Linear algebra for data science

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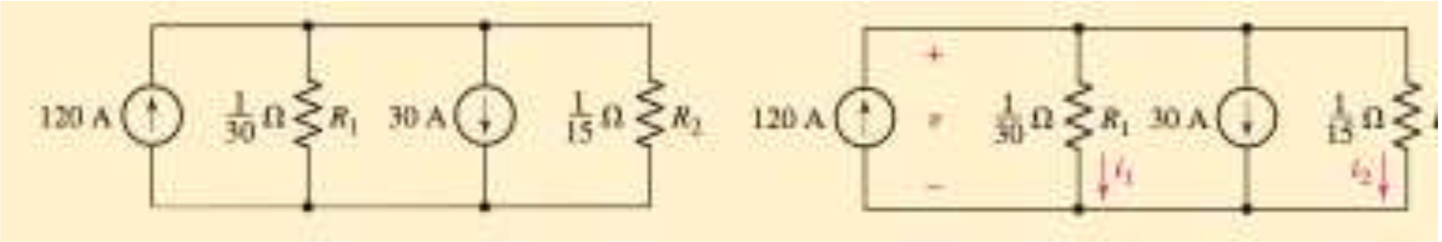
Recall Basic Electrical Engineering



Find currents and voltages

Electrical Networks

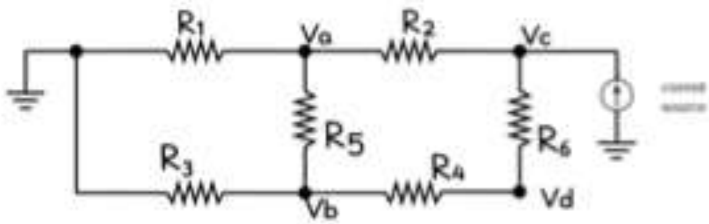


Figure: A resistive network.

In conductance form ($G_i = 1/R_i$) these can be written compactly as

$$\begin{aligned}
(G_1 + G_2 + G_5)V_a - G_5 V_b - G_2 V_c &= 0, \\
-G_5 V_a + (G_3 + G_4 + G_5)V_b - G_4 V_d &= 0, \\
-G_2 V_a + (G_2 + G_6)V_c - G_6 V_d &= I_s, \\
-G_4 V_b - G_6 V_c + (G_4 + G_6)V_d &= 0.
\end{aligned}$$

Thus the matrix equation $A\mathbf{v} = \mathbf{b}$ is

$$\begin{pmatrix} G_1 + G_2 + G_5 & -G_5 & -G_2 & 0 \\ -G_5 & G_3 + G_4 + G_5 & 0 & -G_4 \\ -G_2 & 0 & G_2 + G_6 & -G_6 \\ 0 & -G_4 & -G_6 & G_4 + G_6 \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I_s \\ 0 \end{pmatrix}.$$

Counting

- In how many ways can Roger and Rafa win 10 grand slams between themselves

$$Ro + Ra = 10$$

- In how many ways can write 10 as a sum of two ‘real’ numbers

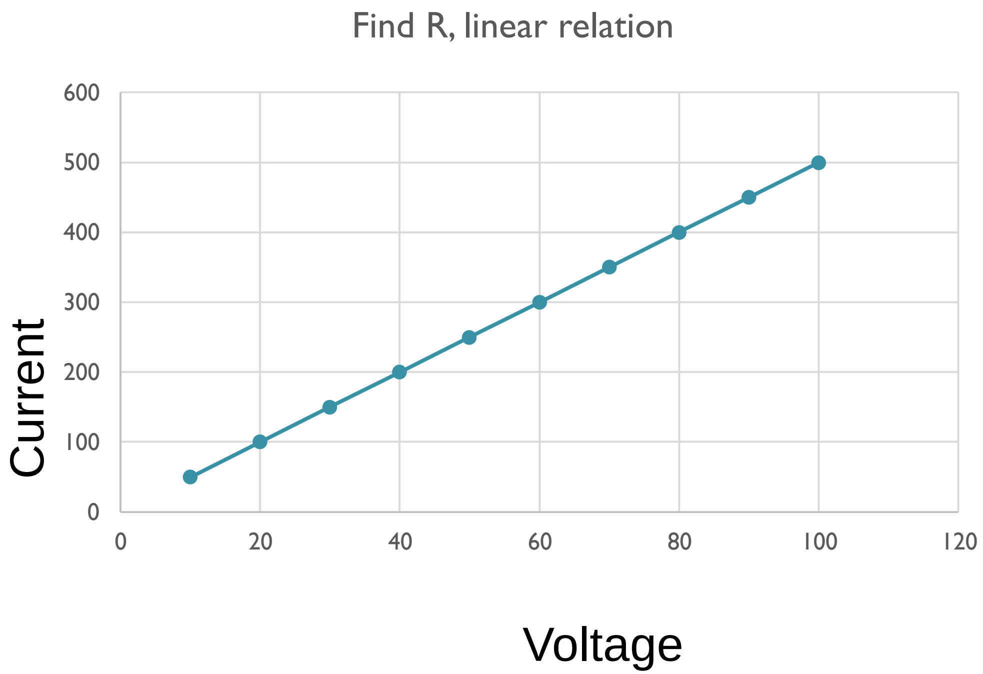
$$x_1 + x_2 = 10$$

A simple college experiment

Given an unknown resistance? How would you find its value?

