AMATH 301
Homework 8
Due: Friday, March 1, 2019

Differentiation

In order to verify the half-life of Plutonium-239, scientists start with a sample 1000 kg of Plutonium-239 in the year 1970. They measure the remaining amount of Plutonium each year for 40 years. The data is contained in the file `Plutonium.mat` which is included with the homework. `Plutonium.mat` contains two vectors: `tdata` and `Pdata`. The vector `tdata` contains the years at which the mass of the plutonium was measured since the beginning of the experiment, and the vector `Pdata` contains the measured mass of remaining Plutonium-239. Let the function $P(t)$ denote the amount of remaining Plutonium as a function of time.

Scorelator Problems

Problem 1 (Scorelator).

Use the second-order central finite difference scheme for the first derivative to approximate the $\frac{\mathrm{d}P}{\mathrm{d}t}$ at time $t = 1980$. Save the result to **A1.dat**.

Problem 2 (Scorelator).

Use the second-order left endpoint (forward) finite difference scheme to approximate the first derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$ at time $t = 1970$. Save the result to **A2.dat**.

Problem 3 (Scorelator).

Use the second-order right endpoint (backward) finite difference scheme to approximate the first derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$ at time $t = 2010$. Save the result to **A3.dat**.

Problem 4 (Scorelator).

Use (multiple) second-order finite difference schemes to approximate the first derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$ at all of the times $t = 1970, 1971, 1972, \ldots, 2010$. Save the results to **A4.dat**.

Problem 5 (Scorelator).

The decay rate of Plutonium-239 is given by $\frac{-1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}$. Use the approximations of the derivative that you computed in Problem 4 to estimate the decay rate at each time $t = 1970, 1971, 1972, \ldots, 2010$. Save the average of these decay rates in **A5.dat**.

Problem 6 (Scorelator).

If λ is the average (mean) decay rate that you found in Problem 5, then the half-life of Plutonium-239, denoted by $t_{1/2}$, is given by the formula

$$t_{1/2} = \frac{\ln(2)}{\lambda},$$

where ln denotes the natural logarithm (log with base e).

Calculate the half-life and save it in **A6.dat**.

Problem 7 (Writeup).

Using our estimated decay rate λ, the theoretical form of $P(t)$ is

$$P(t) = 1000e^{-\lambda(t-1970)}.$$

This decaying exponential decreases in value by half every $t_{1/2}$ years.

Create a figure with an upper and lower axes (`subplot`). In the upper axes plot the data (t, P_t) as discrete points along with the continuous *exponential decay* curve $P(t)$ given above. Add a legend to distinguish the 'data' and ''exponential decay'. Also add a label on the y-axis to indicate the quantity being plotted (Mass of Plutonium). Add a title: "Radioactive Decay of Plutonium". In the lower axes, plot the error: the theoretical curve minus the data values at the times in the vector `t`. Add labels on the x and y-axes — because both axes should have the same horizontal limits, a label is only needed on the x-axis of the lower plot.

Save your figure as `plutonium_decay.png`.

Integration

Consider the integral,

$$P = \int_a^b e^{-(x-\mu)^2/2\sigma^2} \mathrm{d}x.$$

This integral is very important in probability: it is proportional to the chance that a normally-distributed random variable (the bell curve) with mean μ and variance σ^2 will fall between a and b. For this assignment, we will use the values $\mu = 1$ and $\sigma^2 = 4$ for the distribution, and set $a = 2$ and $b = 4$.

In the following problems, we will approximate this integral with several quadrature (numerical integration) schemes: left- and right-rectangle rules, the trapezoid rule, and Simpson's rule.

Problem 8 (Scorelator).

Use the left-rectangle rule to approximate P with a step size of $\Delta x = 1$. Save your answer to the file **B1.dat**.

Repeat this procedure for $\Delta x = 2^0, 2^{-1}, 2^{-2}, \ldots, 2^{-16}$. You should end up with 17 approximations (the first approximation is the same as from **B1.dat**). Store these approximations in a 17×1 column vector and save to the file **B2.dat**.

