FAST COMPRESSIVE IMAGING USING SCRAMBLED BLOCK HADAMARD ENSEMBLE

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ABSTRACT

With the advent of a single-pixel camera, compressive imaging applications have gained wide interests. However, the design of efficient measurement basis in such a system remains as a challenging problem. In this paper, we propose a highly sparse and fast sampling operator based on the scrambled block Hadamard ensemble. Despite its simplicity, the proposed measurement operator offers universality and requires a near-optimal number of samples for perfect reconstruction. Moreover, it can be easily implemented in the optical domain thanks to its integer-valued elements. Several numerical experiments show that its imaging performance is comparable to that of the dense, floating-coefficient scrambled Fourier ensemble at much lower implementation cost.

Index Terms— Compressed sensing, random projections, sparsity,Hadamard transform

1. INTRODUCTION

Over the past few years, there have been increased interests in the study of compressed sensing (CS)—a new framework for simultaneous sampling and compression of signals. In CS, the bandlimited model in the classical Nyquist sampling theorem is replaced by the *sparse model*, assuming that a signal can be efficiently represented using only a few significant coefficients in a certain transform domain. The groundbreaking work by Candes *et al.* [1] and Donoho [2] showed that such a signal can be precisely reconstructed from only a small set of random linear measurements (much lower than the Nyquist rate), implying the potential of dramatic reduction of sampling rates, power consumption and computation complexity in digital data acquisitions.

Due to the large amount of data in image and video signals, CS is very attractive in imaging applications, especially for low-power and low resolution imaging devices or when the measurement is very costly (e.g., Terahertz applications). Since the discovery of the CS theory, several compressive imaging algorithms have been developed for Fourier transform domain measurements in applications such as the MRI [3]. For spatial domain measurement, a single pixel camera [4] has been built that it is applicable for sparse signals in *any* transform domain. Besides, with only one photon detector, it is quite promising for imaging applications at wavelength where current CMOS and CCD cameras are impossible [4].

Despite the above-mentioned works, there still exists a huge gap between the CS theory and imaging applications. In particular, it is still unknown how to construct a fast and efficient sensing operator, especially when the measurement is taken in the spatial domain. Note that in the single-pixel camera, the sampling operator is based on the *random* binary pattern, which requires huge memory and high computation cost. For example, to get a 512×512 image with 64k measurements (i.e., 25% sampling rate), a random binary operator requires nearly gigabytes storage and giga-flop operations, which makes the recovery almost impossible. A popular choice in existing compressive imaging research is the scrambled Fourier ensemble (SFE) [5]. Although it is much more computationally efficient than the binary random operator, it still requires huge memory and expensive implementation cost.

In this paper, we propose a new fast measurement operator for compressive imaging by taking advantages of random permutations and block Hadamard transform. The development is based on our recent work of structurally random matrix in [6]. The proposed sampling operator offers several attractive features. From the theoretical perspective, the sampling operator is *universal* with a variety of sparse signal and the number of measurements required for exact reconstruction is *nearly optimal*. From the practical perspective, the block Hadamard transform can be easily implemented in the optical domain (e.g, using the single pixel camera). It also offers *fast computation* along with very *small memory requirement*. In fact, we shall demonstrate that a 32×32 scrambled block Hadamard ensemble can offer comparable compressive imaging performances to the dense and floating-coefficient SFE at much lower cost.

The rest of this paper is organized as follows. Section 2 provides a brief review of the CS principle. Section 3 describes our proposed sensing operator and presents the theoretical analysis by exploiting the combinatorial central limit theorem [7] and Bernstein-type bounds for random permutations [8]. Simple scrambling algorithms are also proposed for practical considerations. Section 4 reports the simulation results followed by conclusions in Section 5.

2. BACKGROUND

Consider a length-N, real valued signal x[1] and suppose that the basis Ψ provides a K sparse representation of x. In terms of matrix notation, we have $x = \Psi f$, in which f can be well approximated using only $K \ll N$ non-zero entries and Ψ is called as the *sparse basis matrix* [2]. For images, typical choices of Ψ include the DCT and the wavelet. The CS theory states that such a signal x can be reconstructed by taking only $M = \mathcal{O}(K \log N)$ linear, non-adaptive

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[1]For 2D images, we use x to denote the 1D ordering of N pixels through raster scan.

measurements as follows [1, 2]:

$$y = \Phi x = \Phi \Psi f, \tag{1}$$

where y represents an $M \times 1$ sampled vector and Φ is an $M \times N$ measurement matrix that is *incoherent* with Ψ, i.e., the maximum magnitude of the element in $\Phi\Psi$ is small [9].

