2 Infinite Series

2.1 Definitions & convergence

Definition 2.1.1. *Let $\{a_n\}$ be a sequence of real numbers.*

a) *An expression of the form*

$$a_1 + a_2 + \ldots + a_n + \ldots$$

is called an infinite series.

b) *The number a_n is called as the n^{th} term of the series.*

c) *The sequence $\{s_n\}$, defined by $s_n = \sum_{k=1}^{n} a_k$, is called the sequence of partial sums of the series.*

d) *If the sequence of partial sums converges to a limit L, we say that the series converges and its sum is L.*

e) *If the sequence of partial sums does not converge, we say that the series diverges.*

Examples 2.1.2.

1) *If $0 < x < 1$, then $\sum_{n=0}^{\infty} x^n$ converges to $\dfrac{1}{1-x}$.*

Solution. *Let us consider the sequence of partial sums $\{s_n\}$, where $s_n = \sum_{k=1}^{n} x^k$. Here*

$$s_n = \sum_{k=1}^{n} x^k = \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}, \ n \in \mathbb{N}.$$

As, $0 < x < 1$, $x^{n+1} \to 0$ as $n \to \infty$. Hence $s_n \to \dfrac{1}{1-x}$. Thus the given series converges to $\dfrac{1}{1-x}$. ///

2) *The series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.*

Solution. *Consider the sequence of partial sums $\{s_n\}$, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Now, let us examine the subsequence s_{2^n} of $\{s_n\}$. Here*

$$\begin{aligned} s_2 &= 1 + 1/2 = 3/2, \\ s_4 &= 1 + 1/2 + 1/3 + 1/4 > 3/2 + 1/4 + 1/4 = 2. \end{aligned}$$

Suppose $s_{2^n} > (n+2)/2$, then

$$\begin{aligned} s_{2^{n+1}} &= s_{2^n} + \sum_{k=1}^{2^n} \frac{1}{2^n + k} \\ &> \frac{n+2}{2} + \sum_{k=1}^{2^n} \frac{1}{2^{n+1}} \\ &= \frac{n+2}{2} + \frac{2^n}{2^{n+1}} = \frac{(n+1)+2}{2}. \end{aligned}$$

Thus the subsequence $\{s_{2^n}\}$ is not bounded above and as it is also increasing, it diverges. Hence the sequence diverges, i.e., the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. ///

3) *(Telescopic series:) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.*

Solution. *Consider the sequence of partial sums $\{s_n\}$. Then*

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{k} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} \to 1.$$

Summarizing this observation, one has the following theorem on **Telescopic series**

Theorem 2.1.3. *Suppose $\{a_n\}$ is a sequence of non-negative real numbers such that $a_n \to L$. Then the series $\sum (a_n - a_{n+1})$ converges to $a_1 - L$.*

Lemma 2.1.4.

1) *If $\sum_{n=1}^{\infty} a_n$ converges to L and $\sum_{n=1}^{\infty} b_n$ converges to M, then the series $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $L + M$.*

2) *If $\sum_{n=1}^{\infty} a_n$ converges to L and if $c \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} c a_n$ converges to cL.*

Lemma 2.1.5. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Suppose $\sum_{n=1}^{\infty} a_n = L$. Then the sequence of partial sums $\{s_n\}$ also converges to L. Now

$$a_n = s_n - s_{n-1} \to L - L = 0. \quad ///$$

Example 2.1.6. *If $x > 1$, then the series $\sum_{n=1}^{\infty} x^n$ diverges.*

Solution. *Assume to the contrary that the series $\sum_{n=1}^{\infty} x^n$ converges. Then the n^{th} term, i.e., $x^n \to 0$. But as $x > 1$, $x^n \geq 1$ for all $n \in \mathbb{N}$ and hence $\lim_{n\to\infty} x^n \geq 1$, which is a contradiction. Hence the series $\sum_{n=1}^{\infty} x^n$ diverges.* ///

As a first result we have the following comparison theorem:

Theorem 2.1.7. *Let $\{a_n\}, \{b_n\}$ be sequences of positive reals such that $a_n \leq b_n$. If $\sum b_n$ converges then $\sum a_n$ converges.*

