2 Continuity, Differentiability and Taylor's theorem

2.1 Limits of real valued functions

Let $f(x)$ be defined on (a, b) except possibly at x_0.

Definition 2.1.1. *We say that* $\lim_{x \to x_0} f(x) = L$ *if, for every real number* $\epsilon > 0$, *there exists a real number* $\delta > 0$ *such that*

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon. \tag{2.1}$$

Equivalently,

Remark 2.1. *The above definition is equivalent to: for any sequence* $\{x_n\}$ *with* $x_n \to x_0$, *we have* $f(x_n) \to L$ *as* $n \to \infty$.

Proof. Suppose $lim_{x \to x_0} f(x)$ exists. Take $\epsilon > 0$ and let $\{x_n\}$ be a sequence converging to x_0. Then there exists N such that $|x_n - x_0| < \delta$ for $n \geq N$. Then by the definition $|f(x_n) - L| < \epsilon$. i.e., $f(x_n) \to L$.
For the other side, assume that $x_n \to c \implies f(x_n) \to L$. Suppose the limit does not exist. i.e., $\exists\, \epsilon_0 > 0$ such that for any $\delta > 0$, there is $x \in |x - x_0| < \delta$, for which $|f(x) - L| \geq \epsilon_0$. Then take $\delta = \frac{1}{n}$ and pick x_n in $|x_n - x_0| < \frac{1}{n}$, then $x_n \to x_0$ but $|f(x_n) - L| \geq \epsilon_0$. Not possible.

Theorem 2.1.2. *If limit exists, then it is unique.*

Proof. Proof is easy.

Examples: (i) $\lim_{x \to 1}(\frac{3x}{2} - 1) = \frac{1}{2}$. Let $\epsilon > 0$. We have to find $\delta > 0$ such that (2.1) holds with $L = 1/2$. Working backwards,

$$\frac{3}{2}|x - 1| < \epsilon \text{ whenever } |x - 1| < \delta := \frac{2}{3}\epsilon.$$

(ii) Prove that $\lim_{x \to 2} f(x) = 4$, where $f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$

Problem: Show that $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist.

Consider the sequences $\{x_n\} = \{\frac{1}{n\pi}\}, \{y_n\} = \{\frac{1}{2n\pi + \frac{\pi}{2}}\}$. Then it is easy to see that $x_n, y_n \to 0$ and $\sin\left(\frac{1}{x_n}\right) \to 0, \sin\left(\frac{1}{y_n}\right) \to 1$. In fact, for every $c \in [-1, 1]$, we can find a sequence z_n such that $z_n \to 0$ and $\sin(\frac{1}{z_n}) \to c$ as $n \to \infty$.

By now we are familiar with limits and one can expect the following:

Theorem 2.1.3. *Suppose* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$*, then*

1. $\lim_{x\to c}(f(x) \pm g(x)) = L \pm M$.
2. $f(x) \leq g(x)$ *for all* x *in an open interval containing* c*. Then* $L \leq M$.
3. $\lim_{x\to c}(fg)(x) = LM$ *and when* $M \neq 0$, $\lim_{x\to c} \frac{f}{g}(x) = \frac{L}{M}$.
4. *(Sandwich): Suppose that* $h(x)$ *satisfies* $f(x) \leq h(x) \leq g(x)$ *in an interval containing* c*, and* $L = M$*. Then* $\lim_{x\to c} h(x) = L$.

Proof. We give the proof of (iii). Proof of other assertions are easy to prove. Let $\epsilon > 0$. From the definition of limit, we have $\delta_1, \delta_2, \delta_3 > 0$ such that

$$|x - c| < \delta_1 \implies |f(x) - L| < \frac{1}{2} \implies |f(x)| < N \text{ for some } N > 0,$$

$$|x - c| < \delta_2 \implies |f(x) - L| < \frac{\epsilon}{2M}, \text{ and}$$

$$|x - c| < \delta_3 \implies |g(x) - M| < \frac{\epsilon}{2N}.$$

Hence for $|x - c| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$, we have

$$\begin{aligned}|f(x)g(x) - LM| &\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\ &\leq |f(x)||g(x) - M| + M|f(x) - L| \\ &< \epsilon.\end{aligned}$$

To prove the second part, note that there exists an interval $(c-\delta, c+\delta)$ around c such that $g(x) \neq 0$ in $(c - \delta, c + \delta)$.
Examples: (i) $\lim_{x\to 0} x^m = 0$ $(m > 0)$. (ii) $\lim_{x\to 0} x \sin x = 0$.

Remark: Suppose $f(x)$ is bounded in an interval containing c and $\lim_{x\to c} g(x) = 0$. Then $\lim_{x\to c} f(x)g(x) = 0$.

Examples: (i) $\lim_{x\to 0} |x| \sin \frac{1}{x} = 0$. (ii) $\lim_{x\to 0} |x| \ln |x| = 0$.

One sided limits: Let $f(x)$ is defined on (c, b). The right hand limit of $f(x)$ at c is L, if given $\epsilon > 0$, there exists $\delta > 0$, such that

$$x - c < \delta \implies |f(x) - L| < \epsilon.$$

Notation: $\lim\limits_{x \to c^+} f(x) = L$. Similarly, one can define the left hand limit of $f(x)$ at b and is denoted by $\lim\limits_{x \to b^-} f(x) = L$.

Both theorems above holds for right and left limits. Proof is easy.

