SC 639

Assignment 1

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Solution to Question 1: We have that $\alpha, \beta \epsilon$ set of rational numbers. We have that the set of rational numbers along with addition and multiplication defined as a field. So we have that addition and multiplication follow all standard properties of a field. We use this argument to prove further.

- **Solution to Part (a)**:We have $q \epsilon Q$ as, ($\alpha, \beta \epsilon$ set of Rational Numbers.) Hence, let q_1, q_2, q_3 be such that $q_1, q_2, q_3 \epsilon Q$ and we have, ($\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \epsilon$ set of Rational Numbers.)

$$q = \alpha + \beta\sqrt{2}$$
$$q_1 = \alpha_1 + \beta_1\sqrt{2}$$
$$q_2 = \alpha_2 + \beta_2\sqrt{2}$$
$$q_3 = \alpha_3 + \beta_3\sqrt{2}$$

  - **Addition Properties**:
    - **Commutative Property**: We have,

$$q_1 + q_2 = (\alpha_1 + \beta\sqrt{2}) + (\alpha_2 + \beta\sqrt{2})$$
$$q_1 + q_2 = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)\sqrt{2} \tag{1}$$

Now we have,

$$q_2 + q_1 = (\alpha_2 + \beta_2\sqrt{2}) + (\alpha_1 + \beta_1\sqrt{2})$$
$$q_2 + q_1 = (\alpha_2 + \alpha_1) + (\beta_2 + \beta_1)\sqrt{2} \tag{2}$$

As $\alpha_1, \alpha_2, \beta_1, \beta_2$ belong to a field of rationals and satisfy all properties of a field, from equation (1), (2) we can conclude that **addition is commutative**.

    - **Associative Property**: We have,

$$q_1 + (q_2 + q_3) = (\alpha_1 + \beta_1\sqrt{2}) + [(\alpha_2 + \beta_2\sqrt{2}) + (\alpha_3 + \beta_3\sqrt{2})]$$
$$q_1 + (q_2 + q_3) = (\alpha_1 + \alpha_2 + \alpha_3) + (\beta_1 + \beta_2 + \beta_3)\sqrt{2} \tag{3}$$

Now we have,

$$(q_1 + q_2) + q_3 = [(\alpha_1 + \beta_1\sqrt{2}) + (\alpha_2 + \beta_2\sqrt{2})] + (\alpha_3 + \beta_3)$$
$$(q_1 + q_2) + q_3 = (\alpha_1 + \alpha_2 + \alpha_3) + (\beta_1 + \beta_2 + \beta_3)\sqrt{2} \tag{4}$$

As $\alpha_1, \alpha_2, \beta_1, \beta_2$ belong to a field of rationals and satisfy all properties of a field, from equation (3), (4) we can conclude that **addition is commutative**.

- **Existence and Uniqueness of Additive Identity**: Let, $q = \alpha + \beta\sqrt{2}$ where $\alpha = \beta = 0$ and $\alpha, \beta \epsilon$ set of rational numbers, which is the unique scalar 0.
$$q_1 + 0 = (\alpha_1 + \beta_1\sqrt{2}) + 0 + 0\sqrt{2}$$
$$q_1 + 0 = (\alpha_1 + \beta_1\sqrt{2})$$
Hence, it is verified that $q_1 + 0 = q_1$ Hence, we conclude that there **exists a unique additive identity** i.e **0**.
- **Existence and Uniqueness of Additive Inverse**: For every $q \epsilon Q\sqrt{2}$ we can find a unique scalar $q_1 \epsilon Q$ such that $q_1 = -q$, i.e if $q = \alpha + \beta\sqrt{2}$ then, $q_1 = -\alpha + (-\beta\sqrt{2})$.
$$q + (-q) = (\alpha + \beta\sqrt{2}) + (-\alpha + -\beta\sqrt{2})$$
$$q + (-q) = 0$$
We conclude that there **exists a unique additive inverse** i.e there exists a scalar -α $\forall$ $\alpha \epsilon Q$.

