ALTERNATING-CURRENT CIRCUITS

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THIRD EDITION

NEW YORK · JOHN WILEY & SONS, INC.
LONDON · CHAPMAN & HALL, LIMITED

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PREFACE TO THIRD EDITION

Experience with the second edition has suggested a number of changes and additions of importance to students majoring in either power or communication work. Although these additions add some material to the text, it is possible to omit them without breaking the continuity or theoretical structure of the subject matter. The addition of the loss and phase characteristics of elementary four-terminal networks at the close of Chapter IV, for example, is a subject of considerable importance to communication majors but is not essential to an understanding of the material that follows.

In Chapter V the material pertaining to the Q of electrical circuits has been revised to make this subject more directly useful and understandable for the reader. The nodal method of circuit analysis is included for the first time because of its importance to the engineer engaged in the analysis and design of vacuum tube circuits.

An additional method of wave analysis which reduces materially the amount of labor required to analyze certain classes of waves has been incorporated in Chapter VI.

A method of designing tuned coupled circuits to have a specified decibel variation over a specified band width has been included in Chapter VII. Also incorporated is the subject of polarity markings employing a dot-marked terminal.

The three-origin vector diagram of a polyphase circuit which is particularly useful to power engineers has been added to Chapter VIII.

Minor changes have been made throughout the text in an attempt to aid the student to grasp more readily the principles presented.

The end-of-chapter problems have been rather completely revised, and additional problems have been included.

We wish to express our appreciation to all who have submitted criticisms and suggestions concerning the first and second editions, particularly to Professor E. M. Strong of Cornell University, Professor W. B. Shepperd of Pennsylvania State College, and Professors H. R. Reed and H. W. Price of the University of Maryland.

R. M. K.
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PREFACE TO FIRST EDITION

This book is primarily written as a textbook for courses in alternating-current circuits as offered to junior electrical engineering students by most engineering colleges. It is assumed that the student has finished the usual courses in differential and integral calculus, or at least has some knowledge of differentiation and integration. An endeavor has been made to arrange the material in a logical sequence so as to lead the student gradually from the simple to the more complex analyses in alternating-current circuits.

The method of presentation is an outgrowth of the teaching experience which the authors have had at several institutions, and an effort has been made to produce a teachable book. In carrying out this idea, free use has been made of illustrative examples and line drawings. Also a number of illustrative oscillograms of actual circuit performance have been included. In order that many of these oscillograms may be made the basis of further study, rather complete legends have been given.

Problems for which answers are given have been included in many places in the text material immediately following the presentation of certain principles. These problems are primarily intended as exercises which will help the student determine for himself whether he has a workable knowledge of the principles involved. The sequence of the problems at the end of the chapters corresponds to the order in which the text material is presented. The end-of-chapter problems, therefore, form a suitable list from which regular assignments may be made.

To make the book more useful to the student, and to the practicing engineer, it was deemed desirable to include a greater amount of material than can be covered in the average college course as now given, provided such subject matter could be omitted without loss in continuity and without affecting the students' preparation for studying succeeding chapters.

With the exception of the fundamentals of symmetrical components in Chapter XIV which are necessary to understand Chapter XV, any part or all of any chapter after Chapter X can be omitted without affecting the students' preparation for studying succeeding chapters. Beginning with Chapter XI the remainder of the text is, for the most part, made up of extensions and applications of the principles studied

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in the first ten chapters. In consequence, selected parts of the last six chapters may be studied to the extent of the time available. Chapter VIII will also be found to contain a rather large amount of material that is of interest to students and desired by a number of teachers but may be omitted without affecting the reader's preparation for understanding subsequent chapters.

We acknowledge our indebtedness to the earlier writers in this field and to our many colleagues who have assisted and encouraged us in the production of this book. In particular we wish to thank Mr. J. L. Potter for his advice and assistance.

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CONTENTS

I.	General Concepts and Definitions	1
II.	Instantaneous Current and Power (Sinusoidal Voltage Applied)	12
III.	Effective Current and Voltage — Average Power	42
IV.	Vector Algebra (As Applied to A-C Circuit Analysis)	63
V.	Sinusoidal Single-Phase Circuit Analysis	97
VI.	Non-Sinusoidal Waves	161
VII.	Coupled Circuits	211
VIII.	Balanced Polyphase Circuits	262
IX.	Unbalanced Polyphase Circuits	308
X.	Alternating-Current Measurements	364
XI.	Determination of Circuit Parameters	393
XII.	Transmission Line Calculations	410
XIII.	Electric Wave Filters	436
XIV.	Symmetrical Components	490
XV.	Power System Short-Circuit Calculations	521
XVI.	Transient Conditions	548
	Index	587

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CHAPTER I

GENERAL CONCEPTS AND DEFINITIONS

Early History. The first successful electrical power system in the United States was probably Edison's direct-current installation in New York City. This station was performing creditably in 1885. Alternating-current power systems began commercially with the Great Barrington (Massachusetts) installation in 1886.

During the decade 1907–17, which followed the invention of the three-electrode vacuum tube, sustained oscillatory currents at high frequencies became a reality. These high-frequency oscillatory or alternating currents are essential to all modern radio, television, and radar forms of communication.

The outstanding advantage of a-c systems (as contrasted with d-c systems) is the relative ease with which alternating potential differences can be generated, amplified, and otherwise transformed in magnitude. The result is that, at the present time, approximately 95 per cent of the electrical energy consumed in the United States is generated, transmitted, and actually utilized in the form of alternating current. In the power field the annual energy consumption amounts to about 300 billion kilowatthours. In the communication field several thousand broadcast stations (of the AM, FM, and television variety) employ alternating potential differences to generate their carrier waves.

Generation of Alternating Potential Differences. When magnets are moved relative to electrical conductors as shown in Fig. 1, there is induced in the conductors a potential difference or emf. In accordance with Faraday's law, $e = -N\dfrac{d\phi}{dt}$ or its equivalent $e = N'Blv$ and the emf varies with time. For the instant depicted in Fig. 1, the application of one of the rules for finding the magnitude and direction of an induced emf will show that the emf induced in the armature conductors is zero, since at that instant no flux is being cut by these conductors. One-eighth revolution later, however, the induced emf is of maximum magnitude and of such a direction as to establish a voltage rise *from* terminal *a to* terminal *d*. One-quarter of a revolution after the position shown in Fig. 1 the induced emf will again be zero. Three-eighths of a revolution from the reference position the emf will again be of maxi-

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mum magnitude but so directed as to establish a voltage rise *from* terminal *d to* terminal *a*.

Thus the terminals *a* and *d* of the generator become alternately positive and negative relative to each other, and a time-varying potential

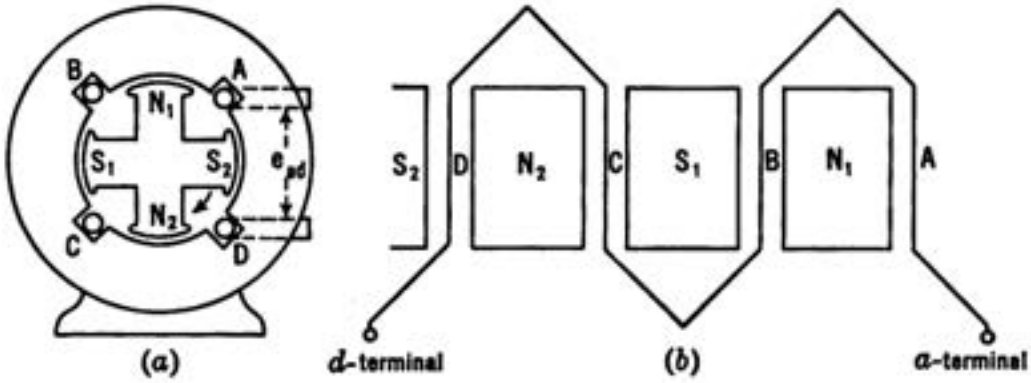

FIG. 1. (*a*) A four-pole, four-conductor a-c generator of the revolving field type. (*b*) Developed diagram showing method of connecting conductors *A*, *B*, *C*, and *D*. Pole faces are toward the reader.

difference of the general nature shown in Oscillogram 1 (page 7) is developed.

In communication systems, vacuum tubes (working in conjunction with suitable electrical circuits) produce alternating currents of higher

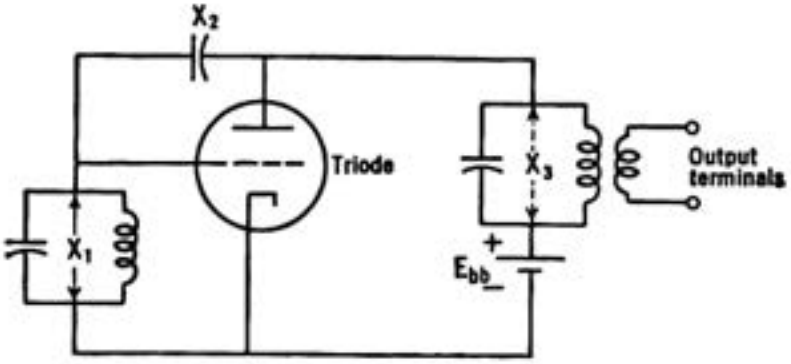

FIG. 2. Circuit arrangement of a simple triode oscillator.

frequencies than those obtainable with rotating equipment. A common triode oscillator circuit is shown schematically in Fig. 2. The a-c energy developed across the output terminals is actually derived from the d-c supply voltage labeled E_{bb}, but it is not expected that the reader will understand the conversion from direct current to alternating current which takes place in Fig. 2 until after he has studied the subject of electrical resonance. The only purpose in mentioning the triode oscillator at this stage is to acquaint the reader with the fact that high-

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frequency alternating currents can be produced with very simple circuit configurations. Many simple circuit configurations other than that shown in Fig. 2 may be used for this purpose.

Definition of Alternating Current. An alternating current, as the name implies, goes through a series of different values both positive and negative in a period of time T, after which it continuously repeats this same series of values in a cyclic manner as indicated in Fig. 3c.

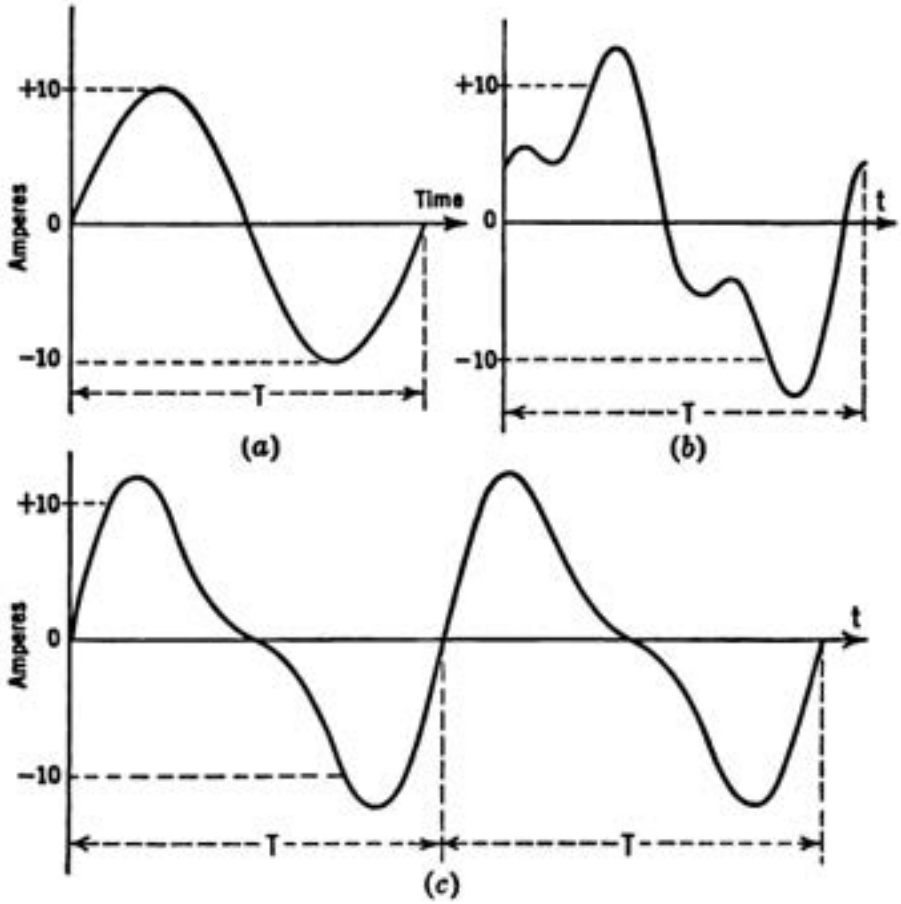

FIG. 3. Wave forms of three a-c variations. T is the period (or duration) of one cycle.

In the current A.I.E.E. "Definitions of Electrical Terms," an *alternating current* is defined in terms of a periodic current, and the latter in terms of an oscillating current.

"An *oscillating current* is a current which alternately increases and decreases in magnitude with respect to time according to some definite law.

"A *periodic current* is an oscillating current the values of which recur at equal intervals of time. Thus

$$i = I_0 + I_1 \sin(\omega t + \alpha_1) + I_2 \sin(2\omega t + \alpha_2) + \cdots \qquad (1)$$

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where i = the instantaneous value of a periodic current at time t

$I_0, I_1, I_2, \alpha_1, \alpha_2$ = constants (positive, negative, or zero)

$\omega = \dfrac{2\pi}{T}$ (T being the period).

"An *alternating current* is a periodic current, the average value of which over a period is zero. The equation for an alternating current is the same as that for a periodic current except that $I_0 = 0$."

Examples. In Fig. 3*a*, $i = 10 \sin \omega t$ amperes; in Fig. 3*b*, $i = 10 \sin \omega t + 4 \sin (3\omega t + 90°)$ amperes; and, in Fig. 3*c*, $i = 10 \sin \omega t + 4 \sin 2\omega t$ amperes.

Period and Cycle. The *period* of an alternating current or voltage is the smallest value of time which separates recurring values[1] of the alternating quantity. The period of time which separates these recurring values is shown in Fig. 3 as T, the symbol normally employed to designate the *period* of one cycle of an alternating quantity.

One complete set of positive and negative values of an alternating quantity is called a *cycle*. Thus Figs. 3*a* and 3*b* each depict one cycle. A cycle is sometimes specified in terms of angular measure since, as will be shown presently, ω in equation (1) actually represents angular velocity. One complete cycle is then said to extend over 360° or 2π radians of angular measure.

Frequency. Frequency is the number of cycles per second. Unless otherwise stated, the term "cycles" implies "cycles per second."

In the rotating machine of Fig. 1, it is apparent that a complete cycle is produced in the armature conductors when these conductors are cut by the flux from a pair of poles or, in this case, one-half revolution of the rotating field. Each conductor will be cut by two pairs of poles in one revolution of the field structure, and two complete cycles of emf will be developed in the armature winding per revolution.

In general, for a p-pole machine the number of cycles per revolution is $p/2$, and, if the speed of rotation in revolutions per second is represented by rps, the equation for frequency is:

$$f = \frac{p\ (\text{rps})}{2} \quad \text{cycles per second} \tag{2}$$

Since T is the time (or duration) of one cycle, it is plain that:

$$f = \frac{1}{T} \quad \text{cycles per second} \tag{3}$$

if T is expressed in seconds.

[1] The mathematical meaning of "recurring values" is implied in this definition, namely, that at least one complete *set* of values intervenes between two recurring values.

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Example. Let it be required to find the *frequency* and the *period* of the emf generated in the armature winding of Fig. 1 if the speed of rotation is 1800 rpm.

$$f = \frac{4}{2} \times \frac{1800}{60} = 60 \text{ cycles per second}$$

$$T = \frac{1}{f} = \frac{1}{60} \text{ second}$$

The common power plant frequencies in use today are 60, 50, and 25 cycles, the first mentioned being by far the most prevalent in this country. Abroad 50 cycles is very common, and some foreign railways use frequencies considerably less than 25 cycles. A 25-cycle variation causes a noticeable flicker in incandescent lamps; hence it is undesirable for lighting. Formerly 25 cycles was used for power work but, with the advent of a better understanding of the laws governing a-c power transmission and the design of machinery, this frequency is rapidly being superseded. In general, 60-cycle apparatus is lighter and costs less than 25-cycle equipment. The difference is similar to that between high- and low-speed d-c machines.

Audio frequencies range from approximately 16 cycles to approximately 20,000 cycles, voice frequencies occupying the range from about 200 to 2500 cycles. Carefully engineered audio systems, like some theater installations, are designed to accommodate frequencies from 30 to 12,000 cycles.

Radio frequencies range from about 50,000 cycles to 10^{10} cycles, the AM program broadcast band being from 540 to 1600 kilocycles, and the FM and television broadcast bands being from about 50 to 200 megacycles. Radar systems often operate with a carrier frequency of 3000 or 10,000 megacycles.

At the 1947 Atlantic City Conference it was agreed to express frequencies as employed by radio engineers in kilocycles per second at and below 30,000 kilocycles per second and in megacycles per second above this frequency. The present FCC standard band designations follow.

VLF	(very low frequency)	less than 30 kilocycles per second
LF	(low frequency)	30–300 kilocycles per second
MF	(medium frequency)	300–3000 kilocycles per second
HF	(high frequency)	3000–30,000 kilocycles per second
VHF	(very high frequency)	30,000 kilocycles per second–300 megacycles per second
UHF	(ultra high frequency)	300–3000 megacycles per second
SHF	(super high frequency)	3000–30,000 megacycles per second
EHF	(extremely high frequency)	30,000–300,000 megacycles per second

Wave Form. The shape of the curve resulting from a plot of instantaneous values of voltage or current as ordinate against time as

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abscissa is its *wave form* or *wave shape*. It has been shown that the passage of a pair of poles past a given reference point on the stator of Fig. 1 produced a complete cycle of generated or induced emf. This corresponded to 2π electrical radians, or 360 electrical degrees. In other words, one cycle occurs in or occupies 2π radians, or 360°. The abscissa, instead of being expressed in terms of time in seconds, can be and is quite frequently expressed in terms of radians or degrees. Thus one cycle occurs in 2π radians, or 360°.

Angular Velocity or Angular Frequency. In the preceding article a complete cycle was seen to correspond to 2π radians. The time for a complete cycle was defined as the period T. Hence the angular velocity ω in radians per second is $2\pi/T$. Therefore:

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{4}$$

Equation (4) specifies angular velocity in terms of frequency, and this velocity is called electrical[2] angular velocity or *angular frequency*.

If equations (2) and (4) are combined:

$$\omega = 2\pi f = 2\pi \frac{p}{2} \text{ (rps)} = \frac{p}{2} [2\pi \text{ (rps)}] \tag{5}$$

Equation (5) shows that electrical angular velocity equals (pairs of poles) times (mechanical angular velocity) in generators of the type shown in Fig. 1.

Alternating Voltages and Currents Represented by Sine Waves. Whereas the foregoing has referred to waves of any shape, the usual attempt is to secure a sine wave. Oscillogram 1 is a photographic record of the potential difference produced by a so-called sine-wave generator.

Many of the alternating waves met with in practice approximate a sine wave very closely. Alternating-voltage and -current calculations are therefore based on sine waves. (The method whereby non-sinusoidal waves are expressed so as to be calculated according to the laws of sine waves is explained in Chapter VI.) A true sine wave is shown in Fig. 4. The equation for it is

$$i = I_m \sin \omega t \tag{6}$$

[2] Mechanical angular velocity, 2π(rps) radians per second, is not to be confused with electrical angular velocity. In Fig. 1 the two are related by the factor $p/2$, but in vacuum tube oscillators of the type shown in Fig. 2 the electrical angular velocity or angular frequency is defined almost solely by the inductance and capacitance employed at the X_1 and X_3 positions in the circuit.

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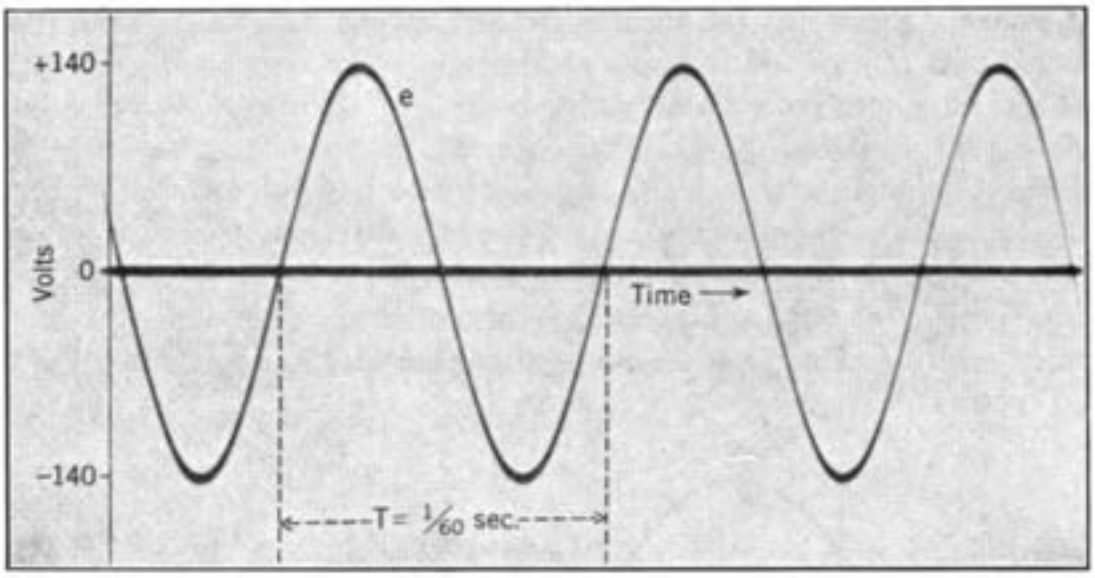

OSCILLOGRAM 1. Emf of a sine-wave generator.

where ωt is expressed in radians and is called the time angle, i is the instantaneous value of current, and I_m is the maximum value of the sinusoidal variation. Since ωt represents an angle, equation (6) may be expressed in terms of radians or degrees. Thus

$$i = I_m \sin \alpha \qquad (7)$$

where α is in degrees or radians. Equation (6) expresses the current as a sinusoidal variation with respect to time, whereas equation (7) expresses it as a function of angular measure.

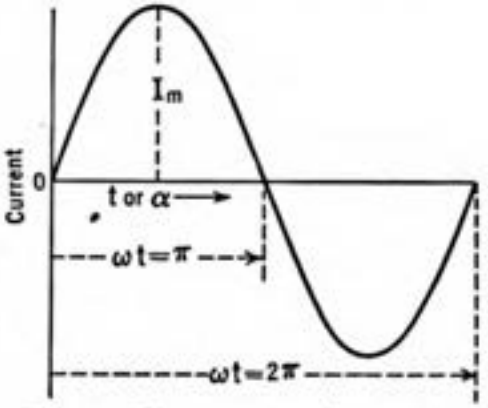

FIG. 4. Sine wave may be expressed as $I_m \sin \alpha$ or as $I_m \sin \omega t$.

Alternating Potential Difference. Alternating voltage or potential difference may take the form of a generated (or induced) emf or the form of a potential drop, sometimes abbreviated p.d. In the interest of clear thinking these two forms of voltage should be distinguished from one another. Instantaneous values of generated or induced emf's will be designated by e, and instantaneous values of potential drops by the symbol v. Similarly E_m and V_m will be used to distinguish a maximum value of induced voltage from a maximum value of potential drop. Corresponding distinctions will be made between other particular values of induced voltages and voltage drops.

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Phase. Phase (as the term is defined by the A.I.E.E.) is the fractional part of a period through which time or the associated time angle ωt has advanced from an arbitrary reference. In the case of a simple sinusoidal variation, the origin is usually taken as the last previous passage through zero from the negative to the positive direction. Thus one phase of a sine wave is $\frac{1}{12}$ of a period (or 30° from the origin) where the ordinate is one-half the maximum ordinate; another phase is $\frac{1}{4}$ of a period (or 90° from the origin) where the ordinate has its maximum positive value; and so on for any other fractional part of T (or of $\omega T = 2\pi$).

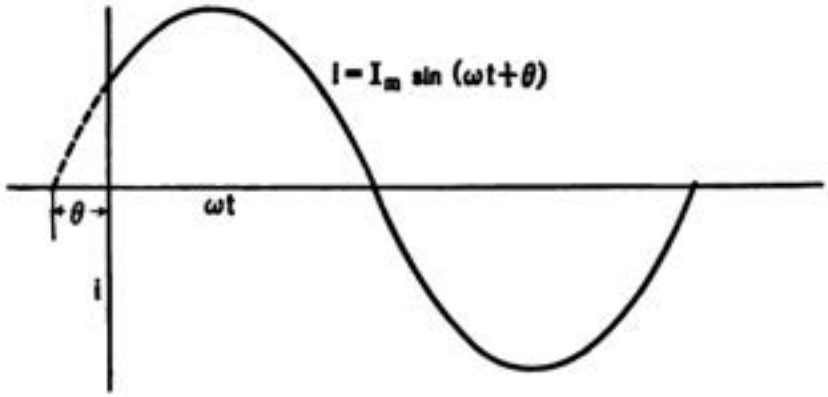

FIG. 5. Phase angle θ of a sine wave.

In accordance with the above definition, the phase angle of a single wave is the angle from the zero point on the wave to the value at the point from which time is reckoned. Thus $i = I_m \sin(\omega t + \theta)$ represents a sine wave of current with a phase angle θ. The phase of the wave from which time is reckoned (i.e., when $t = 0$) is $i = I_m \sin \theta$. The angle θ is the phase angle of the current with respect to the point where $i = 0$ as a reference. These principles are illustrated in Fig. 5.

The phase angle when used in connection with a single alternating quantity merely provides a simple analytical method of starting the variation in question at any point along the wave. As such it is of little importance in steady-state analysis in contrast with its great usefulness in the analysis of transient conditions.

Phase Difference. The phase angle is a very important device for properly locating different alternating quantities with respect to one another. For example, if the applied voltage is

$$v = V_m \sin \omega t \tag{8}$$

and it is known from the nature and magnitude of the circuit parameters that the current comes to a corresponding point on its wave before the

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voltage wave by θ degrees, the current can be expressed as

$$i = I_m \sin (\omega t + \theta) \tag{9}$$

Figure 6 illustrates the phase positions of v and i for $\theta = 45°$. The current in this case is said to lead the voltage by 45°, or the voltage is said to lag the current by 45°. A given alternating quantity lags another if it comes to a certain point on its wave later than the other one comes to the corresponding point on its wave. Another way of saying the same thing is that the positive maximum of the leading

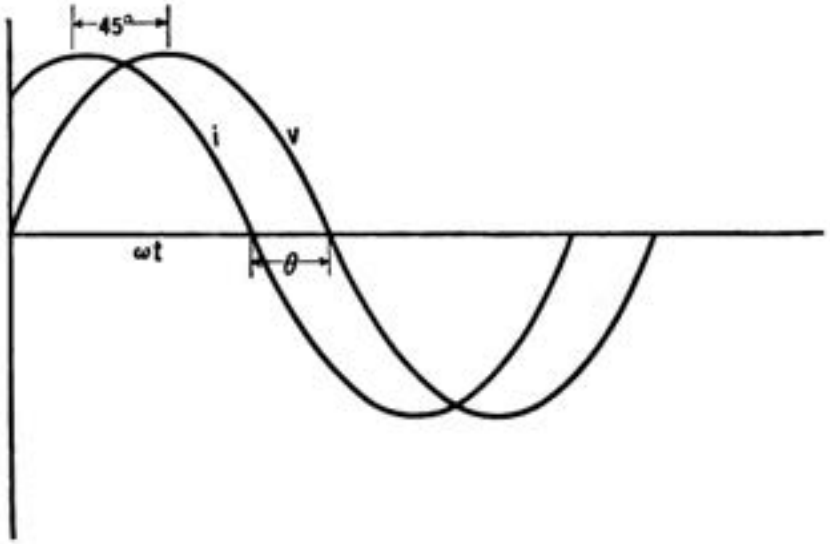

FIG. 6. Illustrating a case where the i wave leads the v wave by $\theta = 45°$.

quantity occurs before the positive maximum of the lagging quantity. Thus it is said that there is a phase difference of 45° between the two waves. The angle of phase difference is the difference of the phase angles of the two waves. Thus, if $e = 100 \sin (\omega t + 45°)$ and $i = 10 \sin (\omega t - 15°)$, the angle of phase difference is $45° - (-15°) = 60°$.

Oscillogram 2 illustrates the actual phase relation between an applied sinusoidal voltage and the resulting current that flows in a particular circuit. Inspection of the oscillogram will show that the current lags the voltage in this particular case by approximately 60°. Oscillogram 3 illustrates a case where the current and power waves are distinctly non-sinusoidal.

PROBLEMS

1. (*a*) What is the frequency of a 20-pole alternator when running at 360 rpm? (*b*) At what speeds should a 4-pole alternator run to yield 25, 30, 50, and 60 cycles?

2. How many poles are required on an alternator which runs at 150 rpm to develop 50 cycles per second?

3. At what speed must a 28-pole alternator run to develop 60 cycles per second?

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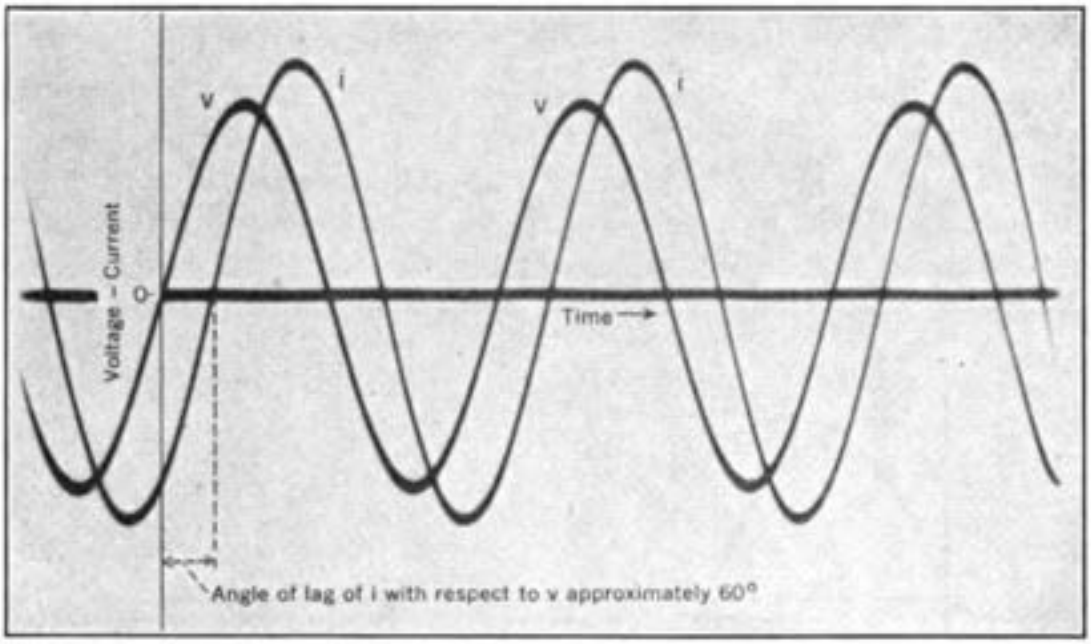
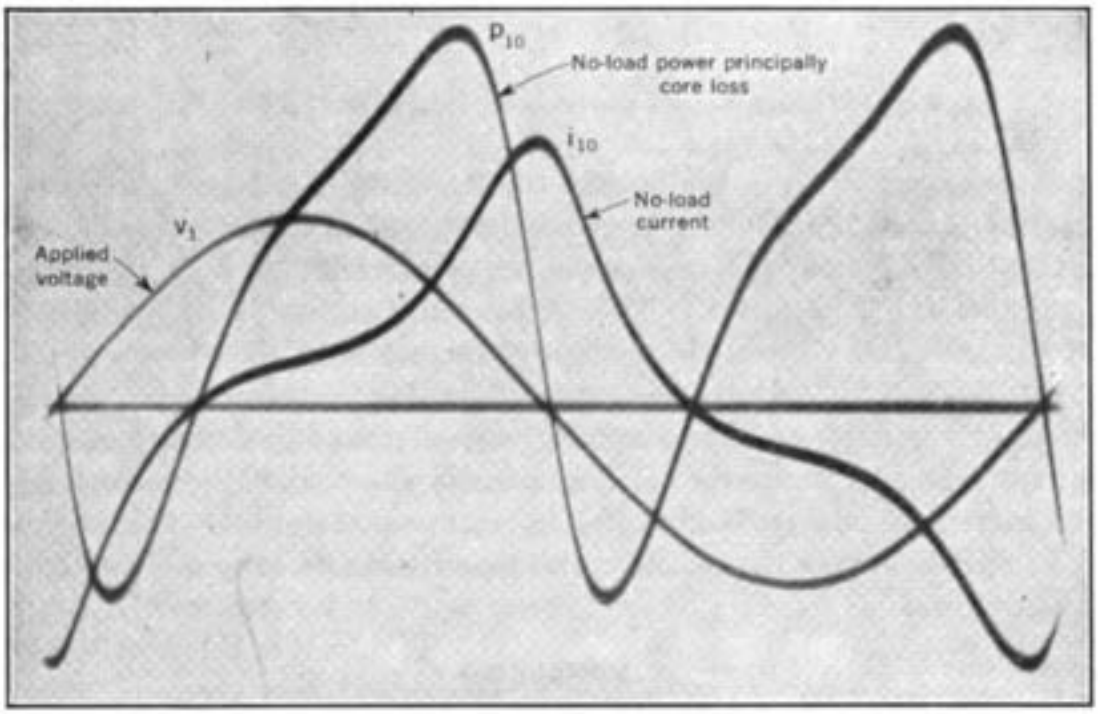


Oscillogram 2. Photographic record of voltage and current for a circuit containing resistance and inductance.



Oscillogram 3. Oscillographic records of the no-load current and no-load power taken by the primary of an iron-core transformer. The applied voltage variation, v_1, is also shown.

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4. What is the mechanical angular velocity of the machine in Problem 3? What is the electrical angular velocity?

5. Express as a sine function of time a 50-cycle alternating current which has a maximum value of 10 amperes. What is the angular velocity of this current wave?

6. Express an alternating current of 10 amperes maximum value which has an angular velocity of 377 radians per second as a cosine function of time. What is the frequency of this wave?

7. Express the equation of the current wave of Problem 5 if time is reckoned from the positive maximum value of the wave. Also express it for each possibility when time is reckoned from the negative 5-ampere value of the wave.

8. The time variation of a voltage wave is given by $e = 100 \sin 157t$ volts, where t is expressed in seconds.

(*a*) What is the maximum value of the voltage?

(*b*) What is the frequency of the voltage variation?

(*c*) If $e = 100 \sin (157t + 30°)$, what is the maximum value of the voltage? the frequency?

9. What are the maximum and minimum rates of change of the voltage depicted in Oscillogram 1, page 7, if the maximum voltage is 140 volts? Express results in volts per second.

10. At what instantaneous value of current is the 60-cycle current wave $i = 10 \sin (\omega t - 30°)$ amperes changing at the rate of 3265 amperes per second? (*b*) At 2260 amperes per second?

11. Find the maximum value of a 50-cycle current wave that is changing at 2000 amperes per second at an instantaneous value 30° from the maximum value of the wave.

12. If $v = 100 \sin (\omega t - 30°)$ and $i = 10 \sin (\omega t - 60°)$, what is the angle of phase difference between the current and voltage waves? Which wave leads?

13. Find the angle of phase difference between $v = 100 \cos (\omega t - 30°)$ and $i = -10 \sin (\omega t - 60°)$. Which wave lags?

14. A voltage has for its equation $v = 100 \cos \omega t$. Write the equation of a current wave of 10 amperes maximum which leads the specified voltage wave by $\frac{1}{8}$ of a cycle. Let angular measure be expressed in radians in this particular case.

15. In the presence of an audio signal (or tone) having an angular frequency of ω radians per second, the resistance of a particular carbon telephone transmitter may be represented by the expression:

$$(R_t - r \sin \omega t) \text{ ohms}$$

This transmitter is connected to a battery voltage of E volts through connecting line-wire resistance of R_l ohms. The current flowing in the circuit is

$$i = \frac{E}{(R_l + R_t) - r \sin \omega t}$$

Show that the maximum magnitude of the fundamental-frequency component of current (i.e., the current component having an angular frequency of ω radians per second) is $Er/(R_l + R_t)^2$ if the current is expressed as:

$$i = I_{dc} + I_{m1} \sin \omega t + I_{m2} \cos 2\omega t$$

where it is assumed that any contributions to I_{dc}, I_{m1}, and I_{m2} due to terms of higher degree than $\sin^2 \omega t$ are negligibly small.

16. Find the numerical values of I_{dc}, I_{m1}, and I_{m2} in Problem 15 if $R_l = 50$ ohms, $R_t = 50$ ohms, $r = 10$ ohms, and $E = 20$ volts.

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CHAPTER II

INSTANTANEOUS CURRENT AND POWER (SINUSOIDAL VOLTAGE APPLIED)

The Basic Problems. The basic problems encountered in a-c circuit analysis are:

1. Given the voltage applied to a pair of terminals; find the current and power passing these terminals.
2. Given the current passing a pair of terminals; find the voltage established across these terminals and the power passing these terminals.

These basic problems may be solved in a variety of ways. They will be approached in this chapter by way of instantaneous values of voltage and current, since this method of approach lends itself readily to simple notation and at the same time emphasizes the physical significance of the relationships involved.

In order to keep the basic problem at an elementary level, only sinusoidal time variations and steady-state values of voltage and current will be considered. Under these conditions, the relationship between the voltage across the driving terminals of a linear circuit element and the current through the element is a very simple one, namely, the impedance function.

A large portion of this chapter is devoted to the derivation of impedance functions. In these derivations, as well as in the calculation of instantaneous power, we shall require a clear understanding of the relationship between the equational or analytical forms of sinusoidal time-varying quantities and the graphs of these quantities when plotted against time (or time angle) as the independent variable. Sinusoidal variations are completely described as functions of time when the maximum values, angular frequencies, and zero time references are specified.

Examples. If a voltage is described as having sinusoidal wave form, a maximum value of 200 volts, and an angular velocity of 377 radians per second (60 cycles per second), and it is desired to reckon time from the point of zero voltage where dv/dt is positive, as illustrated in Fig. 1*a*, the mathematical expression for the alternating voltage as a function of time, t, is:

$$v = 200 \sin 377t \text{ volts} \tag{1}$$

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If it is desired to reckon time from some other point along the voltage wave, it is simply necessary to add to the angle $377t$ in the above equation an angle equal to the angular displacement between $v = 0$ (dv/dt positive) and the point on the voltage wave from which it is desired to reckon time. If it is assumed that time is

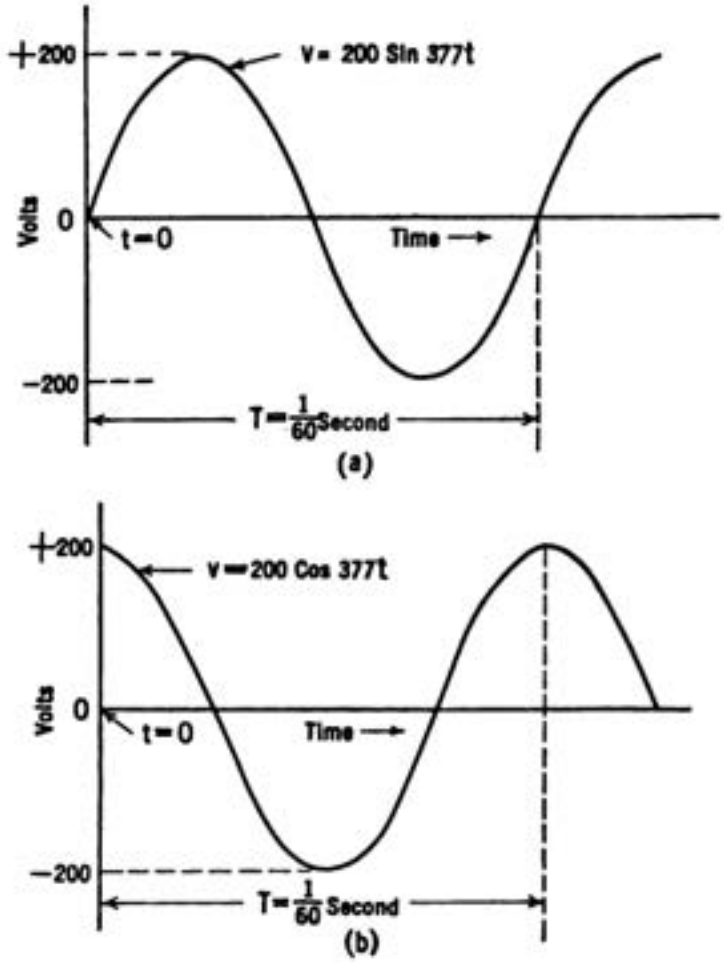

FIG. 1. Graphical representations of equations (1) and (2)

to be reckoned from the point of positive maximum voltage, the angular displacement referred to above is $+90°$, and the expression for voltage becomes:

$$v = 200 \sin (377t + 90°) = 200 \cos 377t \text{ volts} \tag{2}$$

This type of variation is shown in Fig. 1*b*.

Equations (1) and (2) describe exactly the same type of voltage variation except for the $t = 0$ reference.

The current that flows in a circuit as a result of applying a sinusoidal voltage is governed in magnitude and phase by the circuit parameters (resistance R, self-inductance L, capacitance C, and mutual inductance M) and the angular velocity or frequency of the applied voltage. In one sense of the word the angular velocity is an a-c circuit parameter. If the circuit parameters are constant, the current that flows will be

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of sinusoidal wave form but will, in general, differ in phase from the sinusoidal applied voltage.

Mathematically a particular type of function is required to relate voltage and current in an a-c circuit. The one generally employed is called the impedance function or simply the impedance of the circuit. The impedance function must tell two important facts: (1) the ratio of V_m to I_m;[1] and (2) the phase angle between the waves of voltage and current. A special type of notation is required to signify the two properties of the impedance function in abbreviated form. One such type of notation is:

$$Z \underline{/\text{angle}}$$

The above expression does not signify the multiplication of Z and $\underline{/\text{angle}}$. Z is the magnitude of the impedance and in a particular case is represented by a certain number of ohms. It defines the ratio of

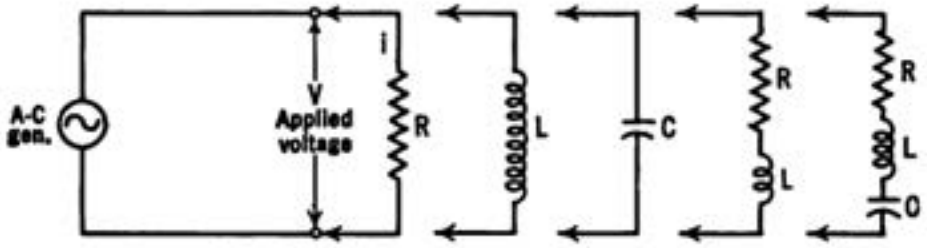

FIG. 2. Elementary circuit arrangements of R, L, and C.

V_m to I_m. The angle associated with Z, if it is positive, defines the *lead* of the voltage with respect to the current. In accordance with the convention thus adopted a positive angle specifies the number of degrees or radians by which the current lags the voltage.

The determination of the complete impedance function for various combinations of R, L, and C is the first step in a-c circuit analysis. The combinations considered in the present chapter are shown in diagrammatic fashion in Fig. 2.

The *R* Branch. The consideration of a circuit element which possesses only ohmic resistance is, of course, a hypothetical venture because some self-inductance is inevitably associated with any circuit configuration. However, the case may be approached in practice to a degree comparable to the accuracy of ordinary measurements. It is well known that *resistance* impedes the motion of electricity and causes an irreversible transformation of electrical energy into heat energy in accordance with Joule's law.

[1] It will be shown in Chapter III that the magnitude of the impedance Z defines the ratio of $V_{effective}$ to $I_{effective}$ as well as the ratio V_m to I_m.

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Impedance. The impedance of a simple R branch may be expressed as:

$$R\,\underline{/0^\circ} \text{ ohms}$$

The reason follows directly from Kirchhoff's emf law. If a voltage, $v = V_m \sin \omega t$, is applied to a branch of R resistance, Fig. 3, the equation for dynamic equilibrium is:

$$v = Ri = V_m \sin \omega t \qquad (3)$$

from which

$$i = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t \quad (4)$$

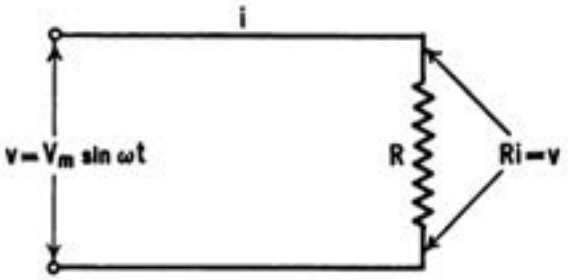

FIG. 3. The R branch.

From the above equation it is evident that $V_m/I_m = R$ and that the current wave is in time phase with the voltage wave. It is possible to express these facts in the single statement:

$$\mathbf{Z}_R = R\,\underline{/0^\circ}$$

In general, R is expressed directly in ohms, in which case $\mathbf{Z}_R$ is in ohms.

Power. The determination of the rate at which electrical energy is generated or absorbed is, in general, an important problem. Instantaneous power is symbolized by the lower-case letter p.

$$p = ei \quad \text{(generated power)}$$

$$p = vi \quad \text{(absorbed power)}$$

The present discussion confines itself to the determination of instantaneous absorbed power wherein positive values of p indicate that the circuit under consideration is receiving energy from the supplying source. Negative values of p indicate that the reactive elements of the circuit, if such are present, are actually releasing energy at a rate which is greater than the rate at which energy is being received.

In the present case, that of the simple R branch, all the energy produced by the instantaneous power absorbed is converted into heat. Presumably no reactive elements, inductance coils or condensers, are present. The instantaneous power is given by the product of equations (3) and (4).

$$p = vi = V_m I_m \sin^2 \omega t \qquad (5)$$

Since $\sin^2 \omega t = \frac{1}{2} - \frac{1}{2} \cos 2\omega t$, it follows that:

$$p = \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos 2\omega t \qquad (6)$$

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Figure 4 illustrates the component parts of equation (6). It will be observed from the above equation that the instantaneous power wave is a double-frequency variation, with respect to the frequency of the current or the voltage, which has an average positive value of $\frac{V_m I_m}{2}$. The cos $2\omega t$ term causes the instantaneous power to acquire periodically zero and $V_m I_m$ values. At no time does the power reach instantaneous negative values.

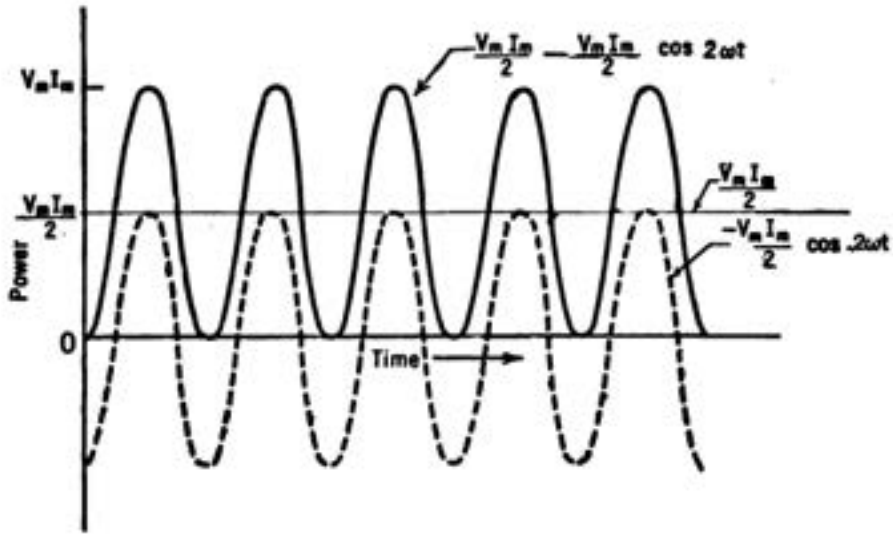

FIG. 4. Graphical representation of equation (6).

Photographic records of v, i, and p in a branch which approximates the purely resistive case are shown in Oscillogram 1. The oscillogram illustrates in a graphical manner the relations which have been derived for the R branch and substantiates the physical fact that voltage and current are in time phase in a resistive circuit.

The L Branch. If a circuit element of pure inductance, Fig. 5, is considered, the equation for dynamic equilibrium is:

$$v = L\frac{di}{dt} = V_m \sin \omega t \tag{7}$$

$$di = \frac{V_m}{L} \sin \omega t \, dt \tag{8}$$

After both sides of the above equation are integrated it follows that:

$$i = -\frac{V_m}{\omega L} \cos \omega t + c_1 \tag{9}$$

The constant of integration c_1 will be considered to be equal to zero

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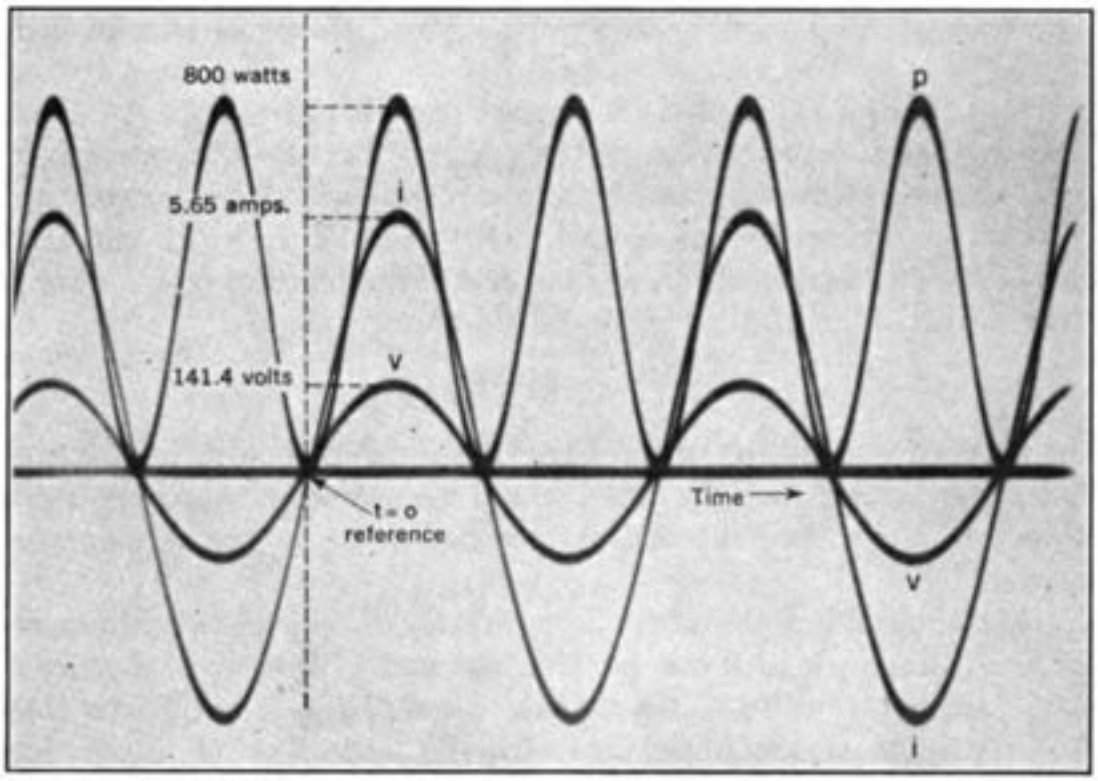

OSCILLOGRAM 1. Voltage, current, and power variations in a resistive circuit element. $R = 25$ ohms. If time is reckoned from the point of zero voltage (dv/dt positive): $V = 141.4 \sin 377t$ volts, $i = 5.65 \sin 377t$ amperes, $p = 400 - 400 \cos 754t$ watts, average power = 400 watts.

since only the steady-state current symmetrical about the zero axis is to be considered.[2]

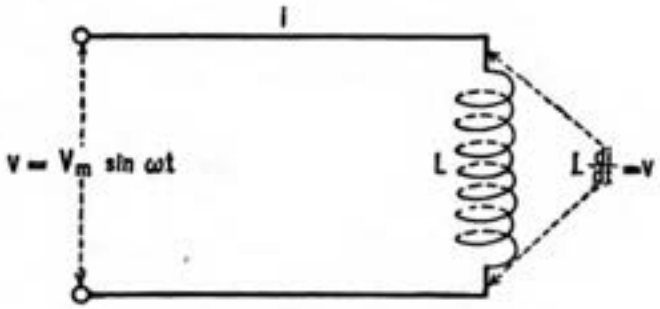

FIG. 5. The L branch.

Under the above conditions equation (9) reduces to:

$$i = \frac{V_m}{\omega L} \sin (\omega t - 90°) = I_m \sin (\omega t - 90°) \tag{10}$$

[2] In a general analysis, c_1 would be evaluated in terms of the boundary conditions under which the circuit is initially closed. Determined in this manner, c_1 would define the transient component of the current. c_1 is neglected here because transient components of the current are not to be considered at this time. In a physically realizable circuit the transient component is of short duration.

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Impedance. Inductance opposes the *rate of change* of current, and for this reason it is sometimes called *electrical inertia.* Since the inductance, L, limits the rate at which the current can change, it follows logically that L actually governs the maximum value of the current in an a-c circuit which is energized by a voltage of specified angular velocity.

It will be observed from equation (10) that $V_m/I_m = \omega L$ and that i lags v by one-quarter of a cycle or 90°. The impedance of a pure L branch is according to the convention previously adopted:

$$\mathbf{Z}_L = \omega L\underline{/90^\circ}$$

The reason for using the positive angle in connection with impedances that cause lagging currents will become more evident when the rules of vector algebra and the conventions pertaining to vector diagrams are considered.

The magnitude of the above impedance, ωL, is called *inductive reactance.* Inasmuch as the inductive reactance is directly proportional to the angular velocity of the driving voltage, $2\pi f$, it is obvious that the magnitude of the impedance offered to the flow of alternating current by a coil of fixed self-inductance, L, is directly proportional to frequency. When ω is expressed in radians per second and L is expressed in henrys, the inductive reactance, X_L, is in ohms.

$$X_L = \omega L = 2\pi f L \tag{11}$$

Example. The inductive reactance of a 10-millihenry inductance coil in a 60-cycle circuit is:

$$X_L = 2\pi \times 60 \times 0.010 = 3.77 \text{ ohms}$$

and

$$\mathbf{Z} = 3.77\underline{/90^\circ} \text{ ohms}$$

The inductive reactance of the same coil in a 60,000-cycle circuit is:

$$X_L = 2\pi \times 60{,}000 \times 0.010 = 3770 \text{ ohms}$$

If a 60-cycle sinusoidal voltage of maximum value equal to 100 volts is applied to the 10-millihenry inductance coil,

$$v = 100 \sin 377t \text{ volts}$$

and

$$i = \frac{100}{3.77} \sin (377t - 90^\circ) \text{ amperes}$$

Power and Energy. The instantaneous power delivered to the pure inductance branch as obtained by multiplying equation (7) by equation (10) is:

$$p = vi = [V_m \sin \omega t]\,[I_m \sin (\omega t - 90^\circ)] \tag{12}$$

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from which:

$$p = V_m I_m (-\sin \omega t \cos \omega t) \tag{13}$$

or

$$p = -\frac{V_m I_m}{2} \sin 2\omega t \tag{14}$$

Figure 6 illustrates the v, i, and p variations in a purely inductive branch. It will be observed that the power variation is again a double-frequency variation with respect to the frequency of the driving voltage. The fact that equation (14) indicates negative power during the first one-quarter of a cycle of the driving voltage, that is, from $t = 0$ to $t = T/4$, is the direct result of the choice of the time reference.[3]

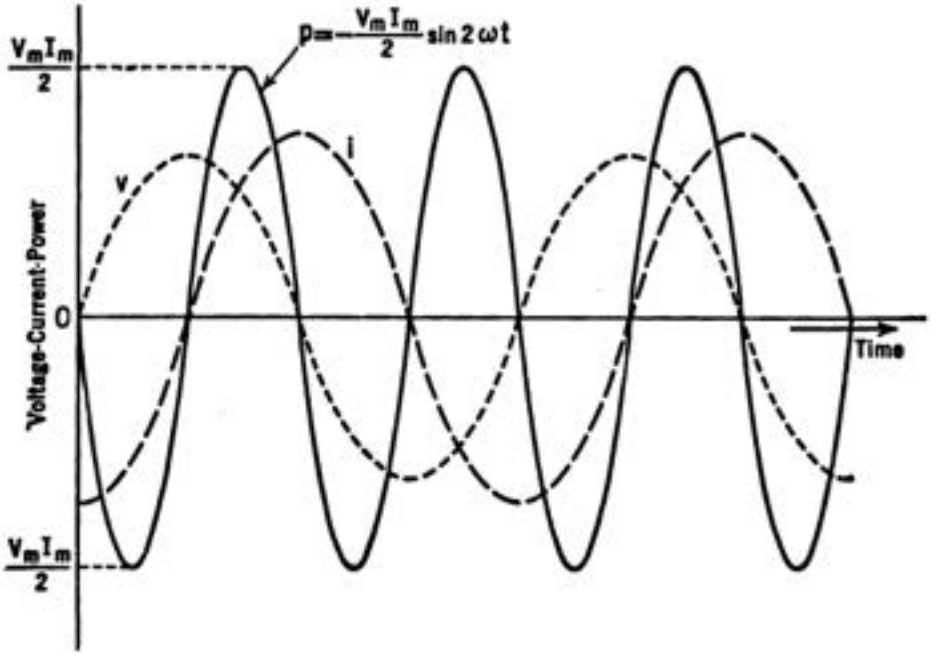

FIG. 6. Voltage, current, and power variations in a purely inductive branch.

Since steady-state conditions have been assumed, the circuit has presumably adjusted itself to the relative phase relations indicated by equations (7), (10), and (14).

Under the conditions which have been assumed, namely, a steady-state sinusoidal driving voltage and a purely inductive circuit, the power variation is symmetrical about the zero power axis. The average power absorbed is equal to zero. The implication is that the inductive

[3] The assumption of sinusoidal driving voltage, $v = V_m \sin \omega t$, automatically imposes the condition of $t = 0$ at the point of $v = 0$ (dv/dt positive). The beginner should not confuse the $t = 0$ reference of a steady-state variation with the time at which the circuit is initially energized.

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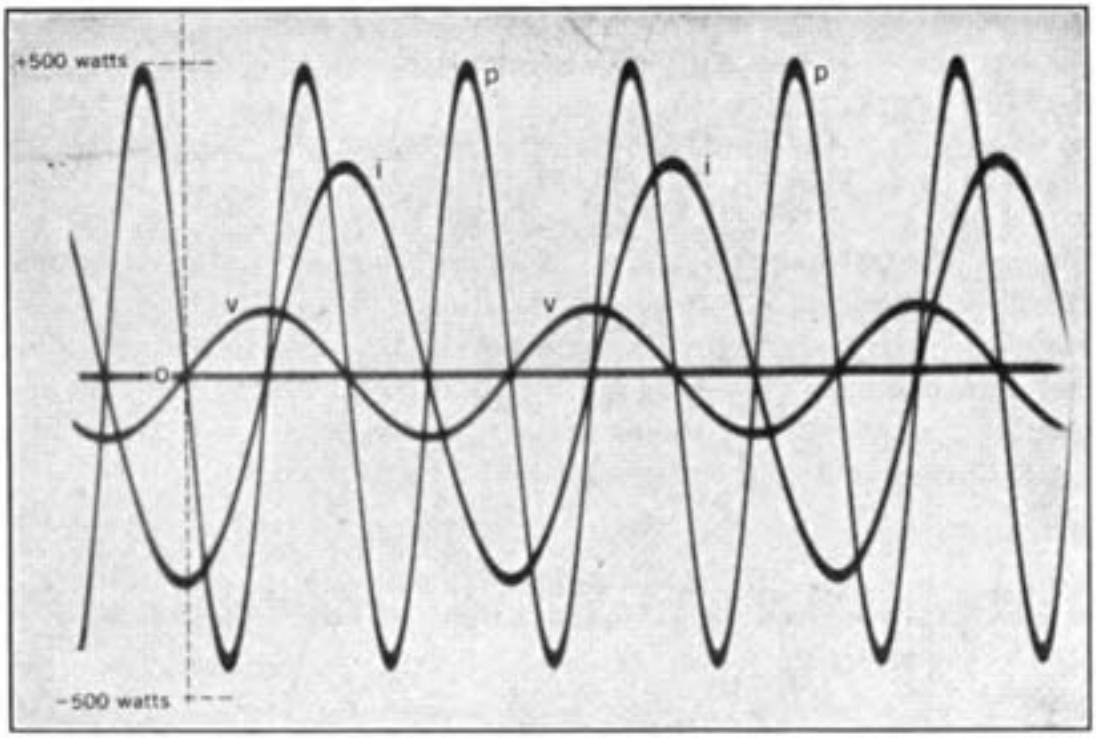

OSCILLOGRAM 2. Voltage, current, and power variations in a highly inductive circuit element. $L = 0.056$ henry, $f = 60$ cycles, $X_L = 21.2$ ohms, $R = 1.0$ ohm, $V_{max} = 141.4$ volts, $I_{max} = 6.66$ amperes, $P_{av} = 25$ watts approximately. Note the lag of the i wave with respect to the v wave; also the large negative power loops. Positive power peaks of approximately 500 watts are present even though the average power dissipated in the circuit element is only about 25 watts.

element receives energy from the source during one-quarter of a cycle of the applied voltage and returns exactly the same amount of energy to the driving source during the next one-quarter of a cycle. The exact amount of energy delivered to the circuit during a quarter of a cycle may be obtained by integrating any positive loop of the power wave, for example, integrating p between the limits of $t = T/4$ and $t = T/2$.

$$W_L = \int_{T/4}^{T/2} -\frac{V_m I_m}{2} \sin 2\omega t \, dt$$

$$= \frac{V_m I_m}{2\left(\frac{4\pi}{T}\right)} \left[\cos \frac{4\pi}{T} t\right]_{T/4}^{T/2}$$

$$= \frac{V_m I_m}{2\omega}$$

Since $V_m = \omega L I_m$,

$$W_L = \frac{(\omega L I_m) I_m}{2\omega} = \frac{L I_m^2}{2} \tag{15}$$

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If L is expressed in abhenrys and I_m in abamperes, the above energy is in ergs. If L and I_m are expressed in henrys and amperes respectively, W_L is given in joules.

Oscillogram 2 illustrates the relative phase relations in a circuit which approaches, to a fair degree of accuracy, the purely inductive arrangement that has been described mathematically.

The C Branch. If it is assumed that a sinusoidal voltage, $V_m \sin \omega t$, is applied to an ideal condenser as indicated in Fig. 7, the expression for steady-state equilibrium is:

$$v = \frac{q}{C} = V_m \sin \omega t \qquad (16)$$

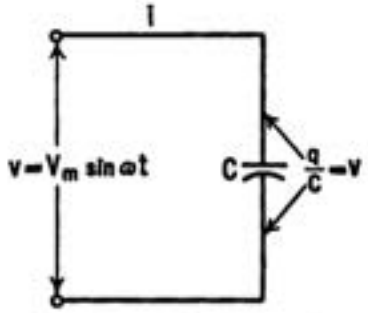

FIG. 7. The C branch.

When the above equation is differentiated with respect to time, it follows that:

$$\frac{dq}{dt} = V_m \omega C \cos \omega t \qquad (17)$$

or

$$i = \frac{V_m}{\frac{1}{\omega C}} \sin (\omega t + 90°) = I_m \sin (\omega t + 90°) \qquad (18)$$

Impedance. The ratio of V_m to I_m in the pure C branch is $1/\omega C$, and the current leads the applied voltage by one-quarter of a cycle or 90°. In accordance with the convention which has been adopted, the impedance of the C branch is:

$$\mathbf{Z}_C = \frac{1}{\omega C} \underline{/-90°}$$

The magnitude of the impedance, $1/\omega C$, is called *capacitive reactance*, and it is evident from the nature of the expression that capacitive reactance is inversely proportional to the frequency of the driving voltage and also inversely proportional to the capacitance of the condenser, C. A series circuit in which no condenser is present has infinite capacitance and, hence, zero capacitive reactance.

The impedance of a condenser causes the current to lead the voltage by 90°, whereas the impedance of an inductance causes the current to lag the voltage 90°. The effects of the two types of reactive elements as regards the phase of the resulting current are exactly opposite.

If, in the expression for capacitive reactance, ω is expressed in radians per second and C is expressed in farads, the resulting capacitive re-

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actance is in ohms. If the capacitance of the condenser is expressed in microfarads (abbreviated μf), the expression for capacitive reactance takes the form:

$$X_C = \frac{10^6}{\omega C_{\mu f}} \text{ ohms}$$

Example. The capacitive reactance of a 15-μf condenser in a 25-cycle circuit is:

$$X_C = \frac{10^6}{2\pi \times 25 \times 15} = 425 \text{ ohms}$$

and

$$Z_C = 425\angle -90^\circ \text{ ohms}$$

The capacitive reactance of the same condenser to a 250-cycle driving voltage is:

$$X_C = \frac{10^6}{2\pi \times 250 \times 15} = 42.5 \text{ ohms}$$

If a 25-cycle sinusoidal voltage of maximum value equal to 200 volts is applied to the 15-μf condenser:

$$v = 200 \sin (157t) \text{ volts}$$

and

$$i = \frac{200}{425} \sin (157t + 90^\circ) \text{ amperes}$$

Power and Energy. The instantaneous power delivered to the C branch is:

$$p = vi = [V_m \sin \omega t]\,[I_m \sin (\omega t + 90^\circ)] \tag{19}$$

from which:

$$p = V_m I_m (\sin \omega t \cos \omega t) \tag{20}$$

or

$$p = \frac{V_m I_m}{2} \sin 2\omega t \tag{21}$$

The phase relations of v, i, and p in a purely capacitive branch are shown in Fig. 8. The double-frequency power variation is, as in the pure L branch, symmetrical about the zero power axis. In the present case the condenser receives energy from the source during the first quarter of a cycle of the voltage variation and returns the same amount during the second quarter cycle, etc. The average power absorbed over an integral number of half cycles is, obviously, equal to zero.