Problem: Show that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.
Solution: Consider the unit circle centered at $O(0,0)$ and passing through $A(1,0)$ and $B(0,1)$. Let Q be the projection of P on $x-$axis and the point T is such that A is the projection of T on $x-$axis. Let OT be the ray with $\angle AOT = \theta, 0 < \theta < \pi/2$. Let P be the point of intersection of OT and circle. Then ΔOPQ and ΔOTA are similar triangles and hence, Area of $\Delta OAP <$ Area of sector $OAP <$ area of ΔOAT. i.e.,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

dividing by $\sin\theta$, we get $1 > \dfrac{\sin\theta}{\theta} > \cos\theta$. Now $\lim\limits_{\theta \to 0^+} \cos\theta = 1$ implies that $\lim\limits_{\theta \to 0^+} \dfrac{\sin\theta}{\theta} = 1$.
Now use the fact that $\dfrac{\sin\theta}{\theta}$ is even function.
At this stage, it is not difficult to prove the following:

Theorem 2.1.4. *$\lim_{x \to a} f(x) = L$ exists $\iff$ $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.*

Limits at infinity and infinite limits

Definition 2.1.5. *$f(x)$ has limit L as x approaches $+\infty$, if for any given $\epsilon > 0$, there exists $M > 0$ such that*

$$x > M \implies |f(x) - L| < \epsilon.$$

Similarly, one can define limit as x approaches $-\infty$.

Problem: (i) $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$, $\quad (ii)$ $\lim_{x \to -\infty} \frac{1}{x} = 0$. $\quad (iii)$ $\lim\limits_{x \to \infty} \sin x$ does not exist.
Solution: (i) and (ii) are easy. For (iii), Choose $x_n = n\pi$ and $y_n = \frac{\pi}{2} + 2n\pi$. Then $x_n, y_n \to \infty$ and $\sin x_n = 0$, $\sin y_n = 1$. Hence the limit does not exist.

Above two theorems on limits hold in this also.

Definition 2.1.6. *(Horizontal Asymptote:) A line $y = b$ is a horizontal asymptote of $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.*

Examples: (i) $y = 1$ is a horizontal asymptote for $1 + \frac{1}{x+1}$

Definition 2.1.7. *(Infinite Limit): A function $f(x)$ approaches ∞ ($f(x) \to \infty$) as $x \to x_0$ if, for every real $B > 0$, there exists $\delta > 0$ such that*

$$0 < |x - x_0| < \delta \implies f(x) > B.$$

Similarly, one can define for $-\infty$. Also one can define one sided limit of $f(x)$ approaching ∞ or $-\infty$.

Examples (i) $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$, $\quad (ii)$ $\lim_{x\to 0} \frac{1}{x^2} \sin(\frac{1}{x})$ does not exist.

For (i) given $B > 0$, we can choose $\delta \leq \frac{1}{\sqrt{B}}$. For (ii), choose a sequence $\{x_n\}$ such that $\sin \frac{1}{x_n} = 1$, say $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$ and $\frac{1}{y_n} = n\pi$. Then $\lim_{n\to\infty} f(x_n) = \dfrac{1}{x_n^2} \to \infty$ and $\lim_{n\to\infty} f(y_n) = 0$, though $x_n, y_n \to 0$ as $n \to \infty$.

Definition 2.1.8. *(Vertical Asymptote:) A line $x = a$ is a vertical asymptote of $y = f(x)$ if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.*

Example: $f(x) = \frac{x+3}{x+2}$.
$x = -2$ is a vertical asymptote and $y = 1$ is a horizontal asymptote.

2.2 Continuous functions

Definition 2.2.1. *A real valued function $f(x)$ is said to be continuous at $x = c$ if*
(i) $c \in domain(f)$
(ii) $\lim_{x\to c} f(x)$ exists
(iii) The limit in (ii) is equal to $f(c)$.

In other words, for every sequence $x_n \to c$, we must have $f(x_n) \to f(c)$ as $n \to \infty$. i.e., for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Examples: (i) $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at 0.

Let $\epsilon > 0$. Then $|f(x) - f(0)| \leq |x^2|$. So it is enough to choose $\delta = \sqrt{\epsilon}$.

(ii) $g(x) = \begin{cases} \frac{1}{x}\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.
Choose $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$. Then $\lim x_n = 0$ and $f(x_n) = \frac{1}{x_n} \to \infty$.

The following theorem is an easy consequence of the definition.

Theorem 2.2.2. *Suppose f and g are continuous at c. Then*
(i) $f \pm g$ is also continuous at c
(ii) fg is continuous at c
(iii) $\dfrac{f}{g}$ is continuous at c if $g(c) \neq 0$.

Theorem 2.2.3. *Composition of continuous functions is also continuous i.e., if f is continuous at c and g is continuous at $f(c)$ then $g(f(x))$ is continuous at c.*

Corollary: If $f(x)$ is continuous at c, then $|f|$ is also continuous at c.

Theorem 2.2.4. *If f, g are continuous at c, then* $\max(f, g)$ *is continuous at c.*

Proof. Proof follows from the relation

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$

and the above theorems.

Types of discontinuities

Removable discontinuity: $f(x)$ is defined every where in an interval containing a except at $x = a$ and limit exists at $x = a$ OR $f(x)$ is defined also at $x = a$ and limit is NOT equal to function value at $x = a$. Then we say that $f(x)$ has removable discontinuity at $x = a$. These functions can be extended as continuous functions by defining the value of f to be the limit value at $x = a$.

Example: $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Here limit as $x \to 0$ is 1. But $f(0)$ is defined to be 0.

