Tut Sheet - 1

Q3 (a) Given $\{a_n\}$ be a seq. of real numbers.

Let $\{a_{2n}\}_{n=1}^{\infty}$ converge to limit L

$\Rightarrow$ $\{a_{2n-1}\}_{n=1}^{\infty}$ also converges to L

Now let $\varepsilon = \frac{L}{2}$

then, $\exists$ $n_1 \in N$. s.t $|a_{2n} - L| < \varepsilon$ for $\forall n \geq n_1$

Also $\exists$ $n_2 \in N$ s.t. $|a_{2n-1} - L| < \varepsilon$ for $\forall n \geq n_2$

$a_n = a_{2n} + a_{2n-1}$.

Let $n_0 = \max.\{2n_1, 2n_2 - 1\}$.

$|a_n - L| < \varepsilon \quad \forall \; n \geq n_0$.

(b) $\varepsilon > 0$, $|b_n - 0| < \varepsilon/M.$ for $\forall$ $n > n_0$

$|a_n| \leq M. \; \forall \; n \in N.$

$|a_n b_n| \leq \frac{M\varepsilon}{M} = \varepsilon. \quad \forall \; n \geq n_0$

As $\varepsilon \longrightarrow 0$

$\therefore$ $a_n b_n \longrightarrow 0$.

If $\{b\} \not\rightarrow 0$

$\therefore$ $a_n b_n$ will not converge.

Scanned by CamScanner

$r > r - 1/n.$

$r > r_n > r - 1/n$

$\downarrow \quad \Downarrow \quad \downarrow$

$r \quad r$ (By Sandwich theorem)

OR

$r - r_n. \quad < \quad r - \left(r - \frac{1}{n}\right)$

$\Rightarrow |r - r_n| < \frac{1}{n}$

Now $\varepsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ such that.

$\frac{1}{\varepsilon} = n_0$

Q4 (i) $a_n \leq M$ for $\forall n \in \mathbb{N}$

$a_n < a_{n+1}$. for $\forall n \in \mathbb{N}$.

$S_n = \frac{1}{n} \sum_{1}^{n} a_i$

$n+1$

Q5

$s_n \le a_n \;\; \forall\, n \;\; \le \frac{1}{n} \sum_{1}^{n} \max a_i$

(ii). $a_n \to a \Rightarrow s_n \to a$.

$s_n \le a_n \le a$

$\Rightarrow a$ is UB for s_n.

$\Rightarrow \varepsilon > 0$

$|a_n - a| \le \varepsilon$ for $\forall\, n \ge n_0$.

~~$a_n \Longrightarrow s$~~

~~$|s_n - a_n| = |s_n - a + a - a_n|$~~

~~$\le |s_n - a| + |a_n - a|$~~

~~$|s_n - a_n| \le 2\varepsilon$.~~

$|s_n - a| = \left|\frac{1}{n} \sum_{1}^{n} (a_i - a)\right|$

$|s_n - a| \le \left|\frac{1}{n} \cancel{\sum_{1}^{n} (a_n} - a)\right|$

$|s_n - a| \le |a_n - a.| \le 2\varepsilon$

$\Rightarrow |s_n - a| \le 2\varepsilon$.

Q5. $\lim a_n = a$. $\lim b_n = b$

$t_n = \max(a_n, b_n)$ $\quad s_n = \min(a_n, b_n)$

★ $\max\{x, y\} = \frac{x + y + |x - y|}{2}$

Q6 $a_n > 0$ $\quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$. S.t. a_n is decreasing

$$a_{n+1} - a_n = \frac{a_n}{2} + \frac{1}{a_n} - a_n.$$

$$= \frac{1}{a_n} - \frac{a_n}{2}$$

$$= \frac{2 - a_n^2}{2a_n} \qquad a_n \geq \sqrt{2}$$

< 0

2) $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$.

$$= \frac{1}{2}\left(\sqrt{a_n} - \sqrt{\frac{2}{a_n}}\right)^2 + \sqrt{2}.$$

$$\geq \sqrt{2}.$$

3). Let $a = \lim (a_n)$

$\Rightarrow a_n \longrightarrow a$ & $a_{n+1} \longrightarrow a$

$\Rightarrow a = \frac{1}{2}\left(a + \frac{2}{a}\right)$

$\Rightarrow a = \pm\sqrt{2}$. but $a \neq -\sqrt{2}$

$\therefore a = \sqrt{2}$

Q7 For $a \in R$ $x_1 = a$
$$x_{n+1} = \frac{1}{4}(x_n^2 + 3).$$

Observation: $a > 0$

Suppose $x_n \to x$. The possible values of $x = 3, 1$

Case (I), $a > 3$.
$\Rightarrow$ it diverges to ∞.

Case (II) $a = 3$
$\Rightarrow$ it is a constant sequence

Case (III) $1 < a < 3$.
decreasing & bounded below by 1

case (IV) $a = 1$.
$\Rightarrow$ const. seq.

case (V) $0 < a < 1$.
increasing seq & bounded above by 1

Q8. $$\sum_{k=0}^{n} \frac{1}{(n+k)^2}$$

$$\sum_{k=0}^{n} \frac{1}{(n+n)^2} \le \sum_{k=0}^{n} \frac{1}{(n+k)^2} \le \sum_{k=0}^{n} \frac{1}{n^2}$$

$$\frac{1}{4n} < \sum_{k=0}^{n} \frac{1}{(n+k)^2} \le \frac{1}{n}.$$

$\frac{1}{4n} \to 0$, $\frac{1}{n} \to 0$

$\Downarrow$

0.

Q9 1) $a_n = \sum_{k=1}^{n} \frac{1}{k!}$

$$a_{n+1} - a_n = \frac{1}{(n+1)!} - \frac{1}{n!}$$

$$= - \left[\frac{1}{n!} - \frac{1}{(n+1)!} \right]$$

$$= - \frac{1}{(n+1)!}$$

$$\Rightarrow |a_n - a_m| \le \sum_{k=m}^{n} \frac{1}{(k+1)!} \le \frac{n-m+1}{\cancel{k+1}\, m!} < \varepsilon$$

2) $a_1 = 1.$

$$a_{n+1} = \left(1 + \frac{(-1)^n}{2^n}\right) a_n.$$

$$a_{n+1} = \left(\left[\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2^2}\right)\left(1 - \frac{1}{2^3}\right) \cdots \left(1 + \frac{(-1)^n}{2^n}\right)\right]^{\frac{1}{n}}\right)^n$$

$$\le \left(\frac{1}{n}\left(n + \sum_{k=1}^{n} \frac{(-1)^k}{2^k}\right)\right)^n$$

$$= \left(1 + \frac{1}{n} \sum_{k=1}^{n} \frac{(-1)^k}{2^k}\right)^n$$

$$< \left(1 + \frac{1}{n}\right)^n < C$$

$$|a_{n+1} - a_n| = \frac{|a_n|}{2^n} < \frac{C}{2^n}$$

Scanned by CamScanner

Q10. $a_n = (-1)^n\left(1+\frac{1}{n}\right), n \in N.$

$\alpha_k = \sup\{a_k, a_{k+1}, ---\}$

$= \sup.\{(-1)^n\left(1+\frac{1}{k}\right) ; n \geq k\}$

$= \begin{cases} 1+\frac{1}{k} & \text{if } k \text{ is even.} \\ 1+\frac{1}{k+1} & \text{if } k \text{ is odd.} \end{cases}$

$\alpha_k \longrightarrow 1$

$\lim_{n\to\infty} \sup a_n = 1.$

$\beta_k = \inf\{a_k, a_{k+1}, ---\}.$

$\inf.\{(-1)^n\left(1+\frac{1}{n}\right) ; n \geq k\}$

$= \begin{cases} -1-\frac{1}{k} & \text{if } k \text{ is odd.} \\ -1-\frac{1}{(k+1)} & \text{if } k \text{ is even.} \end{cases}$

$\beta_k \to -1.$

$\lim_{n\to\infty} \inf a_n = -1.$

Sheet - 2

Q1 $\sum a_n \rightarrow$ convergent series.

$a_n > 0$

$a_n \geq a_{n+1}$ s.t. $\lim_{n\to\infty} 2^n a_{2^n} = 0$

Q2 $0 < a_n \leq 1$.

$0 < x < 1$

s.t. $\sum a_n x^n$ converges.

$\sum a_n x^n \leq \sum x^n$.

and. $\sum x^n$ converges.

Q3 1) $\sum_{n=2}^{\infty} \frac{1}{n \cdot (\log n)^p}$

$\Rightarrow \sum \frac{\cancel{2^n}}{\cancel{2^n} (\log 2^n)^p} = \sum \frac{1}{n^p (\log 2)^p}$

2) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^x}$. $x \in R$

$a_n = \frac{1}{(\log n)^x}$.