You have a voltage source and a current measuring device

Voltage	Current
10	50
20	100
30	150
40	200
50	250
60	300
70	350
80	400
90	450

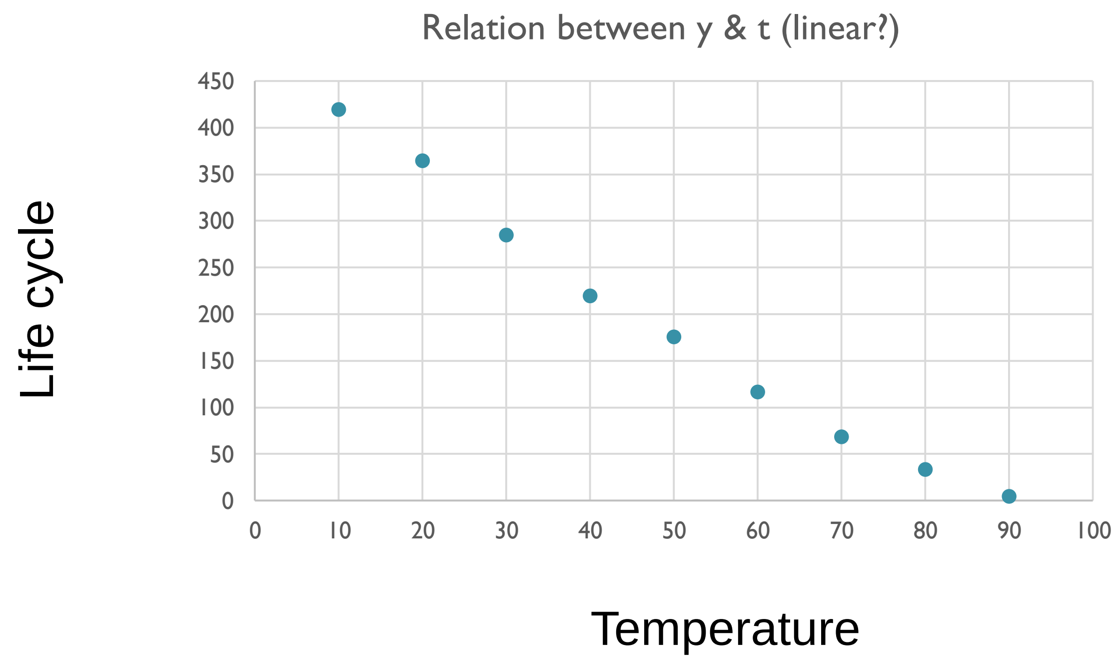


Prediction?

A company manufactures an electronic device to be used in a very wide temperature range. The company knows that increased temperature shortens the life time of the device, and a study is therefore performed in which the life time is determined as a function of temperature. The following data is found:

What is the life for 65 degrees?

Temperature in Celcius (t)	10	20	30	40	50	60	70	80	90
Life time in hours (y)	420	365	285	220	176	117	69	34	5



Curve/Line fitting

- What is the equation of the line passing through (0,6), (1,0), (2,0).

- Is it possible to fit a curve?

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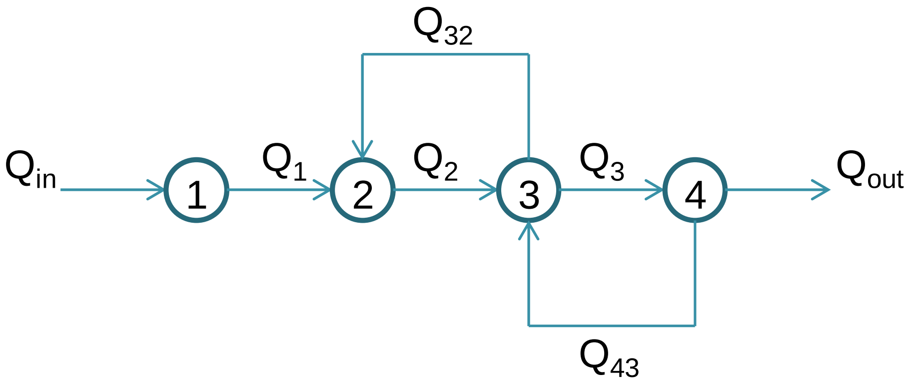
Outcome

Module learning outcomes:

1. Participants will be able to identify relationships between variables in large datasets
2. Participants will be able to identify information sufficiency in terms of both equations and variables
3. Participants will be able to understand basic linear algebra concepts that underlie the complicated data analytics algorithms

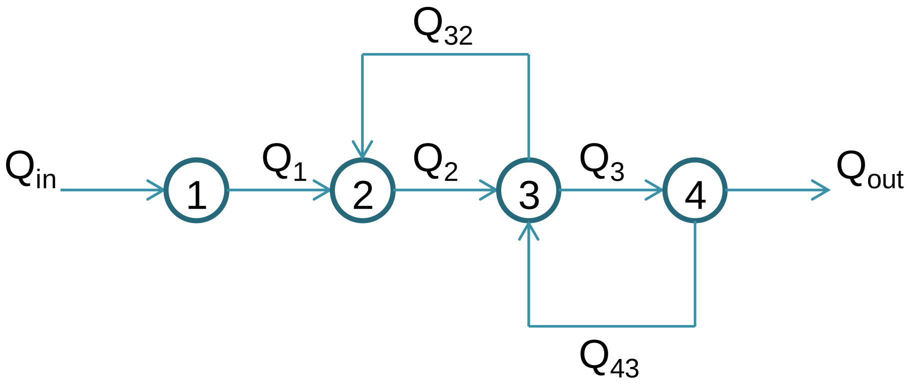
Linear Algebra

System of Equations



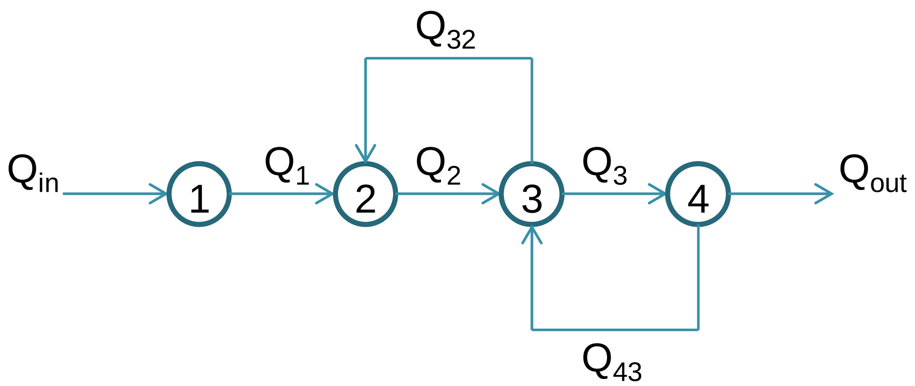
How many equations could be formed from this?

System of Equations



- $Q_{in} = Q_1$
- $Q_1 + Q_{32} = Q_2$
- $Q_2 + Q_{43} = Q_{32} + Q_3$
- $Q_3 = Q_{out} + Q_{43}$

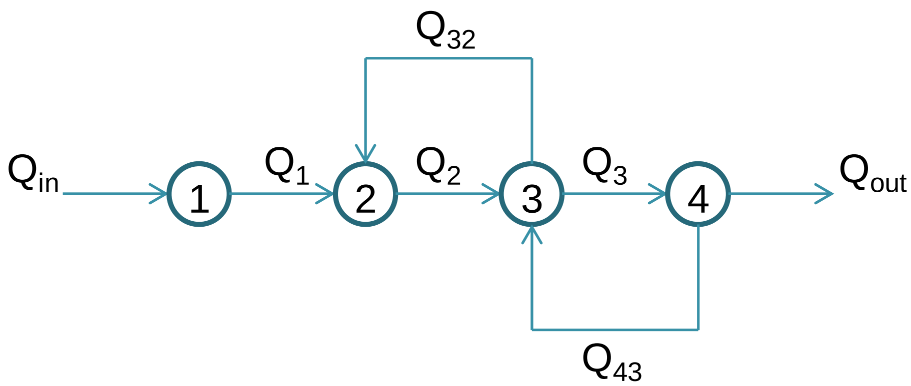
System of Equations



- $Q_{in} = Q_1$
- $Q_1+Q_{32} = Q_2$
- $Q_2+Q_{43} = Q_{32}+Q_3$
- $Q_3 = Q_{out}+Q_{43}$

- Can we solve the equations when –
  - $Q_{in} = 10$, $Q_{32} = 5$ and $Q_{43} = 3$
  - $Q_{in} = 10$, $Q_{out} = 10$ and $Q_3 = 7$
  - $Q_{in} = 10$, $Q_{out} = 11$ and $Q_3 = 7$