Although the sampling process is simply linear projection, the reconstruction algorithm is highly *non-linear*. The l_1 optimizer is widely used to minimize $\|f\|_{l_1}$ under the constraint of (1) [5, 10, 11]. For 2D images, another popular reconstruction algorithm is through the minimization of total variation (min-TV) [1,5,11], which offers reconstructed images with better visual quality at much higher computational cost. Several fast greedy algorithms have also been proposed, such as the orthogonal matching pursuit (OMP) [12] and iterative thresholding [13, 14].

In this paper, we focus on the construction of fast sampling operator for imaging applications. Below, we compile a wishlist for the measurement matrix Φ.

- **Near optimal performance**: The number of measurements for perfect reconstruction is close to the theoretical bound $\mathcal{O}(K \log N)$;
- **Universality**: Φ can be paired with a variety of sparse basis matrix Ψ for natural images;
- **Fast computation**: Due to the large data size in imaging applications, a fast computable Φ is desirable for both sensing and reconstruction algorithms;
- **Memory efficient**: The storage of Φ requires small memory size;
- **Hardware friendly**: Φ can be easily implemented in the optical and analog domain;

None of existing measurement operators can meet all of above mentioned properties. Specifically, the Gaussian or Bernoulli i.i.d matrices [1] offer optimal performance and universality. But, they are impractical for imaging applications due to huge memory requirement and high computational complexity. In existing compressive imaging research, the scrambled Fourier ensemble (SFE) [5, 13] is often employed because of its fast computation structure. However, since SFE is a dense matrix, it still needs large buffer size. In addition, the optical domain implementation is expensive and it cannot be used for devices such as the single-pixel camera. Recently, by exploiting combinatorial algorithms, several researchers have developed binary sparse measurement matrices [15] with low complexity and near optimal performance. Unfortunately, these operators are not universal since they are only incoherent with the identity matrix. As a result, they can only be used for measurement in a certain transform domain (e.g.,the wavelet), rather than the spatial domain.

3. SCRAMBLED BLOCK HADAMARD ENSEMBLE

In this paper, we develop a new sampling operator called *scrambled block Hadamard ensemble* (SBHE). Simply speaking, Φ takes the partial block Hadamard transform and randomly permuting its columns, i.e.,

$$\Phi = \mathbf{Q}_M \mathbf{W} \mathbf{P}_N, \tag{2}$$

where the $N \times N$ matrix $\mathbf{W}$ is a block diagonal matrix that can be written as

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_B & & & \\ & \mathbf{W}_B & & \\ & & \ddots & \\ & & & \mathbf{W}_B \end{bmatrix} \tag{3}$$

in which $\mathbf{W}_B$ represents the $B \times B$ Hadamard matrix, $\mathbf{P}_N$ is a scrambling operator which randomly reorders the N columns of $\mathbf{W}$ and $\mathbf{Q}_M$ is an operator which picks up M rows of $\mathbf{W}\mathbf{P}_N$ uniform at random. Note that (2) replaces the Fourier matrix in SFE with the blocked Hadamard matrix $\mathbf{W}$. As the elements of $\mathbf{W}_B$ are binary, Φ given in (2) can be easily implemented in the optical domain. Besides, the block diagonal structure of $\mathbf{W}$ enables fast and parallel computation at the complexity of $\mathcal{O}(N \log B)$ along with small memory requirement. In the rest of this section, Sections 3.1-3.2 focus on the theoretical analysis of Φ while Section 3.3 discusses the practical design of the scrambling operator $\mathbf{P}_N$.

