EE2703: Applied Programming Lab
Assignment 4: Fourier Approximations

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1 Introduction

Two functions, $exp(x)$ and $cos(cos(x))$ over the interval $[0, 2\pi)$ will be modeled using the fourier series:

$$a_0 + \sum_{n=1}^{\infty}\{a_n\cos(nx) + b_n\sin(nx)\}$$

Where,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)cos(nx)dx$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)sin(nx)dx$$

2 Subquestions

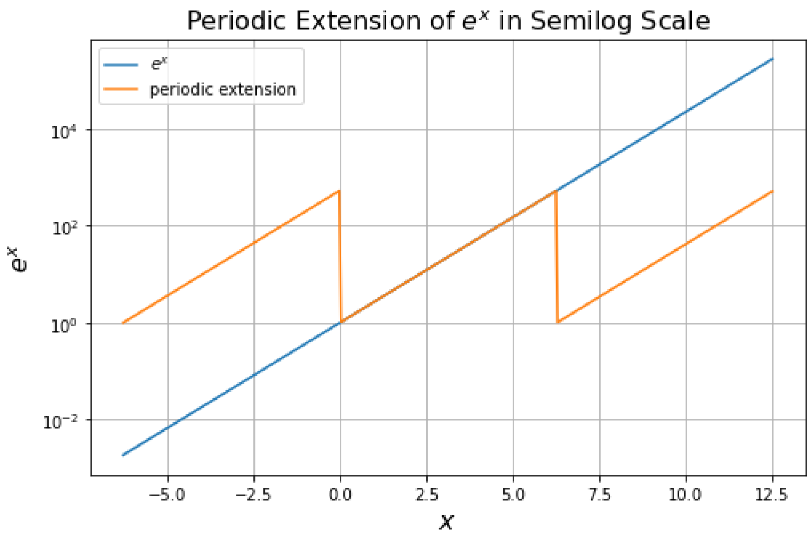
2.1 Define & Plot Python Functions

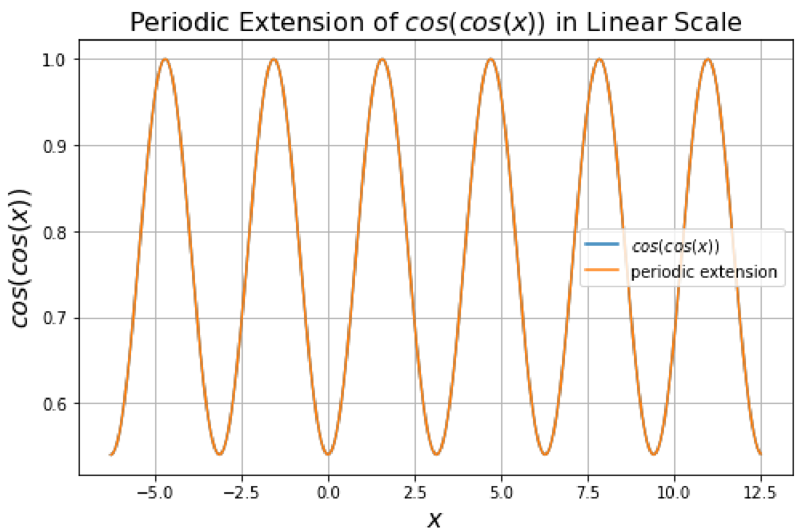
Two Python functions are defined:

```
def exp(x):  # exponential function, supports value or vector
    return np.exp(x)

def coscos(x):  # cos of cos function, supports value or vector
    return np.cos(np.cos(x))
```

The functions have been plotted with their periodic extensions (2π period):





2.2 Evaluate Integrals

The integrands have been defined as functions of x, k, and f, where f is the Python function to be approximated, i.e. exp or coscos:

```
def u(x, k, f):  # f is either exp or coscos
    return f(x) * np.cos(k * x)

def v(x, k, f):  # f is either exp or coscos
    return f(x) * np.sin(k * x)
```