Proof. Let $s_n = a_1 + a_2 + ... + a_n$ and $t_n = b_1 + b_2 + ... + b_n$ be the partial sum of $\sum a_n, \sum b_n$ respectively. Then $s_n \leq t_n$. Since $\sum b_n$ converges, we have $\{t_n\}$ converges and is bounded. Now since $\{s_n\}$ is monotonically increasing sequence that is bounded above, we get the convergence of $\{s_n\}$ and hence the convergence of $\sum a_n$. ///

Theorem 2.1.8. *Let $\{a_n\}_1^\infty$ be an decreasing sequence of positive numbers. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=0}^{\infty} 2^n a_{2^n}$ converges.*

Proof. Let s_n and t_n be the sequence of partial sums of $\sum a_n$ and $\sum 2^n a_{2^n}$ respectively. Then s_n and t_n are monotonically increasing sequences. We know that such sequences converge if they are bounded from above. proof follows from the observation that

$$
\begin{aligned}
s_{2^n} = \sum_{k=1}^{2^n} a_n &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + ... + (a_{2^{n-1}+1} + ... + a_{2^n}) \\
&\geq a_1 + a_2 + 2a_4 + 4a_8 + 8a_{16} + ...2^{n-1} a_{2^n} \\
&= a_1 + \frac{1}{2} t_n. \qquad (2.1)
\end{aligned}
$$

Therefore, if $\{s_n\}$ converges then $\{s_{2^n}\}$ converges and hence bounded from above. Now convergence of $\{t_n\}$ follows from 2.1, $\{t_n\}$.

On the other hand,

$$
\begin{aligned}
s_{2^n-1} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + ...a_{15}) + (a_{2^{n-1}} + ... + a_{2^n-1}) \\
&\leq a_1 + 2a_2 + 4a_4 + 8a_8 + ... + 2^{n-1} a_{2^{n-1}} = a_1 + t_{n-1}
\end{aligned}
$$

So if $\{t_n\}$ converges, then $\{s_{2^n-1}\}$ converges. Now the conclusion follows from $s_n \leq s_{2^{n+1}-1}$ and the fact that $\{s_n\}$ is monotonically increasing sequence. ///

Examples 2.1.9.

1) *Consider the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$. Then, we have $\sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum\limits_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}}$ which converges for $p > 1$ and diverges for $p \leq 1$.*

2) *Consider the series $\sum\limits_{n=2}^{\infty} \frac{1}{n \log n}$. Here $\sum\limits_{n=2}^{\infty} 2^n \frac{1}{2^n \log 2^n} = \frac{1}{\log 2} \sum\limits_{n=2}^{\infty} \frac{1}{n}$ which diverges. Hence the given series diverges.*

2.2 Absolute convergence

Definition 2.2.1. *a) Let* $\sum_{n=1}^{\infty} a_n$ *be a series of real numbers. If* $\sum_{n=1}^{\infty} |a_n|$ *converges, we say that* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

b) If $\sum_{n=1}^{\infty} a_n$ *converges but* $\sum_{n=1}^{\infty} |a_n|$ *diverges, we say that* $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*

Examples 2.2.2.

1) The series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n!}$ *converges absolutely.*

2) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ *converges absolutely.*

Theorem 2.2.3. *If* $\sum_{n=1}^{\infty} a_n$ *converges absolutely, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Let $t_n = \sum_{k=1}^{n} |a_k|$. As the series converges absolutely, the sequence $\{t_n\}_1^{\infty}$ is Cauchy. Thus, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|t_m - t_n| < \epsilon \;\forall\; m, n \geq N.$$

Let $m > n$. Then

$$|s_m - s_n| = \left| \sum_{i=n+1}^{m} a_i \right| \leq \sum_{i=n+1}^{m} |a_i| = |t_m - t_n| < \epsilon.$$

Thus the sequence $\{s_n\}_1^{\infty}$ is Cauchy and hence converges. Thus $\sum_1^{\infty} a_n$ converges. ///

Theorem 2.2.4. *Let* $\sum_1^{\infty} a_n$ *be a series of real numbers. Let* $p_n = \max\{a_n, 0\}$ *and* $q_n = \min\{a_n, 0\}$.

a) If $\sum a_n$ *converges absolutely, then both* $\sum p_n$ *and* $\sum q_n$ *converges.*

b) If $\sum a_n$ *diverges then one of the* $\sum p_n$ *or* $\sum q_n$ *diverges.*

b) If $\sum a_n$ *converges conditionally then both* $\sum p_n$ *and* $\sum q_n$ *diverges.*

Proof.

a) Observe that $p_n = (a_n + |a_n|)/2$ and $q_n = (a_n - |a_n|)/2$. Thus the convergence of the two series follows from the hypothesis.

b) Proof is easy.

c) We leave to this as an exercise.