$\sum 2^n a_{2^n} = \sum \frac{2^n}{(n \log 2)^x} = \sum \frac{2^n}{n^x \cdot \log^x 2}$

Root test. $\left(\frac{2^n}{n^x}\right)^{1/n} = \frac{2}{(n^{1/n})^x} \to 2.$

$\Rightarrow$ The series $\sum \frac{2^n}{n^x}$ diverges

$\Rightarrow$ The series $\frac{1}{\log^x 2} \sum \frac{2^n}{n^x}$ diverges.

Q4 (c) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

$$\sum 2^n a_{2^n} = \sum \frac{2^n (2^n)^{1/2}}{(2^n)^2+1} = \sum \frac{2^{3n/2}}{2^{2n}+1}$$

$$\sum \frac{2^{3n/2}}{2^{2n}+1} \leq \sum \frac{2^{3n/2}}{2^{2n}} = \sum \frac{1}{2^{n/2}}$$

We know that $\sum \frac{1}{2^{n/2}}$, converges

$\Rightarrow \sum \frac{2^{3n/2}}{2^{2n}+1}$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges.

OR.

By Comparison test:

(f) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$

~~$2^n a_n = \sum 2^n \cdot 2^n$~~

$a_n = \frac{\sqrt[n]{n}}{n^2}$ $b_n = \frac{1}{n^2} \to 0$

$\frac{a_n}{b_n} = n^{1/n} \to 1$ i.e. independent of n

$\Rightarrow \sum a_n$ converges because $\sum b_n$ converges

Q5 (a) $\sum_{1}^{\infty} \cancel{\frac{n!}{10^n}} \frac{n^{\sqrt{2}}}{2^n}$.

Root test: $\left(\frac{n^{\sqrt{2}}}{2^n}\right)^{1/n} = \frac{n^{\frac{\sqrt{2}}{n}}}{2} \Rightarrow \frac{(n^{1/n})^{\sqrt{2}}}{2}$.

$\to \frac{1}{2}$.

$\therefore$ The series converges.

(b) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ ~~Root test.~~ $\cancel{\frac{(n!)^{1/n}}{10}}$

Ratio test.

$\frac{a_n}{a_{n+1}} = \frac{n!}{10^n} \frac{10^{n+1}}{\times (n+1)!}$

$= \frac{10}{n+1} \to 0$

Q6. b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

~~Ro~~ Root test: $\limsup a_n^{1/n}$

$= \frac{n!^{1/n}}{n}$

Ratio test: $\frac{a_{n+1}}{a_n}$

$= \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$

$= \frac{\cancel{(n+1)}\, n^n}{(n+1)^n \cancel{(n+1)}}$

$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$ $\quad e^{n \times \frac{n}{n+1}-n-1}$ $\quad e^{n}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

c) $\sum_{1}^{\infty} \frac{n^{n^2}}{(n+1)^{n^2}}$

Root test: $\frac{n^n}{(n+1)^n} = \frac{1}{e} < 1$

$\Rightarrow \frac{n^{n^2}}{(n+1)^{n^2}}$ converges

Q7. a) $\sum_{n=0}^{\infty} \sin \frac{\pi}{(2^n)} < \sum_{n=0}^{\infty} \frac{\pi}{2^n}$

We know $\sum_{n=0}^{\infty} \frac{\pi}{2^n}$ converges

Q8 a) $\sum \frac{n^2 n!}{(n^3+1) n^n}$

Ratio test $\frac{n^2 n!}{(n^3+1)(n^n)}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 (n+1)!}{((n+1)^3+1)\ \cancel{(n+1)}(n+1)^n} \times \frac{n^3+1 \times n^n}{n^2 n!}$$

$$\frac{(n^3+1)}{((n+1)^3+1)} \times \left(\frac{n}{n+1}\right)^{n-2}$$

Sheet - 3

Q3 $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 1-x & x \notin \mathbb{Q} \end{cases}$ is cts. only at $\frac{1}{2}$.

$\Rightarrow$ Choose sequences $\{p_n\}$ & $\{q_n\}$ s.t. $\{p_n\} \subseteq \mathbb{Q}$, $\{q_n\} \subseteq \mathbb{Q}^c$, $p_n \to \frac{1}{2}$ & $q_n \to \frac{1}{2}$.

$$f(p_n) = p_n \to \tfrac{1}{2}$$
$$f(q_n) = 1 - q_n \to \tfrac{1}{2}.$$

The main series consist of irrational & rational numbers.
Both the ~~subs~~ sequences $\{p_n\}$ and $\{q_n\}$ converges to same limit equal to $\frac{1}{2}$
Also $f(p_n)$ & $f(q_n) \to \frac{1}{2}$
$\therefore$ The main series, i.e. $\{x_n\}$
$f(x_n)$ should also converge to $\frac{1}{2}$

At other points, whenever $x \neq \frac{1}{2}$ say $x = a$,
The series $\{p_n\} \to a$ & $\{q_n\} \to 1-a$

$\therefore$ Two subsequences converge at different values. $\therefore$ The ~~ser~~ seq. $\{x_n\}$ is non-converging

Let $\varepsilon > 0$
Choose $\{(x_{n_k})\}$ from $\{x_n\}$ s.t. $x_{n_k} \in \mathbb{Q}$ $\forall k \in \mathbb{N}$. Similarly choose subseq. $\{y_{n_s}\}$ from $\{x_n\}$ s.t. $y_{n_s} \in \mathbb{Q}^c$ $\forall s \in \mathbb{N}$.

$$x_{n_k} \to \tfrac{1}{2} \Rightarrow f(x_{n_k}) \to \tfrac{1}{2}$$
$$y_{n_s} \to \tfrac{1}{2} \Rightarrow f(y_{n_s}) \to \tfrac{1}{2}$$

Now, $\exists k_0 \in \mathbb{N}$ s.t.

$$|f(x_{n_k}) - 1/2| < \varepsilon \quad \forall \; k \geq k_0.$$

Also $\exists \; s_0 \in \mathbb{N}$ s.t.

$$|f(y_{n_s}) - \tfrac{1}{2}| < \varepsilon \quad \forall \; s \geq s_0.$$

let $n_0 = \max.(n_{k_0}, n_{s_0})$

$$\Rightarrow |f(x_n) - 1/2| < \varepsilon \quad \forall \; n \geq n_0.$$

Q4 (o) $e^{x^2} \sin(x^2)$ $\quad (0,1)$.

Define.

$$\tilde{f}(x) = \begin{cases} \lim\limits_{x \to 0^+} e^{x^2} \sin(x^2) & x = 0. \\ f(x) & x \in (0,1). \\ \lim\limits_{x \to 1^-} e^{x^2} \sin(x^2) & x = 1. \end{cases}$$

$$|x - y| < \delta \Rightarrow |\tilde{f}(x) - \tilde{f}(y)| < \varepsilon$$

For $x, y \in (0,1)$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

~~(g)~~ $\sqrt{x} \sin x$. $\quad x > 0$

$x_n = 2n^2\pi + \frac{1}{n}$

$y_n = 2n^2\pi$. $\quad \Rightarrow x_n - y_n \longrightarrow 0$.

$f(x_n) = \sqrt{2n^2\pi + \frac{1}{n}} \sin \frac{1}{n}$

$f(y_n) = \sqrt{2n^2\pi} \sin(2n^2\pi) = 0$.

$\Rightarrow f(x_n) - f(y_n) \not\longrightarrow 0$ as $n \to \infty$.

~~d)~~ $\sin(x^2)$

$x_n = \sqrt{2n\pi + \frac{\pi}{2}}$ $\quad y_n = \sqrt{2n\pi}$.

We know. $x_n - y_n = \sqrt{2n\pi + \frac{\pi}{2}} - \sqrt{2n\pi} \to 0$.

~~by~~ but $\sin x_n^2 = 1$

$\sin y_n^2 = 0$

$\Rightarrow$ whenever $|x_n - y_n| < \delta$.

~~In particul~~ $\Rightarrow |x_n - y_n| \to 0$.

$|f(x_n) - f(y_n)|$

$= |\sin x_n^2 - \sin y_n^2| \to 1 \neq 0$

Therefore it is not uniformly continuous.

Q5 (a) $f(x)\begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0. \end{cases}$ $\quad f'(x) = \begin{cases} e^{-1/x^2} x.\frac{2}{x^3} & x \neq 0. \\ 0 & x = 0 \end{cases}$

~~e^{-1/x^2}~~ ~~x^3~~ ~~$e^{-1/x^2}/x$~~

$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{x\to 0} \frac{f(h) - f(0)}{h}$.

$= ~~f(e^{-1}~~ \lim_{x\to 0} \frac{e^{-1/h^2}}{h}$.

$= \lim_{x\to\infty} xe^{-x^2} \to 0$

$\Rightarrow$ Yes it is differentiable.

(b) $e^{-|x|}$, $x \in \mathbb{R}$.

$$f(x) = \begin{cases} e^{x} & x < 0. \\ e^{-x} & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} e^{x}. & x < 0 \\ -e^{-x} & x > 0 \end{cases}.$$

OR $\lim_{h \to 0^+} \frac{e^{-h} - 1}{h} = -1$.

& $\lim_{h \to 0^-} \frac{e^{h} - 1}{h} = 1$.

