4 in two unknowns a and b. We can obtain a and b by solving these equations for a and b.

The equations (3) and (4) are called normal equations.

From equation (4) we have

$$a = \frac{\sum y_i}{n} - b\frac{\sum x_i}{n} = \bar{y} - b\bar{x}$$

substituting this value of a in (4), we get

$$(\bar{y} - b\bar{x})\, \Sigma x_i + b\, \Sigma x_i^2 = \Sigma x_i y_i.$$

Now $\Sigma x_i = nx$

$$\therefore\ (\bar{y} - b\bar{x})\,(n\bar{x}) + b\Sigma x_i^2 = \Sigma x_i y_i$$

This gives

$$b = \frac{\mathrm{Cov}(x,y)}{\sigma^2 x} = \frac{m_{11}}{\sigma^2 x}$$

Substituting these values in (1) the equation of line of regression is written as

$$Y = \bar{y} - b\bar{x} + bX$$

or $$Y = \bar{y} + b(X - \bar{x}) \qquad .....(5)$$

Likewise the equation of line of regression of X on Y obtained by the method of least squares is

$$X = \bar{x} + b'(Y - \bar{y}) \qquad .....(6)$$

Where $b^1 = \dfrac{Cov(x,y)}{\sigma^2 y}$

From the equations of lines of regression as given in (5) and (6) it is evident that both the lines pass through the point $(\bar{x}, \bar{y})$. Thus $(\bar{x}, \bar{y})$ is the point of intersection of the two lines of regression.

The lines of regression also can be expressed as

$$(Y - \bar{y}) = b(X - \bar{x})$$

which is the line of regression of Y on X and $(X - \bar{x}) = b'(Y - \bar{y})$ which is the line of regression of X on Y. In this form the equations are easy to memorize.

6.4 INTERPRETATION OF COEFFICIENT OF REGRESSION

Consider line of regression of Y on X. In this form the equation is of the form Y = a + bX. Here *b* is the coefficient of regression of Y on X. From the equation of line of regression it is clear that for unit change in value of X, the value of Y will change by *b* units. This *b* is the rate of change of value of Y for unit change in X. If *b* is positive the increase in value of X will be associated with increase in value of Y i.e. the correlation between X and Y will be positive. On the contrary if *b* is negative increase in value of X will correspond to decrease in value of Y, showing that there is negative correlation between X and Y.

In general the coefficient of regression gives the rate of change of dependent variable per unit change in value of independent variable and the algebraic sign of the coefficient of regression determines whether the correlation is positive or negative.

6.5 PROPERTIES OF COEFFICIENT OF REGRESSION

Let a sample of n pairs of values (x_i y_i) of the variables X and Y given the variances $\sigma x^2$and σy^2 and the coefficient of correlation *r*. Let *b* and *b'* be the coefficients of regression of Y on X and X on Y respectively,

Then we know that Cov. (X, Y) = r σx σy

and the regression coefficients *b* and *b'* are given by

$$b = \frac{\text{Cov}(X,Y)}{\sigma x^2} \text{ and } b' = \frac{\text{Cov}(X,Y)}{\sigma y^2}$$

$$\text{Then } b = \frac{\text{Cov}\cdot(X,Y)}{\sigma x^2} = \frac{r\sigma x\sigma y}{\sigma x^2} = r\frac{\sigma y}{\sigma_x}$$

$$\text{and} \quad b' = \frac{\text{Cov}\cdot(X,Y)}{\sigma y^2} = \frac{r\sigma_x\sigma_y}{\sigma y^2} = r\frac{\sigma_x}{\sigma_y}$$

a) Since σ_x and σ_y are always positive we can say that both the regression coefficients have the same algebraic sign which is the same as that of correlation coefficient. Also from the values of b and b' it follows that

$$bb' = r\frac{\sigma_y}{\sigma_x} r\frac{\sigma_x}{\sigma_y} = r^2$$

The product of regression coefficients is equal to r^2.

Thus the values of b and b′ will be said to be consistent if (i) both have the same algebraic sign and (ii) their product is less than of equal to unity, as

$r^2<1$.

b) The regression coefficient is invariant to the change of origin but not to the change of scale.