///

Tests for absolute convergence

Theorem 2.2.5 (Comparison test). *Let $\sum a_n$ be a series of real numbers. Then, $\sum a_n$ converges absolutely if there is an absolutely convergent series $\sum c_n$ with $|a_n| \le |c_n|$ for all $n \ge N, N \in \mathbb{N}$.*

Proof follows as in Theorem 2.1.7

Examples 2.2.6.

1) The series $\sum_{n=1}^{\infty} \frac{7}{7n-2}$ *diverges because* $\frac{7}{7n-2} = \frac{1}{n-2/7} \ge \frac{1}{n}$ *for all* $n \in \mathbb{N}$ *and* $\sum \frac{1}{n}$ *diverges.*

2) The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ *converges because* $\frac{1}{n!} \le \frac{1}{2^n}$ *and* $\sum_{n=0}^{\infty} \frac{1}{2^n}$ *converges.*

Theorem 2.2.7 (Limit comparison test). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers. Then*

a) if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, $\sum a_n$ *and* $\sum b_n$ *both converge or diverge together;*

b) if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*

c) if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

Proof. (a) As $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, *for* $\epsilon = \frac{c}{2} > 0$, *there exists* $N \in \mathbb{N}$ *such that*

$$n \ge N \implies \left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}.$$

Thus, for $n \ge N$,

$$\frac{-c}{2} \le \frac{a_n}{b_n} - c \le \frac{c}{2}$$

or equivalently

$$\frac{cb_n}{2} \le a_n \le \frac{3cb_n}{2}.$$

Hence the conclusion follows from the comparison test.

b) Given that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. *Hence for* $\epsilon = \frac{1}{2}$, *there exists* $N \in \mathbb{N}$ *such that*

$$n \ge N \implies \frac{a_n}{b_n} < \frac{1}{2}$$

or equivalently,

$$n \ge N \implies a_n \le \frac{b_n}{2}.$$

Thus the desired conclusion follows from the comparison test.

c) Here we are given that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$*. Hence for any real number* $M > 0$*, there exists* $N \in \mathbb{N}$ *such that*

$$n \geq N \implies \frac{a_n}{b_n} \geq M$$

or equivalently,

$$n \geq N \implies a_n \geq M b_n.$$

Thus if $\sum |b_n|$ *diverges, then* $\sum |a_n|$ *diverges by comparison test.* ///

Examples 2.2.8.

1) *Consider the series* $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$*. Here* $a_n = \frac{2n+1}{(n+1)^2}$*. Let* $b_n = \frac{1}{n}$*. Then* $\frac{a_n}{b_n} = \frac{\left(\frac{2n+1}{(n+1)^2}\right)}{\frac{1}{n}} = \frac{2n^2+n}{n^2+2n+1} \to 2$ *as* $n \to \infty$*. Further,* $\sum \frac{1}{n}$ *diverges. Thus by limit comparison theorem, the given series diverges.*

2) *Consider the series* $\sum_{1}^{\infty} \frac{1}{2^n - 1}$*. Here* $a_n = \frac{1}{2^n-1}$*. Let* $b_n = \frac{1}{2^n}$*. Then* $\frac{a_n}{b_n} = \frac{2^n}{2^n-1} \to 1$*. Further,* $\sum \frac{1}{2^n}$ *converges and hence the given series converges.*

3) *Consider the series* $\sum \frac{e^{-n}}{n^2}$*. Here* $a_n = \frac{e^{-n}}{n^2}$ *and* $b_n = \frac{1}{n^2}$*. Then* $\frac{a_n}{b_n} = e^{-n} \to 0$ *as* $n \to \infty$*. Further,* $\sum \frac{1}{n^2}$ *converges and hence the given series converges.*

4) *Consider the series* $\sum \frac{e^{-n}}{n}$*. Here* $a_n = \frac{e^{-n}}{n}$ *and* $b_n = \frac{1}{n^2}$*. Then* $\frac{a_n}{b_n} = ne^{-n} \to 0$ *as* $n \to \infty$*. Further,* $\sum \frac{1}{n^2}$ *converges and hence the given series converges.*

