SC 639

Assignment 2

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1 Solution to Question 1:

1.1 Solution to sub-question (i)

$$T : P_2 \rightarrow P_3, T(p) = xp(x) \tag{1}$$

$$Nullspace(T) := N(T) = \{p \in P_2 | T(p) = 0\} \tag{2}$$

$$T(p) = xp(x) = 0 \tag{3}$$

As p $\in P_2$, we can write a general polynomial p as,

$$p(x) = ax^2 + bx + c \tag{4}$$

where a,b,c belong to the field $\mathcal{F}$. According to eqn(3) we get that,

$$xp(x) = ax^3 + bx^2 + cx = 0 \tag{5}$$

According to the rules of fundamental algebra any N degree polynomial can have atmost N roots. However for the above equation must be zero for all values of x. Hence we can conclude that the above equation holds if, $xp(x)$ is identically equal to zero.

$$a = b = c = 0 \tag{6}$$

$$p(x) = 0 \tag{7}$$

$$Nullspace(T) := N(T) = \{0\} \tag{8}$$

$$Dim(N(T)) = 0 \tag{9}$$

$$Rangespace(T) := Im(T) = \{q \in P_3 \,|\, T(p) = q \,\forall\, p \in P_2\} \tag{10}$$

$$Im(T) = \{q \in P_3 \,|\, q(0) = 0\} = span\{x, x^2, x^3\} \tag{11}$$

By rank-nullity theorem we get,

$$Dim(Im(T)) = 3 \tag{12}$$

1.2 Solution to sub-question (ii)

:

$$S : P_3 \rightarrow P_3, [S(p)](x) = p(x + 1) + p(0) \tag{13}$$

Any vector polynomial in P_3 can be represented as,

$$p(x) = ax^3 + bx^2 + cx + d \tag{14}$$

$$S[p(x)] = a(x + 1)^3 + b(x + 1)^2 + c(x + 1) + d + d \tag{15}$$

$$S[p(x)] = span\{(x + 1)^3, (x + 1)^2, (x + 1), 2\} \tag{16}$$

$$Nullspace(S) := N(S) = \{p \in P_3 | S(p) = 0\} \tag{17}$$

$$S[p(x)] = ax^3 + (3a + b)x^2 + (3a + 2b + c)x + 2d \tag{18}$$

According to the rules of fundamental algebra any N degree polynomial can have atmost N roots. However for the above equation must be zero for all values of x. Hence we can conclude that the above equation holds if, $S[p(x)]$ is identically equal to zero. Hence we have that,

$$a = b = c = d = 0 \tag{19}$$

$$p(x) = 0 \tag{20}$$

$$Nullspace(T) := N(T) = \{0\} \tag{21}$$

$$Dim(N(T)) = 0 \tag{22}$$

$$Rangespace(S) := Im(S) = \{q \in P_3 \,|\, T(p) = q \,\forall\, p \in P_3\} \tag{23}$$

$$Im(S) = span\{(x+1)^3, (x+1)^2, (x+1), 2\} \tag{24}$$

By rank-nullity theorem we get,

$$Dim(Im(T)) = 4 \tag{25}$$

2 Solution to Question 2:

We have been given a linear transform and we have to prove that the *im(T)* and *ker(T)* are disjoint. We have the following linear transform. Let T be T: V $\rightarrow$ W where V and W are vector spaces.

$$ker(T) := \{v \in V \,|\, T(v) = 0\} \tag{26}$$

Hence, we get that,

$$T(a, b, c) = (2a + b + c, 0, 3c + 2b - a) = 0 \tag{27}$$

$$2a + b + c = 0 \tag{28}$$

$$3c + 2b - a = 0 \tag{29}$$

Solving both the equations simultaneously we get,

$$a = \frac{c}{5};\;\; b = \frac{-7c}{5};\;\; c = c \tag{30}$$

$$ker(T) = span\{\frac{1}{5}, \frac{-7}{5}, 1\} \tag{31}$$

$$Im(T) = \{w \in W \,|\, T(v) = w \;\forall v \in V\} \tag{32}$$

$$Im(T) = (2a + b + c, 0, 3c + 2b - a) \tag{33}$$

If im(T) and ker(T) have any element in common then we can compare the components i.e (2a + b + c, 0, 3c + 2b -a) with ($\frac{c}{5}$,$\frac{-7c}{5}$, c) then we get that c = 0. We can see that the intersection of the Im(T) and Ker(T) contains only the '0' element.

$$Ker(T) \cap Im(T) = \{0\} \tag{34}$$

Hence, we conclude that Ker(T) and Im(T) are mutually disjoint subspaces.