– **Multiplication Properties:**

- **Commutative**:
$$q_1q_2 = (\alpha_1 + \beta_1\sqrt{2})(\alpha_2 + \beta_2\sqrt{2})$$
$$q_1q_2 = \alpha_1\alpha_2 + \alpha_1\beta_2\sqrt{2} + \alpha_2\beta_1\sqrt{2} + 2\beta_1\beta_2$$
$$q_1q_2 = (\alpha_1\alpha_2 + 2\beta_1\beta_2) + (\alpha_1\beta_2 + \alpha_2\beta_1)\sqrt{2} \tag{5}$$
Now, we set,
$$q_2q_1 = (\alpha_2 + \beta_2\sqrt{2})(\alpha_1 + \beta_1\sqrt{2})$$
$$q_2q_1 = \alpha_2\alpha_1 + \alpha_2\beta_1\sqrt{2} + \alpha_1\beta_2\sqrt{2} + 2\beta_2\beta_1$$
$$q_2q_1 = (\alpha_2\alpha_1 + 2\beta_2\beta_1) + (\alpha_2\beta_1 + \alpha_1\beta_2)\sqrt{2} \tag{6}$$
As $\alpha_1, \alpha_2, \beta_1, \beta_2$ belong to a field of rationals and satisfy all properties of a field, from equation (6), (5) we can conclude that **multiplication is commutative**.
- **Associative**:
$$q_1(q_2q_3) = \alpha_1 + \beta_1\sqrt{2}(\alpha_2 + \beta_2\sqrt{2})(\alpha_3 + \beta_3\sqrt{2})$$
$$q_1(q_2q_3) = \alpha_1 + \beta_1\sqrt{2}\,(\alpha_2\alpha_3 + \alpha_2\beta_3\sqrt{2} + \alpha_3\beta_2\sqrt{2} + 2\beta_2\beta_3)$$

$q_1(q_2q_3) = (\alpha_1\alpha_2\alpha_3 + 2\alpha_1\beta_2\beta_3 + 2\beta_1\alpha_2\beta_3 + 2\beta_1\alpha_3\beta_2) + (\alpha_1\alpha_2\beta_3 + \alpha_1\alpha_3\beta_2 + \beta_1\alpha_2\alpha_3 + 2\beta_1\beta_2\beta_3)\sqrt{2}$

Similarly,
$$(q_1q_2)q_3 = (\alpha_1\alpha_2 + \alpha_1\beta_2\sqrt{2} + \alpha_2\beta_1\sqrt{2} + 2\beta_1\beta_2)(\alpha_3 + \beta_3\sqrt{2})$$
$(q_1q_2)q_3 = (\alpha_1\alpha_2\alpha_3 + 2\beta_1\beta_2\alpha_3 + 2\alpha_1\beta_2\beta_3 + 2\alpha_2\beta_1\beta_3) + (\alpha_1\beta_2\alpha_3 + \alpha_2\beta_1\alpha_3 + \alpha_1\alpha_2\beta_3 + 2\beta_1\beta_2\beta_3)\sqrt{2}$

As $\alpha_1, \alpha_2, \beta_1, \beta_2$ belong to a field of rationals and satisfy all properties of a field, from the above equations we can conclude that **multiplication is associative**.

* **Existence and Uniqueness of Multiplicative Identity**: Let, $q = \alpha + \beta\sqrt{2}$ where $\alpha = 1$ $\beta = 0$ and $\alpha, \beta \epsilon$ set of rational numbers, which is the unique scalar 1.

$$(q_1)(1) = (\alpha_1 + \beta_1\sqrt{2})(1 + 0\sqrt{2})$$

$$(q_1)(1) = \alpha_1 + \beta_1\sqrt{2}$$

Hence, it is proved that $(q_1)(1) = (q_1)$. Hence, we conclude that there **exists a unique multiplicative identity** i.e **1**.

* **Existence and Uniqueness of Multiplicative Inverse**: For every non zero (non-zero q i.e $\alpha \neq 0$ and $\beta \neq 0$). $q \epsilon Q\sqrt{2}$ we can find a unique scalar $q^{-1} \epsilon Q$ such that $q^{-1} = \frac{1}{q}$, i.e if $q = \alpha + \beta\sqrt{2}$ then, $q^{-1} = \frac{1}{\alpha + \beta\sqrt{2}}$. Multiplying q^{-1} with its conjugate we get

$$q^{-1} = \frac{\alpha - \beta\sqrt{2}}{\alpha^2 - 2\beta^2}$$

Here as α, β are rational $\frac{\alpha}{\alpha^2 - 2\beta^2}$ and $\frac{-\beta\sqrt{2}}{\alpha^2 - 2\beta^2}$ are also rational. So, we can uniquely determine the multiplicative inverse if the denominator is not zero, i.e $\alpha^2 \neq 2\beta^2$

$$qq^{-1} = (\alpha + \beta\sqrt{2})(\frac{\alpha - \beta\sqrt{2}}{\alpha^2 - 2\beta^2}) = 1 = 1 + 0\sqrt{2}$$

$$qq^{-1} = 1$$

We conclude that there **exists a unique multiplicative inverse** i.e there exists a scalar -q^{-1} $\forall$ $q \epsilon Q\sqrt{2}$ satisfying the condition $qq^{-1} = 1$.