Jump discontinuity: The left and right limits of $f(x)$ exists but not equal. This type of discontinuities are also called discontinuities of first kind.

Example: $f(x) = \begin{cases} 1 & x \leq 0 \\ -1 & x \geq 0 \end{cases}$. Easy to see that left and right limits at 0 are different.

Infinite discontinuity: Left or right limit of $f(x)$ is ∞ or $-\infty$.

Example: $f(x) = \frac{1}{x}$ has infinite discontinuity at $x = 0$.

Discontinuity of second kind: If either $\lim\limits_{x \to c^-} f(x)$ or $\lim\limits_{x \to c^+} f(x)$ does not exist, then c is called discontinuity of second kind.
Example: Consider the function

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

Then f does not have left or right limit any point c. Indeed, if $c \in \mathbb{Q}$, then $x_n = c + \frac{1}{n} \in \mathbb{Q}$ and $y_n = c + \frac{\pi}{n} \notin \mathbb{Q}$. For these sequences, we will have, $f(c + \frac{1}{n})$ and $f(c + \frac{\pi}{n})$ converges to different values. If $c \notin \mathbb{Q}$, then choos $x_n \in (c, c + \frac{1}{n}) \cap \mathbb{Q}$ and $y_n = c + \frac{1}{n} \notin \mathbb{Q}$. For these sequences we will again get different limit values.

Properties of continuous functions

Definition 2.2.5. *(Closed set): A subset A of $\mathbb{R}$ is called closed set if A contains all its limit points. (i.e., if $\{x_n\} \subset A$ and $x_n \to c$, then $c \in A$).*

Theorem 2.2.6. *Continuous functions on closed, bounded interval is bounded.*

Proof. Let $f(x)$ be continuous on $[a, b]$ and let $\{x_n\} \subset [a, b]$ be a sequence such that $|f(x_n)| > n$. Then $\{x_n\}$ is a bounded sequence and hence there exists a subsequence $\{x_{n_k}\}$ which converges to c. Then $f(x_{n_k}) \to f(c)$, a contradiction to $|f(x_{n_k}| > n_k$.

Theorem 2.2.7. *Let $f(x)$ be a continuous function on closed, bounded interval $[a, b]$. Then maximum and minimum of functions are achieved in $[a, b]$.*

Proof. Let $\{x_n\}$ be a sequence such that $f(x_n) \to \max f$. Then $\{x_n\}$ is bounded and hence by Bolzano-Weierstrass theorem, there exists a subsequence x_{n_k} such that $x_{n_k} \to x_0$ for some x_0. $a \leq x_n \leq b$ implies $x_0 \in [a, b]$. Since f is continuous, $f(x_{n_k}) \to f(x_0)$. Hence $f(x_0) = \max f$. The attainment of minimum can be proved by noting that $-f$ is also continuous and $\min f = -\max(-f)$.

Remark: Closed and boundedness of the interval is important in the above theorem. Consider the examples (i) $f(x) = \frac{1}{x}$ on $(0,1)$ (ii) $f(x) = x$ on $I\!R$.

Theorem 2.2.8. *Let $f(x)$ be a continuous function on $[a,b]$ and let $f(c) > 0$ for some $c \in (a,b)$, Then there exists $\delta > 0$ such that $f(x) > 0$ in $(c-\delta, c+\delta)$.*

Proof. Let $\epsilon = \frac{1}{2} f(c) > 0$. Since $f(x)$ is continuous at c, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \frac{1}{2} f(c)$$

i.e., $-\frac{1}{2} f(c) < f(x) - f(c) < \frac{1}{2} f(c)$. Hence $f(x) > \frac{1}{2} f(c)$ for all $x \in (c-\delta, c+\delta)$.

Corollary: Suppose a continuous functions $f(x)$ satisfies $\int_a^b f(x)\phi(x)dx = 0$ for all continuous functions $\phi(x)$ on $[a,b]$. Then $f(x) \equiv 0$ on $[a,b]$.

Proof. Suppose $f(c) > 0$. Then by above theorem $f(x) > 0$ in $(c-\delta, c+\delta)$. Choose $\phi(x)$ so that $\phi(x) > 0$ in $(c-\delta/2, c+\delta/2)$ and is 0 otherwise. Then $\int_a^b f(x)\phi(x) > 0$. A contradiction.
Alternatively, one can choose $\phi(x) = f(x)$.

Theorem 2.2.9. *Let $f(x)$ be a continuous function on $I\!R$ and let $f(a)f(b) < 0$ for some a, b. Then there exits $c \in (a,b)$ such that $f(c) = 0$.*

Proof. Assume that $f(a) < 0 < f(b)$. Let $S = \{x \in [a,b] : f(x) < 0\}$. Then $[a, a+\delta) \subset S$ for some $\delta > 0$ and S is bounded. Let $c = \sup S$. We claim that $f(c) = 0$. Take $x_n = c + \frac{1}{n}$, then $x_n \notin S$, $x_n \to c$. Therefore, $f(c) = \lim f(x_n) \geq 0$. On the otherhand, note that $c - \frac{1}{n}$ is NOT supremum. Therefore, there exists a point $y_n \in (c - \frac{1}{n}, c) \cap S$. Then note that $y_n \to c$, $f(c) = \lim f(y_n) \leq 0$. Hence $f(c) = 0$.

Corollary: Intermediate value theorem: Let $f(x)$ be a continuous function on $[a,b]$ and let $f(a) < y < f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = y$

Remark: A continuous function assumes all values between its maximum and minimum.