3 Solution to Question 3:

We have, a linear transformation $\alpha : Y \to V$ such that

$$\alpha(w, w') = w + w' \tag{35}$$

where $w \in W$ and $w' \in W'$ where W, W' are sub-spaces of V.

$$ker(\alpha) = \{w \in W,\ w' \in W' \,|\, \alpha(w, w') = 0\} \tag{36}$$

$$\alpha(w, w') = w + w' = 0 \tag{37}$$

$$w = -w' \tag{38}$$

$$w \in W \implies -w' \in W \implies w' \in W \tag{39}$$

$$w' \in W' \implies -w \in W' \implies w \in W' \tag{40}$$

$$w, w' \in W \cap W' \tag{41}$$

$$ker(\alpha) = \{w(1, -1) \,|\, w \in W \cap W'\} \tag{42}$$

Let A : ker(α) $\to W \cap W'$ be a linear mapping between two subspaces ker(α) and $W \cap W'$. Subspaces ker(α) and $W \cap W'$ are called isomorphic to each other if the A is surjective and injective i.e if the matrix associated with the mapping is invertible. Every vector in $ker(\alpha)$ can be expressed in terms of a constant vector $(1, -1) \in V$ and $w \in W \cap W'$. Hence we can conclude that $Dim(ker(\alpha)) = Dim(W \cap W')$ We begin by defining bases for $ker(\alpha)$ and $W \cap W'$

$$\mathcal{B}(ker(\alpha)) = \{u_1, u_2, \ldots, u_n\} \tag{43}$$

$$\mathcal{B}(W \cap W') = \{v_1, v_2, \ldots, v_n\} \tag{44}$$

If n = 0 then surely both $ker(\alpha)$ and $W \cap W'$ are isomorphic. Let n > 0. Let A be,

$$A(u_i) = v_i \,\forall\, 1 \leq i \leq n \tag{45}$$

It can be easily checked that it is a linear transform because it satisfies,

$$A(av + bw) = aA(v) + bA(w) \tag{46}$$

To prove that the matrix associated with A which is a linear transformation between two finite dimensional vector spaces is invertible we need to prove Nullspace(A) = 0

$$N(A) = \{u \in ker(\alpha) \,|\, Au = 0\} \tag{47}$$

Also, any vector in $ker(\alpha)$ can be written as $u = \sum_{i=1}^{n} c_i u_i$. Hence we have,

$$Au = A \sum_{i=1}^{n} c_i u_i = \sum_{i=1}^{n} c_i A(u_i) = \sum_{i=1}^{n} c_i v_i = 0 \tag{48}$$

We know all v_i's form a basis hence they are linearly independent. Hence for the above condition to be true $c_i = 0 \,\forall\, 1 \leq i \leq n$. Hence we can conclude that,

$$u = \sum_{i=1}^{n} c_i u_i = 0 \tag{49}$$

$$N(A) = 0 \tag{50}$$

Hence we conclude that the matrix associated with A is invertible. Hence proved that there exists a linear transform A from $ker(\alpha)$ to $W \cap W'$ such that the matrix a associated with A is invertible. Hence A is an injective and a surjective linear transform. Hence we conclude that $ker(\alpha)$ and $W \cap W'$ are isomorphic.

4 Solution to Question 4:

We have been given a inner product space defined on **R**, V induced with a norm. We then prove the following: $\langle u, v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$. We have that $\|x\| = \langle x, x\rangle$ and $\langle u, v\rangle = \overline{\langle v, u\rangle}$. We use this property further and expand the LHS.

$$LHS = \frac{1}{4}\langle u+v, u+v\rangle - \frac{1}{4}\langle u-v, u-v\rangle \tag{51}$$

$$LHS = \frac{1}{4}[\langle u, u+v\rangle + \langle v, u+v\rangle] - \frac{1}{4}[\langle u, u-v\rangle - \langle v, u-v\rangle] \tag{52}$$

$$LHS = \frac{1}{4}[\langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle] - \frac{1}{4}[\langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle] \tag{53}$$

$$LHS = \frac{1}{2}[\langle u, v\rangle + \langle v, u\rangle] \tag{54}$$

$$LHS = \frac{1}{2}[\langle u, v\rangle + \overline{\langle u, v\rangle}] \tag{55}$$

As the vector space is defined over the field of **R** we can take $\overline{\langle u, v\rangle} = \langle u, v\rangle$, Hence we get that LHS $= \langle u, v\rangle = RHS$. Hence proved.