Since every regression coefficient is a ratio of covariance to variance it is invariant to the change of origin. The reason for this is that both the covariance and the variance are central moment which are known to be invariant to the change of origin.

Let us have $\quad u = \frac{X}{h}$ and $v = \frac{Y}{k}$

Then $\sigma_x^2 = h^2\,\sigma u^2$, and $\sigma y^2 = k^2\,\sigma v^2$

and Cov. (X, Y) = *hk* Cov. *(U, V)*

Let b_{vu} be the coefficient of regression of v on u.

Then $bvu = \frac{Cov\cdot(u,v)}{\sigma^2 u}$ and the coefficient of regression of Y on X is then given by

$$b = byx = \frac{Cov.(X,Y)}{\sigma^2 x} = \frac{hk\,\text{Cov.}(u,v)}{h^2\sigma_u^{\ 2}} = \frac{k}{h}\,b_{vu}$$

and $b' = b_{xy} = \frac{h}{k}\,b_{uv}$

From this it follows that if the scaling factors *h* and *k* for the two transformation are equal the regression coefficient will be unaltered.

Check Your Progress. - 6.2 to 6.5

1. What is Line of Regression?

____

____

2. What is Regression Co-efficient?

____

____

3. Choose the correct answer from the given.

i) Based on the data $\{(x_i, y_i), i = 1,2......20\}$ the two regression lines are

$y = 1/5 + 3/5\ x$ and $x = 1/5 + 3/5x$. Let m_x, m_y denote sample means.

(a) The two lines actually collapse into one line and the correlation coefficient is 1.

(b) correlation coefficient is 1/20

(c) $m_x = m_y = 1$ (d) $m_x = m_y = 1/2$

ii) Linear regression of Y on X is Y = 1 + X. Correlation coefficient between y and x is 1/2. Then the regression coefficient $b_{x,y}$ of x on y is;

(a) 1 (b) ½ (c) 1/4 (d) 1/2

6.6 SUMMARY

– The mathematical relationship between the two variables under study is called *regression equation* which is essentially a prediction equation.

– The simplest equation for expressing the relationship between the two variables is linear equation. In the case the regression is known as liner and the equation is called the **line of regression.**

– The equation Y = a + bX is called line of regression of Y on X and is used for prediction of Y for given X. Here a and *b* are constants for the given line. The coefficient *b* of X, is called the **regression coefficient** of Y on X.

6.7 CHECK YOUR PROGRESS - ANSWERS

6.2 to 6.5

1. The simplest equation for expressing the relationship between the two variables is linear equation. In the case the regression is known as liner and the equation is called the **line of regression**

2. The equation Y = a + *bX* is called line of regression of Y on X and is used for prediction of Y for given X. Here a and *b are* constants for the given line. The coefficient *b* of X, is called the **regression coefficient** of **Y** on **X.**

3 (i) – d (ii) – c

6.8 ILLUSTRATIVE EXAMPLES

Example 1 : In an agricultural experiment on the study of effect of depth of water in the soil (X) in ft. on the yield of as crop in 1b. per plot (Y) the following data were obtained.

X:	1.8	1.9	2.5	1.4	1.3	2.1	2.3
Y:	200	370	450	160	90	440	380

Obtain the equation of line of regression of Y on X and estimate the yield when the depth of water in the soil is 2 ft.

Solution : The Steps in computations are as follows :

i) Find the sum of squares and products.

$\Sigma x_i = 13.3$ $\Sigma y_i = 2090$ $\Sigma x_i^2 = 26.45$

$\Sigma y_i^2 = 751100$ $\Sigma x_i y_i = 4327$

The number of observations n = 7.