The amount of energy received by the condenser during a quarter cycle may be determined by integrating the power wave over any positive loop; for example, integrating equation (21) between the limits

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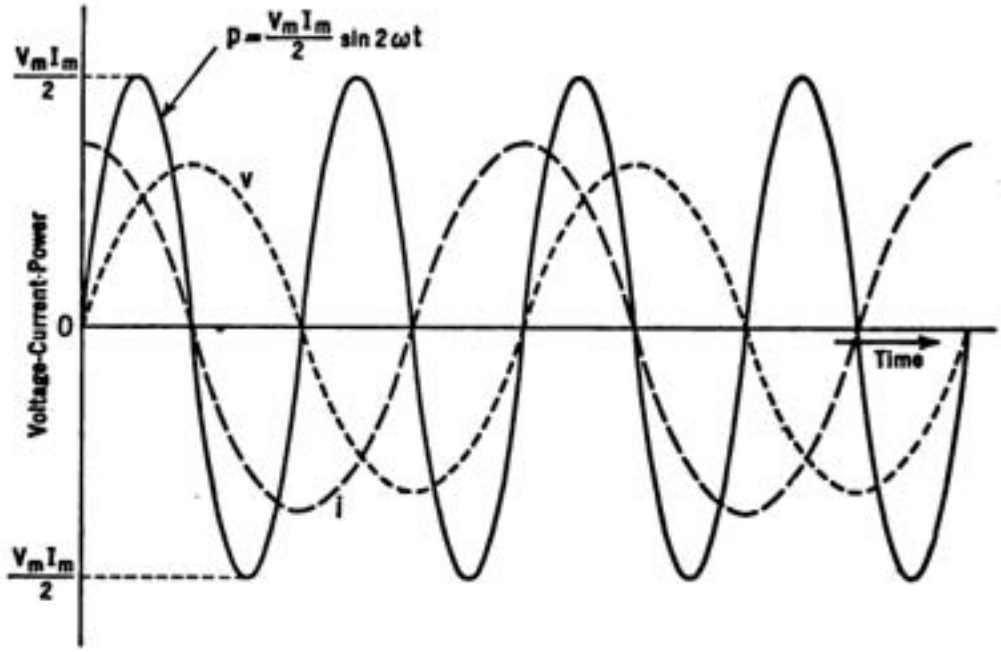

FIG. 8. Voltage, current, and power in a purely capacitive branch.

of $t = 0$ and $t = T/4$.

$$W_C = \int_0^{T/4} \frac{V_m I_m}{2} \sin 2\omega t \, dt$$

$$= \frac{V_m I_m}{2\left(\frac{4\pi}{T}\right)} \left[-\cos \frac{4\pi}{T} t \right]_0^{T/4}$$

$$= \frac{V_m I_m}{\left(\frac{4\pi}{T}\right)} = \frac{V_m I_m}{2\omega}$$

Since $I_m = \omega C V_m$,

$$W_C = \frac{V_m(\omega C V_m)}{2\omega} = \frac{V_m^2 C}{2} \tag{22}$$

If V_m and C are expressed in volts and farads respectively, the above expression for energy is in joules. W_C is the maximum amount of energy stored in the electrostatic field of the condenser at any one time.

Comparison of equations (21) and (14) will show that the capacitive element receives energy from the supplying source during the periods in which the inductive element returns energy to the source, and vice versa. When capacitive elements and inductive elements are both present in a given circuit, there is, in general, a natural tendency for

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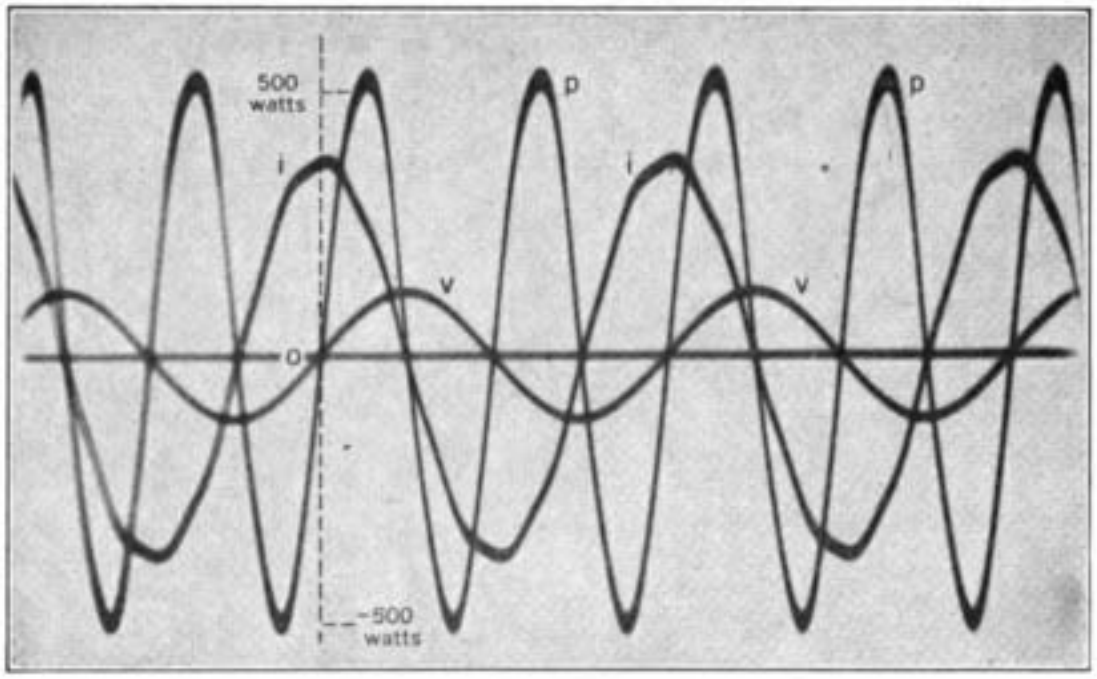
the elements to exchange energy. In certain circuit arrangements relatively large amounts of energy oscillate between the electromagnetic fields of the inductances and the electric fields of the condensers.



OSCILLOGRAM 3. Voltage, current, and power variations in a highly capacitive circuit element. $C = 144\ \mu f$, $f = 60$ cycles, $X_C = 18.4$ ohms, $R = 1.0$ ohm approx., $V_{max} = 141.4$, $I_{max} = 7.6$ amperes, $P_{av} = 25$ watts, approx. Note the lead of the i wave with respect to the v wave.

Oscillogram 3 illustrates the v, i, and p variations in a branch which approaches, to a close degree of accuracy, a purely capacitive circuit element.

The *RL* Branch. If it is assumed that a sinusoidal driving voltage, $V_m \sin \omega t$, is applied to a series combination of a resistive element and an inductive element, Fig. 9, the equation for voltage balance is:

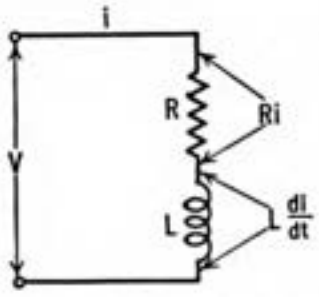


FIG. 9. The *RL* branch.

$$v = Ri + L\frac{di}{dt} = V_m \sin \omega t \qquad (23)$$

This is one form of Kirchhoff's emf law applied to instantaneous voltages. It states that the instantaneous voltage drop across the resistive element plus the instantaneous voltage drop across the inductive element equals the instantaneous voltage drop across the *RL* branch.

A straightforward solution of equation (23) for i in terms of the applied voltage and circuit parameters requires a certain knowledge

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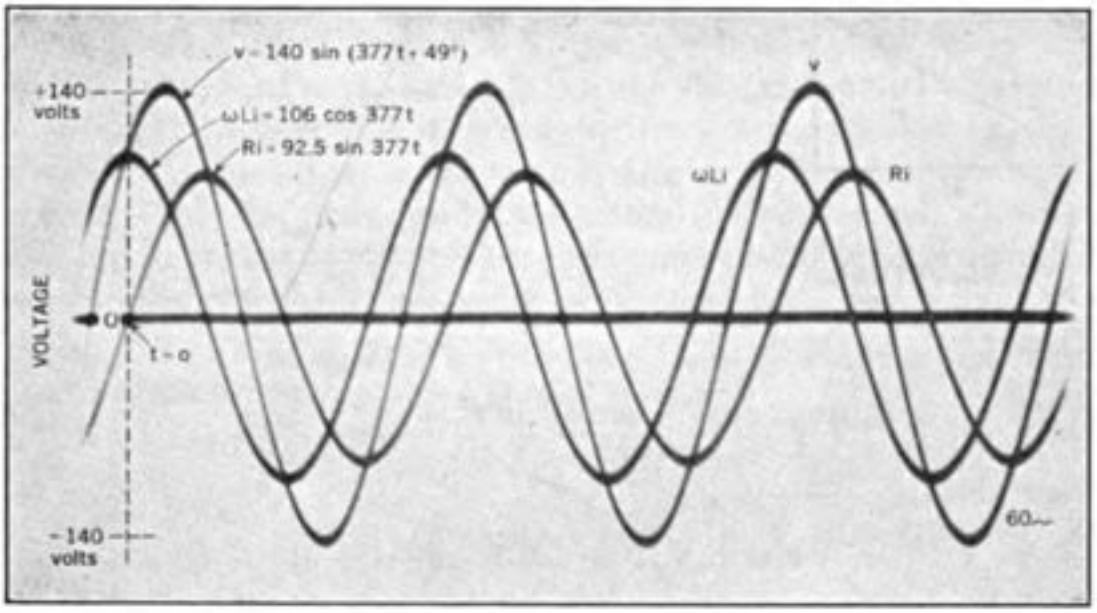
of differential equations on the part of the reader which is not essential to the problem at hand. The problem in which we are particularly interested at this point is the evaluation of the ratio V_m/I_m together with the time-phase difference between the voltage and current in an RL branch. Provided that R and L are constant, a current of sinusoidal wave form will flow in the branch if a sinusoidal voltage is applied. A critical inspection of equation (23) will help to establish the mathematical reasons for this physical fact.



OSCILLOGRAM 4. Illustrating the manner in which the voltage drop Ri across the resistance and the voltage drop ωLi across an inductance coil combine to equal the applied voltage v. $R = 18.5$ ohms connected in series with $X_L = 21.1$ ohms. $RI_{max} = 92.5$ volts, $\omega LI_{max} = 106$ volts, $V_{max} = 140$ volts.

If it is assumed that a sinusoidal current, $i = I_m \sin \omega t$, flows through a series branch consisting of a resistive element, R, and an inductive element, L, then:

$$Ri + L\frac{di}{dt} = \text{voltage applied, } v \tag{24}$$

or

$$RI_m \sin \omega t + \omega LI_m \cos \omega t = v \tag{25}$$

Equations (24) and (25) state that the instantaneous component voltage drops, Ri and $L\,di/dt$, add together to form the combined voltage drop across the RL branch. Oscillogram 4 illustrates the manner in which the Ri component ($RI_m \sin \omega t$) and the $L\,di/dt$ component ($\omega LI_m \cos \omega t$) combine to equal the applied voltage (v) in a particular RL branch.

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Since sine and cosine waves are 90° out of time phase with respect to one another, the RI_m and the ωLI_m components may be related as shown in Fig. 10, that is, as the two right-angle sides of a right triangle.

If both sides of the equation are divided by $\sqrt{R^2 + (\omega L)^2}$, equation (25) takes the following form:

$$I_m\left[\sin \omega t \frac{R}{\sqrt{R^2 + (\omega L)^2}} + \cos \omega t \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}\right] = \frac{v}{\sqrt{R^2 + (\omega L)^2}} \quad (26)$$

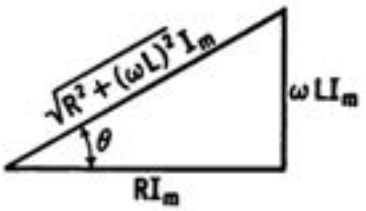

FIG. 10. The addition of RI_m and ωLI_m.

From Fig. 10,

$$\cos \theta = \frac{R}{\sqrt{R^2 + (\omega L)^2}} \quad (27)$$

and

$$\sin \theta = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}} \quad (28)$$

Then:

$$I_m[\sin \omega t \cos \theta + \cos \omega t \sin \theta] = \frac{v}{\sqrt{R^2 + (\omega L)^2}} \quad (29)$$

from which:

$$v = I_m\sqrt{R^2 + (\omega L)^2} \sin (\omega t + \theta)$$

or

$$v = I_m Z \sin (\omega t + \theta) = V_m \sin (\omega t + \theta) \quad (30)$$

It is thus shown that (1) $Z = \sqrt{R^2 + (\omega L)^2} = V_m/I_m$, (2) $\theta = \tan^{-1} \omega L/R$, and (3) v leads i in the RL branch by $\theta°$.

Impedance.

$$\mathbf{Z}_{RL} = \sqrt{R^2 + (\omega L)^2} \angle \tan^{-1} \frac{\omega L}{R} \quad (31)$$

The above expression for $\mathbf{Z}_{RL}$ implies that the numerical ratio of V_m to I_m in the RL branch is $\sqrt{R^2 + (\omega L)^2}$ and that the current lags the applied voltage by the angle whose tangent is $\omega L/R$. In general, R is expressed in ohms, ω in radians per second, and L in henrys, in which case $\sqrt{R^2 + (\omega L)^2}$ is given in ohms. In determining the phase angle it is, of course, only necessary that ωL and R be expressed in similar units.

The expression for the impedance of a pure R branch is at once obtainable from $\mathbf{Z}_{RL}$ by assuming that $L = 0$, in which case $\mathbf{Z}_{RL}$

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reduces to $R\underline{/0°}$. If the assumption is made that $R = 0$, $\mathbf{Z}_{RL}$ reduces immediately to the expression which has previously been derived for the impedance of a pure L branch, namely, $\omega L\underline{/90°}$.

An examination of the two factors which combine to form $\mathbf{Z}_{RL}$ will show that R is the factor which directly impedes or opposes the flow of current, whereas ωL is the factor which impedes or opposes any change in current. For a resulting sinusoidal current these two factors act in time quadrature with respect to one another. For example, when the current is zero the R factor has zero effect and the L factor has its greatest effect because it is when $i = 0$ that $[di/dt]$ for a sine wave is at its maximum value. When the current is at its maximum value, I_m, the R factor has its greatest effect and the L factor has zero effect because $[di/dt]$ for a sine wave is zero at the point of maximum current. It is the time quadrature nature (90° time-phase displacement) of the individual impedance effects that makes possible a simple vector algebra method of analyzing a-c circuits.[4]

Example. If $R = 20$ ohms and $L = 0.056$ henry, the 60-cycle impedance of the RL branch which is formed by placing R in series with L is:

$$\mathbf{Z} = \sqrt{20^2 + (377 \times 0.056)^2}\,\underline{\Big/ \tan^{-1}\frac{21.1}{20}}$$

$$= 29.1\underline{/46.5°} \text{ ohms}$$

If

$$v = 200 \sin (377t) \text{ volts}$$

$$i = \frac{200}{29.1} \sin (377t - 46.5°)$$

$$= 6.87 \sin (377t - 46.5°) \text{ amperes}$$

It will be observed that the instantaneous current is obtained from the instantaneous voltage (200 sin 377t) and the impedance function $(29.1\underline{/46.5°})$ by *two distinct operations* which are performed in a single step. These are:

(*a*) The maximum magnitude of the voltage (200) is divided by the magnitude of the impedance (29.1) to obtain the maximum magnitude of the current, 6.87 amperes.

(*b*) The correct angular displacement of the current wave with respect to the voltage wave is obtained by subtracting the impedance angle (46.5°) from the time angle of the voltage wave, namely, 377t.

Note: In evaluating the correct angular displacement between the instantaneous current and voltage waves in terms of the impedance angle, it is better to combine the angles in such a way as to yield the relation between current and voltage waves which are known to exist from a knowledge of the physical characteristics of the circuit. This process should not be obscured by any elaborate mathematical conventions.

[4] The vector algebra method of analysis is considered in Chapter IV.

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Power. The instantaneous power or, as it is sometimes called, the instantaneous volt-amperes, delivered to the RL branch may be obtained from:

$$p = vi = [V_m \sin(\omega t + \theta)]\,[I_m \sin \omega t] \tag{32}$$

After the sin $(\omega t + \theta)$ term is expanded, the above equation can be written in the following forms:

$$\begin{aligned} p &= V_m I_m \sin \omega t\,[\sin \omega t \cos \theta + \cos \omega t \sin \theta] \\ &= V_m I_m \sin^2 \omega t \cos \theta + V_m I_m (\sin \omega t \cos \omega t) \sin \theta \\ &= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} [\cos 2\omega t] \cos \theta + \frac{V_m I_m}{2} [\sin 2\omega t] \sin \theta \end{aligned} \tag{33}$$

Figure 11 is a graphical representation of the component parts of equation (33) together with the resultant graph of instantaneous power.

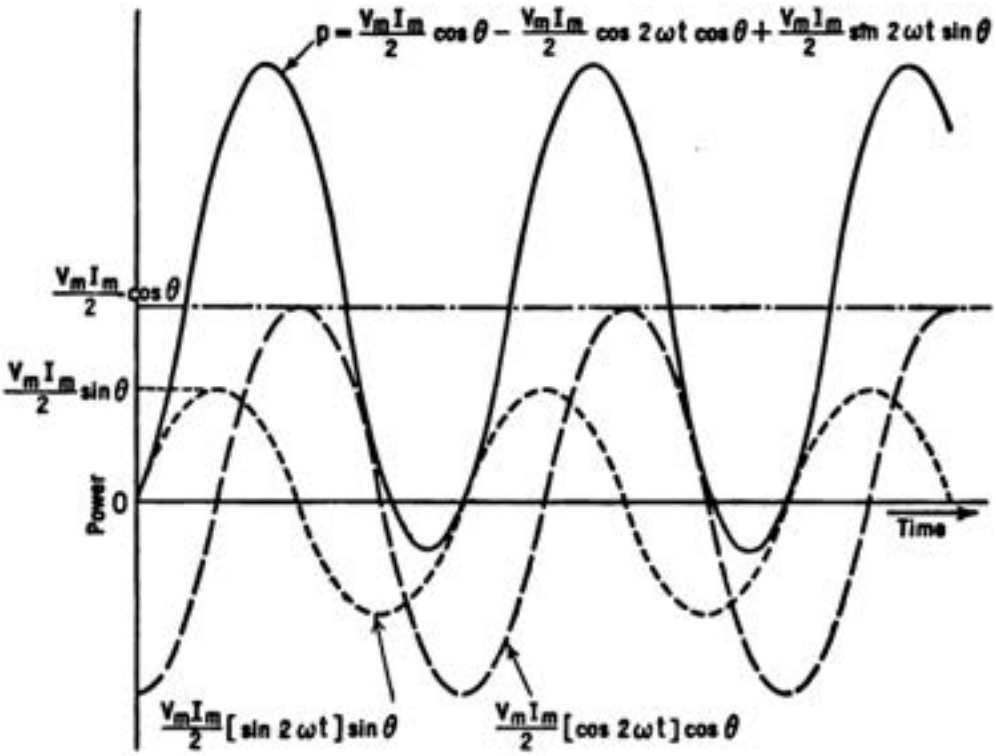

FIG. 11. Graphical representation of equation (33) for the particular case of $\theta = 30°$.

It should be plain that the average value with respect to time of either the $[\cos 2\omega t]$ or the $[\sin 2\omega t]$ term is equal to zero when considered over a time interval equal to an integral number of cycles. The average value with respect to time of the power when considered over an integral number of cycles is, therefore, equal to:

$$P_{av} = \frac{V_m I_m}{2} \cos \theta$$

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The above expression for average power may also be obtained by finding the average value of the right-hand member of equation (32).

$$P_{av} = \frac{1}{T}\int_0^T V_m \sin(\omega t + \theta) I_m \sin \omega t \, dt$$

$$= \frac{V_m I_m}{2} \cos \theta \tag{34}$$

Real Power and Reactive Power or Reactive Volt-Amperes. A detailed analysis of the component parts of equation (33) will aid in understanding why electrical power is treated in terms of real and reactive components and why these two components are sometimes represented as the legs of a right triangle.

Real Power. Instantaneous real power refers to $\left[\frac{V_m I_m}{2}\cos\theta - \frac{V_m I_m}{2}(\cos 2\omega t)\cos\theta\right]$, the first two terms on the right-hand side of equation (33). Reference to Fig. 11 will show that these two terms combine to form an instantaneous power variation which contains no negative values; hence this portion of equation (33) is called the *instantaneous real power.*

Unless qualified to mean *instantaneous* real power, the expression *real power* refers only to $\frac{V_m I_m}{2}\cos\theta$, the average value of the total instantaneous power with respect to time. [See equations (33) and (34).]

Reactive Power or Reactive Volt-Amperes. The third term on the right-hand side of equation (33), $\left[\frac{V_m I_m}{2}(\sin 2\omega t)\sin\theta\right]$, is variously called *instantaneous reactive power, instantaneous quadrature power, instantaneous reactive volt-amperes,* etc., for the reason that the area under the $\left[\frac{V_m I_m}{2}(\sin 2\omega t)\sin\theta\right]$ curve represents the energy which oscillates between the driving source and the reactive (either inductive or capacitive) elements of the receiving circuit. It will be observed from Fig. 11 that the *instantaneous reactive power* is that portion of the total instantaneous power variation which has equal positive and negative loops, and which contains the sine of the phase angle between v and i as a factor.

Unless qualified to mean *instantaneous* reactive power or *instantaneous* reactive volt-amperes, the expressions *reactive power* and *reactive volt-*

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amperes refer simply to $\frac{V_m I_m}{2} \sin \theta$, the maximum instantaneous value of the third term on the right-hand side of equation (33).

Units of reactive volt-amperes in the practical system of units are called *vars*. (See pages 54 and 55.)

Volt-Amperes. Both the real power, $\frac{V_m I_m}{2} \cos \theta$, and the reactive volt-amperes, $\frac{V_m I_m}{2} \sin \theta$, are important quantities, and they are often measured independently, a wattmeter being used to measure $\frac{V_m I_m}{2} \cos \theta$ and a reactive volt-ampere meter, called a varmeter, being used to measure $\frac{V_m I_m}{2} \sin \theta$.

The real power and the reactive power may be combined to yield the volt-amperes of the circuit, namely, $\frac{V_m I_m}{2}$:

$$\sqrt{\left[\frac{V_m I_m}{2} \cos \theta\right]^2 + \left[\frac{V_m I_m}{2} \sin \theta\right]^2} = \frac{V_m I_m}{2}$$

The above relationship is illustrated graphically in Fig. 12 and will be encountered in later chapters in a more universally used form.

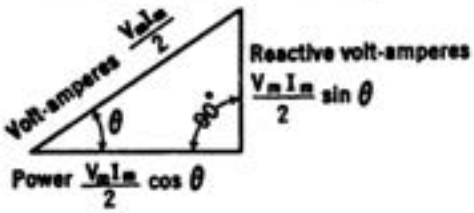

FIG. 12. Relation of power, reactive volt-amperes, and volt-amperes.

Example. Consider the *RL* circuit whose voltage, current, and power variations are depicted in Oscillogram 5. $R = 19.7$ ohms, $\omega L = 21.1$ ohms, and $v = 141.4 \sin 377t$ volts.

Let it be required to evaluate the expressions for the instantaneous current and the instantaneous power from the above data.

$$Z = \sqrt{19.7^2 + 21.1^2} \angle \tan^{-1} \frac{21.1}{19.7} = 28.85 \angle 47^\circ \text{ ohms}$$

The instantaneous current is:

$$i = \frac{141.4}{28.85} \sin (377t - 47^\circ) = 4.9 \sin (377t - 47^\circ) \text{ amperes}$$

The expression for the instantaneous power is, by equation (33):

$$p = 236 - 236 \cos 754t + 253 \sin 754t \text{ watts}$$

In this expression,

[236 − 236 cos 754*t*] is called the *instantaneous* real power
253 sin 754*t* is the *instantaneous* reactive volt-amperes
236 watts is the real power
253 vars is the reactive power or reactive volt-amperes.

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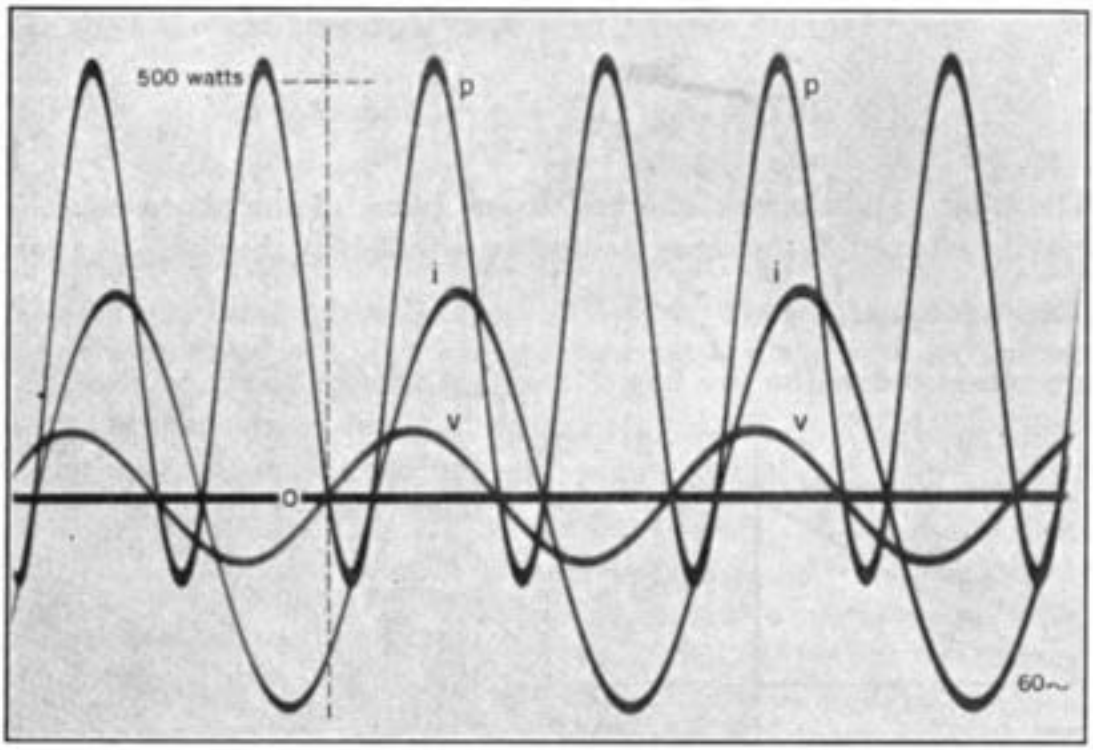

OSCILLOGRAM 5. Voltage, current, and power variations in an *RL* circuit. $R = 19.7$ ohms connected in series with $L = 0.056$ henry, $X_L = 21.1$ ohms, $V_{max} = 141.4$ volts. $I_{max} = 4.90$ amperes, $P_{av} = 236$ watts.

The *RLC* Branch. If a current of sinusoidal wave form, $i = I_m \sin \omega t$, is assumed to flow through the *RLC* branch shown in Fig. 13, it is plain that:

$$v_R = Ri = RI_m \sin \omega t \qquad (35)$$

$$v_L = L\frac{di}{dt} = \omega L I_m \cos \omega t \qquad (36)$$

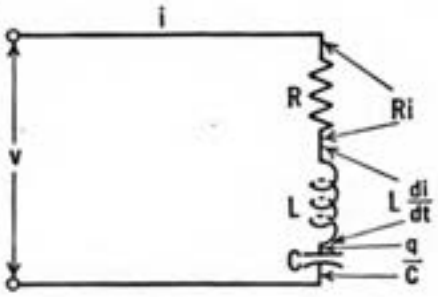

FIG. 13. The *RLC* branch.

and

$$v_C = \frac{q}{C} = \frac{\int i\,dt}{C} = \frac{\int I_m \sin \omega t\,dt}{C} = \frac{-I_m}{\omega C}\cos \omega t \qquad (37)^5$$

The voltage applied to the branch is, physically, the sum of the three component voltages. In the form of an equation:

$$RI_m \sin \omega t + \omega L I_m \cos \omega t - \frac{1}{\omega C} I_m \cos \omega t = v \qquad (38)$$

[5] The reason for neglecting the constant of integration is similar to that given in the footnote on page 17.

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or

$$RI_m \sin \omega t + \left(\omega L - \frac{1}{\omega C}\right) I_m \cos \omega t = v \tag{39}$$

The combination of the sine and cosine terms of the above equation may be effected in the same manner as outlined for the sine and cosine terms of equation (25). In the present case RI_m and $\left(\omega L - \frac{1}{\omega C}\right) I_m$ are considered as the two legs of the right triangle shown in Fig. 14.

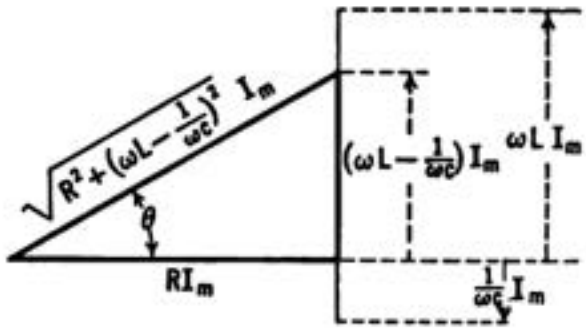

FIG. 14. Illustrating the manner in which the three voltage drops RI_m, $\omega L I_m$, and $\frac{1}{\omega C} I_m$ combine to form the voltage drop $\sqrt{R^2 + (\omega L - 1/\omega C)^2}\, I_m$.

It will be remembered from the discussion of the purely inductive and the purely capacitive branches that these two reactive elements cause exactly opposite phase displacements of the current with respect to the voltage. Since ωL has arbitrarily been considered to be a positive quantity, it becomes necessary to consider $1/\omega C$ a negative quantity. It should be recognized that, of and by itself, there is nothing inherently negative about the quantity $1/\omega C$. The fact that it acts oppositely to the quantity ωL in governing current flow requires that $1/\omega C$ be treated negatively if ωL is treated positively.

Impedance. If equation (39) is manipulated as indicated on page 26, the impedance of the RLC branch is found to be:

$$\mathbf{Z}_{(RLC)} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \angle \tan^{-1} \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R} \tag{40}$$

If R is in ohms, L in henrys, C in farads, and ω in radians per second, $\mathbf{Z}$ is in ohms. Given R, L, C, and ω, the complete impedance function can be evaluated. The general expression for $\mathbf{Z}_{(RLC)}$ is of considerable importance in a-c circuit theory because all the impedance functions which have thus far been treated are directly deducible from this expression.

In a branch where ωL is negligibly small as compared with R and $1/\omega C$, the ωL term may be considered to be equal to zero, in which

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case the branch reduces to a resistance and capacitance branch.

$$\mathbf{Z}_{(RC)} = \sqrt{R^2 + \left(-\frac{1}{\omega C}\right)^2} \Bigg/ \tan^{-1} \frac{\left(-\frac{1}{\omega C}\right)}{R} \tag{41}$$

The negative angle implies that the current wave lags the voltage wave by a *negative* angle. The correct physical interpretation is that the current wave leads the voltage wave by the angle whose tangent is $\left(\frac{1}{\omega C}\right) \Big/ R$.

With respect to its terminals the RLC branch will, in general, simulate the behavior of either the RL or the RC branch. If $\omega L > 1/\omega C$, the RLC branch responds to an impressed voltage at its terminals exactly as would an equivalent RL branch, the inductive reactance of which is $(\omega L - 1/\omega C)$. Similarly, if $1/\omega C > \omega L$, the RLC branch will respond to an impressed voltage at its terminals exactly as would an equivalent RC branch, the capacitive reactance of which is $(1/\omega C - \omega L)$. In either of the cases referred to, there will be interchanges of energy taking place between the two reactive elements.

The singular case, wherein $\omega L = 1/\omega C$, is of particular interest because the impedance here reduces to $R\underline{/0°}$. With respect to its terminals the RLC branch, under the condition of $\omega L = 1/\omega C$, responds as would a purely resistive branch. If R is assumed to be a fixed quantity, the above condition may be obtained by the proper adjustment of L, C, or ω, and when $\omega L = 1/\omega C$ the impedance of the branch will be a minimum.

Example. If $R = 10$ ohms, $L = 0.056$ henry, and $C = 50\ \mu f$, the impedance of the RLC branch at 60 cycles is:

$$\mathbf{Z} = \sqrt{10^2 + \left(377 \times 0.056 - \frac{10^6}{377 \times 50}\right)^2} \Bigg/ \tan^{-1} \frac{(21.1 - 53.0)}{10}$$

$$= 33.4\underline{/\tan^{-1}(-3.19)}$$

$$= 33.4\underline{/-72.6°} \text{ ohms}$$

If

$$v = 200 \sin 377t \text{ volts}$$

$$i = \frac{200}{33.4} \sin (377t + 72.6°) \text{ amperes}$$

Power. Since $i = I_m \sin \omega t$ and $v = V_m \sin (\omega t + \theta)$, the expression for the instantaneous power delivered to the RLC branch takes the

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same form as equation (33), namely,

$$p = \frac{V_m I_m}{2} \cos\theta - \frac{V_m I_m}{2} [\cos 2\omega t] \cos\theta + \frac{V_m I_m}{2} [\sin 2\omega t] \sin\theta \qquad (42)$$

In the present case θ may presumably take any value between $+90°$ and $-90°$. The average power delivered to the *RLC* branch is in any case $\frac{V_m I_m}{2} \cos\theta$. [See equation (34).] The maximum value of the instantaneous reactive volt-amperes $\left[\frac{V_m I_m}{2} \sin 2\omega t \sin\theta\right]$, is directly proportional to $\sin\theta$. Since the $\sin 2\omega t$ factor causes the instantaneous reactive volt-amperes to be alternately positive and negative, the absolute meaning of the sign of the reactive power term is not highly significant.

According to the convention of signs which has been employed in the present discussion, positive reactive volt-amperes — that is, a positive coefficient of $[\sin 2\omega t]$ in equation (42) — indicate inductive reactive volt-amperes, whereas negative reactive power indicates capacitive reactive volt-amperes. These signs are merely the result of choosing ωL positive and $1/\omega C$ negative. Further consideration of signs of reactive power will be given in the next chapter.

The term $\left[\frac{V_m I_m}{2} \sin 2\omega t \sin\theta\right]$ is equal to zero at all times when $\theta = 0$, that is, when $\omega L = 1/\omega C$. In this case the reactive volt-amperes required by the inductive element are furnished by the capacitive element, and vice versa. Relatively large amounts of energy may oscillate between the reactive elements even though the *RLC* branch simulates a purely resistive branch at its terminals.

Oscillogram 6 illustrates the variations of v, i, and p in a particular *RLC* circuit. In the case shown $\omega L < 1/\omega C$ and the lead of the current with respect to the voltage is clearly indicated.

Impedance Functions. It should be understood from the foregoing analyses that impedance functions for any combinations of R, L, and C are independent of the point on the wave from which time is reckoned. In addition, the functions are entirely independent of whether the voltage or current wave is made the dependent wave. Thus in the *RL* branch a current wave $i = I_m \sin \omega t$ was assumed and the voltage wave $v = V_m \sin (\omega t + \theta)$ was found to lead the current by $\theta = \tan^{-1}(\omega L/R)$. If a voltage $v = V_m \sin \omega t$ is assumed impressed upon the circuit, the impedance function is the same, and it states that

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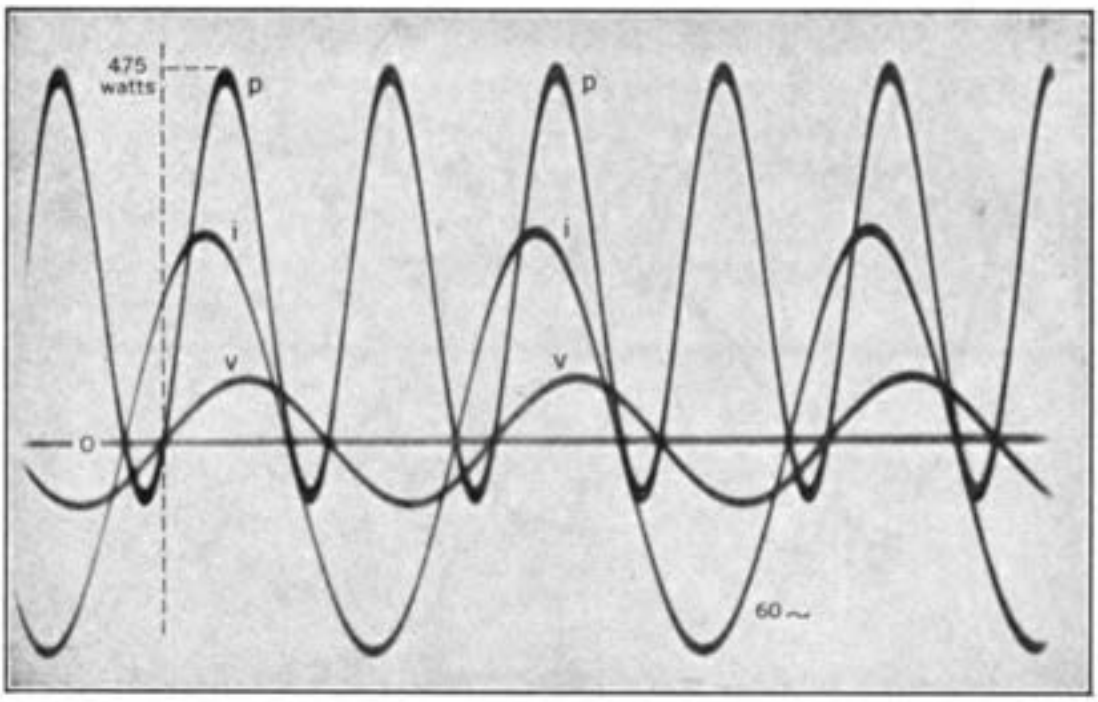

OSCILLOGRAM 6. Voltage, current, and power variations in an *RLC* circuit. $R = 20$ ohms, $L = 0.042$ henry, $C = 78$ μf, $X_L = 15.8$ ohms, $X_C = 34$ ohms, $V_{max} = 141.4$ volts, $I_{max} = 5.23$ amperes, $P_{av} = 275$ watts.

the voltage wave must lead the current by $\tan^{-1}(\omega L/R)$. Hence the current wave may be written as $i = I_m \sin(\omega t - \theta)$. Similar interpretations apply to any combination of R, L, and C. When the impedance function is found, the *relation* between the voltage drop and the current is thereby determined. If one is assumed, the other may be determined from the impedance function as illustrated by the examples in the preceding articles.

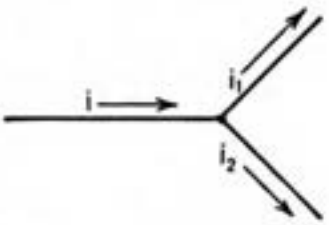

FIG. 15. Instantaneous current toward a junction is equal to the instantaneous current away from the junction.

Instantaneous Currents Combine Algebraically. The concept of adding instantaneous voltage drops across series elements to obtain the total voltage applied to a series circuit has been considered. Kirchhoff's emf law applies to a-c circuits if instantaneous values of voltage or their equivalents are considered. Likewise Kirchhoff's current law applies to a-c circuits provided instantaneous values of current or their equivalents[6] are employed. Figure 15 illustrates the principle in a simple case. Kirchhoff's current law states that the current flowing toward a junction, which in the present case is i, is equal to the current flowing away from the junction, namely, $i_1 + i_2$.

[6] The equivalents referred to are the vector forms that are employed to replace instantaneous values. See Chapters III and IV.

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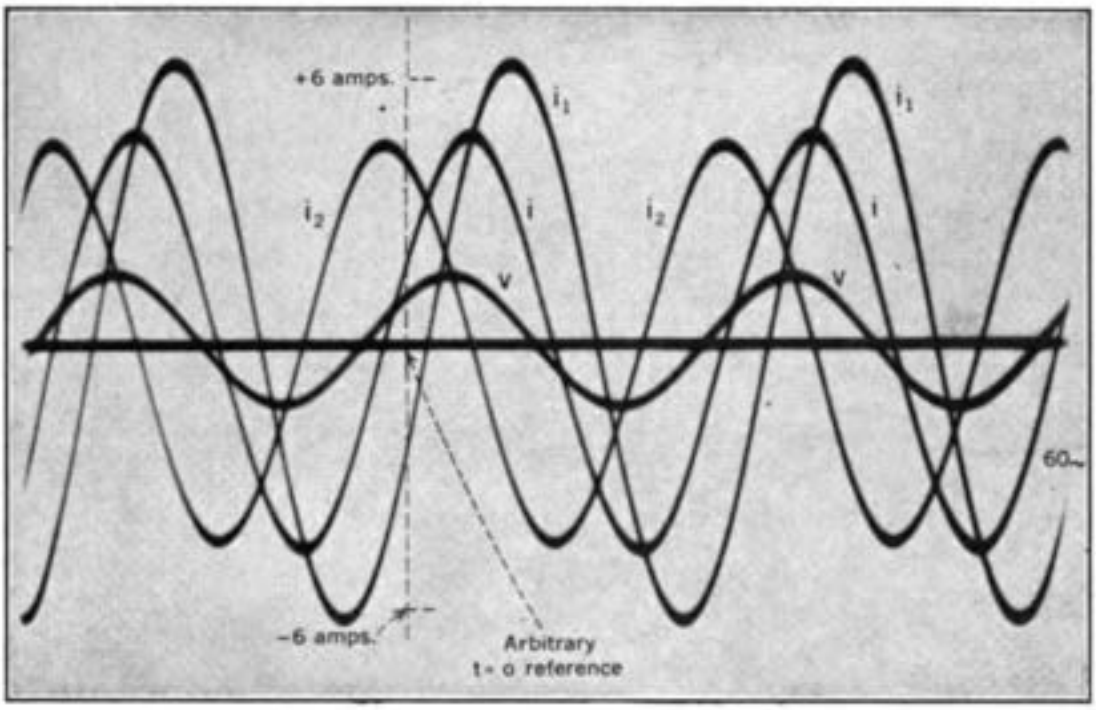

OSCILLOGRAM 7. Photographic records of the applied voltage and the three branch currents of the circuit arrangement shown in Fig. 16.

In general:

$$\sum i_{\text{toward a junction}} = \sum i_{\text{away from the junction}} \tag{43}$$

or, if current *away* from the junction is considered as *negative* current *toward* the junction:

$$\sum i_{\text{toward a junction}} = 0 \tag{44}$$

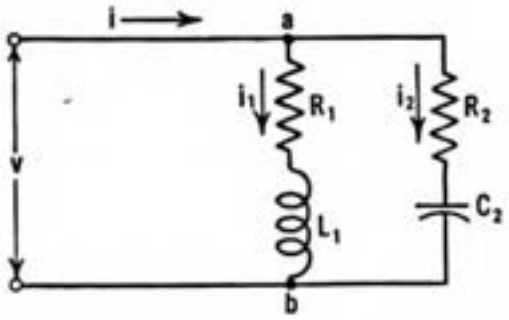

FIG. 16. R_1L_1 branch in parallel with R_2C_2 branch.

If the currents are measured by devices which do not respond to instantaneous values, the combined measurements, in general, will not satisfy the above current law, for the simple reason that the devices employed fail to account for the relative phase positions of the currents involved.

Oscillogram 7 shows how the instantaneous currents i_1 and i_2 of Fig. 16 add algebraically to yield the resultant current i. The analytical method of finding the expression for i from i_1 and i_2 will be explained in Chapter VI.

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PROBLEMS

1. (*a*) Find the instantaneous value of a sinusoidal alternating current having a maximum value of 90 amperes, 60° after the current passes through its zero value going positive; 225° after the current passes through its zero value going positive.

(*b*) Find the difference in time between the 60° value of current and the 225° value of current if the frequency is 50 cycles.

2. The current through a particular filter choke may be represented approximately by the equation:

$$i = 1.0 + 0.50 \sin 1885t - 0.10 \cos 3770t \text{ amperes}$$

or

$$i = 1.0 + 0.50 \sin \alpha - 0.10 \cos 2\alpha \text{ amperes}$$

where $\alpha = 1885t$ radians if t is expressed in seconds.

(*a*) What is the frequency of the sine term? of the cosine term?

(*b*) What are the maximum and minimum values of current?

(*c*) Graph the current i with respect to time t or with respect to angular measure α.

3. A voltage $v = 150 \cos 314t$ volts is applied to a purely resistive branch of $R = 30$ ohms.

(*a*) Write the expression for i as a function of time, employing numerical coefficients.

Ans.: $i = 5 \cos 314t$ amperes.

(*b*) What is the frequency of the voltage and current variations?

Ans.: 50 cycles.

(*c*) Write the expression for p as a function of time, employing numerical coefficients.

Ans.: $p = 750 \cos^2 314t = 375 + 375 \cos 628t$ watts.

(*d*) What is the frequency of the power variation?

Ans.: 100 cycles.

4. A current $i = 5 \sin (110t + 30°)$ amperes flows in a purely resistive branch of 20 ohms.

(*a*) Write the expression for v as a function of time employing numerical coefficients.

(*b*) What is the frequency of the voltage variation?

(*c*) Write the expression for p as a function of time, employing numerical coefficients.

(*d*) What is the frequency of the power variation?

5. A voltage $v = 100 \cos (\omega t + 60°)$ volts is impressed upon a pure resistance circuit of 10 ohms.

(*a*) Write the equation with respect to time of the current wave and employ numerical coefficients.

(*b*) Find the equation with respect to time of the power wave.

(*c*) What is the maximum instantaneous power?

(*d*) What is the minimum instantaneous power?

(*e*) What is the average value of the power wave?

6. (*a*) What is the maximum time rate of change of a 60-cycle alternating current of sine form, the maximum value of which is 10 amperes?

(*b*) If this current flows through a pure inductance of 100 millihenrys, find the maximum value of the voltage across the terminals of the inductance.

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7. A voltage $v = -150 \sin 377t$ volts is applied to a particular circuit element, and it is found, by oscillographic analysis, that $i = 10 \cos 377t$ amperes. Make a sketch of the v and i waves. Find the nature and magnitude of the circuit parameter.

Ans.: $L = 0.0398$ henry.

8. A voltage drop $v = 100 \sin (377t + 30°)$ volts is across a pure inductance of 0.02654 henry.

(*a*) Use numerical coefficients and express the current through the coil as a function of time.

(*b*) Find the equation with respect to time of the power wave. Express the result as a single sine function.

(*c*) What is the average power?

(*d*) What is the first value of time at which maximum energy is stored in the inductance?

(*e*) What is the maximum amount of energy stored in the inductance during a cycle? State units.

9. A current of $5 \sin 300t$ amperes flows through a pure inductive branch of 0.2 henry.

(*a*) Find the impedance function and express numerically.

(*b*) How many joules are stored in the magnetic field about the inductance when $t = 0.05$ second?

(*c*) Write the expression for v as a function of time employing numerical coefficients.

10. A voltage $v = 200 \cos (157t + 30°)$ volts is applied to a particular circuit element, and it is found, by oscillographic analysis, that $i = 5 \sin (157t - 150°)$ amperes. Sketch the v and i waves. Find the nature and magnitude of the circuit parameter.

11. A voltage $v = 100 \sin 377t$ volts is impressed on a pure capacitance of 530.5 μf.

(*a*) Write the expression for i as a function of time employing numerical coefficients.

(*b*) Find the expression for the power wave as a function of time, employing numerical coefficients.

(*c*) How many joules are stored in the condenser when the current is zero? when the current is a maximum?

12. A voltage $v = 200 \sin 377t$ volts is applied to an inductive branch, and the maximum current is found, by oscillographic analysis, to be 10 amperes.

(*a*) Find the value of L in millihenrys.

Ans.: 53.1 millihenrys.

(*b*) If it is known that this inductance coil actually possesses 1.0 ohm resistance, what is the true value of L, assuming that $V_m = 200$ volts and $I_m = 10$ amperes?

Ans.: $L_{true} = \sqrt{20^2 - 1^2}/377 = 53.04$ millihenrys.

13. $R = 10$ ohms and $L = 0.05$ henry are connected in series and energized by a 25-cycle sinusoidal voltage, the maximum value of which is 150 volts.

(*a*) Find the complete impedance expression for the RL branch.

(*b*) Write the expression for the supply voltage as a function of time, making $v = 0$ (dv/dt positive) at $t = 0$.

(*c*) Write the expression for current as a function of time, assuming that the voltage in (*b*) is applied to the branch. Employ numerical coefficients.

(*d*) Write the expression for the instantaneous power delivered to the branch as a function of time. Express the result in three terms — a constant term, a single cosine term, and a single sine term. What is the average power?

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(*e*) What are the reactive volt-amperes or vars?

(*f*) Sketch the v, i, and p variations in rectangular coordinates.

14. $R = 10$ ohms and $L = 0.05$ henry are connected in series and energized by a 25-cycle sinusoidal voltage, the maximum value of which is 150 volts.

(*a*) Find the complete impedance expression for the RL branch.

Ans.: $12.7\angle 38.2°$ ohms.

(*b*) Write the expression for the supply voltage as a function of time, making $v = 75$ (dv/dt positive) at $t = 0$.

Ans.: $v = 150 \sin (157t + 30°)$ volts.

(*c*) Write the expression for current as a function of time, assuming that the voltage in (*b*) is applied to the branch. Employ numerical coefficients.

Ans.: $i = 11.8 \sin (157t - 8.2°)$ amperes.

(*d*) Write the expression for the instantaneous power delivered to the branch as a function of time. Express the result in three terms — a number, one cosine, and one sine term. What is the average power delivered?

Ans.: $p = 695 - 820 \cos 314t + 328 \sin 314t$ watts.

$P_{av} = 695$ watts.

15. A resistive element of 30 ohms is connected in series with an inductance coil, the self-inductance of which is 50 millihenrys and the ohmic resistance of which is 4.5 ohms. A voltage $v = 100 \cos 377t$ volts is connected to the series branch.

(*a*) Evaluate the expression for i.

(*b*) Evaluate the expression for p.

(*c*) Write the expression for the real power as a function of time, employing numerical coefficients. What is the average value of the instantaneous real power?

(*d*) Write the expression for the reactive volt-amperes as a function of time, employing numerical coefficients. What is the average value of the instantaneous reactive power?

(*e*) What is the inductive reactance of the branch in ohms?

16. A current $i = 10 \cos 157t$ amperes flows in an RL circuit containing $R = 15$ ohms and $L = 0.0637$ henry.

(*a*) Write the equation of v as a function of time, employing numerical coefficients.

(*b*) Write the expression for the power wave as a function of time.

17. (*a*) What is the capacitive reactance of an 8-μf condenser at 60 cycles?

(*b*) What is the capacitive reactance of an 800-$\mu\mu$f condenser at 6 megacycles?

18. A resistive element of 151 ohms is connected in series with a condenser of 4 μf capacitance. A 500-cycle sinusoidal voltage, the maximum value of which is 15 volts, energizes the RC branch.

(*a*) Write the expression for the supply voltage, choosing the $t = 0$ reference at the point of maximum positive voltage.

(*b*) Evaluate $\mathbf{Z}_{RC}$ completely.

(*c*) Evaluate the expression for i.

(*d*) Evaluate the expression for p which corresponds to the product of voltage and current employed here, and express all trigonometric terms with exponents no higher than unity.

19. Assume that the current $i = I_m \sin \omega t$ flows through a given RC branch. Show that the voltage across the branch is:

$$v = I_m Z \sin (\omega t + \theta) = V_m \sin (\omega t + \theta)$$

where

$$Z = \frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{-1}{\omega C}\right)^2}$$

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and

$$\theta = \tan^{-1}\frac{\left(\frac{-1}{\omega C}\right)}{R} \qquad \textit{Hint: } \frac{q}{C} = \frac{\int i\,dt}{C}$$

20. A resistance of 10 ohms is in series with a 303-μf capacitor. If the voltage drop across the capacitor is 150 sin (220t − 60°) volts, find the equation with respect to time of the voltage drop across the entire series circuit. Find also the expression for the current at any time t.

21. A 2000-cycle alternating voltage of sine form when impressed across the terminals of a condenser establishes a current of 0.01 ampere (maximum value). If the maximum value of the voltage is 20 volts, find the capacitance of the condenser in microfarads.

22. Consider a series RLC branch wherein R = 10 ohms, L = 0.10 henry, and C is 200 μf. Assume that the current i = 10 sin (157t) amperes flows through the RLC branch.

(*a*) Write the expression for the voltage drop across R, namely, Ri, employing numerical coefficients.

(*b*) Write the expression for the voltage drop across L, namely, $L\,di/dt$, employing numerical coefficients.

(*c*) Write the expression for the voltage drop across C, namely, q/C, employing numerical coefficients.

(*d*) Add (*a*), (*b*), and (*c*) to find the voltage drop across the RLC branch. Express the result as a single sine function of time.

(*e*) What is the numerical value of the impedance of the series RLC branch?

23. Assume that the current $i = I_m \cos \omega t$ flows through a given RLC branch. Show that the voltage across the branch is:

$$v = I_m Z \cos(\omega t + \theta) = V_m \cos(\omega t + \theta)$$

where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

and

$$\theta = \tan^{-1}\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

24. In the following exercise, it is assumed that a coil having L henrys of inductance and R ohms of series resistance is placed in series with a condenser of C farads of capacitance. A current of $i = I_m \sin(t/\sqrt{LC})$ amperes flows in the circuit. Show that the energy $w_L + w_C$ = constant, and evaluate this constant.

25. A resistive element of 20 ohms, an inductance coil of L = 300 millihenrys and R_L = 10 ohms, and a condenser of 50 μf capacitance are connected in series to form an RLC branch. A voltage v = 100 sin 157t volts is applied to the RLC branch.

(*a*) What is the numerical value of $\mathbf{Z}_{RLC}$?

(*b*) Write the expression for i, employing numerical coefficients.

(*c*) Write the expression for p, employing numerical coefficients, and express all trigonometric functions with exponents no higher than unity.

(*d*) What is the average value of the power delivered to the branch?

(*e*) What is the maximum value of the reactive volt-amperes?

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(*f*) Write the expression for the voltage drop across the 20-ohm resistive element as a function of time, employing numerical coefficients.

(*g*) Write the expression for the instantaneous power delivered to the 20-ohm resistor as a function of time, employing numerical coefficients.

26. A voltage $v = 282.8 \sin 500t$ volts is applied to a series circuit, and the resulting current is found to be $i = 5.656 \sin (500t - 36.87°)$ amperes. One element of this series combination is known to be a capacitor which has a capacitance of 100 μf. Determine the magnitudes of the other series elements present.

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CHAPTER III

EFFECTIVE CURRENT AND VOLTAGE — AVERAGE POWER

Except for the maximum values of sinusoidal wave variations attention has been given only to general instantaneous values of current and voltage. The only practicable method of measuring instantaneous values of current, voltage, and power is by means of an oscillograph, a very useful instrument in many respects but one which is relatively inaccurate, cumbersome, and costly. It was shown in the previous chapter that instantaneous values are inconvenient to manipulate analytically, and in general they fail to specify concisely the magnitudes of the quantities involved. In this chapter the values of currents and voltages usually dealt with will be considered.

Ampere Value of Alternating Current. Alternating currents are defined so as to make applicable to them essentially the same laws that govern heating and transfer of power by direct current. An alternating current which produces heat in a given resistance at the same average rate as I amperes of direct current is said to have a value of I amperes. The average rate of heating produced by an alternating current during one cycle is $\frac{1}{T}\int_0^T Ri^2\,dt$. The average rate of producing heat by I amperes of direct current in the same resistance is RI^2. Hence by definition

$$RI^2 = \frac{1}{T}\int_0^T Ri^2\,dt$$

and

$$I = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\text{average } i^2} \tag{1}$$

The current given in equation (1) which defines the alternating current in terms of its average rate of producing heat in a resistance is called the root mean square (abbreviated rms) value. It is also called the effective or virtual value. The graphical evaluation of the rms value of an alternating current is illustrated in Fig. 1. When the equation of the wave is not known or when it is inconvenient to determine it, the graphical means, suggested by Fig. 1, of evaluating equation (1) becomes a useful method to employ.

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Problem 1. Find the effective value of a current that starts at zero, rises instantaneously, then remains at a value of 20 amperes for 10 seconds, then decreases instantaneously, remaining at a value of 10 amperes for 20 seconds, and then repeats this cycle. *Ans.:* 14.14 amperes.

The rms value of an alternating current may be measured with the ordinary dynamometer type of meter. This meter consists of two coils in series, one of them being movable. The force tending to turn

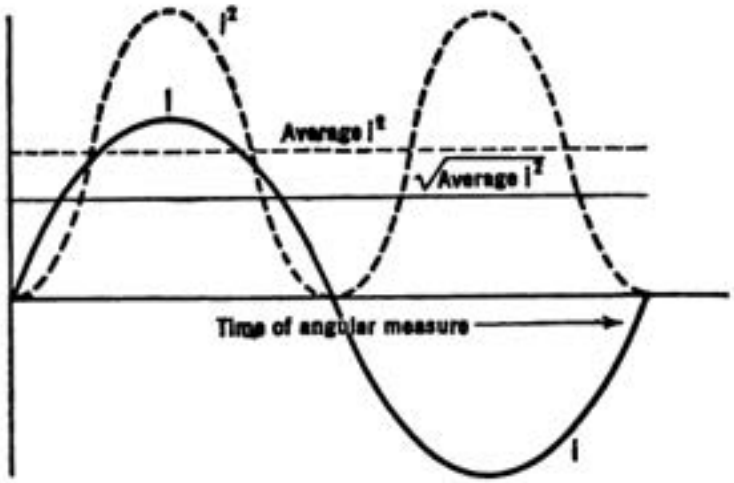

FIG. 1. Graphical evaluation of rms value.

the movable coil from any fixed position is proportional to the product of the currents in the two coils. Since the coils are in series and the same current flows in each, the force for any given position of the coils is proportional to i^2. Since the coil has a relatively high inertia, it cannot follow the variation in the force produced, and therefore takes a position corresponding to the average force or average i^2. If a suitable square root scale is placed under the pointer, the pointer will indicate the square root of the average square, or the rms value. Other types of a-c ammeters are also used to indicate effective values of current. (See Chapter X.)

Alternating Volt. An alternating volt is the value of a wave of alternating potential which maintains an alternating current of 1 rms ampere through a non-inductive resistance of 1 ohm. It therefore follows that the volt value of a wave is measured by the square root of the average square of the instantaneous values of the voltage wave.

Average Values. The average value of any a-c wave which is symmetrical about the zero axis is zero. However, when average value is applied to alternating quantities, it usually means the average of either the positive or negative loop of the wave. This value represents the d-c equivalent for electrolytic action of the alternating wave *abcde*, Fig. 2, if the wave were commutated (or rectified) and made the same

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as the wave *abcfe*. Since the average ordinate multiplied by the base is equal to the area under the curve, it follows directly that

$$\text{Average value} = \frac{2}{T}\int_0^{T/2} i\,dt \tag{2}$$

Equation (2) is applicable only when the wave passes through zero at the time $t = 0$. For any other condition the time t_1 at which the instantaneous value of the wave is zero must be determined and the average value found from

$$\text{Average value} = \frac{2}{T}\int_{t_1}^{(t_1+T/2)} i\,dt \tag{3}$$

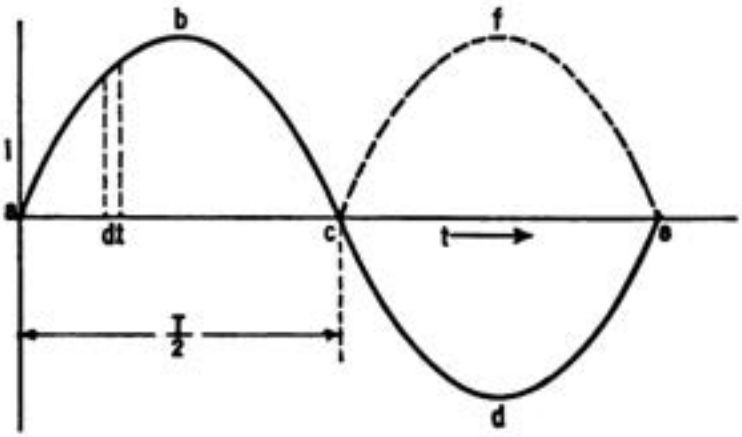

FIG. 2. Rectified a-c wave shown dotted.

If the average values of the positive and negative loops are different, the actual average value taken over a complete cycle represents the value of a d-c component in the wave. For example, the average

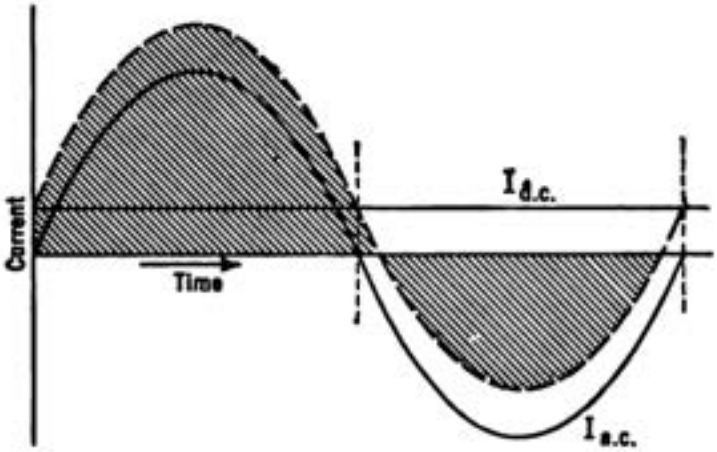

FIG. 3. Displaced a-c wave is equivalent to a symmetrical a-c wave and a d-c component.

value of the cross-hatched wave in Fig. 3 is I_{dc}. Inspection will show that the dotted wave is the sum of the alternating wave I_{ac} and the direct current I_{dc}.

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Effective and Average Values of a Sinusoid. Through the use of equation (1) the effective value of any wave may be found. If the equation of the wave is not known, the integration must be performed graphically. When the equation is known, the analytical solution is generally to be preferred. Consider the sinusoid,

$$i = I_m \sin \omega t$$

$$I^2_{(\text{rms})} = \frac{1}{T}\int_0^T i^2\,dt = \frac{1}{T}\int_0^T I_m^2 \sin^2 \omega t\,dt = \frac{I_m^2}{T}\int_0^T (\tfrac{1}{2} - \tfrac{1}{2}\cos 2\omega t)\,dt$$

$$= \frac{I_m^2}{T}\left[\frac{t}{2} - \frac{1}{4\omega}\sin 2\left(\frac{2\pi}{T}\right)t\right]_0^T = \frac{I_m^2}{2}$$

$$I_{(\text{rms})} = \frac{I_m}{\sqrt{2}} = 0.707 I_m \tag{4}$$

For a *sine wave*, therefore, the rms value is 0.707 times the maximum. In general, $I_{(\text{rms})}$ is written simply as I, and unless otherwise specified the symbol I refers to the effective or rms value of an alternating current.

The average value of a sinusoid over one-half cycle is

$$I_{\text{av}} = \frac{2}{T}\int_0^{T/2} I_m \sin \omega t\,dt = \frac{2}{\pi} I_m = 0.636 I_m \tag{5}$$

Problem 2. A resultant current wave is made up of two components, a 5-ampere, d-c component and a 60-cycle, a-c component which is of sinusoidal wave form and which has a maximum value of 4 amperes.

(*a*) Draw a sketch of the resultant current wave.

(*b*) Write the analytical expression for the resultant current wave, choosing the $t = 0$ reference at a point where the a-c component is at zero value and where di/dt is positive.

Ans.: $i = [5 + 4 \sin (377t)]$ amperes.

(*c*) What is the average value with respect to time of the resultant current over a complete cycle?

Ans.: $I_{\text{av}} = 5$ amperes.

(*d*) What is the effective value of the resultant current?

Ans.: $I_{\text{eff}} = 5.75$ amperes.

Form Factor. Form factor is the ratio of the effective to the average value of a wave. Hence, for a voltage wave, e, which has equal positive and negative loops:

$$\text{Form factor} = \frac{\sqrt{\dfrac{1}{T}\displaystyle\int_0^T e^2\,dt}}{\dfrac{2}{T}\displaystyle\int_0^{T/2} e\,dt} \tag{6}$$

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Equation (6) is subject to the same limitations as those explained for equation (2). Form factor has very little physical significance. It gives no certain indication of wave shape or wave form. Although a peaked wave will usually have a higher form factor than a flat-topped wave, it cannot be conclusively stated that one wave is more peaked than another because it has a higher form factor. That form factor tells nothing of the shape of a wave is evident from the fact that a sine wave and the wave $e = E_m \sin \omega t + (5/12)E_m \sin 5\omega t$, shown in Fig. 4, have the

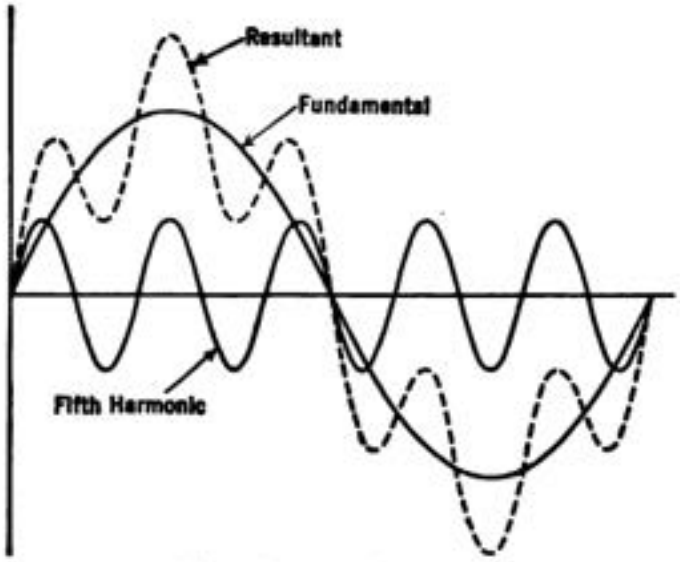

Fig. 4. Form factor of dotted wave is the same as that of a sine wave.

same form factor, namely, 1.11. However, form factor does give some indication of the relative hysteresis loss that will exist when a voltage is impressed on a coil wound on an iron core. Also some use is made of form factor in determining effective voltages induced in such coils when a known non-sinusoidal flux wave is present in the iron core.