$\Rightarrow$ left hand der. $\neq$ right hand derivative

$\therefore$ It is not differentiable.

Q6. (c) $f(x) \begin{cases} x^2 \cdot \ln \frac{1}{|x|}. & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$f(x) = \begin{cases} x^2 \ln\left(-\frac{1}{x}\right) & x < 0 \\ 0 & x = 0 \\ x^2 \cdot \ln \frac{1}{x} & x > 0. \end{cases}$$

$$f'(x) = \begin{cases} 2x \ln\left(-\frac{1}{x}\right) + \cancel{x^2} \frac{1}{\left(-\frac{1}{x}\right)} \frac{1}{\cancel{x^2}}. & x < 0 \\ 0 & x = 0. \\ 2x \ln\left(\frac{1}{x}\right) + \cancel{x^2} \times \frac{1}{1/x} \times \frac{-1}{\cancel{x^2}} & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \ln\left(-\frac{1}{x}\right) - x. & x < 0 \\ 0 & x = 0 \\ 2x \ln\left(\frac{1}{x}\right) - x & x > 0 \end{cases}$$

LHL $= 0$ $\quad f'(0) = 0$ $\quad$ RHL $= 0$

(a) $\begin{cases} x^3 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f'(x) \begin{cases} x^{\cancel{3}} \cos \frac{1}{x} \times \frac{-1}{\cancel{x^2}} + \sin \frac{1}{x} \cdot x \cdot 3x^2 & x \neq 0. \\ 0 & x = 0. \end{cases}$

$\Rightarrow f'(x) = \begin{cases} -x \cos \frac{1}{x} + 3x^2 \sin \frac{1}{x}. & x \neq 0 \\ 0 & x = 0. \end{cases}$

$\underset{x \to 0}{LHL}(f'(0)) = f'(0) = \underset{x \to 0}{RHL} f'(x)$

Q7. $\alpha = \sup_{R} |f'(x)| < 1$ $\quad s_0 \in R.$

define. $s_n = f(s_{n-1})$

To Proof : $\{s_n\}$ is a convergent seq.

Solution : $|f(s_n) - f(s_{n-1})| = |f'(c)||s_n - s_{n-1}|$

$|f(s_n) - f(s_{n-1})| \leq \alpha |s_n - s_{n-1}|$

$\Rightarrow |s_{n+1} - s_n| \leq \alpha |s_n - s_{n-1}|$

$\Rightarrow \{s_n\}$ is a cauchy sequence.

Q f is differentiable on R.

$|f(x) - f(y)| \leq (x-y)^2$

$\frac{|f(x) - f(y)|}{|x-y|} \leq |x-y|$

Scanned by CamScanner

Q9 (i) $\lim_{x \to 0} \frac{e^x - (1+x)}{x^2}$

$$L = \frac{\cancel{1} + \cancel{\frac{x}{1!}} + \frac{x^2}{2!} + \frac{x^3}{3!} - - - - \cancel{(1)} - \cancel{x}}{x^2}$$

$$L = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} - - -$$

as $x \to 0$.

$$L = \frac{1}{2!} = \frac{1}{2}$$

(ii) $L = \lim_{t \to 0} \frac{1 - \cos t - t^2/2}{t^4}$.

Using L'Hospitals rule,

$$L = \lim_{t \to 0} \frac{\frac{d}{dt}(1 - \cos t - t^2/2)}{\frac{d}{dt}(t^4)}$$

$$= \lim_{t \to 0} \frac{\sin t - t}{4t^3}$$

again L'Hospital,

$$L = \lim_{t \to 0} \frac{\cos t - 1}{4 \cdot 3 \cdot t^2}$$

$$L = \lim_{t \to 0} \frac{-\sin t}{4 \cdot 3 \cdot 2 \cdot t}$$

$$L = \lim_{t \to 0} \frac{-\cos t}{4!}$$

$$L = \frac{-1}{4!}$$

$\cancel{\frac{e^x - e}{2}}$

(iii) $\lim_{x\to\infty} x^2(e^{-1/x^2}-1)$

$$= \lim_{x\to\infty} \frac{e^{-1/x^2}-1}{1/x^2} = -1$$

Q10. ~~appr~~ approximate $\sin x$, when error is of magnitude not greater than 5×10^{-4} & $|x| < 3/10$

$$R_n = \frac{f^{n+1}(c)}{(n+1)!}\, x^{n+1}$$

$$|R_n(x)| = \frac{|f^{n+1}(c)|\,|x|^{n+1}}{(n+1)!} \le \frac{1}{(n+1)!}\left(\frac{3}{10}\right)^{n+1} \le 5\times10^{-4}$$

For $n=1$. Error $= \frac{1}{2}\cdot\frac{9}{10^2}$

$n=2$ 〃 $= \frac{1}{6}\times\frac{27}{10^3}$

$n=3$ Error $= \frac{1}{24}\times\frac{81}{10^4}$

$$\therefore P_3(x) = x - \frac{x^3}{3!}$$

Q11 Estimate the error in approximation of $\sinh x$ by $x+\frac{x^3}{3!}$ when $|x| < \frac{1}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$$R_n = \frac{f^{n+1}(c)}{(n+1)!}(x)^{n+1}$$

$$R_3 = \frac{f^4(c)}{4!}x^4 = \frac{e^x - e^{-x}}{24\times2}\times x^4$$

$$= \frac{|\sinh c|}{24\times2^4} < \frac{\sinh 1/2}{24\times2^4}$$

for $|c| < \frac{1}{2}$.

Scanned by CamScanner

Q12 a) $\sum_{n=0}^{\infty} (n+1+2^n)x^n$

$\alpha = \limsup_{n\to\infty} (a_n)^{1/n} = \cancel{\frac{(n+2+2^{n+2})x}{(n+1+2^n)^{1/n}}}$

$\frac{1}{(2^n)^{1/n}} \le \beta = \lim_{n\to\infty} \frac{1}{(n+1+2^n)^{1/n}} \le \frac{1}{(3\cdot 2^n)^{1/n}}$

$\Rightarrow R = \frac{1}{2}$

b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{a^n}, \quad a \neq 0.$

$\alpha = \limsup_{n\to\infty} \left(\frac{x^{2n}}{a^n}\right)^{1/n} = \frac{x^2}{a} = \frac{|y|}{a}$

where $y = x^2$.

it converges for

$\alpha < 1$

$|y| < a$ —(1) $\quad \cancel{|y|\beta < 1}$ $\quad$ ~~where~~ $\cancel{\beta = \frac{1}{a}}$

$\cancel{x\beta\ 1}$

$|x^2| < a$

$x < \sqrt{a}$

$\Rightarrow R = \sqrt{a}$

$\cancel{|y| < \frac{1}{\beta}}$ —(2)

~~Comparing (2) to (1)~~

$\cancel{\beta = \frac{1}{a}}$

$\cancel{R = a}$

for x $R = \sqrt{a}$

Scanned by CamScanner

(c). $$\sum_{n=0}^{\infty} \frac{x^n}{n!\, n^n}$$

$$\alpha = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

it converges for $\alpha < 1$.

$$\Rightarrow \lim_{n \to \infty} \left| \frac{\frac{x^{\cancel{n+1}}}{(n+1)\cancel{!}\,(n+1)^{n+1}}}{\frac{\cancel{x^n}}{\cancel{n!}\; n^n}} \right| < 1.$$

$$\Rightarrow \lim_{n \to \infty} \frac{|x|\, \cancel{(n+1)} \, n^n}{(n+1)^{n+1} \cancel{n}} < 1.$$

$$\lim_{n \to \infty} |x| \left(\frac{n}{(n+1)}\right)^n \frac{}{(n+1)^2} < 1$$

$$\lim_{n \to \infty} |x|\, e^{n\left(\frac{n+1-n}{n}\right)} < 1.$$

$$|x|\, e < 1$$

$$|x| < \frac{1}{0}$$

$$\Rightarrow R = \frac{1}{\cancel{e}}\ \infty$$

(d) $$\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n$$

$$\alpha = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

$$= \lim_{n \to \infty} \left| \frac{\frac{(n+1)!\; x^{\cancel{n+1}}}{(n+1)^{n+1}}}{\frac{\cancel{n!}\; \cancel{x^n}}{n^n}} \right| = \lim_{n \to \infty} \left| x \cdot \left(\frac{n}{n+1}\right)^n \right|.$$

$$= \frac{x}{e}$$

For convergence $x < 1$

$\therefore \frac{|x \cdot|}{e} < 1.$

$|x| < e.$

$\Rightarrow \quad e = R$

Q13 a) $\frac{1}{(1+x)}$

Taylor series : $f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!} (x-a)^{n}$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 \ldots$

$= \sum_{n=0}^{\infty} (-1)^n x^n. \qquad R = 1$

b) $\sinh x = \frac{e^{x} - e^{-x}}{2}$

$= \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \ldots\right)}{2}$

$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \ldots$

$= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$

$R \Rightarrow \frac{x^{2k+3}}{(2k+3)!} \times \frac{(2k+1)!}{x^{2k+1}} < 1.$

$|x| < 2k+2$

3) $e^{x}\sinh x = \frac{e^{x}(e^{x}-e^{-x})}{2}$

$= \frac{e^{2x}-1}{2}$

$= \frac{\cancel{1} + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} \cdots - \cancel{1}}{2}$