System of Equations

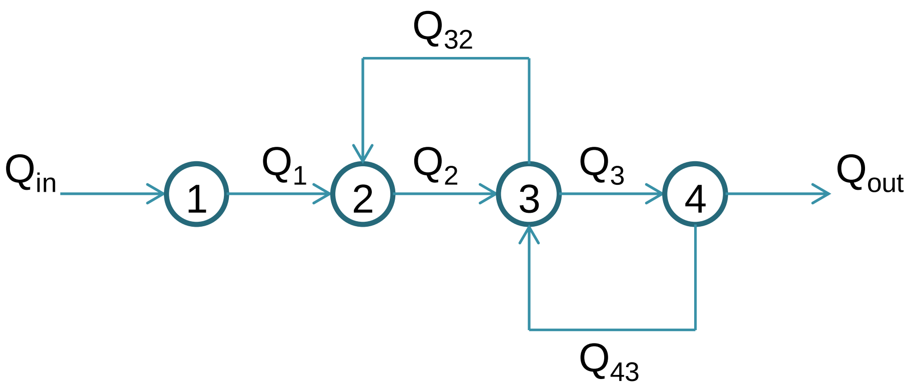


- $Q_{in} = Q_1$
- $Q_1 + Q_{32} = Q_2$
- $Q_2 + Q_{43} = Q_{32} + Q_3$
- $Q_3 = Q_{out} + Q_{43}$

- Can we solve the equations when $Q_{in} = 10$, $Q_{32} = 5$ and $Q_{43} = 3$?

Unique solution

System of Equations



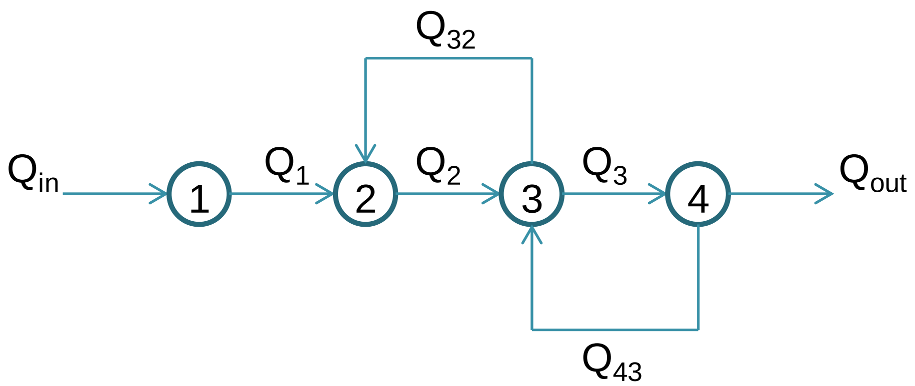
- $Q_{in} = Q_1$
- $Q_1 + Q_{32} = Q_2$
- $Q_2 + Q_{43} = Q_{32} + Q_3$
- $Q_3 = Q_{out} + Q_{43}$

- Can we solve the equations when $Q_{in} = 10$, $Q_{out} = 10$ and $Q_3 = 17$?

Infinite number of solutions

- Loop involving Q_2 and Q_{32} : both are unmeasured
- Given measurements consistent with equations

System of Equations



- $Q_{in} = Q_1$
- $Q_1+Q_{32} = Q_2$
- $Q_2+Q_{43} = Q_{32}+Q_3$
- $Q_3 = Q_{out}+Q_{43}$

- Can we solve the equations when $Q_{in} = 10$, $Q_{out} = 11$ and $Q_3 = 7$?

No solution

- Given information not consistent with equations

System of Equations: Key Concept

Understanding when do we have:

- Unique solution
- No solution
- Infinite number of solutions

Solving Simultaneous Linear Equations

Solve the two linear equations:

$$4x - 2y = 0 \; ; \; -2x + 4y = 6$$

Solving Simultaneous Linear Equations

Elimination (High–school method)

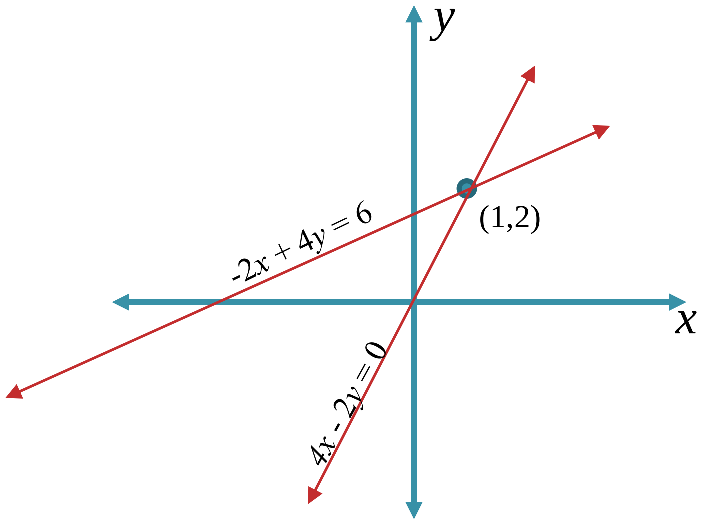
$$\begin{aligned} 4\cancel{x} - 2y &= 0 \\ 2(-\cancel{2}x + 4y &= 6) \\ \hline 6y &= 12 \end{aligned}$$