Problem 3. Find the form factor of the sawtooth wave form shown in Fig. 5. *Hint:* Between the limits of $t = 0$ and $t = T = 3$ seconds, the analytical expression for the voltage is $e = 50t$ volts. In a case of this kind,

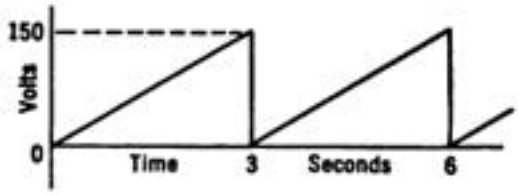

Fig. 5. Sawtooth wave form of voltage for Problems 3 and 4.

$$E_{av} = \frac{1}{T}\int_0^T e\,dt$$

Ans.: 1.155.

Crest or Peak Factor. The crest, peak, or amplitude factor is the ratio of the maximum value of a voltage wave to the effective value. For the dotted wave shown in Fig. 4 the crest factor is 1.85. A knowledge of this factor is necessary when using an ordinary voltmeter

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to measure a voltage employed in insulation testing. The dielectric stress to which insulation is subjected depends upon the maximum value of the voltage attained. Since waves of the same effective value may have different maximum values, it is obvious that a knowledge of crest factor is required when making dielectric tests. The crest factor of a *sine wave* is

$$\frac{E_m}{0.707E_m} = \sqrt{2}$$

Problem 4. Find the crest factor of the sawtooth wave form shown in Fig. 5.
Ans.: 1.732.

Representation of Sine Waves by Vectors. It has previously been stated that an attempt is made to secure sine waves of alternating currents and potentials. Alternating-current computations are based upon the assumption of sine waves of voltage and current. When non-sinusoidal quantities are encountered, they are expressed in terms of a number of sine components of different magnitudes and frequencies, and these components are then handled according to the methods applicable to sine waves. In general, it would be cumbersome continually to handle instantaneous values in the form of equations of the waves. A more convenient means is to employ a vector method of representing these sine waves. The vectors may then be manipulated instead of the sine functions to secure the desired result.

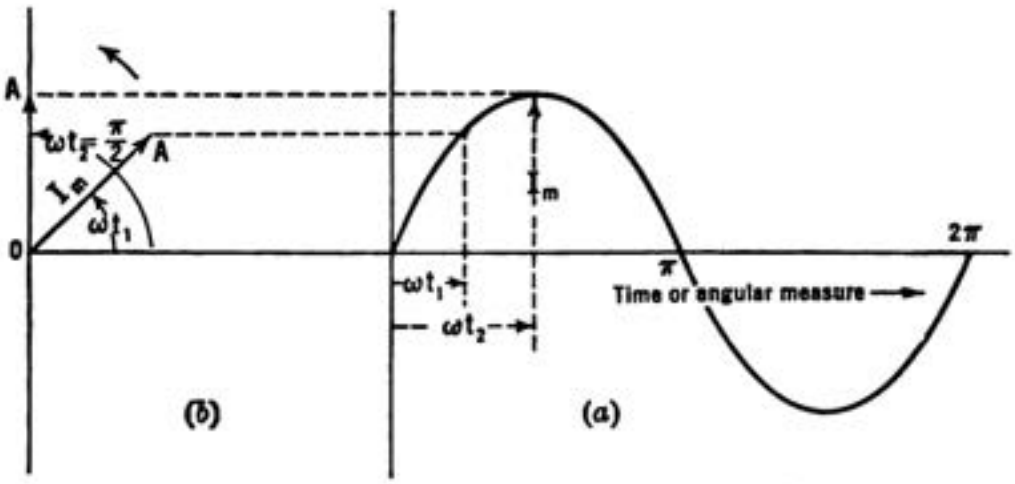

FIG. 6. Projection of a revolving vector represents a sine wave.

The sine wave of current $i = I_m \sin \omega t$ is shown in Fig. 6*a*. All the ordinates of this wave at the various times t may be represented by the projection of the revolving vector OA on the vertical axis of Fig. 6*b*. This projection is $I_m \sin \omega t$ if OA has a magnitude of I_m. This is the equation of the wave shown in Fig. 6*a*.

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If two sine waves are related as shown in Fig. 7, each may be represented by the projections of counterclockwise[1] revolving vectors on the vertical. A little study will show that the angle of phase difference for the two waves must also be the angular displacement between the two

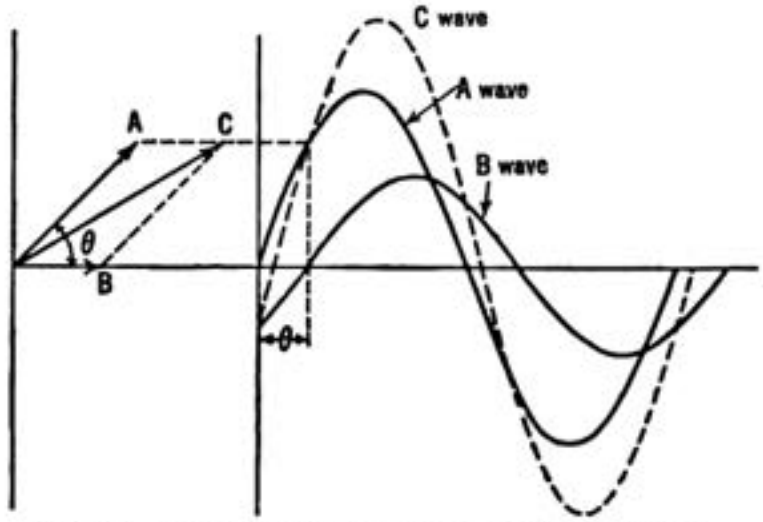

FIG. 7. Addition of sine waves by the use of vectors.

vectors OA and OB representing them. If OA and OB are added vectorially, a resultant OC is obtained whose projection on the vertical will represent the instantaneous values of the algebraic sum of the sine waves A and B.

Example 1. Add the following currents as waves and as vectors:

$$i_1 = 5 \sin \omega t$$
$$i_2 = 10 \sin (\omega t + 60°)$$

As waves:

$$\begin{aligned} \text{Sum} = i_0 = i_1 + i_2 &= 5 \sin \omega t + 10 \sin (\omega t + 60°) \\ &= 5 \sin \omega t + 10 \sin \omega t \cos 60° + 10 \cos \omega t \sin 60° \\ &= 10 \sin \omega t + 8.66 \cos \omega t \end{aligned}$$

Refer to the right triangle shown in Fig. 8a. If the previous equation is multiplied and divided by 13.23, there results

$$\begin{aligned} i_0 &= 13.23 \left[\frac{10}{13.23} \sin \omega t + \frac{8.66}{13.23} \cos \omega t \right] \\ &= 13.23 \, [\cos \alpha \sin \omega t + \sin \alpha \cos \omega t] \\ &= 13.23 \sin (\omega t + \alpha) \\ &= 13.23 \sin (\omega t + 40.9°) \end{aligned}$$

[1] Counterclockwise is assumed the positive direction of rotation of vectors. The counterclockwise direction of rotation has been arbitrarily used by engineers in the United States and many foreign countries. Some foreign countries have used clockwise as positive. To avoid errors the student must always consider counterclockwise as the positive direction of rotation of all vectors in this book. One vector is said to be ahead or leading another when it is farther advanced counterclockwise than the other.

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As vectors: A wave of relative phase represented by sin ωt will be represented by a vector along the reference axis. Positive angles will be assumed to be measured counterclockwise. The two waves are then represented by vectors, as shown in Fig. 8*b*. The sum will be found by adding x and y components.

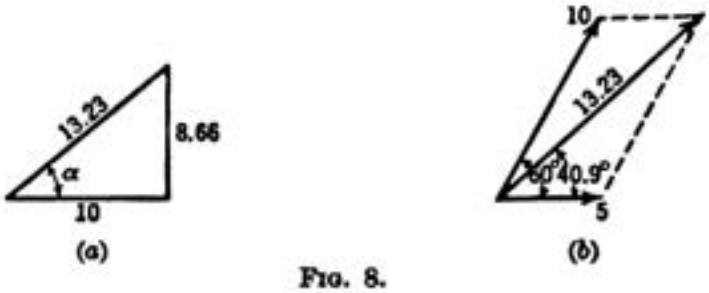

FIG. 8.

$$\sum x = 5 + 10 \cos 60° = 10$$
$$\sum y = 10 \sin 60° = 8.66$$
$$\text{Sum} = \sqrt{\overline{\sum x}^2 + \overline{\sum y}^2} = \sqrt{10^2 + 8.66^2} = 13.23$$
$$\alpha = \tan^{-1} \frac{\sum y}{\sum x} = \tan^{-1} \frac{8.66}{10} = 40.9°$$

Since the resultant is counterclockwise (positive) from the reference, the equation may be written as

$$i_0 = 13.23 \sin (\omega t + 40.9°)$$

Problem 5. Subtract i_2 from i_1 in example 1 by both methods shown above.

Ans.: 8.66 sin (ωt − 90°).

It is apparent that these coplanar vectors are merely convenient representations of sine waves, the independent variable of which is time. As such, they are time vectors and do not have any meaning so far as space relations are concerned. When the lengths of the two vectors represent maximum values of the waves respectively, the resultant vector will represent the maximum value of the resultant of the two waves.

Effective or rms values of voltages and currents are ordinarily used. For sine waves these have been shown to be equal to the maximum value divided by $\sqrt{2}$. Thus maximum values of the vectors could be handled vectorially and the resultant divided by $\sqrt{2}$ to obtain the effective value. Instead, all the initial vectors could have their maximum values multiplied by 0.707 and the resultant of these would then be the resultant maximum divided by $\sqrt{2}$. If the latter procedure is followed, the vectors can be considered to represent effective values. Vectorial representation of effective values is customary, in which case the results are given directly in terms of effective values, the ones usually desired.

In drawing vector diagrams certain conventions must be observed. First, a convenient reference axis should be established. The vectors have their relations to one another fixed but they may be represented with respect to any axis. In Fig. 7, the vectors *OA* and *OB* were con-

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sidered to revolve in order to represent the waves. The resultant OC was obtained by adding the two vectors when OB was along the axis of reference. Obviously, the same result would have been obtained had OA and OB been added when stopped in any other position with respect to the reference axis, provided their magnitudes and the angle θ between them were not changed. Second, it must be observed that *counterclockwise is considered the positive direction of rotation of vectors* and that a vector rotated through an angle of lead or ahead of another vector must be rotated counterclockwise. It then follows that an angle of lag from a given axis must be in the clockwise direction. A vector thus rotated is said to be behind the axis in question.

To illustrate the use of these conventions, the vector diagrams of voltage and current for a pure resistance, pure inductance, and a pure capacitance circuit will be drawn. The waves shown on Oscillogram 1, page 17, for a pure resistance circuit, indicate that the applied voltage is in phase with the current. With current taken as, or along, the reference axis the vector diagram is shown in Fig. 9.

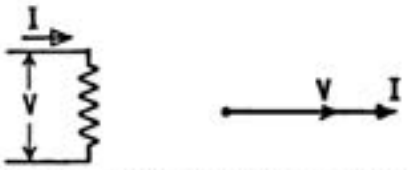

Fig. 9. Resistance branch and vector diagram.

It was shown in Chapter II and experimentally illustrated in Oscillogram 2, page 20, that the wave of voltage drop across a circuit containing only inductance leads the current by 90°. This relation is illustrated vectorially with the current as the reference in Fig. 10 of the present chapter.

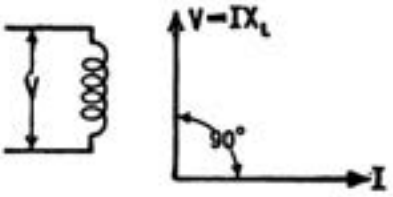

Fig. 10. Inductance branch and vector diagram.

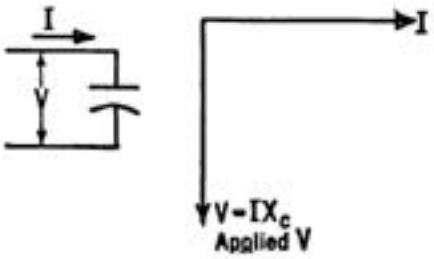

Fig. 11. Capacitance branch and vector diagram.

Reference to Oscillogram 3, page 24, will show that Fig. 11 of the present chapter represents vectorially the relations previously explained for the purely capacitive circuit.

Current was taken as the reference in the three previous diagrams. This was not necessary. The current could just as well have been drawn at any angle with respect to the reference axis, but for any particular case the relation between current and voltage must remain the same, that is, the resistance drop must always be in phase with the current, the drop across the inductance must always lead the current by exactly

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90°, and the drop across the capacitance must always lag the current by exactly 90°. The reference axis that appears to be the most convenient for the particular problem at hand should be chosen.

Vector Diagrams as Determined by Resistance and Reactance Drops and Impedance Functions. If a current $i = I_m \sin \omega t$ is assumed to flow in a circuit containing R and L, Kirchhoff's emf law states that $v = Ri + L\,di/dt$. Therefore $v = RI_m \sin \omega t + I_m L\omega \cos \omega t$. Since $RI_m \sin \omega t$ is of the same phase as $I_m \sin \omega t$, the resistance drop is shown in phase with the current in the vector diagram of Fig. 12. It will also be noted that $I_m L\omega \cos \omega t$ is 90° ahead of $I_m \sin \omega t$. Hence it is so drawn on the vector diagram. The vector sum of these two components is the resultant applied voltage **V**. The angle between **V** and **I** is $\theta = \tan^{-1}\omega L/R$. The same relation between **V** and **I** is obtained from the impedance function $Z\underline{/\theta}$. As explained in Chapter II, a positive angle θ means that the applied voltage leads the current or that the current lags the applied voltage by the phase angle θ. Thus the relation of **V** and **I** shown in the vector diagram could have been shown

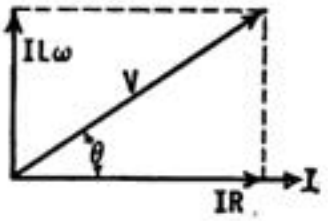

Fig. 12. Addition of voltage drops across L and R.

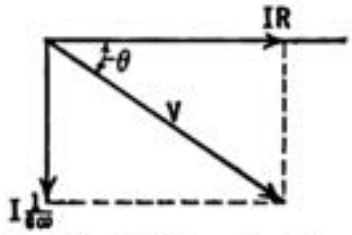

Fig. 13. Addition of voltage drops across C and R.

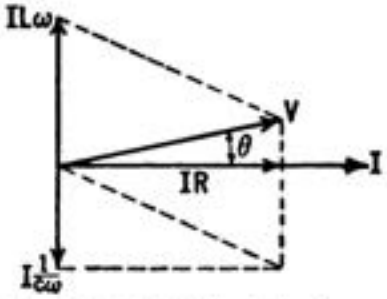

Fig. 14. Addition of voltage drops across L, C, and R.

directly from the impedance function where the angle tells the phase and V/Z gives the magnitude of **I**. It should be noted that effective values were used exclusively in Fig. 12. Through the same procedure the student can show that Fig. 13 represents the vector diagram for an R and C circuit. The vector diagram of the R, L, and C circuit combines the vector diagrams in Figs. 12 and 13 the results of which appear in Fig. 14.

Problem 6. A 60-cycle current of 15 amperes flows in a circuit of 5 ohms resistance, $\frac{10}{377}$ henry inductance, and $\frac{1}{377 \times 15}$ farad capacitance. Draw the vector diagram, and calculate the applied voltage and the phase angle between it and the current. *Ans.:* 106 volts; angle 45°.

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Significance of Currents Flowing in the Direction of Voltage Rises and Drops. If the potential becomes greater in the direction of tracing a circuit, a voltage rise is being encountered. For example, assume the polarities of a circuit at some instant to be as indicated in Fig. 15. When tracing from *a* to *b* through the generator, the tracing is in the direction of increasing potential (from minus to plus) or in the direction of a voltage rise. In a similar way, when tracing through the load from *c* to *d*, the tracing is in the direction of a fall of potential or a voltage drop. Since the generator is the "pump," the current will flow from minus to plus through the generator, whereas in the external circuit it flows from plus to minus. It is evident, then, that a current flowing in the general direction of a potential rise represents electrical power generated or delivered. Also, when the current flows in the direction of a potential fall or drop, as it does through a load, power is being consumed or taken. If, then, a voltage rise is assumed positive, the generated power would be positive. A voltage drop is then negative and, since the same current flows in the direction of the voltage drop through the load, the power determined would be negative. These are the usual conventions employed when power generated and power consumed are simultaneously considered. If a voltage drop is assumed positive, then positive current in conjunction with the positive drop would yield positive power and under such conditions power absorbed is positive. It is immaterial which conventions are used; that which is the most expedient is the one to choose. Physically, the same results are obtained. Although the above conventions are the most common, it is possible to establish other systems.

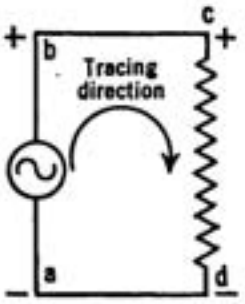

FIG. 15. Polarities of an a-c generator and load at some instant.

If a voltage rise is assumed positive, the question sometimes arises: Will generated power still be positive if the tracing direction is reversed? The answer is yes, as may be shown by the following considerations. Assume the tracing direction in Fig. 15 is *badc*. Then a voltage drop is encountered in the tracing direction through the generator. Since a voltage rise was considered positive, this drop through the generator will be negative. Since current flows through a generator in the general direction of increasing potential, the current will be in a direction opposite to that of the tracing direction. Hence it must be called a negative current. The product of the voltage drop through the generator, which was negative, by the negative current (opposite to the tracing direction) is positive. The sign of power generated is therefore unchanged. Similarly, it may be shown that the sign of the power

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dissipated by the load is unchanged. Hence the choice of the tracing direction does not affect the signs of generated and dissipated power. These are fixed by the signs assumed for voltage rises and drops in conjunction with the current.

Power, Real and Reactive. In Chapter II it was shown that the general expression for average power, when waves of voltage and current are sinusoidal, is $\frac{V_m I_m}{2} \cos \theta$. Since the maximum value of a sine wave divided by the square root of 2 is the effective value, the equation for average power may be written

$$P = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos \theta = VI \cos \theta \tag{7}$$

When V is in volts and I is in amperes, the power is expressed in watts. As previously shown, the power in a single-phase circuit is not constant. The instantaneous power from equation (33), Chapter II, is

$$p = \left[\frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} \cos \theta \cos 2\omega t\right] + \frac{V_m I_m}{2} \sin \theta \sin 2\omega t \tag{8}$$

The first two terms of the right side of equation (8) represent instantaneous real power. When $2\omega t$ is an odd multiple of π, the value of the real power is

$$\frac{2V_m I_m}{2} \cos \theta = 2VI \cos \theta$$

When $2\omega t$ is a multiple of 2π, real power is 0. Hence real power in a single-phase circuit fluctuates between 0 and $2VI \cos \theta$ and has an average value of $VI \cos \theta$ (shown in Chapter II). The third term of the right-hand member of equation (8) represents what has been called instantaneous reactive power, or, preferably, instantaneous reactive volt-amperes. Its equation is

$$p_X = \left(\frac{V_m I_m}{2} \sin \theta\right) \sin 2\omega t \tag{9}$$

Thus instantaneous reactive volt-amperes fluctuate between $+\frac{V_m I_m}{2} \sin \theta$ and $-\frac{V_m I_m}{2} \sin \theta$. Whereas the average value of the instantaneous reactive volt-amperes is zero, the maximum value is $\frac{V_m I_m}{2} \sin \theta$. This is

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the value referred to when reactive volt-amperes are considered.[2] Hence

$$P_X = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \sin \theta = VI \sin \theta \tag{10}$$

It is plain that reactive volt-amperes as determined from equation (10) will be positive when θ is positive. As interpreted from vector diagrams when current is taken as a reference, Fig. 16, θ is positive when the voltage leads the current or for inductive loads. If voltage is taken along the reference, Fig. 17, θ is positive when the current leads the generated voltage. In the former case reactive volt-amperes are positive for inductive loads or lagging currents, whereas in the second case positive reactive volt-amperes are obtained when the load is capacitive or

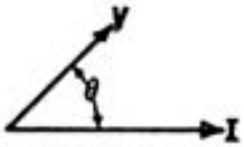

FIG. 16. Angle θ is positive when voltage leads current if current is along the reference axis.

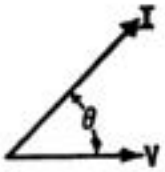

FIG. 17. Angle θ is positive when current leads voltage if voltage is along the reference axis.

where the current leads the voltage. Another basis for determining the sign of reactive power was given in Chapter II. At the 1934 Paris meeting of the Committee on Electrical and Magnetic Units the convention was adopted whereby reactive power caused by a lagging current is to be considered negative. Until this convention comes into general use it will be necessary to distinguish in some suitable and definite manner whether the reactive power in question is caused by a leading or a lagging current. This can be done by following the quantity by the words "leading reactive volt-amperes" or "lagging reactive volt-amperes."

Reactive volt-amperes are expressed in vars, a term coined from the first letters of the words "volt amperes reactive." Reactive volt-amperes considered over a period of time represent oscillations of energy between the source and the load. Their function is to supply the energy for magnetic fields and charging condensers, and to transfer this energy back to the source when the magnetic field collapses or when the condenser discharges. Although reactive volt-amperes, as such, require no average energy input to the generators, they do necessitate a certain amount of generator volt-ampere capacity and thereby limit the

[2] It should be recognized that this discussion refers to components of the resultant power wave. These components do not exist as separate entities but they are convenient components to consider for purposes of analysis. Actually a single wave, as shown in Chapter II, is the only power wave which has a physical existence.

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available power output of the generators. Reactive volt-amperes cannot be transferred without incurring a copper loss. Although this i^2r loss is caused by the transfer of the reactive volt-amperes, it is not a part of the reactive volt-amperes. Reactive volt-amperes are due to quadrature components of voltage and current and as such represent zero average power. These additional losses must be supplied by an average energy input to the alternators.

From equation (33), Chapter II, instantaneous real power was found to be $V_m I_m \cos\theta \sin^2 \omega t$. This may be considered to consist of a voltage $V_m \sin \omega t$ and a current $(I_m \cos\theta) \sin \omega t$, which is in phase with the voltage. The current $I_m \cos\theta$ is called the in-phase component, power component, active component, or energy component of current with respect to voltage. In terms of root mean square values the power is due to a voltage V and a component of current $I \cos\theta$ in phase with V, as shown in Fig. 18. Since the product of the voltage V and energy component of current $I \cos\theta$ is $VI \cos\theta$, the same expression as equa-

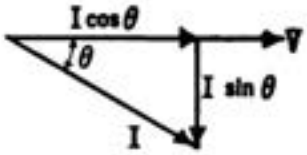

FIG. 18. In-phase and quadrature components of current with respect to voltage.

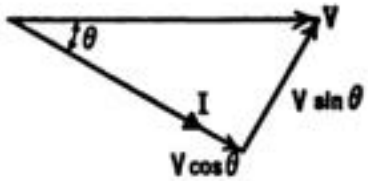

FIG. 19. In-phase and quadrature components of voltage with respect to current.

tion (7) for power, it is evident that power may be determined in this manner. If $\cos\theta$ is grouped with V, then $V \cos\theta$ may be viewed as the in-phase component, active component, energy component, or power component of voltage with respect to current, as shown in Fig. 19. Obviously, power may also be obtained by multiplying the in-phase component of voltage with respect to current by the current. Similarly, $I \sin\theta$ in Fig. 18 is the "out-of-phase component," quadrature component, or reactive component of current with respect to voltage. This component multiplied by the voltage gives reactive volt-amperes, as may be seen by comparison with equation (10). Also, $V \sin\theta$ is the quadrature, reactive, or wattless component of voltage with respect to current. This component of voltage multiplied by current also yields the reactive volt-amperes, or vars.

Volt-Amperes. The product of effective voltage by effective current in an a-c circuit is called volt-amperes. A larger unit is kilovolt-amperes, abbreviated kva. Obviously, a given number of volt-amperes may represent any number of different values of power, depending upon the value of $\cos\theta$ in equation (7). Cosine θ is therefore a factor by which

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volt-amperes are multiplied to give power. Hence cosine θ is called power factor. As an equation

$$\text{Power factor} = \cos\theta = \frac{\text{power}}{\text{volt-amperes}} \tag{11}$$

Reference to equation (10) will show that $\sin\theta$ is the factor by which volt-amperes are multiplied to yield reactive volt-amperes or vars. Hence $\sin\theta$ is called the reactive factor.

$$\text{Reactive factor} = \sin\theta = \frac{\text{reactive volt-amperes}}{\text{volt-amperes}} \tag{12}$$

Since $\sin^2\theta + \cos^2\theta = 1$, reactive factor $= \sqrt{1 - (\text{p.f.})^2}$ and power factor $= \sqrt{1 - (\text{r.f.})^2}$.

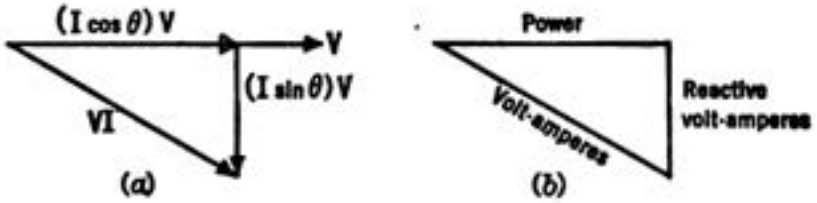

Fig. 20. Relation of power, reactive volt-amperes, and resultant volt-amperes.

If the current and each of its two components in Fig. 18 are multiplied by V, a relationship between power, reactive volt-amperes, and volt-amperes is obtained, as shown in Figs. 20*a* and *b*. Hence

$$\text{Volt-amperes} = \sqrt{(\text{power})^2 + (\text{reactive va})^2}$$

This relation is very useful in problems involving correction of power factor.

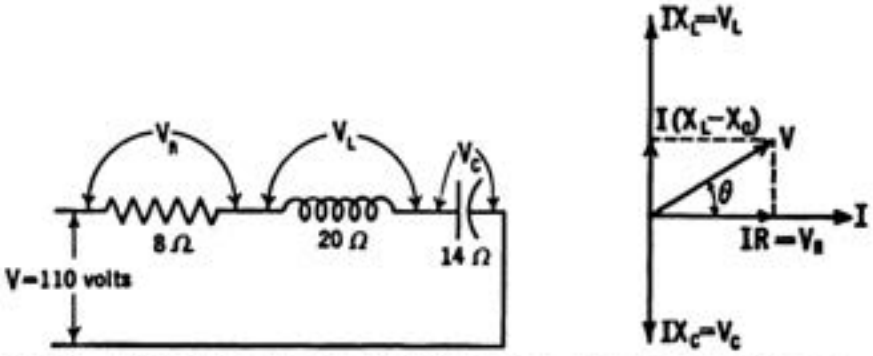

Fig. 21. *R*, *L*, and *C* in series and the corresponding vector diagram.

Example 2. One hundred and ten volts are applied to a series circuit consisting of 8 ohms resistance, 0.0531 henry inductance, and 189.7 μf capacitance. When the frequency is 60 cycles, calculate current, power, power factor, vars, reactive factor, and volt-amperes. Also calculate the voltage drop across each circuit element. The circuit and vector diagrams are shown in Fig. 21.

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$$X_L = 2\pi fL = 2\pi\, 60 \times 0.0531 = 20 \text{ ohms}$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi\, 60 \times 189.7 \times 10^{-6}} = 14 \text{ ohms}$$

$$X = X_L - X_C = 20 - 14 = 6 \text{ ohms}$$

$$R = 8 \text{ ohms}$$

$$Z = \sqrt{R^2 + X^2} = \sqrt{8^2 + 6^2} = 10 \text{ ohms}$$

$$I = \frac{110}{10} = 11 \text{ amperes}$$

$$\text{P.f.} = \cos\theta = \frac{IR}{V} = \frac{IR}{IZ} = \frac{R}{Z} = \frac{8}{10} = 0.8$$

$$P = VI\cos\theta = 110 \times 11 \times 0.8 = 968 \text{ watts}$$

Also $$P = I^2R = 11^2 \times 8 = 968 \text{ watts}$$

$$\text{Reactive va} = VI\sin\theta = 110 \times 11 \times \frac{XI}{ZI} = 110 \times 11 \times \frac{6}{10} = 726 \text{ vars}$$

$$\text{va} = VI = 110 \times 11 = 1210 = \sqrt{968^2 + 726^2}$$

$$V_R = IR = 11 \times 8 = 88 \text{ volts}$$

$$V_L = IX_L = 11 \times 20 = 220 \text{ volts}$$

$$V_C = IX_C = 11 \times 14 = 154 \text{ volts}$$

It will be noted that the arithmetic sum of these three voltages is much greater than the applied voltage. Alternating voltages of the same frequency can be added but they *must* be added *vectorially* with due regard for phase relation. Thus

$$220 - 154 = 66 \text{ volts in quadrature with } \mathbf{I}$$

$$V_R = 88 \text{ volts in phase with } \mathbf{I}$$

Therefore

$$V = \sqrt{88^2 + 66^2} = 110 \text{ volts, which checks the applied voltage.}$$

Example 3. Given the parallel circuit shown in Fig. 22, find I, I_1, I_2, and total power consumed.

Solution. The impedance functions of branches 1 and 2 are

$$\mathbf{Z}_1 = \sqrt{6^2 + 8^2} \angle \tan^{-1}\frac{8}{6} = 10 \angle 53.17^\circ \text{ ohms}$$

$$\mathbf{Z}_2 = \sqrt{5^2 + 5^2} \angle \tan^{-1}\left(\frac{-5}{5}\right) = 7.07 \angle -45^\circ \text{ ohms}$$

$$I_2 = \frac{100}{7.07} = 14.14 \text{ amperes}$$

$$I_1 = \frac{100}{10} = 10 \text{ amperes}$$

$$\mathbf{I} = \mathbf{I}_2 + \mathbf{I}_1$$

The vector diagram is drawn as shown in Fig. 23.

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The currents may be added by using $\sum x$ and $\sum y$ components or by the cosine law. The former will be used and a tabulation of results made. The x-axis will be taken along V. This is arbitrary. Any other position may be used.

Current	x components	y components
I_2	$I_2 \cos 45 = 10$	$I_2 \sin 45 = 10$
I_1	$I_1 \cos(-53.17°) = 6$	$I_1 \sin(-53.17°) = -8$

$$\sum x = 16 \qquad \sum y = 2$$

$$I = \sqrt{16^2 + 2^2} = 16.13 \text{ amperes}$$

$$P = VI_{\text{in-phase}} = 100 \times \sum x = 100 \times 16 = 1600 \text{ watts}$$

An alternative is

$$P = I_1^2R_1 + I_2^2R_2 = 10^2 \times 6 + 14.14^2 \times 5$$
$$= 600 + 1000 = 1600 \text{ watts}$$

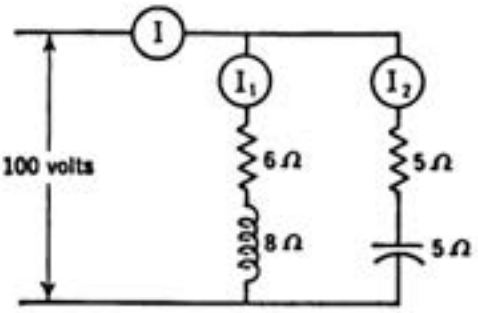

FIG. 22. Parallel branches.

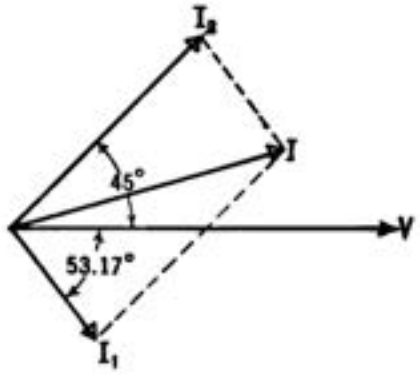

FIG. 23. Vector diagram of Fig. 22.

Vector Combination of Voltages. Thus far, only currents have been added and subtracted vectorially. Since vector combinations are based upon the assumption of sine waves, it is apparent that sinusoidal voltage waves can be added and subtracted vectorially. For example, the coils shown in Fig. 24 are assumed to have induced voltages which

FIG. 24. Coils in which a-c voltages are induced.

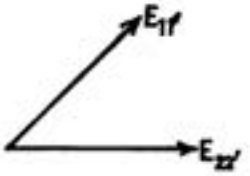

FIG. 25. Voltages induced in coils of Fig. 24.

are phase-displaced by 45°, as shown in Fig. 25. The voltage $E_{12'}$ is desired when 1′ and 2 of Fig. 24 are connected. In general, the difference of potential between two points of a winding or circuit is found by adding all the potential drops (rises are negative drops) encountered in *tracing through* the winding from one point in question to the other.

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This statement follows from the definition of potential difference. The voltages are denoted by subscripts, and the order in which the subscripts are written must be the same as the order in which they are encountered as the circuit is being traced. Thus for Fig. 24, when 1′ and 2 are con-

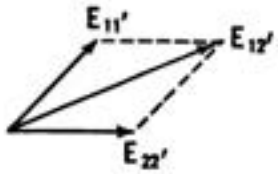

FIG. 26. E_{12}' for Fig. 24 when 1′ and 2 are connected.

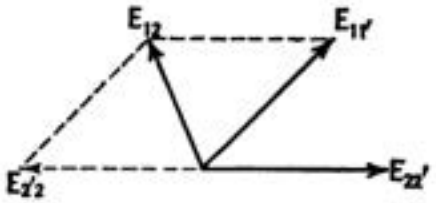

FIG. 27. E_{12} for Fig. 24 when 1′ and 2′ are connected.

nected, $\mathbf{E}_{12'} = \mathbf{E}_{11'} + \mathbf{E}_{22'}$. This vector addition is shown in Fig. 26. If 1′ were connected to 2′ in Fig. 24, the emf $\mathbf{E}_{12}$ would be $\mathbf{E}_{12} = \mathbf{E}_{11'} + \mathbf{E}_{2'2}$. This result is obtained by adding the voltage vectors, $\mathbf{E}_{11'}$ and $\mathbf{E}_{2'2}$, as shown in Fig. 27.

Problem 7. Two coils on the armature of an alternator are displaced 60 electrical degrees. The emf of each coil is 100 volts. What is the resultant emf of the two coils when connected series adding and also when series subtracting?

Ans.: 173.2 volts, 100 volts.

PROBLEMS

8. An elevator motor takes 20 amperes for 15 seconds. Power is then cut off for 45 seconds, after which the cycle is repeated. If rated full-load current of the motor is 12 amperes, will it overheat on a continuation of this cycle? What is the equivalent continuous current which will yield the same average rate of heating?

9. A motor takes 50 amperes for 10 seconds, after which power is off for 20 seconds. It then takes 60 amperes for 5 seconds, after which power is cut off for 1 minute. What will the continuous rated current have to be so that the motor will not overheat?

10. (*a*) What is the average value of the pulsating current shown in Fig. 28?

(*b*) What is the effective value?

11. (*a*) If the current shown in Fig. 28 flows through a d-c ammeter in series with an a-c ammeter, what will be the reading of each instrument, assuming perfect calibration of the instruments?

(*b*) If the resistance of the circuit is constant (the pulsating current being produced by a pulsating voltage), which of the above readings should be employed in finding the power by the I^2R formula?

12. (*a*) If the current shown in Fig. 28 flows through a 5-ohm resistance, what number of joules of heat energy is produced each cycle? what number of gram calories?

(*b*) What power is dissipated in the above resistance over any integral number of cycles?

13. The plate current of a triode operating as an oscillator takes the general form shown in Fig. 29.

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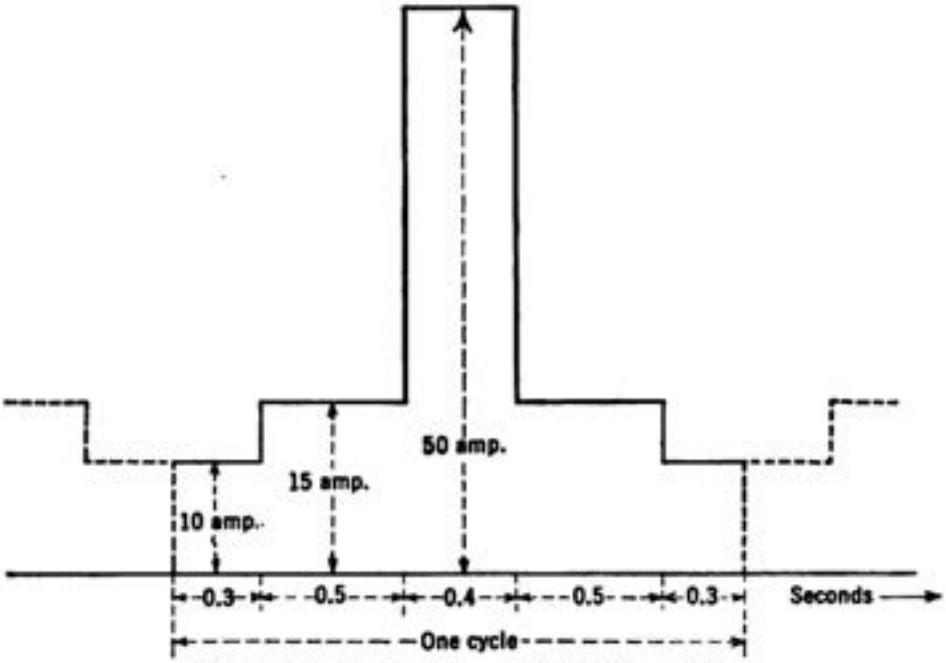

FIG. 28. See Problems 10, 11, 12, and 24.

(a) What is the frequency of oscillation depicted in Fig. 29?

(b) What is the average value of the pulsating current?

(c) What is the effective value of the pulsating current?

Note: The current during the first 2×10^{-4} second shown in Fig. 29 may be represented by the equation $i = 2 \times 10^4 t$ amperes. Utilize symmetry.

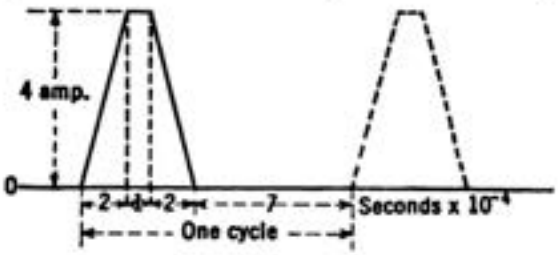

FIG. 29. See Problem 13.

14. A current in a circuit starts at zero and increases linearly until a value of 12 amperes is attained. It then drops to zero in negligible time and repeats the cycle. What will an a-c ammeter in this circuit read?

15. A current starts abruptly at 10 amperes and decreases linearly to zero and then repeats this cycle. Find the rms value without changing the orientation of the wave from that given.

16. Find the rms value of a current in terms of radius ρ whose instantaneous values make semicircles of radius ρ above and below the x-axis.

17. Calculate the form factor of the current wave in Problem 14.

18. Find the rms value of $e = 100 \sin \omega t + 60 \sin (5\omega t + 30°)$ volts by integration.

19. Calculate the form factor of $e = 100 \sin \omega t + 60 \cos 3\omega t$.

20. Find the rms value of $e = 100 \sin \omega t - 40 \sin 3\omega t$ volts.

21. Calculate the form factor of the voltage wave in Problem 20.

22. Find a wave other than that given in the text which is not a sine wave but which has the same form factor as a sine wave.

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23. Calculate the peak factor of (*a*) a sine wave, (*b*) a rectangular wave, (*c*) a symmetrical triangular wave whose positive and negative halves are symmetrical about their respective midordinates if the angle at the peak is 60°, and (*d*) a triangular wave whose angle at the peak is 90°.

24. Calculate the crest factor for the wave shown in Fig. 28.

25. Calculate the crest factor and form factor of a wave whose positive and negative loops are semicircles.

26. The respective branch currents flowing toward a junction of two parallel branches of a circuit are $i_1 = 30 \sin (\omega t + 60°)$ amperes and $i_2 = 20 \sin (\omega t - 20°)$ amperes. Find the resultant current leaving the junction in terms of a single sine wave. Find also the effective value of the current.

27. One branch current of $i_1 = 40 \sin (\omega t - 40°)$ amperes combines with a second branch current to yield a resultant of $50 \sin (\omega t + 80°)$ amperes. Find the equation of the second branch current. Find also the effective value.

28. A motor requires 25 amperes and 220 volts at a lagging power factor of 0.88. Find the power, vars, reactive factor, and the volt-amperes taken.

29. A motor requires 10 amperes and 220 volts at a power factor of 0.8 lag. Find the power, reactive volt-amperes, reactive factor, and the volt-amperes required.

30. The voltage of a circuit is $v = 200 \sin (\omega t + 30°)$, and the current is $i = 50 \sin (\omega t + 60°)$. What are the average power, volt-amperes, and power factor?

31. A motor takes 15 amperes and 220 volts at a lagging power-factor angle of 72° when running at no load. Find the number of watts, vars, and volt-amperes it is taking.

32. How many resultant volt-amperes will be taken from the line when the two motors in Problems 28 and 31 are operating simultaneously as stated from the same line. What is the resultant line current and power supplied?

33. One motor takes 250 amperes at 0.8 power factor lag while another motor takes 50 kw at 0.5 leading power factor from a line of 220 volts. What is the resultant line current for these two motors? What is the power factor of the combined loads? Is it leading or lagging?

34. The voltage of a circuit is $v = 200 \sin \omega t$ volts, and the current is $i = 50 \cos (\omega t - 30°)$ amperes. What are the average power, vars, and power factor?

35. A varmeter in a circuit indicates 600 vars, and a wattmeter in the same circuit shows 800 watts. Find the volt-amperes, power factor, and reactive factor of the circuit.

36. A series circuit has 8 ohms resistance and 20 millihenrys inductance. If 110 volts at 60 cycles are impressed, calculate the current and power.

37. One branch of a parallel circuit consists of 6 ohms resistance, 48 ohms inductive reactance, and 40 ohms capacitive reactance, while the other branch consists of a resistance of 7 ohms and a capacitive reactance of 2 ohms. Find the current delivered to the combination when 100 volts are impressed across the entire circuit. Calculate the total power and that consumed by each branch.

38. (*a*) Find the readings of ammeters I_1, I_2, and I, and of wattmeter W of Fig. 30. Compare the reading of W with $I_1^2R_1 + I_2^2R_2$.

(*b*) Draw the vector diagram of $\mathbf{V}$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}$, $\mathbf{I}_1\mathbf{R}_1$, $\mathbf{I}_1\mathbf{X}_{L1}$, $\mathbf{I}_2\mathbf{R}_2$, and $\mathbf{I}_2\mathbf{X}_{C2}$.

(*c*) Assuming that $\mathbf{V}$ represents a potential drop from *a* to *b* through the circuit branches, find the potential drop from *d* to *c*, or $\mathbf{V}_{dc}$.

(*d*) Assuming that $\mathbf{V}$ represents a potential drop from *b* to *a* through the circuit branches, find the potential drop from *d* to *c*, or $\mathbf{V}_{dc}$.

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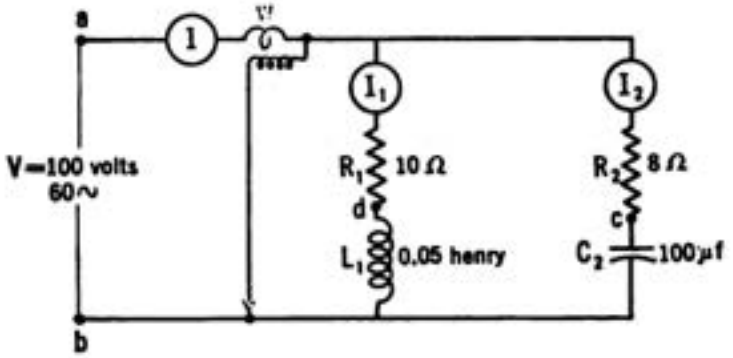

FIG. 30. See Problems 38, 39, and 40.

39. Work Problem 38 if the parameters are changed to $R_1 = 8$ ohms, $L_1 = 0.025$ henry, $R_2 = 10$ ohms, and $C_2 = 120$ μf.

40. Find the readings of the ammeter I and of the wattmeter W in Fig. 30 if an additional branch R_3L_3 is placed in parallel with the R_1L_1 and R_2C_2 branches. $R_3 = 15$ ohms and $L_3 = 0.12$ henry.

41. Find readings of the ammeter I and of the wattmeter W in Fig. 31 for the parameters specified.

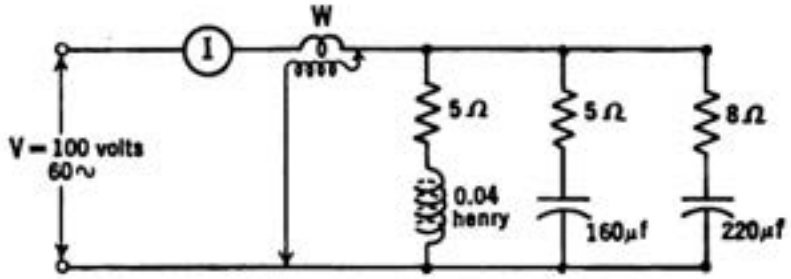

FIG. 31. See Problem 41.

42. A type of alternator much used in laboratories has six coils spaced about the armature at intervals of 30 electrical degrees. The two leads of each coil are brought out to a terminal board, making available six voltages. Because of the 30 electrical degrees of space displacement of the coils on the armature, the individual coil voltages have phase differences of 30°. Let Fig. 32 represent the six coils, and assume that adjacent coils in the figure are electrically adjacent coils on the alternator armature. Assume also that the coil voltages are sinusoidal and that leads 1, 2, 3, 4, 5, and 6 are corresponding ends of the coils, and that $\mathbf{E}_{1'1}$ is 30° behind $\mathbf{E}_{2'2}$, $\mathbf{E}_{2'2}$ is 30° behind $\mathbf{E}_{3'3}$, and so on.

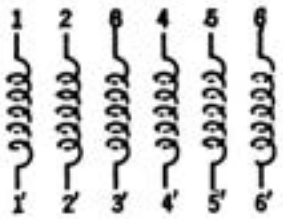

FIG. 32. Six coils of an a-c generator. Adjacent coils are displaced 30 electrical degrees.

(*a*) Draw the vector diagram of $\mathbf{E}_{1'1}, \mathbf{E}_{2'2}, \mathbf{E}_{3'3}, \mathbf{E}_{4'4}, \mathbf{E}_{5'5}$, and $\mathbf{E}_{6'6}$ when $\mathbf{E}_{1'1}$ is laid off along the $+x$-axis. Each coil has an effective emf of 50 volts.

(*b*) Find $\mathbf{E}_{13'}$ when $1'$ is connected to 3.

(*c*) Find $\mathbf{E}_{13}$ when $1'$ is connected to $3'$.

(*d*) Find the greatest voltage that can be obtained by connecting all coils in series.

(*e*) Draw the vector diagram that represents the three voltages, $\mathbf{E}_{12'}$, $\mathbf{E}_{34'}$, and $\mathbf{E}_{56'}$, assuming that $1'$ is connected to 2, $3'$ to 4, and $5'$ to 6.

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CHAPTER IV

VECTOR ALGEBRA
(AS APPLIED TO A-C CIRCUIT ANALYSIS)

The Operator j. Since the complex quantities normally employed in a-c circuit analysis to simplify calculations are added and subtracted like coplanar vectors, they are usually referred to as vectors.

It is well known that a plane vector can be specified in magnitude and direction in terms of its x-axis projection and its y-axis projection. For example, if the x-axis projection of the vector **A** in Fig. 1 is known as x_A and the y-axis projection is known as y_A then the magnitude of the vector **A** is:

$$A = \sqrt{x_A^2 + y_A^2} \tag{1}$$

From the geometry of Fig. 1 it is plain that the angle, θ_A, between the direction of vector **A** and the direction of the positive x-axis is:

$$\theta_A = \tan^{-1}\frac{y_A}{x_A} \tag{2}$$

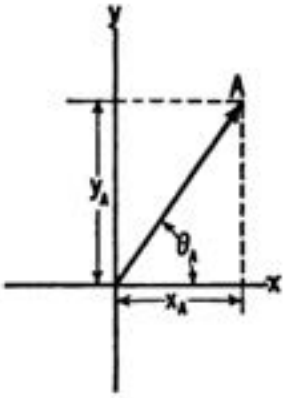

FIG. 1. Resolution of vector **A** into its x-axis and y-axis components.

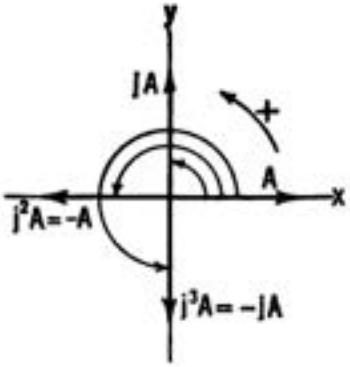

FIG. 2. Effects produced by successive applications of the operator j upon a vector **A**, the original position of which is along the $+x$-axis.

In order to specify a vector in terms of its x and y components some means must be employed to distinguish between the x-axis projection and the y-axis projection. Inasmuch as the $+y$-axis projection is $+90°$ from the $+x$-axis, a convenient operator for the purpose at hand is one which will, when applied to a vector, rotate it 90° counterclockwise without changing the magnitude of the vector.

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Let j be an operator which produces 90° counterclockwise rotation of any vector to which it is applied as a multiplying factor. The physical significance of the operator j can best be appreciated by first considering that it operates on a given vector **A**, the direction of which is along the $+x$-axis. Then, by definition, when the vector **A** of Fig. 2 is multiplied by j a new vector, $j\mathbf{A}$, 90° counterclockwise from **A**, will be obtained. If the operator j is applied to the vector $j\mathbf{A}$ it will, by definition, rotate $j\mathbf{A}$ 90° in the counterclockwise direction. The result is $jj\mathbf{A} = j^2\mathbf{A}$ as shown in Fig. 2. Also from Fig. 2:

$$j^2\mathbf{A} = -\mathbf{A}$$

Hence:

$$j^2 = -1$$

and

$$j = \sqrt{-1} \tag{3}$$

If the operator j is applied to the vector $j^2\mathbf{A}$ the result is $j^3\mathbf{A} = -jA$. The vector $j^3\mathbf{A}$ is 270° counterclockwise from the reference axis, directly opposite the vector $j\mathbf{A}$ in Fig. 2. If the vector $j^3\mathbf{A}$, in turn, is operated on by j, the result is $j^4\mathbf{A} = j^2j^2\mathbf{A} = \mathbf{A}$. It will be observed that successive applications of the operator j to the vector **A** produce successive 90° steps of rotation of the vector in the counterclockwise direction without affecting the magnitude of the vector.

From Fig. 2 it is apparent that multiplying **A** by $-j$ yields $-j\mathbf{A}$, a vector of identical magnitude rotated clockwise 90° from **A**. Hence $-j$ is an operator which produces clockwise rotation of 90°.

The Cartesian Form of Notation. A vector in any quadrant can be completely specified in a cartesian or rectangular form of notation, as shown below.

$$\mathbf{A} = \pm a \pm ja' \tag{4}$$

where a is the x-axis projection and a' is the y-axis projection of the vector. In any case the magnitude of the vector **A** is:

$$A = \sqrt{a^2 + a'^2} \tag{5}$$

The phase position of a first-quadrant vector is conveniently described in terms of the positive acute angle measured in a ccw direction from the $+x$-axis to the position of the vector. In equation form:

$$\theta_{1st} = \tan^{-1}\frac{(+a')}{(+a)} \tag{6}$$

The phase position of a fourth-quadrant vector is conveniently

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described in terms of the negative acute angle measured in a cw direction from the $+x$-axis to the position of the vector.

$$\theta_{4\text{th}} = \tan^{-1}\frac{(-a')}{(+a)} \tag{7}$$

A fourth-quadrant vector can, of course, be specified in terms of the positive angle $(360° - \theta_{4\text{th}})$, where $\theta_{4\text{th}}$ is the magnitude of the angle measured in a negative or clockwise direction from the $+x$-axis to the position of the vector.

Phase positions of second- and third-quadrant vectors are easily located in terms of the a and a' components by first finding the acute angle, the tangent of which is a'/a, without regard to sign, and then subtracting this angle from or adding it to 180°, depending upon whether the a' component is positive or negative.

Figure 3 illustrates how vectors in any quadrant can be specified in magnitude and phase position in terms of real and j components. In determining the phase angle it is necessary to know the individual signs of the a and a' components in order to locate the angle θ correctly.

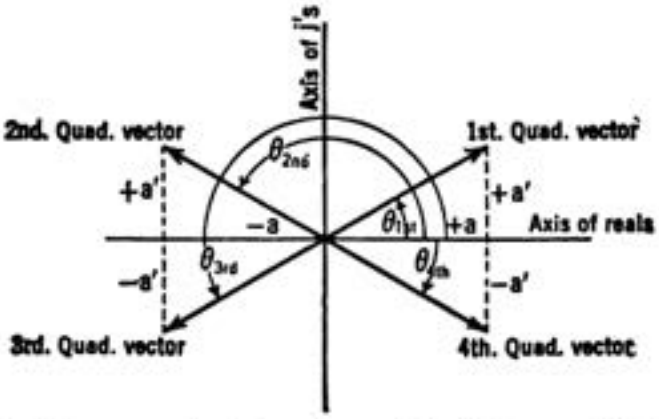

FIG. 3. Vectors in any quadrant can be specified in terms of their real (x-axis) and j (y-axis) components.

The Operator ($\cos\theta \pm j\sin\theta$). Reference to Fig. 3 will show that the x-axis projection of a vector in any quadrant is $A\cos\theta$. The angle θ may be measured either positively or negatively from the $+x$-axis in determining the x-axis projection, since $\cos\theta = \cos(-\theta)$.

The y-axis projection of the vector in any quadrant is $A\sin\theta$ if θ is measured in the ccw direction from the $+x$-axis. The y-axis projection is $-A\sin\theta$ if θ is measured in the cw direction from the $+x$-axis to the position of the vector. Therefore,

$$\mathbf{A} = A\ (\cos\theta \pm j\sin\theta) \tag{8}$$

is equivalent to the form shown in equation (4). The plus sign is

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used if θ is measured counterclockwise from the reference axis, the minus sign if θ is measured clockwise.

Equation (8) shows that $(\cos\theta + j\sin\theta)$ operating on a real magnitude A, that is a vector of A units magnitude along the $+x$-axis, rotates this vector through a $+\theta$-angle from its initial position. Similarly the operator $(\cos\theta - j\sin\theta)$ rotates the original vector through a $-\theta$-angle.

It may be shown that the operator $(\cos\theta \pm j\sin\theta)$ rotates any vector to which it is attached as a multiplying factor through $+\theta$ or $-\theta$ degrees, depending whether the plus or minus sign is employed. Consider a vector in an initial position such that $a = A\cos\alpha$ and $a' = A\sin\alpha$.

$$\mathbf{A}\ (\text{initially}) = a + ja' = A\ (\cos\alpha + j\sin\alpha) \tag{9}$$

Let $\mathbf{A}' = \mathbf{A}$ [operated on by $(\cos\theta + j\sin\theta)$].

$$\mathbf{A}' = A\ (\cos\alpha + j\sin\alpha)\ (\cos\theta + j\sin\theta) \tag{10}$$

$$\begin{aligned}\mathbf{A}' &= A\ (\cos\alpha\cos\theta + j\cos\alpha\sin\theta + j\sin\alpha\cos\theta + j^2\sin\alpha\sin\theta)\\ &= A\ [(\cos\alpha\cos\theta - \sin\alpha\sin\theta) + j\ (\sin\alpha\cos\theta + \cos\alpha\sin\theta)]\\ &= A\ [\cos\ (\alpha + \theta) + j\sin\ (\alpha + \theta)]\end{aligned} \tag{11}$$

Equation (11) shows that $\mathbf{A}'$ is a vector equal in magnitude to the vector $\mathbf{A}$ but advanced θ degrees from the $\mathbf{A}$ position since it now makes an angle of $(\alpha + \theta)$ with the reference axis.

In similar manner it may be shown that the operator $(\cos\theta - j\sin\theta)$ rotates any vector to which it is attached through $-\theta$ degrees.

Exponential Form of the Operator $(\cos\theta \pm j\sin\theta)$. An important relationship is contained in the following equation:

$$(\cos\theta \pm j\sin\theta) = \epsilon^{\pm j\theta} \tag{12}$$

Equation (12), known as Euler's equation, follows directly from an inspection of the Maclaurin series expansions[1] of $\cos\theta$, $\sin\theta$, and $\epsilon^{j\theta}$.

[1] Certain functions, among which are $\cos\ (\theta)$, $\sin\ (\theta)$, and $\epsilon^{\pm j\theta}$, can be expanded into series form by means of Maclaurin's theorem. The theorem states that:

$$f(\theta) = f(0) + \frac{f'(0)\theta}{1} + \frac{f''(0)\theta^2}{\underline{/2}} + \frac{f'''(0)\theta^3}{\underline{/3}} + \cdots \text{etc.}$$

where $f(\theta)$ is the particular function of θ that is to be expanded, $f(0)$ is the value of this particular function when θ is set equal to zero, $f'(0)$ is the value of the first derivative of the function when θ is set equal to zero, f'' is the value of the second derivative of the function when θ is set equal to zero, etc.

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Expanded into series form:

$$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \cdots \tag{13}$$

$$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \cdots \tag{14}$$

$$\epsilon^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{\underline{|2}} + \frac{(j\theta)^3}{\underline{|3}} + \frac{(j\theta)^4}{\underline{|4}} + \frac{(j\theta)^5}{\underline{|5}} + \frac{(j\theta)^6}{\underline{|6}} + \cdots \tag{15}$$

All quantities involving even powers of j reduce to real numbers since $j^2 = -1$, $j^4 = 1$, $j^6 = -1$, etc. All quantities involving odd powers of j reduce to first-degree j terms because $j^3 = -j$, $j^5 = j$, etc. If the j terms are properly evaluated, equation (15) may be arranged as follows:

$$\epsilon^{+j\theta} = \left(1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \cdots\right) + j\left(\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \cdots\right) \tag{16}$$

Therefore
$$\epsilon^{j\theta} = \cos\theta + j\sin\theta \tag{17}$$

and
$$A\epsilon^{j\theta} = A\ (\cos\theta + j\sin\theta) \tag{18}$$

In a similar manner it may be shown that:

$$\epsilon^{-j\theta} = \cos\theta - j\sin\theta \tag{19}$$

Polar Form of the Operator ($\cos\theta \pm j\sin\theta$). The exponential form of the operator $(\cos\theta \pm j\sin\theta)$ is very often written in a simplified form. It has been shown that:

$$\epsilon^{\pm j\theta} = (\cos\theta \pm j\sin\theta) \tag{20}$$

Therefore:

$$A\epsilon^{\pm j\theta} = A\ (\cos\theta \pm j\sin\theta) \tag{21}$$

By definition:

$$\epsilon^{\pm j\theta} = 1\,\underline{/\pm\theta} \tag{22}$$

and

$$A\epsilon^{\pm j\theta} = A\,\underline{/\pm\theta} \tag{23}$$

Therefore:

$$\epsilon^{j\theta} = \underline{/\theta} = (\cos\theta + j\sin\theta) \tag{24}$$

$$\epsilon^{-j\theta} = \underline{/-\theta} = (\cos\theta - j\sin\theta) \tag{25}$$

and

$$A\epsilon^{j\theta} = A\,\underline{/\theta} = A\ (\cos\theta + j\sin\theta) \tag{26}$$

$$A\epsilon^{-j\theta} = A\,\underline{/-\theta} = A\ (\cos\theta - j\sin\theta) \tag{27}$$

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Equations (26) and (27) state the equivalence of the three forms of notation that are commonly employed to define a given vector in magnitude and phase position. Graphical representations of equations (26) and (27) for particular values of A and θ are shown in Fig. 4. The exponential and polar forms are identical by definition and find their greatest use in the processes of multiplication, division, extraction of roots, and raising vectors to given powers. Both these forms express a vector in terms of polar coordinates. $A \underline{/\theta}$ is simply a shorthand or symbolic style of writing $Ae^{j\theta}$. Common usage distinguishes between the two forms by calling $Ae^{j\theta}$ the exponential form and $A \underline{/\theta}$ the polar form.

The rectangular or cartesian form, $A\ (\cos\theta \pm j \sin\theta)$, is indispensable in the processes of addition or subtraction of vectors if the j form of vector algebra is employed.

Problem 1. Write the equivalent polar form of the vector $3 + j4$ where the numbers refer to unit lengths. Illustrate the vector by means of a diagram.

Ans.: $5e^{j53.1°} = 5 \underline{/53.1°}$.

Problem 2. A vector is given in the form of $10e^{-j120°}$. Write the symbolic polar and cartesian forms of the vector, and illustrate, by means of a vector diagram, the magnitude and phase position of the vector.

Ans.: $10 \underline{/-120°} = -5 - j8.66$.

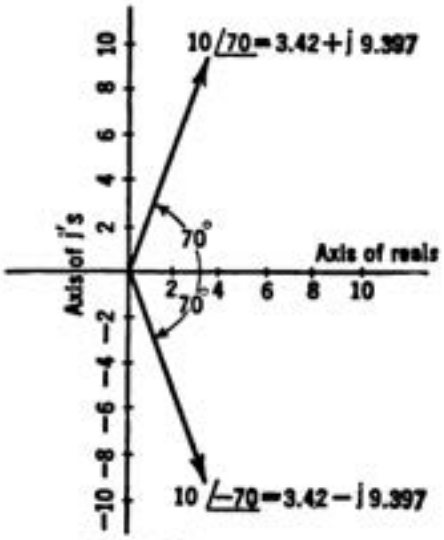

FIG. 4. Vectorial representation of equations (26) and (27) for the case of A = 10 units and $\theta = 70°$.

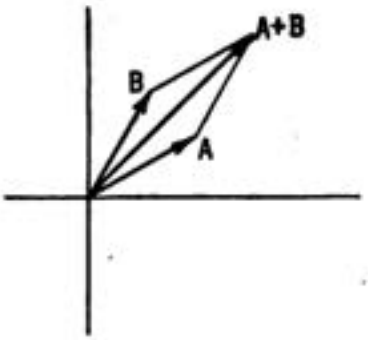

FIG. 5. Addition of the vectors **A** and **B**.

Addition of Vectors. The vector sum of two vectors **A** and **B** is a third vector which is defined in magnitude and phase position by the diagonal of the parallelogram which has for two of its sides the vectors **A** and **B**. The particular diagonal of the parallelogram thus formed, which represents the vector sum, **A** + **B**, is indicated in Fig. 5.

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Each vector may be considered as having a tail and a head. If the arrow heads in Fig. 5 indicate the heads of the vectors, then the vector sum of two vectors is the line which joins the tail of the first vector and the head of the second vector after the second vector has been placed so that its tail coincides with the head of the first vector.

The fact that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{28}$$

is obvious from the definition that has been given for the vector sum of two vectors.

The process of adding two vectors may be extended to include any number of vectors simply by first adding any two of the vectors involved and then adding to this vector sum, which is in itself a vector, the third vector, etc. The order in which the addition is carried out is immaterial. For example:

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{B} + \mathbf{C} + \mathbf{A} = \mathbf{C} + \mathbf{A} + \mathbf{B} \tag{29}$$

Vectors are written in the rectangular $(a + ja')$ form when addition is to be performed, since the exponential or polar forms do not lend themselves to the addition process. If $\mathbf{A} = a + ja'$, $\mathbf{B} = b + jb'$, and $\mathbf{C} = c + jc'$,

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (a + b + c) + j\,(a' + b' + c') \tag{30}$$

The magnitude of the resultant vector is:

$$D = \sqrt{(a + b + c)^2 + (a' + b' + c')^2} \tag{31}$$

The phase position of the resultant vector is:

$$\theta_D = \tan^{-1} \frac{(a' + b' + c')}{(a + b + c)} \tag{32}$$

Any or all of the component parts of the vectors **A**, **B**, and **C** in the above example may be negative. The process that has been given for the addition of three vectors can, of course, be extended.

Example. Let it be required to add

$$\mathbf{A} = 10\,\underline{/36.9^\circ} = 8 + j6 \quad \text{and} \quad \mathbf{B} = 6\,\underline{/120^\circ} = -3 + j5.20$$

$$\mathbf{A} + \mathbf{B} = \mathbf{C} = (8 - 3) + j\,(6 + 5.2)$$

$$\mathbf{C} = 5 + j11.2$$

The magnitude of the **C** vector is:

$$C = \sqrt{5^2 + 11.2^2} = 12.27 \text{ units}$$

The position of the vector **C** with respect to the $+x$-axis is:

$$\theta_C = \tan^{-1} \frac{11.2}{5} = \tan^{-1} 2.24 = 65.95^\circ$$

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Figure 6 illustrates the vector addition of **A** and **B** for the particular values that have been employed in this example.