Theorem 2.2.9 (Ratio test)**.** *Let* $\sum_{1}^{\infty} a_n$ *be a series of real numbers. Let*

$$a = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \text{ and } A = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then

a) $\sum_{1}^{\infty} a_n$ *converges absolutely if* $A < 1$*;*

b) $\sum_{1}^{\infty} |a_n|$ *diverges if* $a > 1$*;*

c) *the test fails in all other cases.*

Proof. a) If $A < 1$*, choose* B *such thata* $A < B < 1$*. Then there exists an* $\epsilon > 0$ *such that* $B = A + \epsilon$ *and also* $N \in \mathbb{N}$ *such that* $\left|\frac{a_{n+1}}{a_n}\right| \leq B$ *for all* $n \geq N$*. Further, for any* $k \in \mathbb{N}$*,*

$$\left|\frac{a_{N+k}}{a_N}\right| = \prod_{i=1}^{k} \left|\frac{a_{N+i}}{a_{N+i-1}}\right| \leq \prod_{i=1}^{k} B = B^k.$$

Thus $|a_{N+k}| \leq B^k|a_N|,\ k \in \mathbb{N}$. *But* $\sum\limits_{k=0}^{\infty}|a_N|B^k < \infty$ *as* $B < 1$. *Thus by comparison test, the series* $\sum\limits_1^{\infty} a_n$ *converges.*

b) If $a > 1$*, choose* b *such that* $1 < b < a$*. There exits* $N \in \mathbb{N}$ *such that* $\left|\frac{a_{n+1}}{a_n}\right| \geq b$ *for all* $n \geq N$*. Further, for any* $k \in \mathbb{N}$*,*

$$\left|\frac{a_{N+k}}{a_N}\right| = \prod_{i=1}^{k}\left|\frac{a_{N+i}}{a_{N+i-1}}\right| \geq \prod_{i=1}^{k} b = b^k.$$

Thus $|a_{N+k}| \geq |a_N|,\ k \in \mathbb{N}$. *But, as* $b > 1$, $\sum\limits_{k=0}^{\infty} a_N b^k$ *diverges. Thus, again, by the comparison test, the series* $\sum\limits_1^{\infty} a_n$ *diverges.*

c) Case1: $a = A = 1$ *Consider the series* $\sum \frac{1}{n}$*. Here* $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$*. But* $\sum \frac{1}{n}$ *diverges. For the series* $\sum \frac{1}{n^2}$*, which converges, again* $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$*.*
Case 2: $A > 1$ *If we consider the series* $\sum 2^n$ *then* $A = 2 > 1$ *and the series diverges. If we take*

$$s = 1 + 2 + \frac{2}{5} + 2(\frac{1}{5})^2 + (\frac{1}{5})^3 + 2(\frac{1}{5})^3 + ...$$

Then it is easy to see that the series converges as

$$s = 1 + (\frac{1}{5}) + (\frac{1}{5})^3 + ... + 2 + 2(\frac{1}{5}) + 2(\frac{1}{5})^2 + 2(\frac{1}{5})^3 + ...$$

But $A = 2$*. Similarly one can construct examples when* $a < 1$*.* ///

Examples 2.2.10.

a) Consider the series $\sum\limits_1^{\infty}\frac{n^n}{n!}$*. Here*

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e,$$

which is greater than 1*. So* $a = A = e > 1$*. Thus the given series diverges.*

b) Consider the series $\sum\limits_0^{\infty}\frac{x^n}{n!}, x \in \mathbb{R}$*. Here*

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} = \frac{x}{n+1} \to 0.$$

Therefore $a = A = 0 < 1$*. Thus, for all* $x \in \mathbb{R}$*, the given series converges.*

Theorem 2.2.11 (Root test)**.** *Let* $\sum\limits_1^{\infty} a_n$ *be a series of real numbers. Let* $A = \limsup\limits_{n\to\infty}\sqrt[n]{|a_n|}$*. Then*

a) the series converges absolutely if $A < 1$*;*

b) the series diverges if $A > 1$;

c) the test fails if $A = 1$.

