Sequences and Series

1 Sequences of real numbers

1.1 Real number system

We are familiar with natural numbers and to some extent the rational numbers. While finding roots of algebraic equations we see that rational numbers are not enough to represent roots which are not rational numbers. For example draw the graph of $y = x^2 - 2$. We see that it cross the x-axis twice. The roots are such that their square is 2, but they cannot be rational numbers according to the following theorem.

Theorem 1.1.1. *Suppose that* $a_0, a_1, ..., a_n (n \geq 1)$ *are integers such that* $a_0 \neq 0, a_n \neq 0$ *and that* r *satisfies the equation*

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0.$$

If $r = \frac{p}{q}$ *where* p, q *are integers with no common factors and* $q \neq 0$*. Then* q *divides* a_n *and* p *divides* a_0*.*

This theorem tells us that only rational candidates for solutions of the above equation have the form $\frac{p}{q}$ where p divides a_0 and q divides a_n.

Proof: Since $\frac{p}{q}$ satisfies the equation, we have

$$a_n p^n + a_{n-1} p^{n-1} q + ... + a_0 q^n = 0$$

i.e., $a_n p^n = -q(a_{n-1} p^{n-1} + ... + a_0 q^{n-1})$. This means q divides a_n as p, q have no common factors. On the other hand we can also write

$$a_0 q^n = -p(a_n p^{n-1} + a_{n-1} p^{n-2} + ... + a_1 q^{n-1}).$$

Thus p divides a_0. ///

Now we see that the possible rational roots of $x^2 - 2 = 0$ are $\pm 1, \pm 2$. But it is easy to check that $\pm 1, \pm 2$ does not satisfy $x^2 - 2 = 0$. So the roots of $x^2 - 2 = 0$ are not rational numbers. This means the set of rational numbers has "gaps". So the natural question to ask is: Can we have a number system without these gaps? The answer is yes and the "complete number system" with out these gaps is the real line $\mathbb{R}$. We will not look into the development of $\mathbb{R}$ as it is not easy to define the real numbers. We assume that there is a set $\mathbb{R}$, whose elements are called real numbers and $\mathbb{R}$ is closed with respect to addition and multiplication. That is, given any $a, b \in \mathbb{R}$, the sum $a+b$ and product ab also represent real numbers. Moreover, $\mathbb{R}$ has an order structure $\leq$ and has no "gaps" in the sense that it satisfies the Completeness Axiom(see below).

Let S be a non-empty subset of $\mathbb{R}$. If S contains a largest element s^0, then we call s^0 the maximum of S. If S contains a smallest element s_0, then we call s_0 the minimum of S. If S is bounded above and S has least upper bound, then we call it the supremum of S. If S is bounded below and S has greatest lower bound, then we call it as infimum of S.

Unlike maximum and minimum, $\sup S$ and $\inf S$ need not belong to the set S. An important observation is if $\alpha = \sup S$ is finite, then for every $\epsilon > 0$, there exists an element $s \in S$ such that $s \geq \alpha - \epsilon$.
Note that any bounded subset of Natural numbers has maximum and minimum.

Completeness Axiom: Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.

The completeness axiom does not hold for $\mathbb{Q}$. That is, every non-empty subset of $\mathbb{Q}$ that is bounded above by a rational number need not have rational least upper bound. For example $\{r \in \mathbb{Q} : r^2 \leq 2\}$.

Archimedean property: For each $x \in \mathbb{R}$, there exists a natural number $N = N(x)$ such that $x < N$.
Proof: Assume by contradiction that this is not true. Then there is no $N \in \mathbb{N}$ such that $x < N$. i.e., x is an upper bound for $\mathbb{N}$. Then, by completeness axiom, let u be the smallest such bound of $\mathbb{N}$ in $\mathbb{R}$. That is $u \in \mathbb{R}$ and so $u - m$ for $2 \leq m \in \mathbb{N}$ is not an upper bound for $\mathbb{N}$. Therefore, there exists $k \in \mathbb{N}$ such that $u - m < k$, but then $u < k + m$, and $k + m \in \mathbb{N}$. a contradiction. ///

Now it is easy to see the following corollary
Corollary: Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $w = \inf S = 0$.
Proof: We note that S is bounded below. Let $\epsilon > 0$ be an arbitrary positive real number. By above Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\epsilon}$. Then we have,

$$0 \leq w \leq \frac{1}{n} < \epsilon.$$

Since ϵ is arbitrary, we have $w = 0$. (why?)

Corollary: If $y > 0$ be a real number, then there exists $n = n(y) \in \mathbb{N}$ such that

$n-1 \leq y < n$.
Finally, we have the following density theorem

Theorem 1.1.2. *Let x, y are real numbers such that $x < y$. Then there exists a rational number q such that $x < q < y$.*

Proof: W.l.g. assume that $x > 0$. Now let $n \in \mathbb{N}$ be such that $y - x > \frac{1}{n}$. Otherwise $n < \frac{1}{y-x}$ for all $n \in \mathbb{N}$. Now consider the set

$$S = \{m \in \mathbb{N} : \frac{m}{n} > x\}.$$

Then S is non-empty (by Archimedean property). By well -ordering of $\mathbb{N}$, S has minimal element say m_0. Then $x < \frac{m_0}{n}$. By the minimality of m_0, we see that $\frac{m_0-1}{n} \leq x$. Then,

$$\frac{m_0}{n} \leq x + \frac{1}{n} < x + (y - x) = y.$$

Therefore,

$$x < \frac{m_0}{n} < y.$$

1.2 Sequences and their limit

Definition 1.2.1. *A sequence of real numbers is a function from $\mathbb{N}$ to $\mathbb{R}$.*