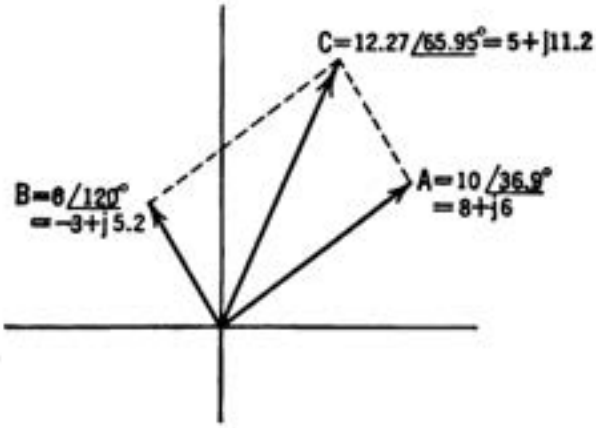

FIG. 6. Vector addition in a particular numerical case.

Problem 3. Add the vectors $14 \angle 60°$ and $20 \angle 15°$. State the result in both rectangular and polar forms, and illustrate, by means of a vector diagram, the operation that has been performed.

Ans.: $26.3 + j17.3 = 31.5 \angle 33.35°$.

Problem 4. Given the following three vectors:

$$\mathbf{A} = 40e^{j120°},\ \mathbf{B} = 20 \angle -40°,\ \mathbf{C} = 26.46 + j0$$

find $\mathbf{A} + \mathbf{B} + \mathbf{C}$ and illustrate the three vectors, together with their vector sum, by means of a vector diagram.

Ans.: $21.78 + j21.78 = 30.8 \angle 45°$.

Subtraction of Vectors. In ordinary algebra the operation or process of subtraction is accomplished by changing the sign of the quantity to be subtracted and proceeding as in addition. In vector algebra the vector which is to be subtracted is rotated through 180° and then added. To rotate a vector through 180° the operator $j^2 = -1$ may be applied or 180° may be added or subtracted from the original phase angle of the vector. Thus a vector $\mathbf{A} = A \angle \theta$ rotated through 180° becomes

$$\mathbf{A}' = j^2 A \angle \theta = -A \angle \theta = A \angle \theta \pm 180°$$

and a vector $\mathbf{B} = b + jb'$ rotated through 180° becomes

$$\mathbf{B}' = j^2(b + jb') = -b - jb'$$

Figure 7*a* illustrates the subtraction of vector **D** from vector **C**. Symbolically, the operation may be indicated as: $\mathbf{C} - \mathbf{D} = \mathbf{E}$. After the vector which is to be subtracted has been rotated through 180°, the vector thus resulting is added to the vector from which the subtraction is being made.

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Figure 7*b* illustrates the subtraction of vector **C** from vector **D**. It will be observed that (**D** − **C**) is of equal magnitude and 180° removed from (**C** − **D**). In general

$$(\mathbf{C} - \mathbf{D}) = -(\mathbf{D} - \mathbf{C}) \tag{33}$$

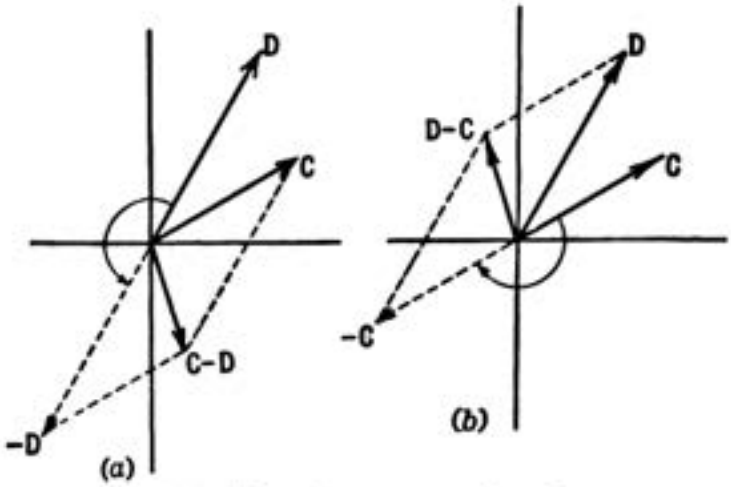

Fig. 7. Illustrating vector subtraction.

The vector difference of two vectors might have been defined in terms of one of the diagonals of the parallelogram formed by the two vectors. Figure 8 illustrates the particular diagonal which represents vector difference. The diagonal concept is useful in certain types of vector diagrams, but for general calculations the method which has previously been described is to be preferred.

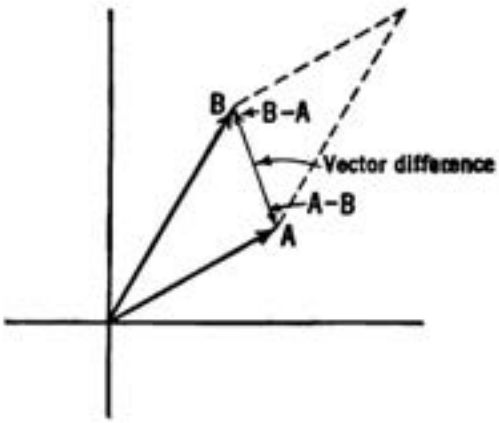

Fig. 8. The diagonal which defines the vector difference between two vectors. (The sense or direction of the diagonal is dependent upon the particular vector difference in question.)

Examples. Given the vectors

$$\mathbf{A} = 30 \angle 60^\circ$$

and

$$\mathbf{B} = 21 (\cos 160^\circ - j \sin 160^\circ)$$

let it be required to subtract vector **B** from vector **A**. The first step is to write the vectors in cartesian form.

$$\mathbf{A} = 30 \angle 60^\circ = 30 (\cos 60^\circ + j \sin 60^\circ) = 15 + j26$$

$$\mathbf{B} = 21 (\cos 160^\circ - j \sin 160^\circ) = -19.75 - j7.18$$

$$\mathbf{A} - \mathbf{B} = (15 + j26) - (-19.75 - j7.18)$$

$$= 34.75 + j33.18 = 48 \angle 43.6^\circ$$

For the particular case considered, the vector difference (**A** − **B**) is somewhat

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greater in magnitude than either of the original vectors. This condition is in general true if the original vectors are separated by more than 90°.

Let it be required to subtract vector **A** from vector **B**.

$$(\mathbf{B} - \mathbf{A}) = (-19.75 - j7.18) - (15 + j26) = -34.75 - j33.18 = 48\,\underline{/223.6^\circ}$$

Problem 5. Draw a vector diagram showing the vectors **A** and **B** of the above illustrative example, together with the vectors (**A** − **B**) and (**B** − **A**).

Problem 6. Given the following three vectors:

$$\mathbf{A} = 42\epsilon^{j200^\circ}$$
$$\mathbf{B} = 20\,\underline{/-40^\circ}$$
$$\mathbf{C} = 24.25 + j14$$

find (**A** + **C**) − **B** analytically and draw the vector diagram.

Ans.: $32.95\,\underline{/157.7^\circ}$.

Multiplication of Vectors and Complex Quantities. In a-c circuit analysis it is often desirable to operate on a vector current with an impedance function so that the resulting voltage may be obtained. Similarly, it is sometimes desirable to operate on a vector voltage with an admittance function, i.e., the reciprocal of the impedance function, to obtain the resulting current. The process of operating on a time vector current (or voltage) with a complex impedance (or admittance) function is called complex or vector multiplication.

The complex product of two vectors, **A** and **B**, in so far as a-c circuit analysis is concerned, is a third vector which has a magnitude equal to AB and a phase position with respect to the reference axis which is equal to the sum of the individual phase angles of **A** and **B**, namely, $(\alpha_A + \alpha_B)$. It will be shown presently why this particular definition of a complex product is especially suited to the vector manipulations that are universally employed in a-c circuit theory. A graphical interpretation of the definition is given in Fig. 9 for the particular case of $\mathbf{A} = 2\,\underline{/40^\circ}$ and $\mathbf{B} = 3\,\underline{/100^\circ}$.

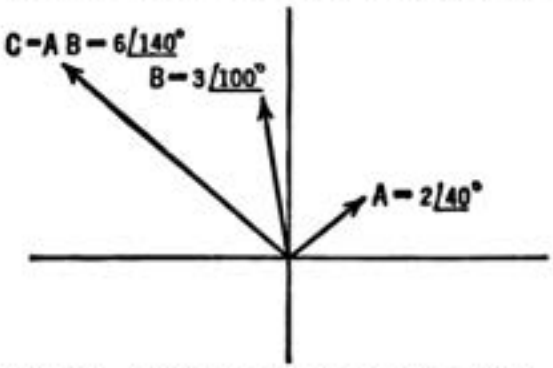

Fig. 9. Illustrating vector multiplication.

Analytically, the complex or vector product (as it is usually called) of two vectors can be formed most conveniently when the vectors are expressed in exponential or polar form. For example, the product of the **A** and **B** vectors shown in Fig. 9 is simply

$$\mathbf{AB} = 2\epsilon^{j40^\circ} \cdot 3\epsilon^{j100^\circ} = 6\epsilon^{j(40^\circ+100^\circ)} = 6\epsilon^{j140^\circ}$$

or

$$\mathbf{AB} = 2 \times 3\,\underline{/40^\circ + 100^\circ} = 6\,\underline{/140^\circ}$$

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From the definition which has been given for the complex product it is evident that the order in which the multiplication is carried out is immaterial. That is:

$$\mathbf{AB} = \mathbf{BA} \tag{34}$$

Furthermore, the definition which has been given is capable of extension to any number of vectors or complex quantities. For example,

$$\mathbf{ABC} = ABC\,\underline{/\alpha_A + \alpha_B + \alpha_C} \tag{35}$$

and

$$\mathbf{ABC} = \mathbf{BCA} = \mathbf{CAB}, \text{etc.} \tag{36}$$

The vector product of two vectors expressed in rectangular style can be formed by taking the cross-products of the component parts as in ordinary algebra. The proper interpretation must, of course, be given to the terms which involve j. If the vectors are given as $\mathbf{A} = a + ja'$ and $\mathbf{B} = b + jb'$, the vector product is formed exactly in accordance with the rules of ordinary algebra.

$$\mathbf{F} = \mathbf{AB} = (a + ja')\,(b + jb') = (ab - a'b') + j\,(a'b + ab') \tag{37}$$

The magnitude of the resulting vector is

$$\begin{aligned} F &= \sqrt{(ab - a'b')^2 + (a'b + ab')^2} \\ &= \sqrt{a^2b^2 - 2aba'b' + a'^2b'^2 + a'^2b^2 + 2a'bab' + a^2b'^2} \\ &= \sqrt{(a^2 + a'^2)\,(b^2 + b'^2)} = \sqrt{A^2B^2} = AB \end{aligned} \tag{38}$$

The magnitude of $\mathbf{F}$ is thus shown to be equal to the product of the magnitudes of the vectors whose product is being formed. It remains to be shown that the phase angle of $\mathbf{F}$ as defined by the real and j components of equation (37) agrees with the definition that has been given for the vector product of two vectors. From equation (37) the phase angle of $\mathbf{F}$ takes the following form:

$$\alpha_F = \tan^{-1}\frac{(a'b + ab')}{(ab - a'b')} = \tan^{-1}\frac{\dfrac{a'b + ab'}{AB}}{\dfrac{ab - a'b'}{AB}}$$

It is evident from the definitions that have been given to a, a', b, and b' that

$$\frac{a'}{A} = \sin\alpha_A,\ \frac{a}{A} = \cos\alpha_A,\ \frac{b'}{B} = \sin\alpha_B,\quad \text{and}\quad \frac{b}{B} = \cos\alpha_B$$

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Therefore:

$$\alpha_F = \tan^{-1} \frac{\sin \alpha_A \cos \alpha_B + \cos \alpha_A \sin \alpha_B}{\cos \alpha_A \cos \alpha_B - \sin \alpha_A \sin \alpha_B}$$

$$\alpha_F = \tan^{-1} \frac{\sin (\alpha_A + \alpha_B)}{\cos (\alpha_A + \alpha_B)} = \tan^{-1} \tan (\alpha_A + \alpha_B)$$

$$\alpha_F = \alpha_A + \alpha_B \tag{39}$$

Equations (38) and (39) show that the vector product of two vectors may be formed by ordinary algebraic multiplication when the factors are expressed in cartesian form.

Example. Given the vectors:

$$\mathbf{A} = 2 (\cos 40° + j \sin 40°) = 1.532 + j1.286$$

$$\mathbf{B} = 3 (\cos 100° + j \sin 100°) = -0.521 + j2.954$$

let it be required to find the vector product of **A** and **B** by the algebraic multiplication of the cartesian forms.

$$\begin{aligned} \mathbf{F} = \mathbf{AB} &= (1.532 + j1.286)(-0.521 + j2.954) \\ &= -0.799 + j4.525 - j0.670 + j^2 3.798 \\ &= (-0.799 - 3.798) + j(-0.670 + 4.525) \\ &= -4.597 + j3.855 \end{aligned}$$

$$\mathbf{F} = \sqrt{-4.597^2 + 3.855^2} \angle \tan^{-1} \frac{3.855}{-4.597}$$

$$= 6.0 \angle 180° - 40° = 6 \angle 140°$$

The graphical representations of the vectors **A**, **B**, and **F** are given in Fig. 9.

Problem 7. Find the complex product of

$$\mathbf{A} = 5 - j4 \quad \text{and} \quad \mathbf{B} = 2 + j3$$

by algebraic multiplication of the cartesian forms and draw the vector diagram. Change **A** and **B** to polar form and perform the multiplication process, **BA**.

Ans.: $22 + j7 = 23.09 \angle 17.65°$.

Problem 8. Given the following three vectors:

$$\mathbf{A} = 20 + j20, \ \mathbf{B} = 30 \angle -120°, \ \mathbf{C} = 5 + j0$$

perform the following indicated operations:

(*a*) $\mathbf{A} + \mathbf{B} + \mathbf{C}$, (*b*) $(\mathbf{A} + \mathbf{B})\mathbf{C}$, (*c*) $\mathbf{ABC}$.

Draw a vector diagram of **A**, **B**, and **C**, together with the vectors which represent the results of the above indicated operations.

Ans.: (*a*) $11.67 \angle -31°$, (*b*) $39.05 \angle -50.2°$, (*c*) $4242 \angle -75°$.

Division of Complex Quantities (or Vectors). For the purposes of a-c circuit theory the division of one complex quantity by another is carried

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out algebraically, as shown below, when the quantities are expressed in exponential form.

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A\epsilon^{j\alpha_A}}{B\epsilon^{j\alpha_B}} = \frac{A}{B}\epsilon^{j\alpha_A}\,\epsilon^{-j\alpha_B} = \frac{A}{B}\epsilon^{j(\alpha_A-\alpha_B)} \tag{40}$$

That is, the process of dividing one vector, **A**, by a second vector, **B**, results in a third vector, the magnitude of which is the quotient of the magnitudes of the vectors **A** and **B**, namely A/B. The phase position of the resulting vector with respect to the reference axis is the algebraic difference between the individual phase angles of the vectors **A** and **B** with respect to the reference axis, namely, $\alpha_A - \alpha_B$. It should be noted that the angle of the vector in the denominator is always subtracted from the angle of the vector in the numerator. Due regard is taken for the inherent signs of the individual phase angles, α_A and α_B, during the process of forming the algebraic difference. In symbolic polar form division is carried out as shown below:

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A\,\underline{/\alpha_A}}{B\,\underline{/\alpha_B}} = \frac{A}{B}\,\underline{/\alpha_A - \alpha_B} \tag{40a}$$

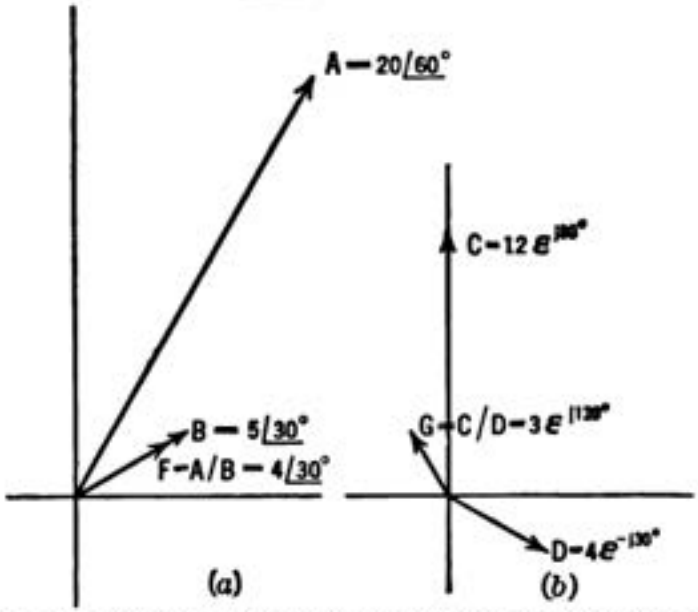

FIG. 10. Vector division in two particular numerical cases.

Examples. The processes of division in two particular cases are shown below.

$$\mathbf{F} = \frac{\mathbf{A}}{\mathbf{B}} = \frac{20\,\underline{/60^\circ}}{5\,\underline{/30^\circ}} = 4\,\underline{/30^\circ}$$

$$\mathbf{G} = \frac{\mathbf{C}}{\mathbf{D}} = \frac{12\epsilon^{j90^\circ}}{4\epsilon^{-j30^\circ}} = 3\epsilon^{j120^\circ}$$

The graphical interpretations of the above operations are contained in Fig. 10*a* and Fig. 10*b*.

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The process of division can be carried out very conveniently when the vectors are expressed in exponential or polar form. However, it is entirely possible and in some cases desirable to perform the operation with the vectors expressed in rectangular form. If $\mathbf{A} = a + ja'$ and $\mathbf{B} = b + jb'$, then

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{a + ja'}{b + jb'} = \frac{(a + ja')\,(b - jb')}{(b + jb')\,(b - jb')} \tag{41}$$

Both numerator and denominator of the above expression are multiplied by $(b - jb')$, the conjugate of $(b + jb')$. The conjugate of a given vector is a second vector, the real component of which is identical with the real component of the given vector and the j part of which is equal in magnitude but reversed in sign from the j component of the given vector.

The purpose of multiplying both numerator and denominator of equation (41) by the conjugate of the denominator is to clear the denominator of its j component. This rationalization process reduces the quotient $\mathbf{A}/\mathbf{B}$ to a more intelligible form. If the operations indicated in equation (41) are performed, the equation reduces to:

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{(ab + a'b') + j(a'b - ab')}{(b^2 + b'^2)} \tag{42}$$

By a process which is somewhat similar to that employed on page 73 it may be shown that:

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A}{B}\Bigg/\tan^{-1}\left[\frac{\sin\,(\alpha_A - \alpha_B)}{\cos\,(\alpha_A - \alpha_B)}\right] = \frac{A}{B}\underline{/\alpha_A - \alpha_B} \tag{43}$$

Example. If $\mathbf{A} = 10 + j17.3$ and $\mathbf{B} = 4.33 + j2.5$, let it be required to find $\mathbf{A}/\mathbf{B}$ by the method given in equations (41) and (42).

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{10 + j17.3}{4.33 + j2.5} = \frac{(10 + j17.3)\,(4.33 - j2.5)}{(4.33 + j2.5)\,(4.33 - j2.5)}$$

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{(43.3 + 43.3) + j(75 - 25)}{4.33^2 + 2.5^2}$$

Reduced to polar form:

$$\frac{\mathbf{A}}{\mathbf{B}} = \sqrt{3.465^2 + 2.0^2}\Bigg/\tan^{-1}\frac{2.0}{3.465} = 4.0\underline{/30^\circ}$$

Problem 9. Given $\mathbf{A} = 40\underline{/105^\circ}$ and $\mathbf{B} = 5 + j8.66$, find $\mathbf{A}/\mathbf{B}$, and draw a vector diagram illustrating $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}/\mathbf{B}$.

Ans.: $4\underline{/45^\circ}$.

Problem 10. Given the following three vectors:

$$\mathbf{A} = 20 + j20, \quad \mathbf{B} = 30\underline{/-120^\circ}, \quad \mathbf{C} = 5 + j0$$

perform the following indicated operations:

(*a*) $\dfrac{\mathbf{A} + \mathbf{B}}{\mathbf{C}}$ (*b*) $\dfrac{\mathbf{BC}}{\mathbf{A}}$

Draw a vector diagram of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, together with the vectors which represent the results of the above indicated operations.

Ans.: (*a*) $1.56\underline{/-50.2^\circ}$. (*b*) $5.3\underline{/195^\circ}$.

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Raising a Vector to a Given Power. A vector or preferably a complex quantity may be raised to a given power n, where n is an integer, by multiplying the vector by itself n times. For example, if $\mathbf{A} = A\,\underline{/\alpha_A}$,

$$\mathbf{A}^n = A^n\,\underline{/n\alpha_A} \tag{44}$$

The nth power of $\mathbf{A}$ is a vector whose magnitude is A^n and whose phase position with respect to the reference is $n\alpha_A$. The concept of successive applications of a given operator follows directly from the successive multiplication of the operator by itself. Obviously the process involved is accomplished most easily with the vector or operator in exponential or polar form.

From the rules which have been given for multiplication it is evident that:

$$\mathbf{A}^n\mathbf{B}^n = A^nB^n\,\underline{/n\alpha_A + n\alpha_B} \tag{45}$$

Example. An operator which is commonly used successively is the one which rotates a given vector through $+120°$. This operator is:

$$\mathbf{a} = 1\,(\cos 120° + j\sin 120°) = -0.50 + j0.866$$

In polar form:

$$\mathbf{a} = 1\,\underline{/120°}$$
$$\mathbf{a}^2 = 1\,\underline{/240°}$$
$$\mathbf{a}^3 = 1\,\underline{/360°} = 1\,\underline{/0°}$$
$$\mathbf{a}^4 = 1\,\underline{/480°} = 1\,\underline{/120°}$$

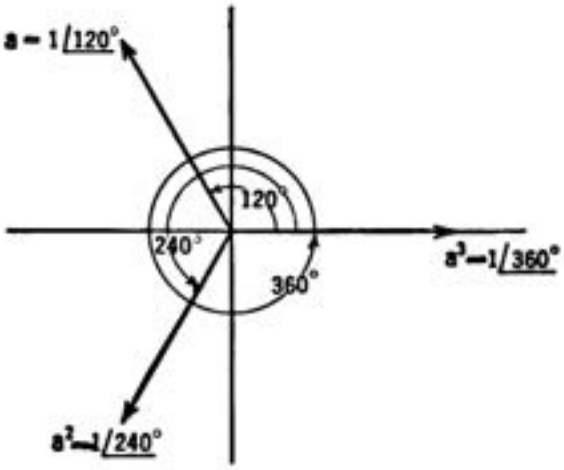

FIG. 11. Illustrating the operator $\mathbf{a} = (-0.50 + j0.866)$, together with $\mathbf{a}^2$ and $\mathbf{a}^3$.

The above operators are widely used in three-phase circuit problems because, under balanced conditions, the individual phase voltages (and currents) are displaced from one another by 120°. Figure 11 illustrates $\mathbf{a}$, $\mathbf{a}^2$, and $\mathbf{a}^3$ diagrammatically. Incidentally, the three values indicated in Fig. 11, $\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$, and $(1 + j0)$, are the three roots of $\sqrt[3]{1}$ because each of these roots cubed equals unity.

Problem 11. Raise the vector $(8.66 + j5.0)$ to the second power; to the fifth power. *Ans.:* $100\,\underline{/60°}$; $100{,}000\,\underline{/150°}$.

Extracting the Roots of a Vector. The inverse of the process of raising a vector to a given power is employed in the extraction of the roots of a particular vector. If $\mathbf{A} = A\,\underline{/\alpha_A}$ it follows that one of the n roots of $\sqrt[n]{\mathbf{A}}$ is $\sqrt[n]{A}\,\underline{/\frac{\alpha_A}{n}}$ because the latter value multiplied by itself

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n times will equal **A**. The remaining $(n - 1)$ roots are found by adding $2\pi q$ radians or $360q$ degrees to α_A before the division by n is performed. q is any integer and is used as 1, 2, 3, $\cdots$, and $(n - 1)$ to obtain the remaining roots. It should be noted that the addition of any multiple of 360° to the angle of the vector does not change the vector although it does provide a systematic method of evaluating the $(n - 1)$ remaining roots. In this method only positive magnitudes are employed, as:

$$\sqrt[n]{\mathbf{A}} = \sqrt[n]{A} \angle \frac{\alpha_A + 2\pi q}{n} \qquad [q = 0, 1, 2, \cdots (n - 1)] \quad (46)$$

The cartesian form of the above equation is:

$$\sqrt[n]{\mathbf{A}} = \sqrt[n]{A}\left[\cos\left(\frac{\alpha_A + 2\pi q}{n}\right) + j \sin\left(\frac{\alpha_A + 2\pi q}{n}\right)\right] \quad (47)$$

Example. Let it be required to find the square roots of **A** where $\mathbf{A} = 3.08 + j8.455$. For convenience the vector is first transformed into polar form.

$$\mathbf{A} = \sqrt{3.08^2 + 8.455^2} \angle \tan^{-1}\frac{8.455}{3.08} = 9.0 \angle 70°$$

The first root is: $\sqrt{9.0} \angle \frac{70°}{2} = 3 \angle 35°$.

The second root is: $\sqrt{9.0} \angle \frac{70° + 360°}{2} = 3 \angle 215°$.

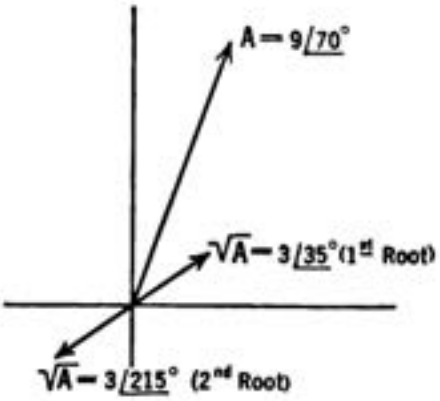

FIG. 12. Vector $9\angle 70°$ and its two roots.

Figure 12 illustrates the vector **A** together with its two roots. It will be noted that either root multiplied by itself yields the vector **A**.

Problem 12. Find the cube roots of the vector $(8 + j0)$, and draw a complete vector diagram of the vector and its three roots.

Ans.: $2 \angle 0°$, $2 \angle 120°$, $2 \angle 240°$.

The Logarithm of a Vector. Certain definitions in long-line and recurrent network theory utilize logarithms of vector quantities. The general concept of the logarithm of a vector is similar to that of the logarithm of an ordinary number. The logarithm of a vector **A** is the inverse of the exponential of **A**. In other words, the logarithm of the vector $\mathbf{A} = Ae^{j\theta}$ to the base ϵ is defined as the power to which ϵ must be raised to equal $Ae^{j\theta}$. By definition:

$$\log_\epsilon Ae^{j\theta} = \log_\epsilon A + \log_\epsilon \epsilon^{j\theta} = \log_\epsilon A + j\theta \log_\epsilon \epsilon = \log_\epsilon A + j\theta \quad (48)$$

It will be noted that the logarithm of the vector $\mathbf{A} = A \angle \theta$ is itself a

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vector. In rectangular form, when the logarithm is taken to the base ϵ, the real component is $\log_\epsilon A$; that is, the logarithm to the base ϵ of the magnitude of the vector **A** and the j component is θ(radians) in magnitude. In this connection, θ, the phase angle of the vector **A**, *must be considered in radians.*

Example. If $\mathbf{A} = 52\,\underline{/70^\circ}$, let it be required to find $\log_\epsilon \mathbf{A}$.

$$\log_\epsilon 52\,\underline{/70^\circ} = \log_\epsilon 52 + j\,\frac{70^\circ}{57.3^\circ} = 3.95 + j1.22$$

Problem 13. Perform the following indicated operations:

$$\frac{15\,\underline{/70^\circ}}{(3 - j4)} + \log_\epsilon (8 + j5)$$

Draw a vector diagram including each of the three original vectors together with the $\log_\epsilon (8 + j5)$ and the vector which represents the result of the indicated operations.

Ans.: $0.60 + j3.07$.

Impedance Expressed in Polar Form. It was shown in Chapter III that the currents and voltages in an a-c circuit can be conveniently represented as coplanar vectors. With the aid of vector algebra it is a simple matter to represent these currents and voltages analytically. However, the great benefit to be derived from the use of vector or complex algebra is the simple algebraic relations that can be established between the voltages and currents by using the impedance function as a complex quantity. Although the impedance function may take the form of a vector, it is not a vector in the same sense that alternating voltages or currents are vectors. From an algebraic point of view the impedance function is merely a complex quantity which properly relates vector voltages and vector currents one to the other. As such it is a most important operator in circuit analysis.

The physical considerations concerning the impedance function have been explained in Chapters II and III. If the polar form of the impedance function which was used throughout Chapter II is manipulated in accordance with the rules of vector algebra, the results obtained will agree with physical facts. For example, it has been shown that the impedance function of a series RLC branch is:

$$\mathbf{Z} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\,\underline{\Big/\tan^{-1}\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}} \tag{49}$$

The abbreviated form is:

$$\mathbf{Z} = Z\,\underline{/\theta} \tag{50}$$

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where $+\theta$ represents a lead of the voltage with respect to the current or a lag of the current with respect to the voltage. If a vector voltage $\mathbf{V} = V\underline{/\alpha}$ is applied to the above branch the resulting current is:

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V\underline{/\alpha}}{Z\underline{/\theta}} = \left[\frac{V}{Z}\right]\underline{/(\alpha - \theta)} \tag{51}$$

The vector quotient $\mathbf{V}/\mathbf{Z}$ results in a vector current which is V/Z in magnitude and θ degrees behind $\mathbf{V}$ regardless of the position that $\mathbf{V}$ has with respect to the reference axis. Thus $\mathbf{I}$ is correctly defined in magnitude and phase position.

In a similar manner it may be shown that $\mathbf{IZ} = \mathbf{V}$. If it be assumed that a current $\mathbf{I} = I\underline{/\beta}$ flows through an RLC branch, the impedance of which is $\mathbf{Z} = Z\underline{/\theta}$,

$$\mathbf{IZ} = (I\underline{/\beta})\,(Z\underline{/\theta}) = [IZ]\underline{/(\beta + \theta)} = \mathbf{V} \tag{52}$$

The vector product $\mathbf{IZ}$ yields a vector voltage $\mathbf{V}$, which is (IZ) in magnitude and θ degrees in advance of the current $\mathbf{I}$. It will be remembered that θ has been defined as

$$\tan^{-1}\frac{\left(\omega L - \dfrac{1}{\omega C}\right)}{R}$$

If $\omega L < 1/\omega C$, θ is a negative angle, in which case $\mathbf{V}$ actually lags $\mathbf{I}$.

Example. A given RL branch has $R = 3.5$ ohms and $L = 0.092$ henry. Find the vector current which flows through the branch if a 60-cycle voltage, $\mathbf{V} = 110\underline{/30^\circ}$ volts, is applied to the RL branch. (The phase angle which is associated with V is wholly arbitrary in a simple series circuit. For simplicity it might have been taken as zero degrees.)

$$\mathbf{Z} = \sqrt{R^2 + (\omega L)^2}\,\underline{\Big/\tan^{-1}\frac{(\omega L)}{R}}$$

$$\mathbf{Z} = \sqrt{3.5^2 + (377 \times 0.092)^2}\,\underline{\Big/\tan^{-1}\frac{(377 \times 0.092)}{3.5}}$$

$$\mathbf{Z} = 34.8\underline{/84.25^\circ}\text{ ohms}$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{110\underline{/30^\circ}}{34.8\underline{/84.25^\circ}} = 3.16\underline{/-54.25^\circ}\text{ amperes}$$

Figure 13 is a vector diagram of $\mathbf{V}$ and $\mathbf{I}$ for the particular RL branch that has been considered.

Problem 14. An RLC series branch consists of $R = 12.9$ ohms, $L = 0.056$ henry, and $C = 78\ \mu f$. (*a*) What is the vector or complex impedance of the RLC branch at 60 cycles? (*b*) If a 60-cycle vector current, $\mathbf{I} = 10\underline{/30^\circ}$ amperes, flows through the branch, find the vector voltage $\mathbf{V}$ across the terminals of the series branch. Draw

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a vector diagram illustrating the vector positions of **I** and **V** and the magnitude of the phase angle of **V** with respect to **I**.

Ans.: (*a*) $12.9 + j(21.1 - 34) = 12.9 - j12.9 = 18.24 \angle -45^\circ$ ohms.
(*b*) $182.4 \angle -15^\circ$ volts.

Impedance Expressed in Cartesian Form. The cartesian form of the impedance function of a given branch or circuit is, in general,

$$\mathbf{Z} = R + j(X_L - X_C) \tag{53}$$

where R is the equivalent resistance of the branch or circuit with respect to the terminals considered and $(X_L - X_C)$ is the equivalent reactance of the branch or circuit with respect to the terminals considered.

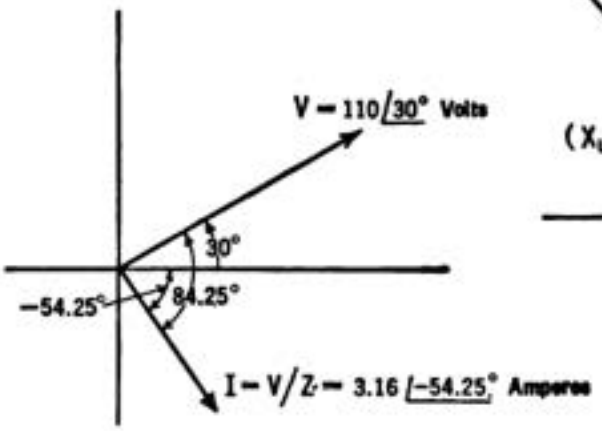

FIG. 13. Vector diagram of **V** and **I**, in a particular RL series circuit.

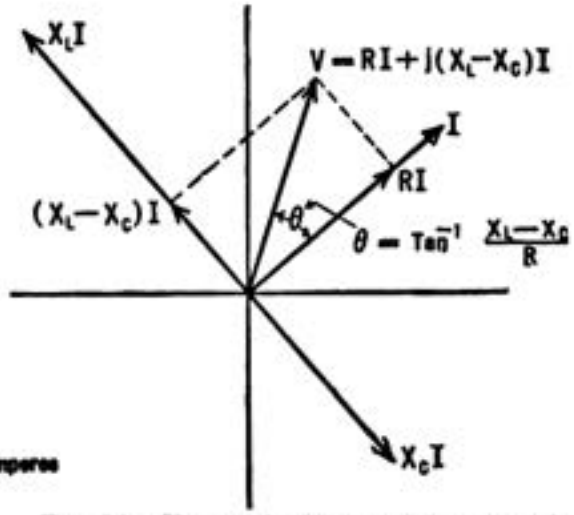

FIG. 14. Vectorial addition of drops equals applied voltage.

In accordance with previous definitions, $X_L = 2\pi fL$ and $X_C = \dfrac{1}{2\pi fC}$. A simple method of showing the validity of equation (53) is to employ a vector diagram in which are represented the RI, X_LI, and X_CI voltage drops which combine vectorially to equal the applied voltage **V**.

In order to agree with physical facts: (1) the RI drop must be in phase with **I**; (2) the X_LI drop must be 90° in advance of **I**; (3) the X_CI drop must be 90° behind **I**.

Reference to Fig. 14 will show that the voltage

$$\mathbf{V} = RI + (X_L - X_C)I \text{ vectorially} \tag{54}$$

or

$$\mathbf{V} = R\mathbf{I} + j(X_L - X_C)\mathbf{I} \tag{55}$$

from which the complex impedance function is:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R + j(X_L - X_C) \tag{56}$$

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Obviously the relations stated in equations (54), (55), and (56) are independent of the vector diagram position of **I**.

The cartesian or rectangular form of the complex expression for **Z** can be transformed to the polar form of **Z** by the method of complex algebra, and the transformation is, of course, reversible.

$$R + j(X_L - X_C) = \sqrt{R^2 + (X_L - X_C)^2} \angle \tan^{-1} \frac{(X_L - X_C)}{R} \quad (57)$$

The rectangular form of the impedance function is, in general, essential in combining impedances because impedances cannot be added or subtracted in polar form.

Example. The terminals of an a-c generator which has an internal resistance of 2 ohms and an equivalent internal inductive reactance of 6 ohms are connected to a particular *RLC* series branch, the *R* of which is 10 ohms, the ωL of which is 20 ohms, and the $1/\omega C$ of which is 40 ohms. If the magnitude of the internally generated emf is 500 volts, find the current that flows in the series circuit and the terminal voltage of the generator.

The internal impedance of the generator is:

$$\mathbf{Z}_g = 2 + j6 = 6.32 \angle 71.6^\circ \text{ ohms}$$

The total impedance of the series circuit is:

$$\begin{aligned} \mathbf{Z}_t &= \mathbf{Z}_g + \mathbf{Z}_{RLC} \\ &= (2 + j6) + [10 + j(20 - 40)] \\ &= 12 - j14 = 18.44 \angle -49.4^\circ \text{ ohms} \end{aligned}$$

The generated emf, $\mathbf{E}_g$, is arbitrarily chosen to coincide with the reference axis. Therefore,

$$\mathbf{E}_g = 500 + j0 = 500 \angle 0^\circ \text{ volts}$$

The current that flows in the series circuit is:

$$\mathbf{I} = \frac{\mathbf{E}_g}{\mathbf{Z}_t} = \frac{500 \angle 0^\circ}{18.44 \angle -49.4^\circ} = 27.1 \angle 49.4^\circ \text{ amperes}$$

The terminal voltage of the generator considered as a voltage drop across the external circuit is:

$$(1) \quad \mathbf{V}_g = \mathbf{E}_g - \mathbf{I}\mathbf{Z}_g \quad \text{or} \quad (2) \quad \mathbf{V}_g = \mathbf{I}\mathbf{Z}_{RLC}$$

$$\begin{aligned} \mathbf{V}_g &= (500 \angle 0^\circ) - (27.1 \angle 49.4^\circ)(6.32 \angle 71.6^\circ) \\ &= 500 \angle 0^\circ - 171.3 \angle 121^\circ \\ &= (500 + j0) - (-88.3 + j147) \\ &= 588.3 - j147 = 606 \angle -14^\circ \text{ volts} \end{aligned}$$

$$\mathbf{V}_g = (27.1 \angle 49.4^\circ)(22.36 \angle -63.4^\circ) = 606 \angle -14^\circ \text{ volts}$$

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A vector diagram illustrating $\mathbf{E}_g$, $\mathbf{I}$, $\mathbf{I}R_g$, $\mathbf{I}X_g$, and $\mathbf{V}_g$ is given in Fig. 15. It will be observed that the terminal voltage of the generator ($\mathbf{V}_g$) is greater in magnitude than the internally generated emf (E_g) owing to the manner in which the vector voltage $\mathbf{I}X_g$ subtracts from ($\mathbf{E}_g - \mathbf{I}R_g$) to form vector $\mathbf{V}_g$.

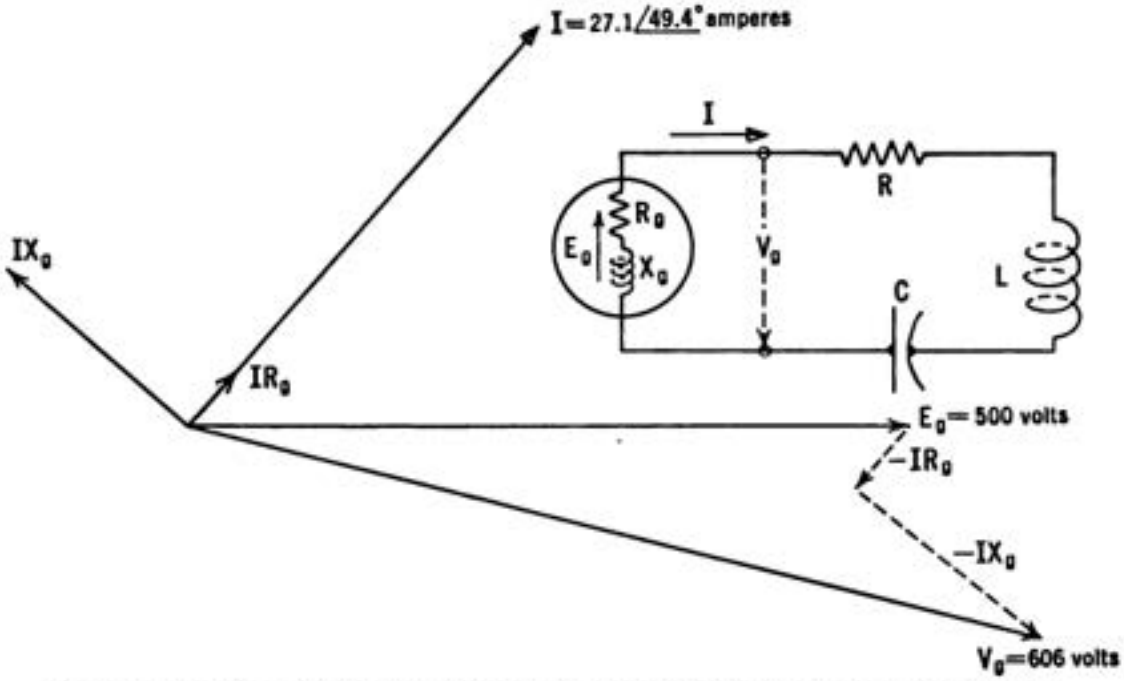

Fig. 15. Voltage relations for a generator supplying a leading power-factor load.

Problem 15. (*a*) Draw a vector diagram illustrating $\mathbf{E}_g$, $\mathbf{I}$, $\mathbf{IR}$, $\mathbf{IX}_L$, $\mathbf{IX}_C$, and $\mathbf{V}_g$ of the above numerical example and show how $\mathbf{IR}$, $\mathbf{IX}_L$, and $\mathbf{IX}_C$ combine vectorially to form $\mathbf{V}_g$.

(*b*) Calculate the total power generated and the total power absorbed by the external *RLC* branch. Compare $V_g I \cos\theta \Big]_{\mathbf{I}}^{\mathbf{V}_g}$ plus I^2R_g with $E_g I \cos\theta \Big]_{\mathbf{I}}^{\mathbf{E}_g}$.

Ans.: Total power = 8810 watts; branch power = 7345 watts.

Addition and Subtraction of Voltages and Currents. Correctly written, the complex expressions for voltages and currents specify both the magnitudes and relative phase positions of these quantities. Therefore, in complex form:

1. Voltage drops in series may be added to obtain the combined voltage drop of the series elements considered. If the combined voltage drop and one component are known, the remaining voltage drop may be determined by subtracting the component in question from the combined voltage drop.

2. Generated emf's connected in additive or subtractive series may be added or subtracted, depending upon the relative polarities of the terminals which are joined together to form the series connection. Series connections of generated emf's will be considered in more detail when polyphase systems are studied.

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3. Two or more currents flowing away from a junction may be added to find the current flowing toward the junction, or vice versa.

Circuit Directions of Voltages and Currents. It has been shown that the average power absorbed by a branch or circuit is:

$$P = VI \cos \theta \Big]_{\mathrm{I}}^{\mathrm{V}} \tag{58}$$

where V is the magnitude of the voltage drop across the branch or circuit,

I is the magnitude of the current flowing through the branch or circuit in the same circuit direction as that which has been taken for the $+V$ direction.

$\theta \Big]_{\mathrm{I}}^{\mathrm{V}}$ is the angle of lag (or lead) of **I** with respect to **V**. In a normal dissipative type of branch or circuit, θ will not be as great as $\pm 90°$.

Similarly, the average power generated by a generating device is:

$$P = EI \cos \theta \Big]_{\mathrm{I}}^{\mathrm{E}} \tag{59}$$

where E is the magnitude of the generated voltage,

I is the magnitude of the current flowing in the same circuit direction as that which has been taken for the $+E$ direction.

$\theta \Big]_{\mathrm{I}}^{\mathrm{E}}$ is the angle of lag (or lead) of **I** with respect to **E**. In case the generating device is actually delivering power, $\theta \Big]_{\mathrm{I}}^{\mathrm{E}}$ will be less than 90° in magnitude. This, in general, is the condition that exists when only one generator is present. Average negative generated power indicates that the generating device in question is actually absorbing power from some other generator.

A single generator connected to a dissipative branch is shown in Fig. 16. If the $+E$ circuit direction is assumed to be from b to a *through* the generator, the positive circuit direction of the current is from b to a *through* the generator, and from a to b *through* the dissipative branch. The positive circuit direction of a voltage drop through a dissipative branch defines the positive circuit direction of the current through the branch, or vice versa. In Fig. 16, therefore, the $+V$ direction is from a to b through the external branch. With the aid of these elementary concepts, the correct phase relations of all quantities involved

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may be conveniently determined. If E_g is taken as reference,

$$\mathbf{I} = \frac{E_g \angle 0^\circ}{\mathbf{Z}_{gen} + \mathbf{Z}_{load}} = \mathbf{I} \angle \alpha \tag{60}$$

$$\mathbf{V} = E_g \angle 0^\circ - \mathbf{I}\mathbf{Z}_{gen} = V \angle \beta \tag{61}$$

Average generated power $= E_g I \cos \alpha$ (62)

Average power absorbed by the external branch $= VI \cos (\beta - \alpha)$ (63)

Thus it will be seen that the current in a series loop may be associated with the generated voltage to obtain the generated power and with a particular voltage drop across a given part of the circuit to obtain the power absorbed by this particular part of the circuit. Unless otherwise specified, the current in a series loop having only one generator is assumed to flow in the positive direction of voltage rise through the generator and in the positive direction of voltage drop through the load portion of the circuit.

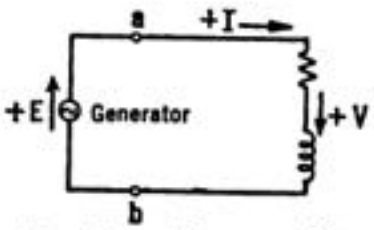

FIG. 16. Illustrating an arbitrarily assigned positive circuit direction of the generated voltage, E, together with the resulting positive circuit directions of I and V.

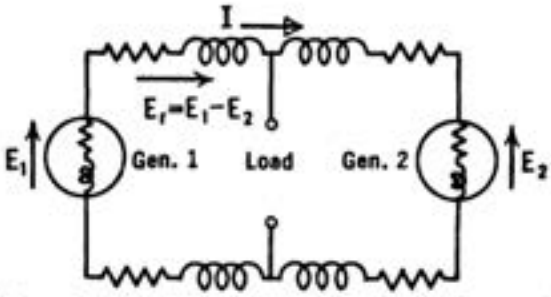

FIG. 17. Two generated emf's connected in parallel with respect to the load terminals. $\mathbf{E}_1$ and $\mathbf{E}_2$ are in subtractive series with respect to the series loop which joins the two generators.

Example of Two Generators. Figure 17 illustrates two a-c generators which are connected in parallel with respect to the load terminals but are connected in subtractive series with respect to the series loop joining the two generators. If no load is placed across the load terminals, the series loop is the only path in which current flows. If it is assumed that the generators are driven by separate prime movers and controlled by separate voltage regulators, it is entirely possible for the voltages to differ in magnitude and phase position.

Let $E_1 = 1350 \angle 0^\circ$ volts and $E_2 = 1300 \angle -10^\circ$ with respect to the load terminals. The impedance of each generator is $(1 + j3)$ ohms and each of the series loop connecting lines has $(2 + j1)$ ohms impedance. Find the magnitude and phase position of the current which circulates in the series loop under the above conditions.

The resultant generated emf which acts to send current through the series loop in the $+E_1$ direction is:

$$\mathbf{E}_r = \mathbf{E}_1 - \mathbf{E}_2 = (1350 + j0) - (1280 - j226) = 70 + j226 \text{ volts}$$

The positive circuit direction of E_r is the same as that which has been arbitrarily assigned to $\mathbf{E}_1$, since the vector difference $\mathbf{E}_1 - \mathbf{E}_2$ has been employed in defining $\mathbf{E}_r$.

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The current that flows in the direction of $\mathbf{E}_r$ is:

$$\mathbf{I} = \frac{\mathbf{E}_r}{\mathbf{Z}_{loop}} = \frac{70 + j226}{6 + j8}$$
$$= \frac{(70 + j226)\,(6 - j8)}{(6 + j8)\,(6 - j8)}$$
$$= (22.28 + j7.96) = 23.65 \underline{/19.65^\circ} \text{ amperes}$$

The power generated by the $\mathbf{E}_1$ generator is:

$$P_{g1} = 1350 \times 23.65 \cos 19.65^\circ = 30{,}110 \text{ watts}$$

The power generated by the $\mathbf{E}_2$ is:

$$P_{g2} = 1300 \times 23.65 \cos [180^\circ - (10^\circ + 19.65^\circ)]$$
$$= -26{,}750 \text{ watts}$$

In calculating the power generated by the $\mathbf{E}_2$ machine, either the voltage or the current is reversed in phase position so that the $\mathbf{E}_2$ and I circuit directions coincide. The physical interpretation of the negative generated power found for machine 2 is that machine 2 is actually receiving power from machine 1. A vector diagram of $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_r$, and I is shown in Fig. 18.

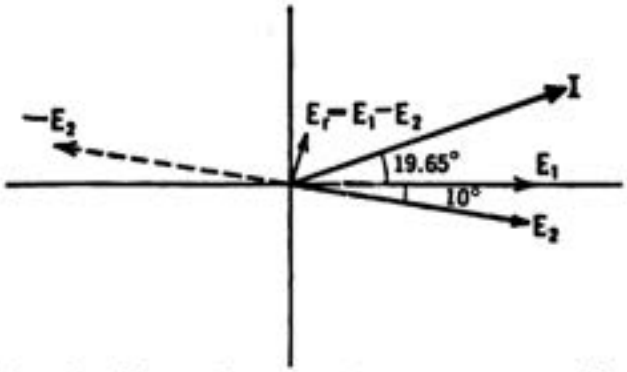

FIG. 18. Vector diagram of two-generator problem.

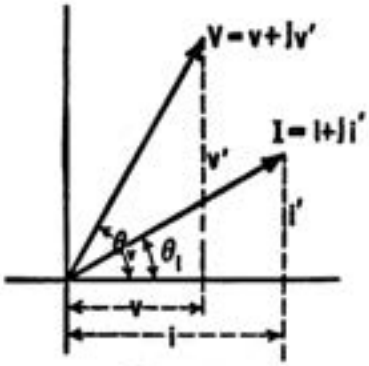

FIG. 19.

In general, a circulating current between the two generators may exist as a result of difference in the *magnitude* of the two generated voltages, or a difference in *phase*, or *both*.

A further insight into the power relations of the circuit arrangement shown in Fig. 17 may be obtained by adding to the power absorbed by machine 2 the total I^2R loss of the series loop and comparing the result with the total power generated by machine 1.

$$(23.65^2 \times 6) + 26{,}750 = 30{,}110 \text{ watts}$$

The physical interpretation of the above equation is that machine 1 generates 30,110 watts, of which 3360 are dissipated in the form of heat in the resistance of the series loop and 26,750 watts are absorbed by machine 2 in the form of electromagnetic motor power.

Power Calculations Employing Complex Forms. If voltage and current are expressed in rectangular complex form, the average absorbed or generated power may be calculated in terms of the components of the

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voltage and current which are involved. Reference to Fig. 19 will show that:

$$P = VI \cos \theta \Big]_{\mathbf{I}}^{\mathbf{V}} \tag{64}$$

or

$$\begin{aligned} P &= VI \cos (\theta_v - \theta_i) = VI \cos (\theta_i - \theta_v) \\ &= VI [\cos \theta_v \cos \theta_i + \sin \theta_v \sin \theta_i] \\ &= (V \cos \theta_v)(I \cos \theta_i) + (V \sin \theta_v)(I \sin \theta_i) \end{aligned} \tag{65}$$

In rectangular form:

$$\mathbf{V} = V \cos \theta_v + jV \sin \theta_v = v + jv' \tag{66}$$

$$\mathbf{I} = I \cos \theta_i + jI \sin \theta_i = i + ji' \tag{67}$$

If the above components of **V** and **I** in equation (65) are employed, it follows that:

$$P = vi + v'i' \text{ (absorbed power)} \tag{68}$$

If the voltage in question is a generated voltage:

$$P = ei + e'i' \text{ (generated power)} \tag{69}$$

Due regard must be taken for the sign of each component in equations (68) and (69) when these power equations are employed.

Example. If, at a certain stage in the solution of a problem, it is found that $\mathbf{E} = (200 + j40)$ volts and that the current flowing in the positive circuit direction of **E** is $\mathbf{I} = (30 - j10)$ amperes, the power generated is:

$$\begin{aligned} P = ei + e'i' &= (200)(30) + (40)(-10) \\ &= 6000 - 400 = 5600 \text{ watts} \end{aligned}$$

The same result could, of course, be obtained by first evaluating the magnitudes of E, I, and $\theta \Big]_{\mathbf{I}}^{\mathbf{E}}$ and then making use of the more familiar relation:

$$P = EI \cos \theta \Big]_{\mathbf{I}}^{\mathbf{E}}$$

Reactive Volt-Ampere Calculations Employing Complex Forms. Reactive volt-amperes or reactive power, P_X, may also be calculated in terms of the rectangular components of the voltage and current involved. If the vector voltage and the vector current shown in Fig. 19 are considered:

$$\mathbf{V} = v + jv'$$

$$\mathbf{I} = i + ji'$$

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As defined in Chapters II and III:

$$P_X = VI \sin \theta \Big]_{\mathbf{I}}^{\mathbf{V}} \tag{70}$$

In accordance with a convention which is in common use, θ is the angle of *lead* of the voltage with respect to the current. If this convention of signs is employed, reactive power is a positive quantity for lagging currents and a negative quantity for leading currents. (See Chapter III, page 54.) If the angle θ in equation (70) is considered as the angle of lead of the voltage with respect to the current, then:

$$\begin{aligned} P_X &= VI \sin (\theta_v - \theta_i) \\ &= VI (\sin \theta_v \cos \theta_i - \cos \theta_v \sin \theta_i) \\ &= (V \sin \theta_v)(I \cos \theta_i) - (V \cos \theta_v)(I \sin \theta_i) \end{aligned} \tag{71}$$

From the definitions which have been attached to v, v', i, and i', equation (71) reduces directly to:

$$P_X = v'i - vi' \tag{72}$$

Example. If $\mathbf{V} = 200 \angle 30^\circ = (173.2 + j100)$ volts and $\mathbf{I} = 10 \angle 60^\circ = (5 + j8.66)$ amperes, find the real power, the reactive volt-amperes, and the total volt-amperes involved.

$$P = vi + v'i' = 866 + 866 = 1732 \text{ watts}$$

$$P_X = v'i - vi' = 500 - 1500 = -1000 \text{ vars}$$

The minus sign in connection with P_X merely indicates that the reactive power in question is the result of a leading current.

The volt-amperes associated with **V** and **I** can be obtained directly from the product of V and I, or as follows:

$$\begin{aligned} Va &= \sqrt{P^2 + P_X^2} = \sqrt{1732^2 + (-1000)^2} \\ &= 2000 \text{ volt-amperes} \end{aligned}$$

The Conjugate Method of Calculating Real and Reactive Power. The question naturally arises as to the significance of the product of vector voltage and vector current. The answer is to be found in the definition that has been given to the product of two complex numbers. The magnitude of the product of voltage and current, even in complex form, represents the volt-amperes which are associated with **V** and **I**. The component parts of the cartesian expression for **VI** are, however, meaningless. For this reason, vector voltage times vector current cannot be used directly to calculate real power or reactive volt-amperes.

A method of conjugates is sometimes employed in the determination of real power and reactive volt-amperes. It affords a convenient means of calculating these quantities when both the voltage and current are expressed in cartesian form.

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If the conjugate of the current, that is, the cartesian expression of the current with the sign of the j component reversed, is multiplied by the voltage in cartesian form, the result is a complex quantity the real part of which is the real power and the j part of which is the reactive volt-amperes.

Let $$\mathbf{V} = v + jv' \quad \text{and} \quad \mathbf{I} = i + ji'$$

The conjugate of $\mathbf{I}$ is $(i - ji')$ and:

$$(v + jv')(i - ji') = (vi + v'i') + j(v'i - vi') \tag{73}$$

or

$$(v + jv')\,(i - ji') = P + jP_X \tag{74}$$

If the conjugate of $\mathbf{V}$ is multiplied by $\mathbf{I}$ in complex form, the result is:

$$(v - jv')\,(i + ji') = (vi + v'i') - j(v'i - vi') \tag{75}$$

or

$$(v - jv')\,(i + ji') = P - jP_X \tag{76}$$

The real power, as obtained by the method of conjugates, is the same regardless of whether $\mathbf{V}$ or $\mathbf{I}$ is conjugated. The sign of the reactive volt-amperes, however, is dependent upon the choice of the $\mathbf{V}$ or $\mathbf{I}$ conjugate as shown by equations (74) and (76). To be consistent with convention of signs employed in equation (70), the conjugate of the *current* must be employed. To be in accordance with the convention adopted by the Committee on Electrical and Magnetic Units, the conjugate of *voltage* must be employed. (See page 54.) In either case $P_x = v'i - vi'$, the sign of which is a matter of definition.

Example. Given $\mathbf{V} = 173.2 + j100$ volts and $\mathbf{I} = 5.0 + j8.66$ amperes, find the real power and the reactive volt-amperes by the method of conjugates. Employing the conjugate of the voltage:

$$\begin{aligned} P_{va} &= (173.2 - j100)\,(5.0 + j8.66) \\ &= 866 + j1500 - j500 + 866 \\ &= 1732 + j1000 \end{aligned}$$

This above result may be interpreted, in light of equations (75) and (76), to mean that $P = 1732$ watts and that $P_X = +1000$ vars. The positive sign indicates capacitive vars when the conjugate of the voltage is used.

Transmission Expressed as a Complex Number. The term "transmission" will be used here as a general designation of the effect (say the current or power) in a receiver station produced by a generator at the sending station. (See Fig. 20.)[2] In low-power communication

[2] If any series or shunt reactance is associated with either the generator or load, it may be placed within the four-terminal network for the purposes of analysis, thus making Fig. 20 a more general case than is apparent from the diagram.

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networks, particular attention centers on the *change* in magnitude (and the phase shift) of the receiver current *relative* to the receiver current which could be obtained under optimum conditions of operation. These relative changes are due to two causes, namely:

(1) R_2 in Fig. 20 not being equal to the generator resistance R_1, the latter being fixed by the generator characteristics.

(2) The network intervening between the sending-station generator E_1 and the receiver-station resistance R_2.

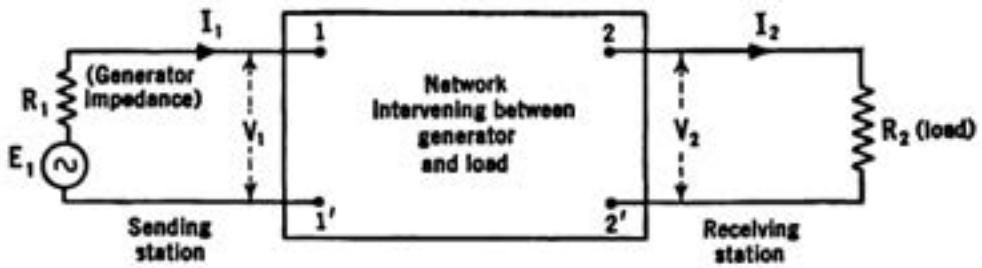

FIG. 20. Four-terminal network intervening between a generator and a resistive load.

The intervening network will usually take the form of a transmission line, transformer, selective filter, attenuator, or amplifier. Various combinations of these basic four-terminal networks may be employed between the generator terminals ($11'$ in Fig. 20) and the receiver terminals $22'$, but until the detailed operation of these devices has been studied we shall represent them simply as a box having four terminals as shown in the figure.

It will be accepted here, subject to later proof, that the most efficient possible transmission between E_1 and R_2 will occur when the impedance looking to right of terminals $11'$ is equal to R_1, that is, when $V_1/I_1 = R_1$. (Amplifiers are excluded from the foregoing statement because these devices draw power from sources other than the E_1 generator.) Under this condition of operation the generator resistance is said to match the impedance looking to the right of terminals $11'$ and

$$I_{1(\text{matched})} = \frac{E_1}{2R_1}$$

The ratio of the powers entering and leaving the network under the condition that $\mathbf{V}_1/\mathbf{I}_1 = R_1$ is

$$\frac{\text{Power entering terminals } 11'}{\text{Power leaving terminals } 22'} = \frac{V_1 I_1}{V_2 I_2} = \frac{(E_1/2)(E_1/2R_1)}{(R_2 I_2)(I_2)} \tag{77}$$

If now we define the *transfer impedance* from E_1 to R_2 under any condi-

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tion of operation as

$$\mathbf{Z}_T = \frac{\mathbf{E}_1}{\mathbf{I}_2} \tag{78}$$

we note that the value of $\mathbf{Z}_T$ which will make the power ratio of equation (77) unity is

$$\mathbf{Z}_{T(\text{opt})} = 2\sqrt{R_1R_2} \tag{79}$$

In other words, for fixed values of R_1 and R_2, all the power entering terminals $11'$ in Fig. 20 will leave terminals $22'$ if the intervening network is such that $\mathbf{E}_1/\mathbf{I}_2 = 2\sqrt{R_1R_2}$.

In describing the transmission characteristics of an arbitrary four-terminal network of the kind shown in Fig. 20, it is desirable that the receiver current, $\mathbf{I}_2$, be measured relative to its optimum value, $\mathbf{E}_1/2\sqrt{R_1R_2}$. Both the magnitude and phase of $\mathbf{I}_2$ relative to this base can be measured in terms of the real and j components of the transmission constant, $\boldsymbol{\gamma}$, if the latter is defined as

$$\boldsymbol{\gamma} = \alpha + j\beta = \log_e \frac{\mathbf{Z}_T}{2\sqrt{R_1R_2}} = \log_e \frac{\mathbf{E}_1/2\sqrt{R_1R_2}}{\mathbf{I}_{2(\text{general})}} = \log_e \frac{\mathbf{I}_{2(\text{opt})}}{\mathbf{I}_{2(\text{general})}} \tag{80}$$

where $\mathbf{Z}_T = \mathbf{E}_1/\mathbf{I}_2$ for any arbitrary intervening network.

α is the attenuation (to be described in more detail later).

β is the phase shift (also to be described in detail later).

The transmission constant is thus defined as a logarithmic measure of $\mathbf{Z}_T$ relative to $\mathbf{Z}_{T(\text{opt})}$. Since $E_1/2\sqrt{R_1R_2}$ in Fig. 20 is considered to be a constant, it is plain that α is a logarithmic measure of $I_{2(\text{opt})}/I_{2(\text{general})}$ and that β is the phase angle difference between $\mathbf{I}_{2(\text{general})}$ and $\mathbf{I}_{2(\text{opt})}$. The phase angle of $\mathbf{I}_{2(\text{opt})}$ would normally be zero, since the reference would normally be $\mathbf{E}_1 = E_1\underline{/0^\circ}$ and $\mathbf{I}_{2(\text{opt})}$ is in phase with $\mathbf{E}_1$, being equal to $\mathbf{E}_1/2\sqrt{R_1R_2}$.

Attenuation, α. It will be noted from equation (80) that the attenuation can be written as

$$\alpha = \log_e \frac{\sqrt{I^2_{(\text{opt})}}}{\sqrt{I^2_{(\text{general})}}} = \frac{1}{2}\log_e \frac{I_2{}^2{}_{(\text{opt})}R_2}{I_2{}^2{}_{(\text{general})}R_2} \text{ nepers} \tag{81}$$

Attenuation in this case is an inverse logarithmic measure of the power received by R_2 under general conditions of operation to that which is received by R_2 under optimum conditions of operation. The fact that logarithmic measure is employed in the definition of $\boldsymbol{\gamma}$ makes $\alpha = 0$ if $I_2{}^2{}_{(\text{general})}R_2$ is equal to $I_2{}^2{}_{(\text{opt})}R_2$, and as the former decreases in value owing to losses in the intervening network α grows larger logarithmically. If $\log_e$ is employed as in equation (81), the units of α are called nepers.

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Another common definition of attenuation as it applies to general transmission characteristics is

$$\alpha_{db} = 10 \log_{10} \frac{I_2^2{}_{(opt)} R_2}{I_2^2{}_{(general)} R_2} \text{ decibels} \tag{82}$$

Plainly

$$\frac{(\text{No. of})\alpha_{db}}{\alpha_{nepers}} = \frac{10 \log_{10} K}{\frac{1}{2}\log_e K} = \frac{20 \log_{10} K}{2.303 \log_{10} K} = 8.686$$

where K is any power ratio. The above relationship indicates that the number of decibels per neper is 8.686. It is a matter of indifference which unit of attenuation is used, since engineers generally understand that the decibel is by definition a unit of attenuation which is 8.686 times smaller in magnitude than the neper, there being 8.686 decibels of attenuation for each neper of attenuation in any particular specification of attenuation.

Phase Shift, β. In taking the logarithm indicated in equation (80) it will be noted that

$$\boldsymbol{\gamma} = \alpha + j\beta = \log_e \frac{\mathbf{I}_{2(opt)}}{\mathbf{I}_{2(general)}} = \log_e \frac{I_{2(opt)}}{I_{2(general)}} + j[\theta_{(opt)} - \theta_{(general)}] \tag{83}$$

Thus, if $\mathbf{E}_1$ is selected as a reference, $\theta_{(opt)} = 0$ and $\beta = -\theta_{(general)}$. Regardless of the reference selected, β specifies the phase difference between $\mathbf{I}_2$ under optimum conditions and $\mathbf{I}_2$ under general operating conditions.

If the evaluation of $\alpha + j\beta$ is to be carried no further than that shown in equation (83), it is a matter of choice whether β is stated in radians or degrees. If $\boldsymbol{\gamma}$ is to be expressed in polar form, however, β must be expressed in radians.

Examples. In Fig. 20, let $R_1 = 100$ ohms, $R_2 = 25$ ohms, and assume that terminal 1 is connected directly to terminal 2 and terminal 1′ directly to 2′. Let it be required to find the attenuation and phase shift relative to the optimum operating conditions.