$e^{x}\sinh x = \sum_{n=0}^{\infty} \frac{2^{n} x^{n+1}}{(n+1)!}$ $R = \frac{1}{2}$ ∞

4) $x\sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right)$

$= \frac{x^2}{1!} - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^7}{7!} \cdots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2(n+1)}}{(2n+1)!}$ $R = \infty$

Q14 (a) $\tan^{-1}x$:

~~$d(\tan^{-1}x)$~~ $f'(x) = \frac{1}{1+x^2}$ $f''(x) = \frac{-1\times 2x}{(1+x^2)^2}$

$\Rightarrow f'(x) = 1 + \frac{-\cancel{2}x \cdot x\, x}{(1+x^2)^2 \times \cancel{2}!} +$

$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 \cdots$

$f(x) = \int (1 - x^2 + x^4 - x^6 \cdots)$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots$

$= \sum (-1)^n \frac{x^{2n+1}}{2n+1}$ $B = 1 = R$

Scanned by CamScanner

* $(n!)^{1/n}$ diverges to ∞

b). $f(x) = \sin^{-1} x$:

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$= (1-x^2)^{-1/2}$$

$$= \cancel{1} \cancel{+} \cancel{\frac{x^{2}}{2}} \cancel{+} \cancel{\frac{3}{8}} x^{4}$$

$$= 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5}{16}x^6 - - -$$

$$= 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}\times\frac{3}{2}\cdot x^4}{2!} + \frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times x^6}{3!} - - .$$

$$= \sum \frac{\cancel{1\times3\times5\ldots}\cdot(2n-1)!}{n!\,2^n\,n!} x^{2n}$$

(c) $\sinh^{-1} x$.

$\sinh y = x$. $\cosh^2\theta - \sinh^2\theta = 1$

$$\frac{e^y - e^{-y}}{2} = x.$$

$$\left(\frac{e^y + e^{-y}}{2}\right) y' = 1$$

$$y' = \frac{2}{e^y + e^{-y}} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$$

Now $\frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16}$

$$\sinh^{-1} x = \int \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} - - - \right.$$

Sheet - 4

Q1 (a) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2-y^2}$ $\quad * \; y = \frac{x}{1+x}$

$$\frac{x^3 + \frac{x^3}{(1+x)^3}}{x^2 - \frac{x^2}{(1+x)^2}} = \frac{x\{(1+x)^3+(1)\}}{((1+x)^2-1)(1+x)}$$

(You can also use polar coordinates)

$$= \lim_{x\to0} \frac{x(1+x^3+3x^2+3x+1)}{(x^2+2x)(1+x)}$$

$$= \lim_{x\to0} \frac{x^4+3x^3+3x^2+2x}{x^3+3x^2+2x}$$

$$= \lim_{x\to0} \frac{x^3+3x^2+3x+2}{x^2+3x+2}$$

$$L = 1$$

but try with $y = mx$.

$L = 0$

$\therefore$ limit does not exist

(b) $L = \lim_{(x,y)\to(0,0)} xy \frac{(x^2-y^2)}{(x^2+y^2)}$

with $y = mx$.

$$L = \lim_{x\to0} mx^2 \frac{(1-m^2)}{(1+m^2)} = 0$$

~~with~~ $y = u^2 \cancel{\cos\theta\sin\theta}$ ~~$(\cos^2\theta - \sin^2\theta)$~~

(d) $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2} < \frac{xy}{x^2+y^2}$

put $y = mx$. $\lim_{(x,y)\to(0,0)} \frac{\sin(mx^2)}{(1+m^2)mx^2} \times m$

$\Rightarrow \lim_{x,y\to 0,0} \frac{\sin(mx^2)}{(mx^2)} \times \frac{m}{1+m^2}$

$\Rightarrow \frac{m}{1+m^2} \neq 0$

$\therefore$ this limit does not exist.

Q2. (a) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = L$

Put $y = mx^{1/3}$

$\Rightarrow L = \lim_{x,y\to 0,0} \frac{x \cdot m^3 x \cdot x}{x^2 + m^6 x x^2}$

$\Rightarrow L = \frac{m^3}{1+m^6} \neq 0$

$\therefore$ It is not continuous.

(c) $\frac{\sin^2(x-y)}{|x|+|y|}$ $(x,y) = (0,0)$

$0 < x < \delta$ & $0 < y < \delta$

take $\varepsilon = 2\delta$

We know. $x - y \leq |x| + |y|$

$\Rightarrow L = \lim_{(x,y)\to(0,0)} \frac{\sin^2(x-y)}{|x|+|y|} \leq \lim_{(x,y)\to(0,0)} \frac{\sin^2(|x|+|y|)}{|x|+|y|}$

~~Also~~

$\leq \lim_{(x,y)\to(0,0)} \frac{(|x|+|y|)^2}{|x|+|y|}$

$\Rightarrow$ continuous $\leq |x| + |y| \leq 2\delta\ 2\varepsilon$

d) $L = \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$

Put $x - y = xy$.

$x = xy + y$

$y = \dfrac{x}{x+1}$

$\Rightarrow \cancel{\dfrac{x^2\left(\frac{x}{x+1}\right)^2}{x^2\left(\frac{x}{x+1}\right)^2 + }}$ $\Rightarrow L = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2 + m^2x^2y^2}$

$= \dfrac{1}{1+m^2} \neq 0.$

$\Rightarrow$ discontinuous.

Q3 (a) $f(x) = \begin{cases} x\sin\frac{1}{x} + y\sin\frac{1}{y} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$

$f_x(x,y) = \lim\limits_{h\to0} \dfrac{f(x+h,y) - f(x,y)}{h}$

$\Rightarrow f_x(0,0) = \lim\limits_{h\to0} \dfrac{f(h,0) - f(0,0)}{h}$

$= 0.$

similarly $f_y = 0$

If f is differentiable at $(0,0)$

$f(h,k) - f(0,0) = hf_x(0,0) + kf_y(0,0) + \varepsilon_1 h + \varepsilon_2 k.$

$\Rightarrow f(h,k) = \varepsilon_1 h + \varepsilon_2 k$ where ε_1 & $\varepsilon_2 \to 0$ as $h,k \to 0$

If we choose $h = k$,

$f(h,h) = h(\varepsilon_1 + \varepsilon_2)$

$2h\sin\frac{1}{h} = \cancel{h}(\varepsilon_1 + \varepsilon_2)$

$2\sin\frac{1}{h} = \varepsilon_1 + \varepsilon_2 \to 0$ as $h \to 0$

which is absolutely absurd, which means our assumption was wrong.

$\therefore f(x,y)$ is not differentiable at $(0,0)$

(b) $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ $\quad x^2+y^2 \neq 0$.

0 $\quad$ otherwise.

We know, if $\frac{\Delta f - df}{\rho} \to 0$ when $\rho \to 0$

then f is differentiable.

$$\therefore \frac{f(h,k) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \frac{f(h,k)}{\sqrt{h^2+k^2}} = \frac{hk}{h^2+k^2} = \frac{hk}{\rho^2} \not\to 0$$

as $\rho \to 0$

$\therefore$ it is not differentiable.

(c) $f(x,y) = \frac{x^6 - 2y^4}{x^2+y^2}$ $\quad x^2+y^2 \neq 0$.

$$f_x(0,0) = \frac{f(h,0)}{h} = h^3 = 0.$$

$$f_y(0,0) = \frac{f(0,k)}{k} = -2k = 0 = 0$$

$$\frac{\Delta f - df}{\rho} = \frac{f(h,k)}{\rho} = \frac{h^6 - 2k^4}{\rho^3}, \text{ let } h=k.$$

$\therefore \quad \frac{\Delta f - df}{\rho} = \frac{h^6 - 2h^4}{2^{3/2} h^3} \to 0$ as $h, k \to 0$.

$\Rightarrow f(x,y)$ is differentiable.

Q4. Wt $f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2}$ $\quad f(x,0)=0$

$\hat{p} = (p_1, p_2)$ s.t. $\sqrt{p_1^2 + p_2^2}$

$D_{\hat{p}} f(0,0) = \lim_{h\to 0} \frac{f(hp_1, hp_2)}{h}$

$= \frac{\cancel{h}p_2}{|\cancel{h}p_2|}\sqrt{p_1^2+p_2^2}$

$= \frac{p_2}{|p_2|} = \operatorname{sgn}(p_2)$

$\Rightarrow$ take $\hat{p}_0 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$D_{\hat{p}} f(0,0) = p_1 f_x(0,0) + p_2 f_y(0,0)$

$\Rightarrow$ LHS $= 1$ RHS $= 0$.

$\Rightarrow$ LHS $\neq$ RHS.

$\Rightarrow f$ is not differentiable at $(0,0)$

Q5 $f(x,y) = \big||x| - |y|\big| - |x| - |y|$

f is continuous at $(0,0)$ as modulus fn^c are continuous.