3.1. Gaussian behavior of $\Phi\Psi$

It is obvious that small B is advantageous for computational and storage purposes. But how small could B be if we want to get comparable reconstruction performance as that of the SFE? Later in Section 4, we shall demonstrate that $B = 32$ already works well even for an image with one mega pixels ($N = 2^{20}$). To explain this result, let us consider the equivalent sampling operator in the frequency domain $\Phi_f = \Phi\Psi$. We shall show that Φ_f behaves like a Gaussian matrix under some mild conditions:

Proposition 1 *For the SBHE sampling operator Φ given in (2) and a sparse basis transform Ψ, let $\Phi_f = \Phi\Psi$. Then, each element $\Phi_f(i,j)$ is asymptotically normal with zero mean and variance of $\frac{1}{N}$, i.e., $\Phi_f(i,j) \sim \mathcal{N}(0, \frac{1}{N})$ if the following conditions are met:*

1. *$\max_{i,j} |\Psi(i,j)| \le \dfrac{\alpha_0}{\sqrt{N}}$ for a constant α_0;*
2. *The block dimension $B \to \infty$ as $N \to \infty$;*

The proof of this theorem is based on the combinatorial central limit theorem in [7]. Details are omitted here due to lack of space. Note that in the above proposition, Condition 1 requires that Ψ be a dense matrix, whose elements' magnitudes are almost evenly distributed. Examples of such Ψ include the 2D FFT ($\alpha_0 = 1$) and the 2D DCT ($\alpha_0 \le 4$). While Condition 2 states $B \to \infty$ as $N \to \infty$, it is important to point out that B can be much smaller than N, e.g., on the order of $\sqrt{N}$ or even $\log N$. As a quick demo, Figure 1 shows the quantile-quantile plot for $\Phi_f(4, 200)$ when $B = 16$ and $N = 512^2 = 262144$ with Ψ corresponding to the 2D DCT. In this case, B is on the order of $\log N$. But $\Phi_f(4, 200)$ already behaves like a Gaussian random variable. In our opinion, such a normal behavior is one of the primary reason for excellent imaging performance of the SBHE even when B is small.

However, it should be pointed out that the entries $\Phi_f(i,j)$ are *not independent* as they are derived from the same scrambling operator. It is thus difficult to analyze exactly the minimum number of samples required. Fortunately, we can give a bound under l_1 reconstruction, as shown in the next subsection.

3.2. Performance bound of the l_1 reconstruction

Assume that Ψ is an orthogonal basis sparse matrix and define $\mathbf{U}$ as

$$\mathbf{U} = \mathbf{W}\mathbf{P}\Psi, \tag{4}$$

where $\mathbf{W}$ and $\mathbf{P}$ are the same as in (2). Note that $\Psi_f = \mathbf{Q}_M\mathbf{U}$ is a uniform random subset of $\mathbf{U}$. It is clear that $\mathbf{U}$ is an orthonormal matrix if $\mathbf{W}$ is normalized. According to [9], the performance bounds of l_1 reconstruction depends on $\mu = \max_{i,j} |\mathbf{U}(i,j)|$. To get a bound of μ, we took advantage of the recent result on Bernstein-type inequality for random permutations [8], as stated in the following theorem:

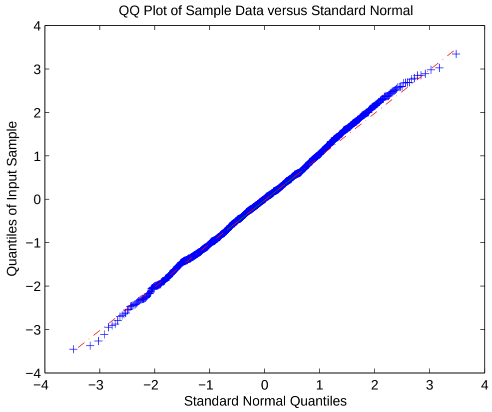

Fig. 1. Quantile-quantile plot of $\Phi_f(4, 200)$. Here, $B = 16$, $N = 512^2 = 262144$ and Ψ corresponds to the 2D DCT.

Theorem 1 *Let $\alpha_{k,l}$ be a collection of numbers from [0,1] and define $\mathcal{S} = \sum_{k=1}^{N} \alpha_{k\pi(k)}$, where π is drawn from the uniform distribution over the set of all permutations of $1, ..., N$. Then [8],*

$$\mathbb{P}\{|\mathcal{S} - \mathbb{E}(\mathcal{S})| \geq \varepsilon\} \leq 2\exp\left(-\frac{\varepsilon^2}{4\mathbb{E}(\mathcal{S}) + 2\varepsilon}\right)$$

for any $\epsilon > 0$.