- **Multiplication distributive wrt to addition**:

$$q_1(q_2 + q_3) = (\alpha_1 + \beta_1\sqrt{2})[(\alpha_2 + \beta_2\sqrt{2}) + (\alpha_3 + \beta_3\sqrt{2})]$$

$$q_1(q_2 + q_3) = (\alpha_1 + \beta_1\sqrt{2})[(\alpha_2 + \alpha_3) + (\beta_2 + \beta_3)\sqrt{2}]$$

$$q_1(q_2 + q_3) = \alpha_1(\alpha_2 + \alpha_3) + \alpha_1(\beta_2 + \beta_3)\sqrt{2} + (\alpha_2 + \alpha_3)\beta_1\sqrt{2} + 2\beta_1(\beta_2 + \beta_3)$$

$$q_1(q_2 + q_3) = (\alpha_1\alpha_2 + \alpha_1\alpha_3 + 2\beta_1\beta_2 + 2\beta_1\beta_3) + (\alpha_1\beta_2 + \alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_1)\sqrt{2}$$

Now, we have

$$q_1q_2 + q_1q_3 = [(\alpha_1\alpha_2 + 2\beta_1\beta_2) + (\alpha_1\beta_2 + \alpha_2\beta_1)\sqrt{2}] + [(\alpha_1\alpha_3 + 2\beta_1\beta_3) + (\alpha_1\beta_3 + \alpha_3\beta_1)\sqrt{2}]$$

$$q_1q_2 + q_1q_3 = (\alpha_1\alpha_2 + 2\beta_1\beta_2 + \alpha_1\alpha_3 + 2\beta_1\beta_3) + (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1\beta_3 + \alpha_3\beta_1)\sqrt{2}]$$

As $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ belong to a field of rationals and satisfy all properties of a field, from the above equations we can conclude that **multiplication is distributive wrt addition**.
As all the properties are satisfied by $(Q\sqrt{2})$ in the condition that $\alpha, \beta \in$ set of rational numbers **it is a field**.

- **Solution to Part (b)**: To prove if a given set is a field we need to prove all the above properties. Here, we are required to have α, β as integers. Here, we can verify the properties just like we did it for rationals. We find that in this case the set and the operations defined over the set satisfy all the properties of a field **except one** from the Multiplicative Properties. We prove it here. In this case the property of **Existence of Uniqueness of Multiplicative Inverse** is **not satisfied**.

  For every non zero (i.e $\alpha \neq 0$ and $\beta \neq 0$). $q \epsilon Q\sqrt{2}$ we can find a unique scalar $q^{-1} \epsilon Q$ such that $q^{-1} = \frac{1}{q}$, i.e if $q = \alpha + \beta\sqrt{2}$ then, $q^{-1} = \frac{1}{\alpha+\beta\sqrt{2}}$. Multiplying q^{-1} with its conjugate we get,

  $$q^{-1} = \frac{\alpha - \beta\sqrt{2}}{\alpha^2 - 2\beta^2}$$

  Here as α, β are integers while $\frac{\alpha}{\alpha^2-2\beta^2}$ and $\frac{-\beta\sqrt{2}}{\alpha^2-2\beta^2}$ are rational. Hence, we cannot express q^{-1} in the format $\alpha + \beta\sqrt{2}$ such that α, β ϵ set of integers. So, we cannot uniquely determine and express the multiplicative inverse. In other words $q^{-1} \notin Q(\sqrt{2})$. Hence as the multiplicative inverse doesn't belong to the field, the property of Existence of Uniqueness of Multiplicative Inverse is violated. Hence, we conclude that Q($\sqrt{2}$) in the condition that $\alpha, \beta \in$ set of integers **is not a field.**

Solution to Question 2:Here we have $\mathcal{P}$ as the vector space and V as the subset. In each case we have to prove that V is a vector space. Here as it is not mentioned otherwise we assume that the vector space $\mathcal{P}$ is defined over a field of R (real numbers). Also, we assume that the binary operations defined on the field of reals R are the usual addition and multiplication as it is not mentioned otherwise.

- **Solution to part(a)**: x is a vector (polynomial) such that $x \,\epsilon\, V$ and x has a degree 3. x can be written as (we take t as the dependent variable) where a, b, c, $d \,\epsilon$ R:

$$x(t) = a + bt + ct^2 + dt^3$$

In order to prove whether a given subset is a subspace or not, instead of checking closure under scalar multiplication and vector addition, it suffices to check if its closed under linear combination.
Let $x_1, x_2 \,\epsilon\, V$ and $\alpha, \beta \,\epsilon\, R$ define $x_3 = \alpha x_1 + \beta x_2$. We set as follows,

$$x_1(t) = a_1 + b_1 t + c_1 t^2 + d_1 t^3$$

$$x_2(t) = a_2 + b_2 t + c_2 t^2 + d_2 t^3$$

$$x_3(t) = \alpha x_1(t) + \beta x_2(t)$$

$$x_3(t) = \alpha(a_1 + b_1 t + c_1 t^2 + d_1 t^3) + \beta(a_2 + b_2 t + c_2 t^2 + d_2 t^3)$$

$$x_3(t) = (\alpha a_1 + \beta a_2) + (\alpha b_1 + \beta b_2)t + (\alpha c_1 + \beta c_2)t^2 + (\alpha d_1 + \beta d_2)t^3$$

Putting $(\alpha a_1 + \beta a_2) = \gamma_1$, $(\alpha b_1 + \beta b_2) = \gamma_2$, $(\alpha c_1 + \beta c_2) = \gamma_3$, $(\alpha a_1 + \beta a_2) = \gamma_4$ we get,

$$x_3(t) = \gamma_1 + \gamma_2 t + \gamma_3 t^2 + \gamma_4 t^3$$

We can see that x_3 is of degree 3. Hence, we can see that $x_3 \,\epsilon\, V$. Hence we can conclude that V is closed under linear combination. Hence, V is a subspace.