Notation. *It is customary to denote a sequence as $\{a_n\}_{n=1}^{\infty}$.*

Examples 1.2.2. *(i) $\{c\}_{n=1}^{\infty}, c \in \mathbb{R}$, (ii) $\{\frac{(-1)^{n+1}}{n}\}_{n=1}^{\infty}$, (iii) $\{\frac{n-1}{n}\}_{n=1}^{\infty}$ and (iv) $\{\sqrt{n}\}_{n=1}^{\infty}$.*

Definition 1.2.3. *A sequence $\{a_n\}_{n=1}^{\infty}$ converges to limit L if for every $\epsilon > 0$ (given) there exists a positive integer N such that $n \geq N \implies |a_n - L| < \epsilon$.*

Notation. $L = \lim_{n\to\infty} a_n$ *or* $a_n \to L$.

Examples 1.2.4.

(i) It is clear that the constant sequence $\{c\}_{n=1}^{\infty}, c \in \mathbb{R}$, has c as it's limit.

(ii) Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Solution. *Let $\epsilon > 0$ be given. In order to show that $1/n$ approaches 0, we must show that there exists an integer $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon.$$

But $1/n < \epsilon \Leftrightarrow n > 1/\epsilon$. Thus, if we choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$, then for all $n \geq N$, $1/n < \epsilon$.

(iii) *Consider the sequence $\{(-1)^{n+1}\}_{n=1}^{\infty}$. It is intuitively clear that this sequence does not have a limit or it does not approach to any real number. We now prove this by definition. Assume to the contrary, that there exists an $L \in \mathbb{R}$ such that the sequence $\{(-1)^{n+1}\}_{n=1}^{\infty}$ converges to L. Then for $\epsilon = \frac{1}{2}$, there exists an $N \in \mathbb{N}$ such that*

$$|(-1)^{n+1} - L| < \frac{1}{2},\ \forall\ n \geq N. \tag{1.1}$$

For n even, (1.1) says

$$|-1 - L| < \frac{1}{2},\ \forall\ n \geq N. \tag{1.2}$$

while for n odd, (1.1) says

$$|1 - L| < \frac{1}{2},\ \forall\ n \geq N. \tag{1.3}$$

which is a contradiction as $2 = |1+1| \leq |1-L| + |1+L| < 1$.

Lemma 1.2.5. *If $\{a_n\}_1^{\infty}$ is a sequence and if both $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} a_n = M$ holds, then $L = M$.*

Proof. Suppose that $L \neq M$. Then $|L - M| > 0$. Let $\epsilon = \frac{|L-M|}{2}$. As $\lim\limits_{n\to\infty} a_n = L$, there exists $N_1 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N_1$. Also as $\lim\limits_{n\to\infty} a_n = M$,there exists $N_2 \in \mathbb{N}$ such that $|a_n - M| < \epsilon$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, $|a_n - L| < \epsilon$ and $|a_n - M| < \epsilon$. Thus $|L - M| \leq |a_n - L| + |a_n - M| < 2\epsilon = |L - M|$, which is a contradiction. ///

Theorem 1.2.6 (Sandwich theorem for sequences)**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.*

Proof. Let $\epsilon > 0$ be given. As $\lim\limits_{n\to\infty} a_n = L$, there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |a_n - L| < \epsilon. \tag{1.4}$$

Similarly as $\lim\limits_{n\to\infty} c_n = L$, there exists $N_2 \in \mathbb{N}$

$$n \geq N_2 \implies |c_n - L| < \epsilon. \tag{1.5}$$

Let $N = \max\{N_1, N_2\}$. Then, $L - \epsilon < a_n$ (from (1.4)) and $c_n \leq L + \epsilon$ (from (1.5)). Thus

$$L - \epsilon < a_n \leq b_n \leq c_n \leq L + \epsilon.$$

Thus $|b_n - L| < \epsilon$ for all $n \geq N$. Hence the proof. ///

Examples 1.2.7.

(i) Consider the sequence $\left\{\frac{cos\ n}{n}\right\}_{n=1}^{\infty}$. *Then* $\frac{-1}{n} \leq \frac{cos\ n}{n} \leq \frac{1}{n}$. *Hence by Sandwich theorem* $\lim_{n\to\infty} \frac{cos\ n}{n} = 0$.

(ii) As $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$ *and* $\frac{1}{n} \to 0$ *as* $n \to \infty$, $\frac{1}{2^n}$ *also converges to* 0 *by Sandwich theorem.*

(iv) If $b > 0$, *then* $\lim_{n\to\infty} \sqrt[n]{b} = 1$.

Solution. *First assume that* $b > 1$. *Let* $a_n = b^{\frac{1}{n}} - 1$. *As* $b > 1$, $a_n > 0$ *for all* $n \in \mathbb{N}$. *Further,*

$$b = (1 + a_n)^n \geq 1 + na_n.$$

Then $0 \leq a_n \leq \frac{b-1}{n}$. *Thus* $a_n \to 0$, *i.e.,* $b^{\frac{1}{n}} \to 1$ *as* $n \to \infty$.
Now if $b < 1$, *then take* $c = \frac{1}{b}$ *and it is easy to show the result.* ///

Examples 1.2.8.