The integrals to calculate the values of a_n and b_n have been evaluated using the following loop:

```
F = [exp, coscos]
a_0 = np.zeros((2))
a_n = np.zeros((2, 25))
b_n = np.zeros((2, 25))

for i in range(2):  # iterate over exp and coscos
    a_0[i] = quad(F[i], 0, 2 * np.pi)[0] / (2 * np.pi)

for j in range(25):  # integration
    a_n[i, j] = quad(u, 0, 2 * np.pi, args=(j + 1, F[i]))[0] / np.pi
    b_n[i, j] = quad(v, 0, 2 * np.pi, args=(j + 1, F[i]))[0] / np.pi
```

2.3 Plot the Fourier Coefficients

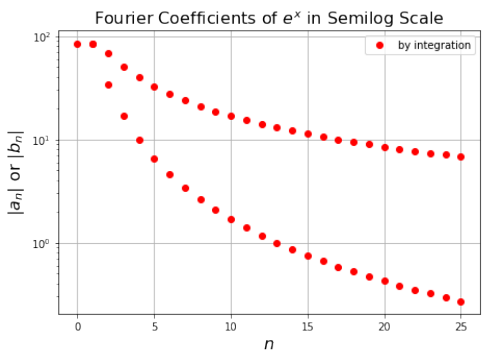
The answer vector c has been generated by:

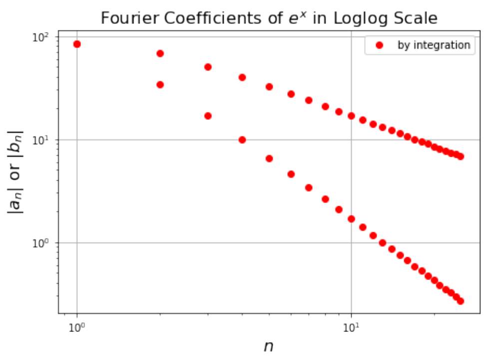
```
c_n = np.zeros((2, 51))  # c_n[0] and c_n[1] are the answer vectors
c_n[:, 0] = a_0
c_n[:, 1::2] = a_n  # alternate elements starting at 1
c_n[:, 2::2] = b_n  # alternate elements starting at 2
```

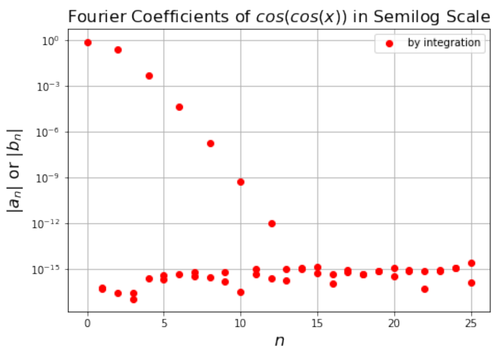
To plot in semilog scale, for example, we use:

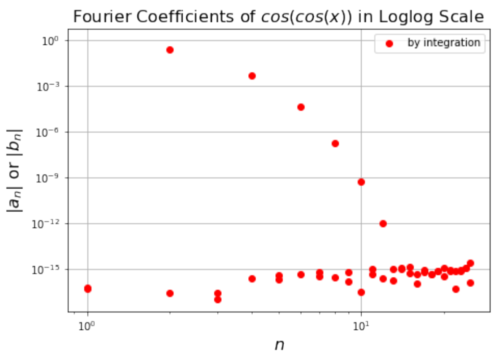
```
plt.semilogy(0, np.abs(a_0[f_i]), 'ro', label="by integration")
plt.semilogy(range(1, 26), np.abs(a_n[f_i]), 'ro')
plt.semilogy(range(1, 26), np.abs(b_n[f_i]), 'ro')
```

The coefficients $|a_n|$ and $|b_n|$ have been plotted:









(a) The b_n coefficients for $cos(cos(x))$ are of the order of magnitude -16, i.e. nearly zero.
This is because $cos(cos(x))$ is an even function, and thus does not have any odd component.