Based on the above theorem and the union bound for supreme of random variables, we can show that if Conditions 1 in Proposition 1 is met, then $\mu = \max_{i,j} |\mathbf{U}(i, j)|$ satisfies

$$\mathbb{P}\left\{\mu \leq \mathcal{O}\left(\frac{\sqrt{\log N}}{\sqrt[4]{NB}}\right)\right\} = 1 - O\left(\frac{1}{N}\right). \tag{5}$$

We omit details here due to lack of space. Finally, combined with the result in [9], we arrive at the following theorem:

Theorem 2 *Suppose that the sampling operator Φ is constructed from the SBHE in (2) and assume that the sparse basis matrix Ψ is orhornormal sastisfying Conditions 1 in Proposition 1. For a given signal $x = \Psi f$, if f is supported on a fixed (but arbitrary) set T with K none zero entries, the l_1 optimizer can recover x exactly with high probability if the number of measuremens M satisfies*

$$M \geq C\left(K\sqrt{N/B}(\log N)^2\right). \tag{6}$$

for some constant C.

Proofs of Eq.(5) and Theorem 2 will be reported in the journal version of this paper. Note that when $B = N$, M is nearly optimal except for the $\log N$ factor. When $B < N$, (6) implies that M is inverse to $\sqrt{B}$ for a strictly sparse signal. For a compressible (weakly sparse) signal, this bound can be further improved as our simulations showed that B can be actually very small ($B = 32$) for imaging applications. Also, just as Proposition 1, Theorem 2 needs Ψ to be a dense matrix. We found that this restriction could be removed in practice as well.

3.3. Practical scrambling operator

In the ideal case, $\mathbf{P}_N$ needs to scramble the input signal in a chaotic way so that the sampling operator Φ is incoherent with the spare basis matrix Ψ. However, in practice, due to limitations of buffer size and optical devices, $\mathbf{P}_N$ cannot be selected as a pure random operator. Here, we will consider the design of two simple scrambling operators.

Linear congruential permutation(LCP) is a simple pixel level scrambler. For an input signal x, it outputs x_p with $x_p(i) = x(\pi(i))$ for $1 \leq i \leq N$, where $\pi(i)$ can be expressed as $\pi(i) = [A(i - 1) \mod N] + 1$ and A is a positive integer relative prime with N and $A < N$. Note that as there is only one parameter A, an LCP can be easily stored and implemented. It can be used for optical device such as the single-pixel camera, where pixel-by-pixel scrambling is allowed and $\mathbf{P}_N$ needs to be efficiently stored.

In applications when pixel-level scrambling is not allowed (e.g., conventional analog-TV systems), we propose to use *row permutation and row inversion*(RPRI) scrambler. Specifically, the operator $\mathbf{P}$ randomly permuted the rows of a 2D image, and reverse the even-numbered rows. Note that this kind of scrambling was widely used for commerical secure analog video broadcast sytems, e.g., [16]. Section 4 will demonstrate that for some images, the RPRI is as efficient as an ideal scrambler.

4. SIMULATION RESULTS

The performance of the SFE and our proposed SBHE have been evaluated and compared using the following reconstruction algorithms and software packages: (1) min-TV solver in the l_1 magic package [11]; (2) l_1-optimization solver using the gradient projection for sparse reconstruction (GPSR) algorithm [10]; (3) Iterative thresholding proposed by us in [14]. The sparse basis matrices in the GPSR and iterative thresholding are the Daubechies-8 wavelet and its undecimated version, respectively. All experiments have been carried out in Matlab 7.4 on a 2.66GHz dual-core desktop computer. For the SBHE, we will only present results with $B = 32$, but other larger values like 64, 128 and 512 usually yield similar results. For practical considerations, the random scrambling operator $\mathbf{P}_N$ has been implemented using the LCP and the RPRI methods, as mentioned in Section 3.3. For the SFE, the random permutation is based on Matlab's "randperm" function.

Table 1 tabulates the PSNR values for four 256×256 ($N = 2^{16}$) natural images *Lena*, *Cameraman*, *Peppers* and *Boats*. We also include results reported in [13] as benchmarks, where random Fourier sampling matrices were applied directly in the wavelet domain. For each image and each M in Table 1, we highlighted the best result in bold letters. Figure 2 further shows the reconstructed *Boats* images with $M = 20000$ measurements. Note that for these images, the percentage of nonzero elements in our SBHE is only $\frac{32}{256\times256} = \frac{1}{2^{11}}$! But Table 1 indicates that a highly sparse SBHE with an LCP scrambler produces similar imaging performance as that of a dense SFE with an ideal scrambler, regardless of the reconstruction algorithms. In most cases, results of the SBHE are even slightly better. The visual qualities of reconstructed images are also comparable, as can be seen in Figure 2. The RPRI scrambler is generally worse than the LCP. But when combined with our proposed iterative thresholding method [14], it still offers comparable performance to that of [13].