- **Solution to part (b)**: Now $x(t) = x(1-t) \,\forall\, t$. In order to prove whether a given subset is a subspace or not, instead of checking closure under scalar multiplication and vector addition, it suffices to check if its closed under linear combination.
Let x_1,$x_2 \,\epsilon\, V$ and $\alpha, \beta \,\epsilon\, R$. Now for x_1 and x_2 as they $x_1, x_2 \,\epsilon\, V$ we have the following properties,

$$x_1(t) = x_1(1-t) \tag{7}$$

$$x_2(t) = x_2(1-t) \tag{8}$$

We now set as follows,

$$x_3(t) = (\alpha x_1 + \beta x_2)(t)$$

$$x_3(t) = \alpha x_1(t) + \beta x_2(t)$$

$$x_3(1-t) = \alpha x_1(1-t) + \beta x_2(1-t)$$

From equation (7) and (8) we get the following,

$$x_3(1-t) = \alpha x_1(t) + \beta x_2(t)$$

$$x_3(1-t) = x_3(t)$$

Hence, we can conclude that $x_3 \,\epsilon\, V$. Hence we can conclude that V is closed under linear combination. Hence we conclude that V is a subspace.

Solution to Question 3:We have to prove that $dim(U_1 + U_2) \leq$ dim(U$_1$) + $dim(U_2)$. U_1, U_2 are subspaces of U. Consider a basis for $U_1 \cap U_2$. Let $dim(U_1 \cap U_2) = k$ where $k > 0$

Proof: Let

$$B = (y_1, y_2, \ldots, y_n)$$

Now we extend the basis to form the basis for U_1 and U_2. Let the basis sets be B_1 and B_2. Let $dim(U_1) = m + k$ and $dim(U_2) = n + k$; where m,n $>$ 0. We get,

$$B_1 = (y_1, y_2, \ldots, y_k, u_1, \ldots, u_m)$$

$$B_2 = (y_1, y_2, \ldots, y_k, v_1, \ldots, v_n)$$

Now we take $B_1 \cup B_2$ we get,

$$B_1 \cup B_2 = (y_1, y_2, \ldots, y_k, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n)$$

We now prove that $B_1 \cup B_2$ spans the set $U_1 \cup U_2$.
For every $x \epsilon U_1 + U_2$ it can be written as $x = u + v$ where $u \epsilon U_1$ and $v \epsilon U_2$.

$$x = u + v$$

Now, u can be written as a linear combination of the basis vectors in set B_1 and v can be written as a linear combination of the basis vectors in set B_2, we get,

$$u = \sum_{i=1}^{k} \alpha_i y_i + \sum_{i=1}^{m} \beta_i u_i$$

$$v = \sum_{i=1}^{k} \gamma_i y_i + \sum_{i=1}^{n} \delta_i v_i$$

Hence, x can be written as,

$$x = \sum_{i=1}^{k} \alpha_i y_i + \sum_{i=1}^{m} \beta_i u_i + \sum_{i=1}^{k} \gamma_i y_i + \sum_{i=1}^{n} \delta_i v_i$$

$$x = \sum_{i=1}^{k} (\alpha_i + \gamma_i) y_i + \sum_{i=1}^{m} \beta_i u_i + \sum_{i=1}^{n} \delta_i v_i$$

As we had chosen x arbritarilty, we can say that x is a linear combination of $B_1 \cup B_2$ $\forall$ $x \epsilon U_1 + U_2$. Also, B_1 and B_2 are linearly independent sets and hence we can conclude that $B_1 \cup B_2$ is linearly independent. Also, by the above argument we can conclude that $B_1 \cup B_2$ spans the entire set $U_1 + U_2$. Hence, we conclude that $B_1 \cup B_2$ is a basis for $U_1 + U_2$. We get that,. cardinality of $B_1 \cup B_2 = m + n + k$. Hence,

$$dim(U_1 + U_2) = m + n + k$$

$$dim(U_1) = m + k$$

$$dim(U_2) = n + k$$

$$dim(U_1) + dim(U_2) = m + n + 2k$$

As we have $k \geq 0$, we have,

$$dim(U_1 + U_2) \leq dim(U_1) + dim(U_2)$$

Hence proved.