$D_{\hat{p}} f(0,0) = \lim_{h\to 0} \frac{f(hp_1, hp_2)}{h}$

$\Rightarrow \lim_{h\to 0} \frac{\big||hp_1| - |hp_2|\big| - |hp_1| - |hp_2|}{h}$

$= \left(\big||p_1| - |p_2|\big| - |p_1| - |p_2|\right)\frac{|h|}{h}$

Scanned by CamScanner

This limit will only exist if sign doesn't change. which will only happen if

$$||p_1| - |p_2|| = |p_1| + |p_2|.$$

which only occur when $p_1 = 0$ or $p_2 = 0$

No differentiable

Q6 $f(x,y) = \frac{1}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$

gradient of f:

$$\nabla f = \left(\frac{1}{\cancel{2}}\frac{(\cancel{2}x)}{x^2+y^2} + \frac{1 \times -\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}, \frac{1}{\cancel{2}}\frac{\cancel{2}y}{x^2+y^2} + \frac{1/x}{1+\left(\frac{y}{x}\right)^2}\right)$$

$$= \left(\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}, \frac{y+x}{y^2+x^2}\right)$$

$$= \left(\frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2}\right)$$

at $(1,3)$

$$\nabla f(1,3) = \left(\frac{-2}{10}, \frac{4}{10}\right)$$

$$= \left(\frac{-1}{5}, \frac{2}{5}\right)$$

$\hat{p}$ ~~unit vector~~ $= \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$$D_{\hat{p}} f(x,y) = p_1 f_x(x,y) + p_2 f_y(x,y)$$

$$= \frac{-1}{\sqrt{5}}\left(\frac{x-y}{x^2+y^2}\right) + \frac{2}{\sqrt{5}}\left(\frac{x-y}{x^2+y^2}\right).$$

$$= \frac{x-y}{\sqrt{5}(x^2+y^2)}$$

$e^{-\frac{(x^2+y^2)}{5}}$ $100y$ + $50y^2 e^{-\frac{x^2+y^2}{5}}, \frac{-2y}{5}$ $50y^2 x \frac{-2}{5}$

Q7 $\nabla f(2,1)$ when $f = 50y^2 e^{-\frac{1}{5}(x^2+y^2)}$

$\nabla f(x,y) = \left(50y^2 e^{-\frac{1}{5}(x^2+y^2)} \left(\frac{-2x}{5}\right), e^{-\frac{1}{5}(x^2+y^2)}\left(100y - \frac{20y^3}{5}\right) \right)$

$\nabla f(2,1) = \left(50e^{-1} \left(\frac{-4}{5}\right), e^{-1}(100-20) \right)$

$= \left(\frac{-40}{e}, \frac{80}{e} \right)$

$\therefore$ it will move in the direction

$\hat{p} = \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$

Q8 $z = f(x,y)$ $\nabla z = (5, -12)$

$\rightarrow f_x = 5$ $f_y = -12$

$f(11,11) = f(9,15) + f_x(9,15)(11-9) + f_y(9,15)(11-15)$

$= 17 + 5(2) + (-12)(-4)$

$= 17 + 10 + 48$

$f(11,11) = 75$

Q9 $f(9,3) = f(8,3) + f_x(8,3)(9-8)$

$= 24 - 3.4$

$= 20.6$

$f(8,5) = f(8,3) + f_y(8,3)(5-3)$

$= 24 + (4.2)2 = 24 + 8.4$

$= 32.4$

$f(9,5) = f(8,3) + f_x(8,3)(9-8) + f_y(8,3)(5-3)$

$= 24 - 3.4 + 8.4$

$= 29$

Q10 $f(x,y) = e^{-x^2-2y^2}$ $\quad f(0,0) = 1$

$f_x(x,y) = -2x e^{-x^2-2y^2}$ $\quad f_x(0,0) = 0$

$f_y(x,y) = -4y(e^{-x^2-2y^2})$ $\quad f_y(0,0) = 0$

$f_{xx}(x,y) = (4x^2 - 2)e^{-x^2-2y^2}$ $\quad f_{xx}(0,0) = -2.$

$f_{yy}(x,y) = (16y^2 - 4)e^{-x^2-2y^2}$ $\quad f_{yy}(0,0) = -4.$

$f_{xy}(x,y) = 8xye^{-x^2-2y^2}$ $\quad f_{xy}(0,0) = 0$

Now $f(0+h, 0+k) = f(h,k)$

$$= \frac{f(0,0)}{0!} + \frac{(hf_x(0,0) + kf_y(0,0))}{1!}$$

$$+ \frac{1}{2!}\left(h^2 f_{xx}(0,0) + k^2 f_{yy}(0,0) + 2hk f_{xy}(0,0)\right)$$

$$= 1 + \frac{(h \times 0 + k \times 0)}{1!} + \frac{1}{2!}\left(h^2(-2) + k^2(-4) + 0\right)$$

$$= 1 - h^2 - 2k^2$$

$\therefore$ Taylor polynomial (Quadratic)

$$= 1 - x^2 - 2y^2$$

Sheet - 5

Q1 $f = e^x \sin y$ $f_x = e^x \sin y$ $f_y = e^x \cos y$

$f_{xy} = +e^x \cos y$ $f_{xx} = e^x \sin y$ $f_{yy} = -e^x \sin y$

$f_{yx} = e^x \cos y$

Quadratic approximation:

$$f(0+h, 0+k) = f(h,k) = \frac{f(0,0)}{0!} + \frac{h f_x(0,0) + k f_y(0,0)}{1!}$$

$$+ \frac{1}{2!}\left[h^2 f_{xx}(0,0) + k^2 f_{yy}(0,0) + 2hk f_{xy}(0,0)\right]$$

$$f(h,k) = \frac{0}{0!} + \left(\frac{0 + k(1)}{1!}\right) + \frac{1}{2!}\left[0 + 0 + 2hk.(1)\right]$$

$$= k + hk$$

$\therefore$ Quadratic polynomial approximation

$$f(x,y) = y + xy$$

$$R_n(x,y) = \frac{1}{3!}\left[h^3 f_{xxx}(x,y) + 3h^2k f_{xxy}(x,y) + 3hk^2 f_{xyy}(x,y) + k^3 f_{yyy}(x,y)\right]$$

Error $= R_n(0.1, 0.2)$

$f_{xxx} = e^x \sin y = 0$

$f_{xyy} = -e^x \sin y = 0$

$f_{xxy} = e^x \cos y = 1$

$f_{yyy} = -e^x \cos y = -1$

$\Rightarrow M = f_{xxy}(0,0) = 1$

$$\text{Error} = \frac{M}{3!}\left(|x - x_0| + |y - y_0|\right)^3 = \frac{1}{6}(0.3)^3 = \frac{0.027}{6} = 0.0045$$

Scanned by CamScanner

Q2. $f(x,y) = \sin x \sin y$

$f_x(x,y) = \cos x \cdot \sin y$

$f_y(x,y) = \sin x \cdot \cos y$

Critical point $\Rightarrow f_x(x,y) = 0$

$\cos x \sin y = 0$

$y = n\pi \qquad x = (2n+1)\frac{\pi}{2}$

Also $f_y(x,y) = 0$

$\sin x \cdot \cos y = 0$

$x = n\pi \qquad y = (2n+1)\frac{\pi}{2}$

$\Rightarrow$ All critical points:

$(0,0)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Q3. $f(x,y) = xye^{-(x^2+y^2)}$

$f_x(x,y) = e^{-(x^2+y^2)}(y) + xye^{-(x^2+y^2)}(-2x)$

$= e^{-x^2-y^2}(y - 2x^2y)$

$f_y(x,y) = e^{-x^2-y^2}(x - 2xy^2)$

For critical points, $f_x = 0$ & $f_y = 0$.

$\Rightarrow e^{-x^2-y^2}(y - 2x^2y) = 0$

$y = 0, \; x = \frac{1}{\sqrt{2}}, \; x = \frac{-1}{\sqrt{2}}$

$\Rightarrow e^{-x^2-y^2}(x - 2y^2x) = 0$

$x = 0, \; y = \frac{1}{\sqrt{2}}, \; y = \frac{-1}{\sqrt{2}}$

Critical points $= (0,0)$. $(0, \pm\frac{1}{\sqrt{2}})$ $(\pm\frac{1}{\sqrt{2}}, 0)$. $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$

$(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}})$

$A = f_{xx} = e^{-x^2-y^2}(-4x) + (y - 2x^2y)(-2x)e^{-x^2-y^2}$.

$= e^{-x^2-y^2}(-4x - 2xy + 4x^3y)$

$C\cancel{B} = f_{yy} = e^{-x^2-y^2}(-4y - 2xy + 4y^3x)$.

$B = f_{xy} = e^{-x^2-y^2}(1-2x^2) + (x - 2xy^2)(-2y)e^{-x^2-y^2}$

$= e^{-x^2-y^2}(1 - 2x^2 - 2xy + 4xy^3)$

case -(I) $(0,0)$ $A=0$, $B=0$, $C=0$.