Problem: (fixed point): Let $f(x)$ be a continuous function from $[0,1]$ into $[0,1]$. Then show that there is a point $c \in [0,1]$ such that $f(c) = c$.

Define the function $g(x) = f(x) - x$. Then $g(0) \geq 0$ and $g(1) \leq 0$. Now Apply Intermediate value theorem.

Application: Root finding: To find the solutions of $f(x) = 0$, one can think of defining a new function g such that $g(x)$ has a fixed point, which in turn satisfies $f(x) = 0$.
Example: (1) $f(x) = x^3 + 4x^2 - 10$ in the interval $[1, 2]$. Define $g(x) = \left(\frac{10}{4+x}\right)^{1/2}$. We can check that g maps $[1, 2]$ into $[1, 2]$. So g has fixed point in $[1, 2]$ which is also solution of $f(x) = 0$. Such fixed points can be obtained as limit of the sequence $\{x_n\}$, where $x_{n+1} = g(x_n), x_0 \in (1, 2)$. Note that

$$g'(x) = \frac{\sqrt{10}}{(4 + x)^{3/2}} < \frac{1}{2}.$$

By Mean Value Theorem, $\exists\ z$ (see next section)

$$|x_{n+1} - x_n| = |g'(z)||x_n - x_{n-1}| \leq \frac{1}{2}|x_n - x_{n-1}|$$

Iterating this, we get

$$|x_{n+1} - x_n| < \frac{1}{2^n}|x_1 - x_0|.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. (see problem after Theorem 1.4.4).

Uniformly continuous functions

Definition: A function $f(x)$ is said to be uniformly continuous on a set S, if for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in S, \quad |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Here δ depends only on ϵ, not on x or y.

Proposition: If $f(x)$ is uniformly continuous function $\iff$ for ANY two sequences $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$, we have $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$.

Proof. Suppose not. Then there exists $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| > \eta$ for some $\eta > 0$. Then it is clear that for $\epsilon = \eta$, there is no δ for which $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Because the above sequence satisfies $|x - y| < \delta$, but its image does not.
For converse, assume that for any two sequences $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$ we have $|f(x_n) - f(y_n)| \to 0$. Suppose f is not uniformly continuous. Then by the definition there exists ϵ_0 such that for any $\delta > 0$, there exist $z, w \in |x - y| < \delta$ for which $|f(z) - f(w)| > \epsilon_0$. Now take $\delta = \frac{1}{n}$ and choose z_n, w_n such that $|z_n - w_n| < \frac{1}{n}$. Then $|f(z_n) - f(w_n)| > \epsilon_0$. A contradiction.

Examples: (i) $f(x) = x^2$ is uniformly continuous on bounded interval $[a, b]$.
Note that $|x^2 - y^2| \leq |x + y||x - y| \leq 2b|x - y|$. So one can choose $\delta < \frac{\epsilon}{2b}$.

(ii) $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.
Take $x_n = \frac{1}{n+1}, y_n = \frac{1}{n}$, then for n large $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| = 1$.

(iii) $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.
Take $x_n = n + \frac{1}{n}$ and $y_n = n$. Then $|x_n - y_n| = \frac{1}{n} \to 0$, but $|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} > 2$.

Remarks:

1. It is easy to see from the definition that if f, g are uniformly continuous, then $f \pm g$ is also uniformly continuous.

2. If f, g are uniformly continuous, then fg need not be uniformly continuous. This can be seen by noting that $f(x) = x$ is uniformly continuous on $\mathbb{R}$ but x^2 is not uniformly continuous on $\mathbb{R}$.

Theorem 2.2.10. *A continuous function $f(x)$ on a closed, bounded interval $[a, b]$ is uniformly continuous.*

Proof. Suppose not. Then there exists $\epsilon > 0$ and sequences $\{x_n\}$ and $\{y_n\}$ in $[a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \epsilon$. But then by Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to x_0. Also $y_{n_k} \to x_0$. Now since f is continuous, we have $f(x_0) = \lim f(x_{n_k}) = \lim f(y_{n_k})$. Hence $|f(x_{n_k}) - f(y_{n_k})| \to 0$, a contradiction.

Corollary: Suppose $f(x)$ has only removable discontinuities in $[a, b]$. Then $\tilde{f}$, the extension of f, is uniformly continuous.
Example: $f(x) = \frac{\sin x}{x}$ for $x \neq 0$ and 0 for $x = 0$ on $[0, 1]$.

Theorem 2.2.11. *Let f be a uniformly continuous function and let $\{x_n\}$ be a cauchy sequence. Then $\{f(x_n)\}$ is also a Cauchy sequence.*

Proof. Let $\epsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists N such that

$$m, n > N \implies |x_n - x_m| < \delta.$$

Therefore $|f(x_n) - f(x_m)| < \epsilon$.

Example: $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

The sequence $x_n = \frac{1}{n}$ is cauchy but $f(x_n) = n^2$ is not. Hence f cannot be uniformly continuous.

2.3 Differentiability

Definition 2.3.1. *A real valued function $f(x)$ is said to be differentiable at x_0 if*

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \textit{exists.}$$

This limit is called the derivative of f at x_0, denoted by $f'(x_0)$.

Example: $f(x) = x^2$

$$f'(x) = \lim_{h\to 0} \frac{2xh + h^2}{h} = 2x.$$

Theorem 2.3.2. *If $f(x)$ is differentiable at a, then it is continuous at a.*

Proof. For $x \neq a$, we may write,

$$f(x) = (x - a)\frac{f(x) - f(a)}{(x - a)} + f(a).$$

Now taking the limit $x \to a$ and noting that $\lim(x - a) = 0$ and $\lim \frac{f(x)-f(a)}{(x-a)} = f'(a)$, we get the result.