Solution to Question 4:Here, we have $S = \{1, x, x^2\}$ and $T = \{1 - x, 1 + x, 1 + x^2\}$. First we prove S is linearly independent. Set $\alpha_1(1) + \alpha_2(x) + \alpha_3(x^2) = 0 \ \forall x$. However, the equation is of degree 2 and hence it can at most have 2 roots. However if the equation equals to 0 $\forall x \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence we can conclude that set S is linearly independent.
Now we prove $span\{S\} = span\{T\}$. In order to show 2 sets A and B equal to each other; is via proving that $A \subset B$ and $B \subset A$. We prove it as follows:

1. $span\{A\} \subset span\{T\}$: Let v be a given arbitrary element in $span\{A\}$ and $\exists$ set of scalars α_1,α_2,α_3 ϵ R such that,

$$v = \alpha_1(1) + \alpha_2(x) + \alpha_3(x^2)$$

   Putting $\alpha_1 = \beta_1 + \beta_2 + \beta_3$, $\alpha_2 = \beta_2 - \beta_1$, $\alpha_3 = \beta_3$ we get,

$$v = \beta_1 + \beta_2 + \beta_3 + (\beta_2 - \beta_1)x + \beta_3 x^2$$

$$v = \beta_1 + \beta_2 + \beta_3 + \beta_2 x - \beta_1 x + \beta_3 x^2$$
$$v = (\beta_1 - \beta_1 x) + (\beta_2 + \beta_2 x) + (\beta_3 + \beta_3 x^2)$$
$$v = \beta_1(1 - x) + \beta_2(1 + x) + \beta_3(1 + x^2)$$

   Thus v ϵ $span\{T\}$. Since v was chosen arbitrarily we conclude that $span\{A\} \subset span\{T\}$.

2. $span\{T\} \subset span\{A\}$: Let u be a given arbitrary element in $span\{T\}$ and $\exists$ set of scalars γ_1,γ_2,γ_3 ϵ R such that,

$$v = \gamma_1(1 - x) + \gamma_2(1 + x) + \gamma_3(1 + x^2)$$

$$v = \gamma_1 - \gamma_1 x + \gamma_2 + \gamma_2 x + \gamma_3 + \gamma_3 x^2$$
$$v = \gamma_1 + \gamma_2 + \gamma_3 - \gamma_1 x + \gamma_2 x + \gamma_3 x^2$$
$$v = \gamma_1 + \gamma_2 + \gamma_3 + (\gamma_2 - \gamma_1)x + \gamma_3 x^2$$

   Putting $\delta_1 = \gamma_1 + \gamma_2 + \gamma_3$, $\delta_2 = \gamma_2 - \gamma_1$, $\delta_3 = \gamma_3$ we get,

$$v = \delta_1 + \delta_2 x + \delta_3 x^2$$

   Thus v ϵ $span\{S\}$. Since v was chosen arbitrarily we conclude that $span\{T\} \subset span\{A\}$.
   As we get that $span\{A\} \subset span\{T\}$ and $span\{T\} \subset span\{A\}$, we can conclude that $span\{A\} = span\{T\}$. Hence proved.

Solution to Question 5:

1. **Question (a)**: We have V_1, V_2 as subspaces. In order to prove whether a given subset is a subspace or not, instead of checking closure under scalar multiplication and vector addition, it suffices to check if its closed under linear combination.

   Let x, y be $x, y \,\epsilon\, V_1 \cap V_2$ by definition we know $V_1 \cap V_2 := \{x \mid x \,\epsilon\, V_1$ and $x \,\epsilon\, V_2\}$ hence $x \,\epsilon\, V_1$ and $x \,\epsilon\, V_2$ similarly, $y \,\epsilon\, V_1$ and $y \,\epsilon\, V_2$. As V_1 and V_2 are subspaces, they will be closed under linear combination.
   Hence we can conclude that,

   $$x, y \,\epsilon\, V_1 \;;\; \alpha, \beta \,\epsilon\, R \;\Rightarrow\; \alpha x + \beta y \,\epsilon\, V_1$$

   $$x, y \,\epsilon\, V_2 \;;\; \alpha, \beta \,\epsilon\, R \;\Rightarrow\; \alpha x + \beta y \,\epsilon\, V_2$$

   Hence $\alpha x + \beta y \,\epsilon\, V_1$ and $\alpha x + \beta y \,\epsilon\, V_2 \Rightarrow \alpha x + \beta y \,\epsilon\, V_1 \cap V_2$. Hence we got that when

   $$x, y \,\epsilon\, V_1 \cap V_2,\; \alpha x + \beta y \,\epsilon\, V_1 + V_2.$$

   Hence $V_1 \cap V_2$ is closed under linear combination. Hence we can conclude that $V_1 \cap V_2$ is a subspace.