(i) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(ii) If $x > 0$ *then* $\lim_{n\to\infty} \frac{n^x}{(1+x)^n} = 0$.

(iii) If $p > 0$, *then* $\lim_{n\to\infty} \frac{\log(n)}{n^p} = 0$.

Solution. *(i) Let* $a_n = n^{\frac{1}{n}} - 1$. *Then* $0 \leq a_n \leq 1$ *for all* $n \in \mathbb{N}$. *Further,*

$$n = (1 + a_n)^n = \frac{n(n-1)}{2} a_n^2.$$

Thus $0 \leq a_n \leq \sqrt{\frac{2}{(n-1)}}$ $(n \geq 2)$. *As* $\sqrt{\frac{2}{(n-1)}} \to 0$ *as* $n \to \infty$, *by Sandwich theorem,* $a_n \to 0$, *i.e.,* $n^{\frac{1}{n}} \to 1$ *as* $n \to \infty$.

(ii) Let k *be an integer such that* $k > x$, $k > 0$. *Then for* $n > 2k$,

$$(1+p)^n > \ {}_nC_k p^k = \frac{n!}{k!(n-k)!} p^k = \frac{p^k}{k!} \prod_{i=1}^{k} [n - i + 1] > \frac{n^k}{2^k} \frac{p^k}{k!}.$$

Hence,

$$0 < \frac{n^x}{(1+x)^n} < \frac{2^k k!}{x^k} n^{x-k}\ (n > 2k).$$

As $x - k < 0$, $n^{x-k} \to 0$. *Thus* $\frac{n^x}{(1+x)^n} \to 0$ *as* $n \to \infty$.

(iii) For any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m \leq n^p < (m+1)$ or equivalently $m^{\frac{1}{p}} \leq n < (m+1)^{\frac{1}{p}}$. Let $\epsilon > 0$. Since $n^{\frac{1}{n}} \to 1$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $n^{\frac{1}{n}} \in (e^{-\epsilon}, e^{\epsilon})$, $\forall\, n \geq N$ (or) $\dfrac{\log n}{n} \in (-\epsilon, \epsilon)$, $\forall\, n \geq N$. That is $\frac{\log n}{n} \to 0$ as $n \to \infty$. This implies that

$$\lim_{m\to\infty} \frac{1}{p}\frac{\log(m+1)}{m} = 0.$$

As $\dfrac{1}{n^p} < \dfrac{\log n}{n^p} < \dfrac{1}{p}\dfrac{\log(m+1)}{m}$. Now the conclusion follows from Sandwich theorem.

Definition 1.2.9. *(**Subsequence**): Let $\{a_n\}$ be a sequence and $\{n_1, n_2, ...\}$ be a sequence of positive integers such that $i > j$ implies $n_i > n_j$. Then the sequence $\{a_{n_i}\}_{i=1}^{\infty}$ is called a subsequence of $\{a_n\}$.*

Theorem 1.2.10. *If the sequence of real numbers $\{a_n\}_1^{\infty}$, is convergent to L, then any subsequence of $\{a_n\}$ is also convergent to L.*

Proof. Let $\{n_i\}_{i=1}^{\infty}$ be a sequence of positive integers such that $\{a_{n_i}\}_{i=1}^{\infty}$ forms a subsequence of $\{a_n\}$. Let $\epsilon > 0$ be given. As $\{a_n\}$ converges to L, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon,\ \forall\, n \geq N.$$

Choose $M \in \mathbb{N}$ such that $n_i \geq N$ for $i \geq M$. Then

$$|a_{n_i} - L| < \epsilon,\ \forall\, i \geq M.$$

Hence the proof. ///

Definition 1.2.11. *(**Bounded sequence**): A sequence $\{a_n\}$ is said to be bounded above, if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. Similarly, we say that a sequence $\{a_n\}$ is bounded below, if there exists $N \in \mathbb{R}$ such that $a_n \geq N$ for all $n \in \mathbb{N}$. Thus a sequence $\{a_n\}$ is said to be bounded if it is both bounded above and below.*

Lemma 1.2.12. *Every convergent sequence is bounded.*

Proof. Let $\{a_n\}$ be a convergent sequence and $L = \lim\limits_{n\to\infty} a_n$. Let $\epsilon = 1$. Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for all $n \geq N$. Further,

$$|a_n| = |a_n - L + L| \leq |a_n - L| + L < 1 + L, \forall\, n \geq N.$$

Let $M = \max\{|a_1|, |a_2|, ..., |a_{n-1}|, 1 + |L|\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence $\{a_n\}$ is bounded. ///

1.3 Operations on convergent sequences

Theorem 1.3.1. *Let* $\{a_n\}_1^\infty$ *and* $\{b_n\}_1^\infty$ *be two sequences such that* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$. *Then*

(i) $\lim_{n\to\infty}(a_n + b_n) = L + M$.

(ii) $\lim_{n\to\infty}(ca_n) = cL$, $c \in \mathbb{R}$..

(iii) $\lim_{n\to\infty}(a_n b_n) = LM$.