5 Solution to Question 5:

We have been given a inner product space defined on $\mathbf{R}$, V induced with a norm. We then prove the following: $|\|v\| - \|w\|| \leq \|v - w\|$. We proceed as follows. We start with $\|v - w\| = \sqrt{\langle v - w, v - w\rangle}$. Hence we have,

$$\|v - w\| = \sqrt{\langle v, v\rangle - \langle v, w\rangle - \langle w, v\rangle + \langle w, w\rangle} \tag{56}$$

As the vector space is defined over R we have, $\langle v, u\rangle = \overline{\langle u, v\rangle} = \langle u, v\rangle$

$$\|v - w\| = \sqrt{\|v\|^2 - 2\langle v, w\rangle + \|w\|^2} \tag{57}$$

Now using Cauchy-Schwarz inequality we have, $\langle u, v\rangle \leq \|u\|.\|v|$. Hence

$$-2\langle v, w\rangle \geq -2\|v\|.\|w\| \tag{58}$$

$$\|v\|^2 - 2\langle v, w\rangle + \|w\|^2 \geq \|v\|^2 - 2\|v\|.\|w\| + \|w\|^2 \tag{59}$$

$$\|v - w\|^2 \geq (\|v\| - \|w\|)^2 \tag{60}$$

. We have that, $\|v - w\| \geq 0$ as it is a norm. But $\|v\| - \|w\|$ can be greater than or less than 0. Hence, Taking square roots on both sides we get, (The square root of the right side can be positive or negative hence we take the absolute value)

$$\sqrt{\|v - w\|^2} \geq \sqrt{(\|v\| - \|w\|)^2} \tag{61}$$

$$\|v - w\| \geq |\|v\| - \|w\|| \tag{62}$$

Hence proved.

6 Solution to Question 6:

We have been given a inner product space defined on **R**, V induced with a norm. We also have been given that $\|v+w\| = \|v\| + \|w\|$. Solving this by definition of inner product and norm we get that,

$$\langle v, w\rangle = \|v\|.\|w\| \tag{63}$$

We then prove the following:

$$\|av + bw\| = a\|v\| + b\|w\| \ \forall \ 0 \le a, b \in R \tag{64}$$

We proceed as follows. We consider and use the below mentioned properties of the inner product.

$$\langle \alpha_1 v_1 + \alpha_2 v_2, w\rangle = \alpha_1\langle v_1, w\rangle + \alpha_2\langle v_2, w\rangle \tag{65}$$

which can be extended to both the terms as follows

$$\langle \alpha_1 v_1 + \alpha_2 v_2, \beta_1 w_1 + \beta_2 w_2\rangle = \alpha_1\beta_1\langle v_1, w_1\rangle + \alpha_1\beta_2\langle v_1, w_2\rangle + \alpha_2\beta_1\langle v_2, w_1\rangle + \alpha_2\beta_2\langle v_2, w_2\rangle \tag{66}$$

. From eqn(64) we have,

$$LHS = \|av + bw\| \tag{67}$$

$$LHS = \sqrt{\langle av + bw, av + bw\rangle} \tag{68}$$

$$LHS = \sqrt{\langle av + bw, av\rangle + \langle av + bw, bw\rangle} \tag{69}$$

$$LHS = \sqrt{a\langle v, av\rangle + b\langle w, av\rangle + a\langle v, bw\rangle + b\langle w, bw\rangle} \tag{70}$$

$$LHS = \sqrt{a^2\langle v, v\rangle + ab\langle v, w\rangle + ab\langle w, v\rangle + b^2\langle w, w\rangle} \tag{71}$$

As the vector space is defined over R we have, $\langle v, w\rangle = \overline{\langle w, v\rangle} = \langle w, v\rangle$ therefore,

$$LHS = \sqrt{a^2\|v\|^2 + 2ab\langle v, w\rangle + b^2\|w\|^2} \tag{72}$$

From eqn(63) we get that(we have that a,b ≥ 0 and $\|v\|$, $\|w\|$ are non-negative hence we don't need to take the absolute value in taking the square root),

$$LHS = \sqrt{a^2\|v\|^2 + 2ab\|v\|.\|w\| + b^2\|w\|^2} \tag{73}$$

$$LHS = \sqrt{(a\|v\| + b\|w\|)^2} \tag{74}$$

$$LHS = a\|v\| + b\|w\| \ \forall \ 0 \le a, b \in R \tag{75}$$

Hence we get that, LHS = RHS. Hence proved.