ii) Compute the means of X and Y

$\bar{x} = \frac{13.3}{7} = 1.9$ and $\bar{y} = \frac{2090}{7} = 298.57$

iii) Compute the variance of X

$$\sigma_x^2 = \left[\frac{\sum x^2 i}{n} - (\bar{x})^2\right]$$

$$= \left(\frac{26.45}{7}\right) - (1.9)^2$$

$= 3.7786 - 3.61$

$= 0.1685$

iv) Compute the covariance

$$\text{Cov}(X,Y) = \frac{\sum x_i y_i}{n} - \bar{x}\,\bar{y}$$

$$= \frac{4327}{7} - (1.9)(298.57)$$

$= 618.143 - 567.28$

$= 50.863$

v) Let the equation of line of regression of Y on X be Y = a + bX

$$\text{Then } b = \frac{\text{Cov.}(X,Y)}{\sigma x^2}$$

$$= \frac{50.863}{0.1685}$$

$= 301.85$

$a = \bar{y} - b\bar{x}$

] $= 298.57 - (301.85)(1.9)$

$= -274.945$

Hence the equation of line of regression is

$Y = 301.85 \times - 274.945$

The regression estimate of Y when X = 2 is

$Y = 328.76$

Thus the linear regression estimate of yield of crop when the depth of water is 2 ft., is 328.76 Ib.

Example 2 : The following table shows the means and the standard deviations of prices of shares of two companies.

Company	Mean Price	Standard deviation
A	Rs. 39.50	Rs. 10.80
B	Rs. 47.50	Rs. 16.80

The coefficient of correlation between the prices of two shares is 0.42. Find the most likely price of shares of company A when the price of share of company B is Rs. 55.

Solution : Let the prices of share of company A and company B in Rs. be X and Y respectively. Then we are given that the means of X and Y are x = 39.50 and y = 47.50 and the standard deviations are σ_x = 10.80 and σ_y = 16.80.

Also the coefficient of correlation is r = 0.42.

Now to estimate the price of shares of company. A i.e. value of X we are given the price of shares of company B i.e. Y = 55.

For this we have to use the equation of line of regression of X on Y.

Let this equation be $X = a' + b' Y$.

Here $\quad b' = r\dfrac{\sigma_x}{\sigma_y}$

$= 0.42 \times \dfrac{10.8}{16.8}$

$=0.27$

and $\quad a' = \bar{x} - b' \bar{y}$

$= 39.50 - 0.27 \times 47.50$

$\therefore\ a' = 26.675$

$= 26.68$

$\therefore$ The equation of line of regression is

$X = 26.68 + 0.27\ Y$

The most likely price of shares of company A is the linear regression estimate of X.

For Y = 55 this estimate of X is given by

$x = 26.68 + 0.27 \times 55 = 41.53$

The most likely price of shares of company A is Rs. 41.53 when that of company B is Rs. 55.

Example 3 : Given the two linear regression equation

8X + 10Y + 66 = 0 and 40X – 18Y = 214 and V (X) = 9,

find the means of X and Y, the correlation coefficient between X and Y and V(Y).

Solution : We know that the coordinates of point of intersection of the two lines of regression are x and y, the means of X and Y.

The regression equations are

$8X - 10Y = -66$

and $\quad 40X - 18Y = 214$

Solving these equations we get X = 13 and Y = 17.

Hence the means of X and Y are x = 13 and y = 17.

Now to find the correlation coefficient we have to find the regression coefficients *b* and *b′*.

For this we have to choose one of the lines as that of regression of Y on X and the other is then the line of regression of X on Y.

Let 8X - 10Y + 66 = 0 be the line of regression of Y on X.

This gives $Y = \frac{8}{10}x + 6.6 = 0.8x + 6.6$

The coefficient of X in this equation is b = 0.8. Then the other equation is that of line of regression of X on Y which can be written as

$$X = \frac{18}{40}Y + \frac{214}{40}$$

Here the regression coefficient $b' = \frac{18}{40} = 0.45$

Now $r^2 = b\,b' = 0.8 \times 0.45 = 0.36$

$r = \pm 0.6$

The correlation coefficient has the algebraic sign same as that of b $\therefore$ r = 0.6

[**Note** : We choose arbitrarily the lines as that of regression of Y on X or X on Y. If the product $b\,b'$ is less than unity, our choice is correct. Otherwise we have to take the other chose. Fortunately there are only two choices].