Next use the right-rectangle rule to compute approximations to P, again with step sizes of $\Delta x = 1, 2^{-1}, 2^{-2}, \ldots, 2^{-16}$. Store these approximations in a 17×1 column vector and save to the file **B3.dat**.

Problem 9 (Scorelator).

Use the trapezoid rule to approximate P with step sizes of $\Delta x = 1, 2^{-1}, 2^{-2}, \ldots, 2^{-16}$. Store these approximations in a 17×1 column vector and save to the file **B4.dat**.

Problem 10 (Scorelator).

Use Simpson's rule to approximate P with step sizes of $\Delta x = 1, 2^{-1}, 2^{-2}, \ldots, 2^{-16}$. Store these approximations in a 17×1 column vector and save to the file **B5.dat**.

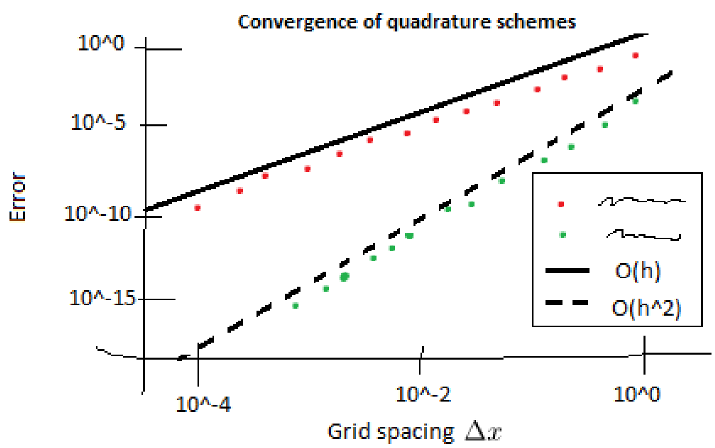
Note: Simpson's rule is not implemented in MATLAB. You will need to calculate the result from scratch.

Problem 11 (Scorelator).

Use the `integral` function to calculate the "true" value of P. Save the answer to **B6.dat**.

Problem 12 (Writeup).

Produce a figure showcasing the error and rates of convergence of the four quadrature schemes from Problems 8–10 in a log-log plot. A crude drawing of what the figure might look like when plotting two of the trends is shown below.



For each quadrature scheme, plot the absolute error (absolute value of the approximation minus the true value of P) versus the step size Δx. Use a distinct color and marker type for each of these discrete trends.

Plot the data on a log-log scale. You should observe that as the grid size Δx decreases, so does the error. Furthermore, the error should appear to follow straight-line trends that correspond to the order of convergence of the methods.

Plot an $\mathcal{O}(h)$ (order-of-h) trend line by plotting Δx versus $c\Delta x$. The number c controls the vertical position of the trend line: increasing c moves the trend line up, whereas decreasing it moves it down. The value of c should be chosen so that the trend line appears close to the data that appears to follow this order of convergence, but not on top of it. Next plot a $\mathcal{O}(h^2)$ trend line by plotting Δx versus $\tilde{c}(\Delta x)^2$ for a suitable value of $\tilde{c}$. Plot these lines for any error trends that appear to follow this rate of convergence. Do the same if $\mathcal{O}(h^3)$ or $\mathcal{O}(h^4)$ trends appear in the error of any of the quadrature schemes. Use a solid, dashed, dotted, and then dash-dotted lines of medium thickness (linewidth 2, for example) for these trend lines, in increasing order. For this problem, you will be graded on how easily the plot can be interpreted and how well it illustrates the order of convergence of the methods.

Add a horizontal line at the level 10^{-16}, which corresponds to "machine precision", or the maximum accuracy you can expect of numbers on a computer under typical circumstances.

Add appropriate labels to the x- and y-axis, a legend labeling all of the data and trends in the figure, and a title. Save your figure as `quadrature_errors.png`.