(b) $cos(cos(x))$ is a periodic function, comprising not many frequencies. e^x, on the other hand, is a non-periodic function, and thus its periodic extension has multiple discontinuities. Hence, high frequency components are required to represent this function as a sum of trigonometric functions.

(c) The *loglog* plot is linear for e^x because the *log* of its coefficients vary linearly with $log(n)$, i.e. the coefficients depend on $\frac{1}{n^a}$ for some a.
On the other hand, the *semilog* plot is linear for $cos(cos(x))$ because the *log* of its coefficients vary linearly with n, i.e. the coefficients decay exponentially.

2.4 Least Squares

The equation to be solved by scipy.linalg.lstsq is:

$$Ac = B$$

The matrix A has been generated by:

```
A = np.ones((400, 51))  # first column should be ones
for k in range(1, 26):
    A[:, 2 * k - 1] = np.cos(k * x)
    A[:, 2 * k] = np.sin(k * x)
```

The solution matrices are c[0] for e^x and c[1] for $coscos(x)$:

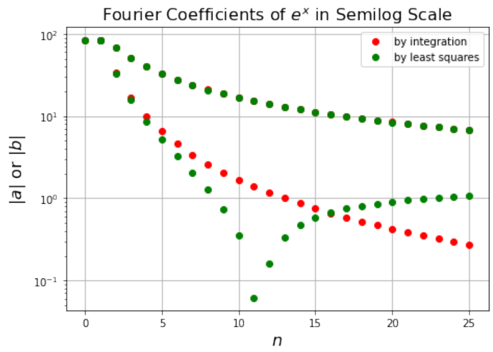
```
for i in range(2):
    b = F[i](x)
    c[i] = lstsq(A, b)[0]
```

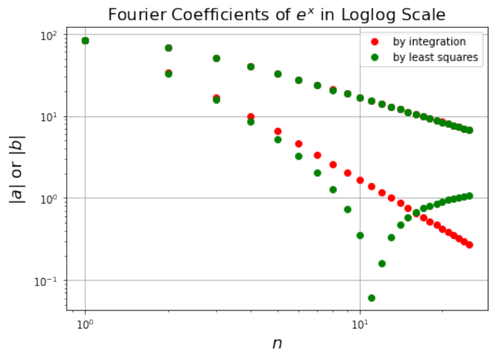
2.5 Plot the Best Fit Coefficients

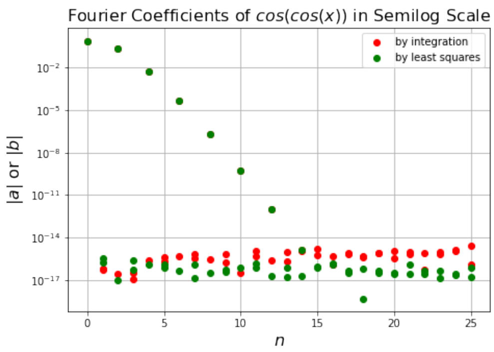
To plot in semilog scale, for example, we use:

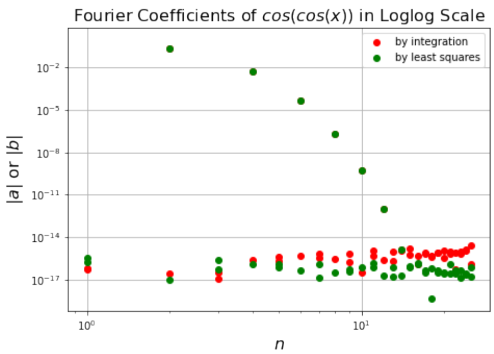
```
plt.semilogy(0, np.abs(c[f_i][0]), 'go', label="by least squares")
plt.semilogy(range(1, 26), np.abs(c[f_i][1::2]), 'go')
plt.semilogy(range(1, 26), np.abs(c[f_i][2::2]), 'go')
```

The coefficients $|a_n|$ and $|b_n|$ by least squares method have been contrasted with those by integration:









2.6 Compare Least Squares and Direct Integration

The maximum errors between the least squares and direct integration methods has been calculated:

```
np.max(np.abs(c[0] - c_n[0]))  # for e^x
np.max(np.abs(c[1] - c_n[1]))  # for coscos(x)
```

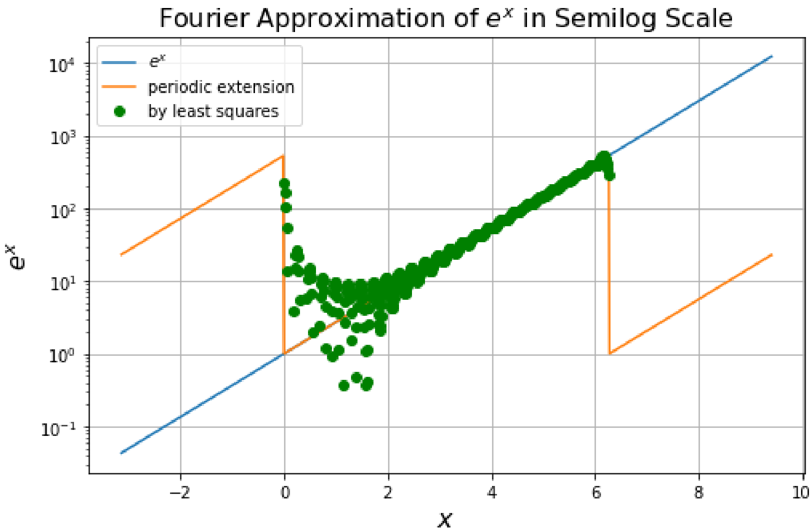
The largest deviation of coefficients for	e^x	is :	1.3327
The largest deviation of coefficients for	$cos(cos(x))$	is :	2.6684e-15

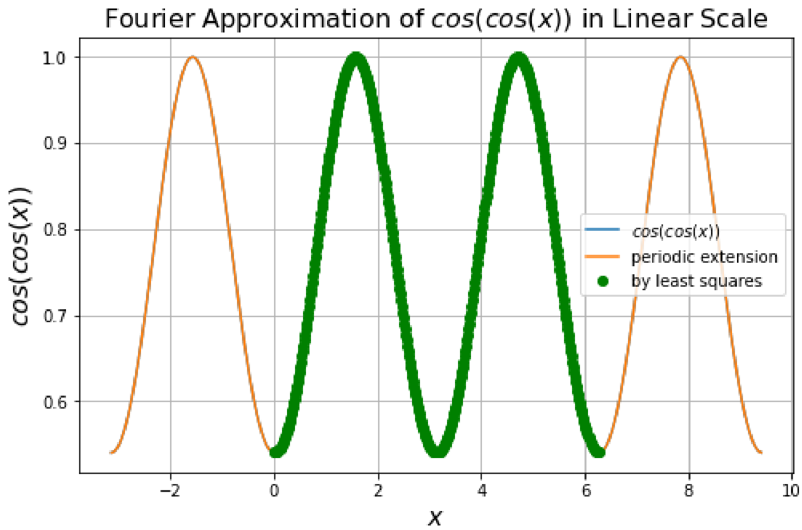
The error is significant for e^x but negligible for $cos(cos(x))$.
This is because there are multiple discontinuities in the periodic extension of e^x, and thus would require a much higher number of coefficients to be somewhat accurately represented.
This lack in accuracy is more apparent close to those discontinuities.

2.7 Plot the Fourier Approximations

The functions as represented by the Fourier coefficients calculated through the least squares method have been plotted:

```
plt.plot(x[200:600], np.dot(A, c[1]), 'go', label="by_least_squares")
```





The $cos(cos(x))$ plot agrees nearly perfectly, but the e^x plot has a large deviation. This is because unlike $cos(cos(x))$, which has periodic components, the e^x function is not inherently comprised of periodic trigonometric functions. The periodic extension of e^x has multiple discontinuities, which causes the Fourier representation to overshoot around them due to the Gibbs phenomenon.