Now the coefficient of regression of Y on X is $r\frac{\sigma_y}{\sigma_x}$

$\therefore \quad b = r\frac{\sigma_y}{\sigma_x} = 0.6 \times \frac{\sigma_y}{3} = 0.8$

$\therefore \quad \sigma y = \frac{3 \times 0.8}{0.6} = 4$, Hence V (Y) = 16

6.9 QUESTIONS FOR SELF - STUDY

1. Explain the concept of regression and its utility.
2. Why do we have two lines of regression (i) of Y on X and (ii) X on Y?
3. What do you mean by a linear regression coefficient of Y on X? How will you interpret the value of it?
4. If b_{yx} and b_{xy} are the coefficients of regression of Y on X and X on Y respectively, show that $b_{yx}, b_{xy} = r^2$
5. Bring out the inconsistency, if any, in the following :

   i) b = 1.6, b' = –0.5   ii) b = 3.2, b' = –0.5

   iii) $b = b^1 = 1.50$ and r = – 0.7

5. The following table gives the infant mortality rate (X) and birth rate (Y) for eight years.

X:	22.9	17.8	20.8	21.3	20.7	20.9	17.5	13.9
Y:	44	46	56	42	32	47	38	45

   Obtain the line of regression of birth rate on infant mortality rate and estimate the birth rate for the infant mortality rate 15.

6. The following data give live weight of a pig (X) and weight of a side of bacon (Y)

X :	125	155	190	203	217
Y :	36	46	51	65	72

Estimate the line of regression of weight of pig on weight of a side of bacon and calculate the weight of pig if the weight of bacon is 101.

7. The number of defective items produced per unit time, Y, by a certain machine is thought to vary directly with the speed of the machine, X measured in r p.m. Observations for 10 hours selected at random from a month give the following results.

X:	13.2	14.9	16.4	8.1	13.1	10.8	10.9	17.4	10.2	15.8
Y:	9.4	12.2	11.4	6.0	9.6	7.5	5.7	12.3	7.0	9.0

Estimate the line of regression of Y on X and the number of defectives per hour when the speed of the machine is 10 r.p.m.

8. The following random sample gives the number of hours of study (X) for an examination and the grades Y obtained by 12 students.

X:	3	3	3	4	4	5	5	5	6	6	7	8
Y:	45	60	55	60	75	70	80	75	90	80	75	85

Obtain the line of regression of grades on hours of study.

9. The average price of 200 shares was Rs. 150 and the average gain per share was Rs. 7. The coefficient of regression of gain per share (Y) on the price (X) was 0.50, Estimate the gain per share for the price Rs. 200.

10. Twelve observations on the price (X) of shares and the volume of sales (Y) at Bombay stock exchange gave the following results.

$\Sigma x = 580$, $\Sigma y = 370$, $\Sigma xy = 11494$, $\Sigma x^2 = 41568$ and $\Sigma y^2 = 17206$.

Obtain the equation of line of regression of volume of sales on price of shares. Predict the volume of sales (in thousands of shares) for shares of price Rs. 40/– .

11. Give the following data, obtain the linear regression estimate of X for Y = 10.

$\Sigma xi = 7.6$, $\Sigma yi = 14.8$, $\sigma_x = 3.6$, $\sigma_y = 25$, $r = 0.8$

12. The two regression lines are $2x - 3y = 0$ and $4y - 5x + 7 = 0$. Find the means of X and Y. If standard deviation of X is 3, find that of Y.

13. Find the means of X and Y and the correlation between X and Y, if the equations of lines of regression are $2y - x - 50 = 0$ and $3y - 2x - 10 = 0$.

14. The equations of two lines of regression are

$3X + 2Y - 26 = 0$ and $6X + Y - 31 = 0$

Find the means of X and Y. Estimate Y for X = 2.

15. Given the means of X and Y, 5 and 10. The line of regression of Y on X is parallel to the line $20Y = 9X + 40$ and correlation coefficient is 0.6. Estimate the value of X when Y = 30.

16. In the regression analysis of a problem the equations of lines of regression were found to be $10X - 4Y = 80$ and $10Y - 9X + 40 = 0$. The variance of Y was 36. Find the means of X and Y, σx^2 and the coefficient of correlation.