$AC - B^2 = 0 \rightarrow$ No conclusions.

case (II) $(0, \frac{1}{\sqrt{2}})$ $A = e^{-1}($

Scanned by CamScanner

Q4 $f(x,y) = (y-4x^2)(y-x^2)$
$= y^2 - yx^2 - 4x^2y + 4x^4$
$= y^2 - 5yx^2 + 4x^4$

$f_x(x,y) = -10xy + 16x^3$

$A = f_{xx}(x,y) = -10y + 48x^2$

$f_y(x,y) = 2y - 5x^2$

$C = f_{yy}(x,y) = 2$

$B = f_{xy}(x,y) = -10x$

$\cancel{f_x(x,y)}$ $f_x(0,0) = 0$ & $f_y(0,0) = 0$
$\Rightarrow (0,0)$ is a critical pt.

Now $A(0,0) = 0$
$B(0,0) = 0$
$C(0,0) = 2$

~~hu~~ $AC - B^2 = 0 \Rightarrow$ No conclusions

Scanned by CamScanner

Q5 $f(x,y) = x^2 + y^2 - 2xy$

$f_x(x,y) = 2x - 2y$.

$f_y(x,y) = 2y - 2x$.

$A = f_{xx} = 2$.

$C = f_{yy} = 2$.

$B = f_{xy} = -2$

critical points:

$f_x = 0$ & $f_y = 0$.

it gives $\Rightarrow$ $x = y$

$AC - B^2 = 0$ $\Rightarrow$ No conclusion can be made using Taylor's Theorem.

$\Rightarrow$ Second Derivative test is not useful.

But critical points are all points of the sort (x,x) and at ~~f(x,~~ point (x,x)

$f(x,x) = 0$.

it is clearly visible $f(x,y)$ cannot attain a value less than 0

$\Rightarrow$ 0 has to be the minimum for $f(x,y)$ and it is attained whenever $x=y$

Q7 $f(x,y) = 3x + 4y$. $\qquad x^2 + 4xy + 5y^2 = 10$.

$L(x,y,\lambda) = 3x + 4y + \lambda(x^2 + 4xy + 5y^2 - 10)$

$L_x = 0 \Rightarrow 3 + 2\lambda x + 4\lambda y = 0$.

$4 + 4\lambda x + 10\lambda y = 0$

$-2 + 2\lambda y = 0$.

$\lambda y = 1 \qquad \lambda x = -\frac{7}{2}$.

$\frac{49}{4\lambda^2} + \frac{\cancel{4x}^{2} - 7}{2\lambda^2} + \frac{5}{\lambda^2} = 10$

$\Rightarrow 10\lambda^2 = \frac{49 - 36}{4} = \frac{13}{4}$

$x^2 = \frac{13}{40}$. $\quad x = -\sqrt{\frac{13}{40}}, +\sqrt{\frac{13}{40}}$.

$$x, y = \left(\frac{7}{2}\sqrt{\frac{40}{13}}, -\sqrt{\frac{40}{13}}\right)$$

$$x, y = \left(-\frac{7}{2}\sqrt{\frac{40}{13}}, \sqrt{\frac{40}{13}}\right)$$

Q8 $\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad 3x + y - 9 = 0$

$(x_2, y_2) \qquad (x, y_1)$

$f(x,y) = \cancel{3x+y-9} \; (x-p)^2 + (y-q)^2$.

Subject : $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

& $3p + q - 9 = 0$.

$L(x, y, \alpha, B) = (x-p)^2 + (y-q)^2 + \alpha\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + B(3p + q - 9)$

$L_x = 0$.

$2(x-p) + \frac{2\alpha x}{4} = 0$

$4x - 4p + \alpha x = 0$. $\quad -(1)$

$L_y = 0$

$2(y-q) + \frac{2\alpha y}{9} = 0$

$\Rightarrow 9y - 9q + \alpha y = 0 \quad -(2)$.

$L_p = 0$.

$-2(x-p) + 3B = 0$

$2p - 2x + 3B = 0 \quad -(3)$

Scanned by CamScanner

$L_q \Rightarrow 2(q-y) + B = 0.$

$$2q - 2y + B = 0 \quad - (4)$$

$$3p + q - 9 = 0 \quad - (5)$$

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \quad - (6).$$

$$x = \frac{4p}{4+\alpha} \quad - (7),$$

$$y = \frac{9q}{9+\alpha} \quad - (8).$$

~~Q8. $2x + y - 9 + \lambda E$.~~

Q9. max. volume; 1st octant; vertex at 0,0,0.

~~max.~~ $f(x,y,z) = xyz.$

$$L(x,y,z,\lambda) = xyz + \lambda\left(x + y + \frac{z}{2} - 1\right).$$

$$L_x = 0 \Rightarrow yz + \lambda = 0.$$

$$L_y = 0 \Rightarrow xz + \lambda = 0.$$

$$\Rightarrow x = y.$$

$$L_z = 0 \Rightarrow xy + \frac{\lambda}{2} = 0$$

$$2xy = yz$$

$$2x = z.$$

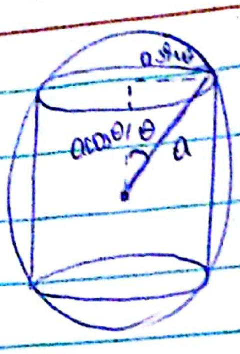
$$(x + x + x - 1) = 0$$

$$x = \frac{1}{3}, \; y = \frac{1}{3} \quad z = \frac{2}{3}.$$

$$\lambda = -\frac{1}{9}.$$

$$V = \frac{2}{27}.$$

Q10



$r = a\sin\theta$

$h = 2a\cos\theta$

$S = 2\pi r^2 + 2\pi r h$

$= 2\pi a^2 \sin^2\theta + 2\pi \times 2a^2 \sin\theta\cos\theta$

$f(a,\theta) = \pi a^2 (2\sin^2\theta + \underbrace{4\sin\theta\cdot\cos\theta}_{2\sin 2\theta})$

Subject $f_\theta = 0$

$\Rightarrow \pi a^2 (4\sin\theta\cos\theta + (-4\sin^2\theta) + 4\cos^2\theta) = 0$

$4\sin\theta\cos\theta = -4\cos^2\theta + 4\sin^2\theta$

$\sin 2\theta = -\cancel{8}2(\cos 2\theta)$

$\tan 2\theta = -\cancel{8}2$

$-2 = \frac{2x}{1-x^2}$ $\quad x^2 - x - 1 = 0$

$\tan\theta = \frac{1 \pm \sqrt{5}}{2}$ $\quad \sin\theta = 1+\sqrt{5}$

Another way:

max. $2\pi r^2 + 2\pi r h = f$

constraint: $r^2 + \left(\frac{h}{2}\right)^2 = a^2 = g$

$\therefore \nabla f = \lambda \nabla g$

and $g = 0$

Q1 $V = 500\ m^3$.

height $= h$.
width $= w$.
length $= l$.

min $f(h,w,l) \Rightarrow 2hw + 2hl + wl$
Subject $\Rightarrow hwl = 500$

$L(h,w,l,\lambda) = 2hw + 2hl + wl + \lambda hwl$.

$f_h = 2w + 2l + \lambda wl = 0$.

$f_w = 2h + l + \lambda hl = 0$.

$f_\lambda = 2h + w + \lambda hw = 0$.

$\Rightarrow (w-l) + \lambda h(w-l) = 0$.

$\Rightarrow (w = l.)$ or $(1 + \lambda h) = 0$

$\Rightarrow h = \frac{500}{w^2}$.

$\Rightarrow 2 \times \frac{500}{w^2} + w + \lambda \times \frac{500}{\cancel{w}} \times \cancel{w} = 0$

$\Rightarrow \frac{1000}{w^2} + w + 500\lambda = 0$. — (1).

$4\cancel{w} + \lambda w^2 = 0$.

$4 = -\lambda w$.

$\lambda = \frac{-4}{w}$.

$\Rightarrow \frac{1000}{w^2} + w^3 - \frac{2000w}{w} = 0$.

$w^3 - 2000w + 1000$.