$\Rightarrow y = 2$

$\Rightarrow x = 1$

(1,2) is the solution

Graphical method



(1,2) is the solution

Geometry of Linear Equations

- We may view a system of linear equations in three different ways –
  - Matrix form – $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ where **A** forms a matrix with the coefficients of the unknowns and $\boldsymbol{x}$ forms a matrix with the unknowns and **b**, a matrix with the values in the R.H.S
  - Row picture – viewing one equation at a time
  - Column picture – two separate equations as one vector equation

Geometry of Linear Equations

Geometry for a system of 2 equations:

Matrix form

- Consider two linear equations:

$$4x - 2y = 0$$
$$-2x + 4y = 6$$

- A matrix is a rectangular arrangement of numbers in rows and columns
- Rows run horizontally and columns run vertically

- Order of a matrix: $m \times n$ where m is the # of rows and n is the # of columns

Geometry of Linear Equations

<u>Matrix form</u>

$$\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

- This is of the form $\mathbf{A}\boldsymbol{x} = \mathbf{b}$

  where –

  - **A** – matrix with coefficients of the unknowns
  - ***x*** – unknowns
  - **b** – R.H.S of the equations

- n equations and n unknowns ⇒ $n \times n$ matrix (square matrix)

- m equations and n unknowns ⇒ $m \times n$ matrix (rectangular matrix)

Geometry of Linear Equations

<u>Row picture</u>

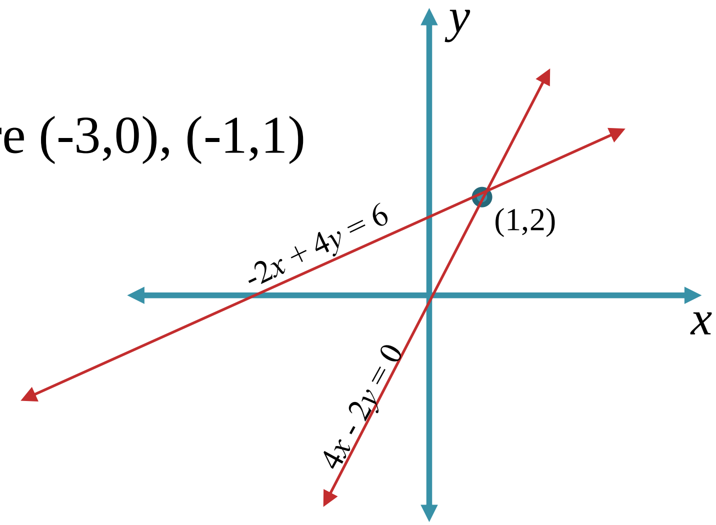
$$4x - 2y = 0 \; ; \; -2x + 4y = 6$$

- Taking one row at a time and plotting it in the x–y plane

Few points that satisfy $4x - 2y = 0$ are (0,0), (1,2) and (1/2, 1)

Few points that satisfy $-2x + 4y = 6$ are (-3,0), (-1,1) and (1,2)

So the solution of the system is (1,2)



Geometry of Linear Equations

Geometry for a system of 3 equations

- Consider a system of 3 equations:

$$
\begin{aligned}
x + 2y + z &= 6 \\
6x - 2y &= 4 \\
-3x - y + 4z &= 8
\end{aligned}
$$

Matrix form

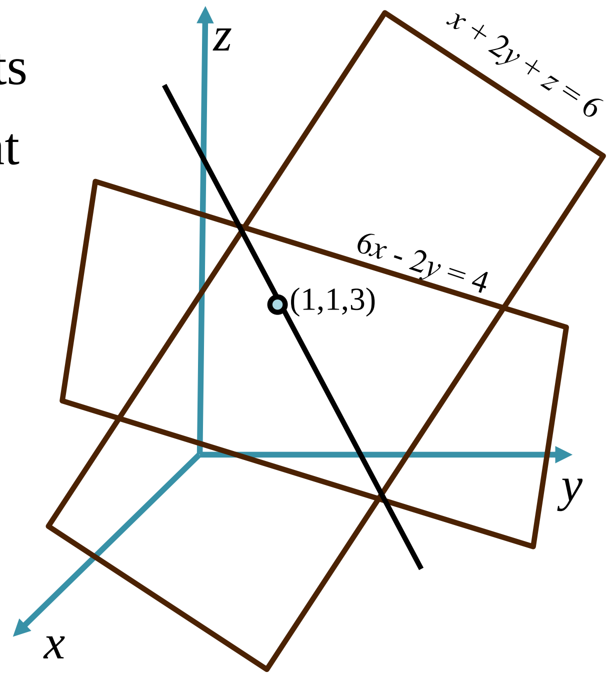
$$
\begin{bmatrix} 1 & 2 & 1 \\ 6 & -2 & 0 \\ -3 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 8 \end{bmatrix}
$$