(iv) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ *if* $M \neq 0$.

Proof. (i) Let $\epsilon > 0$. Since a_n converges to L there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon/2 \;\forall\; n \geq N_1.$$

Also, as b_n converges to M there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - M| < \epsilon/2 \;\forall\; n \geq N_2.$$

Thus

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_N - M| < \epsilon \;\forall\; n \geq N = \max\{N_1.N_2\}.$$

(ii) Easy to prove. Hence left as an exercise to the students.

(iii) Let $\epsilon > 0$. Since a_n is a convergent sequence, it is bounded by M_1 (say). Also as a_n converges to L there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon/2M \;\forall\; n \geq N_1.$$

Similarly as b_n converges to M there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - M| < \epsilon/2M_1 \;\forall\; n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then

$$\begin{aligned}|a_n b_n - LM| &= |a_n b_n - a_n M + a_n M - LM| \leq |a_n(b_n - M)| + |M(a_n - L)| \\ &= |a_n||b_n - M| + M|a_n - L| < \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

(iv) In order to prove this, it is enough to prove that if $\lim_{n\to\infty} a_n = L$, $L \neq 0$, then $\lim_{n\to\infty} 1/a_n = 1/L$. Without loss of generality, let us assume that $L > 0$. Let $\epsilon > 0$ be given. As $\{a_n\}$ forms a convergent sequence, it is bounded. Choose $N_1 \in \mathbb{N}$ such that $a_n > L/2$ for all $n \geq N_1$. Also, as a_n converges to L, there exists $N_2 \in \mathbb{N}$ such that $|a_n - L| < L^2\epsilon/2$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then

$$n \geq N \implies \left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|a_n L|} < \frac{2}{L^2}\frac{L^2\epsilon}{2} = \epsilon. \qquad ///$$

Examples 1.3.2.

(i) Consider the sequence $\left\{\frac{5}{n^2}\right\}_1^\infty$. *Then* $\lim_{n\to\infty} \frac{5}{n^2} = \lim_{n\to\infty} 5 \cdot \frac{1}{n} \cdot \frac{1}{n} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0.$

(ii) Consider the sequence $\left\{\frac{3n^2 - 6n}{5n^2 + 4}\right\}_1^\infty$. *Notice that* $\frac{3n^2 - 6n}{5n^2 + 4} = \frac{3 - 6/n}{5 + 4/n} \to 3/5.$

1.4 Divergent sequence and Monotone sequences

Definition 1.4.1. *Let $\{a_n\}$be a sequence of real numbers. We say that a_n approaches infinity or diverges to infinity, if for any real number $M > 0$, there is a positive integer N such that*

$$n \geq N \implies a_n \geq M.$$

- If a_n approaches infinity, then we write $a_n \to \infty$ as $n \to \infty$.
- A similar definition is given for the sequences diverging to $-\infty$. In this case we write $a_n \to -\infty$ as $n \to \infty$.

Examples 1.4.2.

(i) The sequence $\{\log(1/n)\}_1^\infty$ diverges to $-\infty$. In order to prove this, for any $M > 0$, we must produce a $N \in \mathbb{N}$ such that

$$\log(1/n) < -M, \quad \forall\, n \geq N.$$

But this is equivalent to saying that $n > e^M$, $\forall\, n \geq N$. Choose $N \geq e^M$. Then, for this choice of N,

$$\log(1/n) < -M, \quad \forall\, n \geq N.$$

Thus $\{\log(1/n)\}_1^\infty$ diverges to $-\infty$.

Definition 1.4.3. *If a sequence $\{a_n\}$does not converge to a value in $\mathbb{R}$ and also does not diverge to ∞ or $-\infty$, we say that $\{a_n\}$oscillates.*

Lemma 1.4.4. *Let $\{a_n\}$and $\{b_n\}$ be two sequences.*

(i) If $\{a_n\}$and $\{b_n\}$ both diverges to ∞, then the sequences $\{a_n + b_n\}$ and $\{a_n b_n\}$ also diverges to ∞.

(ii) If $\{a_n\}$diverges to ∞ and $\{b_n\}$ converges then $\{a_n + b_n\}$ diverges to ∞.

Example 1.4.5. *Consider the sequence $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{\infty}$. We know that $\sqrt{n+1}$ and $\sqrt{n}$ both converges to ∞. But the sequence $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{\infty}$ diverges to 0. To see this, notice that, for a given $\epsilon > 0$, $\sqrt{n+1} - \sqrt{n} < \epsilon$ if and only if $1 < \epsilon^2 + 2\epsilon\sqrt{n}$. Thus, if N is such that $N > \dfrac{1}{4\epsilon^2}$, then for all $n \geq N$, $\sqrt{n+1} - \sqrt{n} < \epsilon$. Thus $\sqrt{n+1} - \sqrt{n}$ converges to 0. This example shows that the sequence formed by taking difference of two diverging sequences may converge.*

Definition 1.4.6. *A sequence $\{a_n\}$ of real numbers is called a nondecreasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and $\{a_n\}$ is called a nonincreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence that is nondecreasing or nonincreasing is called a monotone sequence.*

Examples 1.4.7.

(i) The sequences $\{1 - 1/n\}$, $\{n^3\}$ are nondecreasing sequences.