7 Solution to Question 7:

We have to prove that,

$$A = EAE + EAF + FAE + FAF \tag{76}$$

For E to be a projection E must be idempotent i.e $E^2 = E$. Also, we have the condition that (1 - E) is a projection if and only if E is a projection. So here F is also a projection and follows $F^2 = F$. Also it should be noted that all projections are basically linear transformations and they follow all properties of LTs. We start with pre-multiplying RHS with E,(after this step we have LHS = EA)

$$RHS = E(EAE + EAF + FAE + FAF) \tag{77}$$

$$RHS = E^2AE + E^2AF + EFAE + EFAF \tag{78}$$

$$RHS = EAE + EAF + E(1 - E)AE + E(1 - E)AF \tag{79}$$

$$RHS = EAE + EAF + EAE - E^2AE + EAF - E^2AF \tag{80}$$

$$RHS = EAE + EAF + EAE - EAE + EAF - EAF \cdots\cdots (E^2 = E) \tag{81}$$

$$RHS = EAE + EAF \tag{82}$$

Post-multiplying RHS with F we get, (after this step we have **LHS = EAF**)

$$RHS = (EAE + EAF)F \tag{83}$$

$$RHS = EAEF + EAF^2 \tag{84}$$

$$RHS = EAE(1 - E) + EAF \cdots\cdots (F^2 = F) \tag{85}$$

$$RHS = EAE - EAE^2 + EAF \tag{86}$$

$$RHS = EAE - EAE + EAF \cdots\cdots (E^2 = E) \tag{87}$$

$$RHS = EAF = LHS \tag{88}$$

Hence proved.

7.1 Alternative Approach:

We consider E to be a projection on $\mathcal{M}$ along $\mathcal{N}$ where $\mathcal{V} = \mathcal{M} \oplus \mathcal{N}$ and $\mathcal{V}$ is a vector space and $\mathcal{M}$, $\mathcal{N}$ are subspaces. Hence E can be defined as $Ez = x$ where we have $z \in \mathcal{V}; x \in \mathcal{M}$

$$A = EAE + EAF + FAE + FAF \tag{89}$$

$$Az = (EAE + EAF + FAE + FAF)z \tag{90}$$

$$Az = EAEz + EAFz + FAEz + FAFz \tag{91}$$

$$Az = EAx + EA(1 - E)z + FAx + FA(1 - E)z \tag{92}$$

$$Az = EAx + EAz - EAEz + FAx + FAz - FAEz \tag{93}$$

$$Az = EAx + EAz - EAx + FAx + FAz - FAx \tag{94}$$

$$Az = EAz + FAz \tag{95}$$

$$Az = EAz + (1 - E)Az \tag{96}$$

$$Az = Az \rightarrow LHS = RHS \tag{97}$$

Hence proved.

8 Solution to Question 8:

First we check whether 'b' is in the column space of 'A'. By linear combinations we get,

$$C(A) = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \alpha + \beta \end{bmatrix} \neq b \; (\forall\, \alpha, \beta \in \mathcal{F}) \tag{98}$$

The problem is converted to minimizing the error $e = \|Ax - b\|^2$ where the the error vector e must be perpendicular to the column space. The solution x^* is which minimizes e is the same as locating the point $p = Ax^*$ that is closer to b than any other point in the column space of A. All vectors perpendicular to the column space lie in the left nullspace. Thus the error vector e must be in the nullspace of A^T

$$A^T(b - Ax^*) = 0 \tag{99}$$

$$x^* = (A^TA)^{-1}A^Tb \tag{100}$$

$$p = Ax^* = A((A^TA)^{-1}A^Tb) \tag{101}$$

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} [(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}] \tag{102}$$

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{103}$$

$$p = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{104}$$

$$p = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \tag{105}$$

Now we verify whether the error e = b - p is perpendicular to the column space of A i.e perpendicular to both columns of A; we check this via dot product.

$$e = b - p = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{bmatrix} \tag{106}$$

Now, any vector in the C(A) can be denoted as the linear combination of the columns of A. If the vector e is perpendicular to this linear combination then it can be concluded that e is perpendicular to the columnspace of A.

$$e \cdot L(C(A)) = e^T L(C(A)) = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \alpha + \beta \end{bmatrix} = 0 \tag{107}$$

We conclude that error vector e = b - p is perpendicular to the column space of A