6.10 SUGGESTED READINGS

1. Mathematics and Statistics by M. L. Vaidya, M. K. Kelkar
2. Statistical Analysis by S. P. Azen and A. A. Afifi
3. Pre-degree Mathematics by Vaze, Gosavi

◆◆◆

NOTES

Chapter 7

Index Numbers

7.0 OBJECTIVES

After studying this chapter you will be able to calculate and explain-

- Index numbers
- Types of index numbers
- Construction of Index numbers
- Uses of Index numbers

7.1 INTRODUCTION

Many a time we are interested in knowing the relative changes in values of variables like population, prices, industrial production, agricultural production, exports, imports etc. over a period. One of the ways of measuring these changes is the index numbers. We are quite familiar with the wholesale price index, consumer's price index, Bombay stock exchange index which give us the knowledge of the degree of changes in corresponding variables.

An *index number* can be defined as the device used for measuring the relative changes in value of a variable or of a group of related variables from one period to other or from one place to another place.

7.2 USES OF INDEX NUMBERS

The price and quantity (of consumption) index numbers have in recent years become he important tools in interpretation of the economic conditions of a state. Rapid and erect changes in price index or index of price of shares indicate unstable economic conditions. Whereas stability in these index numbers indicate stable economy. In that sense these index numbers are called **Economic Barometers.** Wages of employees are closely tied with the consumer's price index of that locality. Revision of pay scales, fixation of dearness allowances, minimum wages, pension policies are linked with price index numbers. So the price index numbers are closely watched by the employers as well as employees. The index of industrial production is of great interest to businessmen and the students of national economy because it furnishes the information on the current position of national production. The price index numbers are also useful in determining the purchasing power of money. It can also be used for determining the real income or real wages of employees.

The index numbers like intelligence quotient are useful in Psychology and Education. The population index is of interest to the students of sociology and demography.

Although there can exist many index numbers for different purposes a common man is more concerned with the price index numbers. So for the further discussion let us restrict ourselves to the price index numbers only.

7.3 PRICE INDEX NUMBERS

For measuring the relative change in price of a single commodity or a group of commodities we use price index number. Generally we are interested in measuring changes in prices over a change of time.

The change is measured from some fixed period of reference known as *base period* The period which is compared with this base period is called *current period.* Let us use the notation p_0 and p_1 for the prices in base period and current period respectively. Then we use the ratio of prices for measuring the change because the ratio is independent of units in which the prices are expressed.

The relative importance of different commodities can be considered by using weights. The weights used are proportional to the quantities of consumption or the value of goods and services in the series. Such index numbers are called *weighted Index numbers.* Different systems of weights used give rise to different formula. Some of these are given below.

(a) Laspeyre's Index:

This index number is the ratio of weighted aggregate prices using the quantities consumed in base year *(q_0)* as the weights.

$$P_{01} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100.$$

This can be interpreted as the ratio of the value of basket of goods consumed in base year according to current year and base year prices. This index number always gives an *upward* bias.

(b) Paasche's Index :

This index number is also a ratio of weighted agreegate of prices using the quantities consumed in current year as weights

$$P_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_1} \times 100$$

This index can be interpreted as ratio of value of good consumed in current year to prices in current year and base year. This index as contrary to Laspeyre's Index, is found to give a downward bias,

(c) Fisher's Index:

Since neither Laspeyers formula nor Paasche's formula give a correct idea of change in price level, *Irving Fisher* suggested that the geometric mean of these two index numbers will give the suitable index number. According to him the index number is given by

$$P_{01} = \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} \times \frac{\sum p_1 q_1}{\sum p_0 q_1}} \times 100$$

This gives a more accurate price index but it lacks in interpretation. However, it has many desirable properties and hence it is known as *Fisher's Ideal Index Number.*

Example : Calculate the Laspeyre's, Paasche's and Fisher's index numbers for prices in the year 1987 with 1982 as base year from the following data.