Geometry of Linear Equations

Row picture

$$x + 2y + z = 6\ ;\ 6x - 2y = 4\ ;\ -3x - y + 4z = 8$$

- Each equation describes a plane in 3 dimensions. The intersection of the first plane with the second plane is a line
- The 3rd plane (not shown in the figure) intersects the line of intersection of the other two planes at a point (1,1,3)
- Solution for the system of equation is (1,1,3)



Types of Solutions

A system of linear equations can have –

- Unique solution

  or
- No solution

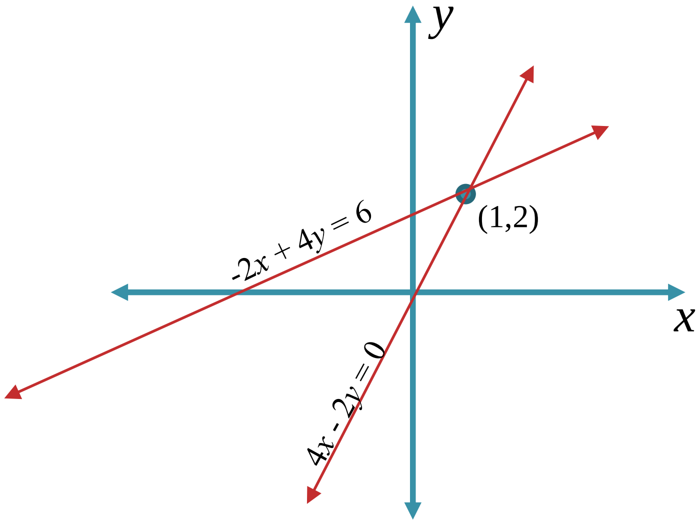
  or
- Infinite number of solutions

Types of Solutions

<u>2 dimensional case</u>

<u>Unique solution</u>

- This is the nice case where the system will have a point of intersection and hence a unique solution. $4x - 2y = 0$ and $-2x + 4y = 6$ has a unique solution (1,2)



One solution $(x, y) = (1,2)$

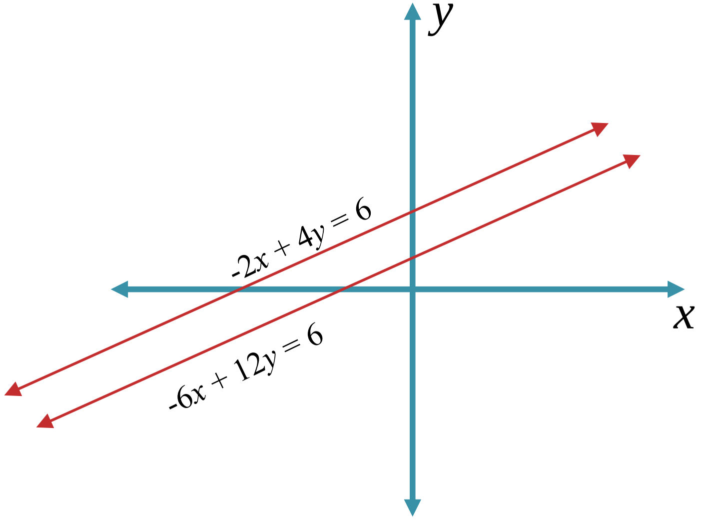
Types of Solutions

<u>No solution</u>

- A system has no solution if the equations are inconsistent. For example, $-2x + 4y = 6$ and $-6x + 12y = 6$ has no solution

$$-6x + 12y = 6 \Rightarrow -2x + 4y = 2$$

which contradicts with the first equation and hence the system has no solution



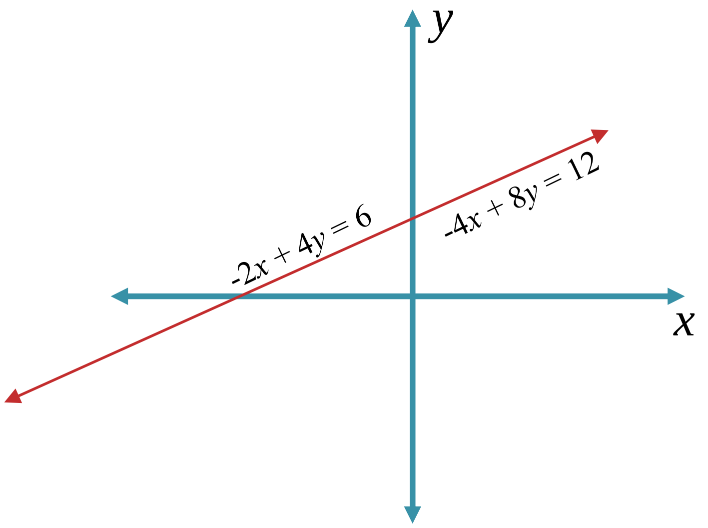
Parallel: No solution

Types of Solutions

<u>Infinite number of solutions</u>

- The other case is when one equation is just some multiple of the other. Then we will get infinite number of solutions