Theorem 2.3.3. *Let f, g be differentiable at $c \in (a, b)$. Then $f \pm g, fg$ and $\dfrac{f}{g}$ $(g(c) \neq 0)$ is also differentiable at c*

Proof. We give the proof for product formula: First note that

$$\frac{(fg)(x) - (fg)(c)}{x - c} = f(x)\frac{g(x) - g(c)}{x - c} + g(c)\frac{f(x) - f(c)}{x - c}.$$

Now taking the limit $x \to c$, we get the product formula

$$(fg)'(c) = f(c)g'(c) + f'(c)g(c).$$

Since $g(c) \neq 0$ and g is continuous, we get $g(x) \neq 0$ in a small interval around c. Therefore

$$\frac{f}{g}(x) - \frac{f}{g}(c) = \frac{g(c)f(x) - g(c)f(c) + g(c)f(c) - g(x)f(c)}{g(x)g(c)}$$

Hence

$$\frac{(f/g)(x) - (f/g)(c)}{x - c} = \left\{ g(c)\frac{f(x) - f(c)}{x - c} - f(c)\frac{g(x) - g(c)}{x - c} \right\} \frac{1}{g(x)g(c)}$$

Now taking the limit $x \to c$, we get

$$(\frac{f}{g})'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}.$$

Theorem 2.3.4. *(Chain Rule): Suppose $f(x)$ is differentiable at c and g is differentiable at $f(c)$, then $h(x) := g(f(x))$ is differentiable at c and*

$$h'(c) = g'(f(c))f'(c)$$

Proof. Define the function h as

$$h(y) = \begin{cases} \frac{g(y) - g(f(c))}{y - f(c)} & y \neq f(c) \\ g'(f(c)) & y = f(c) \end{cases}$$

Then the function h is continuous at $y = f(c)$ and $g(y) - g(f(c)) = h(y)(y - f(c))$, so

$$\frac{g(f(x)) - g(f(c))}{x - c} = h(f(x))\frac{f(x) - f(c)}{x - c}.$$

Now taking limit $x \to c$, we get the required formula.

Local extremum: A point $x = c$ is called local maximum of $f(x)$, if there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies f(c) \geq f(x).$$

Similarly, one can define local minimum: $x = b$ is a local minimum of $f(x)$ if there exists $\delta > 0$ such that

$$0 < |x - b| < \delta \implies f(b) \leq f(x).$$

Theorem 2.3.5. *Let $f(x)$ be a differentiable function on (a, b) and let $c \in (a, b)$ is a local maximum of f. Then $f'(c) = 0$.*

Proof. Let δ be as in the above definition. Then

$$x \in (c, c+\delta) \implies \frac{f(x)-f(c)}{x-c} \leq 0$$

$$x \in (c-\delta, c) \implies \frac{f(x)-f(c)}{x-c} \geq 0.$$

Now taking the limit $x \to c$, we get $f'(c) = 0$.

Theorem 2.3.6. *Rolle's Theorem: Let $f(x)$ be a continuous function on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. If $f(x)$ is constant, then it is trivial. Suppose $f(x_0) > f(a)$ for some $x_0 \in (a,b)$, then f attains maximum at some $c \in (a,b)$. Other possibilities can be worked out similarly.

Theorem 2.3.7. *Mean-Value Theorem (MVT): Let f be a continuous function on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

Proof. Let $l(x)$ be a straight line joining $(a, f(a))$ and $(b, f(b))$. Consider the function $g(x) = f(x) - l(x)$. Then $g(a) = g(b) = 0$. Hence by Rolle's theorem

$$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

Corollary: If f is a differentiable function on (a,b) and $f' = 0$, then f is constant.

Proof. By mean value theorem $f(x) - f(y) = 0$ for all $x, y \in (a,b)$.

Example: Show that $|\cos x - \cos y| \leq |x-y|$.
Use Mean-Value theorem and the fact that $|\sin x| \leq 1$.

Problem: If $f(x)$ is differentiable and $\sup|f'(x)| < C$ for some C. Then, f is uniformly continuous.
Apply mean value theorem to get $|f(x) - f(y)| \leq C|x-y|$ for all x, y.

Definition: A function $f(x)$ is strictly increasing on an interval I, if for $x, y \in I$ with $x < y$ we have $f(x) < f(y)$. We say f is strictly decreasing if $x < y$ in I implies $f(x) > f(y)$.

Theorem 2.3.8. *A differentiable function f is (i) strictly increasing in (a,b) if $f'(x) > 0$ for all $x \in (a,b)$. (ii) strictly decreasing in (a,b) if $f'(x) < 0$.*

Proof. Choose x, y in (a, b) such $x < y$. Then by MVT, for some $c \in (x, y)$

$$\frac{f(x) - f(y)}{x - y} = f'(c) > 0.$$

Hence $f(x) < f(y)$.