If equation (80) is to be employed, we note that

$$\mathbf{I}_{2(opt)} = \frac{\mathbf{E}_1}{2\sqrt{100 \times 25}} = \frac{\mathbf{E}_1}{100}$$

$$\mathbf{I}_{2(actual)} = \frac{\mathbf{E}_1}{125}$$

$$\alpha + j\beta = \log_e \frac{\mathbf{E}_1/100}{\mathbf{E}_1/125} = 0.223 + j0$$

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Thus $\alpha = 0.223$ neper or 1.938 decibels. This attenuation results from R_2 not being equal to R_1.

$\beta = 0$ since no phase difference exists between the two conditions of operation.

As a check on the arithmetic we might employ equation (82) as

$$\alpha_{db} = 20 \log_{10} \frac{\mathbf{I}_{2(opt)}}{\mathbf{I}_{2(actual)}} = 20 \log_{10} \frac{E_1/100}{E_1/125}$$

$$\alpha_{db} = 20 \log_{10} \frac{125}{100} = 20 \times 0.0969 = 1.938 \text{ decibels}$$

As a second example of the use of equation (80) let it be assumed that $R_1 = 25$ ohms, $R_2 = 100$ ohms, and that, for $\mathbf{E}_1 = 10\underline{/0^\circ}$ volts, $\mathbf{V}_2 = 3.53\underline{/-45^\circ}$ volts.

It is required that the transmission constant, $\boldsymbol{\gamma}$, be found from the above data.

$$\mathbf{I}_{2(opt)} = \frac{\mathbf{E}_1}{2\sqrt{R_1R_2}} = \frac{10\underline{/0^\circ}}{2\sqrt{2500}} = 0.1\underline{/0^\circ} \text{ ampere}$$

$$\mathbf{I}_{2(actual)} = \frac{\mathbf{V}_2}{R_2} = \frac{3.53\underline{/-45^\circ}}{100} = 0.0353\underline{/-45^\circ} \text{ ampere}$$

$$\mathbf{Z}_T = \frac{\mathbf{E}_1}{\mathbf{I}_2} = \frac{10\underline{/0^\circ}}{0.0353\underline{/-45^\circ}} = 283\underline{/45^\circ} \text{ ohms}$$

$$\boldsymbol{\gamma} = \alpha + j\beta = \log_\epsilon \frac{\mathbf{Z}_T}{2\sqrt{R_1R_2}} = \log_\epsilon \frac{283\underline{/45^\circ}}{100} = 1.04 + j45^\circ$$

or

$$\boldsymbol{\gamma} = \alpha + j\beta = \log_\epsilon \frac{\mathbf{I}_{2(opt)}}{\mathbf{I}_{2(actual)}} = \log_\epsilon \frac{0.1\underline{/0^\circ}}{0.0353\underline{/-45^\circ}} = 1.04 + j0.785 \text{ radians}$$

Thus

$$\boldsymbol{\gamma} = 1.304\underline{/0.647 \text{ (radians)}} = 1.304\underline{/37.05^\circ}$$

If $\boldsymbol{\gamma}$ is specified in polar form as above, we obtain α and β as the real and j terms directly by changing the polar form of $\boldsymbol{\gamma}$ to rectangular form.

A significant point which should not be overlooked in the foregoing discussion of *attenuation* and *phase shift* is that, as applied to the four-terminal network shown in Fig. 20, these quantities were obtained from

$$\alpha + j\beta = \log_\epsilon \frac{\mathbf{Z}_{T(general)}}{\mathbf{Z}_{T(opt)}} = \frac{1}{2} \log_\epsilon \frac{I_2{}^2{}_{(opt)}R_2}{I_2{}^2{}_{(general)}R_2} + j[\theta_{\mathbf{Z}_{T(general)}}]$$

where $\mathbf{Z}_{T(opt)}$ was an *arbitrarily selected* base which yielded maximum power delivered to the load resistance, R_2. (It was assumed that R_1 was fixed by the characteristics of the E_1 generator and that R_2 was fixed by the characteristics of the receiving device.) The base selected here is that which is normally employed when we wish to take account

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of the possible mismatch between R_1 and R_2 as well as the loss and phase shift introduced by the intervening four-terminal network. It also permits the possible mismatch between R_1 and R_2 to be rectified by the intervening network if the latter is designed for this purpose.

In general circuit analysis, *attenuation* and *phase shift* are used in a wide variety of different ways to describe loss (or gain) and phase difference relative to other *arbitrarily selected* bases. Attenuation and phase shift are meaningful quantities only when the base is clearly understood, since attenuation and phase shift are measures of power loss (or gain) and phase relative to the base which is selected as being most appropriate for the problem at hand.

PROBLEMS

16. Perform the following indicated operations: (*a*) $(5 + j8) + (-2 - j4)$; (*b*) $(-12 + j6) - (30 - j20)$; (*c*) $(16 - j12)(-5 + j8)$; (*d*) $(-5 + j8.66) + (5 - j8.66)$; (*e*) $(2 - j3) \div (-1 + j2)$.

17. Two impedances, $\mathbf{Z}_1 = 2 + j3$ ohms and $\mathbf{Z}_2 = 3 - j7$ ohms, are connected in a circuit so that they are additive. Find the equivalent impedance of the two in polar form.

18. Write the cartesian and polar expressions for a vector, the magnitude of which is 100 units and the phase position of which is:

(*a*) 30° behind the reference axis.
(*b*) 45° behind the reference axis.
(*c*) 120° behind the reference axis.
(*d*) 180° behind the reference axis.
(*e*) 60° ahead of the reference axis.
(*f*) 120° ahead of the reference axis.
(*g*) 210° ahead of the reference axis.

19. Find the magnitude and angular position with respect to the reference axis of the vectors which are represented by:

(*a*) $8.0 + j6.0$.
(*b*) $-10 + j10.0$.
(*c*) $38.3 - j31.14$.
(*d*) $-57.36 + j81.92$.
(*e*) $-76.6 - j64.3$.
(*f*) $-50.0 - j86.6$.

20. (*a*) Rotate the vector $(8.66 + j5.0)$ through $+40°$ by multiplying it by the correct operator.

(*b*) Rotate the vector $(-5.0 - j8.66)$ through $-30°$.

(*c*) Express the results of (*a*) and (*b*) in both cartesian and polar forms.

21. Perform the following indicated operations:

(*a*) $(8 + j6)(10\underline{/-120°})(\cos 36.87° - j \sin 36.87°)(0.1\epsilon^{+j60°})$.

(*b*) $\dfrac{[34.2 + j94][10\epsilon^{-j30°}][30(\cos 60° + j \sin 60°)]}{[20\underline{/40°}][50(\cos 30° + j \sin 30°)]}$.

22. Express each of the following as a single vector in cartesian and polar forms:

(*a*) $[\sqrt{4.5 - j7.79} + \log_\epsilon 10\underline{/172°}]$.

(*b*) $\sqrt{\dfrac{(940 + j342)}{10\epsilon^{j10°}}}$.

(*c*) $\dfrac{(-8.66 + j5.0)(50\underline{/-100°})(2\epsilon^{j70°})}{j5}$.

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(d) $50\epsilon^{-j\omega t}$ at $t = \frac{1}{480}$ second if $\omega = 377$ radians per second.

(e) $\frac{30 + j10}{6 - j3} \cdot \sqrt{4.5 - j7.794}$.

23. Find all possible roots of:

$$\sqrt[3]{\frac{10\underline{/45^\circ}\, 5\epsilon^{j60}\,(-4.047 - j2.94)}{1 - j1.732}}$$

24. The series impedance of a transmission line is $\mathbf{Z}_a = 10\underline{/68^\circ}$ ohms, and the shunt impedance of the line is $\mathbf{Z}_b = 25{,}000\underline{/-90^\circ}$ ohms.

(a) Find the characteristic impedance of the line which is defined as $\mathbf{Z}_0 = \sqrt{\mathbf{Z}_a\mathbf{Z}_b}$.

(b) Find the propagation constant of the line which is defined as $\gamma = \sqrt{\mathbf{Z}_a/\mathbf{Z}_b}$.

25. A vector voltage of $125\underline{/40^\circ}$ volts is impressed across a series combination of 2.0 ohms resistance and 8.0 ohms inductive reactance. Find the magnitude and phase position of the current with respect to the reference axis employed in stating the vector voltage.

26. Two impedances, $\mathbf{Z}_1 = (1 - j3)$ ohms and $\mathbf{Z}_2 = (3 + j6)$ ohms, are connected in parallel. The magnitude of the current through $\mathbf{Z}_1$ is known to be 10 amperes.

(a) Find the complex polar expression for the current through $\mathbf{Z}_2$ with respect to $\mathbf{I}_1 = 10\underline{/0^\circ}$ as a reference.

(b) Find $\mathbf{I}_0 = \mathbf{I}_1 + \mathbf{I}_2$ in cartesian form.

(c) Draw a vector diagram of $\mathbf{V}$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_0$, employing $\mathbf{I}_1$ as reference.

27. The characteristic impedance of a T-section filter is $\mathbf{Z}_{0T} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \frac{\mathbf{Z}_1^2}{4}}$, where $\mathbf{Z}_1$ is the full series arm impedance and $\mathbf{Z}_2$ is the shunt impedance of the filter section. If $\mathbf{Z}_1 = 30\underline{/86.0^\circ}$ ohms and $\mathbf{Z}_2 = 10.0\underline{/-90^\circ}$ ohms, find $\mathbf{Z}_{0T}$ from the above definition of $\mathbf{Z}_{0T}$.

28. Express $\log_\epsilon \sqrt{\frac{125\underline{/-90}}{5\underline{/90}}}$ in rectangular form.

Ans.: $1.61 \mp j\pi/2$.

29. An equation which is useful in filter circuit analysis is:

$$\alpha + j\beta = 2\log_\epsilon\left(\sqrt{1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}} + \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right)$$

If $\mathbf{Z}_1 = 25.14\underline{/-90^\circ}$ ohms and $4\mathbf{Z}_2 = 795\underline{/+90^\circ}$, evaluate α and β.

30. Find α and β in Problem 29 if

$$\mathbf{Z}_1 = 4 \times 10^3\underline{/-90^\circ}$$
$$4\mathbf{Z}_2 = 1000\underline{/90^\circ}$$

31. Given the equation:

$$\mathbf{V}_m = \mathbf{V} - \mathbf{Z}\mathbf{I}$$

where $\mathbf{V} = 100\underline{/0^\circ}$ volts, $\mathbf{Z} = 15\underline{/80^\circ}$ ohms, $\mathbf{I} = 10\underline{/-3^\circ}$ amperes. Express $\mathbf{V}_m$ in polar form.

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32. (*a*) Solve the following equation for a and for b:

$$(12 + a) + jb = 20 + j10$$

(*b*) Solve the following equation for a and for β:

$$(a + 10) + j50 = 100(\cos\beta + j\sin\beta)$$

(*c*) Given: $(100 + j0) + 5R\underline{/-45^\circ} = 200\underline{/-\theta^\circ}$; find R and θ.

33. (*a*) Plot $Ae^{+j\omega t}$ and $Ae^{-j\omega t}$ in polar coordinates for $\omega = 157$ radians per second at $t = 0.005$, $t = 0.010$, $t = 0.015$, $t = 0.020$, and $t = 0.04$ second.

(*b*) Plot $\dfrac{Ae^{j\omega t} + Ae^{-j\omega t}}{2}$ in polar coordinates and also in rectangular coordinates versus ωt for one complete cycle.

(*c*) Show that a simple harmonic oscillating variation, such as $A\cos\omega t$, can be represented by two oppositely rotating vectors, each of which has the same angular velocity as the oscillating vector and each of which has a magnitude equal to one-half the magnitude of the oscillating vector.

34. (*a*) A voltage $\mathbf{V} = 100 - j50$ volts across a circuit causes a current $\mathbf{I} = -2 - j8$ amperes to flow. Calculate the power absorbed by the circuit, employing equation (68).

(*b*) Calculate power if $\mathbf{V} = -50 + j100$ volts and $\mathbf{I} = -6 - j2$ amperes.

(*c*) Calculate power if $\mathbf{V} = -50 + j100$ volts and $\mathbf{I} = -8 + j3$ amperes.

35. (*a*) Calculate the vars for each of the parts of Problem 34, employing equation (72).

(*b*) How should equation (72) be modified to yield the present standard signs for vars?

36. Calculate the power and vars by the method of conjugates for each part of Problem 34.

37. The voltage applied to two parallel branches is $40\underline{/80^\circ}$ volts. The current through branch 1 is $5\underline{/30^\circ}$ amperes, and the current through branch 2 is $(-6 + j8)$ amperes. Find the real power, P, and the reactive volt-amperes, P_x, supplied to the parallel combination by the method of conjugates. *Note:* Check results against $VI = 40 \times 10.62 = \sqrt{P^2 + P_x^2}$.

38. In Fig. 20, page 90, $R_1 = 200$ ohms, $R_2 = 20{,}000$ ohms, and $V_2 = (0.1\underline{/114.6^\circ})\mathbf{E}_1$. Find the attenuation and phase shift which are produced by the combination of the mismatch of R_1 and R_2 and the intervening network.

39. In Fig. 20, $R_1 = 200$ ohms, $R_2 = 20{,}000$ ohms, and $\mathbf{I}_2 = \mathbf{E}_1/4000$ amperes. Find the attenuation and phase shift which are produced by the combination of the mismatch of R_1 and R_2 and the intervening network.

40. For the circuit shown in Fig. 21,

$$\mathbf{Z} = \frac{(R + jX)(-j2X)}{R + jX - j2X}$$

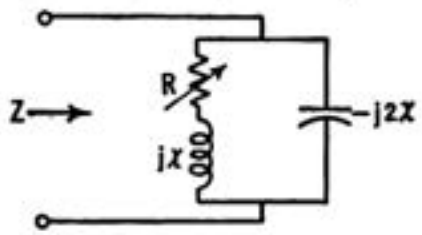

FIG. 21. See Problem 40.

Plot Z and θ of $\mathbf{Z} = Z\underline{/\theta}$ versus R, employing $R = 0$, $R = X/2$, $R = X$, $R = 2X$, $R = 5X$, $R = 10X$, and $R = \infty$.

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CHAPTER V

SINUSOIDAL SINGLE-PHASE CIRCUIT ANALYSIS

Impedances in Series. A series circuit of three impedances is shown in Fig. 1. In a circuit of this kind it is evident that only a single current

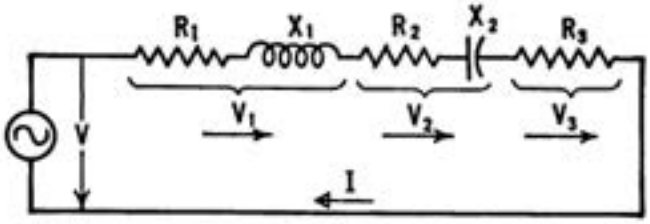

FIG. 1. Impedances in series.

can exist at any instant and that the current throughout all impedances is the same.[1] Kirchhoff's emf law states that

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 \tag{1}$$

or

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_1 + \mathbf{I}\mathbf{Z}_2 + \mathbf{I}\mathbf{Z}_3 \tag{2}$$

and

$$\mathbf{V} = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{I}\mathbf{Z} \tag{3}$$

Equation (3) shows that series impedances are added in complex form to obtain the equivalent impedance. Thus

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 = (R_1 + jX_1) + (R_2 + jX_2) + (R_3 + j0)$$

or

$$\mathbf{Z} = (R_1 + R_2 + R_3) + j(X_1 + X_2) = R + jX \tag{4}$$

Equation (4) shows that the resultant resistance R of a simple series circuit is obtained by arithmetically adding the separate resistances. When it is remembered that inductive reactances are considered positive and capacitive reactances are negative, equation (4) also shows that the resultant reactance X of a series circuit is the algebraic sum of the separate reactances.

If current is taken as the reference, the vector diagram of the circuit of Fig. 1 appears as shown in Fig. 2. Such a vector diagram is called a funicular or string diagram. Another type of vector diagram which

[1] The assumption is made that the current is confined to the series circuit. Maxwellian space displacement currents are neglected.

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represents the same circuit is shown in Fig. 3. This is called a polar diagram. The distinguishing characteristic of a string vector diagram is that certain component vectors are combined head-to-tail to form a resultant vector as, for example, the component voltages $\mathbf{IR}_1$, $\mathbf{IX}_1$, $\mathbf{IR}_2$,

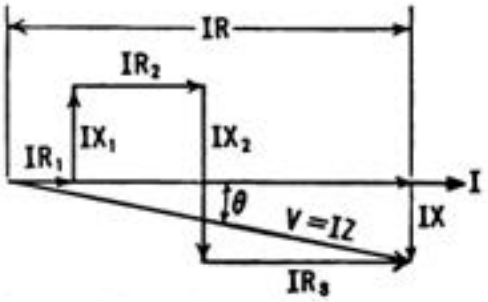

FIG. 2. Funicular or string vector diagram of circuit in Fig. 1.

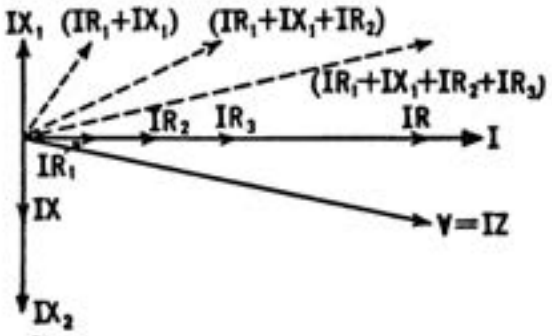

FIG. 3. Polar vector diagram of circuit in Fig. 1.

$\mathbf{IX}_2$, and $\mathbf{IR}_3$ are combined head-to-tail to form the resultant voltage vector $\mathbf{V}$. In a polar vector diagram, all vectors are started from a common origin as shown in Fig. 3.

Either type of diagram may be used since they represent the same thing. The one which appears to be the simpler in any particular case should be used. In certain cases the funicular diagram shows the quantities to better advantage, whereas for others the polar diagram is more suggestive of the relationships and more convenient to use.

In general, for a series circuit of n impedances

$$\mathbf{V} = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \cdots + \mathbf{Z}_n) \tag{5}$$

and $$\mathbf{Z} = (R_1 + R_2 + R_3 + \cdots + R_n) + j(X_1 + X_2 + X_3 + \cdots + X_n) \tag{6}$$

$$\mathbf{Z} = \sqrt{(R_1 + R_2 + R_3 + \cdots + R_n)^2 + (X_1 + X_2 + X_3 + \cdots + X_n)^2} \angle \tan^{-1} \frac{X_1 + X_2 + X_3 + \cdots + X_n}{R_1 + R_2 + R_3 + \cdots + R_n} \tag{7}$$

In Chapter II the impedance angle was shown to be the phase angle between the current and the voltage. In Chapter III power factor was shown to be the cosine of this angle. Hence, for a series circuit, Fig. 2 shows

$$\text{Power factor} = \cos\theta = \frac{IR}{IZ} = \frac{R}{Z}$$

$$= \frac{R_1 + R_2 + R_3 + \cdots + R_n}{\sqrt{(R_1 + R_2 + R_3 + \cdots + R_n)^2 + (X_1 + X_2 + X_3 + \cdots + X_n)^2}} \tag{8}$$

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Example 1. Calculate the current, voltage drops V_1, V_2, and V_3, power consumed by each impedance, and the total power taken by the circuit with the constants shown in Fig. 4. The impressed voltage will be taken along the reference axis.

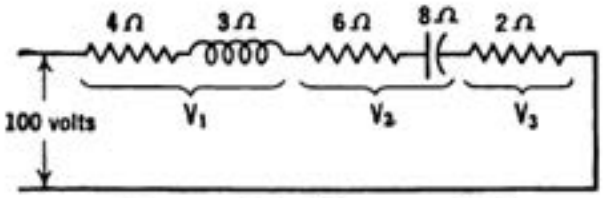

FIG. 4. Circuit for example 1.

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{100 + j0}{4 + j3 + 6 - j8 + 2} = \frac{100\,(12 + j5)}{(12 - j5)\,(12 + j5)} = 7.1 + j2.96 \text{ amperes}$$

$$\mathbf{V}_1 = \mathbf{IZ}_1 = (7.1 + j2.96)\,(4 + j3) = 19.53 + j33.14 \text{ volts}$$

$$\mathbf{V}_2 = \mathbf{IZ}_2 = (7.1 + j2.96)\,(6 - j8) = 66.27 - j39.06 \text{ volts}$$

$$\mathbf{V}_3 = \mathbf{IZ}_3 = (7.1 + j2.96)\,(2 + j0) = 14.2 + j5.92 \text{ volts}$$

Check: $\mathbf{V} = 100 + j0$ volts

Note that the drops are added *vectorially* to check the impressed voltage.

$$P_1 = RI^2 = 4(\sqrt{7.1^2 + 2.96^2})^2 = 4 \times 7.69^2 = 237 \text{ watts}$$

$$P_2 = 6 \times 7.69^2 = 355 \text{ watts}$$

$$P_3 = 2 \times 7.69^2 = 118 \text{ watts}$$

$$\text{Total power} = 710 \text{ watts}$$

The total power is also $(vi + v'i') = 100 \times 7.1 = 710$ watts.

Problem 1. (*a*) Find the current through the circuit in Fig. 5 and the voltage drops $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{cd}$.

Ans.: $\mathbf{I} = 10\,\underline{/0^\circ}$ amperes, $\mathbf{V}_{ab} = 20 - j40 = 44.7\,\underline{/-63.45^\circ}$ volts.
$\mathbf{V}_{bc} = 30 + j110 = 114\,\underline{/74.75^\circ}$ volts.
$\mathbf{V}_{cd} = 20 + j0 = 20\,\underline{/0^\circ}$ volts.

(*b*) Draw a string vector diagram of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{cd}$, including both $\mathbf{V}$ and $\mathbf{I}$ on the diagram.

(*c*) Draw a polar vector diagram of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{cd}$, $\mathbf{V}$, and $\mathbf{I}$.

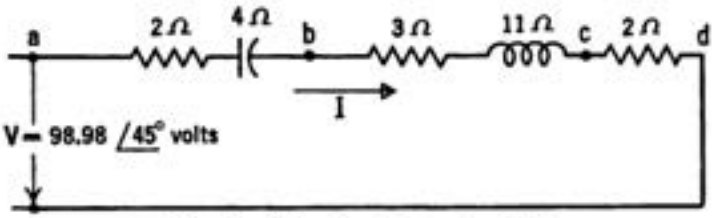

FIG 5. See Problems 1 and 2.

Problem 2. Calculate the total power dissipated in Fig. 5 from (I^2R), from $(VI \cos \theta)$, and from $(vi + v'i')$. *Ans.:* $P = 700$ watts.

Series Resonance. A series circuit containing R, L, and C is in resonance when the resultant reactance is zero. Since the drop across

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the inductance leads the current by 90° whereas that across the condenser lags the current by 90°, the two drops are opposite. If they are made equal as in Fig. 6, the reactive voltage drops neutralize and the impressed voltage is equal only to the resistance drop. This condition is called series resonance. Inspection of the vector diagram of Fig. 6 shows that the applied voltage is in phase with the current. The power factor is unity, and the circuit is in resonance. Thus for series resonance

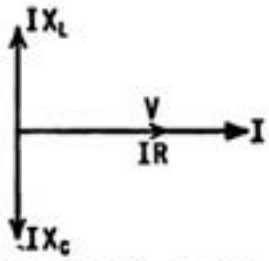

Fig. 6. Vector diagram of series circuit in resonance.

$$IX_L = IX_C \quad \text{or} \quad X_L = X_C \tag{9}$$

Since $2\pi fL = 1/2\pi fC$ at the point of series resonance, the series resonant frequency is

$$f_m = \frac{1}{2\pi\sqrt{LC}} \tag{10}$$

where f_m is in cycles per second when L is expressed in henrys and C in farads. It is apparent that series resonance can be produced in a series circuit by varying either L, C, or f. The current is always given by

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}} \tag{11}$$

For any value of current the drop across the resistance is

$$V_R = IR = \frac{VR}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}} \tag{12}$$

Similarly, the drops across the inductance and capacitance are respectively

$$V_L = IX_L = \frac{VX_L}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}} \tag{13}$$

and

$$V_C = IX_C = \frac{VX_C}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}} \tag{14}$$

The general characteristics of a circuit in resonance are the same regardless of which parameter is varied to produce resonance. For instance,

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in all cases the power factor at resonance is 1. The power is simply the impressed voltage times the current. The current is V/R, the maximum possible value for the resistance which is in the circuit. The general shape of the current curve before, at, and after resonance is shown in Fig. 7. Resonance occurs at the point C. Limited as it is only by the resistance of the circuit, the current at the resonant point C will be large if the resistance is small. When the resultant reactance is large as it is at point A there will be only a small current flowing. Hence

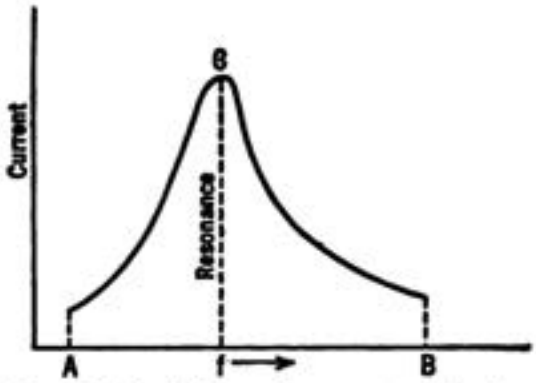

FIG. 7. Variation of current with frequency in the range series resonance.

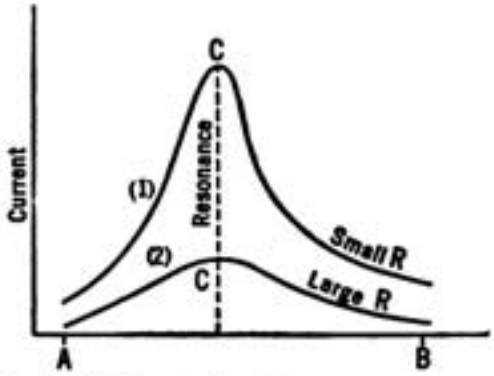

FIG. 8. Effect of resistance on current variation in the range of series resonance.

there is a rapid rise in current from point A to point C. Conversely, when the resistance is large, the amount of the change in current from point A to C will be small. In the former case the current peak will be sharper than in the latter, as illustrated in Fig. 8. Hence the small resistance is said to give sharp tuning and the large resistance broad tuning. More accurately, the ratio of L to R governs the sharpness of tuning. This is shown later. The preceding statements are true for all methods of securing resonance. The various ways of securing resonance will now be considered in somewhat more detail.

Varying Inductance. When L is varied to produce resonance, a series of curves shown in Fig. 9 is obtained. Equations (11), (12), (13), and (14) are the equations of the current and potential drop curves shown. It will be noted that V_C becomes a maximum at resonance whereas *the maximum value of V_L occurs after resonance.* This result is expected. Since $V_C = IX_C$ and X_C is constant, the maximum drop across the condenser will occur when the current is a maximum. In the case of $V_L = IX_L$, both I and X_L are increasing before resonance and the product must be increasing. At resonance, I is not changing but X_L is increasing, and hence the drop is increasing. The drop continues to increase until the reduction in the current offsets the increase in X_L. This point can be determined from $dV_L/dX_L = 0$. Differentiating equation (13) and setting

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the result equal to zero yield

$$\frac{dV_L}{dX_L} = \frac{[R^2+(X_L-X_C)^2]^{\frac{1}{2}}V - VX_L\frac{1}{2}[R^2+(X_L-X_C)^2]^{-\frac{1}{2}}2(X_L-X_C)}{R^2+(X_L-X_C)^2} = 0$$

and
$$X_L = \frac{R^2+X_C^2}{X_C}$$

or
$$L = \frac{1}{2\pi f}\left(\frac{R^2+X_C^2}{X_C}\right) = C(R^2+X_C^2) \tag{15}$$

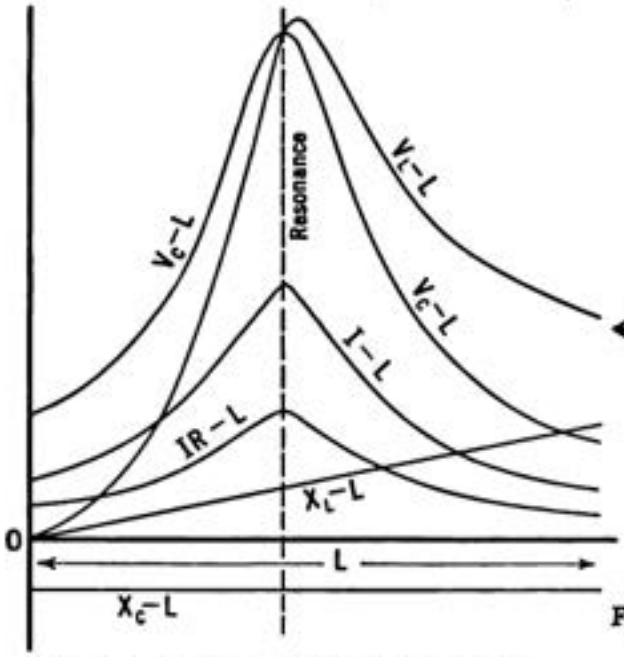

Fig. 9. Series resonance by varying L.

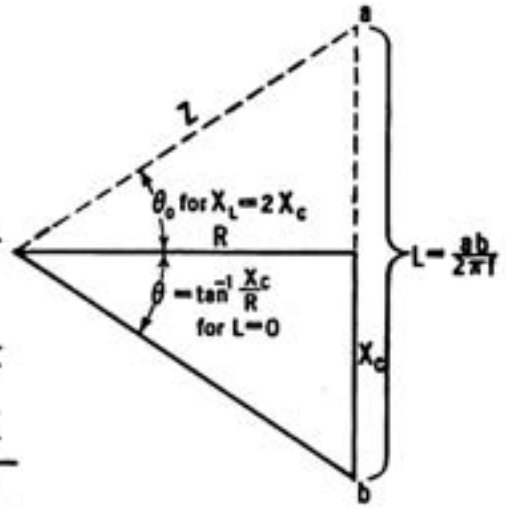

Fig. 10. Impedance diagram showing the power factor angle θ as L is varied in an RLC series circuit.

Example 2. As L is varied to produce resonance in a series circuit containing $R = 100$ ohms, $X_C = 200$ ohms, and $f = 60$ cycles, find the voltage drop across L at resonance and also when the drop across L is a maximum if 1000 volts are impressed.

For resonance $X_L = X_C = 200$. $\mathbf{Z} = 100 + j200 - j200 = 100 + j0$ ohms.

$I = \dfrac{1000}{100} = 10$ amperes.

V_L (at resonance) $= IX_L = 10 \times 200 = 2000$ volts.

For maximum V_L $\quad 2\pi fL = \dfrac{R^2+X_C^2}{X_C} = \dfrac{100^2+200^2}{200} = 250$ ohms.

I (for maximum V_L) $= \dfrac{1000}{\sqrt{100^2+(250-200)^2}} = 8.94$ amperes.

Maximum $V_L = 8.94 \times 250 = 2235$ volts.

The variation in phase angle between **V** and **I** as L is varied is easily obtained from the impedance diagram in Fig. 10. The angle can be seen to vary from $\tan^{-1}\dfrac{X_C}{R}$

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(a negative angle) when L is zero to $+90°$ when L becomes ∞. Hence the power factor varies from $\frac{R}{\sqrt{R^2 + X_C^2}}$ (when L is 0) to 0 (when L becomes infinite).

Problem 3. (*a*) Find the value of inductive reactance and the value of inductance which will make the power factor of the above series circuit equal to 0.866, current leading.

Hint: Problems of this type are most easily solved when it is recognized that $\frac{\sum X}{\sum R} = \pm \tan \theta$. *Ans.:* $X_L = 142.3$ ohms, $L = 0.377$ henry.

(*b*) Find the value of inductive reactance which will make the p.f. equal to 0.866, current lagging.

Ans.: $X_L = 257.7$ ohms.

Varying Capacitance. When C is varied to produce resonance, curves as shown in Fig. 11 are obtained. As before, the equations of these curves are equations (11), (12), (13), and (14). Here the drop across the inductance is a maximum when the current is a maximum, since X_L is constant. The maximum drop across the condenser occurs before resonance. At resonance, X_C is decreasing whereas the current is not changing (slope being zero). The drop IX_C must, therefore, be decreasing. Consequently, the drop must have been a maximum before resonance. At resonance the drops across the inductance and the capacitance are equal and opposite. The conditions for maximum V_C may be determined analytically by setting the first derivative of equation (14) with respect to C or X_C equal to zero, similarly to the procedure illustrated when L was varied. This derivation is left to the student.

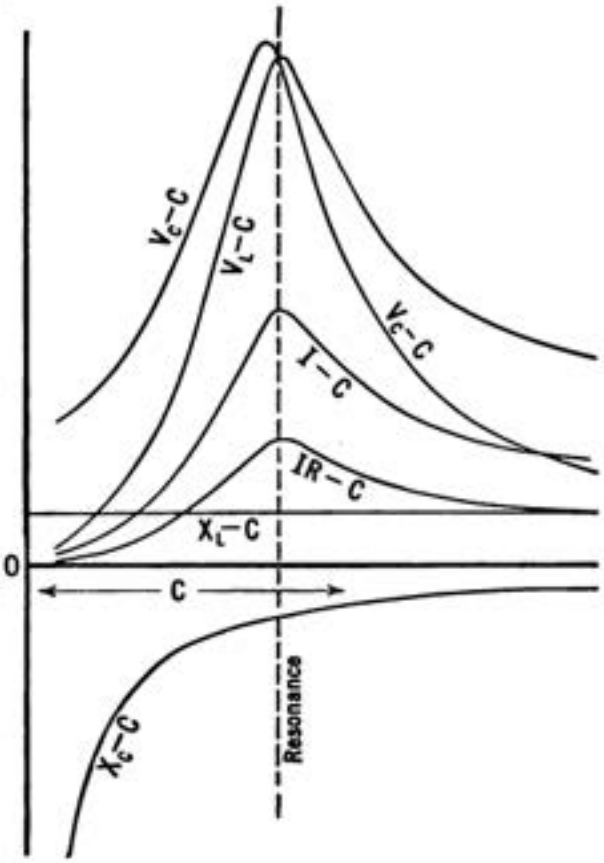

FIG. 11. Series resonance by varying capacitance.

The impressed voltage equals the IR drop, the power factor is unity, and the current is a maximum at resonance. For zero capacitance the capacity reactance is infinite and the current is therefore zero. For

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infinite capacitance the capacity reactance is zero and the current is $\frac{V}{\sqrt{R^2 + X_L^2}}$. The phase angle between the current and the applied voltage varies between the limits indicated in Fig. 12. The power factor varies from $\frac{R}{\sqrt{R^2 + X_L^2}}$, when C is infinite, to zero when C is zero.

Resonance is usually obtained by varying capacitance since it is only necessary to make alternate plates of a condenser movable to secure variable capacitance. This is easily and simply accomplished, and the variation of capacitance can be made extremely smooth and gradual.

Problem 4. When varying C to produce resonance in a circuit containing 100 ohms resistance and 200 ohms inductive reactance at 60 cycles, find the maximum drop across the capacitance if the impressed voltage on the circuit is 100 volts.

Ans.: 223.5 volts.

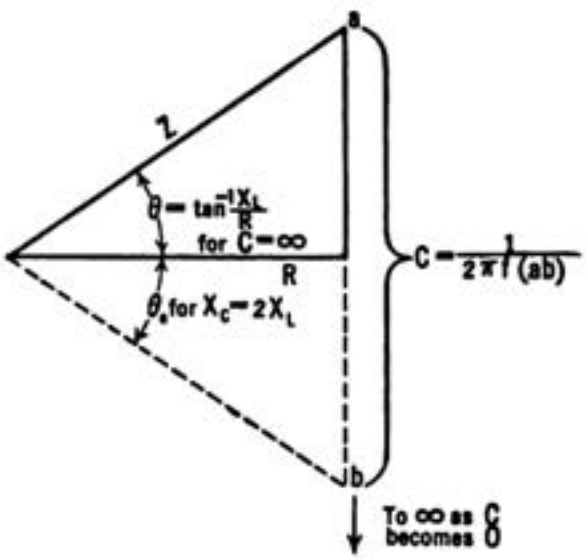

FIG. 12. Impedance diagram indicating range of power factor angle θ as C is varied in an RLC series circuit.

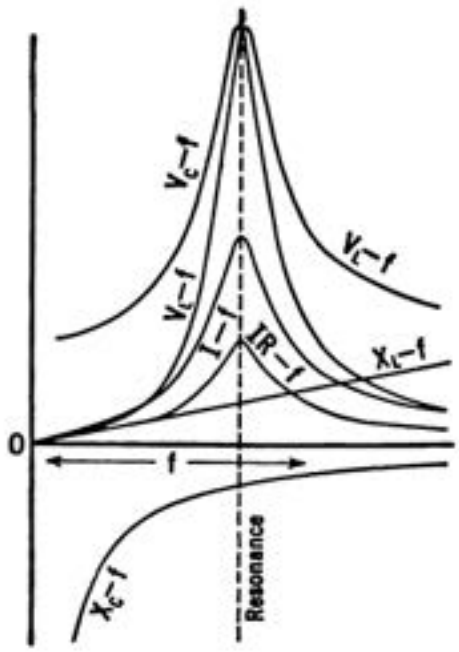

FIG. 13. Series resonance by varying frequency.

Varying Frequency. When frequency is varied to produce resonance, the curves shown in Fig. 13 are obtained. Here neither the inductance nor the capacitance has the maximum drop across it at resonance. Inspection of Figs. 9, 11, and 13 will show that this method of securing resonance partakes of both the methods previously discussed. The student can explain these curves by considering the principles previously presented. The current is zero for both zero frequency and infinite fre-

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quency. The phase angle between current and voltage varies between $-90°$ to $+90°$, as may be seen by studying the impedance triangles portrayed in Fig. 14. It will be observed that, for all methods of producing resonance, the current is a maximum and dependent only upon the impressed voltage and the resistance of the circuit, that the power factor is 1, and that the power is a maximum and equal to the volt-amperes at the point of resonance.

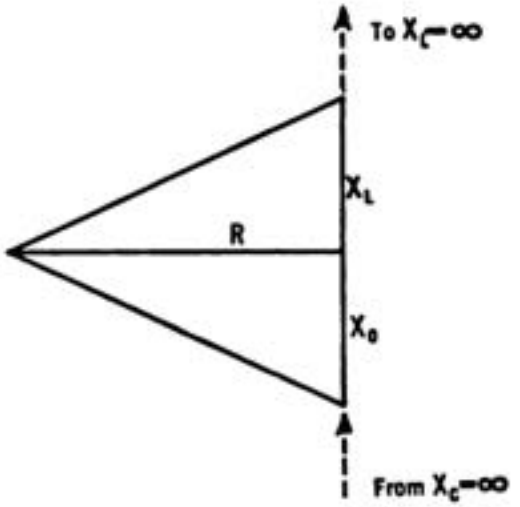

FIG. 14. Impedance triangle indicating variation of phase angle from $-90°$ to $+90°$ as frequency is varied in an *RLC* series circuit.

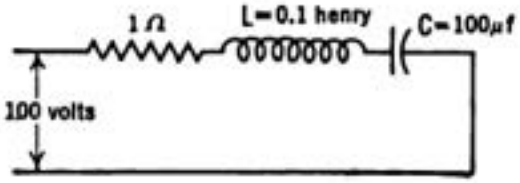

FIG. 15. Circuit for example 3.

Example 3. For the circuit arrangement and constants shown in Fig. 15 calculate the frequency, power, power factor, and voltage drop across each part of the circuit at resonance.

$$f_m = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi}\sqrt{\frac{1}{0.1 \times 0.000100}} = 50.4 \text{ cycles}$$

Check:

$$\begin{cases} X_L = 2\pi\, 50.4 \times 0.1 = 31.6 \text{ ohms} \\ X_C = \dfrac{1}{2\pi\, 50.4 \times 0.0001} = 31.6 \text{ ohms} \end{cases}$$

$$I = \frac{100}{\sqrt{1^2 + (31.6 - 31.6)^2}} = 100 \text{ amperes}$$

$$P = 100 \times 100 = 10{,}000 \text{ watts}$$

$$\text{P.f.} = \frac{\text{watts}}{\text{va}} = \frac{10{,}000}{100 \times 100} = 1$$

$$V_R = 100 \times 1 = 100 \text{ volts}$$

$$V_L = 100 \times 31.6 = 3160 \text{ volts}$$

$$V_C = 100 \times 31.6 = 3160 \text{ volts}$$

Problem 5. (*a*) What is the resonant frequency of a series circuit consisting of 2 ohms resistance, 150 microhenrys, and 200 $\mu\mu$f capacitance? (*b*) What is the resonant frequency if $R = 3$ ohms, $L = 300$ microhenrys, and $C = 100$ $\mu\mu$f? (*c*) What is the impedance of each of the combinations at 1000 kilocycles?

Ans.: (*a*) 920 kilocycles, (*b*) 920 kilocycles, (*c*) 147 ohms and 294 ohms.

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The Series *RLC* Circuit as a Selector. Even though the *RLC* circuit passes all waves of finite frequency to some extent, it has been shown to have the lowest impedance for the resonant frequency. As Fig. 7 shows, the *RLC* circuit passes frequencies near the resonant frequency more readily than other frequencies. The circuit thus has selective properties. The band of frequencies which is passed quite readily is called

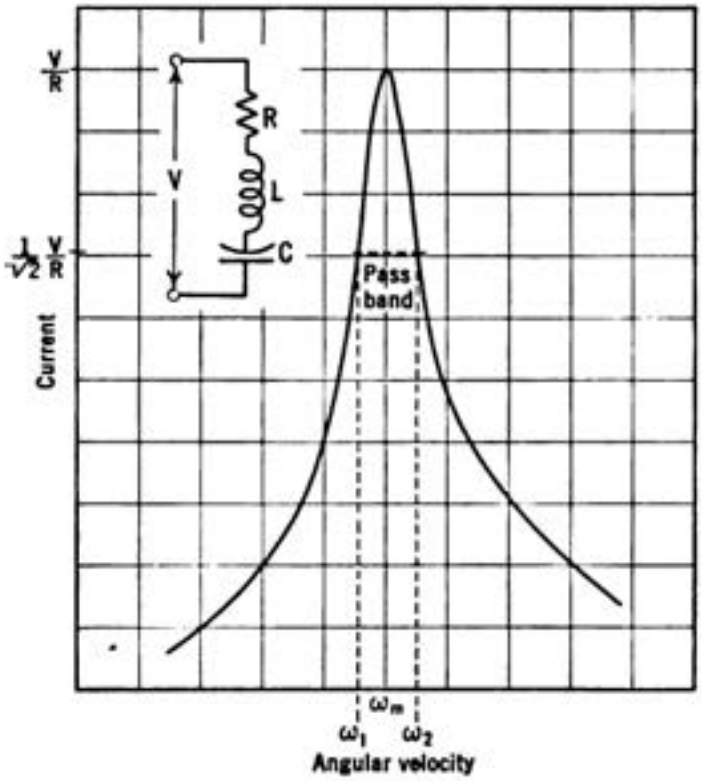

FIG. 16. The *RLC* series branch, as a band selector, graphed for $R = 10$ ohms, $L = 0.01$ henry, and $C = 4.0$ μf.

the pass band. The pass band is sometimes arbitrarily considered to be the range of frequency over which the current is equal to or greater than $V/\sqrt{2}R$, as indicated in Fig. 16. Within this range, the power (I^2R) is equal to or greater than $V^2/2R$. This range will now be determined. From equation (11)

$$I = \frac{V}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \tag{16}$$

The maximum current (V/R) and the maximum power V^2/R occur at the resonant frequency or when

$$\omega_m = \frac{1}{\sqrt{LC}} \tag{17}$$

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where ω_m is 2π times the resonant frequency f_m. Let ω_x be the angular velocities at which

$$I = \frac{V}{\sqrt{2}R} \quad \text{(called the half-power points)}$$

Then

$$\frac{V}{\sqrt{2}R} = \frac{V}{\sqrt{R^2 + (\omega_x L - 1/\omega_x C)^2}} \tag{18}$$

From which $R = \pm(\omega_x L - 1/\omega_x C)$.

When solved for ω_x the above equation yields

$$\omega_x = \pm\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \tag{19}$$

In a selective *RLC* branch, $(R/2L)^2$ is usually much smaller than $1/LC$. Hence, neglecting this term, equation (19) becomes

$$\omega_x \approx \pm R/2L \pm \sqrt{1/LC} \tag{20}$$

But $\sqrt{1/LC}$ is the angular velocity ω_m corresponding to the resonant frequency. Therefore

$$\omega_x \approx \pm\frac{R}{2L} \pm \omega_m \tag{21}$$

and, if only positive values of ω_m are considered,

$$\omega_x = \omega_m \pm \frac{R}{2L} \tag{22}$$

Let

$$\omega_1 = \omega_m - \frac{R}{2L} \tag{23}$$

and

$$\omega_2 = \omega_m + \frac{R}{2L} \tag{24}$$

The width of the pass band as shown on Fig. 16 is

$$\Delta\omega = \omega_2 - \omega_1 = \frac{R}{L} \text{ radians per second} \tag{25}$$

The frequency range for the pass band, as here defined, is $\Delta f = f_2 - f_1 = R/2\pi L$. The per unit band width is defined as $\Delta f/f_m$. If we arbitrarily select a band width other than that shown in Fig. 16, as we shall have occasion to do later, we make appropriate changes in our definition of Δf.

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Example 4. Let it be required to find the decibel (db) current response at the half-power points of Fig. 16 (relative to the response at ω_m) if by definition we take

$$\text{db} = 20 \log \frac{I}{\frac{V}{R}}$$

where I is the current response at any point on the graph shown in the figure.

Since $I = V/\sqrt{2}R$ at the points in question:

$$\text{db} = 20 \log \frac{\frac{V}{\sqrt{2}R}}{\frac{V}{R}} = -20 \log 1.414 = -3$$

The above arithmetic shows why the half-power points are sometimes referred to in the literature as the -3 db points.

The Q of a Series Circuit. The degree of selectivity of a circuit, that is, the narrowness of the band width shown in Fig. 16, is usually expressed in terms of the symbol Q. Although several different forms of the definition of Q appear in the literature, they are all intended to convey the same meaning. We shall employ the following definition since it ties in closely with experimental procedures:

$$Q = \frac{\omega_m}{\omega_1 - \omega_2} = \frac{\omega_m}{\Delta\omega} = \frac{f_m}{\Delta f} \tag{26}$$

See Fig. 16 for the meanings of ω_1, ω_2, and ω_m.

In the case of the series RLC circuit

$$Q_s = \frac{\omega_m L}{R_s} = \frac{1}{\omega_m C R_s} = \frac{1}{R_s}\sqrt{\frac{L}{C}} \tag{27}$$

where R_s is the total equivalent series resistance of the circuit. Since the equivalent series circuit resistance of the condenser is usually negligibly small in comparison with the series circuit resistance of the coil, it is customary to speak of the Q of the coil alone, the assumption being that the coil will be resonated at some specified frequency with a condenser of suitable size.

Example 5. The per unit band width between the half-power (or -3 db) points in Fig. 16 is to be 0.02. Find the Q of the coil required.

$$\text{Per unit band width} = \frac{\Delta\omega}{\omega_m} = \frac{1}{Q}$$

$$Q = \frac{1}{0.02} = 50$$

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If the coil to be employed has an inductance of 10 millihenrys and the resonant frequency is 20 kc, find the values of R_s and C.

$$R_s = \frac{\omega_m L}{Q_s} = \frac{2\pi \times 20{,}000 \times 0.01}{50} = 8\pi = 25.1 \text{ ohms}$$

$$C = \frac{1}{\omega_m^2 L} = \frac{1}{4\pi^2(20{,}000)^2 \times 0.01} = 0.00633 \times 10^{-6} \text{ farad}$$

The use of Q (or the reciprocal of Q) in circuit analysis will take on more importance and significance in radio-frequency circuits where Q_s is essentially constant than in low-frequency circuits where R_s is essentially constant. [It should be noted that R_s has been tacitly assumed constant in equation (27) as well as in Fig. 16.] In analyzing tuned radio-frequency circuits near resonant frequency, $\omega_m = 1/\sqrt{LC}$, we obtain greater accuracy by writing

$$Z = R_s + j\left(\omega L - \frac{1}{\omega C}\right)$$

as

$$Z = \omega_m L\left[\frac{R_s}{\omega_m L} + j\left(\frac{\omega}{\omega_m} - \frac{\omega_m}{\omega}\right)\right]$$

or

$$Z = \omega_m L\left(\frac{1}{Q} + jF\right) = \sqrt{\frac{L}{C}}\left(\frac{1}{Q} + jF\right) \tag{28}$$

since Q is considerably more constant over a reasonable frequency range centered on ω_m than is R_s. It is plain that $F = (\omega/\omega_m - \omega_m/\omega)$.

If L, C, and Q in equation (28) are essentially constant, then $F = (\omega/\omega_m - \omega_m/\omega)$ is the only variable involved, and it should be plain that the current response versus ω will take the same shape as that shown in Fig. 16 since in one case the response is based upon

$$I = \frac{V}{\sqrt{R_s^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and the other upon

$$I = \frac{V}{\sqrt{\frac{L}{CQ^2} + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

In the low-frequency case we assume that R_s is constant, which is essentially true, and in the high-frequency case we assume that L/CQ^2 is essentially constant. Cases arise where neither assumption is justified, but cases of this kind are reserved for more advanced courses.

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Circle Diagram of Series Circuit. Circle diagrams are often employed as an aid in analyzing the operating characteristics of circuits which under some conditions are used in representing transmission lines and some types of a-c machinery. The basis of representing a series circuit by means of a circle diagram will be derived with reference to Fig. 17.

The resistance R of the circuit in Fig. 17 will be considered a variable, whereas the applied voltage and reactance will be assumed constant.

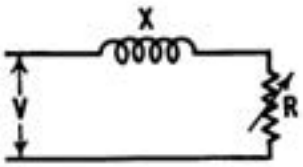

FIG. 17. Series circuit with variable R.

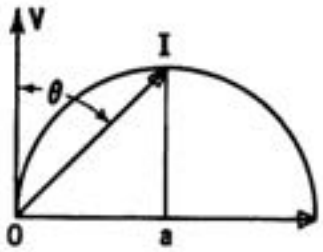

FIG. 18. Circle diagram of Fig. 17 for constant V and X but with variable R.

The power factor angle is designated by θ. If R is zero, I is obviously equal to V/X, and this value of I will lag **V** by 90° if X is inductive. (See Fig. 18.) As R is increased from its zero value, the magnitude of **I** becomes less than V/X and θ becomes less than 90° and finally, when R equals ∞, I equals zero and θ equals zero. The fact that the locus of the vector **I** traces out a semicircle, as indicated in Fig. 18, may be seen from the following derivation.

In general,

$$I = \frac{V}{Z} \tag{29}$$

and

$$\sin\theta = \frac{X}{Z} \tag{30}$$

or

$$Z = \frac{X}{\sin\theta} \tag{31}$$

Substituting (31) in (29),

$$I = \frac{V}{X}\sin\theta \tag{32}$$

For constant V and X, equation (32) is the polar equation of a circle of diameter V/X. Figure 18 shows a plot of equation (32) with respect to V as a reference and for positive angles θ, representing inductive loads, measured clockwise. These conventions are employed because they are the ones most commonly used for such circle diagrams in

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a-c machinery. Since Ia in Fig. 18 is $OI \cos \theta$, it is apparent that Ia is proportional to the power consumed by the circuit. If the diagram is drawn to a certain current scale as I amperes per inch, the watt scale must be VI watts per inch.

A simple transmission line circuit in which the capacitance and leakance are assumed negligible may be represented by Fig. 19, where R and X are, respectively, the series resistance and reactance of the line and R_L is the load resistance. If R is constant and R_L is varied the current follows the equation $I = (V/X) \sin \theta$ as in the previous case. The distance Ia in Fig. 18 again represents the total power consumed by the circuit, but the total power dissipated is consumed in both R and R_L. The power dissipated by each resistance can easily be represented on the diagram.

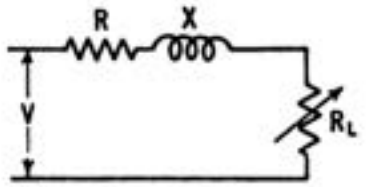

FIG. 19. Series circuit, R and X assumed constant, R_L variable.

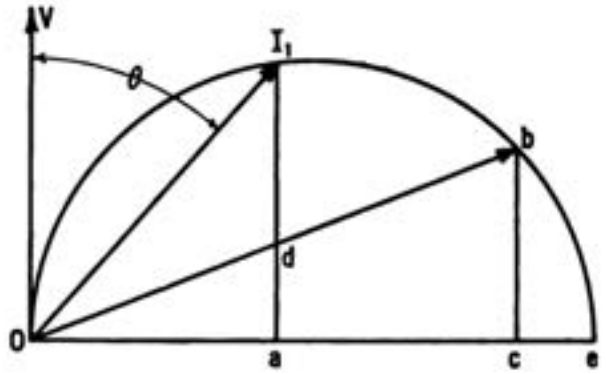

FIG. 20. Circle diagram of Fig. 19 for constant V, R, and X and variable R_L.

If the resistance R_L is assumed to be zero, all power must be dissipated in the resistance R. For this condition the power is represented by bc in Fig. 20 and Ob represents the corresponding current. For some finite value of R_L other than zero, the current is OI_1 and the total power consumed is proportional to I_1a. Of this total, da is the amount consumed in R and I_1d is dissipated by R_L. To prove that da represents the power dissipated in R it is only required to show that da and bc are proportional to the respective squares of the currents OI_1 and Ob for the two conditions.

From similar triangles $$\frac{da}{bc} = \frac{Oa}{Oc}$$

Since $$Oa = OI_1 \cos aOI_1$$

$$= OI_1 \frac{OI_1}{Oe} = \frac{(OI_1)^2}{Oe}$$

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and
$$Oc = Ob \cos cOb = Ob \frac{Ob}{Oe} = \frac{(Ob)^2}{Oe}$$

$$\frac{da}{bc} = \frac{\dfrac{(OI_1)^2}{Oe}}{\dfrac{(Ob)^2}{Oe}} = \frac{(OI_1)^2}{(Ob)^2}$$

Therefore, for any current such as OI_1, I_1d represents the power consumed in R_L, da shows the watts lost in R, and the total power input to the circuit is given by I_1a. If I^2R_L is considered as the output of the circuit (the power transmitted by the line), the efficiency must be

$$\text{Efficiency} = \frac{\text{output}}{\text{input}} = \frac{I_1d}{I_1a}$$

The power factor at the input end is $\cos\theta$. It is also I_1a/OI_1.

The maximum power that can be transmitted by a circuit like Fig. 19 under conditions of constant R and X occurs when the extremity of OI_1 in Fig. 20 coincides with the point of tangency to the circle of a line drawn parallel to Ob. It is a matter of simple geometry to show that V times I_1d under these conditions yields the result for maximum power as given by equation (59) if $X_r = 0$ [which requires that k in equation (59) be zero]. Since I_1d may be employed as a quantitative measure of the power delivered to the load resistance R_L, it is plain from Fig. 20 that this load power varies from zero (when $R_L = 0$) to a maximum and back to zero again (when $R_L = \infty$).

The details of circle diagram constructions which apply to circuits of the kind shown in Fig. 19 may be readily comprehended from a numerical problem like the following.

Problem 6. Refer to Fig. 19. R and X are constant at the values $R = 2$ ohms and $X = 3.464$ ohms. V is constant at 346.4 volts.

(*a*) Lay off $OV = V$ graphically in a vertical position to any convenient voltage scale as, for example, 100 volts per inch.

(*b*) Lay off Oe (of Fig. 20) equal to V/X in a horizontal position to a scale of not more than 20 amperes per inch. (A scale of 10 amperes per inch will give more accurate results.)

(*c*) Lay off Ob (of Fig. 20) equal to I when $R_L = 0$.

Ans.: $I = 346.4/4 = 86.6$ amperes, 60° behind V.

(*d*) Draw a tangent to the semicircle which is parallel to Ob and construct OI_1 from O to this point of tangency. What is the magnitude of the current and what is the p.f. at this point of operation? *Ans.:* $I = 50$ amperes, p.f. $= 0.86$.

(*e*) What is the maximum power that can be delivered to R_L?

Ans.: $P_{max} = V \times I_1d_{max} = 10{,}000$ watts.

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Parallel Branches. When impedances are connected in parallel, as in Fig. 21, the same voltage V is impressed across each impedance. The current in each impedance is therefore

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1}, \quad \mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2}, \quad \text{and} \quad \mathbf{I}_3 = \frac{\mathbf{V}}{\mathbf{Z}_3}$$

From Kirchhoff's current law,

$$\begin{aligned} \mathbf{I} &= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 \\ &= \frac{\mathbf{V}}{\mathbf{Z}_1} + \frac{\mathbf{V}}{\mathbf{Z}_2} + \frac{\mathbf{V}}{\mathbf{Z}_3} = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}\right) \\ &= \mathbf{V}(\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3) = \mathbf{V}\mathbf{Y}_0 \end{aligned} \tag{33}$$

where the symbol $\mathbf{Y}$ represents the reciprocal of impedance and is called admittance. Equation (33) shows that the resultant current flowing through several impedances in parallel is the product of the voltage

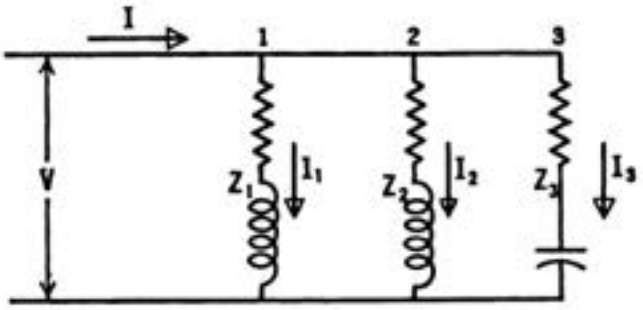

FIG. 21. Impedances in parallel.

and the sum of the reciprocals of the several branch impedances. In other words, the voltage is multiplied by the sum of the admittances of the several branches. Equation (33) shows that *admittances* are *added* for *parallel* branches. For branches in series it will be remembered that *impedances* are added. Since both admittance and impedance are complex quantities, all additions involving either of them *must* be made in *complex* form. Arithmetic addition should not be attempted. In only one case is arithmetic addition correct, and in this case the addition in complex form will give the same result. If equation (33) is solved for impedance $\mathbf{Z}_0$ by obtaining the ratio of $\mathbf{V}$ to $\mathbf{I}$, we obtain

$$\mathbf{Z}_0 = \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3} = \frac{1}{\mathbf{Y}_0} \tag{34}$$

Equation (34) shows that the resultant impedance of several parallel branches is the reciprocal of the resultant admittance. Since the unit of impedance is the ohm and admittance is the reciprocal of im-

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pedance, the unit of admittance is the reciprocal ohm or mho (ohm spelled backwards).

The Parallel Equivalent of a Series Impedance. Cases arise where it becomes desirable to change a series branch impedance as shown in

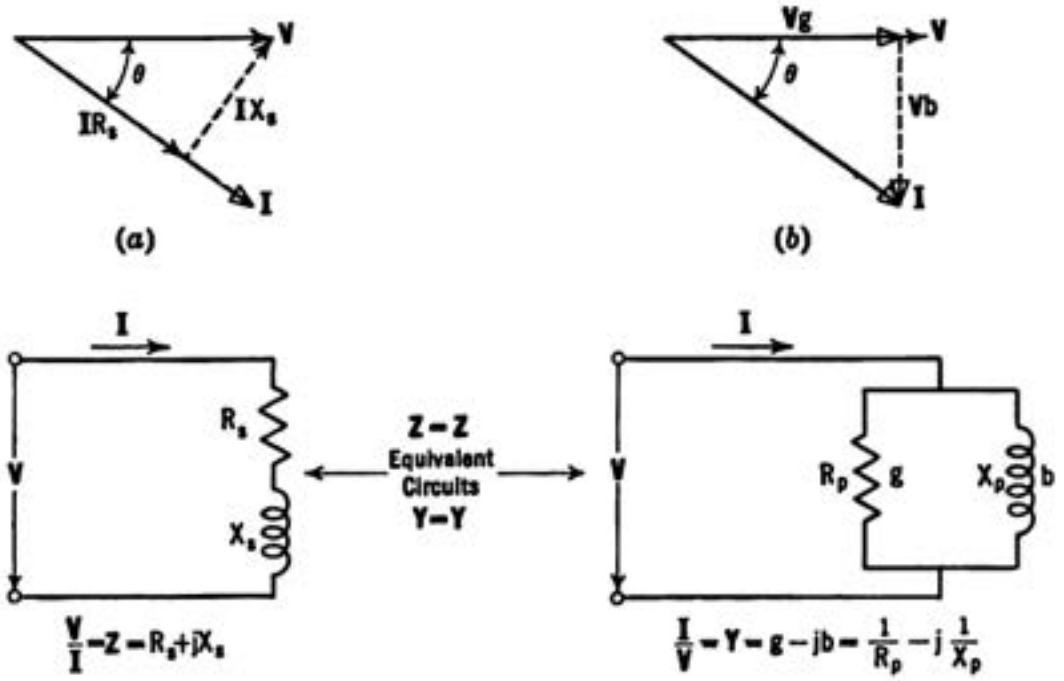

FIG. 22. The parallel equivalent of a series impedance, $R_s + jX_s$.

Fig. 22*a* to its parallel equivalent (shown in Fig. 22*b*). For equivalence, **Y** of Fig. 22*a* must equal **Y** of Fig. 22*b*. Therefore

$$\mathbf{Y} = \frac{1}{R_s + jX_s} = \frac{1}{R_p} + \frac{1}{jX_p}$$

or, upon rationalizing,

$$\frac{R_s}{R_s^2 + X_s^2} - j\frac{X_s}{R_s^2 + X_s^2} = \frac{1}{R_p} - j\frac{1}{X_p} \tag{35}$$

$R_s/(R_s^2 + X_s^2)$ is called the conductance of the series impedance Z_s and is denoted by the symbol g. $X_s/(R_s^2 + X_s^2)$ is called the susceptance of the series impedance Z_s and is denoted by the symbol b. Employing the symbols g and b, we have

$$\mathbf{Y} = g - jb = \frac{1}{R_p} - j\frac{1}{X_p} \tag{36}$$

The physical significance of g and b may be interpreted as follows. If equation (36) is multiplied by **V** to obtain the current **I**, we have

$$\mathbf{I} = \mathbf{V}g - j\mathbf{V}b = \frac{\mathbf{V}}{R_p} - j\frac{\mathbf{V}}{X_p}$$

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It will be seen that $\mathbf{V}g$ shown on the vector diagram, Fig. 22*b*, is the component of current in phase with the voltage and is the current $\mathbf{V}/R_p$ in the resistive branch of the parallel equivalent of $\mathbf{Z}_s$. Also $\mathbf{V}b$ shown on the vector diagram is the component of current in quadrature with the voltage and is the component $\mathbf{V}/X_p$ in the inductive branch of the parallel equivalent of $\mathbf{Z}_s$. Hence the conductance $1/R_p$ of the resistive branch of the equivalent parallel circuit is the conductance g of the admittance $\mathbf{Y} = g - jb = 1/\mathbf{Z}_s$, and the susceptance $1/X_p$ of the inductive branch is the susceptance b of the admittance $\mathbf{Y} = 1/\mathbf{Z}_s$. It is important to observe that conductance g in the circuits of Fig. 22 is the reciprocal of R_p but *not* of R_s. Similarly susceptance is the reciprocal of X_p but *not* of X_s.

Since g and b are components of admittance and either g, b, or Y multiplied by voltage yields a current, they are all expressed in the same units, namely, mhos.

If the admittances in equation (33) are expressed in terms of their conductances and susceptances, we have

$$\begin{aligned}\mathbf{I} &= \mathbf{V}(g_1 - jb_1 + g_2 - jb_2 + g_3 - jb_3)\\ &= \mathbf{V}[(g_1 + g_2 + g_3) - j(b_1 + b_2 + b_3)] = \mathbf{V}(g_0 - jb_0) \qquad (37)\end{aligned}$$

Equation (37) shows that conductances may be added arithmetically to obtain the resultant conductance while susceptance must be added algebraically to obtain the resultant susceptance. That algebraic addition of susceptances is required is evident from the expression $X/(R^2 + X^2)$ for susceptance when it is remembered that X may be positive or negative depending upon whether it is inductive or capacitive, respectively.