(ii) The sequences $\{1/n\}$, $\{1/n^2\}$ are nonincreasing sequences.

Lemma 1.4.8. *(i) A nondecreasing sequence which is not bounded above diverges to ∞.*

(ii) A nonincreasing sequence which is not bounded below diverges to $-\infty$.

Example 1.4.9. *If $b > 1$, then the sequence $\{b^n\}_1^{\infty}$ diverges to ∞.*

Theorem 1.4.10.

(i) A nondecreasing sequence which is bounded above is convergent.

(ii) A nonincreasing sequence which is bounded below is convergent.

Proof. (i) Let $\{a_n\}$be a nondecreasing, bounded above sequence and $a = \sup\limits_{n \in \mathbb{N}} a_n$. Since the sequence is bounded, $a \in \mathbb{R}$. We claim that a is the limit point of the sequence $\{a_n\}$. Indeed, let $\epsilon > 0$ be given. Since $a - \epsilon$ is not an upper bound for $\{a_n\}$, there exists $N \in \mathbb{N}$ such that $a_N > a - \epsilon$. As the sequence is nondecreasing, we have $a - \epsilon < a_N \leq a_n$ for all $n \geq N$. Also it is clear that $a_n \leq a$ for all $n \in \mathbb{N}$. Thus,

$$a - \epsilon \leq a_n \leq a + \epsilon, \ \forall\, n \geq N.$$

Hence the proof.

The proof of (ii) is similar to (i) and is left as an exercise to the students. ///

Examples 1.4.11.

(i) If $0 < b < 1$*, then the sequence* $\{b^n\}_1^\infty$ *converges to* 0.

Solution. *We may write* $b^{n+1} = b^n b < b^n$*. Hence* $\{b^n\}$ *is nonincreasing. Since* $b^n > 0$ *for all* $n \in \mathbb{N}$*, the sequence* $\{b^n\}$ *is bounded below. Hence, by the above theorem,* $\{b^n\}$ *converges. Let* $L = \lim_{n\to\infty} b^n$*. Further,* $\lim_{n\to\infty} b^{n+1} = \lim_{n\to\infty} b \cdot b^n = b \cdot \lim_{n\to\infty} b^n = b \cdot L$*. Thus the sequence* $\{b^{n+1}\}$ *converges to* $b \cdot L$*. On the other hand,* $\{b^{n+1}\}$ *is a subsequence of* $\{b^n\}$*. Hence* $L = b \cdot L$ *which implies* $L = 0$ *as* $b \neq 1$.

(ii) The sequence $\{(1+1/n)^n\}_1^\infty$ *is convergent.*

Solution. *Let* $a_n = (1+1/n)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k$*. For* $k = 1, 2, ..., n$*, the* $(k+1)^{th}$ *term in the expansion is*

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdots k}\frac{1}{n^k} = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right). \tag{1.6}$$

Similarly, if we expand a_{n+1}*, then we obtain* $(n+2)$ *terms in the expansion and for* $k = 1, 2, 3, ...,$ *the* $(k+1)^{th}$ *term is*

$$\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right). \tag{1.7}$$

It is clear that (1.7) is greater than or equal to (1.6) and hence $a_n \leq a_{n+1}$ *which implies that* $\{a_n\}$*is nondecreasing. Further,*

$$a_n = (1+1/n)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k < 2 + \sum_{k=2}^{n} \frac{1}{k!} < 1 + e < 3.$$

Thus $\{a_n\}$*is a bounded monotone sequence and hence convergent.*

Theorem 1.4.12. *Every sequence has a monotone subsequence.*

Proof. Pick x_{N_1} such that $x_n \leq x_{N_1}$ for all $n > N_1$. We call such x_N as "peak". If we are able to pick infinitely many $x'_{N_i}s$, then $\{x_{N_i}\}$ is decreasing and we are done. If there are only finitely many $x'_N s$ and let x_{n_1} be the last peak. Then we can choose n_2 such that $x_{n_2} \geq x_{n_1}$. Again x_{n_2} is not a peak. So we can choose x_{n_3} such that $x_{n_3} \geq x_{n_2}$. Proceeding this way, we get a non-decreasing sub-sequence.

The following theorem is Bolzano-Weierstrass theorem. Proof is a consequence of Theorem1.4.12

Theorem 1.4.13. *Every bounded sequence has a convergent subsequence.*

Theorem 1.4.14. *Nested Interval theorem: Let $I_n = [a_n, b_n], n \geq 1$ be non-empty closed, bounded intervals such that*

$$I_1 \supset I_2 \supset I_3 ... \supset I_n \supset I_{n+1} ...$$

and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$. Then $\cap_{n=1}^{\infty} I_n$ contains precisely one point

Proof. Since $\{a_n\}, \{b_n\} \subset [a_1, b_1]$, $\{a_n\}, \{b_n\}$ are bounded sequences. By Bolzano-Weierstrass theorem, there exists sub sequences a_{n_k}, b_{n_k} and a, b such that $a_{n_k} \to a, b_{n_k} \to b$. Since a_n is increasing $a_1 < a_2 < ...... \leq a$ and $b_1 > b_2 > .... \geq b$. It is easy to see that $a \leq b$. Also since $0 = \lim a_n - b_n = a - b$, we have $a = b$.
It is easy to show that there is no other point in $\cap_{n=1}^{\infty} I_n$.