2.4 Taylor's theorem and Taylor Series

Let f be a k times differentiable function on an interval I of $I\!R$. We want to approximate this function by a polynomial $P_n(x)$ such that $P_n(a) = f(a)$ at a point a. Moreover, if the derivatives of f and P_n also equal at a then we see that this approximation becomes more accurate in a neighbourhood of a. So the best coefficients of the polynomial can be calculated using the relation $f^{(k)}(a) = P_n^{(k)}(a), k = 0, 1, 2, ..., n$. The best is in the sense that if $f(x)$ itself is a polynomial of degree less than or equal to n, then both f and P_n are equal. This implies that the polynomial is $\displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. Then we write $f(x) = P_n(x) + R_n(x)$ in a neighbourhood of a. From this, we also expect the $R_n(x) \to 0$ as $x \to a$. In fact, we have the following theorem known as **Taylor's theorem**:

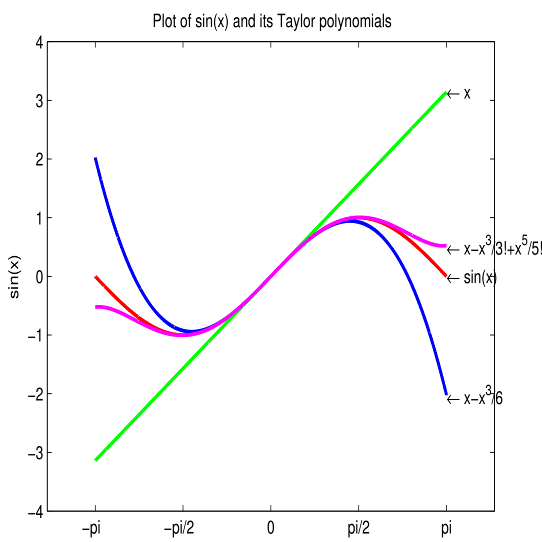

Figure 1: Approximation of sin(x) by Taylor's polynomials

Theorem 2.4.1. *Let $f(x)$ and its derivatives of order m are continuous and $f^{(m+1)}(x)$*

exists in a neighbourhood of $x = a$. Then there exists $c \in (a, x)$(or $c \in (x, a)$) such that

$$f(x) = f(a) + f'(a)(x-a) + ... + f^{(m)}(a)\frac{(x-a)^m}{m!} + R_m(x)$$

where $R_m(x) = \dfrac{f^{(m+1)}(c)}{(m+1)!}(x-a)^{m+1}$.

Proof. Define the functions F and g as

$$F(y) = f(x) - f(y) - f'(y)(x-y) - .... - \frac{f^{(m)}(y)}{m!}(x-y)^m,$$

$$g(y) = F(y) - \left(\frac{x-y}{x-a}\right)^{m+1} F(a).$$

Then it is easy to check that $g(a) = 0$. Also $g(x) = F(x) = f(x) - f(x) = 0$. Therefore, by Rolle's theorem, there exists some $c \in (a, x)$ such that

$$g'(c) = 0 = F'(c) + \frac{(m+1)(x-c)^m}{(x-a)^{m+1}}F(a).$$

On the other hand, from the definition of F,

$$F'(c) = -\frac{f^{(m+1)}(c)}{m!}(x-c)^m.$$

Hence $F(a) = \dfrac{(x-a)^{m+1}}{(m+1)!}f^{(m+1)}(c)$ and the result follows.

Examples:

(i) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!}e^c, c \in (0, x)$ or $(x, 0)$ depending on the sign of x.

(ii) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + ... + \dfrac{x^n}{n!}\sin(c + \dfrac{n\pi}{2}), c \in (0, x)$ or $(x, 0)$.

(iii) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + ... + \dfrac{x^n}{n!}\cos(c + \dfrac{n\pi}{2}), c \in (0, x)$ or $(x, 0)$.

Problem: Find the order n of Taylor Polynomial P_n, about $x = 0$ to approximate e^x in $(-1, 1)$ so that the error is not more than 0.005

Solution: We know that $p_n(x) = 1 + x + ... + \frac{x^n}{n!}$. The maximum error in [-1,1] is

$$|R_n(x)| \leq \frac{1}{(n+1)!}\max_{[-1,1]}|x|^{n+1}e^x \leq \frac{e}{(n+1)!}.$$

So n is such that $\frac{e}{(n+1)!} \leq 0.005$ or $n \geq 5$.

Problem: Find the interval of validity when we approximate $\cos x$ with 2nd order polynomial with error tolerance 10^{-4}.

Solution: Taylor polynomial of degree 2 for $\cos x$ is $1 - \frac{x^2}{2}$. So the remainder is $(\sin c)\frac{x^3}{3!}$. Since $|\sin c| \leq 1$, the error will be atmost 10^{-4} if $|\frac{x^3}{3!}| \leq 10^{-4}$. Solving this gives $|x| < 0.084$

Taylor's Series

Suppose f is infinitely differentiable at a and if the remainder term in the Taylor's formula, $R_n(x) \to 0$ as $n \to \infty$. Then we may formally write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

For fixed x the above infinite sum is a series of real numbers. One can check the convergence of such series by the convergence tests. This series is called Taylor series of $f(x)$ about the point a.
It is also natural (why) for one to expect that $R_n(x) \to 0$ as $n \to \infty$ for the Taylor series to be well defined. In some cases it is easy to verify this. For example,
Suppose there exists $C = C(x) > 0$, independent of n, such that $|f^{(n)}(x)| \leq C(x)$. Then $|R_n(x)| \to 0$ if $\lim\limits_{n\to\infty} \dfrac{|x-a|^{n+1}}{(n+1)!} = 0$. For any fixed x and a, we can always find N such that $|x-a| < N$. Let $q := \frac{|x-a|}{N} < 1$. Then

$$\begin{aligned}\left|\frac{(x-a)^{n+1}}{(n+1)!}\right| &= \left|\frac{|x-a|}{1}\right|\left|\frac{|x-a|}{2}\right| \dots \left|\frac{|x-a|}{N-1}\right|\left|\frac{|x-a|}{N}\right| \dots \left|\frac{|x-a|}{n+1}\right| \\ &< \left|\frac{|x-a|^{N-1}}{(N-1)!}\right| q^{n-N+2}\end{aligned}$$

$\to 0$ as $n \to \infty$ thanks to $q < 1$.