Commodity	**Base year (1982)**		**Current Year (1987)**	
	Price (P_o)	**Quantity (q_o)**	**Price (P_1)**	**Quantity (q_1)**
Rice	4	15	6	20
Wheat	3	40	5	35
Jawar	5	20	5	25
Pulses	6	10	8	10

Solution : Computations for the required index numbers are shown in the following table.

Commodity	P_o	Q_o	P_1	q_1	p_oq_o	p_0q_1	p_1q_0	p_1q_1
Rice	4	15	6	20	60	80	90	120
Wheat	3	40	5	35	120	105	200	175
Jawar	5	20	5	25	100	125	100	125
Pulses	6	10	8	10	60	60	80	80
TOTAL					340	370	470	500

Laspeyre's Index

$$p_1 = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$$

$$= \frac{470}{340} \times 100$$

$$=138.235$$

Paasche's Index

$$P_a = \frac{\sum p_1 q_1}{\sum p_0 q_1} \times 100$$

$$= \frac{500}{370} \times 100$$

$$=135.135$$

Fisher’s Index $P_F = \sqrt{P_L \times P_a}$

$$= \sqrt{138.235 x 135.135}$$

$$= 136.676$$

7.4 PROBLEMS IN CONSTRUCTION OF AN INDEX NUMBER

Construction of any index number itself a difficult task. It poses many problems in the process of construction of an index number. Following are the common problems which need a careful and thoughtful consideration while constructing an index number.

(i) Specification of the Purpose and Scope of Index Number :

Every index number is constructed with some definite purpose and its uses are also limited. There does not exist an all purpose index number. The wholesale price index umber cannot be used for comparing the retail price levels in two periods. The consumer's price index for textile workers cannot be used for comparing cost of living of higher income group. So it is very important and necessary to specify the purpose at the outset only. It governs the further details of construction of an index number. It also defines the proper use of index number under consideration.

(ii) Selection of Items :

Selection of the items and their number is governed by the purpose itself. Only relevant items should be included in the series which have direct influence on the index number. The number of items to be included should be enough to make it representative and it should not be too large also as it would create difficulties in collecting the price data and information on weights. The items should suit the tastes, habits and customs of the class of people for whom the index number is constructed. Consideration should be given to the quality of the items like rice, wheat, recreation as these differ from class to class.

(iii) Selection of Weights :

Generally, the index number is a weighted index numbers as it is more realistic. Weights allow different items to influence the indes number to different extents. The weights should be proportional to the relative importance of the item. The method in which we use ushc weights is called explicit weighting. In some cases, we may include in the series more varieties of the same item which is more important and less varieties of items of less importance, this is indirect or implicit weighting. In explicit weighting, we generally use the quantity of consumption or the value of goods consumed as weights For collecting these data we have to conduct a sample enquiry.

(iv) Selection of Base Year:

The usefulness of the index number depends to some extent on the choice of base year. So proper care should be taken in selecting the base year also. The base year should to be too distant in the past. The pattern of consumption is likely to change with the time. Some items may become out of use and some new items may come in to use. this may lose the comparability of the periods. This necessitates revision of base year from time to time. The other important point in choice of base year is that it must be the period of economic stability. The events like wars, famines, epidemics are likely to create erratic changes in prices which are bound to be temporary. They reflect the instability of the economy and these conditions do not prevail for a long time. So it is necessary that the base year chosen must be an year of economic stability.

(v) Selection of Sources of Price Data :

After the commodities have been selected it is necessary to collect the data on prices of these commodities for constructing a price index number. This prices are collected from the markets or shops from which useal purchases are done. The concessions or discounts should not be taken into consideration. The prices should be for those qualities of the commodity which are commonly consumed by the class of people under consideration. The price may be collected by inviting quotations from the reliable sources and agents. To ensure reliability the quotations for some commodities may be invited from two or more agents. The price data also can be obtained from published reports of official agencies.

(vi) Selection of Average or Formula :

Since we are interested in constructing a single index which will summaries the changes in values of number of related variables we have to select the proper average that will serve the purpose.