  $-2x + 4y = 6$; $-4x + 8y = 12$

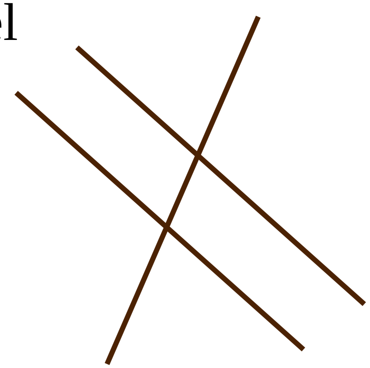


Whole line of solutions

Types of Solutions

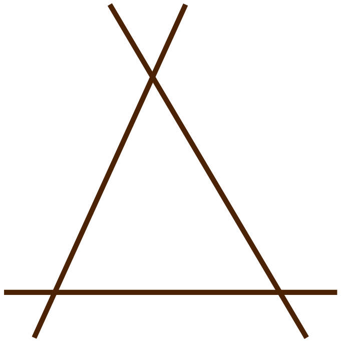
Singular case for three dimensions

- Two planes may be parallel



Two parallel planes: No solution

- One plane is parallel to the line of intersection of the other two planes

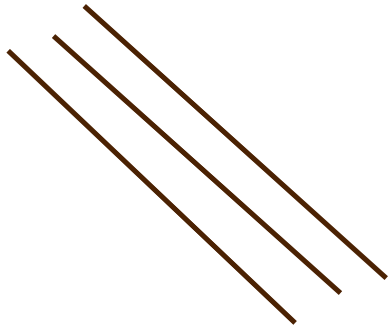


No intersection: No solution

Types of Solutions

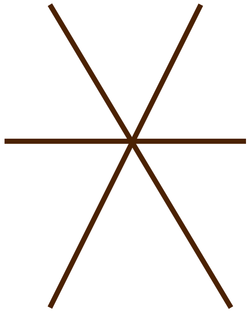
<u>Singular case for three dimensions</u>

- Three parallel planes



All planes parallel: No solution or a whole plane of solutions

- One equation is just the sum of the other two equations, the three planes have a whole line in common

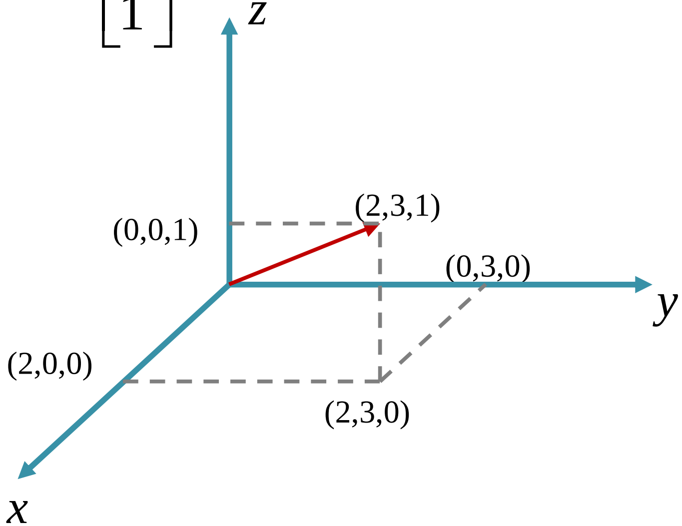


Line of intersection: Infinite # of solutions

Geometry of Linear Equations

Vector

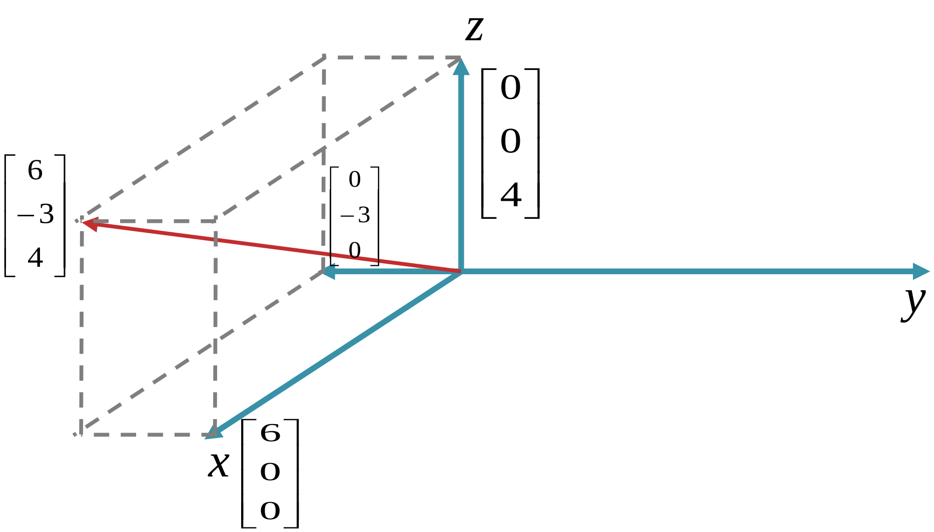
- A vector is defined as an ordered collection of numbers
- Elements of a vector arranged as a column $\longrightarrow$ column vector
- Elements of a vector arranged as a row $\longrightarrow$ row vector
- If a vector v contains three real numbers say, $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, then v belongs to the vector space R^3
- The vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} b \\ a \end{bmatrix}$ are not the same