In case of $a = 0$, the formula obtained in Taylor's theorem is known as *Maclaurin's formula* and the corresponding series that one obtains is known as *Maclaurin's series.*

Example: (i) $f(x) = e^x$.
In this case $R_n(x) = \dfrac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = \dfrac{x^{n+1}}{(n+1)!} e^c = \dfrac{x^{n+1}}{(n+1)!} e^{\theta x}$, for some $\theta \in (0,1)$.

Therefore for any given x fixed, $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left(\frac{x^{n+1}}{(n+1)!} \right) e^{\theta x} = 0.$

Example: (ii) $f(x) = \sin x$.

In this case it is easy to see that $|R_n(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!} |\sin(c + \frac{n\pi}{2})|$. Now use the fact that $|\sin x| \leq 1$ and follow as in example (i).

Maxima and Minima: Derivative test

Definition 2.4.2. *A point $x = a$ is called critical point of the function $f(x)$ if $f'(a) = 0$.*

Second derivative test: A point $x = a$ is a local maxima if $f'(a) = 0, f''(a) < 0$.

Suppose $f(x)$ is continuously differentiable in an interval around $x = a$ and let $x = a$ be a critical point of f. Then $f'(a) = 0$. By Taylor's theorem around $x = a$, there exists, $c \in (a, x)$ (or $c \in (x, a)$),

$$f(x) - f(a) = \frac{f''(c)}{2}(x - a)^2.$$

If $f''(a) < 0$. Then by Theorem2.2.8, $f''(c) < 0$ in $|x - a| < \delta$. Hence $f(x) < f(a)$ in $|x - a| < \delta$, which implies that $x = a$ is a local maximum.

Similarly, one can show the following for local minima: $x = a$ is a local minima if $f'(a) = 0, f''(a) > 0$.

Also the above observations show that if $f'(a) = 0, f''(a) = 0$ and $f^{(3)}(a) \neq 0$, then the sign of $f(x) - f(a)$ depends on $(x - a)^3$. i.e., it has no constant sign in any interval containing a. Such point is called point of inflection or saddle point.

We can also derive that if $f'(a) = f''(a) = f^{(3)}(a) = 0$, then we again have $x = a$ is a local minima if $f^{(4)}(a) > 0$ and is a local maxima if $f^{(4)}(a) < 0$.

Summarizing the above, we have:

Theorem 2.4.3. *Let f be a real valued function that is differentiable $2n$ times and $f^{(2n)}$ is continuous at $x = a$. Then*

1. *If $f^{(k)}(a) = 0$ for $k = 1, 2, ...2n - 1$ and $f^{(2n)}(a) > 0$ then a is a point of local minimum of $f(x)$*

2. *If $f^{(k)}(a) = 0$ for $k = 1, 2, ...2n-1$ and $f^{(2n)}(a) < 0$ then a is a point local maximum of $f(x)$.*

3. If $f^{(k)} = 0$ for $k = 1, 2, ...2n-2$ and $f^{(2n-1)}(a) \neq 0$, then a is point of inflection. i.e., f has neither local maxima nor local minima at $x = a$.

L'Hospitals Rule:
Suppose $f(x)$ and $g(x)$ are differentiable n times, $f^{(n)}, g^{(n)}$ are continuous at a and $f^{(k)}(a) = g^{(k)}(a) = 0$ for $k = 0, 1, 2, ..., n-1$. Also if $g^{(n)}(a) \neq 0$. Then by Taylor's theorem,

$$\begin{aligned}\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f^{(n)}(c)}{g^{(n)}(c)} \\ &= \frac{f^{(n)}(a)}{g^{(n)}(a)}\end{aligned}$$

In the above, we used the fact that $g^{(n)}(x) \neq 0$ "near $x = a$" and $g^{(n)}(c) \to g^{(n)}(a)$ as $x \to a$.
Similarly, we can derive a formula for limits as x approaches infinity by taking $x = \frac{1}{y}$.

$$\begin{aligned}\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{y\to 0} \frac{f(1/y)}{g(1/y)} \\ &= \lim_{y\to 0} \frac{(-1/y^2)f'(1/y)}{(-1/y^2)g'(1/y)} \\ &= \lim_{x\to\infty} \frac{f'(x)}{g'(x)}\end{aligned}$$

2.5 Power series & Taylor Series

Given a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is called *power series* with center c. It is easy to see that a power series converges for $x = c$. Power series is a function of x provided it converges for x. If a power series converges, then the domain of convergence is either a bounded interval or the whole of $I\!R$. So it is natural to study the largest interval where the power series converges.
Remark: If $\sum a_n x^n$ converges at $x = r$, then $\sum a_n x^n$ converges for $|x| < |r|$.
Proof: We can find $C > 0$ such that $|a_n x^n| \leq C$ for all n. Then

$$|a_n x^n| \leq |a_n r^n| |\frac{x}{r}|^n \leq C |\frac{x}{r}|^n.$$

Conclusion follows from comparison theorem.

Theorem 2.5.1. *Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Suppose $\beta = \limsup \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ (We define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then*

1. $\sum\limits_{n=0}^{\infty} a_n x^n$ *converges for* $|x| < R$

2. $\sum\limits_{n=0}^{\infty} a_n x^n$ *diverges for* $|x| > R$.