$w = 10$

$$f(0,1)\begin{cases} x & x\in\mathbb{Q} \\ 1-x & x\notin\mathbb{Q}\end{cases}$$

Find integral

Q. $f[0,1]\to\mathbb{R}$

a) $f(x)=\begin{cases} 1 & x<1 \\ 2 & x=1.\end{cases}$

Lower sums $= \sum_{k=0}^{n} m_k(x_k - x_{k-1})$

$= 1$

Upper sum : $\sum_{k=0}^{n-1} M_k(x_k - x_{k-1})$

$= 2\left(1-\frac{n-1}{n}\right)+1\left(\frac{n-1}{n}\right) = \frac{2}{n}\cdot\frac{n-1}{n}$

$1+\frac{n}{n}$

b) $f(x) = \begin{cases} \sin x. & x \text{ is of the form } 1/n. ; n\in\mathbb{N} \\ \cos x & \text{otherwise}\end{cases}$

let $\varepsilon>0$, $[0,1/N]$ $[1/N,1]$

$P=\{x_0, x_1, x_2 ---- x_n\}$

$U(P;f) - L(P;f)$

$= \sum_{k=1}^{n}(M_k-m_k)(x_k-x_{k-1}) \leq \sum_{k=1}^{n}(x_k-x_{k-1}) = \frac{1}{N} < \varepsilon$

(*) d) $\begin{cases}\cos\frac{\pi}{x} & ; x\notin\mathbb{Q} \\ 0 & ; x\in\mathbb{Q}\end{cases}$

lower sums $= 0$

~~upper sum~~

$=$

$\left[\frac{1}{2}, \frac{2}{3}-\varepsilon\right]$ or $\left[\frac{2}{3}+\varepsilon, \frac{1}{2}\right]$ or $\left[\frac{2}{3}+\varepsilon, 1\right]$

$U(P;f) = \sum_{k=1}^{n} M_k(x_k-x_{k-1})$

$L(P;f)=0.$

$\geq \cos\frac{\pi}{\left(\frac{2}{3}-2\right)} > \varepsilon$

(c) $\begin{cases} 1+x & x \in \mathbb{Q} \\ -1-x & x \notin \mathbb{Q} \end{cases}$ $f[0,1] - \mathbb{R}$

Let $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$

$\left[0, \frac{1}{N}\right]$ $U(P,f) - L(P;f) = \sum (M_k - m_k)\Delta x_k$

$\leq \sum \Delta x_k \leq \frac{1}{N} < \varepsilon$

(e) $f:[0,1] - \mathbb{R}$

$f(x) := \begin{cases} x\left[\frac{1}{x}\right] & ; x \in (0,1] \\ 0 & ; x = 0. \end{cases}$

For any $N \in \mathbb{N}$

$[0, 1/N]$ & $[1/N, 1]$.

$P = \{x_0, x_1, x_2, x_3 \cdots x_n\}$

$U(P;f) - L(P;f) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1})$

$= \cancel{1(1 - x_{n-1})}$

$\leq \sum_{k=1}^{n} x_k - x_{k-1} = \frac{1}{N} < \varepsilon$

Q2 (a) If f integrable ~~iff~~ f^2 is also integrable.

f is bdd, i.e. $f(x) \leq K$, let $\varepsilon > 0$

$\therefore \exists$ a partition P_ε s.t. $U(P_\varepsilon;f) - L(P_\varepsilon;f) < \varepsilon/2k$

$P_\varepsilon = \{x_0, x_1, x_2 \cdots x_n\}$

For $x,y \in [x_{i-1}, x_i]$

$f^2(x) - f^2(y) = (f(x) + f(y))(f(x) - f(y))$

$\leq 2k(f(x) - f(y))$

$\leq 2k(M_i - m_i)$

$M'_i - m'_i \leq 2k(M_i - m_i)$

where $M'_i = \sup_{x \in [x_{i-1}, x_i]} f^2(x)$ & $m'_i = \inf_{x \in [x_{i-1}, x_i]} f^2(x)$

$U(P_\varepsilon; f^2) - L(P_\varepsilon; f^2) = \sum_{i=1}^{n} (M'_i - m'_i)(x_i - x_{i-1})$

$\leq 2k. \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon$

(b) If f & g are integrable, will fg be integrable?

f & g are bounded.

$\therefore$ $f(x) < k_1$ & $g(x) < k_2$

Let $\varepsilon_2, \varepsilon_1 > 0$

$\exists$ a partition P_1 s.t. & P_2 s.t.

$U(P_1; f) - L(P_1; f) < \varepsilon_1$ $\quad$ $U(P_2; g) - L(P_2; g) < \varepsilon_2$

Let $P = P_1 \cup P_2$ & $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$

for $x, y \in [x_{i-1}, x_i]$

Now, $g(x)f(x) - g(y)f(y) - g(x)f(y) + g(x)f(y)$

$= g(x)(f(x) - f(y)) + \cancel{g(x)} f(y)(g(x) - g(y))$

$\leq k_2(f(x) - f(y)) + k_1(g(x) - g(y))$

$\leq k_2(M'_i - m'_i) + k_1(M_i - n_i)$

Scanned by CamScanner

Q3 If $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$

then $\exists$ c $\in [a,b]$ s.t. $f(c) = g(c)$.

$\int_a^b f(x) - g(x) = (b-a)$ (Mean Value Theorem)

Q4 f cts. $f(x) \geq 0 \ \forall x.$ & $\underbrace{\int_a^b f(x)\,dx = 0.}$

$f = 0.$

$\exists$ $c \in [a,b]$ s.t. $f(c) \neq 0.$

$\exists$ $\delta > 0$ s.t. $f(x) > 0 \ \forall x \in (c-\delta, c+\delta)$

$\int_a^b f(x)\,dx \geq \int_{c-\delta}^{c+\delta} f(x)\,dx. > 0$

$f : [0,1] \to \mathbb{R}$

If f is not cts.

eg. $f(x) = \begin{cases} 1 & x \in \{0, 1/2, 1\} \\ 0 & \text{otherwise.} \end{cases}$

Q5. $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} \cdots \frac{1}{2n} \right)$ Darboux sum Reimann sum

$\sum_{k=1}^{n} f(\xi_k)\, \Delta x_i \to \int_a^b f(x)\,dx.$

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+k/n} \cdot \frac{1}{n}.$

$f(x) = \frac{1}{1+x}$, $x \in [0,1]$

$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n} \dots 1\right\}$

Since f is cts, $\therefore$ it is ~~ex~~ integrable

$\therefore \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+k/n} \cdot \frac{1}{n} \rightarrow \int_0^1 \frac{dx}{1+x}$

(By darboux theorem)

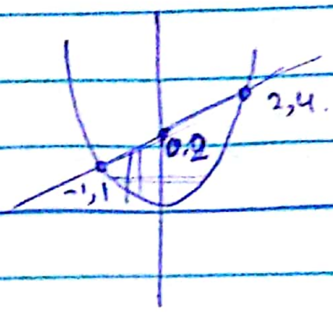
$\therefore \int_0^1 \frac{dx}{1+x} = \ln|1+x| \Big|_0^1$

$= \ln|2| - \ln|1|$

$= \ln 2$

Sheet - ~~8~~ 7.

Q.1 (a) $\iint x^2\,dA.$



$$\int_{x=-1}^{x=2}\int_{y=x^2}^{y=x+2} x^2\,dy\,dx.$$

$$= \int_{x=-1}^{x=2} x^2\, y\Big|_{y=x^2}^{x+2}\,dx \quad = \int_{x=-1}^{x=2} x^2(2+x-x^2)\,dx.$$

$$= \int_{x=-1}^{x=2} (2x^2+x^3-x^4)\,dx.$$

$$= \left.\frac{2x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}\right|_{x=-1}^{x=2}$$

$$= \frac{16}{3}+\frac{16}{4}-\frac{32}{5}+\frac{2}{3}-\frac{1}{4}-\frac{1}{5}$$

$$= 6.+\frac{15}{4}-\frac{33}{5}$$

$$= \frac{63}{20}.$$

(b) $\iint (x^2+y^2)\,dA.$ $= \int_{x=0}^{x=1}\int_{y=0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx.$

$$= \int_{x=0}^{x=1} \left(x^2y+\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=\sqrt{1-x^2}}\,dx$$

$$= \int_{x=0}^{x=1} \frac{2x^2\sqrt{1-x^2}}{3} + \frac{(1\cancel{\ })\sqrt{1-x^2}}{3}$$

$$= \frac{\pi}{12} + \frac{2}{3}\int_0^1 x^2\sqrt{1-x^2}$$

$$= \cancel{\frac{\pi}{12} + \frac{2}{3}\left[x^2 \right.}$$

Q2 (a) $\int_0^3 \int_{-y}^{y} (x^2 + y^2)\, dx\, dy$

$$\int_0^3 \left(\frac{x^3}{3} + xy^2\right)_{-y}^{y} dy$$

$$= \int \frac{y^3}{3} + y^3 + \frac{y^3}{3} + y^3$$

$$= \int_0^3 \frac{8y^3}{3}\, dy = \left| \frac{\overset{2}{\cancel{8}}y^4}{\cancel{4}\times 3} \right|_0^3$$

$$= \underline{\underline{54}}$$

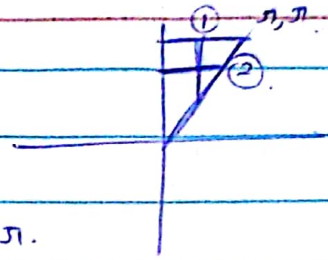
(b) $\int_0^1 \int_2^{4-2x} dy\, dx$

$$= \int_0^1 (4-2x-2)\, dx = \int_0^1 (2-2x)\, dx$$

$$= \left| 2x - x^2 \right|_0^1$$

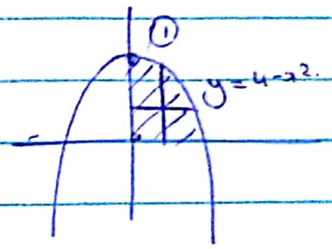
$$= 1$$

(c) $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx = \cancel{\int_x^{\pi}\int_0^{\pi} \frac{\sin y}{y}\,dx\,dy}$