Finally, to demonstrate the practicality of the SBHE for large dimensional images, we applied it for a 1024×1024 image *Man* with $M = 10^5$ measurements and reconstructed it with the iterative thresholding algorithm [14]. Figure 3 shows the reconstructed images. As one can observe, even for this one-mega pixel image, an SBHE with $B = 32$ and an LCP scrambler works as well as the dense SFE, both in terms of objective PSNR and the visual quality, which suggests that the SBHE is very promising in compressive

Table 1. Objective coding performance (PSNR in dB)

No. of Samples M	[13]	l_1 optimization [10]			min-TV optimization [11]			Iterative thresholding [14]		
		SFE	SBHE ($B = 32$)		SFE	SBHE ($B = 32$)		SFE	SBHE ($B = 32$)	
		Randperm	LCP	RPRI	Randperm	LCP	RPRI	Randperm	LCP	RPRI
Lenna										
10000	26.5	21.5	21.1	20.7	27.5	**28.0**	27.6	27.2	27.7	26.2
15000	28.7	23.9	23.7	23.0	29.7	30.0	28.9	30.4	**30.6**	29.4
20000	30.4	26.0	26.0	25.5	31.4	31.8	30.7	**33.1**	**33.1**	31.9
25000	32.1	28.2	27.9	27.7	32.9	33.3	32.6	**35.5**	35.3	34.3
Cameraman										
10000	26.2	20.9	20.7	20.1	27.0	**27.1**	26.1	26.5	26.8	25.3
15000	28.7	23.2	23.0	20.8	29.3	29.5	26.6	29.5	**29.7**	26.2
20000	30.9	25.3	25.1	23.6	31.0	31.3	29.3	31. 8	**31.9**	29.7
25000	33.0	27.3	27.0	25.4	32.2	33.3	31.1	**34.4**	34.3	31.7
Peppers										
10000	21.6	20.3	20.0	20.3	28.6	29.0	26.2	28.1	**28.2**	24.8
15000	25.3	22.7	22.4	22.6	31.2	**31.7**	29.8	31.2	31.6	30.1
20000	27.5	25.7	25.3	25.3	32.9	33.4	30.8	**33.9**	**33.9**	31.1
25000	29.4	28.1	27.6	28.1	34.3	34.9	31.6	**36.0**	35.8	31.3
Boats										
10000	26.7	21.6	21.4	20.9	27.7	**28.0**	27.2	27.2	27.4	26.1
15000	29.8	23.9	23.8	23.5	29.7	30.3	29.8	30.4	**30.9**	30.3
20000	31.8	26.0	25.9	25.5	31.5	32.0	31.4	33.1	**33.7**	32.6
25000	33.7	28.1	28.1	27.6	33.1	33.6	32.9	35.7	**36.2**	34.9

imaging applications.

5. CONCLUSIONS

This paper has proposed the scrambled block Hadamard ensemble (SBHE) as a new sampling operator for compressive imaging applications. The SBHE is highly sparse and fast computable along with optical-domain friendly implementations. Both theoretical analysis and numerical simulation results have been presented to demonstrate the promising potential of the SBHE. In particular, we showed that a highly sparse SBHE can produce similar compressive imaging performance as that of a dense scrambled Fourier ensemble at much lower implementation cost.

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Fig. 2. Reconstructed 256×256 *Boats* with $M = 20,000$ samples. First row: min-TV reconstruction [11]. Second Row: Iterative thresholding reconstruction [14]. In each row, left: results of the dense SFE and ideal random permutation; middle: results of the SBHE with $B = 32$ and an LCP scrambler; right: results of the SBHE with $B = 32$ and an RPRI scrambler;

Fig. 3. Portions of reconstructed 1024×1024 image *Man* using iterative thresholding. Left: SFE with ideal random permutation, PSNR=27.6dB; Middle: SBHE with $B = 32$ and LCP scrambler, PSNR=27.8dB; SBHE $B = 32$ and RPRI crambler: PSNR=26.5dB.