Linear Systems Theory

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Module 2

Lecture 2

Math Preliminaries: Linear Algebra 1

Notation

- **Scalars:** Small alphabets
  Eg. a, b, c
- **Vectors:** Bold small alphabets
  Eg. $\mathbf{a}, \mathbf{x}, \mathbf{u}$
- **Vectors Elements:** Small alphabets with single subscript
  Eg. a_i is the i^{th} element of vector $\mathbf{a}$
- Length of a vector: Denoted by $|.|$
  Eg. $|\mathbf{a}|$
- **Matrices:** Bold capital alphabets
  Eg. $\mathbf{A}, \mathbf{B}$; i^{th} Row : $\mathbf{A}_{ri}, \mathbf{B}_{ri}$; i^{th} Column : $\mathbf{A}_{ci}, \mathbf{B}_{ci}$
- **Matrix Elements:** Small alphabets with double subscript
  Eg. a_{ij} is the i^{th} row and j^{th} column element of matrix $\mathbf{A}$

Vector Spaces

Vector Space

- Vector space $(V, \mathbb{F})$ is a set of vectors, and a field of scalars $\mathbb{F}$ equipped with two (binary) operations:
    1. Vector Addition: $\mathbf{x} + \mathbf{y} \in V \; \forall \; \mathbf{x}, \mathbf{y} \in V$
    2. Scalar Multiplication: $a\mathbf{x} \in V \; \forall \; \mathbf{x} \in V, \; a \in \mathbb{F}$
- Vector addition and scalar multiplication must satisfy certain rules
- Eight such rules to be satisfied in Vector spaces

Rules of Vector Addition and Scalar Multiplication

For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $c, c_1, c_2 \in \mathbb{F}$

- Commutativity: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- Associativity: $x + (y + z) = (x + y) + z$
- Zero vector: There is a unique zero vector such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$
- Negative vector: For each $\mathbf{x}$ there is a unique vector $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
- Identity: 1 times x equals x
- Associativity: $(c_1c_2)x = c_1(c_2x)$
- Distributivity: $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
- Distributivity: $(c_1 + c_2)x = c_1x + c_2x$

Examples

Vector spaces

- Zero vector is a vector space
- $\mathbb{R}^n$ is a vector space, (over $\mathbb{R}$)
- $\mathbb{R}^{m\times n}$ - Space of matrices is a vector space, (over $\mathbb{R}$)
- The set of functions: continuous functions that map an interval of the real line to $\mathbb{R}^n$; $(F([t_1, t_2], \mathbb{R}^n), \mathbb{R})$

Not Vector spaces

- $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \middle| x \geq 0 \right\}$
- $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \middle| y = 1 \right\}$
- $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \middle| y = 2x + 1 \right\}$

Inner Product Vector Space

- Vector spaces in which inner or dot product is defined apart from vector addition and scalar multiplication.
- Any vectors vectors in inner product vector space can be combined to get a scalar.
- In an inner product vector space V:

$$\mathbf{x}.\mathbf{y} = c \; \forall \; \mathbf{x}, \mathbf{y} \in V$$

Normed Vector Spaces

- An inner product vector space in which a norm is defined.
- A norm is a function that assigns a positive length or size to each vector in a vector space.
- p-norm: $p : V \to \mathbb{R}$ denoted as $||x||_p$; $\mathbf{x} \in V$.
- In a p-normed vector space $(V, ||.||)$:

$$||\mathbf{x}||_p = p(\mathbf{x}) \; \forall \; \mathbf{x} \in V$$

- Norm in a vector space has to satisfy certain properties

Properties of a Norm p

- $\|\mathbf{x}\|_p \geq 0 \; \forall \; \mathbf{x} \in V$
- $\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p \; \forall \; \mathbf{x}, \mathbf{y} \in V$ - Triangular inequality
- $\|c\mathbf{x}\|_p = |c|\|\mathbf{x}\|_p \; \forall \; \mathbf{x} \in V; c \in \mathbb{R}$
- $\|\mathbf{x}\|_p = 0$ iff $\mathbf{x} = \mathbf{0}$

Important Norms:

The most important norms for $\mathbf{x} \in \mathbb{R}^n$ and $x \in \mathbb{R}$ are:

- 1-norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$
- 2-norm or Euclidean norm: $\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$
- In general, p-norm: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$
- ∞-norm: $\|\mathbf{x}\|_\infty = \max_i |x_i|$
- It can be verified that properties of norms are satisfied by all these

Equivalence of Norms

Two norms $||\cdot||_a$ and $||\cdot||_b$ defined on $(V, \mathbb{F})$ are equivalent if one can be bounded wrt the other.

$\exists \alpha, \beta \in \mathbb{R}^+$ such that

$$\alpha ||x||_a \leq ||x||_b \leq \beta ||x||_a$$

- Let X be a set. A metric on X is a function $d : X \times X \to \mathbb{R}^+$ such that
- Properties of distance metric given $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$:
  1. $d(\mathbf{x}, \mathbf{y}) \geq 0$
  2. $d(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y}$
  3. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$
  4. $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$

Euclidean Space

- The *Euclidean norm* of an element $x \in \mathbb{R}^n$ is the number $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$.
- The *Euclidean distance* between two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2$
- In general, if $\| \cdot \|$ is a norm on a vector space V, then the function $d : V \times V \to \mathbb{R}^+$ defined by $d(\mathbf{x}, \mathbf{y}) = \|x - y\|_2$ is a metric on V.
- A normed vector space is a metric space, but a metric space need not be a vector space.
- The concept of a vector space is a generalization of the concept of a normed vector space.

Overview

Summary: Mod 2 Lecture 2

- Vector spaces
- Inner product spaces
- Normed vector spaces
- Metric spaces

Contents: Mod 2 Lecture 3

- Span of a vector space
- Basis of a vector space
- Vector subspace