3. *No conclusion if* $|x| = R$.

Proof. Proof of (i) follows from the root test. For a proof, take $\alpha_n(x) = a_n x^n$ and $\alpha = \limsup \sqrt[n]{|\alpha_n|}$. For (ii), one can show that if $|x| > R$, then there exists a subsequence $\{a_n\}$ such that $a_n \not\to 0$. Notice that $\alpha = \beta|x|$. For (iii), observe as earlier that the series with $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$ will have $R = 1$.
Similarly, we can prove:

Theorem 2.5.2. *Consider the power series* $\sum\limits_{n=0}^{\infty} a_n x^n$. *Suppose* $\beta = \lim \left|\frac{a_{n+1}}{a_n}\right|$ *and* $R = \frac{1}{\beta}$ *(We define* $R = 0$ *if* $\beta = \infty$ *and* $R = \infty$ *if* $\beta = 0$*). Then*

1. $\sum\limits_{n=0}^{\infty} a_n x^n$ *converges for* $|x| < R$

2. $\sum\limits_{n=0}^{\infty} a_n x^n$ *diverges for* $|x| > R$.

3. *No conclusion if* $|x| = R$.

Definition 2.5.3. *The real number* R *in the above theorems is called the Radius of convergence of power series.*

Examples: Find the interval of convergence of (i) $\sum \frac{x^n}{n}$ (ii) $\sum \frac{x^n}{n!}$ (iii) $\sum 2^{-n} x^{3n}$

1. $\beta = \lim |\frac{a_{n+1}}{a_n}| = 1$, and we know that the series does not converge for $x = 1$, but converges at $x = -1$.

2. $\beta = \lim |\frac{a_{n+1}}{a_n}| = 0$. Hence the series converges everywhere.

3. To see the subsequent non-zero terms, we write the series as $\sum 2^{-n}(x^3)^n = \sum 2^{-n} y^n$. For this series $\beta_y = \limsup \sqrt[n]{|a_n|} = 2^{-1}$. Therefore, $\beta_x = 2^{-1/3}$ and $R = 2^{1/3}$.

Example 2.5.4. *Let* $a_n = \begin{cases} 2^n & n \text{ is even} \\ 2^{n-1} & n \text{ is odd} \end{cases}$. *Then* $\limsup \frac{a_{n+1}}{a_n} = 4$ *and the limit of* $\frac{a_{n+1}}{a_n}$ *does not exist. But* $\limsup |a_n|^{1/n} = 1/2$. *So the radius of convergence of the series* $\sum a_n x^n$ *is* $1/2$.

The following theorem is very useful in identifying the domain of convergence of some Taylor series.

Theorem 2.5.5. *(Term by term differentiation and integration): Suppose* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *converges for* $|x| < R$*. Then*

1. $\sum_{n=0}^{\infty} n a_n x^{n-1}$ *converges in* $|x| < R$ *and is equal to* $f'(x)$*.*

2. $\sum_{n=0}^{\infty} \frac{1}{n+1} a_n x^{n+1}$ *converges in* $|x| < R$ *and is equal to* $\int f(x)dx$*.*

From this theorem one concludes that a power series is infinitely differentiable with in its radius of convergence. Now it is natural to ask weather this series coincides with the Taylor series of the resultant function. The answer is yes and it is simple to prove that if $f(x) = \sum a_n x^n$, then $a_n = \frac{f^{(n)}}{n!}$. The above theorem is useful to find Taylor series of some functions.

Example: The Taylor series of $f(x) = \tan^{-1} x$ and a domain of its convergence.

$$\begin{aligned} \tan^{-1} x = & \int \frac{dx}{1+x^2} = \int 1 - x^2 + x^4 + ... \\ = & x - \frac{x^3}{3} + \frac{x^5}{5} + ... \end{aligned}$$

It is easy to check that the Radius of convergence of this series is equal to 1 (Try!). At the end point $x = 1$ we get interesting sum

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \tan^{-1}(1) = \frac{\pi}{4}.$$

Though the function $\tan^{-1} x$ is defined on all of $\mathbb{R}$, we see that the power series converges on $(-1, 1)$. We can apply Abel's theorem on alternating series to show that the series converges at $x = 1, -1$.

For approximation, we can use the error approximation of alternating series discussed in the previous section. The total error if we approximate $\tan^{-1} x$ by $s_n(x)$, then the maximum error is $\frac{|x|^{n+1}}{n+1}$.

We note that the power series may converge to a function on small interval, even though the function is defined on a much bigger interval. For example the function $\log(1 + x)$ has power series that converges on $(-1, 1)$, but $\log(1 + x)$ is defined on $(-1, \infty)$. This is obvious due to the fact that the domain of convergence of power series is symmetric about the center. For instance, for a function defined on $(-1, 3)$ the radius of convergence of its power series(about 0) cannot be more than 1.
Another interesting application is to integrate the functions for which we have no "clue".

For example,

$$(1)\ erf(x) = \int_0^x e^{-t^2} dt = \int_0^x (1 - \frac{t^2}{1!} + \frac{t^4}{2!} + ... = x - \frac{x^3}{3} + \frac{x^5}{10} + ...$$

$$(2)\ \int_0^x \frac{\sin t}{t} = \int_0^x 1 - \frac{t^2}{3!} + \frac{t^4}{5!} = x - \frac{x^3}{3! \cdot 3} + \frac{x^5}{5! \cdot 5} + ...$$