2. **Question (b)**: Now we prove the general case. We have $V_1, V_2 \ldots V_n$ as subspaces. In order to prove whether a given subset is a subspace or not, instead of checking closure under scalar multiplication and vector addition, it suffices to check if its closed under linear combination. We prove that $V_1 \cap V_2 \ldots \cap V_n$ is a subspace.

   Let $x, y \,\epsilon\, V_1 \cap V_2 \ldots \cap V_n$ by definition, we get,

   $$x \,\epsilon\, V_1 \cap V_2 \ldots \cap V_n := \{x \mid x \,\epsilon\, V_1, x \,\epsilon\, V_2, \ldots, x \,\epsilon\, V_n\}$$

   $$y \,\epsilon\, V_1 \cap V_2 \ldots \cap V_n := \{y \mid y \,\epsilon\, V_1, y \,\epsilon\, V_2, \ldots, y \,\epsilon\, V_n\}$$

   Hence, $x, y \,\epsilon\, V_1$, $x, y \,\epsilon\, V_2$ …, $x, y \,\epsilon\, V_n$. As V_1, V_2, …, V_n are subspaces, they will be closed under linear combination.
   Hence we can conclude that,

   $$x, y \,\epsilon\, V_1 \;;\; \alpha, \beta \,\epsilon\, R \;\Rightarrow\; \alpha x + \beta y \,\epsilon\, V_1$$

   $$x, y \,\epsilon\, V_2 \;;\; \alpha, \beta \,\epsilon\, R \;\Rightarrow\; \alpha x + \beta y \,\epsilon\, V_2$$

   $$\vdots$$

   $$x, y \,\epsilon\, V_n \;;\; \alpha, \beta \,\epsilon\, R \;\Rightarrow\; \alpha x + \beta y \,\epsilon\, V_n$$

   Hence we get that,

   $$\alpha x + \beta y \,\epsilon\, V_1\,, \alpha x + \beta y \,\epsilon\, V_2\,, \ldots, \alpha x + \beta y \,\epsilon\, V_n$$

   $$\alpha x + \beta y \,\epsilon\, V_1 \cap V_2 \ldots \cap V_n$$

   Hence, we can conclude that $V_1 \cap V_2 \ldots \cap V_n$ is closed under linear combination. Hence we can conclude that $V_1 \cap V_2 \ldots \cap V_n$ is a subspace.

Solution to Question 6:We have a linear transformation T from R^3 →R^2 defined as,

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$$

We have B as the standard ordered basis for R^3 and B' as the standard ordered basis of R^2.

- **Solution to Question (a)**: We have to find matrix of T relative to the pair B, B'. We have,

$$B = \{(1,0,0),(0,1,0),(0,0,1)\}$$
$$B' = \{(1,0),(0,1)\}$$

Now, we have $Tx = y$ where x ϵ R^3 and y ϵ R^2. Now, we know that the matrix form of $[T]$ is a $2X3$ matrix,

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_3 - x_1 \end{bmatrix}$$

To find matrix representation in respective basis, we have to represent the transformed vector as a linear combination of its own basis. Let's take a general representation $T : V \to W$ with $(v_1, v_2, \ldots, v_n)$ basis of V and $(w_1, w_2, \ldots, w_m)$ basis of W.

$$Tv_k = \sum_{i=1}^{m} a_{ik} w_k$$
$$[T_{ij}] = a_{ij}$$

Applying the same concept in our example we get,

$$T(1,0,0) = (1,-1) = 1(1,0) + (-1)(0,1)$$
$$T(0,1,0) = (1,0) = 1(1,0) + (0)(0,1)$$
$$T(0,0,1) = (0,2) = 0(1,0) + (2)(0,1)$$
$$[T] = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Hence we got the matrix of T relative to the pair B, B'.

- **Solution to Question (b)**: We have $B = \{\alpha_1, \alpha_2, \alpha_3\}$ and $B' = \{\beta_1, \beta_2\}$ where, $\alpha_1 = (1,0,1),\ \alpha_2 = (1,1,1),\ \alpha_3 = (1,0,0),\ \beta_1 = (0,1),\ \beta_2 = (1,0)$
Again, to find matrix representation in respective basis, we have to represent the transformed vector as a linear combination of its own basis. Let's take a general representation $T : V \to W$ with $(v_1, v_2, \ldots, v_n)$ basis of V and $(w_1, w_2, \ldots, w_m)$ basis of W.

$$Tv_k = \sum_{i=1}^{m} a_{ik} w_k$$
$$[T_{ij}] = a_{ij}$$

Applying the same concept in our example we get,

$$T(1,0,-1) = (1,-3) = (-3)(0,1) + 1(1,0)$$

$$T(1,1,1) = (2,1) = (1)(0,1) + 2(1,0)$$
$$T(1,0,0) = (1,-1) = (-1)(0,1) + 1(1,0)$$
$$[T] = \begin{pmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

Hence we got the matrix of T relative to the pair B, B'.