Among the various averages the median and the mode are out of consideration. Only the arithmetic mean and the geometric mean can be used. Among them the A. M. is easy to understand and to compute also. So in many cases we used the weighted A. M. of price relatives for constructing the index number. In practice, Laspeyre's formula is widely used as it uses the base year quantities as weights. The data on base year quantities are easily available then those on current year quantities.

7.2 to 7.4 Check Your Progress.

1. What is Index number?

 ____

 ____

2. What is Price index number?

 ____

 ____

3. List the common problems which need to be considered while constructing an index number.

 ____

 ____

4. Choose the correct answer from given list.

i) Laspeyre's index number is given by the formula

a) $\frac{\sum p_0 q_1}{\sum p_1 q_0} \times 100$ b) $\frac{\sum p_0 q_0}{\sum p_1 q_1} \times 100$

c) $\frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$ d) $\frac{\sum p_1 q_1}{\sum p_0 q_0} \times 100$

ii) Paasche's index number is given by the formula.–

a) $\frac{\sum p_0 q_1}{\sum p_1 q_0} \times 100$ b) $\frac{\sum p_1 q_1}{\sum p_0 q_1} \times 100$

c) $\frac{\sum p_1 q_0}{\sum p_0 q_1} \times 100$ d) $\frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$

7.5 SUMMARY

- An I*ndex number* can be defined as the device used for measuring the relative changes in value of a variable or of a group of related variables from one period to other or from one place to another place. Index numbers are also called as economic barometers:
- For measuring the relative change in price of a single commodity or a group of commodities we use price index number.

Common problems in construction of an Index number are :

(i) Specification of the Purpose and Scope of Index Number.

(ii) Selection of Items .

(iii) Selection of Weights .

(iv) Selection of Base Year.

(v) Selection of Sources of Price Data.

(vi) Selection of Average or Formula.

7.6 CHECK YOUR PROGRESS - ANSWERS

7.2 to 7.4

1. An I*ndex number* can be defined as the device used for measuring the relative changes in value of a variable or of a group of related variables from one period to other or from one place to another place. Index numbers are also called as economic barometers.
2. For measuring the relative change in price of a single commodity or a group of commodities we use price index number.

3. Common problems in construction of an Index number are :

(i) Specification of the Purpose and Scope of Index Number:

(ii) Selection of Items

(iii) Selection of Weights

(iv) Selection of Base Year

(v) Selection of Sources of Price Data

(vi) Selection of Average or Formula

4. (i) – c (ii) – b

7.7 QUESTIONS FOR SELF - STUDY

1. Explain the meaning and the utiling of Index numbers.
2. State Laspeyre's, Paasche's and Fisher's formulae of index numbers and mention their specialties.
3. Give interpretation of Laspeyre's and Pasche's Index number of price:
4. Discuss various problems in construction of an Index number.
5. Compute Laspeyre's, Passche's and Fisher's Index number for price from the following data.

Commodity	**Base Year**		**Current Year**	
Price	**Quantity**	**Price**	**Quantity**	**Price**
A	8	50	10	60
B	10	40	12	50
C	5	100	9	70
D	6	10	8	20

6. Calculate Laspeyre's and Pasche's Index number for price from the following data and comment on your results.

(a)

Commodity	p_o	q_o	p_1	q_1
A	1	10	1.5	8
B	5	12	6.0	9
C	8	5	12.0	3

(b)

Commodity	p_o	q_o	p_1	q_1
1	2	7	3	8
2	5	9	4	7
3	3	5	6	9

(c)

Commodity	p_o	q_o	p_1	q_1
I	19	3	6	
II	3	8	4	3
III	2	7	6	5

7. Calculate appropriate Index number in each of the following.

(a)

Commodity	**A**	**B**	**C**	**D**
Po	5	7	8	4
P1	6	5	9	2
q_0	6	8	10	5

(b)

Commodity	**I**	**II**	**III**
P_1	1	4	7
P0	2	5	5
q_1	5	7	11

7.8 SUGGESTED READINGS

1. Mathematics and Statistics by M. L. Vaidya, M. K. Kelkar
2. Statistical Analysis by S. P. Azen and A. A. Afifi
3. Pre-degree Mathematics by Vaze, Gosavi

◆◆◆

NOTES