Remark 1.1. *closedness of I_n cannot be dropped. for example the sequence $\{(0, \frac{1}{n})\}$. Then $\cap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$ because there cannot be any element x such that $0 < x < \frac{1}{n}$ else Archimedean property fails.*

Corollary 1.4.15. $\mathbb{R}$ *is uncountable.*

Proof. It is enough to show that $[0, 1]$ is uncountable. If not, there exists an onto map $f : \mathbb{N} \to [0, 1]$. Now subdivide $[0, 1]$ into 3 equal parts so that choose J_1 such that $f(1) \notin J_1$. Now subdivide J_1 into 3 equal parts and choose J_2 so that $f(2) \notin J_2$. Continue the process to obtain J_n so that $f(n) \notin J_n$. These J_n satisfy the hypothesis of above theorem, so $\cap_{n=1}^{\infty} = \{x\}$ and $x \in [0, 1]$. By the construction, there is no $n \in \mathbb{N}$ such that $f(n) = x$. contradiction to f is onto.

1.5 Cauchy sequence

Definition 1.5.1. *A sequence $\{a_n\}$is called a Cauchy sequence if for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n, m \geq N \implies |a_n - a_m| < \epsilon$.*

Example 1.5.2. *Let $\{a_n\}$be a sequence such that $\{a_n\}$converges to L (say). Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that*

$$|a_n - L| < \frac{\epsilon}{2} \; \forall \; n \geq N.$$

Thus if $n, m \geq N$, we have

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\{a_n\}$is Cauchy.

Lemma 1.5.3. *If $\{a_n\}$is a Cauchy sequence, then $\{a_n\}$is bounded.*

Proof. Since $\{a_n\}$forms a Cauchy sequence, for $\epsilon = 1$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < 1, \ \forall \ n, m \geq N.$$

In particular,

$$|a_n - a_N| < 1, \ \forall \ n \geq N.$$

Hence if $n \geq N$, then

$$|a_n| \leq |a_n - a_N| + |a_N| < 1 + |a_N|, \ \forall \ n \geq N.$$

Let $M = \max\{|a_1|, |a_2|, ..., |a_{N-1}|, 1 + |a_N|\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence $\{a_n\}$is bounded. ///

Theorem 1.5.4. *If $\{a_n\}$is a Cauchy sequence, then $\{a_n\}$is convergent.*

Proof. Let a_{n_k} be a monotone subsequence of the Cauchy sequence $\{a_n\}$. Then a_{n_k} is a bounded, monotone subsequence. Hence $\{a_{n_k}\}$ converges to L(say). Now we claim that the sequence $\{a_n\}$ itself converges to L. Let $\epsilon > 0$. Choose N_1, N_2 such that

$$n, n_k \geq N_1 \implies |a_n - a_{n_k}| < \epsilon/2$$

$$n_k \geq N_2 \implies |a_{n_k} - a| < \epsilon/2.$$

Then

$$n, n_k \geq \max\{N, N_1\} \implies |a_n - a| \leq |a_n - a_{n_k}| + \| a_{n_k} - a| < \epsilon.$$

Hence the claim. ///

Therefore, we have the following Criterion:
Cauchy's Criterion for convergence: A sequence $\{a_n\}$ converges if and only if for every $\epsilon > 0$, there exists N such that

$$|a_n - a_m| < \epsilon \ \forall \ m, n \geq N.$$

Problem: Let $\{a_n\}$be defined as $a_1 = 1, a_{n+1} = 1 + \frac{1}{a_n}$. The show that $\{a_n\}$is Cauchy.
Solution: Note that $a_n > 1$ and $a_n a_{n-1} = a_{n-1} + 1 > 2$. Then

$$|a_{n+1} - a_n| = |\frac{a_{n-1} - a_n}{a_n a_{n-1}}| \leq \frac{1}{2}|a_n - a_{n-1}| \leq \frac{1}{2^{n-1}}|a_2 - a_1|, \ \forall n \geq 2.$$

Hence

$$|a_m - a_n| \le |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + ... + |a_{n+1} - a_n| \le |a_2 - a_1| \frac{\alpha^{n-1}}{1-\alpha}, \alpha = \frac{1}{2}$$

So given, $\epsilon > 0$, we can choose N such that $\frac{1}{2^{N-1}} < \frac{\epsilon}{2}$.

Indeed the following holds,

Theorem 1.5.5. *Let $\{a_n\}$ be a sequence such that $|a_{n+1} - a_n| < \alpha |a_n - a_{n-1}|$ for all $n \ge N$ for some N and $0 < \alpha < 1$. Then $\{a_n\}$ is a Cauchy sequence.*

Theorem 1.5.6. *For any sequence $\{a_n\}$ with $a_n > 0$*

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

provided the limit on the right side exists.