9 Solution to Question 9:

We have,

$$V = W_1 \oplus W_2 \oplus W_3 \tag{108}$$

And we have three linear operators E_1, E_2, E_3 and we have to prove that, each E_i is an projection and $E_iE_j = 0$ if $i \neq j$ and range of E_i is W_i. We know, that A linear transformation E is a projection on some subspace if and only if it is idempotent, that is, $E^2 = E$. We then proceed as follows. As V is a direct sum of W_1,W_2 and W_3 any vector v in V can be written **uniquely** as follows:

$$v = w_1 + w_2 + w_3 \cdots (v \in V; w_1 \in W_1; w_2 \in W_2; w_3 \in W_3) \tag{109}$$

We now change the indexes to W_i,W_j and W_k from W_1,W_2 and W_3 for convenience. We can now let $W_j \oplus W_k = W$ where W is some subspace. Hence we can take , $w_j + w_k = w$ for some $w \in W$. We first prove the theorem along the forward direction i.e when we have $V = W_1 \oplus W_2 \oplus W_3$. **For a general projection F on M along N is defined as $Fv = x$ where we have $(v \in V; x \in M)$ where** $V = M \oplus N$. If E_i is a projection on W_i along $W_j \oplus W_k$ we have,

$$E_i^2 v = E_iE_iv = E_iw_i = w_i = E_iv \tag{110}$$

$$E_i^2 = E_i \tag{111}$$

Hence we got that E_i is idempotent. We now prove it in the backward direction. Suppose we have $E_i^2 = E_i$. Let $W_j \oplus W_k$ be a set of all vectors v $\in$ V such that $E_iv = 0$. Let W_i be a set of all vectors $v \in V$ such that $E_iv = v$. For an arbitary v we have,

$$v = E_iv + (1 - E_i)v \tag{112}$$

If we write $E_iv = w_i$ and $w = w_j + w_k = (1 - E_i)v$, then

$$E_iv = E_i^2 v = E_iE_iv = E_iw_i = w_i \tag{113}$$

$$E_iw = E_i(w_j + w_k) = E_i(1 - E_i)v = E_iv - E_i^2 v = 0 \tag{114}$$

so that $w_i \in W_i$ and $w = w_j + w_k \in W_j \oplus W_k$. This proves $V = W_i \oplus W$ where $W = W_j \oplus W_k$. Hence we have $V = W_i \oplus W_j \oplus W_k$ and that the projection on W_i along $Wj \oplus W_k$ is E_i. Hence we prove that a general E_i is a projection and it proves that E_1,E_2 and E_3 are projections such that,

- E_1 : It is a projection on W_1 along $W_2 \oplus W_3$ such that $E_1v = v$ where $v \in W_1 \subset V$ and $E_1v = 0$ where $v \in (W_2 \oplus W_3) \subset V$
- E_2 : It is a projection on W_1 along $W_1 \oplus W_3$ such that $E_2v = v$ where $v \in W_2 \subset V$ and $E_2v = 0$ where $v \in (W_1 \oplus W_3) \subset V$
- E_3 : It is a projection on W_1 along $W_1 \oplus W_2$ such that $E_3v = v$ where $v \in W_3 \subset V$ and $E_3v = 0$ where $v \in (W_1 \oplus W_2) \subset V$

From the above mentioned proof and the way the projections are defined we can see that the range of E_i is W_i. Now we prove that $E_iE_j = 0$ when $i \neq j$. Take any general vector $v \in V$

$$E_j(v) = v \;\; v \in W_j \tag{115}$$

or we can have,

$$E_j(v) = 0 \;\; v \in W_i \oplus W_k \tag{116}$$

Now we take the case presented in equation eqn(115) we have, $v \in W_j$

$$E_i(E_j(v)) = E_i(v) \;\; (v \in W_j) \tag{117}$$

But by definition of E_i we have $E_i(v) = v$ if $v \in W_i \subset V$ and $E_i(v) = 0$ if $v \in W_j \oplus W_k \subset V$
Hence, we have from eqn(117),

$$E_i(v) = 0 \quad (v \in W_j) \tag{118}$$

$$E_i(E_j(v)) = 0 \quad (v \in W_j) \tag{119}$$

Now lets take the case when $v \in W_j \oplus W_k$,

$$E_i(E_j(v)) = E_i(0) = 0 \quad v \in W_j \oplus W_k \tag{120}$$

This one is trivially satisfied. Now to verify these properties over E_1, E_2 and E_3 we can vary the indices as the indices i,j,k take the values from 1,2,3. So we can conclude that,

$$E_i E_j = 0 \;\forall\; i, j \in \{1, 2, 3\} \tag{121}$$

Hence we have verified that E_1, E_2 and E_3 are indeed projections, their ranges are W_i,W_j and W_k respectively and $E_i E_j = 0 \;\forall\; i, j \in \{1, 2, 3\}$.