Example 6. For the circuit of Fig. 23*a* with the parameters shown, the following are desired: (*a*) conductance and susceptance of each branch; (*b*) the resultant conductance and susceptance; (*c*) the vector diagram.

$$\mathbf{I}_1 = \frac{100 + j0}{6 + j8} = 6 - j8 = 10\underline{/-53.2^\circ} \text{ amperes}$$

$$\mathbf{I}_2 = \frac{100 + j0}{4 - j3} = 16 + j12 = 20\underline{/36.9^\circ} \text{ amperes}$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 22 + j4 = 22.35\underline{/10.3^\circ} \text{ amperes}$$

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} = \frac{1}{(6 + j8)}\,\frac{(6 - j8)}{(6 - j8)} = 0.06 - j0.08 \text{ mho}$$

from which

$$g_1 = 0.06 \text{ mho}, \quad b_1 = 0.08 \text{ mho}$$

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or, as an alternative method,

$$g_1 = \frac{R_1}{Z_1^2} = \frac{6}{100}, \quad b_1 = \frac{X_1}{Z_1^2} = \frac{8}{100}$$

$$\mathbf{Y}_2 = \frac{1}{Z_2} = \frac{1}{(4 - j3)} \frac{4 + j3}{(4 + j3)} = 0.16 + j0.12 \text{ mho}$$

from which

$$g_2 = 0.16 \text{ mho}, \quad b_2 = -0.12 \text{ mho}$$

or, as an alternative method,

$$g_2 = \frac{R_2}{Z_2^2} = \frac{4}{25}, \quad b_2 = \frac{X_2}{Z_2^2} = \frac{-3}{25}$$

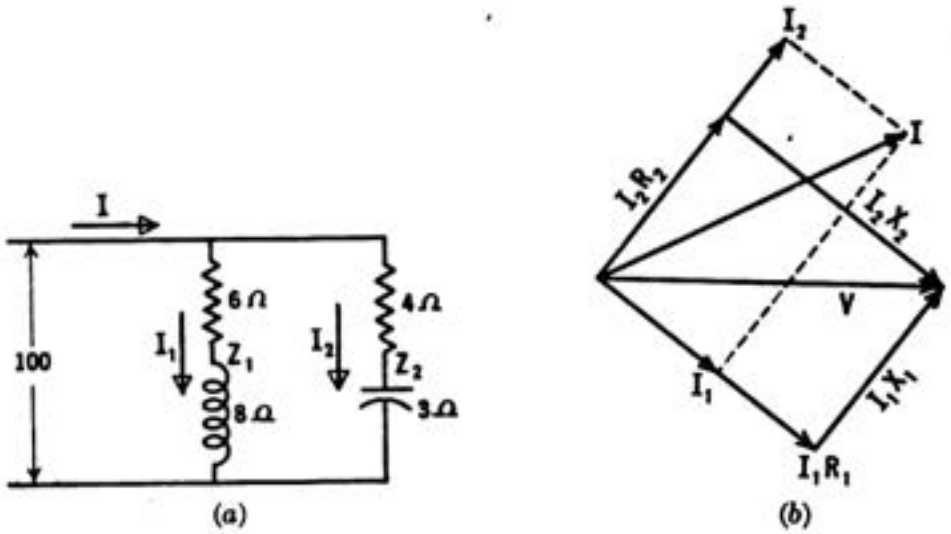

FIG. 23. (a) Circuit for example 6. (b) Vector diagram of (a).

The vector diagram is shown in Fig. 23b.

Another way to obtain the resultant current is shown below:

$$g = g_1 + g_2 = 0.06 + 0.16 = 0.22 \text{ mho}$$
$$b = b_1 + b_2 = 0.08 - 0.12 = -0.04 \text{ mho}$$
$$\mathbf{Y} = g - jb = 0.22 - j(-0.04) = 0.22 + j0.04 \text{ mho}$$
$$\mathbf{I} = \mathbf{VY} = 100\,(0.22 + j0.04) = 22 + j4 = 22.35 \angle 10.3^\circ \text{ amperes}$$

Or admittances may be added as follows:

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = 0.06 - j0.08 + 0.16 + j0.12 = 0.22 + j0.04$$

and

$$I = \mathbf{VY} = 22 + j4 \text{ amperes}$$

The calculation of admittances from the reciprocals of impedances and their addition in complex form is generally the most direct procedure. Experience has shown that students make fewer errors in signs when following this procedure.

Instead of representing admittance in general as $g - jb$ and then using $g = R/Z^2$ and $b = X/Z^2$, many prefer to call it $g + jb$ and then

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to use $g = R/Z^2$ and b as $-X/Z^2$. Both give the same result for admittance. In either case, X is substituted as a positive value for inductance and negative for capacitance. In a dissipative circuit conductance is always positive. To *avoid confusion* in signs it is *best* to obtain admittance from $1/(R + jX)$ rather than from calculations of conductance and susceptance. Knowing how to calculate and use conductances and susceptances expedites the solution of some types of problems, although they may be solved by other means. The special case of two parallel impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ occurs often in electrical engineering. For this case, $\mathbf{Y}_1 = 1/\mathbf{Z}_1$ and $\mathbf{Y}_2 = 1/\mathbf{Z}_2$. Hence

$$\mathbf{Y} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} \quad \text{and} \quad \mathbf{Z} = \frac{1}{\mathbf{Y}} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

This expression, which is analogous to the much used expression for the resultant of two parallel resistances in direct currents, is very useful in alternating currents. When all reactances are zero, the expression reduces to the d-c case of $R_1R_2/(R_1 + R_2)$.

Problem 7. Three impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ are connected in parallel across a 60-cycle voltage the magnitude of which is 40 volts.

$$\mathbf{Z}_1 = 10 + j0, \quad \mathbf{Z}_2 = 20 + j20, \quad \mathbf{Z}_3 = 30 - j40 \text{ ohms}$$

(*a*) Find g_1, b_1, g_2, b_2, g_3, and b_3.

(*b*) Find the resultant g and the resultant b of the three parallel branches.

Ans.: $g = 0.137$, $b = 0.009$ mhos.

(*c*) What is the in-phase component of the resultant current; the quadrature component of the resultant current?

Ans.: $Vg = 5.48$ amperes, $Vb = 0.36$ amperes.

Resonance in Parallel Branches. Parallel branches containing inductance and capacitance are in resonance when the reactive current in the inductive branch is equal to the reactive current in the capacitive branch. The resultant reactive current for the circuit as a whole is therefore zero. For resonance

$$Vb_L = Vb_C$$

$$b_L = b_C \tag{38}$$

Hence the resultant current flowing is in phase with the applied voltage, and the power factor of the whole circuit is 1. This is sometimes called unity-power-factor resonance. Figure 24 shows a circuit and the corresponding vector diagram for this condition. From an inspection of the vector diagram it will be noted that the reactive components of current contribute nothing to the total current. Only the components of current in phase with the voltage exist in the resultant current. It

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might be inferred from this that the resultant current is a minimum at resonance. This is true if the *conductances* are *constant*. It is approximately true if the conductances are negligibly small, as they usually are in selective circuits as used in radio. An example will be considered later wherein minimum current does not occur at resonance.

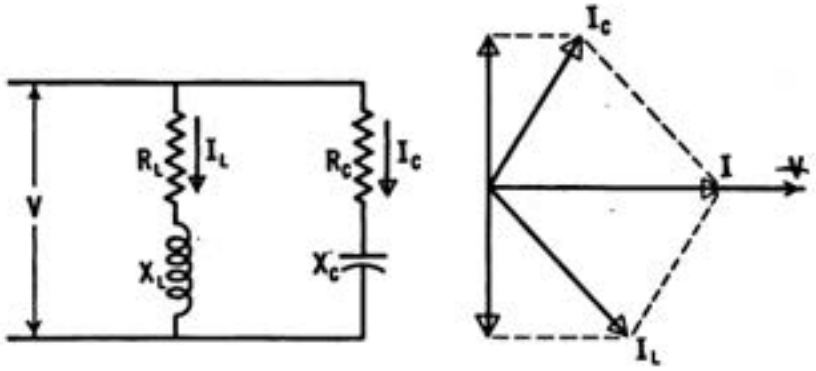

FIG. 24. Circuit and corresponding vector diagram for parallel resonance.

The parameters possible of variation to make equation (38) true may be seen when the susceptances are replaced by their equivalent values, as shown in equation (39).

$$\frac{2\pi fL}{R_L^2 + (2\pi fL)^2} = \frac{\dfrac{1}{2\pi fC}}{R_C^2 + \left(\dfrac{1}{2\pi fC}\right)^2} \tag{39}$$

The quantities that may be varied are L, C, f, R_L, or R_C.

Resonance by Varying L. In the following discussion L will be varied by a means which will not change the resistance of the inductive circuit. Let OV, Fig. 25, be the voltage impressed on a circuit like the one shown in Fig. 24. A current, I_C, will then flow in the condenser branch whose parameters are held constant. When L is zero, the current through the inductive branch is V/R_L and it is in phase with the applied voltage. The applied voltage is equal to I_LR_L under these conditions. When L is increased from zero, the current through the inductive branch lags V by the $\tan^{-1}(X_L/R_L)$, as illustrated in Fig. 25 by OI_L. For any value of I_L, the I_LR_L drop and the I_LX_L drop must add at right angles to give the applied voltage. These component drops are OA and AV, respectively. Since they are always at right angles and their sum must be OV, the locus of the I_LR_L drop must be a semicircle OAV. Since I_L is proportional to the I_LR_L drop and in phase with it, the locus of I_L must also be a semicircle.

When the I_LR_L drop coincides with the diameter of its circle, the

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current I_L must also coincide with the diameter of its own circle. The diameter of the latter must, therefore, be V/R_L. Hence the dotted circle drawn with V/R_L as a diameter must be the locus of I_L. Since

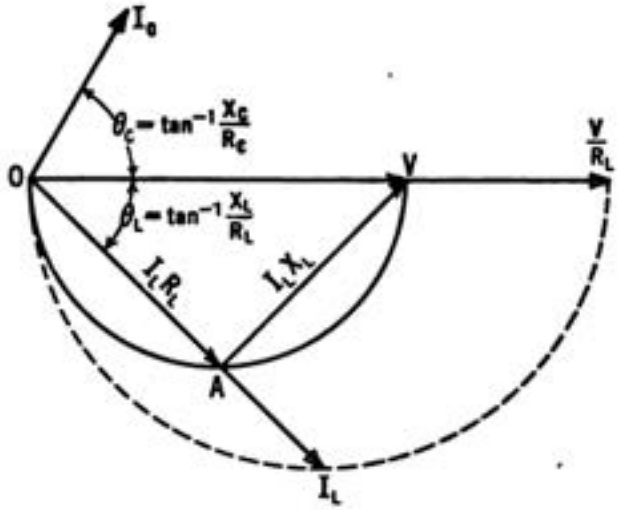

FIG. 25. Locus of I as L is varied in the circuit shown in Fig. 24.

the resultant current is $I_C + I_L$, this addition is performed by drawing the semicircle OI_LB with the left extremity of its diameter starting at

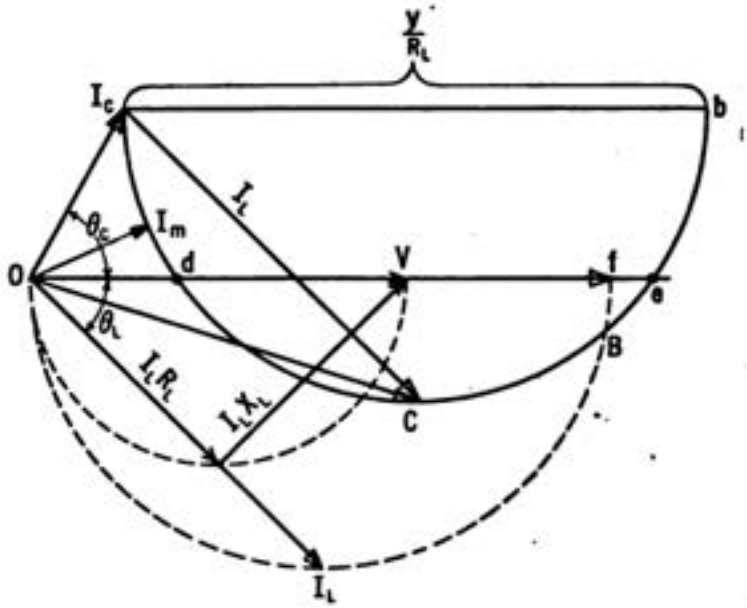

FIG. 26. Locus of OC, the resultant current to the circuit of Fig. 24 as L is varied.

I_C as shown in Fig. 26. For example, a particular sum of I_C and I_L is represented by OC. As L is varied, the locus of the resultant current is, therefore, the circle I_CCb. Hence, as L is increased from 0 to ∞, the resultant current varies from Ob to Oe, which is one point of reso-

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nance; thence to Od, which is a second resonant point; and then to OI_C. Neither of the resonant points gives either a maximum or minimum current, but they do yield unity power factor. The minimum current is OI_m, the value where the resultant current is normal to the circle I_CCb. For any particular problem the values of I_C, θ_C, and I_Cb, which is equal to V/R_L, can be calculated directly from the parameters. Any other values of current can then be calculated trigonometrically from the geometry of the figure. A few facts should be observed. First, if $V/2R_L$ (the radius of the circle I_CCb) is less than $I_C \sin \theta_C$, parallel resonance cannot be obtained regardless of the value of L.

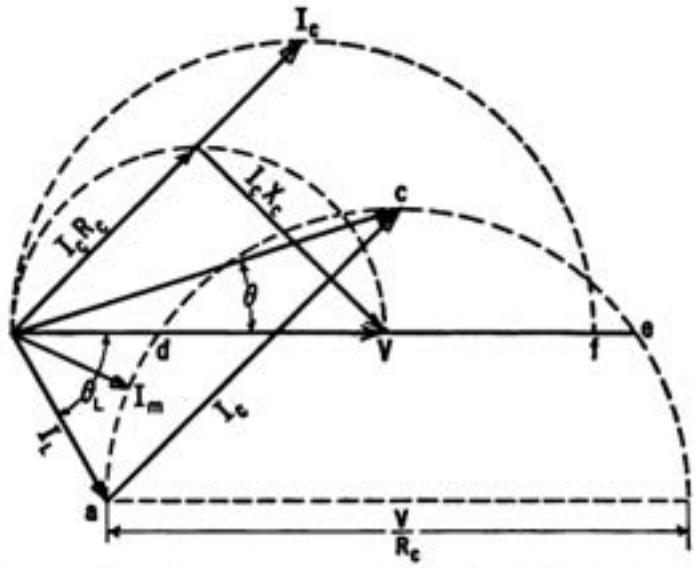

FIG. 27. Locus of resultant current to the circuit of Fig. 24 is the circle *adce* as C is varied.

This is in contrast to the series circuit, where some value of L will yield resonance for any value of R or C. Second, if $V/2R_L = I_C \sin \theta_C$, there will be only one resonant point. Third, if $V/2R_L > I_C \sin \theta_C$, there will be two resonant points. Fourth, if the resistance of the inductance were zero, minimum current would occur at resonance. Note that for this condition the conductances would be constant for the two branches.

Resonance by Varying C. Through a similar procedure to that outlined above, the student can develop the graphical representation for the case where resonance is produced by varying C while R_L, L, R_C, and f are held constant. The graphical representation is shown in Fig. 27. The locus of the resultant current is the circle *adce*. Again it will be noted that resonance which occurs at d and e is not the condition for minimum current. Minimum current occurs at I_m, where the resultant current is normal to the circle *adce*. If R_C is zero, the radius

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of the circle *adce* becomes infinite, or, what is the same thing, the current I_C is in quadrature with the voltage V. Under this condition there is but one point of resonance and it corresponds to minimum current. The conductance of the condenser circuit is zero, whereas that of the inductive branch is constant. This constant conductance makes the current at resonance a minimum, and hence the impedance a maximum. Since most selective circuits employ constant inductance and variable capacitance and the resistances of the capacitive branches are very small, maximum impedance or minimum current at resonance is practically realized in these circuits. Since at resonance the current is simply the conductance times the voltage impressed, it is evident that the power factor is 1. An inspection of Fig. 27 will reveal the manner in which the phase angle θ between the resultant current and the applied voltage varies as the resultant current follows the circle *adce*. Between points d and e, leading power factor obtains.

Resonance by Varying Frequency. From equation (39) the frequency for parallel resonance is found to be

$$f_m = \frac{1}{2\pi\sqrt{LC}}\left[\frac{R_L^2C - L}{R_C^2C - L}\right]^{1/2} \tag{40}$$

When $R_L^2C > L$ and $R_C^2C < L$, the quantity $\left[\frac{R_L^2C - L}{R_C^2C - L}\right]^{1/2}$ is imaginary and therefore no real frequency will yield resonance. The same situation results if both inequality signs are reversed. If R_L and R_C are equal, equation (40) for resonance becomes

$$f_m = \frac{1}{2\pi\sqrt{LC}}$$

which is the same as that for series resonance. This equation is also correct when $R_L = R_C = 0$ and may therefore be used as a close approximation when R_L and R_C are very small. It should be apparent that there are values of R_L, C, R_C, and L in a parallel circuit for which parallel resonance is impossible, regardless of frequency. This is in contrast to the series circuit containing R, L, and C where there is always some real resonant frequency for any values of the three parameters. The trends of various quantities as frequency is varied from a value too small to produce resonance to a value higher than that required for resonance are shown in Fig. 28 for a condition where resonance is obtainable.

Resonance by Varying R_L or R_C. When equation (40) is solved for

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R_L, the following equations result.

$$R_L = \sqrt{\frac{LC\omega^2 (R_C{}^2C - L) + L}{C}} \tag{41}$$

$$R_L = \sqrt{LC\omega^2 R_C{}^2 - L^2\omega^2 + \frac{L}{C}} \tag{42}$$

$$R_L = \sqrt{\frac{X_L}{X_C} R_C{}^2 - X_L{}^2 + \frac{L}{C}} \tag{43}$$

When the parameters are such as to make the expressions under the above radical positive, R_L takes on definite positive values. It is

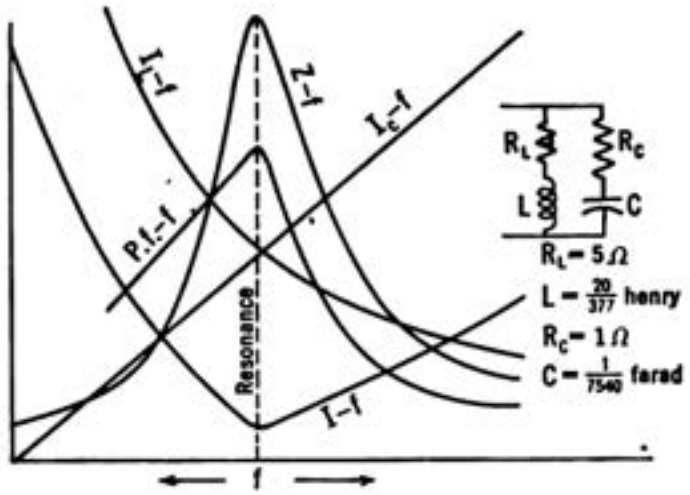

FIG. 28. Parallel resonance by varying frequency.

thus shown that within limits there are definite values of R_L which will bring the circuit to resonance at some particular values of frequency, L, C, and R_C. Also, for resonance,

$$R_C = \sqrt{\frac{R_L{}^2}{\omega^2 LC} - \frac{1}{\omega^2 C^2} + \frac{L}{C}} \tag{44}$$

Equation (44) shows that, for those values of parameters which make the quantity under the radical positive, resonance may be produced by choosing the proper value of R_C.

In contrast to the series circuit, where resistances have no part in determining the frequency of resonance, the resistances of a parallel circuit are of signal importance in determining the frequency of resonance, even to the extent of making resonance either possible or impossible to attain. Physically this can be understood when it is remembered that, with a certain quadrature component of current in

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the condensive branch, some sufficiently large value of R_L will prevent a resultant current in the inductive branch from flowing, which is as much as the quadrature current in the condensive circuit even when the inductance is zero. Under such conditions it is apparent that inserting inductance will do nothing but make the current in the inductive branch still smaller and hence contribute nothing toward making resonance possible. Such a case was discussed with reference to Fig. 26 when $I_C \sin \theta_C$ was greater than $V/2R_L$. Figure 26, which is simply a vector diagram, shows that $I_L \sin \theta_L$ can never be made as large as $I_C \sin \theta_C$ if $V/2R_L$ is less than $I_C \sin \theta_C$. A similar situation obtains for the condensive branch.

Problem 8. Draw the vector diagram and show the locus of I_L as X_L is varied, when $R_C = 1$ ohm, $X_C = 10$ ohms, $R_L = 6$ ohms, and the impressed voltage 100 volts for a circuit as shown in Fig. 24. Repeat the problem when R_L is changed to 4 ohms. What is the largest possible quadrature component of current in the inductive branch as X_L is varied in each case? In which case can resonance be produced? Why?

Ans.: 8.33 amperes, 12.5 amperes, resonance for 4-ohm case only.

A Simple Form of Wave Trap. Resonance phenomena as presented in the foregoing articles form the basis upon which many circuits used in both wire and wireless communication operate. They are especially adapted to selective circuits such as those for filters and oscillators. A parallel combination of capacitance and inductance, along with its incidental resistance, can be made into an effective band eliminator, suppressor, or wave trap. The impedance of such a branch (from a to b in Fig. 29), where the resistance of the capacitance is negligibly small and R_L is very small compared to L, is most easily found by taking the reciprocal of the resultant admittance. Since the branches are tuned for parallel resonance, the resultant admittance is conductance only. Thus

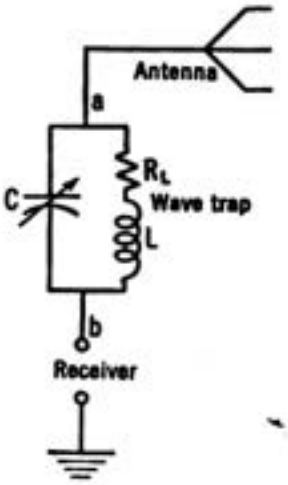

Fig. 29. Simple form of wave trap.

$$Y_m = \frac{R_L}{Z_L^2} \tag{45}$$

and

$$Z_m = \frac{1}{Y_m} = \frac{Z_L^2}{R_L} \tag{46}$$

Since $R_L^2 \ll \omega^2 L^2$,

$$Z_m = \frac{\omega^2 L^2}{R_L} \tag{47}$$

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In a previous article it was shown that when $R_L = R_C = 0$ the resonant frequency is practically

$$f_m = \frac{1}{2\pi\sqrt{LC}} \quad \text{or} \quad \omega = \omega_m = 2\pi f_m = \frac{1}{\sqrt{LC}} \tag{48}$$

Substituting (48) in (47) gives the impedance at resonance:

$$Z_m = \frac{L}{CR_L} \tag{49}$$

When used as a wave trap, the parallel combination of inductance and capacitance is placed in series with the antenna lead as shown in Fig. 29. At the resonant frequency the dynamic resistance of the wave trap is very nearly equal to L/CR_L [equation (49)]. Experience has shown that within the standard broadcast band the dynamic resistance at the frequency f_m can be made about 10 times the impedance at frequencies ± 20 kc from f_m. Thus the wave trap acts as a band suppressor or eliminator.

Problem 9. A typical coil used in the broadcast band for a wave trap like that in Fig. 29 has $L = 250 \times 10^{-6}$ henry and a ratio of reactance to resistance at 10^6 cycles of 170. Assuming the resistance of the condenser to be zero, calculate the following:

(*a*) C to produce resonance at 1000 kc from equation (39).
(*b*) C to produce resonance at 1000 kc from equation (48).
(*c*) Impedance of the wave trap from *a* to *b* when adjusted for parallel resonance at 1000 kc.
(*d*) Impedance of the wave trap to 990 kc when in resonance for 1000 kc.
(*e*) The ratio of the impedances for (*c*) to (*d*).

Ans.: 101.3 $\mu\mu$f, 101.3 $\mu\mu$f, 267,000 ohms, 75,100 ohms, 3.56.

A Singular Case of Parallel Resonance. For some values of the parameters R_L, R_C, L, and C connected as in Fig. 24, the circuit is in resonance for all frequencies. This may be shown as follows. From equation (39) the condition for parallel resonance is

$$\frac{\omega L}{R_L^2 + \omega^2 L^2} = \frac{\dfrac{1}{\omega C}}{R_C^2 + \dfrac{1}{\omega^2 C^2}}$$

$$= \frac{1}{\omega C}\,\frac{\omega^2 C^2}{R_C^2\omega^2 C^2 + 1} = \frac{\omega C}{1 + \omega^2 C^2 R_C^2}$$

or

$$\frac{1}{\dfrac{R_L^2}{L} + \omega^2 L} = \frac{1}{\dfrac{1}{C} + \omega^2 C R_C^2} \tag{50}$$

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To be independent of frequency an inspection of equation (50) will show that the following two conditions must be imposed simultaneously.

Condition 1 $$\frac{R_L^2}{L} = \frac{1}{C} \quad \text{or} \quad R_L = \sqrt{\frac{L}{C}}$$

Condition 2 $$CR_C^2 = L \quad \text{or} \quad R_C = \sqrt{\frac{L}{C}}$$

Hence for resonance at all frequencies

$$R_L = R_C = \sqrt{\frac{L}{C}} \tag{51}$$

Since the circuit is in resonance (resultant susceptance = 0), its admittance must be the resultant conductance and, because $R_L = R_C$, $\omega = 1/\sqrt{LC}$. Therefore

$$Y_m = g_m = \frac{R_L}{Z_L^2} + \frac{R_C}{Z_C^2} = \frac{\sqrt{\frac{L}{C}}}{\frac{L}{C} + \omega^2 L^2} + \frac{\sqrt{\frac{L}{C}}}{\frac{L}{C} + \frac{1}{\omega^2 C^2}} = \sqrt{\frac{C}{L}}$$

and

$$Z_m = \sqrt{\frac{L}{C}} \tag{52}$$

Equation (52) shows that the impedance of the circuit is also independent of frequency. The preceding demonstration has shown that, when $R_L = R_C = \sqrt{L/C}$, a circuit arrangement like that in Fig. 24 is in resonance for all frequencies and offers the same impedance $\sqrt{L/C}$ to all frequencies.

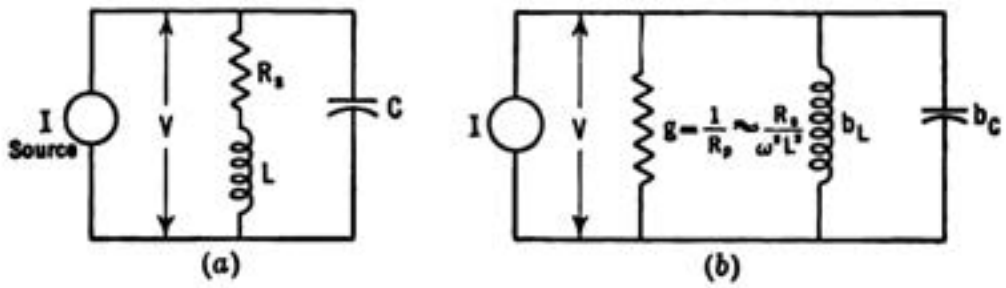

FIG. 30. Circuit shown in (b) is the equivalent of that shown in (a).

The Q of Parallel Circuits. In vacuum tube circuit analysis one frequently encounters the circuit arrangement which reduces essentially to that shown in Fig. 30a, namely, a coil and condenser connected in parallel and energized with a current source. In the practical cases

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which will be encountered, the resistance of the coil, R_s, is very small compared to ωL; therefore

$$R_s^2 \ll \omega^2 L^2$$

Under these conditions the transformation of the series R_s and L to a parallel combination of g and b_L as suggested in Fig. 22 transforms Fig. 30*a* to that shown in Fig. 30*b*, where

$$g = \frac{1}{R_p} \approx \frac{R_s}{\omega^2 L^2}$$

$$b_L \approx \frac{1}{\omega L} \quad \text{and} \quad b_C = \omega C$$

It should be noted that b_L and b_C are magnitudes of the inductive and capacitive susceptances, respectively. Where purely reactive branches are placed in parallel, as in Fig. 30*b*, it is convenient to write $\mathbf{Y} = g + j(b_C - b_L)$ and thereby obtain an expression which is directly analogous to $\mathbf{Z} = R + j(X_L - X_C)$. In Fig. 30*b* we find

$$V = \frac{I}{\sqrt{g^2 + (b_C - b_L)^2}} = \frac{I}{\sqrt{g^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}} \tag{53}$$

Comparing the above equation with equation (16), we observe a correspondence which allows us to interpret Fig. 16 as the voltage response versus ω. This response has a maximum value of I/g, and the analysis following equation (16) can with a few obvious changes in notation be employed to determine the band width of the selective circuit shown in Fig. 30.

Since g in equation (53) corresponds to R in equation (16), and C to L, and L to C, we may write for the parallel circuit

$$\Delta\omega = \omega_2 - \omega_1 = \frac{g}{C} \tag{54}$$

either by analogy with equation (25) or by direct computation.

Employing the same definition of Q as given on page 108 (namely, $Q = \omega_m/\Delta\omega$) and remembering that $\omega_m \approx 1/\sqrt{LC}$ when the resistances of the parallel branches are small relative to the reactances, we find that for the parallel circuit

$$Q_p = \frac{\omega_m}{\Delta\omega} = \frac{\omega_m C}{g} = \frac{1}{g\omega_m L} = \frac{1}{g}\sqrt{\frac{C}{L}} \tag{55}$$

In elementary analytical calculations, it is quite customary to treat

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both R_s of equation (27) and g of equation (55) as constants, that is, independent of frequency. Neither of these approximations, however, agrees with the physical facts as accurately as treating Q as constant over a reasonable frequency range centered on the resonant frequency, f_m, since R_s increases with increases in ω. Over certain ranges of the radio-frequency band, R_s varies almost linearly with respect to ω, and under these conditions we may set $R_s = k\omega$ with the following results.

$$Q_s = \frac{\omega L}{R_s} = \frac{\omega L}{k\omega} = \text{constant}$$

$$Q_p = \frac{1}{g\omega L} = \frac{\omega^2 L^2}{R_s \omega L} = \text{constant}$$

Example 7. In Fig. 30*a* it will be assumed that the coil has a series resistance, R_s, of 25.1 ohms and a self-inductance of 10 millihenrys. This coil is to be resonated at 20 kc with the condenser C.

Let it be required to find the equivalent parallel circuit resistance, $1/g$, the tuning capacitance, the Q of the parallel circuit, and maximum voltage response per milliampere of current I.

$$g \approx \frac{R_s}{\omega_m^2 L^2} = \frac{25.1}{(2\pi \times 20{,}000)^2 \times (0.01)^2} = 1.59 \times 10^{-5} \text{ mho}$$

$$R_p = \frac{1}{g} \approx 62{,}900 \text{ ohms}$$

$$C \approx \frac{1}{L\omega_m^2} = \frac{1}{0.01(2\pi \times 20{,}000)^2} = 0.00633 \times 10^{-6} \text{ farad}$$

$$Q = \frac{\omega_m C}{g} = \frac{\omega_m \omega_m^2 L^2}{R_s \omega_m^2 L} = \frac{\omega_m L}{R_s} = 50$$

$$\text{Maximum voltage response} = \frac{0.001}{g} = 62.9 \text{ volts per milliampere}$$

A certain class of vacuum tube, namely, the pentode, can under certain operating conditions be made to function as current source supplying up to several milliamperes of alternating current simply by energizing one of its electrodes (the control grid) with a small a-c voltage. Since this small a-c voltage is often considerably less than 1 volt in magnitude, it is plain that large voltage amplifications may be obtained from the circuit configuration shown in Fig. 30*b* if the current source takes the form of a pentode. Moreover this circuit has a reasonable degree of selectivity since the band width between the $0.707V_{max}$ points on the response curve is

$$\Delta\omega = \frac{g}{C} = \frac{1.59 \times 10^{-5}}{0.00633 \times 10^{-6}} = 2510 \text{ radians per second}$$

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On this basis of reckoning, the per unit band width is

$$\frac{\Delta\omega}{\omega_m} = \frac{2510}{2\pi \times 20{,}000} = 0.02$$

Series-Parallel Circuits. The series-parallel circuit illustrated in Fig. 31 is a combination of the series and parallel circuits which have been discussed previously. The principles previously considered apply to the analysis of series-parallel circuits. These are (1) impedances in series are added in complex form and (2) admittances of those branches which are in parallel must be added in complex form. To illustrate, consider Fig. 31. The admittances of impedances Z_4 and Z_5 are added in complex form, and the reciprocal of the resultant admittance is then the equivalent impedance of section B. An alternative method of finding the impedance of section B, as was previously shown, is to use $\mathbf{Z}_B = \mathbf{Z}_4\mathbf{Z}_5/(\mathbf{Z}_4 + \mathbf{Z}_5)$. Through a similar procedure the impedance of section A is determined. The impedances of section A, section B, and Z_1 are in series and are, therefore, added in complex form. This procedure yields the equivalent or resultant impedance $\mathbf{Z}_e$ of the series-parallel circuit. The current $\mathbf{I}$ may then be found from $\mathbf{V}/\mathbf{Z}_e$.

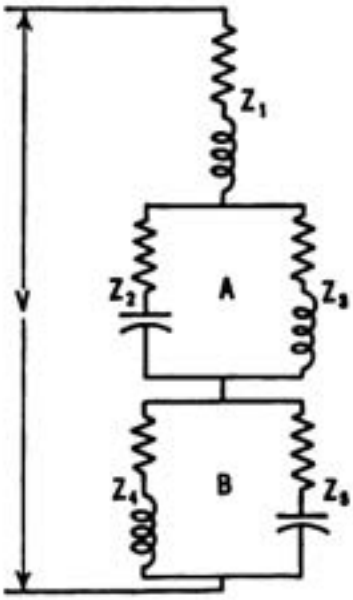

FIG. 31. Impedances in series-parallel.

Determination of Branch Currents and Voltages. After the resultant current is determined, the process is reversed to determine branch voltages and currents. The general procedure is to subtract the voltage drop calculated for the known current and the impedance through which it flows from the applied voltage to obtain the voltage drop across the remainder of the circuit, or to calculate the drops across various sections from the resultant current and the equivalent impedance of the branch through which the current flows. For example, in Fig. 31, the drop across section A is the product of equivalent impedance $\mathbf{Z}_A$ of that section and the current $\mathbf{I}$. The current through each of the parallel impedances is then determined by dividing this drop by the impedance of the particular branch or, if the admittances have been determined, by multiplying the voltage drop across the branch by the particular branch admittance. A similar procedure can be followed for section B, and so on.

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Example 8. Calculate current, power, and power factor for each impedance shown in Fig. 32, and the total current and power and the power factor of the whole combination.

$$\mathbf{Y}_{ab} = \frac{1}{6 - j8} = 0.06 + j0.08 \text{ mho}$$

$$\mathbf{Y}_{cd} = \frac{1}{4 + j3} = 0.16 - j0.12 \text{ mho}$$

$$\mathbf{Y}_{fg} = \mathbf{Y}_{ab} + \mathbf{Y}_{cd} = 0.22 - j0.04 \text{ mho}$$

$$\mathbf{Z}_{fg} = \frac{1}{\mathbf{Y}_{fg}} = \frac{1}{(0.22 - j0.04)} \frac{(0.22 + j0.04)}{(0.22 + j0.04)} = 4.4 + j0.8 \text{ ohms}$$

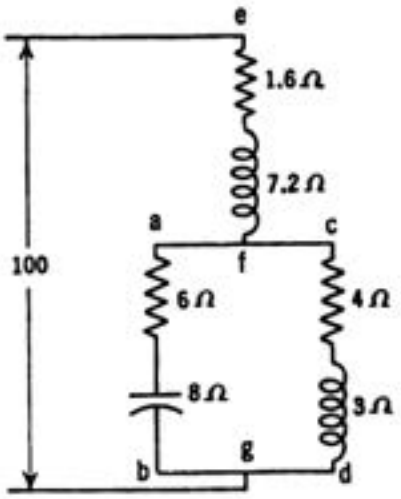

FIG. 32. Circuit for example 8.

An alternative method is:

$$\mathbf{Z}_{fg} = \frac{\mathbf{Z}_{ab}\mathbf{Z}_{cd}}{\mathbf{Z}_{ab} + \mathbf{Z}_{cd}} = \frac{(6 - j8)(4 + j3)}{6 - j8 + 4 + j3} = 4.4 + j0.8 \text{ ohms}$$

$$\mathbf{Z}_{eg} = \mathbf{Z}_{ef} + \mathbf{Z}_{fg} = 1.6 + j7.2 + 4.4 + j0.8 = 6 + j8 \text{ ohms}$$

$$\mathbf{I} = \frac{100\angle 0^\circ}{6 + j8} = 6 - j8 = 10\angle -53.2^\circ \text{ amperes}$$

$$P = vi + v'i' = 6 \times 100 + 0 \times 8 = 600 \text{ watts}$$

$$\text{P.f.} = \frac{600}{100 \times 10} = 0.6 \quad \text{or} \quad \frac{R}{Z} = \frac{6}{10} = 0.6 \text{ lag}$$

$$\mathbf{V}_{ef} = \mathbf{I}_{ef}\mathbf{Z}_{ef} = (6 - j8)(1.6 + j7.2) = 67.2 + j30.4$$
$$= 73.8\angle 24.4^\circ \text{ volts}$$

$$\mathbf{V}_{fg} = \mathbf{V} - \mathbf{I}_{ef}\mathbf{Z}_{ef} = 100 - 67.2 - j30.4 = 32.8 - j30.4$$
$$= 44.7\angle -42.8^\circ \text{ volts}$$

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Or, more directly,

$$\mathbf{V}_{fg} = \mathbf{I}\mathbf{Z}_{fg} = (6 - j8)(4.4 + j0.8) = 32.8 - j30.4$$
$$= 44.7\angle{-42.8^\circ} \text{ volts}$$

$$\mathbf{I}_{ab} = \mathbf{V}_{fg}\mathbf{Y}_{ab} = (32.8 - j30.4)(0.06 + j0.08)$$
$$= 4.4 + j0.8 = 4.48\angle{10.3^\circ} \text{ amperes}$$

$$\mathbf{I}_{cd} = \mathbf{V}_{fg}\mathbf{Y}_{cd} = (32.8 - j30.4)(0.16 - j0.12)$$
$$= 1.6 - j8.8 = 8.95\angle{-79.7^\circ} \text{ amperes}$$

or $$\mathbf{I}_{cd} = \mathbf{I} - \mathbf{I}_{ab} = 6 - j8 - 4.4 - j0.8 = 1.6 - j8.8$$
$$= 8.95\angle{-79.7^\circ} \text{ amperes}$$

The powers in the various branches may now be determined in terms of principles previously considered.

$$P_{ab} = vi + v'i' = (32.8)(4.4) + (-30.4)(0.8)$$
$$= 144.32 - 24.32 = 120 \text{ watts}$$

$$P_{cd} = (32.8)(1.6) + (-30.4)(-8.8)$$
$$= 52.48 + 267.52 = 320 \text{ watts}$$

$$P_{ef} = (67.2)(6) + (30.4)(-8) = 403.2 - 243.2 = 160$$

or $$P_{ef} = I^2r = (6^2 + 8^2)(1.6) = 160 \text{ watts}$$

$$P_{eg} = 100 \times 6 = 600 \text{ watts}$$

Check: $$P = P_{ab} + P_{cd} + P_{ef} = 120 + 320 + 160 = 600 \text{ watts}$$

$$Pf_{ab} = \frac{R_{ab}}{Z_{ab}} = \frac{6}{\sqrt{6^2 + 8^2}} = 0.6 \text{ lead}$$

$$Pf_{cd} = \frac{R_{cd}}{Z_{cd}} = \frac{4}{\sqrt{4^2 + 3^2}} = 0.8 \text{ lag}$$

Problem. 10. Study through the details of the above example and draw a vector diagram of $\mathbf{V}$, $\mathbf{I}$, $\mathbf{V}_{ef}$, $\mathbf{I}_{ab}$, $\mathbf{I}_{cd}$, and $\mathbf{V}_{fg}$. Employ a voltage scale of 25 volts per inch and a current scale of 2 amperes per inch.

Series-Parallel Tuning. It has been shown that for certain conditions parallel resonance yields maximum impedance and that series resonance gives minimum impedance. These facts suggest that a combination of these two phenomena may be used to exaggerate the effect of some certain frequency and minimize the effect of another. An arrangement that does this is shown in Fig. 33. This procedure is known as series-parallel tuning. To illustrate, assume that two waves, one of 10,000 cycles and the other of 20,000 cycles, are impressed at *ab* and that it

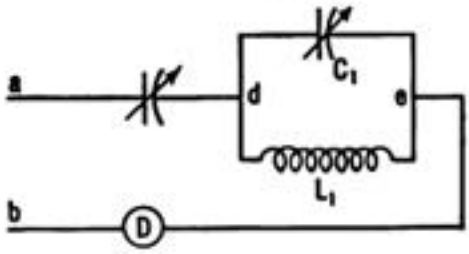

FIG. 33. Series-parallel tuning circuit.

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is desired to detect the 10,000-cycle wave at D. Obviously as much 10,000-cycle current through D as can be obtained is desired, and as little as possible of the 20,000-cycle wave is to be tolerated. Hence the parallel branches of capacitance and inductance are adjusted for parallel resonance at 20,000 cycles. Then the 20,000-cycle wave encounters a high impedance, and little current due to it will flow through D. For the 10,000-cycle wave a little thought will show that the parallel circuit acts as an inductance. If a capacitance is placed in series with the parallel circuit de and its reactance for the 10,000-cycle frequency is made equal to the equivalent inductive reactance of the parallel circuit de for this same frequency, the circuit from a to b will be in series resonance for the 10,000-cycle wave. The current through D for the 10,000-cycle wave, therefore, will be large, whereas parallel resonance from d to e for the 20,000-cycle frequency will allow only a small 20,000-cycle current to flow through D.

Example 9. Assume L_1 to have 0.005 henry inductance and 50 ohms resistance. Neglect resistance of the condensers. Parallel resonance for 20,000 cycles obtains when

$$b_L = b_C$$

$$\frac{\omega 0.005}{50^2 + \omega^2(0.005)^2} = \omega C_1$$

where

$$\omega = 2\pi\, 20{,}000 = 12.57 \times 10^4 \text{ radians per second}$$

$$C_1 = \frac{0.005}{50^2 + 0.005^2\omega^2} = 1.257 \times 10^{-8} \text{ farad}$$

$$\mathbf{Y}_{de} = g = \frac{50}{50^2 + 0.005^2\omega^2} = \frac{50}{397{,}300} \text{ mho}$$

$$\mathbf{Z}_{de} = \frac{397{,}300}{50} = 7946 \text{ ohms}$$

For 10,000 cycles,

$$\mathbf{Y}_{C1} = j2\pi\, 10{,}000 \times 1.257\, 10^{-8} = j79 \times 10^{-5} \text{ mho}$$

$$\mathbf{Y}_{L1} = \frac{1}{50 + j0.005 \times 2\pi\, 10{,}000} = 49.3 \times 10^{-5} - j310 \times 10^{-5} \text{ mho}$$

$$\mathbf{Y}_{de} = \mathbf{Y}_{C1} + \mathbf{Y}_{L1} = 49.3 \times 10^{-5} - j231 \times 10^{-5} \text{ mho}$$

$$\mathbf{Z}_{de} = \frac{10^5}{49.2 - j231} = 88.1 + j413 \text{ ohms}$$

Since 413 ohms is the equivalent reactance of the divided circuit, a condensive reactance of 413 ohms is required to produce series resonance. Then $\mathbf{Z}_{ab} = 88.1$ ohms for 10,000 cycles.

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For 20,000 cycles,

$$Z_{cd} = -\frac{j413}{2} = -j206.5 \text{ ohms}$$

$$Z_{ab} = 7946 - j206.5 \quad \text{or} \quad 7946 \text{ ohms approximately}$$

$$\frac{Z_{ab20,000}}{Z_{ab10,000}} = \frac{7946}{88.1} = 90.2$$

Hence for equal impressed voltages across ab, the value of the 20,000-cycle current will be about $\frac{1}{90}$ of the value of the 10,000-cycle current.

The student should devise the explanation to show that if the 10,000-cycle wave is to be suppressed and the 20,000-cycle wave detected, an inductance would have to be substituted for the condenser between a and d.

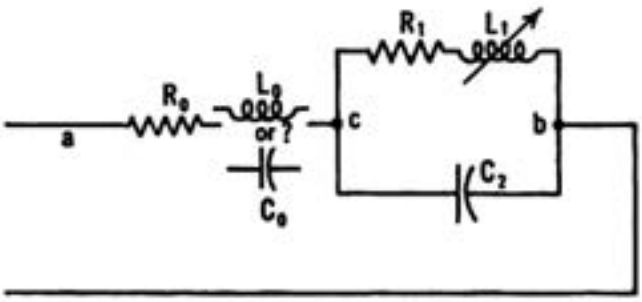

FIG. 34. See Problem 11.

Problem 11. The circuit ab of Fig. 34 is to pass a 45,000-cycle current with minimum impedance and is to block a 15,000-cycle current as effectively as possible. $R_0 = 20$ ohms, $R_1 = 40$ ohms, and $C_2 = 0.05$ μf are fixed. R_2, the resistance of the C_2 branch, is assumed to be negligibly small. L_1 is capable of being varied over the required range, it being assumed that the resistance of branch 1 is 40 ohms when L_1 is set at the desired value. Either a fixed C_0 *or* a fixed L_0 (of negligibly small resistance presumably) is to be placed in series with R_0 to accomplish the above-stated tuning effect.

(*a*) Solve for L_1, which will put the parallel circuit bc into parallel resonance at 15,000 cycles.

(*b*) Calculate the equivalent impedance from b to c at 45,000 cycles with L_1 set at its 15,000-cycle resonant value. Is bc predominantly capacitive or predominantly inductive at 45,000 cycles?

(*c*) What type of reactance (inductive or capacitive) must be placed in series with R_0 to put the branch ab into series resonance? Calculate the value of L_0 or C_0 which is required to put the branch ab into series resonance at 45,000 cycles.

(*d*) Assuming that the branch ab has been put into series resonance at 45,000 cycles, what is the actual impedance from a to b at 45,000 cycles? at 15,000 cycles?

Outline the above procedure for the reverse tuning effect, that is, for circuit ab to pass 15,000 cycles and block 45,000 cycles.

Ans.: (*a*) $L_1 = 2.17$ or 0.0835 millihenry. Use 2.17 for lower conductance.
(*b*) $Z_{bc} = 0.69 - j79.9$ ohms, predominantly capacitive.
(*c*) $L = 0.283$ millihenry.
(*d*) $Z_{ab45,000} = 20.69$ ohms. $Z_{ab15,000} = 1103$ ohms.

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Impedance Matching and Maximum Power Transfer. A common problem in impedance matching is to determine the load impedance which will allow the maximum power to be transferred to the load from some generating device having a constant generated voltage, E_g. Let Fig. 35 represent such an arrangement and consider R_1 to represent the sum of the internal resistance of the generating device and the resistance of the connecting lines. Also assume X_1 to be the combined reactance of the line and internal reactance of the generating device.

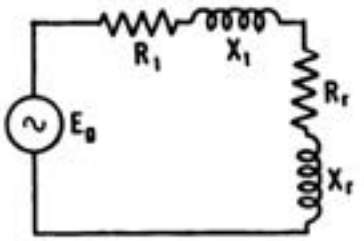

FIG. 35. Generator connected to a load through line impedance.

The solution is obtained by expressing the power at the receiver algebraically and then finding the maximum value of the expression. Let the receiver impedance be represented by R_r and X_r. If the receiver is a two-terminal network, R_r and X_r are its equivalent series parameters. Thus

$$I = \frac{E_g}{\sqrt{(R_1 + R_r)^2 + (X_1 + X_r)^2}}$$

$$P_r = I^2 R_r = \frac{E_g^2 R_r}{(R_1 + R_r)^2 + (X_1 + X_r)^2} \tag{56}$$

In order to make the derivation easily applicable to all conditions, the ratio of X_r/R_r will be represented by k. Then

$$X_r = kR_r$$

and

$$P_r = \frac{E_g^2 R_r}{(R_1 + R_r)^2 + (X_1 + kR_r)^2} \tag{57}$$

Setting $dP_r/dR_r = 0$ and solving for R_r give

$$R_r = \frac{Z_1}{\sqrt{1 + k^2}} \tag{58}$$

where $Z_1 = \sqrt{R_1^2 + X_1^2}$. Substituting equation (58) in equation (57), expanding the terms in the denominator, and simplifying give

$$P_{max} = \frac{E_g^2}{2Z_1\sqrt{1 + k^2} + 2(R_1 + kX_1)} \tag{59}$$

Equation (59) gives the maximum power for any value of k, the ratio of X_r/R_r. To find the value of k that yields the greatest maximum power, it is necessary simply to set $dP_{max}/dk = 0$ and solve for k.

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Then

$$k = \pm \frac{X_1}{R_1} \tag{60}$$

Substituting equation (60) in equation (59) yields

$$P_{\max \max} = \frac{E_g^2}{4R_1 + \frac{2}{R_1}(X_1^2 \pm X_1^2)}. \tag{61}$$

It is obvious from equation (61) that the greatest maximum power will occur when the minus sign is used or when $k = -X_1/R_1$. For this case

$$P_{\max \max} = \frac{E_g^2}{4R_1} \tag{62}$$

Since R_r cannot be negative in a dissipative network, X_r must be minus to make k negative. Hence X_r is condensive if X_1 is inductive, and vice versa. Also for this condition, from equation (58),

$$R_r = \frac{\sqrt{R_1^2 + X_1^2}}{\sqrt{1 + X_1^2/R_1^2}} = \frac{R_1\sqrt{R_1^2 + X_1^2}}{\sqrt{R_1^2 + X_1^2}} = R_1$$

Also for the greatest maximum power $X_r = kR_r = -(X_1/R_1)\ R_r = -(X_1/R_1)\ R_1 = -X_1$. Hence the receiver impedance must equal the generator plus line impedance, and the reactances must be of opposite signs. In short, the receiver impedance must be the conjugate of the combined generator and line impedance. As would be expected, the circuit is tuned for series resonance. Since R_1 and R_r are equal and the current is the same in both, one-half the power input is dissipated in the generator and line, and one-half is given to the receiver. The efficiency of transmission for the greatest maximum power is, therefore, 50 per cent.

Constant potential power systems are not designed to operate on the basis of maximum power transfer, but most low-current circuits are so designed. Impedance matching is, therefore, of considerable importance in all communication networks, and much attention has been given to this phase of circuit analysis by communication engineers.

Problem 12. A generating device has an impedance of $0.5 + j1$ ohms and is connected to a load by a line of $1.5 + j4$ ohms. At what load will maximum power transfer be realized? If the generated voltage is 20 volts, what is the power received by the load when adjusted for maximum power transfer? Find the line loss and loss in the generating device.

Ans.: $Z_{\text{load}} = 2 - j5$ ohms. $P_{\max \max} = 50$ watts at receiver.
$P_{\text{line loss}} = 37.5$ watts. $P_{\text{gen loss}} = 12.5$ watts.

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Problem 13. If a load impedance having a ratio of $X/R = 5$ is used at the end of the line in Problem 12, find the load impedance for maximum power transfer. What is the maximum power the load can receive?

Ans.: For positive k, $P = 3.675$ watts. $\mathbf{Z}_L = 1.056 + j5.28$ ohms.
For negative k, $P = 45.2$ watts. $\mathbf{Z}_L = 1.056 - j5.28$ ohms.

Networks. Resistors, inductors, capacitors, vacuum tubes, and sources of emf may be linked together in all conceivable forms. Most of the combinations, and almost all of those which contain emf's in more than one branch, cannot be solved by simple series-parallel circuit theory alone, as previously outlined in this chapter. Such combinations may be classed as networks. Networks that contain sources of emf or power are sometimes called active, whereas those that do not contain any internal emf's or sources of power are called passive networks. Networks are said to be linear when the current in all branches is directly proportional to the driving voltage or emf impressed. Thus a network containing iron-core inductance coils and resistances that vary with current strength are non-linear. Networks may be composed of bilateral or unilateral elements. *Bilateral* elements are those circuit elements like inductance, resistance, and capacitance which transmit current equally well in either direction. *Unilateral* elements are those circuit elements like rectifiers and vacuum tubes which transmit effectively in only one direction.

Through the application of a few simple network theorems, certain combinations of circuit elements which are not solvable by ordinary series-parallel circuit theory directly may be solved quite readily.

The Superposition Theorem. The current which flows at any point or the voltage between any two points in a linear network, as a result of the simultaneous action of a number of emf's distributed throughout the network, is the sum of the currents or voltages at these points which would exist if each source of emf were considered separately, each of the other sources being replaced at that time by their internal impedances. This theorem states that each emf in a network may be treated as acting independently and the current in any branch of a network due to the simultaneous action of all emf's is the vector sum of the currents in the particular branch produced by each emf acting separately. It is important to keep all circuit elements closed or connected as they are in the network. All the emf's except the one for which currents are being calculated are assumed to be zero. Any impedances associated with the source of emf must be left connected in the network whether the emf is assumed to be zero or whether it is the one considered as an independent driving voltage.

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Example 10. Calculate the current in branch *bc* for the network of Fig 36.

Solution: Assume

$$E_{g2} = 0$$

$$Z_{cf} = \frac{Z_{cef}Z_{cd}}{Z_{cef} + Z_{cd}} = \frac{(3 - j3)(2 + j4)}{(3 - j3 + 2 + j4)} = 3.69 + j0.462 \text{ ohms}$$

$$Z_{bf} = Z_{abc} + Z_{cf} = 1 + j3 + 1 - j3 + 3.69 + j0.462$$

$$= 5.69 + j0.462 \text{ ohms}$$

$$I_{bc1} = \frac{100 + j0}{5.69 + j0.462} = 17.43 - j1.417 \text{ amperes}$$

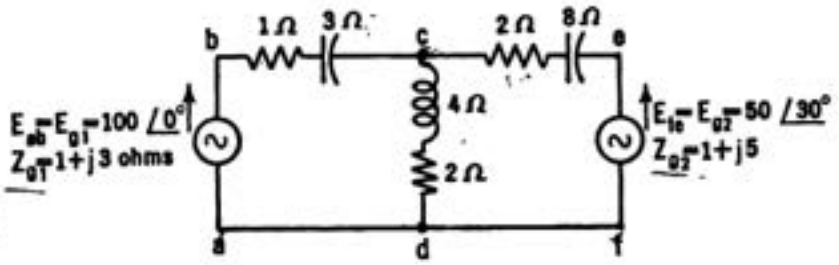

FIG. 36. See example 10.

Now assume

$$E_{g1} = 0$$

$$Z_{ca} = \frac{Z_{cba}Z_{cd}}{Z_{cba} + Z_{cd}} = \frac{(1 - j3 + 1 + j3)(2 + j4)}{2 + 2 + j4} = 1.5 + j0.5 \text{ ohms}$$

$$Z_{ea} = Z_{fec} + Z_{ca} = 1 + j5 + 2 - j8 + 1.5 + j0.5 = 4.5 - j2.5 \text{ ohms}$$

$$I_{ec} = \frac{50\ \underline{/30^\circ}}{4.5 - j2.5} = 5 + j8.34 \text{ amperes}$$

$$I_{cb2} = \frac{I_{ec}Z_{ca}}{Z_{cba}} = \frac{(5 + j8.34)(1.5 + j0.5)}{2} = 1.66 + j7.50 \text{ amperes}$$

$$I_{bc} = I_{bc1} + I_{bc2} = I_{bc1} - I_{cb2} = 17.43 - j1.417 - 1.66 - j7.50$$

$$= 15.77 - j8.917 \text{ amperes}$$

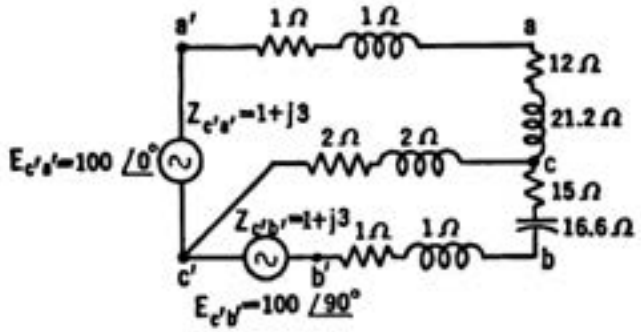

FIG. 37. See Problem 14.

Problem 14. Calculate the current in branch *ac* for Fig. 37.

Ans.: $I_{ac} = 1.76 - j3.14$ amperes.

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Reciprocity Theorem. If any source of emf, **E**, located at one point in a network composed of linear bilateral circuit elements, produces a current **I** at a second point in the network, the same source of emf, **E**, acting at the second point will produce the same current **I** at the first point.

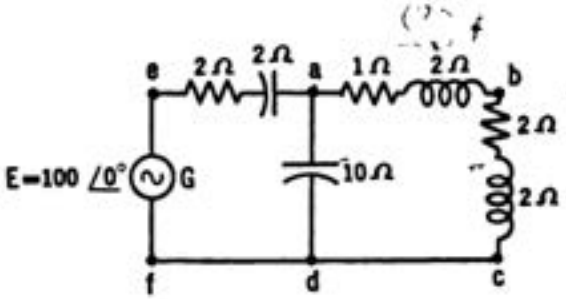

FIG. 38. See example 11.

Example 11. The application of the above theorem may be illustrated as follows. Given the network shown in Fig. 38. The reciprocity theorem states that, if 100 volts are inserted in *bc* and branch *ef* is left closed, the current flowing in *ef* will then be exactly the same as the current that flowed in *bc* when this same voltage was applied at *ef*. To verify this theorem the current in *bc* will be calculated for the 100 volts at *ef*.

$$\mathbf{Z}_{ac} = \frac{\mathbf{Z}_{abc}\mathbf{Z}_{ad}}{\mathbf{Z}_{abc} + \mathbf{Z}_{ad}} = \frac{(3 + j4)(-j10)}{3 + j4 - j10} = 6.67 + j3.33 \text{ ohms}$$

$$\mathbf{Z}_{ec} = \mathbf{Z}_{ea} + \mathbf{Z}_{ac} = 2 - j2 + 6.67 + j3.33 = 8.67 + j1.33 \text{ ohms}$$

$$\mathbf{I}_{ec} = \frac{\mathbf{V}_{fe}}{\mathbf{Z}_{ec}} = \frac{100 + j0}{8.67 + j1.33} = 11.27 - j1.732 \text{ amperes}$$

$$\mathbf{V}_{ac} = \mathbf{I}_{ec}\mathbf{Z}_{ac} = (11.27 - j1.732)(6.67 + j3.33) = 81 + j26 \text{ volts}$$

$$\mathbf{I}_{bc} = \frac{\mathbf{V}_{ac}}{\mathbf{Z}_{abc}} = \frac{81 + j26}{3 + j4} = 13.88 - j9.84 \text{ amperes}$$

Now assume that 100 volts are inserted in branch *cb* and that *ef* remains closed. The current in *ef* will be calculated by a procedure similar to that shown above.

$$\mathbf{Z}_{af} = \frac{(2 - j2)(-j10)}{(2 - j2 - j10)} = 1.352 - j1.892 \text{ ohms}$$

$$\mathbf{Z}_{caf} = 3 + j4 + 1.352 - j1.892 = 4.352 + j2.108 \text{ ohms}$$

$$\mathbf{I}_{caf} = \frac{100 + j0}{4.352 + j2.108} = 18.6 - j9.02 \text{ amperes}$$

$$\mathbf{V}_{af} = (18.6 - j9.02)(1.352 - j1.892) = 8.07 - j47.4 \text{ volts}$$

$$\mathbf{I}_{ef} = \frac{8.07 - j47.4}{2 - j2} = 13.88 - j9.84 \text{ amperes}$$

which is the same as the current $\mathbf{I}_{bc}$ above.

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From the reciprocity theorem it follows that the ratio of the emf in branch 1 of a linear bilateral network to the current it causes in branch 2 is the same as the ratio of a voltage placed in branch 2 to the current it would cause in branch 1. This ratio of voltage in one branch to the current in another branch is called the *transfer impedance*.

Problem 15. Make use of the first set of calculations for Fig. 38 when the emf is inserted in *fe* and with the aid of the reciprocity theorem find the current in *fe* if 100 volts are inserted in branch *ad*. Verify your result by actually calculating the current in *fe* when 100 volts are inserted in branch *ad*.

Ans.: $-2.6 + j8.1$ amperes.

Thévenin's Theorem. If an impedance **Z** is connected between any two points of an energized network, the resulting current **I** through this impedance is the ratio of the potential difference **V** between these points, prior to connection, divided by the sum of the connected impedance **Z** and the impedance $\mathbf{Z}_0$, where $\mathbf{Z}_0$ is the impedance of the rest of the network looking back into the network from the points across which impedance **Z** is connected. In evaluating $\mathbf{Z}_0$ all sources of emf must be assumed to be zero and replaced by their internal impedances.

Example 12. For the network shown in Fig. 39 the voltage drop at *ab* is found as follows:

$$\mathbf{I}_{fecd} = \frac{100\angle 0^\circ}{10\angle -90^\circ} = 10\angle 90^\circ \text{ amperes}$$

$$\mathbf{V}_{cd} = \mathbf{V}_{ab} = (10\angle 90^\circ)(20\angle -90^\circ) = 200\angle 0^\circ \text{ volts}$$

Now suppose that the current through a load impedance $\mathbf{Z}_L = 30\angle 0^\circ$ ohms connected across *ab* is desired. According to Thévenin's theorem, the current is $\mathbf{V}_{ab}$ divided by the sum of the load impedance and the impedance looking into the network at *ab*. Thus the impedance looking into the network at *ab* (designated by $\mathbf{Z}_0$) when the emf in the branch *ef* is assumed zero is:

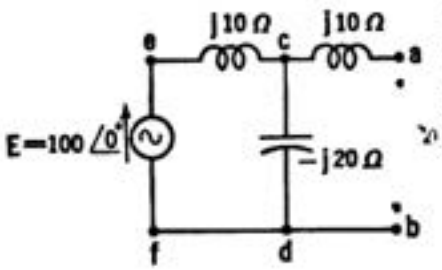

FIG. 39. See example 12.

$$\mathbf{Z}_0 = j10 + \frac{(j10)(-j20)}{j10 - j20} = j30 \text{ ohms}$$

According to Thévenin's theorem:

$$\mathbf{I}_{load} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_0 + \mathbf{Z}_L} = \frac{200\angle 0^\circ}{j30 + 30\angle 0^\circ} = 4.72\angle -45^\circ \text{ amperes}$$

This result may be checked by the usual series-parallel circuit theory as follows:

$$\mathbf{Z}_{ab} = \frac{(30 + j10)(-j20)}{30 + j10 - j20} = 12 - j16 \text{ ohms}$$

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$$\mathbf{Z}_{ab} = j10 + 12 - j16 = 12 - j6 \text{ ohms}$$

$$\mathbf{I}_{ac} = \frac{100 + j0}{12 - j6} = 6.667 + j3.333 \text{ amperes}$$

$$\mathbf{V}_{cb} = (6.667 + j3.333)(12 - j16) = 133.3 - j66.67 \text{ volts}$$

$$\mathbf{I}_{ab} = \frac{133.3 - j66.67}{30 + j10} = 3.333 - j3.333 = 4.72\underline{/-45^\circ} \text{ amperes}$$

which is the same as that obtained by Thévenin's theorem.

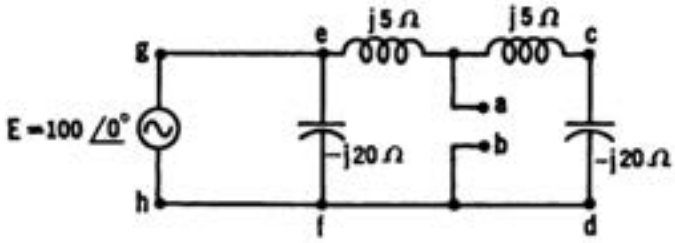

FIG. 40. See Problem 16.

Problem 16. In the circuit of Fig. 40, the impedance of the generator is assumed low enough so that it may be considered to be zero. Find the impedance $\mathbf{Z}_0$ looking into the terminals *ab* as employed in applying Thévenin's theorem. As may be easily shown, the drop across *ab* is $150\underline{/0^\circ}$ volts. Calculate the current in a load impedance $\mathbf{Z}_L = 10 - j7.5$ ohms connected across *ab*.

Ans.: $\mathbf{Z}_0 = j7.5$ ohms, $\mathbf{I}_L = 15\underline{/0^\circ}$ amperes.

The Nodal Method. The method ordinarily employed in analyzing circuits consists in establishing the necessary number of *voltage equilibrium equations* and solving for the currents. In many cases, particularly in vacuum tube circuits, it is desirable to employ *current equilibrium equations* and solve for the voltages. The latter method, known as the nodal method, consists essentially in writing Kirchhoff's current law at the nodes or junctions of the network the required number of times to effect a solution for various voltages in which we might be interested.

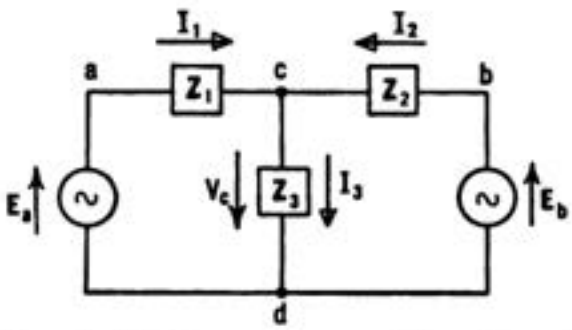

FIG. 41. Voltage sources may be transformed to equivalent current sources shown in Figs. 42 and 43.

In its simplest sense, a *node* of a network is any accessible terminal which is at a significant potential difference with respect to the other terminals. In this sense, the network shown in Fig. 41 *might be* considered a four-node network having nodes *a*, *b*, *c*, and *d*. Only the junction points (*c* and *d*) of

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the network, however, *need be* considered nodes since the number of independent nodes is the number of junctions minus one. This will become more evident as we proceed.

Before the nodal method of analysis can be applied to voltage sources having internal impedance, these voltage sources must be transformed to equivalent current sources in accordance with the following principles. (If a specified voltage source is assumed to have zero impedance, it follows that the potential difference between the terminals of the generator is specified and hence does not enter the analysis as an unknown potential difference.)

In order to illustrate the transformation of a voltage source having internal impedance to an equivalent current source, let us suppose that $\mathbf{Z}_1$ of Fig. 41 is actually the internal impedance of the E_a voltage generator, thus eliminating point a as a node. Let V_c be the potential of node c relative to node d. Applying Kirchhoff's voltage law we have

$$\mathbf{I}_1\mathbf{Z}_1 + \mathbf{V}_c = \mathbf{E}_a \tag{63}$$

or

$$\mathbf{I}_1 = \frac{\mathbf{E}_a}{\mathbf{Z}_1} - \frac{\mathbf{V}_c}{\mathbf{Z}_1} \tag{64}$$

It will be observed in equation (63) that the inclusion of the potential of node d (V_d) is unnecessary and in general any node may be selected as a reference node from which to reckon all other nodal potentials.

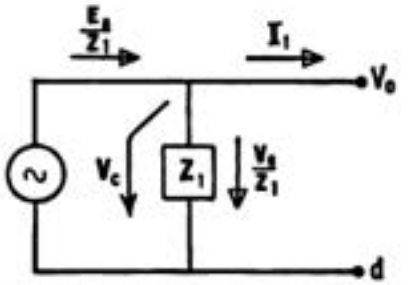

FIG. 42. Equivalent current source of E_a voltage source of Fig. 14.