Vector Addition

Addition of a vector $\begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix}$ is done component by component and can be written as –

$$\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix}$$



Geometry of Linear Equations

Column picture

$$4x - 2y = 0 \;\; ; \;\; -2x + 4y = 6$$

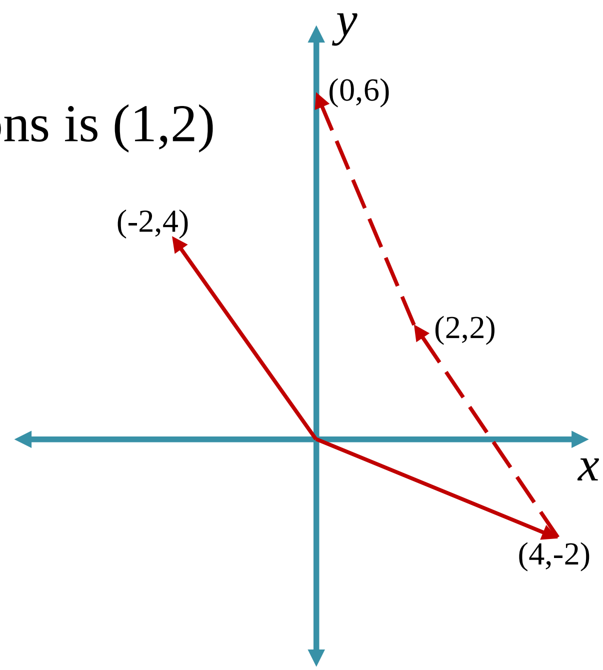
- Column picture: Linear combination of columns

$$x\begin{bmatrix} 4 \\ -2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

We know that the solution for the two equations is (1,2)

Substitute them

$$1\begin{bmatrix} 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



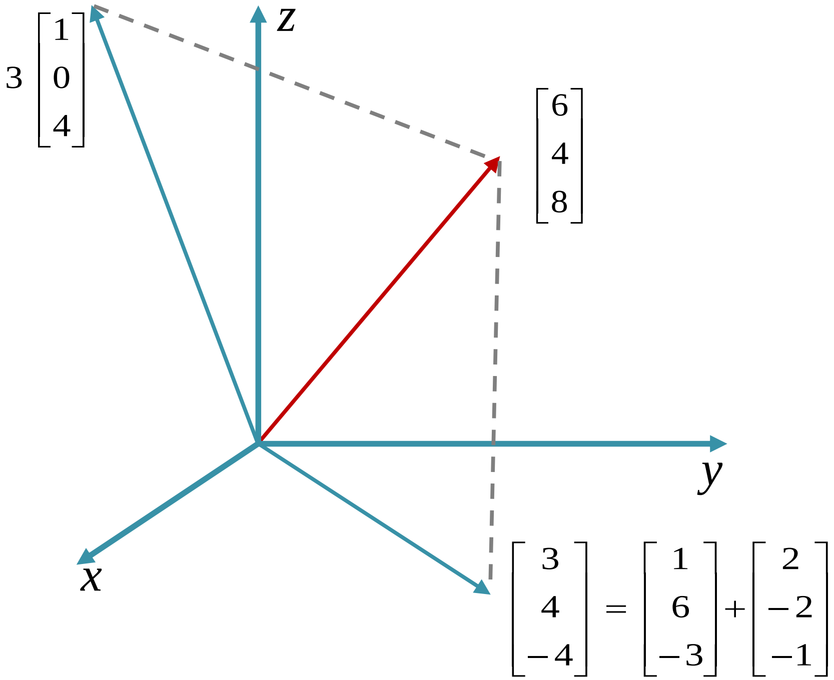
Geometry of Linear Equations

Column picture

$$x + 2y + z = 6\ ;\ 6x - 2y = 4\ ;\ -3x - y + 4z = 8$$

$$x\begin{bmatrix}1\\6\\-3\end{bmatrix} + y\begin{bmatrix}2\\-2\\-1\end{bmatrix} + z\begin{bmatrix}1\\0\\4\end{bmatrix} = \begin{bmatrix}6\\4\\8\end{bmatrix}$$

Solution for the system of equation is (1,1,3)



Thank you

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