Proof. Let $\epsilon > 0$ be arbitrary. Suppose the second limit exists (say l), then there exists $N \in \mathbb{N}$ such that

$$l - \epsilon < \frac{a_{n+1}}{a_n} < l + \epsilon, \quad \forall n \ge N.$$

Taking $n = N, N+1, ..., m-1$ and multiplying we get

$$(l-\epsilon)^{m-N} < \frac{a_m}{a_N} < (l+\epsilon)^{m-N}, \quad \forall m \ge N+1$$

equivalently,

$$(l-\epsilon)^{1-\frac{N}{m}} a_N^{\frac{1}{m}} < (a_m)^{\frac{1}{m}} < (l+\epsilon)^{1-\frac{N}{m}} a_N^{\frac{1}{m}}, \quad \forall m \ge N+1.$$

Now the result follows from the fact that $\lim_{m\to\infty} (l \pm \epsilon)^{1-N/m} a_N^{1/m} = l \pm \epsilon$. ///

Corollary: If $a_n > 0$ and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l < 1$, then $\lim_{n\to\infty} a_n = 0$

Corollary: If $a_n > 0$ and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l > 1$, then $a_n \to \infty$.

Problems: (i) $\lim a^{1/n} = 1$, if $a > 0$.

(ii) $\lim n^\alpha x^n = 0$, if $|x| < 1$ and $\alpha \in \mathbb{R}$.

Solution:

(i) Take $a_n = a$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

(ii) If $x \ne 0$, take $a_n = n^\alpha x^n$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim (1 + \frac{1}{n})^\alpha |x| = |x|$.

1.6 Limit superior and limit inferior

Definition 1.6.1. *Let* $\{a_n\}$*be a bounded sequence. Then limit superior of the sequence* $\{a_n\}$*, denoted by* $\limsup\limits_{n\to\infty} a_n$*, is defined as*

$$\limsup_{n\to\infty} a_n := \inf_{k\in\mathbb{N}} \sup_{n\geq k} a_n.$$

Similarly limit inferior of the sequence $\{a_n\}$, denoted by $\liminf\limits_{n\to\infty} a_n$, is defined as

$$\liminf_{n\to\infty} a_n := \sup_{k\in\mathbb{N}} \inf_{n\geq k} a_n.$$

Example 1.6.2. (i) *Consider the sequence* $\{a_n\} = \{0, 1, 0, 1, ...\}$*. Then* $\beta_n = \sup\{a_m, m \geq n\} = 1$ *and* $\alpha_n = \inf\{a_m, m \geq n\} = 0$*. Therefore,* $\liminf a_n = 0, \limsup a_n = 1$*.*

(ii) *Consider the sequence* $\{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, ...\}$*. Then for large* k

$$1 \geq \sup\{a_m, m \geq k\} \geq \lim \frac{k-1}{k}$$

$$0 < \inf\{a_m, m \geq k\} \leq \lim \frac{1}{k}$$

Then by sandwich theorem, we see that $\limsup a_n = 1$ *and* $\liminf a_n = 0$*.*

Lemma 1.6.3.

(i) If $\{a_n\}$*is a bounded sequence, then* $\limsup\limits_{n\to\infty} a_n \geq \liminf\limits_{n\to\infty} a_n$*.*

(ii) If $\{a_n\}$*and* $\{b_n\}$ *are bounded sequences of real numbers and if* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$*, then*

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$$

and

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

(iii) Let $\{a_n\}$*and* $\{b_n\}$ *are bounded sequences of real numbers. Then*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

and

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

Example 1.6.4. *Consider the sequences* $\{(-1)^n\}$ *and* $\{(-1)^{n+1}\}$. *Here* $a_n = (-1)^n$ *and* $b_n = (-1)^{n+1}$. *Also* $\limsup\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} b_n = 1$. *But* $a_n + b_n = 0$ *for all* $n \in \mathbb{N}$ *and hence* $\limsup\limits_{n\to\infty} (a_n + b_n) = 0$. *Thus a strict inequality may hold in (iii) the above Lemma.*

Theorem 1.6.5. *If* $\{a_n\}$*is a bounded sequence, then there exists subsequences* $\{a_{n_k}\}$ *and* $\{b_{n_k}\}$ *such that*

$$\limsup a_n = \lim a_{n_k} \ \ \textit{and} \ \ \liminf a_n = \lim b_{n_k}.$$

Proof. Since $\{a_n\}$is bounded, $\limsup a_n = \alpha$ exists. Then from the definition, for each $k \in \mathbb{N}$ there exits a_{n_k} such that

$$\alpha - \frac{1}{k} < a_{n_k} < \alpha + \frac{1}{k}.$$

Therefore, $a_{n_k} \to \alpha$ as $k \to \infty$. Similarly, one can obtain b_{n_k}. ///

Theorem 1.6.6. *If there exists a subsequence* $a_{n_k} \to t$. *Then* $t \le s := \limsup a_n$.

Proof. Suppose NOT. Then choose $\epsilon > 0$ such that $t - \epsilon > s$. Then we can find N such that

$$n \ge N \implies a_n < t - \epsilon$$

Therefore $|a_n - t| > \epsilon$ for all $n \ge N$. Hence such a sequence cannot have a convergent subsequence. ///

Remark 1.2. *From the above two theorems we can say that the limsup is the supremum of all limits of subsequences of a sequence.*

Remark 1.3. *In case of unbounded sequences, either* $\limsup$ *or* $\limsup$ *or both can approach* ∞. *Even in this case, one can show the existence of subsequences that approach infinity.*

Remark 1.4. *If we can find the limits of* **all** *subsequences of* $\{a_n\}$. *Then* $\limsup$ *is nothing but the supremum of all these limits. Similarly,* $\liminf$ *is the infimum of all these limits.*

Problem Find $\limsup$ and $\liminf$ of $\{a_n\}$where $a_n = \left(1 + (-1)^n + \frac{1}{2^n}\right)^{\frac{1}{n}}$.
Solution: The sequence $\{a_n\}$is bounded and has two convergent subsequences $\{\frac{1}{2}\}$ and $\{(2 + \frac{1}{2^n})^{\frac{1}{n}}\}$. So the two limits are $\frac{1}{2}$ and 1. Therefore, $\limsup a_n = 1$ and $\liminf a_n = \frac{1}{2}$.