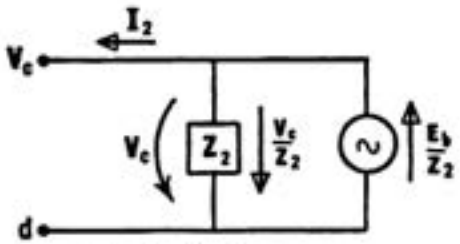

FIG. 43. Equivalent current source of E_b voltage source of Fig. 41.

If $\mathbf{E}_a$ and $\mathbf{Z}_1$ are specified quantities, equation (64) states that the current flowing into node c (I_1) is equal to a specified current ($\mathbf{E}_a/\mathbf{Z}_1$) minus a current ($\mathbf{V}_c/\mathbf{Z}_1$). The specified current ($\mathbf{E}_a/\mathbf{Z}_1$) may be considered as a current source across nodes c and d, provided that a $\mathbf{Z}_1$ path is placed in parallel with this source to account for the ($\mathbf{V}_c/\mathbf{Z}_1$) current in equation (64). Thus the voltage source $\mathbf{E}_a$ in series with $\mathbf{Z}_1$ shown in Fig. 41 may be replaced with the circuit configuration shown in Fig. 42. In a similar manner the $\mathbf{E}_b$ source and the impedance $\mathbf{Z}_2$ may be replaced with the configuration shown in Fig. 43.

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If now these equivalent current sources are used in Fig. 41 instead of the voltage sources, Fig. 41 takes the form shown in Fig. 44. Employing Fig. 44, the current equation for the node c can be written in terms of voltage drops and admittances as follows:

$$\underbrace{\mathbf{Y}_1\mathbf{V}_c + \mathbf{Y}_3\mathbf{V}_c + \mathbf{Y}_2\mathbf{V}_c}_{\text{(current leaving node } c)} = \underbrace{\mathbf{Y}_1\mathbf{E}_a + \mathbf{Y}_2\mathbf{E}_b}_{\text{(current entering node } c)} \tag{65}$$

$\mathbf{V}_c$ can be obtained from equation (65) directly in terms of known quantities and all currents thereby calculated.

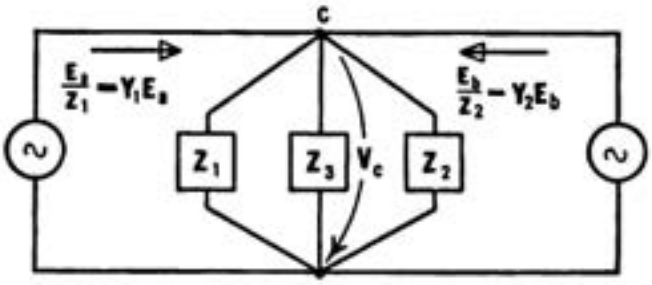

FIG. 44. Transformation of the circuit shown in Fig. 41.

Example 13. Assume the data for Fig. 41 to be as follows: $\mathbf{E}_a = 100\angle 0°$ volts, $\mathbf{E}_b = 50\angle 90°$ volts, $\mathbf{Z}_1 = 5\angle 0°$ ohms, $\mathbf{Z}_2 = 10\angle 36.9°$ ohms, and $\mathbf{Z}_3 = 20\angle 53.1°$ ohms. Find the voltage $\mathbf{V}_c$ and currents $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.

From equation (65),

$$\mathbf{V}_c(\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3) = \mathbf{Y}_1\mathbf{E}_a + \mathbf{Y}_2\mathbf{E}_b$$

$$\mathbf{V}_c\left(\frac{1}{5\angle 0°} + \frac{1}{10\angle 36.9°} + \frac{1}{20\angle 53.1°}\right) = \frac{100\angle 0°}{5\angle 0°} + \frac{50\angle 90°}{10\angle 36.9°}$$

$$\mathbf{V}_c(0.2 + 0.08 - j0.06 + 0.03 - j0.04) = 20 + 5\angle 53.1°$$

$$\mathbf{V}_c = 71.6\angle 27.76° \text{ volts}$$

$$\mathbf{I}_3 = \mathbf{V}_c\mathbf{Y}_3 = (71.6\angle 27.76°)(0.05\angle -53.1°) = 3.58\angle -25.34° \text{ amperes}$$

As seen from Fig. 42,

$$\mathbf{I}_1 = \mathbf{E}_a\mathbf{Y}_1 - \mathbf{V}_c\mathbf{Y}_1 = (0.2\angle 0°)(100\angle 0° - 71.6\angle 27.76°) = 7.35 - j6.66 \text{ amperes}$$

and, from Fig. 43,

$$\mathbf{I}_2 = \mathbf{E}_b\mathbf{Y}_2 - \mathbf{V}_c\mathbf{Y}_2 = (0.1\angle -36.9°)(50\angle 90° - 71.6\angle 27.76°)$$

$$= -4.05 + j5.134 \text{ amperes}$$

The nodal method of analysis is usually superior to the mesh-current method if the number of nodes (after transformation to current sources) does not exceed the number of meshes or loops. If N represents the

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number of nodes in a network, only $N - 1$ independent node equations are required, and these are obtained by applying Kirchhoff's current law to $N - 1$ nodes.

To arrive at the method of formulating a general system of nodal equations, assume that Fig. 45*a* is the network to be solved. First,

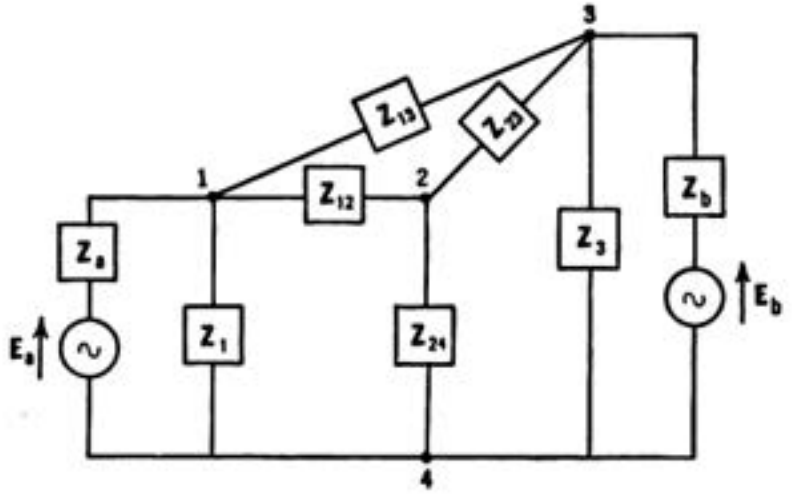

FIG. 45*a*. A network having two voltage sources.

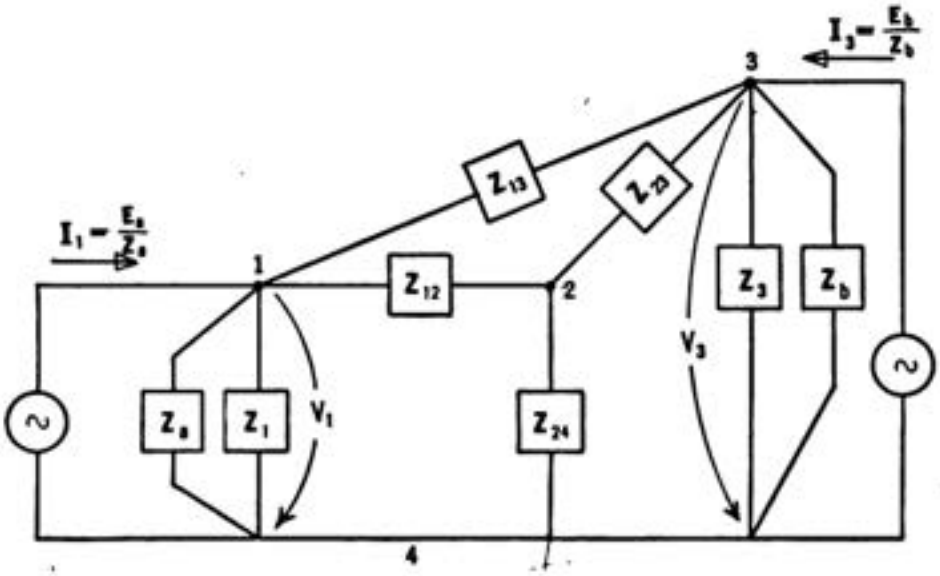

FIG. 45*b*. Transformation of circuit shown in Fig. 45*a*.

replace the voltage sources by constant-current sources as shown in Fig. 45*b*. Assume one node as the reference node, node 4 in this case. The output of the constant-current generator a is $\mathbf{E}_a/\mathbf{Z}_a = \mathbf{I}_1$. Similarly the output of constant-current generator b is $\mathbf{E}_b/\mathbf{Z}_b = \mathbf{I}_3$. To obtain the current in any impedance, the voltage drop across the impedance is multiplied by the admittance. The voltage drop can always be obtained in terms of the nodal voltages. Remembering that

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the voltage drop from node 1 to node 2 is the sum of the drops encountered in going from node 1 to 2 by any path, we may write $\mathbf{V}_{12} = \mathbf{V}_{14} + \mathbf{V}_{42} = \mathbf{V}_1 - \mathbf{V}_2$. Hence $\mathbf{I}_{12} = (\mathbf{V}_1 - \mathbf{V}_2)\mathbf{Y}_{12}$. Application of Kirchhoff's current law to node 1 yields

$$\mathbf{Y}_a\mathbf{V}_1 + \mathbf{Y}_1\mathbf{V}_1 + \mathbf{Y}_{12}(\mathbf{V}_1 - \mathbf{V}_2) + \mathbf{Y}_{13}(\mathbf{V}_1 - \mathbf{V}_3) = \mathbf{I}_1 \tag{66}$$

or

$$(\mathbf{Y}_a + \mathbf{Y}_1 + \mathbf{Y}_{12} + \mathbf{Y}_{13})\mathbf{V}_1 - \mathbf{Y}_{12}\mathbf{V}_2 - \mathbf{Y}_{13}\mathbf{V}_3 = \mathbf{I}_1 \tag{67}$$

The sum of all the admittances from node 1 to all other nodes is called the self-admittance and is designated by $\mathbf{Y}_{11}$. The admittance of the impedance connecting node 1 to any other node, say n, is called the mutual admittance, $\mathbf{Y}_{1n}$. Thus $\mathbf{Y}_{12}$, $\mathbf{Y}_{13}$, etc., are mutual admittances. When these notations are used, equation (67) becomes

$$\mathbf{Y}_{11}\mathbf{V}_1 - \mathbf{Y}_{12}\mathbf{V}_2 - \mathbf{Y}_{13}\mathbf{V}_3 = \mathbf{I}_1 \tag{68}$$

Similarly, for node 3,

$$\mathbf{Y}_{33}\mathbf{V}_3 - \mathbf{Y}_{32}\mathbf{V}_2 - \mathbf{Y}_{31}\mathbf{V}_1 = \mathbf{I}_3 \tag{69}$$

where $\mathbf{Y}_{33} = \mathbf{Y}_{13} + \mathbf{Y}_3 + \mathbf{Y}_{23} + \mathbf{Y}_b$. And, for node 2,

$$\mathbf{Y}_{22}\mathbf{V}_2 - \mathbf{Y}_{21}\mathbf{V}_1 - \mathbf{Y}_{23}\mathbf{V}_3 = 0 \tag{70}$$

An extension of equations (68), (69), and (70) will yield the general system of nodal equations for an n-node system as follows.

$$\left.\begin{array}{l}
\mathbf{Y}_{11}\mathbf{V}_1 - \mathbf{Y}_{12}\mathbf{V}_2 - \mathbf{Y}_{13}\mathbf{V}_3 - \cdots - \mathbf{Y}_{1n}\mathbf{V}_n = \mathbf{I}_1 \\
-\mathbf{Y}_{21}\mathbf{V}_1 + \mathbf{Y}_{22}\mathbf{V}_2 - \mathbf{Y}_{23}\mathbf{V}_3 - \cdots - \mathbf{Y}_{2n}\mathbf{V}_n = \mathbf{I}_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-\mathbf{Y}_{n1}\mathbf{V}_1 - \mathbf{Y}_{n2}\mathbf{V}_2 - \mathbf{Y}_{n3}\mathbf{V}_3 - \cdots + \mathbf{Y}_{nn}\mathbf{V}_n = \mathbf{I}_n
\end{array}\right\} \text{General system of nodal equations}$$

As previously defined, $\mathbf{I}_1, \mathbf{I}_2, \cdots$, and $\mathbf{I}_n$ are the output currents of the constant-current generators directed toward the various nodes. The nodal voltages in the general system of equations above may be solved for by determinants. After some practice with these systematic forms of solution, the determinant forms can be established from an inspection of the network after all specified voltage generators have been transformed to equivalent current generators. The writing of the current equations as shown above can therefore be dispensed with and the analysis reduced to a simple routine procedure.

In order to appreciate fully the usefulness of the nodal method, one should apply it to vacuum tube circuits where the plate-to-cathode path of the tube functions as a current sink (or negative current source). This application, however, presupposes an elementary knowledge of the functioning of a vacuum tube, and for this reason the following

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example may be omitted without loss of continuity by those readers who have no knowledge of the performance of a vacuum tube.

Example 14. *The Equivalent Plate Circuit of a Vacuum Tube.* For the present, we may accept the fact that the plate current, i_b, of a vacuum tube as shown in Fig. 46*a* is a function of both the plate voltage, e_b, and the control grid voltage, e_c. Both of these potentials are relative to the cathode labeled k, as indicated in Fig. 46*a*.

If only small *changes* from the d-c operating values of current and voltage are involved, we may write

$$\Delta i_b = \frac{\partial i_b}{\partial e_b}\Delta e_b + \frac{\partial i_b}{\partial e_c}\Delta e_c \tag{71}$$

and, if the change in plate current Δi_b is called i_p, if the change in plate voltage Δe_b is called e_p, and if the change in grid voltage Δe_c is called e_g, we have

$$i_p = \frac{e_p}{r_p} + g_m e_g \tag{72}$$

where $r_p = \partial e_b/\partial i_b$ is called the variational or plate resistance of the vacuum tube, and $g_m = \partial i_b/\partial e_c$ is called the mutual conductance or transconductance. For

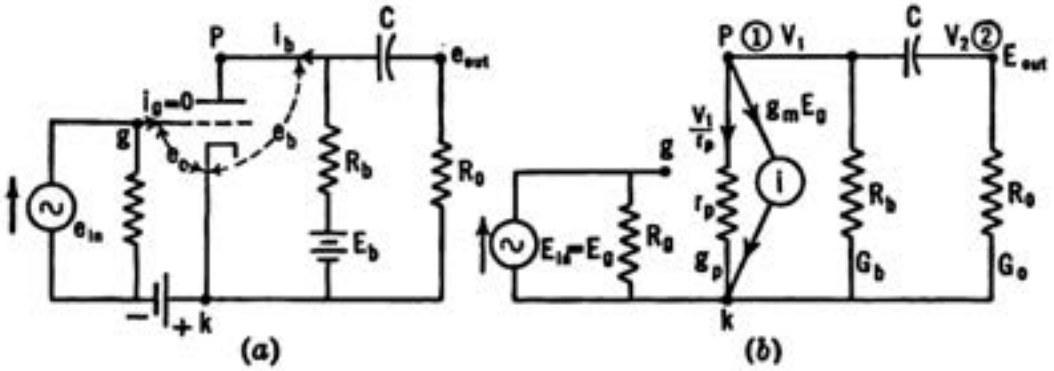

FIG. 46. The a-c equivalent of (*a*) is shown in (*b*).

a particular condition of d-c operation both r_p and g_m are usually known. The plate current of the vacuum tube so biased that the control grid current is zero is given by equation (72), and it is this equation which permits the use of the equivalent circuit shown in Fig. 46*b* for the plate-to-cathode portion of the vacuum tube shown in Fig. 46*a*.

In Fig. 46 we may replace the instantaneous values of the e's and the i's with effective values if a sinusoidal time variation of e_{in} is assumed and if e_{in} is at no time so large as to permit the control grid to draw current. It will be observed that the vacuum tube functions as a current sink ($g_m e_g$) in parallel with a resistance path, namely, the r_p path in Fig. 46.

In order to illustrate further the application of the nodal method in a numerical case let it be required to find E_{out} in Fig. 46 if:

$e_{in} = 0.707 \sin 3770t$ volt or	$E_{in} = E_g = 0.5\angle 0°$ volt
$g_m = 2000$ micromhos	$g_m = 200 \times 10^{-5}$ mho
$r_p = 20{,}000$ ohms	$g_p = 5 \times 10^{-5}$ mho
$R_b = 50{,}000$ ohms	$G_b = 2 \times 10^{-5}$ mho
$R_0 = 200{,}000$ ohms	$G_0 = 0.5 \times 10^{-5}$ mho
$C = 0.00265\ \mu f$	$\mathbf{Y}_{12} = \mathbf{Y}_{21} = j\omega C = j10^{-5}$ mho

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Applying Kirchhoff's current law to node 1 in Fig. 46b, we obtain

$$g_p\mathbf{V}_1 + G_b\mathbf{V}_1 + j\omega C(\mathbf{V}_1 - \mathbf{V}_2) = -g_m\mathbf{E}_g = \mathbf{I}_1$$

or

$$\mathbf{Y}_{11}\mathbf{V}_1 - \mathbf{Y}_{12}\mathbf{V}_2 = -g_m\mathbf{E}_g = -100 \times 10^{-5} \text{ ampere}$$

where, in this particular case,

$$\begin{aligned}\mathbf{Y}_{11} \text{ (the self-admittance of node 1)} &= g_p + G_b + j\omega C \\ &= (7 + j1)10^{-5} \text{ mho}\end{aligned}$$

$$\begin{aligned}\mathbf{Y}_{12} = \mathbf{Y}_{21} \text{ (the mutual admittance between nodes 1 and 2)} \\ &= j\omega C \\ &= j10^{-5} \text{ mho}\end{aligned}$$

Applying the current law to node 2:

$$G_0\mathbf{V}_2 + j\omega C(\mathbf{V}_2 - \mathbf{V}_1) = 0$$

or

$$-\mathbf{Y}_{21}\mathbf{V}_1 + \mathbf{Y}_{22}\mathbf{V}_2 = 0$$

where

$$\begin{aligned}\mathbf{Y}_{22} \text{ (the self-admittance of node 2)} &= G_0 + j\omega C \\ &= (0.5 + j1) \times 10^{-5} \text{ mho}\end{aligned}$$

The detailed applications of the current law can be dispensed with as soon as the systematized procedure implied by the subscripts attached to the Y's is understood. The determinant form of the solution for $\mathbf{V}_2$ is:

$$\mathbf{V}_2 = \frac{\begin{vmatrix} \mathbf{Y}_{11} & \mathbf{I}_1 \\ -\mathbf{Y}_{21} & 0 \end{vmatrix}}{\begin{vmatrix} \mathbf{Y}_{11} & -\mathbf{Y}_{12} \\ -\mathbf{Y}_{21} & \mathbf{Y}_{22} \end{vmatrix}} = \frac{\begin{vmatrix} (7 + j1) & -100 \\ -j1 & 0 \end{vmatrix} \times 10^{-10}}{\begin{vmatrix} (7 + j1) & -j1 \\ -j1 & (0.5 + j1) \end{vmatrix} \times 10^{-10}}$$

$$\mathbf{V}_2 = \frac{-j100}{3.5 + j7.5} = \frac{100\underline{/-90^\circ}}{8.27\underline{/65^\circ}} = 12.08\underline{/-155^\circ} \text{ volts}$$

The amplification of the circuit arrangement shown in Fig. 46 is

$$\frac{\mathbf{E}_{out}}{\mathbf{E}_{in}} = \frac{12.08\underline{/-155^\circ}}{0.5\underline{/0^\circ}} = 24.16\underline{/-155^\circ}$$

which indicates that the magnitude of the output voltage is 24.16 times that of the input voltage and that the output voltage lags the input voltage by 155° or 155/360 part of a cycle.

Equivalence of Special Circuits (Wyes and Deltas). Figures 47 and 48 show two types of circuits which are very commonly encountered in the reduction of electrical networks. The first is called a delta system; Fig. 48 is called a wye. It is possible to substitute a wye-connected system of impedances for a delta system, and vice versa, if proper values are given to the substituted impedances. Suppose that it is desired to substitute a wye for a given delta. The two systems will be exactly equivalent if the impedance between any pair of lines *A*, *B*, and *C*, Fig. 49, for the delta is the same as that between the cor-

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responding pair for the wye when the third line is broken. If this condition is imposed, the following equations are obtained.

Line A open: $$\mathbf{Z}_C + \mathbf{Z}_B = \frac{\mathbf{Z}_3(\mathbf{Z}_1 + \mathbf{Z}_2)}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{73}$$

Line B open: $$\mathbf{Z}_A + \mathbf{Z}_C = \frac{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}_3)}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{74}$$

Line C open: $$\mathbf{Z}_A + \mathbf{Z}_B = \frac{\mathbf{Z}_2(\mathbf{Z}_1 + \mathbf{Z}_3)}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{75}$$

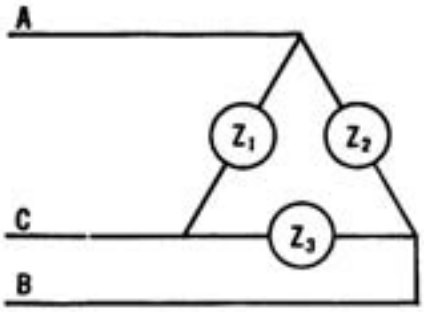

Fig. 47. Delta.

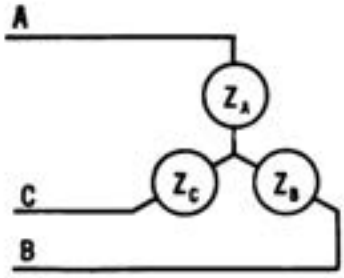

Fig. 48. Wye.

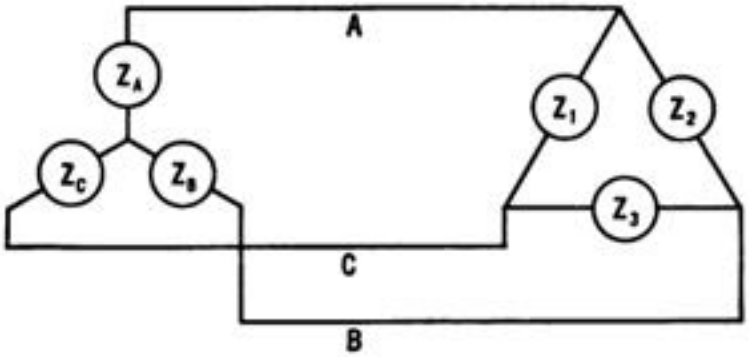

Fig. 49. Circuit for establishment of equivalence between wye and delta systems of impedances.

Solution of these three equations simultaneously for $\mathbf{Z}_A$, $\mathbf{Z}_B$, and $\mathbf{Z}_C$ in terms of the impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ gives the following:

$$\mathbf{Z}_A = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{76}$$

$$\mathbf{Z}_B = \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{77}$$

$$\mathbf{Z}_C = \frac{\mathbf{Z}_1\mathbf{Z}_3}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{78}$$

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From equations (76), (77), and (78), the values of the wye impedances $\mathbf{Z}_A$, $\mathbf{Z}_B$, and $\mathbf{Z}_C$ that will replace a system of delta impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ may be found. These results are easily remembered when it is observed that the denominators are all the same and equal to the sum of the three delta impedances. The numerator for $\mathbf{Z}_A$ is the product of the two delta impedances which connect to $\mathbf{Z}_A$. Similarly the numerator for $\mathbf{Z}_B$ is the product of $\mathbf{Z}_2$ and $\mathbf{Z}_3$.

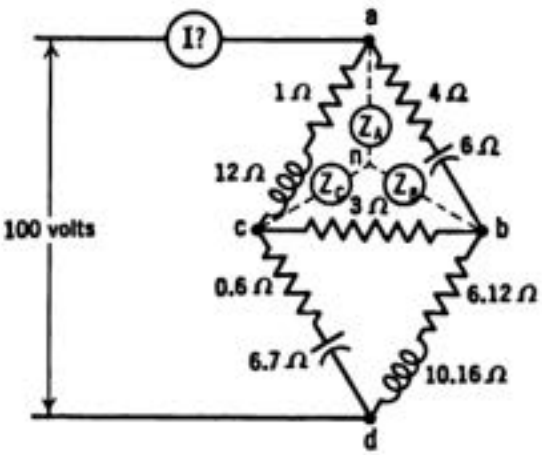

FIG. 50. See example 15.

It should be noticed that the special case of balanced delta impedances yields wye impedances, which are also balanced and equal to

$$\mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta^2}{3\mathbf{Z}_\Delta} = \frac{\mathbf{Z}_\Delta}{3}$$

and

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y$$

Example 15. Find I for the circuit and constants shown in Fig. 50. First a wye is substituted for the delta abc. The wye and its corresponding impedances are shown dotted.

$$\mathbf{Z}_A = \frac{(1 + j12)(4 - j6)}{(4 - j6) + (3 + j0) + (1 + j12)} = 8.6 - j1.2 \text{ ohms}$$

$$\mathbf{Z}_B = \frac{(4 - j6)(3)}{8 + j6} = -0.12 - j2.16 \text{ ohms}$$

$$\mathbf{Z}_C = \frac{(1 + j12)(3)}{8 + j6} = 2.4 + j2.7 \text{ ohms}$$

After the above impedances are substituted, the circuit appears as shown in Fig. 51. It is apparent that a series-parallel circuit results, the method of solution of which has been given in a previous article. Combining the parallel branches results in the circuit shown in Fig. 52. Thus

$$\mathbf{Z}_{ncd} = 3 - j4 \text{ ohms}$$

$$\mathbf{Z}_{nbd} = 6 + j8 \text{ ohms}$$

$$\mathbf{Z}_{nd} = \frac{(6 + j8)(3 - j4)}{(3 - j4) + (6 + j8)} = \frac{50(9 - j4)}{(9 + j4)(9 - j4)}$$

$$= \frac{450 - j200}{81 + 16} = 4.645 - j2.065 \text{ ohms}$$

$$\mathbf{I} = \frac{100 + j0}{13.245 - j3.265} = 7.14 + j1.76 = 7.355\,\underline{/14^\circ} \text{ amperes}$$

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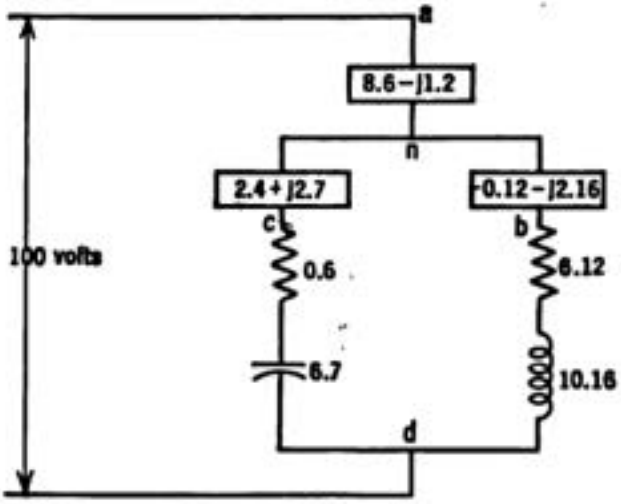

FIG. 51. Reduction from Fig. 50.

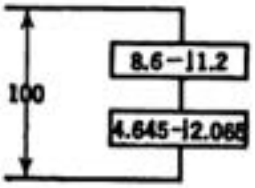

FIG. 52. Reduction from Fig. 51.

To find the currents in the various branches, the steps are retraced as follows:

$$\mathbf{V}_{nd} = \mathbf{I}\mathbf{Z}_{nd} = (7.14 + j1.76)(4.645 - j2.065)$$
$$= 36.73 - j6.57 \text{ volts}$$

$$\mathbf{I}_{ncd} = \frac{(36.73 - j6.57)(3 + j4)}{(3 - j4)(3 + j4)} = 5.45 + j5.09 \text{ amperes}$$

$$\mathbf{I}_{nbd} = \frac{(36.73 - j6.57)(6 - j8)}{(6 + j8)(6 - j8)} = 1.69 - j3.33 \text{ amperes}$$

$$\mathbf{V}_{an} = \mathbf{I}\mathbf{Z}_{an} = (7.14 + j1.76)(8.6 - j1.2) = 63.51 + j6.57 \text{ volts}$$

$$\mathbf{V}_{nc} = \mathbf{I}_{ncd}\mathbf{Z}_{nc} = (5.45 + j5.09)(2.4 + j2.7)$$
$$= -0.64 + j26.9 \text{ volts}$$

$$\mathbf{V}_{nb} = \mathbf{I}_{nbd}\mathbf{Z}_{nb} = (1.69 - j3.33)(-0.12 - j2.16)$$
$$= -7.403 - j3.25 \text{ volts}$$

$$\mathbf{V}_{ac} = \mathbf{V}_{an} + \mathbf{V}_{nc} = 63.51 + j6.57 - 0.64 + j26.9$$
$$= 62.87 + j33.47 \text{ volts}$$

$$\mathbf{V}_{ab} = \mathbf{V}_{an} + \mathbf{V}_{nb} = 63.51 + j6.57 - 7.403 - j3.25$$
$$= 56.11 + j3.32 \text{ volts}$$

$$\mathbf{I}_{ac} = \frac{(62.87 + j33.47)(1 - j12)}{(1 + j12)(1 - j12)} = 3.19 - j4.96 \text{ amperes}$$

$$\mathbf{I}_{ab} = \frac{(56.11 + j3.32)(4 + j6)}{(4 - j6)(4 + j6)} = 3.93 + j6.73 \text{ amperes}$$

Check: $3.19 - j4.96 + 3.93 + j6.73 = 7.12 + j1.77$, which is within slide-rule accuracy of $7.14 + j1.76$ amperes.

$$\mathbf{V}_{cb} = \mathbf{V}_{cn} + \mathbf{V}_{nb} = 0.64 - j26.9 - 7.403 - j3.25$$
$$= -6.763 - j30.15 \text{ volts}$$

$$\mathbf{I}_{cb} = \frac{-6.763 - j30.15}{3} = -2.254 - j10.05 \text{ amperes}$$

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$$\mathbf{I}_{cd} = \mathbf{I}_{ac} - \mathbf{I}_{cb} = 3.19 - j4.96 + 2.254 + j10.05$$
$$= 5.444 + j5.09 \text{ amperes}$$

which checks $\mathbf{I}_{acd}$.

$$\mathbf{I}_{bd} = \mathbf{I}_{cb} + \mathbf{I}_{ab} = -2.254 - j10.05 + 3.93 + j6.73$$
$$= 1.68 - j3.32 \text{ amperes}$$

There are a few occasions when it is convenient and desirable to substitute an equivalent delta for a wye. This is simply the problem of finding the values of $\mathbf{Z}_1$, and $\mathbf{Z}_2$, and $\mathbf{Z}_3$ that will replace the values of $\mathbf{Z}_A$, and $\mathbf{Z}_B$, and $\mathbf{Z}_C$ in Fig. 49. The solution is obtained when equations (73), (74), and (75) are solved algebraically for the impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ in terms of the impedances $\mathbf{Z}_A$, $\mathbf{Z}_B$, and $\mathbf{Z}_C$. It will usually be found simpler to solve for these quantities from equations (76), (77), and (78), which were derived from equations (73), (74), and (75). The solution gives

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_A\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_C + \mathbf{Z}_C\mathbf{Z}_A}{\mathbf{Z}_B} \tag{79}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_A\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_C + \mathbf{Z}_C\mathbf{Z}_A}{\mathbf{Z}_C} \tag{80}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_A\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_C + \mathbf{Z}_C\mathbf{Z}_A}{\mathbf{Z}_A} \tag{81}$$

Equations (79), (80), and (81) are easy to write when it is observed that the numerator of each is the same and equal to the sum of all possible products of the three impedances when taken two at a time. The denominator of $\mathbf{Z}_1$ is the wye impedance that has no connection to either extremity of $\mathbf{Z}_1$. Similar relations obtain for $\mathbf{Z}_2$ and $\mathbf{Z}_3$.

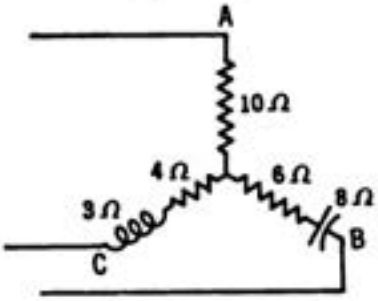

Fig. 53. See example 16.

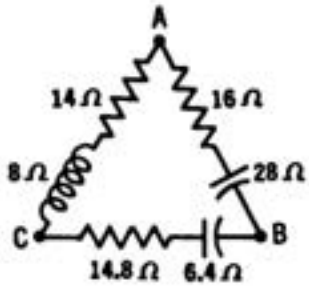

Fig. 54. Equivalent delta of Fig. 53.

Example 16. Find the delta that will replace the wye system shown in Fig. 53.

$$\mathbf{Z}_{AB\Delta} = \frac{(10)(6 - j8) + (6 - j8)(4 + j3) + (10)(4 + j3)}{4 + j3}$$
$$= \frac{148 - j64}{4 + j3} = 16 - j28 \text{ ohms}$$

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$$Z_{BC\Delta} = \frac{148 - j64}{10} = 14.8 - j6.4 \text{ ohms}$$

$$Z_{CA\Delta} = \frac{148 - j64}{6 - j8} = 14 + j8 \text{ ohms}$$

From these three impedances the equivalent delta is found as shown in Fig. 54.

Two commonly used types of networks are the T and π configurations shown, respectively, in Fig. 55*a* and Fig. 55*b*. Viewed as three-terminal networks, these configurations will be recognized as the wye and delta, respectively. The same formulas derived for changing a wye to an equivalent delta are therefore applicable for changing a T to an equivalent π. Likewise formulas for changing a delta to an equivalent wye may be used to change a π to an equivalent T.

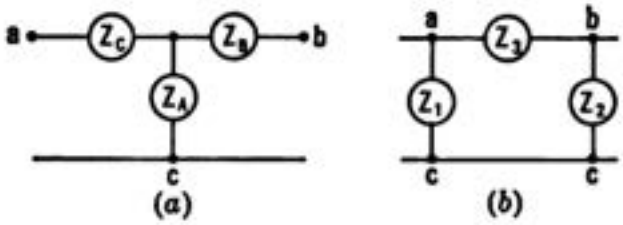

FIG. 55. (*a*) T network, (*b*) π network.

T- and π-sections are used extensively in transmission line and filter-section calculations. In cases of this kind, the T- and π-sections shown in Fig. 55 are usually considered as four-terminal networks because these sections are inserted into a two-wire circuit and are considered to have a pair of "input" terminals and a pair of "output" terminals. The manipulation of T- and π-sections as four-terminal networks will be considered in detail in Chapters XII and XIII.

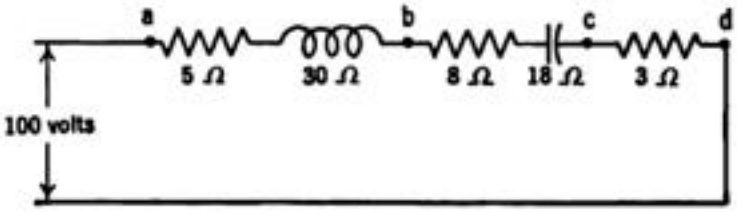

FIG. 56. See Problems 17, 18, and 23.

PROBLEMS

17. Calculate the current through the impedances of Fig. 56. Find voltage drops across *ab*, *bc*, and *cd*. Draw the vector diagram showing the current and the voltage drop across each resistance or reactance. Calculate the power factor of the whole circuit.

18. Find all possible values of pure reactance which, when placed in series with the circuit of Fig. 56, will make the overall power factor 0.6. Find the power dissipated in the circuit for this condition.

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19. A particular 110-volt, 60-cycle, $\frac{1}{4}$-hp, single-phase induction motor has an efficiency of 60 per cent and a power factor of 0.6 lagging at full load. This motor is to be used temporarily on a 220-volt, 60-cycle line. A resistor (non-inductive) of suitable current capacity and of proper resistance is to be placed in series with the motor.

(*a*) What value of resistance is required if the motor is to have 110 volts across its terminals at rated full load?

(*b*) Draw the complete vector diagram ($\mathbf{V}_{motor}$, $\mathbf{IR}_{external}$, $\mathbf{I}$, and $\mathbf{V}_{line}$) with $\mathbf{V}_{motor}$ as reference.

20. Two single-phase motors are connected in parallel across a 110-volt, 60-cycle source of supply. Motor 1 is a split-phase induction type which takes a lagging current, and motor 2 is a capacitor type which takes a leading current. Find the total power, the combined line current, and the resultant power factor of the two motors operating in parallel from the following data:

Motor	Horsepower Output	Per Unit Efficiency	Per Unit Power Factor
1	$\frac{1}{3}$	0.60	0.70 (lagging)
2	$\frac{1}{2}$	0.75	0.95 (leading)

21. A series circuit on which 100 volts is impressed consists of a 10-ohm resistance, a 5-ohm condenser, a resistance R in which is lost 50 watts, and a reactance X taking 100 inductive vars. Calculate all values of R and X to satisfy the conditions stated and the corresponding currents for each of the combinations.

22. A toaster operates at 115 volts, 60 cycles, and 10 amperes and absorbs 1150 watts at its terminals. A choke coil is to be wound with a ratio of X_L to R of 5, so that, if placed in series with the toaster on a 230-volt, 60-cycle line, the toaster will have 115 volts across its terminals.

(*a*) What is the impedance of the choke coil required? State $\mathbf{Z}$ in polar and in rectangular complex form.

(*b*) Draw the complete vector diagram with V_{lamp} as reference.

(*c*) What is the power factor of the combined lamp and choke coil in series?

23. Find the inductance or capacitance which may be inserted in the circuit of Fig. 56 to put the entire circuit in resonance. Frequency 60 cycles.

24. (*a*) If the impressed voltage on a series circuit containing 5 ohms resistance, 100 ohms inductive reactance at 60 cycles, and a variable capacitance is 100 volts, find the maximum drop across the capacitance and the value of the capacitance for this condition.

(*b*) Repeat the calculation if, instead of the 5-ohm resistance, a 100-ohm resistance is used. Compare the results in the two cases.

25. A series circuit dissipates 800 watts and also requires 1000 volt-amperes when the impressed voltage is 100 volts. Find the equivalent series resistance and possible reactances of this circuit.

26. The frequency range of the pass band as previously defined in this chapter for an *RLC* circuit is 100 cycles when a coil having a Q of 50 is used. All resistance of the circuit is assumed in the coil.

(*a*) Find the upper and lower frequency limits of the pass band.

(*b*) If a coil with a Q of 200 is used at the same resonant frequency as in (*a*), what will be the frequency range of the pass band?

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27. Given the *RLC* series circuit shown in Fig. 57.

(*a*) Find the resonant frequency of the series circuit.

(*b*) Find the Q of the series circuit at the resonant frequency.

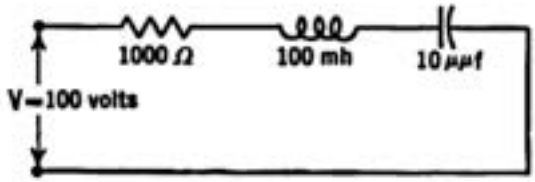

FIG. 57. See Problem 27.

(*c*) At what angular velocities do the half-power points occur?

(*d*) Assuming that L is varied to obtain resonance, at what value of L would V_L be maximum? Assume the frequency in this case to be constant at 159 kc.

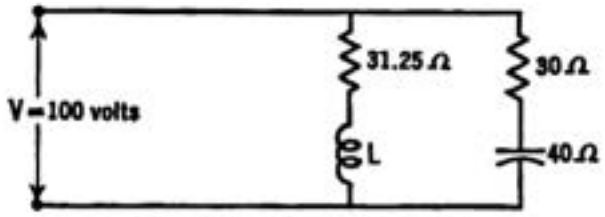

FIG. 58. See Problem 28.

28. Given the circuit shown in Fig. 58.

(*a*) What are the values of X_L that will produce resonance?

(*b*) Find the magnitude of the maximum impedance obtainable with this circuit. Assume that the frequency is held fixed.

(*c*) If R_L is changed to 30 ohms (R_C remaining the same) and L and C are made 9 millihenrys and 10 μf, respectively, what is the impedance looking into the circuit at 100 cycles per second and 10,000 cycles per second?

(*d*) At what frequency will the circuit as designated in part (*c*) be in resonance?

29. In the following exercises, it is assumed that a coil having L henrys of inductance and R_s ohms of series resistance is placed in resonance with a series condenser C, so that $\omega_m = 1/\sqrt{LC}$.

(*a*) Show that $Q_s = \omega_m L/R_s$ is

$$Q_s = \frac{\text{reactive factor (of the coil)}}{\text{power factor (of the coil)}}$$

(*b*) Show that

$$\text{Power factor (of the coil)} = \frac{1}{\sqrt{Q_s^2 + 1}}$$

(*c*) Show that

$$Q_s = \frac{\omega_m w}{RI^2}$$

where w is the reactive energy stored in L and C at any time and RI^2 is the average dissipated power of the circuit. *Note:* $w = (Li^2/2) + (Cv_c^2/2) = \text{constant}$.

30. An impedance $Z_1 = 8 - j5$ is in parallel with an impedance $Z_2 = 3 + j7$. Find the resultant impedance of the combination. What is the overall power factor?

31. If 100 volts are impressed on the parallel impedances of Problem 30, find I_1,

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I_2, and the resultant current. Draw the vector diagram of the circuit, showing each current, and the voltage drop across each parameter.

32. An impedance load consisting of 12 ohms resistance and 16 ohms inductive reactance is connected across a 60-cycle, 100-volt source. Find the capacitance of a condenser which may be paralleled with this load to bring the power factor to 1. Assume negligible resistance for the condenser.

33. Work Problem 32 if a final power factor of 0.8 instead of 1 is desired. Obtain solutions for leading and lagging power factors.

34. Find the value of pure resistance which would be required in parallel with the impedance load of Problem 32 to bring the resultant power factor to 0.8.

35. A capacitor branch having a ratio of X to R of 5 is paralleled with an impedance consisting of 4 ohms resistance and 3 ohms inductive reactance. The power factor of the resulting circuit is 0.8 lead. Find the size of the capacitor in microfarads if the frequency is 60 cycles.

36. A single-phase load on 200 volts takes 5 kw at 0.6 lagging power factor. Find the kva size of condenser which may be connected in parallel with this motor to bring the resultant power factor to 1.

37. Work Problem 36 if it is desired to bring the power factor to 0.9 lag instead of to 1.

38. The load of Problem 36 is operated in parallel with a synchronous motor that takes 8 kw at 0.5 leading power factor. What are the resultant current supplied by the line and the power factor of the combination?

39. Over the period of a year, an industrial establishment takes an average load of 2000 kw continuously at a (current) lagging power factor of 0.80.

(*a*) What is the annual fixed charge on the kva capacity required to serve this establishment if 1 kva of installed capacity (boiler, generator, transmission line, and transformers) costs $200? The fixed charge (consisting of interest, taxes, and depreciation) may be taken as 8 per cent of the investment.

(*b*) Repeat part (*a*) assuming that the power factor of the establishment is unity.

40. What value of resistance should be placed in parallel with a 50-μf condenser to give a combined power factor of 0.6 on a 60-cycle system? (Neglect the resistance of the condenser.)

41. Find the series-circuit resonant frequency of a 100-microhenry inductance and a 400-$\mu\mu$f capacitance.

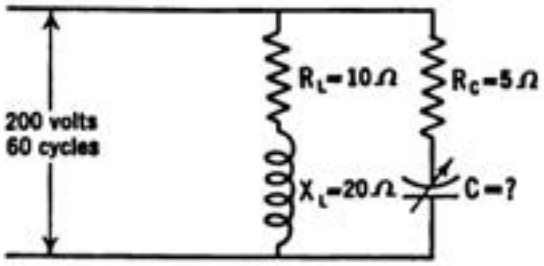

FIG. 59. See Problems 42, 43, and 44.

42. Find C to produce resonance in Fig. 59. How much power is dissipated in R_C at resonance?

43. Find the value of C in Fig. 59 which will yield maximum impedance for the whole circuit.

44. What minimum value of R_C in Fig. 59 would prevent the possibility of attaining resonance by varying C?

45. A fixed condenser is placed in parallel with a fixed resistance and variable

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inductance of negligible resistance as shown in Fig. 60. Show that the general expression for X_L which will produce unity-power-factor resonance is:

$$X_L = \frac{X_C}{2} \pm \sqrt{\frac{X_C^2}{4} - R^2}$$

Hint: For unity p.f., $b_L = b_C$.

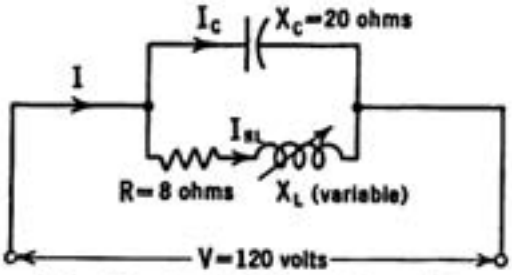

FIG. 60. See Problems 45 and 46.

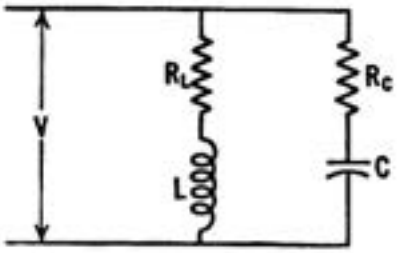

FIG. 61. See Problem 48.

46. Refer to Fig. 60.

(*a*) Draw a to-scale vector diagram of V, I_C and I_{RL} for $X_L = 0$.

(*b*) On the above diagram draw the loci of I_{RL} and I for X_L variable from 0 to ∞.

(*c*) Determine the values of X_L which will produce unity-power-factor resonance either graphically or analytically.

(*d*) Determine the minimum value of I either graphically or analytically, and find the value of X_L which produces this minimum value of I.

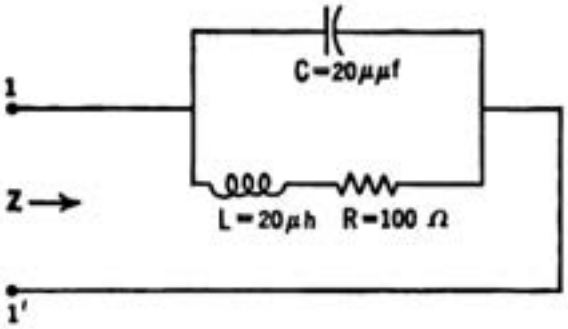

FIG. 62. See Problems 49, 50, 52, 53, and 55.

47. A 2-μf capacitance is connected in parallel with a 20-ohm resistance. Plot the magnitudes of the admittance and impedance of the parallel combination against frequency for frequencies of 0, 10,000, 100,000, and 1,000,000 cycles.

48. (*a*) If $L = 0.050$ henry, $C = 200$ μf, and $R_L = R_C = 1.0$ ohm, find the resonant frequency of the parallel branches shown in Fig. 61.

(*b*) If $R_L = 20$ ohms, $L = 0.050$ henry, $C = 100$ μf, find the value of R_C which will yield parallel resonance of the two branches at a frequency of 45 cycles.

(*c*) If $C = 100$ μf, $R_L = 20$ ohms, and $R_C = 20$ ohms, find the value of L that will place the branches in parallel resonance irrespective of frequency.

49. (*a*) Transform the circuit shown in Fig. 62 to that shown in Fig. 63, employing numerical values of g, b_L, and b_C and assuming that the operating angular frequency is 5×10^7 radians per second. (Results which are accurate to within 1 per cent will be considered satisfactory.)

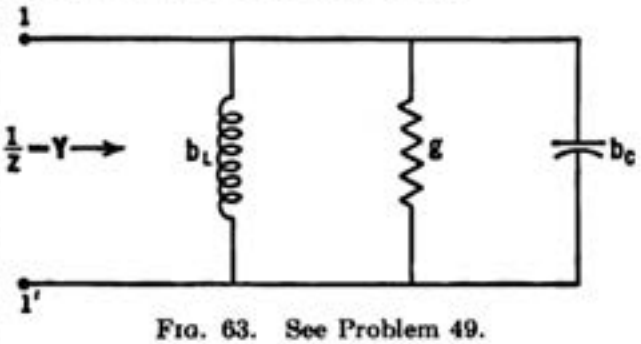

FIG. 63. See Problem 49.

(*b*) If terminals 11′ of Fig. 62 are energized with a current of 2 milliamperes (at $\omega = 5 \times 10^7$ radians per second), what voltage will be developed across these terminals?

(*c*) What is the Q_p of the circuit?

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50. Find the resistance component of Z in Fig. 62 in terms of L, R, C, and ω, assuming that R is constant.

$$Ans.: \quad \frac{R}{(LC\omega^2 - 1)^2 + R^2\omega^2C^2} = R_Z.$$

51. (*a*) Find the angular frequency at which R_Z of Problem 50 has its maximum value, employing literal values of L, C, and R.

(*b*) What is the numerical value of the angular frequency for (*a*)?

Ans.: 4.987×10^7 radians per second.

(*c*) Compare the above result with the approximate value of $1/\sqrt{LC}$.

52. What is the maximum numerical value of the resistance component of Z in Fig. 62 as ω is varied from zero to infinity? (A result which is accurate to within 1 per cent will be considered satisfactory.)

53. The series resistance of the 20 microhenry coil shown in Fig. 62 is $R = 100$ ohms. What is the Q of the coil at $\omega = 0.1/\sqrt{LC}$ and at $\omega = 1/\sqrt{LC}$?

54. A coil having L henrys of inductance and R_s ohms of series resistance is placed in resonance with a parallel condenser, C, having no appreciable series resistance at an angular frequency of ω_m which is essentially equal to $1/\sqrt{LC}$. $R_s^2 \ll \omega_m^2L^2$.

Show that $Q_p = \omega_m C/g$ is essentially equal to

$$Q_p = \frac{\omega_m w}{V^2 g}$$

where V is the effective voltage across the parallel branches, w is the reactive energy stored in L and C at any time, and V^2g is the average dissipated power of the circuit. *Note:* In terms of instantaneous values and letting $v_c = v$, the instantaneous applied voltage,

$$w = \frac{Li_L^2}{2} + \frac{Cv^2}{2} = \text{constant}$$

55. It will be assumed here that the condenser shown in Fig. 62 has a series resistance of 10 ohms.

(*a*) What is the equivalent parallel resistance of the condenser at $\omega_m \approx 1/\sqrt{LC}$?

(*b*) What is the equivalent parallel resistance of the two branches at $\omega_m \approx 1/\sqrt{LC}$?

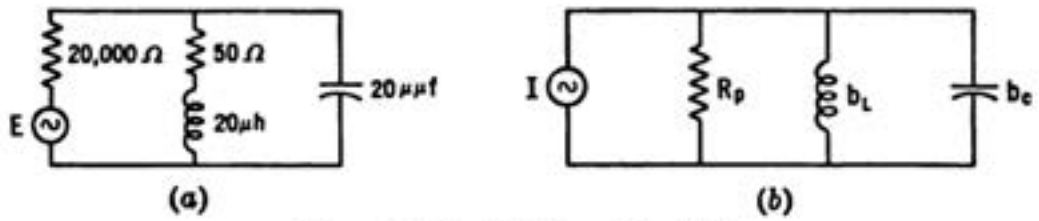

FIG. 64. See Problems 56 and 57.

56. Given the circuit arrangement shown in Fig. 64*a*, where the voltage generator has an internal resistance of 20,000 ohms as indicated.

(*a*) Transform the circuit to that shown in Fig. 64*b*.

(*b*) What is the Q_p of the parallel branches facing the current generator in Fig. 64*b* at $\omega = 5 \times 10^7$ radians per second.

(*c*) Compare the result obtained in (*b*) with the Q of the coil itself at $\omega = 5 \times 10^7$ radians per second. The coil has a resistance of 50 ohms as indicated.

57. (*a*) If the generator voltage in Fig. 64*a* is 200 volts at $\omega = 5 \times 10^7$ radians

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per second, what is the magnitude of the current of the equivalent current generator employed in Fig. 64*b*?

(*b*) What voltage is developed across the parallel branches by the current generator at $\omega = 5 \times 10^7$ radians per second?

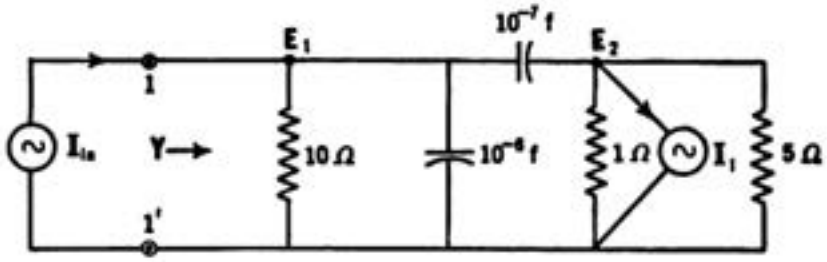

FIG. 65. See Problem 58.

58. Find the admittance Y (looking to the right of terminals $11'$) in Fig. 65, and express the result in terms of a resistance R_p in parallel with a condenser C, where R_p and C are expressed numerically in ohms and microfarads, respectively. $I_1 = 0.1E_1$.

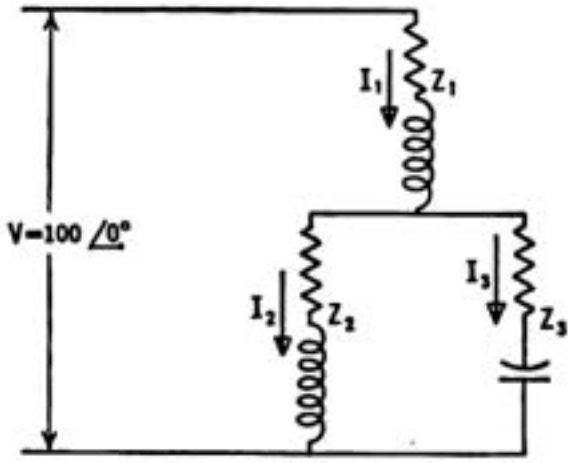

FIG. 66. See Problems 59, 60, and 69.

The I_{in} and I_1 current generators have the polarities indicated, and the operating angular frequency is 10^6 radians per second. *Note:* Current generators are always considered to have infinite internal impedance or zero internal admittance.

59. The parameters in Fig. 66 are:

$$Z_1 = R_1 + jX_{L1} = 10 + j30$$
$$Z_2 = R_2 + jX_{L2} = 5 + j10$$
$$Z_3 = R_3 - jX_{C2} = 4 - j16$$

(*a*) Find $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$, $\mathbf{V}_1$, and $\mathbf{V}_{22'}$ in complex polar form with respect to applied voltage ($100°\underline{/0°}$ volts) as a reference.

(*b*) Draw a complete vector diagram of the above voltages and currents.

(*c*) Find the watts and vars input to the whole circuit.

60. Find the power dissipated in each branch of Fig. 66 for the parameters given in Problem 59.

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61. Find the pure reactance or reactances X in Fig. 67 which will make the overall power factor 0.707.

62. A circuit similar to that shown in Fig. 34, page 132, except that L_1 is constant while C_2 is variable, is to pass a 45,000-cycle current with minimum impedance and to block a 15,000-cycle current as effectively as possible. $R_0 = 20$ ohms, $R_1 = 40$ ohms, and $L_1 = 0.002$ henry are fixed. The resistance, R_2, of the C_2 branch is assumed to be negligibly small. Either a fixed C_0 or a fixed L_0 (of negligibly small resistance) is to be placed in series with R_0 to accomplish the desired tuning effect.

(*a*) Solve for C_2 which will put the parallel circuit *bc* into parallel resonance at 15,000 cycles.

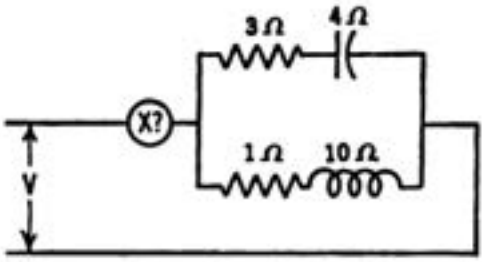

FIG. 67. See Problem 61.

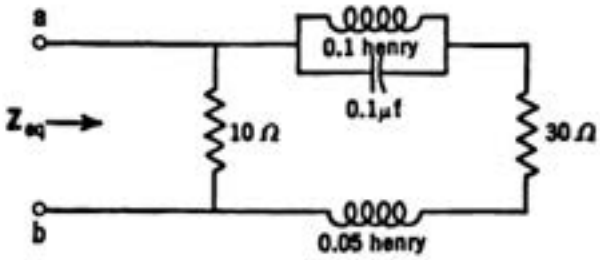

FIG. 68. See Problem 63.

(*b*) Calculate the equivalent impedance from *b* to *c* at 45,000 cycles with C_2 set at its 15,000-cycle resonant value. Is *bc* predominantly capacitive or inductive at 45,000 cycles?

(*c*) Must an inductance L_0 or a capacitance C_0 be used to put the branch *ab* into series resonance for 45,000 cycles? Calculate its value.

(*d*) Assuming that branch *ab* has been put into series resonance at 45,000 cycles, what is the actual impedance from *a* to *b* at 45,000 cycles? at 15,000 cycles?

63. Given the circuit shown in Fig. 68, determine the impedance looking into terminals *ab* at 1592 cycles per second.

64. A generating device has an impedance of $0.5 + j1$ ohms and is connected to a load by a line of $0.25 + j2$ ohms. At what load will maximum power transfer be realized? If the generated voltage is 20 volts, what is the power received by the load when adjusted for maximum power transfer? Find the line loss and the loss in the generating device.

65. (*a*) If the resistance of the load in Problem 64 is fixed at 0.75 ohm and only inductive reactance is permitted in the load, for what value of load reactance will maximum load power to the load be realized?

(*b*) What is the maximum load power under these conditions?

66. Work Problem 64 if the receiver impedance is restricted to pure resistance.

67. If a load impedance having a ratio of $X/R = 5$ is used at the end of the line in Problem 64, find the load impedance for maximum power transfer. What is the maximum power the load can receive?

68. Calculate $\mathbf{I}_2$ in Fig. 69 by the superposition theorem if $E_1 = 100\underline{/0^\circ}$ and $E_2 = 50\underline{/60^\circ}$ volts.

69. The voltage $\mathbf{V} = 100\underline{/0^\circ}$ volts is removed from branch 1 in Fig. 66 and inserted in branch 3. If the upper terminal of $\mathbf{Z}_1$ is connected to the lower common terminal of $\mathbf{Z}_2$ and $\mathbf{Z}_3$, calculate the current $\mathbf{I}_1$. How does this compare with $\mathbf{I}_3$ as calculated in Problem 59? By what theorem could this conclusion be reached?

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70. Calculate $\mathbf{V}_R$ in Fig. 70 if $\mathbf{E}_1 = 200\angle 0°$ volts. Then use Thévenin's theorem to calculate the current in an impedance $\mathbf{Z}_{ab} = 1.46 + j6.78$ ohms if it is connected

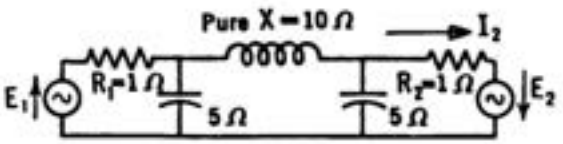

Fig. 69. See Problem 68.

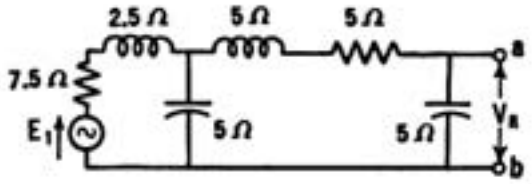

Fig. 70. See Problem 70.

to the terminals ab. Verify the result by calculating the current in ab by the ordinary methods of circuit reduction without employing Thévenin's theorem.

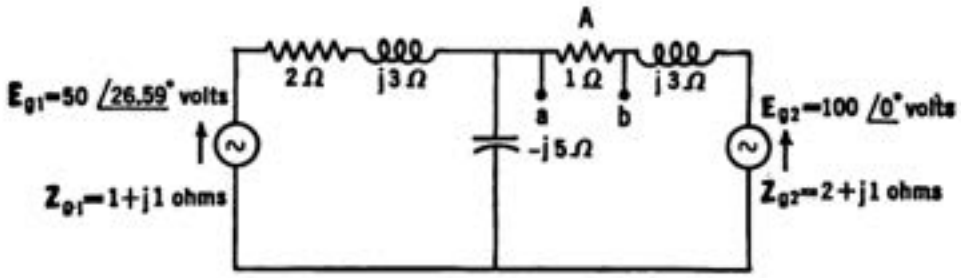

Fig. 71. See Problem 71.

71. Given the circuit shown in Fig. 71.

(*a*) Using the superposition theorem, determine the current through the resistor marked A.

(*b*) Using Thévenin's theorem, determine the current through an impedance $Z_{ab}(= 3 + j3$ ohms) that is presumed to be placed across terminals ab.

72. In Fig. 72*b*: $R_1 = 10^5$ ohms, $R_2 = 5 \times 10^4$ ohms, r_p(of the tube) $= 10^4$ ohms; $C_{gk} = C_{pk} = 40$ $\mu\mu$f, $C_{gp} = 5$ $\mu\mu$f; μ(of the tube) $= 20$; g_m(of the tube) $= \mu/r_p = 2 \times 10^{-3}$ mho.

Find the voltage, V_2, relative to ground if $E_1 = 1\angle 0°$ volt. The operating angular frequency is 10^6 radians per second.

Note: In Fig. 72*b*: $g_1 = 1/R_1$, $g_p = 1/r_p$, $g_2 = 1/R_2$; $j\omega C_{gk} = j\omega C_{pk} = j4 \times 10^{-5}$ mho, and $j\omega C_{gp} = 0.5 \times 10^{-5}$ mho, which is a hint that the problem should probably be solved on the nodal basis, employing E_1 as a known voltage.

73. Find the admittance $\mathbf{Y}$ (looking to the right of the $\mathbf{E}_1$ generator terminals in Fig. 72*b*), and express the result in terms of a resistance R_p in parallel with a condenser C where R_p and C are expressed numerically in ohms and microfarads, respectively.

The parameters and the operating angular frequency are given in Problem 72, and if this problem has been worked $\mathbf{V}_2$ will be a known voltage of $15.6\angle 159.32°$ volts.

74. Reduce the impedances shown in Fig. 73 to a single equivalent series impedance. Find the current in branch ab.

75. Derive the expressions shown in equations (79), (80), and (81), page 149.

76. Find the equivalent delta system of impedances which will replace the wye an, bn, cn, in Fig. 73.

77. Find the voltages $\mathbf{V}_{dc}$, $\mathbf{V}_{ef}$, and $\mathbf{V}_{fd}$ in Fig. 74. What is the phase displacement between these voltages?

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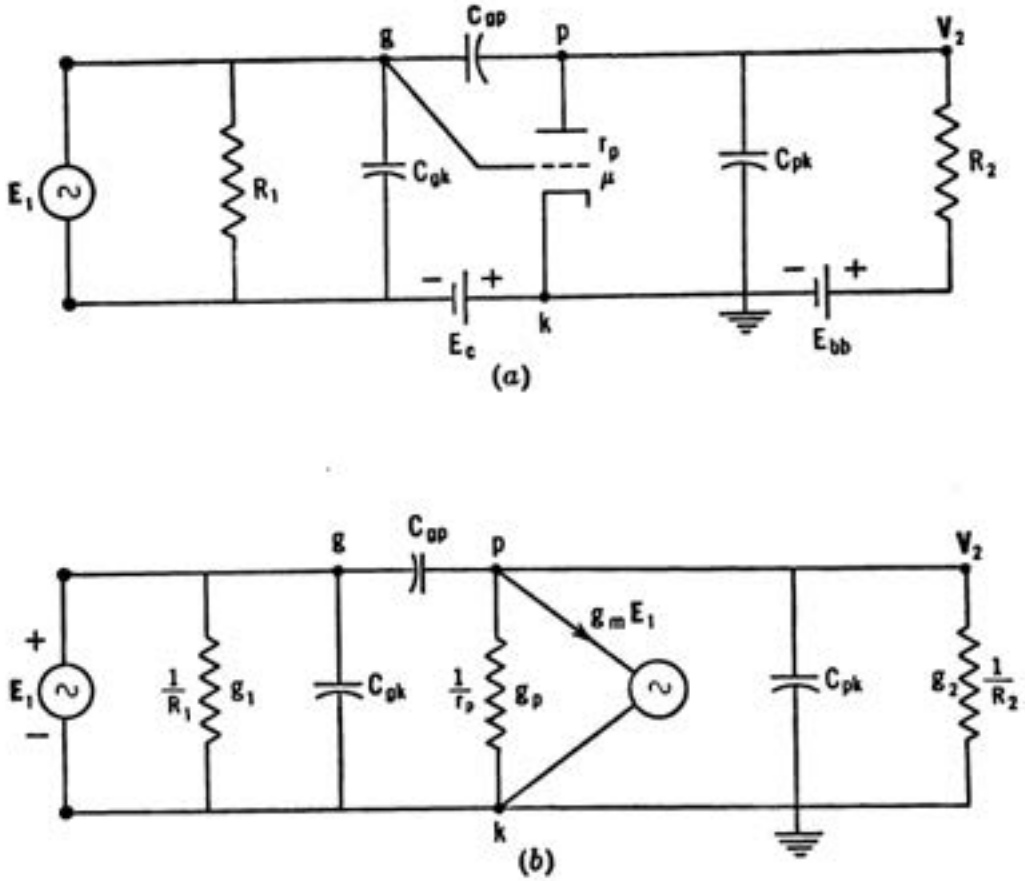

FIG. 72. See Problems 72 and 73.