Solution to Question 7 (Practice Question 6):Here the vector space V is a space of continuous functions such that $V = C(-1, 1)$. We have two functions $f, g \epsilon V$ such that they are defined by $f : x \to x^2$ and $g : x \to |x|x$. We have to prove that f, g are linearly independent.
Proof: A set of vectors $(v_1, v_2, \ldots, v_n)$ is said to be linearly independent if the equation,

$$a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$$

can only be satisfied by $a_1 = 0 \; \forall \, i = 1, 2, \ldots, n$. Similarly, we prove that f, g is linearly independent. We set the following to prove. Here $\alpha, \beta \, \epsilon \, R$.

$$\alpha f(x) + \beta g(x) = 0$$

$$\alpha x^2 + \beta |x| x = 0 \tag{9}$$

We know the properties of $|x|$ are,

$$|x| = x \quad x \geq 0$$

$$|x| = -x \quad x \leq 0$$

Now, we have 3 cases here.

- $x > 0$: Here, we have the following,

  $$\alpha x^2 + \beta x^2 = 0$$

  $$(\alpha + \beta)x^2 = 0 \tag{10}$$

  Here, the equation (10) will only hold if $\alpha, \beta = 0$ or $\alpha = -\beta$. Hence, **the equation doesn't only hold when $\alpha = \beta = 0$.** Hence, with the conditions above f, g is **not linearly independent**.

- $x < 0$: Here, we have the following,

  $$\alpha x^2 - \beta x^2 = 0$$

  $$(\alpha - \beta)x^2 = 0 \tag{11}$$

  Here, the equation (11) will only hold if $\alpha, \beta = 0$ or $\alpha = \beta$. Hence, **the equation doesn't only hold when $\alpha = \beta = 0$.** Hence, with the conditions above f, g is **not linearly independent**.

- $x = 0$: Here, we already have that the equation (9) is 0

Hence, we have proved in each of the three cases that f, g is **linearly dependent**.

Solution to Question 8 (Practice Question 12):B is a linear transformation on a finite-dimensional vector space V and if B commutes with every linear transformation on V , then B is a scalar; that is, there exists a scalar β such that $Bx = \beta x$, $\forall$ x ϵ V. We will prove this by contradiction.
Proof: Let us assume that B is not a scalar linear transformation i.e $Bx \neq \beta x$.
Then, $\exists$ x *epsilon* V such that x, Bx are linearly independent. As x, Bx are linearly independent we can extend them to form a basis of V. Let this basis set be A; we define this as follows:

$$A = x, Bx, v_1, v_2, \ldots, v_n$$

Now, we consider a linear transformation T : $V \to V$ such that,

$$T(x) = x, \;\; T(Bx) = x, \;\; T(v_i) = 0 \quad \forall \; i = 1, 2, \ldots n. \tag{12}$$

Then we get,

$$T(B(x)) = B(T(x)) \quad \ldots \quad \text{(B commutes with every linear transform on V)}$$

$$T(B(x)) = B(x) \quad \ldots \quad \text{(From (12))}$$

But we have that $T(Bx) = Bx$ from (12). So we get that,

$$x = Bx$$

We get a contradiction to the initial argument that x and Bx are linearly independent. Hence. we can conclude that,

$$\text{for every} \quad x \;\epsilon\; V \quad \exists \quad \beta \;\epsilon\; R \quad \text{such that} \quad Bx = \beta x$$

But now, **we also have to prove that β is unique**. We prove it as follows. We first assume that β is not unique $\forall$ x ϵ V. We first define a set of basis vectors for V as X, where X is,

$$X = \{x_1, x_2, \ldots, x_n\}$$

Now, we know $Bx = \beta x$. And as x is an arbitrarily chosen vector from V we can realize it with the linear combination of vectors in the basis set X. We get,

$$Bx = \beta x = \beta(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n)$$

$$Bx = \beta x = (\alpha_1 \beta x_1 + \alpha_2 \beta x_2 + \ldots + \alpha_n \beta x_n)$$

where, α_i ϵ field $\forall$ $i = 1, 2, \ldots, n$ But also, we have as follows,

$$Bx = B(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) \tag{13}$$

As we have considered β not to be unique we consider that $B(x_i) = \gamma_i x_i$ where, γ_i ϵ field $\forall$ $i = 1, 2, \ldots, n$. We have,

$$Bx = \alpha_1 \gamma_1 x_1 + \alpha_2 \gamma_2 x_2 + \ldots + \alpha_n \gamma_n x_n \tag{14}$$

The set $\{x_1, x_2, \ldots, x_n\}$ is linearly independent and any vector x ϵ V can be determined by the linear combination of these vectors uniquely. Hence, we can conclude that the co-efficients of (13) and (14) should be equal.
Hence, we conclude that, $\gamma_i = \beta \quad \forall \quad i = 1, 2, \ldots, n$. Hence, $Bx_i = \beta x_i \; \forall \quad i = 1, 2, \ldots, n$. As β is unique for all the basis vectors, we can conclude that it will br unique for the entire vector space V as every vector $x \epsilon V$ can be realized as a linear combination of basis vectors.
Hence, we conclude that, $Bx = \beta x \; \forall \; x$. Hence proved.