Theorem 1.6.7. *If* $\{a_n\}$*is a convergent sequence, then*

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Proof. Let $L = \lim_{n\to\infty} a_n$. Then given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon, \; \forall\, n \geq N.$$

Equivalently $L - \epsilon < a_n < L + \epsilon$, for all $n \geq N$. Thus, if $n \geq N$, $L + \epsilon$ is an upper bound for the set $\{a_k | k \geq N\}$. If $\alpha_k := \sup\{a_k | k \geq n\}$, then we note that $L - \epsilon < \alpha_N \leq L + \epsilon$ and $\alpha_{N+1} < L + \epsilon$, ...,$\alpha_n < L + \epsilon$ for all $n \geq N$ (As α_n is decreasing). Also $a_n > L - \epsilon$, $n \geq N \implies \alpha_n \geq L - \epsilon$, $n \geq N$. Therefore, $\lim \alpha_n = L$. Hence $\limsup a_n = L$. Similarly, one can prove the $\liminf a_n = L$.

///

Theorem 1.6.8. *If $\{a_n\}$is a bounded sequence and if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$, $L \in \mathbb{R}$, then $\{a_n\}$is a convergent sequence.*

Proof. Notice that

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup\{a_k | k \geq n\})$$

and

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf\{a_k | k \geq n\}).$$

Given that $L = \lim \sup_{n\to\infty} a_n$. Thus for $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$|\sup\{a_n, a_{n+1}, ...\} - L| < \epsilon, \; \forall\, n \geq N_1.$$

This implies

$$a_n < L + \epsilon, \; \forall\, n \geq N_1 \tag{1.8}$$

Similarly there exists $N_2 \in \mathbb{N}$ such that

$$|\inf\{a_n, a_{n+1}, ...\} - L| < \epsilon, \; \forall\, n \geq N_2.$$

This implies

$$L - \epsilon < a_n, \; \forall\, n \geq N_2 \tag{1.9}$$

Let $N = \max\{N_1, N_2\}$. Then from (1.8)and (1.9) we get

$$|a_n - L| < \epsilon, \forall\, n \geq N.$$

Thus the sequence $\{a_n\}$converges. ///

Examples 1.6.9. $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. *Assume that* $e = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$.

Solution. *Let* $a_n = \sum\limits_{k=0}^{n} \frac{1}{k!}$ *and* $b_n = \left(1+\frac{1}{n}\right)^n$. *Now,*

$$b_n = \sum_{k=0}^{n} {}_nC_k \left(\frac{1}{n}\right)^k = 2 + \sum_{k=2}^{n} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \leq a_n.(see\ (1.6)$$

This implies

$$\limsup_{n\to\infty}\ b_n \leq \limsup_{n\to\infty}\ a_n = e.$$

Further, if $n \geq m$, *then*

$$b_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} {}_nC_k \left(\frac{1}{n}\right)^k \geq \sum_{k=0}^{m} {}_nC_k \left(\frac{1}{n}\right)^k = 2 + \sum_{k=2}^{m} \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right).$$

Keeping m *fixed and letting* $n \to \infty$, *we get*

$$\liminf_{n\to\infty}\ b_n \geq \sum_{k=0}^{m} \frac{1}{k!}$$

which implies $a_n \leq \liminf\limits_{n\to\infty}\ b_n$. *Hence*

$$e = \liminf_{n\to\infty}\ a_n \leq \liminf_{n\to\infty}\ b_n.$$

Finally we have the following more precise version of theorem 1.6.6

Theorem 1.6.10. *Let* $\{a_n\}$ *be any sequence of nonzero real numbers. Then we have*

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

Proof. The inequality in the middle is trivial. Now we show the right end inequality. Let $L = \limsup |\frac{a_{n+1}}{a_n}|$. W.l.g assume $L < \infty$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon \ \ \forall n \geq N.$$

Then for any $n > N$, we can write

$$\begin{aligned} |a_n| &= \left|\frac{a_n}{a_{n-1}}\right| \left|\frac{a_{n-1}}{a_{n-1}}\right| ... \left|\frac{a_{N+1}}{a_N}\right| |a_N| \\ &< (L+\epsilon)^{n-N} |a_N| \\ &= (L+\epsilon)^n ((L+\epsilon)^{-N} |a_N|). \end{aligned}$$

Now taking $a = ((L-\epsilon)^{-N}|a_N|)$, we have, $|a_n|^{1/n} < (L+\epsilon)a^{1/n}$ for $n > N$. Since $\lim_{n\to\infty} a^{1/n} = 1$, we conclude that $\limsup |a_n|^{1/n} \leq (L+\epsilon)$. Since ϵ is arbitrary, we get the result. Similarly, we can prove the first inequality.

References

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