$= \int_{y=0}^{\pi}\int_{x=0}^{y} \frac{\sin y}{y}\,dx\,dy = \cancel{\int_x^{\pi} \frac{\sin y}{y}\,dy} = \int_{y=0}^{\pi} \sin y\,dy = \underline{2}$

(d) $\int_0^{2}\int_{y=0}^{4-x^2} \frac{x e^{2y}}{4-y}\,dy\,dx = \cancel{\int_0^{4-x^2}}$



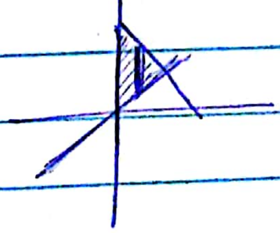
$= \int_{y=0}^{y=4}\int_{x=0}^{x=\sqrt{4-y}} \frac{x e^{2y}}{4-y}\,dx\,dy$

$= \int_{y=0}^{y=4} \left.\frac{x^2 e^{2y}}{2(4-y)}\right|_0^{\sqrt{4-y}} dy = \int_0^4 \frac{e^{2y}}{2}\,dy$

$= \left.\frac{e^{2y}}{4}\right|_0^4$

$= \frac{e^8}{4} - \frac{1}{4} = \frac{e^8-1}{4}$

Q3 (a)



$I = \int_0^1\int_x^{2-x} (x^2+y^2)\,dy\,dx$

$= \int_0^1 \left. x^2y + \frac{y^3}{3} \right|_x^{2-x} dx$

$= \int_0^1 2x^2 - x^3 + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3}$

$$= \int_0^1 \cancel{2}x^2 - \frac{7x^3}{\cancel{3}} + \frac{8}{3} - 4x + 2\cancel{x^2} - \frac{x^3}{\cancel{3}}\, dx$$

$$= \int_0^1 4x^2 - \frac{8x^3}{3} + \frac{8}{3} - 4x \, dx$$

$$= \frac{4x^3}{3} - \frac{8x^4}{12} + \frac{8}{3}x - 2x^2 \Big|_0^1$$

$$= \frac{4}{3} - \cancel{\frac{2}{3}} + \cancel{\frac{8}{3}} - \cancel{\frac{6}{3}}$$

$$= \frac{4}{3}$$

(c)

Q10 $f: 2\pi r h + 2\pi r^2$ $\quad g: r^2 + \left(\frac{h}{2}\right)^2 = a^2$

$$\nabla f = \lambda \nabla g$$

$$(2\pi h + 4\pi r,\ 2\pi r) = \lambda\left(2r, \frac{h}{2}\right)$$

$$2\pi h + 4\pi r = 2\lambda r$$

$$2\pi r = \frac{\lambda h}{2}$$

$$h = \frac{4\pi r}{\lambda} \Rightarrow \frac{2\pi(4\pi \cancel{r})}{\lambda} + 4\pi \cancel{r} = 2\lambda \cancel{r}$$

$$= \frac{8\pi^2}{\lambda} + 4\pi = 2\lambda$$

$$\Rightarrow 4\pi^2 + 2\pi\lambda = \lambda^2$$

$$\Rightarrow \lambda^2 - 2\pi\lambda - 4\pi^2$$

$$\lambda = \frac{2\pi \pm \sqrt{4\pi^2 + 4 \times 4\pi^2}}{2} = \pi(1 \pm \sqrt{5})$$

$$g: r^2 + \left(\frac{\cancel{4}\cancel{\pi} r}{\cancel{2}(\cancel{\pi})(1+\sqrt{5})}\right)^2 = a^2$$

$$r^2 + \left(\frac{2 \times \cancel{2} r^2}{3\cancel{6} + \cancel{2}\sqrt{5}}\right)^{\cancel{2}} = a^2 \Rightarrow r^2 + \frac{2r^2}{3+\sqrt{5}} = a^2$$

$$\left(\frac{5+\sqrt{5}}{3+\sqrt{5}}\right) r^2 = a^2$$

$$r^2 = \frac{3+\sqrt{5}}{\sqrt{5}(1+\sqrt{5})} a^2$$

$$r^2 = \frac{(3+\sqrt{5})(\sqrt{5}-1)}{4\sqrt{5}} a^2$$

$$= \left(\frac{3\sqrt{5} + 5 - 3 - \sqrt{5}}{4\sqrt{5}}\right) a^2 = \frac{2 + 2\sqrt{5}}{4\sqrt{5}} a^2$$

$$r = \frac{a}{2}\sqrt{2 + \frac{2}{\sqrt{5}}}$$

Q7. $f = 3x + 4y \qquad g = x^2 + 4xy + 5y^2 - 10.$

$$\nabla f = \lambda \nabla g.$$

$$(3, 4) = \lambda(2x + 4y,\ 4x + 10y)$$

$$(2x + 4y)\lambda = 3.$$

$$2x + 4y = \frac{3}{\lambda} \qquad \cancel{4x}\ \&\ 2x + 5y = \frac{2}{\lambda}$$

$$y = \frac{-1}{\lambda}$$

$$x = \frac{7}{2\lambda}$$

Applying $g = 0$.

$$\frac{49}{4\lambda^2} + 4^{\cancel{2}}\left(\frac{7}{\cancel{2}\lambda}\right)\left(\frac{-1}{\lambda}\right) + \frac{5}{\lambda^2} = 10.$$

$$\frac{49}{4} - \cancel{14} \overset{-9}{\cancel{+5}} = 10\lambda^2.$$

$$\frac{13}{40} = \lambda^2.$$

$$\lambda = \pm\frac{1}{2}\sqrt{\frac{13}{10}}$$

$$x = \frac{7\sqrt{10}}{\cancel{2}\cdot x\sqrt{13}} = 7\sqrt{\frac{10}{13}} \qquad y = -2\sqrt{\frac{10}{13}}.$$

$$f_{max} = \sqrt{130} \qquad f_{min} = -\sqrt{130}.$$

Q. ~~$\frac{x^2}{4} + \frac{y^2}{9} = 1$~~

~~General point on ellipse :~~

$f = xyz$ $\qquad g = x + y + \frac{z}{2} - 1$

$\nabla f = \lambda \nabla g$

$(yz, xz, xy) = \lambda(1, 1, \frac{1}{2})$

$yz = \lambda$

$xz = \lambda$

$xy = \frac{\lambda}{2}$

$x = \sqrt{\frac{\lambda}{2}}$ $\quad y = \sqrt{\frac{\lambda}{2}}$ $\quad z = \sqrt{2\lambda}$

$g = 0 \Rightarrow \sqrt{2\lambda} + \sqrt{\frac{\lambda}{2}} = 1$

$\frac{3\sqrt{\lambda}}{\sqrt{2}} = 1$

$\lambda = \frac{\sqrt{2}}{3}$ $\quad \lambda = \frac{2}{9}$

$x = \frac{1}{3}$ $\quad y = \frac{1}{3}$ $\quad z = \frac{2}{3}$

$V = \frac{2}{27}$

Q11 $V = 500\,m^3$

$\min f(h,w,l) = 2hw + 2hl + lw$

$g(h,w,l) = hwl - 500$

~~$L(h,w,l,\lambda)$~~ =

$\nabla f = \lambda \nabla g$

$(2w+2l,\ 2h+l,\ 2h+w) = \lambda(wl,\ hl,\ hw)$

$2w + 2l = \lambda wl$

$2h + l = \lambda hl$

$2h + w = \lambda hw$

$w = l$

$4\cancel{w} = \lambda w^{\cancel{2}}$

$w = \frac{4}{\lambda} = l$

$2h + \frac{4}{\lambda} = \lambda \times h \times \frac{4}{\cancel{\lambda}}$

$\frac{\cancel{4}^{2}}{\lambda} = 2h$

$h = \frac{2}{\lambda}$

$g = 0 \Rightarrow \frac{4 \times 4 \times 2}{\lambda^3} = 500$

$\frac{4 \times 4 \times \cancel{2}.4}{1000} = \lambda^3$

$\lambda = 0.4$

$h = 5 \quad l = w = 10$

Q12. $f(x,y,z) = 8xyz^2 - 200(x+y+z) = 8xyz^2 - 20000.$

$g(x,y,z) = x+y+z = 100.$

$\nabla f = \lambda \nabla g.$

$(8yz^2, 8xz^2, \cancel{8}16xyz) = \lambda(1,1,1).$

$8yz^2 = \lambda \qquad 8xz^2 = \lambda \qquad 16xyz = \lambda$

$y = x$

$16x^2z = \lambda$

$8 \cdot xz^2 = \lambda$

$2x = z$

$16 \times x \cdot x \cdot x \times 2x = \lambda.$

$32x^3 = \lambda.$

$x + \cancel{y}x + 2x = 100.$

$4x = 100.$

$x = 25.$

$y = 25 \quad z = 50.$