Proof. *a) If $A < 1$, choose B such that $A < B < 1$. Then there exists $N \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} < B$ for all $n \geq N$. This implies $|a_n| < B^n$ for all $n \geq N$. As $B < 1$, the series converges by comparison test.*

b) If $A > 1$, there exists infinitely many $n \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} > 1$. But this implies that $|a_n| > 1$ for infinitely many values of n and hence $a_N \nrightarrow 0$, i.e., $\sum a_n$ diverges.

c) Consider the series $\sum \frac{1}{n}$. Here $A = 1$ and the series diverges. On the other hand, for the series $\sum \frac{1}{n^2}$, again $A = 1$, but the series converges. ///

Examples 2.2.12.

1) *Consider the series $\sum_1^\infty \frac{x^n}{n}, x \in \mathbb{R}$. Here $a_n = \frac{x^n}{n}$. Therefore, $\sqrt[n]{\left|\frac{x^n}{n}\right|} = \left|\frac{x}{\sqrt[n]{n}}\right| \to |x|$. Thus the series converges for $|x| < 1$ and diverges for $|x| > 1$.*

2) *Consider the series $\sum_1^\infty \frac{x^n}{n^n}, x \in \mathbb{R}$. Here $a_n = \frac{x^n}{n^n}$. Then, $\sqrt[n]{|a_n|} = \left|\frac{x}{n}\right| \to 0$. Thus the series converges for any $x \in \mathbb{R}$.*

3) *Consider the series $\sum a_n$, where $a_n = \begin{cases} \frac{n}{4^n} & n \text{ is odd} \\ \frac{1}{2^n} & n \text{ is even} \end{cases}$. Then $\limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2}$. Therefore the series converges.*

4) *The series $\sum 3^{-n-(-1)^n}$. Then it is not difficult to see that $\limsup |a_n|^{\frac{1}{n}} = 1/3$. However ratio test fails in this case.*

Remark 2.1. *We note that the root test is stronger than the ratio test. for example, take the series $\sum a_n$ where*

$$a_n = \begin{cases} 2^{-n} & n \text{ odd} \\ 2^{-n+2} & n \text{ even} \end{cases}$$

Then it is easy to see that

$$\limsup \frac{|a_{n+1}|}{|a_n|} = 2, \text{but} \quad \limsup |a_n|^{1/n} = 1/2.$$

So the root test implies that the series converges but ratio test is inconclusive.

Alternating series:

Definition 2.2.13. *An alternating series is an infinite series whose terms alternate in sign.*

Theorem 2.2.14. *Suppose $\{a_n\}$ is a sequence of positive numbers such that*

(a) $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ and

(b) $\lim_{n\to\infty} a_n = 0$,

then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Proof. Consider the partial sums with odd index, $s_1, s_3, s_5, \ldots$. Now, for any $n \in \mathbb{N}$,

$$s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} \leq s_{2n-1} \text{ (by (a))}.$$

Thus the sequence $\{s_{2n-1}\}_1^{\infty}$ forms a non-increasing sequence. Also, notice that

$$s_{2n-1} = \sum_{i=1}^{n-1}(a_{2i-1} - a_{2i}) + a_{2n-1}.$$

Since each quantity in the parenthesis is non-negative and $a_{2n-1} > 0$, the sequence $\{s_{2n-1}\}$ is bounded below by 0. Hence $\{s_{2n-1}\}_1^{\infty}$ is convergent.

Now, consider the partial sums with even index, $s_2, s_4, s_6, \ldots$. For any $n \in \mathbb{N}$,

$$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n} \text{ (by (a))}.$$

Thus the sequence $\{s_{2n}\}_1^{\infty}$ forms a non-decreasing sequence. Further,

$$s_{2n} = a_1 - \sum_{i=1}^{n-1}(a_{2i} - a_{2i+1}) - a_{2n} \leq a_1,$$

which means that s_{2n} is bounded above by a_1. Therefore, $\{s_{2n}\}$ is convergent.

Let $L = \lim\ s_{2n}$ and $M = \lim\ S_{2n-1}$. By ((b)),

$$0 = \lim\ a_{2n} = \lim\ (s_{2n} - s_{2n-1}) = L - M.$$

Thus $L = M$ and hence the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. ///

Examples 2.2.15.

1) *Consider the series* $\sum_{n=1}^{\infty}(-1)^{n+1}2^{1/n}$. *Here* $a_n = 2^{1/n} \to 1$ *as* $n \to \infty$. *Hence the above theorem does not apply. Anyhow, one can show that the series diverges.*

2) *Consider the series* $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. *The* a_n'*s of this series satisfies the hypothesis of the above theorem and hence the series converges.*

Examples 2.2.16.