Solution to Question 9:Here we have, a linear operator T on R^3 such that,

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

We analyze if T is invertible or not. We have two ways for checking invertibilty of T.

- **Matrix Approach**: To find matrix representation in respective basis, we have to represent the transformed vector as a linear combination of its own basis. Let's take a general representation $T : V \to W$ with $(v_1, v_2, \dots, v_n)$ basis of V and $(w_1, w_2, \dots, w_m)$ basis of W.

$$Tv_k = \sum_{i=1}^{m} a_{ik} w_k$$
$$[T_{ij}] = a_{ij}$$

  Applying the same concept in our example we consider the standard basis of R^3 we get,

$$T(1, 0, 0) = (3, 1, 2)$$
$$T(0, 1, 0) = (0, -1, 1)$$
$$T(0, 0, 1) = (0, 0, 1)$$

  Hence, we get that the matrix of T is,

$$[T] = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

  If T is a singular matrix then T is not invertible. Here, **det(T)** $=$ **-3** $\to$ T **is invertible**.

- **Analytical Approach**: A linear operator is invertible iff (i) T is 1-1 (injective) and (ii) T is onto (surjective). T is 1-1 if for every y ϵ R^3 there exists at most one x ϵ R^3 such that T(x) = y. To prove this we have to show that two different vectors in R^3 have different pre-images in R^3. Let i_1, i_2 be two different vectors in the Rangespace of T such that $i_1 \neq i_2$. We have,

$$i_1 = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$
$$i_2 = (3y_1, y_1 - y_2, 2y_1 + y_2 + y_3)$$

Applying the linear operator T we can obtain pre-images of i_1, i_2. Let them be j_1, j_2 respectively. We have,

$$j_1 = (x_1, x_2, x_3)$$
$$j_2 = (y_1, y_2, y_3)$$

From the above argument that, $i_1 \neq i_2$, we can conclude that,

$$(x_1, x_2, x_3) \neq (y_1, y_2, y_3) \Rightarrow j_1 \neq j_2$$

Therefore, we conclude that T is one-one transform. Now, we prove that T is **onto**. T is onto if for every y ϵ R^3 there exists at least one x ϵ R^3 such that T(x) = y. We take i_3 such that i_3 belongs to the Rangespace of T. We have,

$$i_3 = (3z_1, z_1 - z_2, 2z_1 + z_2 + z_3)$$

Here, by applying the linear operator we can concldue that there exists a pre-image of i_3 which is j_3 such that $j_3 = (z_1, z_2, z_3)$. Hence, we can conclude that linear operator T is **onto**.

We proved that T is both one-one and onto. Hence, we can conclude that T is invertible. **Now, we find a, find a rule for T^1 like the one which defines T.** Here, we have two methods we can find the inverse of the matrix and then apply the matrix on the basis vectors and find the formula. Also, we can undertake an analytical approach. We have,

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

We replace $3x_1$ by y_1 we get,

$$T(\frac{y_1}{3}, x_2, x_3) = (y_1, \frac{y_1}{3} - x_2, \frac{2y_1}{3} + x_2 + x_3)$$

We then replace x_2 by $\frac{y_1}{3} - y_2$ we get,

$$T(\frac{y_1}{3}, \frac{y_1}{3} - y_2, x_3) = (y_1, y_2, \frac{2y_1}{3} + \frac{y_1}{3} - y_2 + x_3)$$

$$T(\frac{y_1}{3}, \frac{y_1}{3} - y_2, x_3) = (y_1, y_2, y_1 - y_2 + x_3)$$

We then replace x_3 by $y_2 - y_1 + y_3$ we get,

$$T(\frac{y_1}{3}, \frac{y_1}{3} - y_2, y_2 - y_1 + y_3) = (y_1, y_2, y_3)$$

Now, pre-multiplying with T^{-1} we get,

$$(\frac{y_1}{3}, \frac{y_1}{3} - y_2, y_2 - y_1 + y_3) = T^{-1}(y_1, y_2, y_3)$$

Replacing y_1 with x_1, y_2 with x_2, y_3 with x_3 we get,

$$\boxed{T^{-1}(x_1, x_2, x_3) = (\frac{x_1}{3}, \frac{x_1}{3} - x_2, x_2 - x_1 + x_3)} \tag{15}$$

From equation (15) we get the formula for the linear operator T^{-1}.