1) *The series* $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ *converges conditionally.*

2) *The series* $\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{2n-1}$ *converges conditionally.*

The following is a more genreal test than the previous theorem.

Theorem 2.2.17. *(Dirichlet test)*
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that

1. *the sequence $s_n = \sum_{k=1}^{n} a_k$ is bounded,*
2. *the sequence b_n is decreasing and $b_n \to 0$.*

Then the series $\sum a_n b_n$ converges.

Proof. Since A_n is bounded, there exists $M > 0$, such that $|A_n| \leq M$ for all n. Now note that

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \dots . s_{n-1}(b_{n-1} - b_n) + s_n b_n$$

Since b_n is decreasing, $b_n - b_{n+1} \geq 0$. Therefore

$$|a_1 b_1 + a_2 b_2 + \dots + a_n b_n| \leq M b_1$$

Since $b_n \to 0$, for any $\epsilon > 0$, we get N such that $|b_n| \leq \epsilon$ for all $n \geq N$. Now we can easily see,

$$|\sum_{n}^{m} a_k b_k| = M|b_n| \leq M\epsilon$$

Therefore, by Cauchy's test, the series $\sum a_n b_n$ converges. ///

Examples 2.2.18.

1) *Consider the series $\sum \frac{\cos n\pi}{\log n}$. Here take $a_n = \cos n\pi$ and $b_n = \frac{1}{\log n}$. Then*

$$|A_n| \leq |\sum_{k=1}^{n} \cos n\pi| \leq 1$$

(check the first 4 terms and then use periodicity of $\cos x$)
and b_n decreases to 0. Hence the series conveges. In this case we can see that the series does not converge absolutely (apply Cauchy's test).

2) *$\sum \frac{2^{2n} n^2}{e^n n!} \frac{1}{(\log n)^2}$. Take $b_n = \frac{1}{(\log n)^2}$ and $a_n = \frac{2^{2n} n^2}{e^n n!}$. Then b_n decreases to 0. To show the boundedness of the partial sums of $\sum a_n$, we can apply Ratio test to see that the series $\sum a_n$ converges. Hence the sequence of partial sums converge and so will be bounded. Therefore by Dirichlet test the series $\sum a_n b_n$ converges.*

Theorem 2.2.19. *(Integral Test). If $f(x)$ is decreasing and non-negative on $[1, \infty)$, Then*

$$\int_1^\infty f(x)dx < \infty \iff \sum_{n=1}^{\infty} f(n) \text{ converges.}$$

Details of this theorem will be done after convergence of improper integral.

2.3 Problems

1. If the terms of the convergent series $\sum_{n=1}^{\infty} a_n$ are positive and forms a non-increasing sequence, then prove that $\lim_{n\to\infty} 2^n a_{2^n} = 0$.

2. If $0 \le a_n \le 1$ $(n \ge 0)$ and if $0 \le x \le 1$, then prove that $\sum_{n=1}^{\infty} a_n x^n$ converges.

3. Test the convergence of the series (1) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ (2) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^x}$ $x \in R$.

4. Determine which of the following series diverges

$$(a) \sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}} \quad (b) \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^{3/2}} \quad (c) \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1} \quad (d) \sum_{n=1}^{\infty} \frac{1-n}{n2^n} \quad (e) \sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}} \quad (f) \sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}.$$

5. Determine which of the following series converges

$$(a) \sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n} \quad (b) \sum_{n=1}^{\infty} \frac{n!}{10^n} \quad (c) \sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n \quad (d) \sum_{n=1}^{\infty} \frac{(\log n)^n}{n^n} \quad (e) \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$$

6. Test the convergence of the infinite series

$$(1) \sum_{n=0}^{\infty} \sin\left(\frac{\pi}{2^n}\right) \quad (2) \sum_{n=2}^{\infty} \frac{1}{n\ln(n^3)} \quad (3) \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{\sqrt{n}}\right) \quad (4) \sum \frac{1}{\sqrt{n}} \sin(\frac{1}{n})$$

7. Test the convergence of the infinite series

$$(1) \sum \frac{n^2 n!}{(n^3+1)n^n} \quad (2) \sum \frac{\sin\frac{n\pi}{2}}{n} \quad (c) \sum \frac{n(n!)}{(n^2+1)[(2n+1)!]}$$