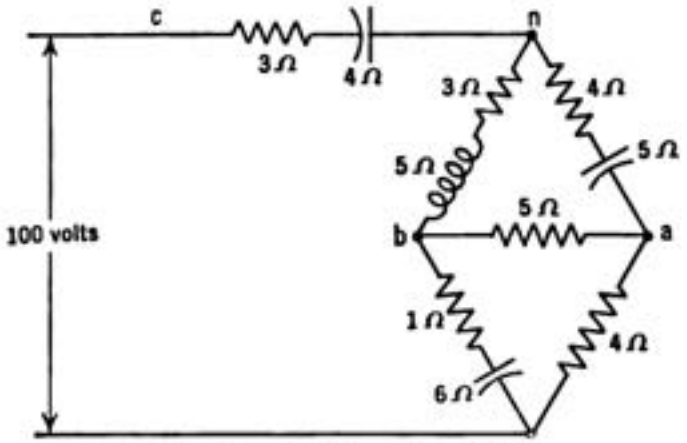

FIG. 73. See Problems 74 and 76.

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78. What relationship between the Z's of Fig. 75 will make $I_3 = 0$ regardless of the magnitude of E_{in}? *Hint:* A simple method of solution is to transform the Z_2-Z_3-Z_4 and Z_6-Z_7-Z_8 deltas to equivalent wyes and make the Z_{24} and Z_{68} legs of the latter the negatives of each other to produce a short circuit across the load.

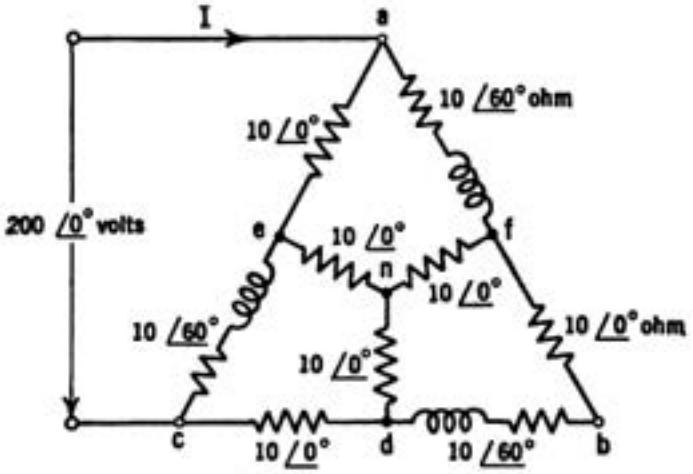

FIG. 74. See Problem 77.

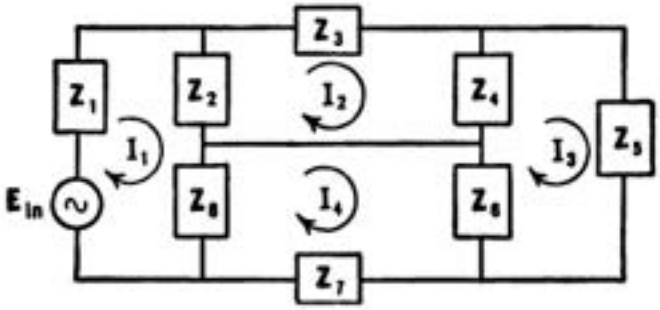

FIG. 75. See Problem 78.

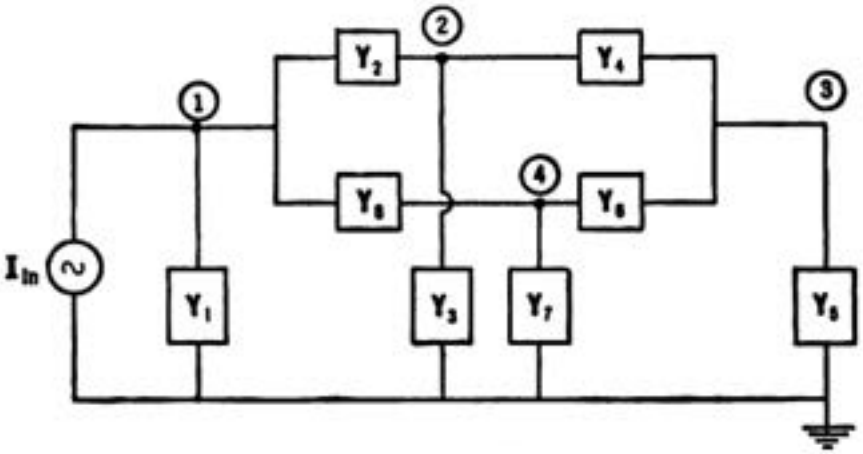

FIG. 76. See Problem 79.

79. What relationship between the Y's of Fig. 76 will make the voltage V_3 (relative to ground) equal to zero regardless of the magnitude of I_{in}?

80. What explanation can be given for the following current ratings of a 15,000-volt, 200-$\mu\mu$f transmitting condenser?

4 amperes at 600 kc
5 amperes at 1000 kc
6 amperes at 1500 kc

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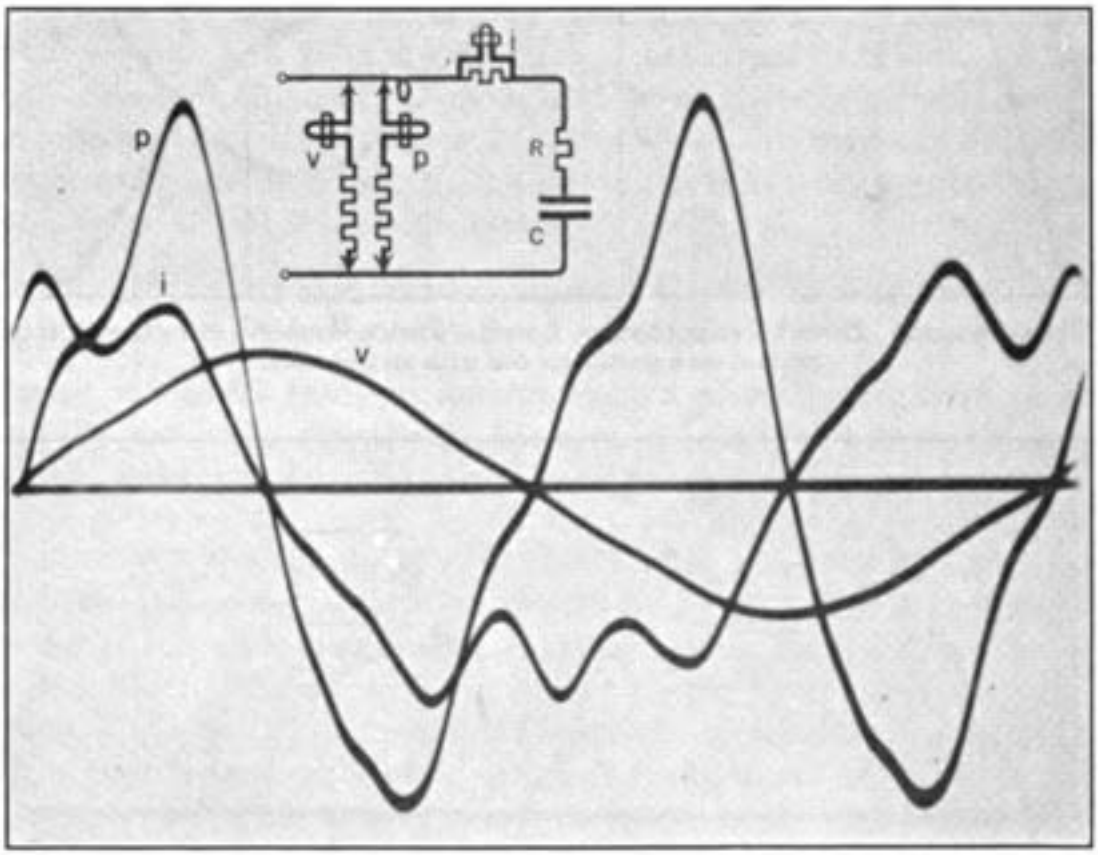
CHAPTER VI

NON-SINUSOIDAL WAVES

Complex Waves. The circuit theory that has been presented in the foregoing chapters has been based upon sine-wave variations of voltage and current, and only sine waves have been considered in the calculations. In many branches of electrical engineering non-sinusoidal waves are as common as sinusoidal waves, and in all branches non-sinusoidal



OSCILLOGRAM 1. v-wave form of voltage generated by a particular alternator. i-wave form of current which flows through a capacitive circuit element. Note the relatively larger harmonics in the current wave. p-wave form of instantaneous power. $E = 120$ volts (eff.), $I = 3.9$ amperes (eff.), $P_{av} = 20$ watts, $f = 60$ cycles.

waves must occasionally be given attention. Examples of non-sinusoidal waves are shown in Oscillograms 1, 2, and 3. Even though the voltage wave in Oscillogram 1 is nearly sinusoidal, the current through the capacitive circuit is greatly distorted. Also in Oscillogram 2 the current is non-sinusoidal even though the impressed voltage is practically

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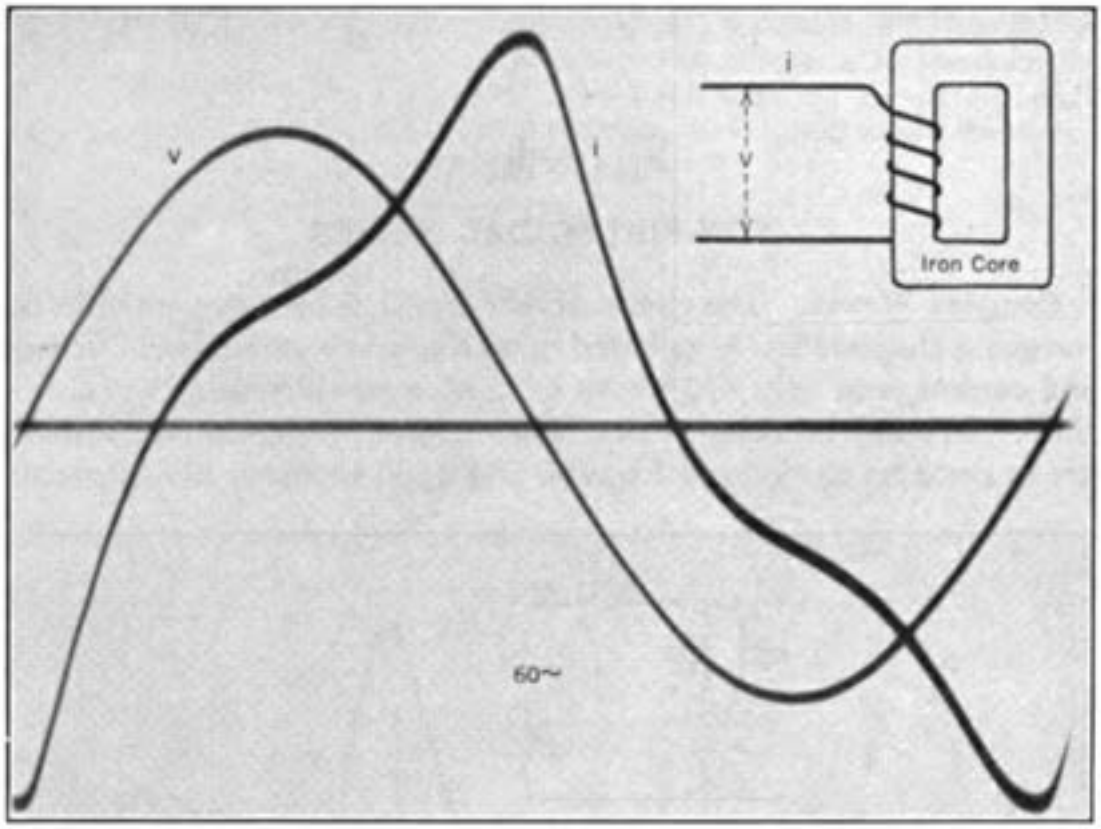
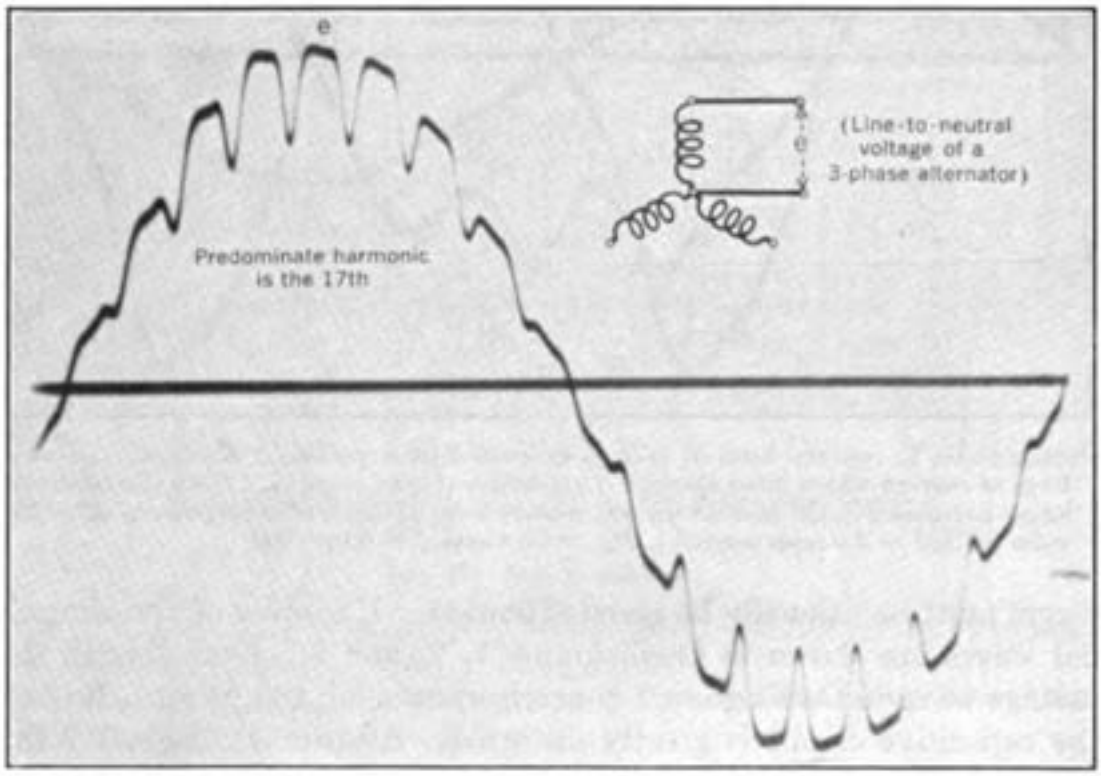


OSCILLOGRAM 2. Distorted current wave, i, results when a sine wave of voltage, v, is impressed on a particular coil with an iron core.



OSCILLOGRAM 3. Wave form produced by an open-slot type of generator.

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a sinusoid. Oscillogram 3 shows the effect on the voltage wave form of an alternator due to open slots. The predominant harmonic in this case can easily be determined by the methods discussed in this chapter. The method of making circuit calculations when non-sinusoidal wave forms are encountered will also be given.

Most non-sinusoidal waves found in electrical engineering can be expressed in terms of sine-wave components of different frequencies. Under these conditions each sine component may be handled according to the laws governing the calculations of sine waves. The results of all component analyses are combined according to certain laws to form the composite or final analysis. There are, however, certain limitations to representing non-sinusoidal waves in terms of sine components.

Any periodic wave which is single-valued and continuous except for a finite number of finite discontinuities, and which does not have an infinite number of maxima or minima in the neighborhood of any point, may be represented by the sum of a number of sine waves of different frequencies. As an equation, the above theorem takes the following form and is known as a Fourier series:

$$y = f(x) = A_0 + A_1 \sin x + B_1 \cos x + A_2 \sin 2x + B_2 \cos 2x + A_3 \sin 3x + B_3 \cos 3x + \cdots + A_n \sin nx + B_n \cos nx \qquad (1)$$

Except in special cases an infinite number of components are theoretically required. Practically, however, only a few terms are necessary in most instances because of the relatively small effect of the terms of higher frequency. Since the wave which is represented by equation (1) is made up of a number of sine waves of different frequencies, it is called a complex wave. It is apparent that each component of this wave is sinusoidal and that each component in itself may be handled by the methods previously outlined for calculating sine waves. The facility with which sinusoidal components of a complex wave may be manipulated is sufficient justification for expressing a non-sinusoidal wave in such terms as equation (1) even though the equation of the wave may be known in terms of some other function of x.

Wave Analysis. Usually, a photographic record of the wave will be obtained through oscillographic analysis or other means. The determination of the Fourier equation which specifies a particular wave is called wave analysis. Wave analysis consists simply of determining the coefficients A_0, A_1, B_1, etc., of equation (1). These coefficients are determined by some operation on equation (1) that will eliminate all terms except the desired quantity. Then the desired coefficient may be evaluated. Thus, to determine A_0, it is necessary simply to

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multiply the equation by dx and to integrate between 0 and 2π, as shown below.

$$\int_0^{2\pi} y\,dx = \int_0^{2\pi} A_0\,dx + \int_0^{2\pi} A_1 \sin x\,dx + \int_0^{2\pi} B_1 \cos x\,dx +$$
$$\int_0^{2\pi} A_2 \sin 2x\,dx + \int_0^{2\pi} B_2 \cos 2x\,dx + \int_0^{2\pi} A_3 \sin 3x\,dx +$$
$$\int_0^{2\pi} B_3 \cos 3x\,dx + \cdots + \int_0^{2\pi} A_n \sin nx\,dx + \int_0^{2\pi} B_n \cos nx\,dx \quad (2)$$

or
$$\int_0^{2\pi} y\,dx = A_0 \int_0^{2\pi} dx = 2\pi A_0$$

and
$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,dx \quad (3)$$

To find A_1, equation (1) may be multiplied by $\sin x\,dx$ and integrated from 0 to 2π. Thus

$$\int_0^{2\pi} y \sin x\,dx = \int_0^{2\pi} A_0 \sin x\,dx + \int_0^{2\pi} A_1 \sin^2 x\,dx +$$
$$\int_0^{2\pi} B_1 \cos x \sin x\,dx + \int_0^{2\pi} A_2 \sin 2x \sin x\,dx + \int_0^{2\pi} B_2 \cos 2x \sin x\,dx +$$
$$\int_0^{2\pi} A_3 \sin 3x \sin x\,dx + \int_0^{2\pi} B_3 \cos 3x \sin x\,dx + \cdots +$$
$$\int_0^{2\pi} A_n \sin nx \sin x\,dx + \int_0^{2\pi} B_n \cos nx \sin x\,dx \quad (4)$$

It is obvious that $\int_0^{2\pi} A_0 \sin x\,dx$ is zero since it represents the area under a sine wave for a complete cycle. There are four other types of terms. They are

(a) $\int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} (\frac{1}{2} - \frac{1}{2}\cos 2x)\,dx = \frac{2\pi}{2} = \pi,$

(b) $\int_0^{2\pi} \sin 2x \sin x\,dx$, which is of the general type:

$\int_0^{2\pi} \sin mx \sin nx\,dx = 0$, when m and n are different integers,[1]

(c) $\int_0^{2\pi} \cos mx \sin nx\,dx = 0$, when m and n are different integers,[2] and

(d) $\int_0^{2\pi} \cos x \sin x\,dx = 0.$

[1] This may be readily proved by substituting for $\sin mx \sin nx$ its equivalent $\frac{1}{2}[\cos(mx - nx) - \cos(mx + nx)]$.

[2] This may be readily proved by substituting for $\cos mx \sin nx$ its equivalent $\frac{1}{2}[\sin(mx + nx) - \sin(mx - nx)]$.

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The student should prove statements a, b, c, and d by carrying out the operations indicated. If the above facts are used, equation (4) reduces to

$$\int_0^{2\pi} y \sin x \, dx = A_1 \pi$$

or

$$A_1 = \frac{1}{\pi} \int_0^{2\pi} y \sin x \, dx \tag{5}$$

To evaluate the coefficient of the cosine term B_1, equation (1) is multiplied by $\cos x \, dx$ and integrated from 0 to 2π. Thus

$$\int_0^{2\pi} y \cos x \, dx = \int_0^{2\pi} A_0 \cos x \, dx + \int_0^{2\pi} A_1 \sin x \cos x \, dx$$
$$+ \int_0^{2\pi} B_1 \cos^2 x \, dx + \int_0^{2\pi} A_2 \sin 2x \cos x \, dx + \int_0^{2\pi} B_2 \cos 2x \cos x \, dx$$
$$+ \int_0^{2\pi} A_3 \sin 3x \cos x \, dx + \int_0^{2\pi} B_3 \cos 3x \cos x \, dx + \cdots$$
$$+ \int_0^{2\pi} A_n \sin nx \cos x \, dx + \int_0^{2\pi} B_n \cos nx \cos x \, dx \tag{6}$$

If the relations stated above in a, b, c, and d are used, equation (6) becomes

$$\int_0^{2\pi} y \cos x \, dx = B_1 \int_0^{2\pi} \cos^2 x \, dx = B_1 \pi$$

or

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} y \cos x \, dx \tag{7}$$

Similarly,

$$A_2 = \frac{1}{\pi} \int_0^{2\pi} y \sin 2x \, dx \tag{8}$$

$$B_2 = \frac{1}{\pi} \int_0^{2\pi} y \cos 2x \, dx \tag{9}$$

$$A_3 = \frac{1}{\pi} \int_0^{2\pi} y \sin 3x \, dx \tag{10}$$

$$B_3 = \frac{1}{\pi} \int_0^{2\pi} y \cos 3x \, dx \tag{11}$$

$$A_n = \frac{1}{\pi} \int_0^{2\pi} y \sin nx \, dx \tag{12}$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} y \cos nx \, dx \tag{13}$$

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Various analytical and graphical methods may be employed to evaluate the coefficients of equations (3), (12), and (13). Two general methods are outlined below.

Analytical Method. If the equation of y in terms of x is known in some mathematical form, the wave may be analyzed analytically. This method is the least laborious but it cannot be employed if the function of x is not known analytically. The function of x employed need not throughout its entire range represent the particular wave to be analyzed. It is necessary to have the function of x only over the interval of periodicity, namely, 2π. Not even a single function of x is necessary. Several different ones may be used and the complete integral from 0 to 2π may be obtained from a sum of the integrals of the several functions, each taken over the interval in which it follows the curve to be analyzed.

The details connected with writing a Fourier series to represent a specified wave form are illustrated by the following examples.

Example 1. Let it be required to write the Fourier series which will represent the sawtooth wave form shown in Fig. 1. It will be observed that this wave form is

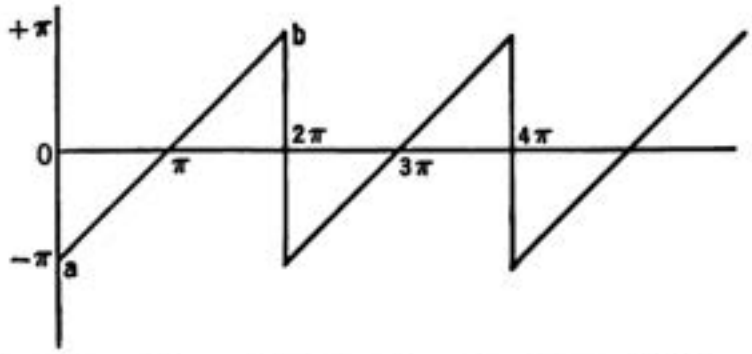

FIG. 1. A type of wave which is easily analyzed by analytical methods.

simply a straight-line variation, ranging from $y = -\pi$ to $y = +\pi$ over one complete cycle. This straight-line variation may be expressed analytically (between $x = 0$ and $x = 2\pi$) as:

$$y = f(x) = x - \pi$$

It should be noted that the above analytical expression for y in terms of x gives no indication of the various harmonics which are present in the wave, whereas a Fourier-series representation of the wave will yield this information.

From equation (3):

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi}(x - \pi)\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2} - \pi x\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - 2\pi^2\right] = 0$$

The fact that A_0 is zero could have been determined by inspection of Fig. 1 since it is obvious from the figure that the negative half of the wave is equal in area to the positive half.

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From equation (12):

$$A_n = \frac{1}{\pi}\int_0^{2\pi}(x-\pi)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{2\pi} x\sin nx\,dx - \int_0^{2\pi}\pi\sin nx\,dx\right]$$

$$\int_0^{2\pi} x\sin nx\,dx = \left[-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right]_0^{2\pi}$$

as may be proved by differentiation of the right member and $\int_0^{2\pi}\pi\sin nx\,dx = 0$ for all integral values of n. Therefore:

$$A_n = \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right]_0^{2\pi} = -\frac{2}{n}$$

whence

$$A_1 = -\tfrac{2}{1};\quad A_2 = -\tfrac{2}{2};\quad A_3 = -\tfrac{2}{3};\quad A_4 = -\tfrac{2}{4};\quad \text{etc.}$$

From equation (13):

$$B_n = \frac{1}{\pi}\int_0^{2\pi}(x-\pi)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{2\pi} x\cos nx\,dx - \int_0^{2\pi}\pi\cos nx\,dx\right]$$

$$\int_0^{2\pi} x\cos nx\,dx = \left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

as may be proved by differentiation of the right member and $\int_0^{2\pi}\pi\cos nx\,dx = 0$ for all integral values of n. Therefore:

$$B_n = \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi} = 0 \quad \text{(for all integral values of } n\text{)}$$

Hence all the coefficients B_1, B_2, B_3, etc., in equation (1) are 0 and the Fourier equation of the wave shown in Fig. 1 becomes:

$$y = -2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots + \frac{1}{n}\sin nx\right)$$

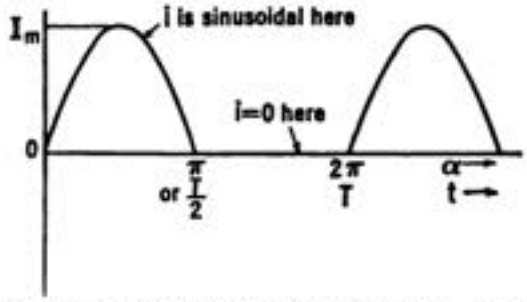

Fig. 2. Half-wave rectification of a sine wave. See example 2.

Example 2. Let it be required to write the first four terms of the Fourier series which will represent the wave form shown in Fig. 2. From Fig. 2, it is plain that i may be expressed analytically between the limits of 0 and 2π as two separate functions. That is:

$$i = I_m \sin\alpha \qquad [\text{between } \alpha \text{ (or } \omega t) = 0 \text{ and } \alpha \text{ (or } \omega t) = \pi]$$

and

$$i = 0 \qquad [\text{between } \alpha \text{ (or } \omega t) = \pi \text{ and } \alpha \text{ (or } \omega t) = 2\pi]$$

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From equation (3):

$$A_0 = \frac{1}{2\pi}\left[\int_0^{\pi} I_m \sin\alpha\, d\alpha + \int_{\pi}^{2\pi} 0\, d\alpha\right] = \frac{I_m}{2\pi}\Big[-\cos\alpha\Big]_0^{\pi}$$

$$= \frac{I_m}{\pi} = 0.318 I_m$$

From equation (5):

$$A_1 = \frac{1}{\pi}\left[\int_0^{\pi} (I_m \sin\alpha)\sin\alpha\, d\alpha + \int_{\pi}^{2\pi} (0)\sin\alpha\, d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[\int_0^{\pi} (\tfrac{1}{2} - \tfrac{1}{2}\cos 2\alpha)\, d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[\frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha\right]_0^{\pi} = \frac{I_m}{\pi}\left[\frac{\pi}{2}\right] = 0.500 I_m$$

From equation (12) it follows that A_2, A_3, A_4, etc., are all zero because:

$$A_n = \frac{1}{\pi}\left[\int_0^{\pi} (I_m \sin\alpha)\sin n\alpha\, d\alpha\right] = 0 \text{ (for } n \neq 0 \text{ and } n \neq 1)$$

The above evaluation of A_n is evident if $(\sin\alpha \sin n\alpha)$ is replaced by its equivalent $\frac{1}{2}[\cos(n-1)\alpha - \cos(n+1)\alpha]$. Thus A_2, A_3, A_4, etc., are zero because:

$$A_n = \frac{1}{\pi}\int_0^{\pi} \tfrac{1}{2}[\cos(n-1)\alpha - \cos(n+1)\alpha]\, d\alpha$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)\alpha}{(n-1)} - \frac{\sin(n+1)\alpha}{(n+1)}\right]_0^{\pi} = 0 \qquad \begin{cases}\text{for } n \neq 0\\ \text{and } n \neq 1\end{cases}$$

From equation (7):

$$B_1 = \frac{1}{\pi}\left[\int_0^{\pi} (I_m \sin\alpha)\cos\alpha\, d\alpha + \int_{\pi}^{2\pi} (0)\cos\alpha\, d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[\int_0^{\pi} \frac{\sin 2\alpha}{2}\, d\alpha\right] = \frac{I_m}{\pi}\left[-\frac{\cos 2\alpha}{4}\right]_0^{\pi} = 0$$

From equation (13):

$$B_n = \frac{1}{\pi}\left[\int_0^{\pi} (I_m \sin\alpha)\cos n\alpha\, d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[\int_0^{\pi}\left(\frac{\sin(\alpha + n\alpha)}{2} + \frac{\sin(\alpha - n\alpha)}{2}\right) d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[-\frac{\cos(1+n)\alpha}{2(1+n)} - \frac{\cos(1-n)\alpha}{2(1-n)}\right]_0^{\pi} \qquad \begin{cases}\text{for } n \neq 0\\ \text{and } n \neq 1\end{cases}$$

For $n = 2$:

$$B_2 = \frac{I_m}{\pi}\left[-\frac{\cos 3\alpha}{6} - \frac{\cos(-\alpha)}{-2}\right]_0^{\pi} = \frac{I_m}{\pi}\left[+\frac{1}{6} + \frac{1}{6} - \frac{1}{2} - \frac{1}{2}\right]$$

$$= -\frac{2I_m}{3\pi} = -0.212 I_m$$

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Similarly for $n = 3$,

$$B_3 = 0$$

and for $n = 4$,

$$B_4 = -0.0424I_m$$

The Fourier series which represents the wave form shown in Fig. 2 is therefore:

$$i = 0.318I_m + 0.500I_m \sin \alpha - 0.212I_m \cos 2\alpha - 0.0424I_m \cos 4\alpha - \cdots$$

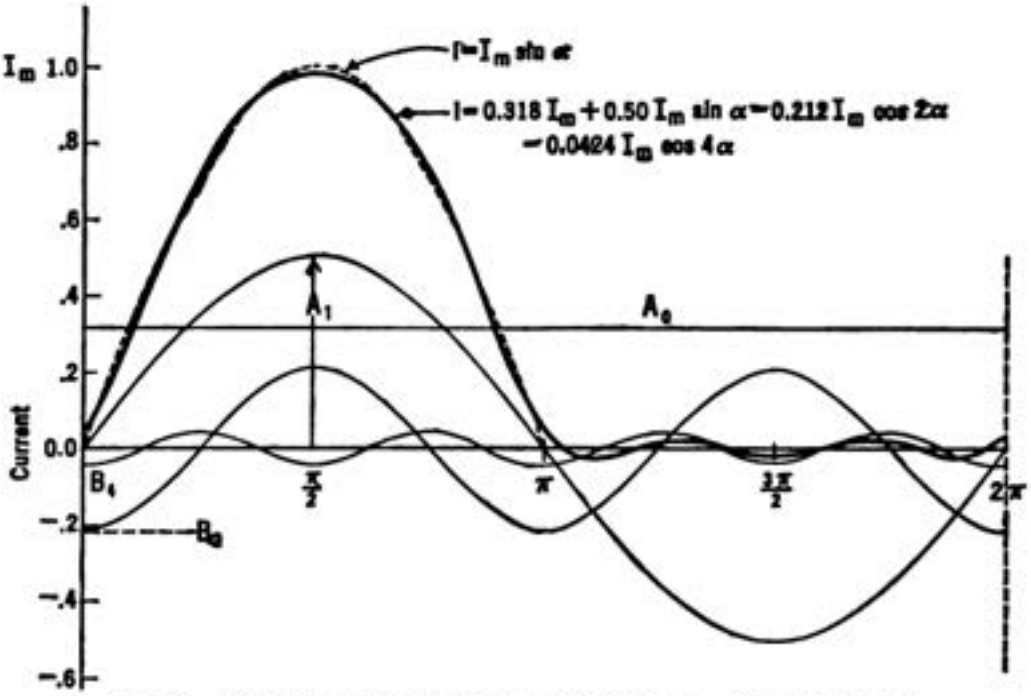

FIG. 3. Components of half-wave rectification. See example 2.

If the above four terms are combined graphically as shown in Fig. 3, the resultant wave approaches the original wave form shown in Fig. 2 to a fair degree of accuracy. The inclusion of more terms in the Fourier series will, of course, improve the correspondence between the resultant wave of Fig. 3 and the original wave form.

Problem 1. (*a*) Write the Fourier series which represents the wave form shown in Fig. 4 out to and including the A_3 term of the series. *Note:* $e = 100$ between $\alpha = 0$ and $\alpha = \pi$, and $e = 0$ between $\alpha = \pi$ and $\alpha = 2\pi$.

Ans.: $e = 50 + 63.7 \sin \alpha + 21.2 \sin 3\alpha$ volts.

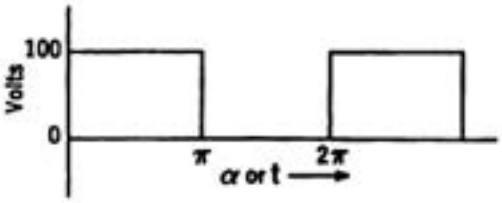

FIG. 4. See Problem 1.

(*b*) Show by means of a sketch the manner in which the above three components combine to approximate the flat-topped wave shown in Fig. 4.

Fourier Analysis of Symmetrical Triangular and Rectangular Waves. Symmetrical waves of triangular and rectangular shape are shown in Figs. 5 (solid lines) and 6 respectively. Since these wave forms are often used in the analyses of certain basic problems, it is convenient to have the Fourier equations of these waves readily available.

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Triangular Wave. To facilitate analyzing, the triangular wave may be considered to be composed of several pieces, namely, the straight lines *oa*, *ac*, and *cd*. If the point slope form of equation for a straight line is applied, the equations of these lines will be found to be:

$$y_{oa} = \frac{2x}{\pi}; \quad y_{ac} = -\frac{2x}{\pi} + 2; \quad y_{cd} = \frac{2x}{\pi} - 4$$

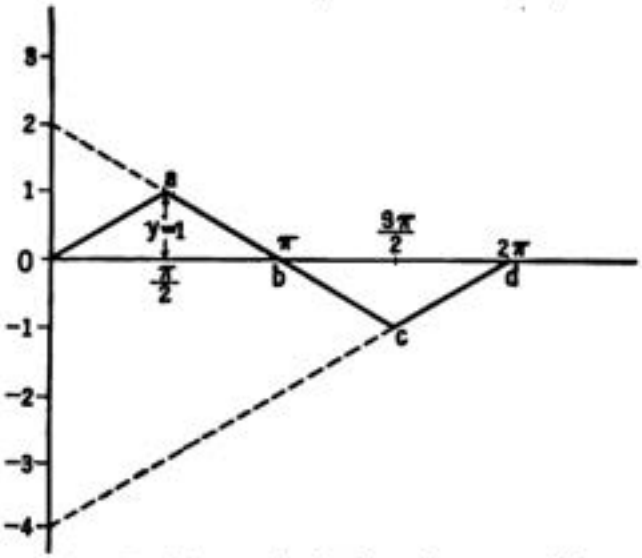

FIG. 5. Symmetrical triangular wave with a maximum value of 1.

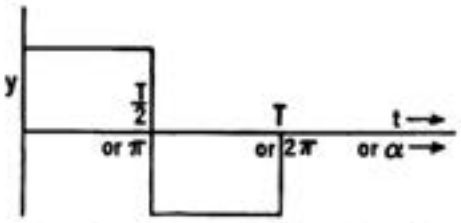

FIG. 6. Symmetrical rectangular wave.

Applying equations (3), (12), and (13) gives:

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,dx = \frac{1}{2\pi}\left\{\int_0^{\pi/2}\frac{2x}{\pi}dx + \int_{\pi/2}^{3\pi/2}\left(\frac{-2x}{\pi}+2\right)dx + \int_{3\pi/2}^{2\pi}\left(\frac{2x}{\pi}-4\right)dx\right\}$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} y\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2}\frac{2x}{\pi}\sin nx\,dx + \int_{\pi/2}^{3\pi/2}\left(\frac{-2x}{\pi}+2\right)\sin nx\,dx + \int_{3\pi/2}^{2\pi}\left(\frac{2x}{\pi}-4\right)\sin nx\,dx\right\}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2}\frac{2x}{\pi}\cos nx\,dx + \int_{\pi/2}^{3\pi/2}\left(\frac{-2x}{\pi}+2\right)\cos nx\,dx + \int_{3\pi/2}^{2\pi}\left(\frac{2x}{\pi}-4\right)\cos nx\,dx\right.$$

Evaluation of the above for various values of n by ordinary calculus methods gives the equation of the wave in terms of a Fourier series as follows:

$$y = \frac{8}{\pi^2}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \cdots \text{etc.}\right) \quad (14)$$

It will be shown later how it is possible to determine from inspection, that, in certain classes of waves as typified by the above example, the terms represented by B_n must be zero.

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The results of the above analysis may be generalized and the equation of a symmetrical triangular wave written as

$$y = A_1 \sin \omega t - \frac{A_1}{3^2} \sin 3\omega t + \frac{A_1}{5^2} \sin 5\omega t - \frac{A_1}{7^2} \sin 7\omega t + \cdots \text{etc.} \quad (14a)$$

where x of equation (14) has been replaced by ωt and A_1 equals $8/\pi^2$ times the maximum ordinate of the triangular wave. Theoretically, there is an infinite number of terms and the progression continues as the first four terms indicate.

Rectangular Wave. The rectangular wave is much used in the analysis of a-c machinery and has for its Fourier equation:

$$y = A_1 \sin \omega t + \frac{A_1}{3} \sin 3\omega t + \frac{A_1}{5} \sin 5\omega t + \frac{A_1}{7} \sin 7\omega t + \cdots \text{etc.} \quad (15)$$

where $A_1 = \frac{4}{\pi}$ times the height of the rectangle. Again there is an infinite number of terms which may be written as indicated by the first four terms shown. Figure 7 shows a graphical representation of the first three terms and illustrates that a fair approximation to the resultant wave is obtained by the addition of very few terms.

Problem 2. Analyze the rectangular wave shown in Fig. 6 by the analytical method to prove the validity of equation (15).

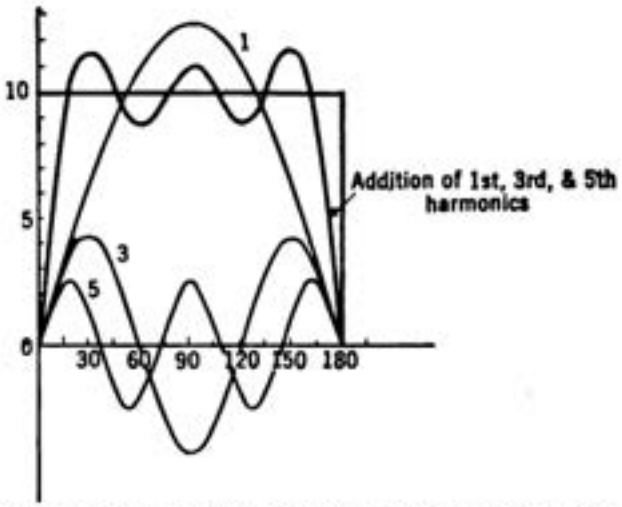

FIG. 7. The addition of only three harmonics gives a fair approximation of the rectangular wave.

Graphical Method. A second method of evaluating equations (3), (12), and (13) involves the evaluation of the integrals by a step-by-step method. The equation of y in terms of x is usually unknown, and for the majority of waves encountered it would be very cumbersome and laborious to establish equations which would yield pieces of the wave. It is under these conditions that the step-by-step method (sometimes called the graphical method) or its equivalent is employed. The details of this method follow.

Suppose the wave of Fig. 8 is to be analyzed. Equation (3) is simply the average height of the curve over 2π radians. It is found by dividing the area under the curve by the base. Any method of determining the area, such as counting squares or by use of a planimeter, may be em-

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ployed. If the areas of the positive and negative loops are the same, A_0 is zero. Hence for waves having adjacent loops of the same shape and area with respect to some horizontal axis, the constant A_0 when present simply indicates how much the whole wave has been raised or lowered from symmetry about the axis of abscissas. For graphical analysis, equation (5) may be written

$$A_1 = \frac{1}{\pi}\sum_0^{2\pi} y \sin x \, \Delta x \tag{16}$$

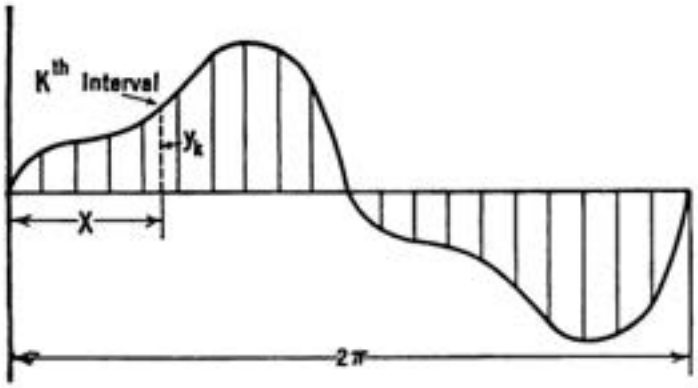

FIG. 8. Preparation of a wave for analysis by the graphical method.

Let 2π radians in Fig. 8 be divided into m equal parts. Then $\Delta x = \frac{2\pi}{m}$ and x, the distance to the midpoint of the kth interval, is $k\frac{2\pi}{m} - \frac{1}{2}\left(\frac{2\pi}{m}\right)$ or $(k - \frac{1}{2})\frac{2\pi}{m}$. Equation (16) now becomes

$$A_1 = \frac{1}{\pi}\sum_0^m \left[y_k \sin (k - \tfrac{1}{2})\frac{2\pi}{m}\right]\frac{2\pi}{m}$$

$$= \frac{2\pi}{m}\,\frac{1}{\pi}\sum_0^m y_k \sin\left[(k - \tfrac{1}{2})\frac{2\pi}{m}\right]$$

$$= \frac{2}{m}\sum_0^m y_k \sin\left[(k - \tfrac{1}{2})\frac{2\pi}{m}\right] \tag{17}$$

Similarly,

$$B_1 = \frac{2}{m}\sum_0^m y_k \cos (k - \tfrac{1}{2})\frac{2\pi}{m} \tag{18}$$

and

$$A_2 = \frac{2}{m}\sum_0^m y_k \sin 2\,(k - \tfrac{1}{2})\frac{2\pi}{m} \tag{19}$$

The first form of equation (17) shows that A_1 is $1/\pi$ times the area under a *new curve*, which would be obtained by plotting corresponding ordinates

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of the original curve multiplied by the sine of the angle to the ordinate in question. For A_n the ordinates of the new curve would be obtained by multiplying selected ordinates of the original curve by the sine of n times the fundamental angular distance to the respective ordinates. An analogous procedure is employed for cosine terms. Looked at in another way, equation (17) indicates that A_1 is twice the average ordinate of the *new* curve, which would be obtained by plotting corresponding ordinates of the original curve multiplied by the sine of the angle to the ordinate in question. Multiplying and dividing equation (16) or (17) by 2 makes this statement evident. Thus

$$A_1 = 2\left[\frac{1}{2\pi}\sum_{0}^{2\pi} y \sin x \, \Delta x\right]$$

Similar interpretations may be drawn regarding the other coefficients of the Fourier series. The summations are best carried out in tabular form, and for this purpose a more or less standardized system is used. The tables which are used are called analyzing tables. One form of analyzing tables for odd harmonics up to and including the seventh are shown in heavy type on the following pages. (The light type refers to specific values for an illustrative example.)

It will be shown in a subsequent article that waves having symmetrical positive and negative loops cannot contain even harmonics. Under these conditions it is unnecessary to evaluate A_2, B_2, A_4, B_4, etc. Also, when the wave being analyzed consists of odd harmonics only, it is necessary only to take the summation over the first 180°. Since the summation over the second 180° would be the same as that over the first 180°, the total summation over 360° can be obtained by multiplying the summation over 180° by 2. If m is taken as the number of intervals in 360°, the summation over 180° may be multiplied by $4/m$ instead of multiplying the summation over 360° by $2/m$ as shown in equation (17). Whereas equations (17), (18), and (19) indicate that the midordinate of the interval selected should be used, it is customary to use the ordinate and the angle corresponding to those given in the tables. When the intervals are as small as 5°, the difference between the two schemes is negligible.

Example 3. Given the experimentally determined wave form shown in Oscillogram 4. Find the Fourier equation, employing analyzing tables similar to those given on pages 175–178.

Solution. Ordinates at every 5° are constructed as shown in Oscillogram 4. The magnitude of each is scaled and set in the column for ordinates opposite the corresponding angle in the column for angles. The product of the ordinates and the corresponding sines and cosines of n times the angles are obtained and tabulated as shown in the analyzing tables on pages 175–178.

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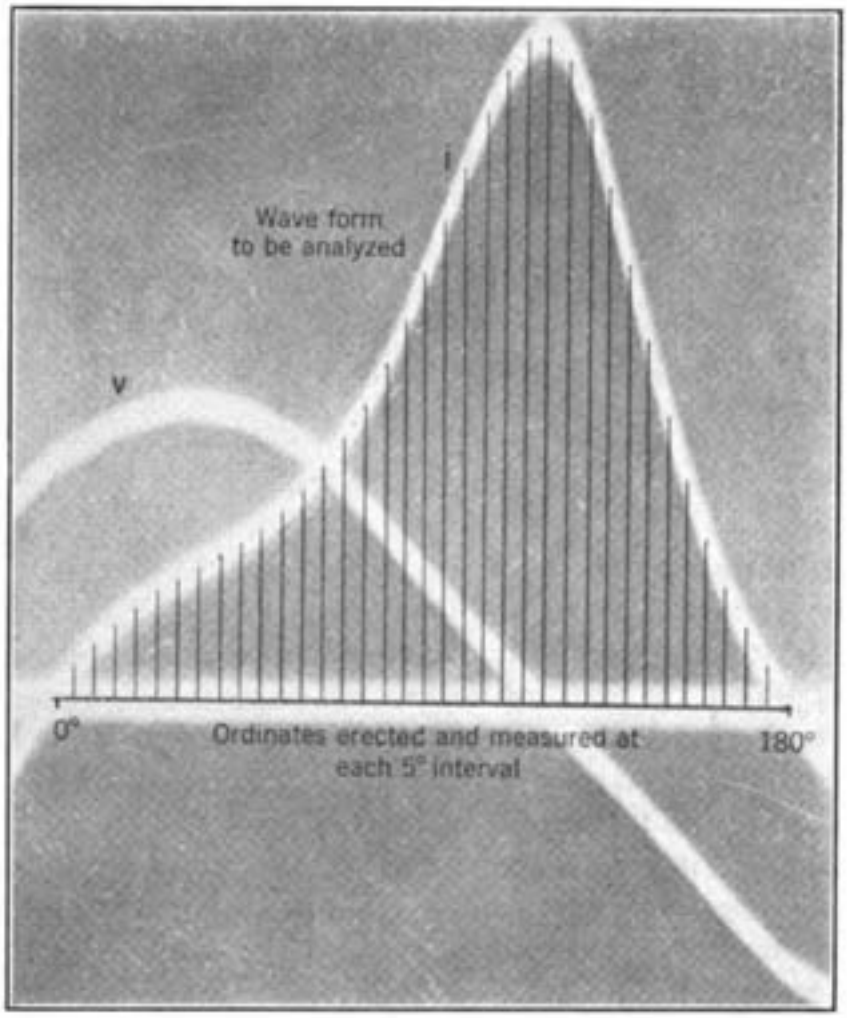

OSCILLOGRAM 4. See example 3.

For the particular wave which is being analyzed:

$A_1 = 82.45$ units	$A_5 = -5.38$ units
$B_1 = -22.11$ units	$B_5 = -3.65$ units
$A_3 = -0.92$ unit	$A_7 = 2.01$ units
$B_3 = 26.2$ units	$B_7 = -1.29$ units

The Fourier equation of the wave is, therefore,

$$i = 82.45 \sin \omega t - 22.11 \cos \omega t - 0.92 \sin 3\omega t$$
$$+ 26.2 \cos 3\omega t - 5.38 \sin 5\omega t - 3.65 \cos 5\omega t$$
$$+ 2.01 \sin 7\omega t - 1.29 \cos 7\omega t$$

The fundamental frequency in this particular case is 60 cycles per second. Therefore ω is equal to 377 radians per second.

The actual number of terms in the Fourier equation in any particular case can usually be reduced because it is always possible to combine sine and cosine waves of the same frequencies. For example, consider the general wave

$$y = A_1 \sin \omega t + B_1 \cos \omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t$$
$$+ A_3 \sin 3\omega t + B_3 \cos 3\omega t$$

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FUNDAMENTAL

1	2	3	4	5	6	7	8	9
sin x	Products (y sin x)		Ordinate No.	Angle x to ordinate	Meas. ordinate (y)	cos x	Products (y cos x)	
	+	−					+	−
.0872	0.5		1	5°	5.9	.9962	5.9	
.1736	1.7		2	10°	10.0	.9848	9.8	
.2588	3.5		3	15°	13.4	.9659	13.0	
.3420	5.6		4	20°	16.4	.9397	15.4	
.4226	8.2		5	25°	19.4	.9063	17.6	
.5000	10.9		6	30°	21.8	.8660	18.9	
.5736	13.5		7	35°	23.6	.8192	19.3	
.6428	16.6		8	40°	25.9	.7660	19.9	
.7071	19.9		9	45°	28.1	.7071	19.9	
.7660	23.5		10	50°	30.7	.6428	19.7	
.8192	27.8		11	55°	33.9	.5736	19.4	
.8660	32.4		12	60°	37.4	.5000	18.7	
.9063	38.1		13	65°	42.0	.4226	17.7	
.9397	43.9		14	70°	46.7	.3420	16.0	
.9659	51.0		15	75°	52.8	.2588	13.7	
.9848	59.1		16	80°	60.0	.1736	10.4	
.9962	67.5		17	85°	67.7	.0872	5.9	
1.0000	76.4		18	90°	76.4	.0000	0.0	
.9962	86.2		19	95°	86.5	−.0872		7.5
.9848	94.1		20	100°	95.5	−.1736		16.6
.9659	101.5		21	105°	105.1	−.2588		27.2
.9397	106.0		22	110°	112.8	−.3420		38.6
.9063	106.4		23	115°	117.4	−.4226		49.6
.8660	102.7		24	120°	118.5	−.5000		59.3
.8192	93.5		25	125°	114.2	−.5736		65.5
.7660	80.4		26	130°	104.9	−.6428		67.4
.7071	64.6		27	135°	91.4	−.7071		64.6
.6428	50.4		28	140°	78.3	−.7660		60.0
.5736	37.2		29	145°	65.0	−.8192		53.2
.5000	25.6		30	150°	51.1	−.8660		44.3
.4226	16.9		31	155°	40.0	−.9063		36.3
.3420	10.1		32	160°	29.4	−.9397		27.6
.2588	5.5		33	165°	21.3	−.9659		20.6
.1736	2.4		34	170°	14.0	−.9848		13.8
.0872	0.6		35	175°	7.1	−.9962		7.1
.0000	0.0		36	180°	0.0	−1.0000		0.0
Sum of products	1484.2	0					261.2	659.2
	1484.2						−398.0	

$$A_1 = \frac{1484}{36} \times 2 = 82.45$$

$$B_1 = \frac{-398.0}{36} \times 2 = -22.11$$

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THIRD HARMONIC

1	2	3	4	5	6	7	8	9
sin 3x	Products (y sin 3x)		Ordinate No.	Angle x to ordinate	Meas. ordinate (y)	cos 3x	Products (y cos 3x)	
	+	−					+	−
.2588	1.5		1	5°	5.9	.9659	5.7	
.5000	5.0		2	10°	10.0	.8660	8.7	
.7071	9.5		3	15°	13.4	.7071	9.5	
.8660	14.2		4	20°	16.4	.5000	8.2	
.9659	18.8		5	25°	19.4	.2588	5.0	
1.0000	21.8		6	30°	21.8	.0000	0.0	
.9659	22.8		7	35°	23.6	−.2588		6.1
.8660	22.4		8	40°	25.9	−.5000		12.9
.7071	19.9		9	45°	28.1	−.7071		19.9
.5000	15.4		10	50°	30.7	−.8660		26.6
.2588	8.8		11	55°	33.9	−.9659		32.8
.0000	0.0		12	60°	37.4	−1.0000		37.4
−.2588		10.9	13	65°	42.0	−.9659		40.6
−.5000		23.4	14	70°	46.7	−.8660		40.5
−.7071		37.4	15	75°	52.8	−.7071		37.4
−.8660		52.0	16	80°	60.0	−.5000		30.0
−.9659		65.5	17	85°	67.7	−.2588		17.5
−1.0000		76.4	18	90°	76.4	−.0000		0.0
−.9659		83.7	19	95°	86.5	.2588	22.4	
−.8660		82.8	20	100°	95.5	.5000	47.8	
−.7071		74.4	21	105°	105.1	.7071	74.4	
−.5000		56.4	22	110°	112.8	.8660	97.7	
−.2588		30.4	23	115°	117.4	.9659	113.6	
.0000		0.0	24	120°	118.5	1.0000	118.5	
.2588	29.6		25	125°	114.2	.9659	110.4	
.5000	52.5		26	130°	104.9	.8660	90.9	
.7071	64.6		27	135°	91.4	.7071	64.6	
.8660	67.9		28	140°	78.3	.5000	39.2	
.9659	62.8		29	145°	65.0	.2588	16.8	
1.0000	51.1		30	150°	51.1	.0000	0.0	
.9659	38.7		31	155°	40.0	−.2588		10.4
.8660	25.5		32	160°	29.4	−.5000		14.7
.7071	15.1		33	165°	21.3	−.7071		15.1
.5000	7.0		34	170°	14.0	−.8660		12.1
.2588	1.8		35	175°	7.1	−.9659		6.9
.0000	0.0		36	180°	0.0	−1.0000		0.0
Sum of products	576.7	593.3					833.4	360.9
	−16.6						+472.5	

$$A_3 = \frac{2(-16.6)}{36} = -0.92$$

$$B_3 = \frac{2(472.5)}{36} = 26.2$$

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FIFTH HARMONIC

1	2	3	4	5	6	7	8	9
sin 5x	Products (y sin 5x)		Ordinate No.	Angle x to ordinate	Meas. ordinate (y)	cos 5x	Products (y cos 5x)	
	+	−					+	−
.4226	2.5		1	5°	5.9	.9063	5.4	
.7660	7.7		2	10°	10.0	.6428	6.4	
.9659	13.0		3	15°	13.4	.2588	3.5	
.9848	16.2		4	20°	16.4	−.1736		2.8
.8192	15.9		5	25°	19.4	−.5736		11.1
.5000	10.9		6	30°	21.8	−.8660		18.9
.0872	2.1		7	35°	23.6	−.9962		23.5
−.3420		8.9	8	40°	25.9	−.9397		24.4
−.7071		19.9	9	45°	28.1	−.7071		19.9
−.9397		28.8	10	50°	30.7	−.3420		10.5
−.9962		33.8	11	55°	33.9	.0872	3.0	
−.8660		32.4	12	60°	37.4	.5000	18.7	
−.5736		24.1	13	65°	42.0	.8192	34.4	
−.1736		8.1	14	70°	46.7	.9848	46.0	
.2588	13.7		15	75°	52.8	.9659	51.0	
.6428	38.6		16	80°	60.0	.7660	46.0	
.9063	61.4		17	85°	67.7	.4226	28.6	
1.0000	76.4		18	90°	76.4	.0000	0.0	
.9063	78.5		19	95°	86.5	−.4226		36.6
.6428	61.4		20	100°	95.5	−.7660		73.2
.2588	27.2		21	105°	105.1	−.9659		101.6
−.1736		19.6	22	110°	112.8	−.9848		111.1
−.5736		67.4	23	115°	117.4	−.8192		96.2
−.8660		102.7	24	120°	118.5	−.5000		59.2
−.9962		114.0	25	125°	114.2	−.0872		10.0
−.9397		98.5	26	130°	104.9	.3420	35.8	
−.7071		64.6	27	135°	91.4	.7071	64.6	
−.3420		26.8	28	140°	78.3	.9397	73.6	
.0872	5.7		29	145°	65.0	.9962	64.8	
.5000	25.6		30	150°	51.1	.8660	44.3	
.8192	32.7		31	155°	40.0	.5736	23.0	
.9848	29.0		32	160°	29.4	.1736	5.1	
.9659	20.6		33	165°	21.3	−.2588		5.5
.7660	10.7		34	170°	14.0	−.6428		9.0
.4226	3.0		35	175°	7.1	−.9063		6.4
.0000	0.0		36	180°	0.0	−1.0000		0.0
Sum of products	552.8	649.6					554.2	619.9
	−96.8						−65.7	

$$A_5 = \frac{-96.8}{36} \times 2 = -5.38$$

$$B_5 = \frac{-65.7}{36} \times 2 = -3.65$$

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SEVENTH HARMONIC

1	2	3	4	5	6	7	8	9
sin 7x	Products (y sin 7x)		Ordinate No.	Angle x to ordinate	Meas. ordinate (y)	cos 7x	Products (y cos 7x)	
	+	−					+	−
.5736	3.4		1	5°	5.9	.8192	4.8	
.9397	9.4		2	10°	10.0	.3420	3.4	
.9659	13.0		3	15°	13.4	−.2588		3.5
.6428	10.5		4	20°	16.4	−.7660		12.6
.0872	1.7		5	25°	19.4	−.9962		19.3
−.5000		10.9	6	30°	21.8	−.8660		18.9
−.9063		21.4	7	35°	23.6	−.4226		10.0
−.9848		24.4	8	40°	25.9	.1736	4.5	
−.7071		19.9	9	45°	28.1	.7071	19.9	
−.1736		5.3	10	50°	30.7	.9848	30.2	
.4226	14.3		11	55°	33.9	.9063	30.8	
.8660	32.4		12	60°	37.4	.5000	18.7	
.9962	41.9		13	65°	42.0	−.0872		3.7
.7660	35.8		14	70°	46.7	−.6428		30.0
.2588	13.7		15	75°	52.8	−.9659		51.0
−.3420		20.5	16	80°	60.0	−.9397		56.5
−.8192		55.5	17	85°	67.7	−.5736		38.8
−1.0000		76.4	18	90°	76.4	.0000		0.0
−.8192		70.9	19	95°	86.5	.5736	49.6	
−.3420		32.6	20	100°	95.5	.9397	89.8	
.2588	27.2		21	105°	105.1	.9659	101.6	
.7660	86.5		22	110°	112.8	.6428	72.5	
.9962	117.0		23	115°	117.4	.0872	10.2	
.8660	102.7		24	120°	118.5	−.5000		59.2
.4226	48.3		25	125°	114.2	−.9063		103.5
−.1736		18.2	26	130°	104.9	−.9848		103.2
−.7071		64.6	27	135°	91.4	−.7071		64.6
−.9848		77.1	28	140°	78.3	−.1736		13.6
−.9063		59.0	29	145°	65.0	.4226	27.5	
−.5000		25.6	30	150°	51.1	.8660	44.3	
.0872	3.5		31	155°	40.0	.9962	39.9	
.6428	18.9		32	160°	29.4	.7660	22.6	
.9659	20.6		33	165°	21.3	.2588	5.5	
.9397	13.6		34	170°	14.0	−.3420		4.8
.5736	4.1		35	175°	7.1	−.8192		5.8
.0000	0.0		36	180°	0.0	−1.0000		0.0
Sum of products	618.5	582.3					575.8	599.0
	36.2						−23.2	

$$A_7 = \frac{36.2}{36} \times 2 = 2.01$$

$$B_7 = \frac{-23.2}{36} \times 2 = -1.29$$

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In Fig. 9 the vector OA of magnitude A_1 may be taken to represent the sin ωt. Remembering that the cosine wave leads the sine wave by 90°, the vector OB may be used to represent the cosine term. The vector sum OC of the two vectors OA and OB, therefore, represents the sum of $A_1 \sin \omega t$ and $B_1 \cos \omega t$ in both magnitude and phase. It leads the sin ωt position by $\tan^{-1} \frac{B_1}{A_1}$ and it also lags the cos ωt by $\tan^{-1} \frac{A_1}{B_1}$. The magnitude OC is $\sqrt{A_1^2 + B_1^2}$. The equation of the combination is $\sqrt{A_1^2 + B_1^2} \sin \left(\omega t + \tan^{-1} \frac{B_1}{A_1}\right)$, or $\sqrt{A_1^2 + B_1^2} \cos \left(\omega t - \tan^{-1} \frac{A_1}{B_1}\right)$.

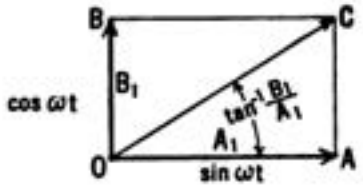

FIG. 9. Vector representation of sin ωt and cos ωt and their sum OC for particular magnitudes A_1 and B_1.

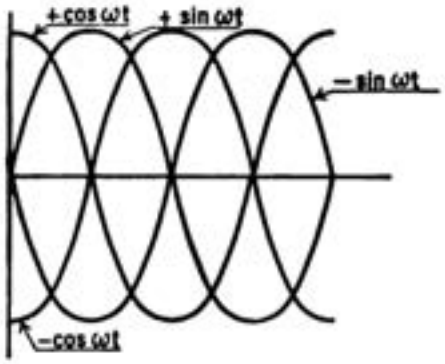

FIG. 10.

The vector representation of the positive and negative sines and cosines forms a convenient way to find trigonometric relations and to make combinations of these waves. For instance, the waves are shown in Fig. 10. The corresponding vector representation of the same waves is shown in Fig. 11. In Fig. 11 it can be seen that the

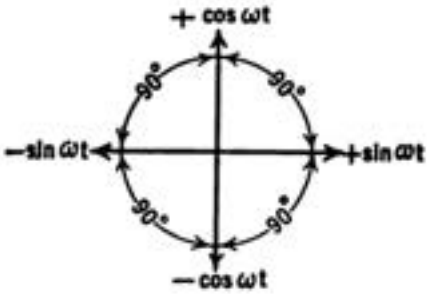

FIG. 11. Vector representation of waves shown in Fig. 10.

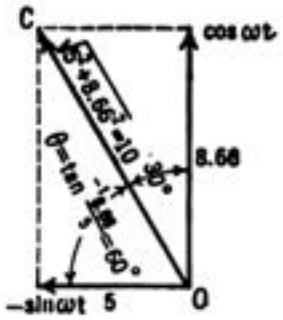

FIG. 12. Combination of $-5 \sin \omega t + 8.66 \cos \omega t$.

sin $(\omega t + 90°)$ gives the $+\cos \omega t$, that $-\cos (\omega t - 90°)$ gives the $-\sin \omega t$, etc. By visualizing Fig. 11, all similar relations become apparent. In like manner, if $[-5 \sin \omega t + 8.66 \cos \omega t]$ is to be reduced to a single trigonometric term, the values would be laid off on Fig. 11 as shown in Fig. 12. The vector addition would then be performed to obtain the resultant OC. OC may be seen to lead the cos ωt by 30° or to lag the $-\sin \omega t$ by 60°. It also leads the $+\sin \omega t$ by 120°. Thus the equation of OC is any one of the following: $10 \cos (\omega t + 30°)$, $-10 \sin (\omega t - 60°)$, or $10 \sin (\omega t + 120°)$. There are also other equivalent expressions for the resultant wave.

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Example 4. Express the equation obtained from the analysis of the wave of Oscillogram 4 in terms of positive sine components only. The results of the analysis show that:

$$A_1 = 82.45 \qquad B_1 = -22.11 \qquad C_1 = \sqrt{82.45^2 + (-22.11)^2} = 85.50 \text{ units}$$
$$A_3 = -0.92 \qquad B_3 = 26.2 \qquad C_3 = \sqrt{(-0.92)^2 + 26.2^2} = 26.2 \text{ units}$$
$$A_5 = -5.38 \qquad B_5 = -3.65 \qquad C_5 = \sqrt{(-5.38)^2 + (-3.65)^2} = 6.50 \text{ units}$$
$$A_7 = 2.01 \qquad B_7 = -1.29 \qquad C_7 = \sqrt{2.01^2 + (-1.29)^2} = 2.39 \text{ units}$$

With respect to the $+\sin \omega t$ position of Fig. 11 as a reference:

$$\alpha_1 = \tan^{-1} \frac{-22.11}{82.45} = \tan^{-1} -0.268 = -15°$$

$$\alpha_3 = \tan^{-1} \frac{26.2}{-0.92} = \tan^{-1} -28.5 = 92°$$

$$\alpha_5 = \tan^{-1} \frac{-3.65}{-5.38} = \tan^{-1} 0.678 = 214.2°$$

$$\alpha_7 = \tan^{-1} \frac{-1.29}{2.01} = \tan^{-1} -0.642 = -32.7°$$

It will be noted that the individual signs of the coefficients B and A must be considered in the evaluation of the phase angles.

The equation for the wave form shown in Oscillogram 4 is:

$$i = 85.50 \sin (\omega t - 15°) + 26.2 \sin (3\omega t + 92°)$$
$$+ 6.50 \sin (5\omega t + 214.2°) + 2.39 \sin (7\omega t - 32.7°)$$

It is desirable to draw figures, similar to that shown in Fig. 12, for each of the harmonics. This exercise is left to the student. The final test of the correctness of any wave analysis is whether the component parts found by the analysis can be combined to yield the original wave.

Problem 3. Evaluate i in the above equation at 30° intervals of ωt throughout one-half cycle, and plot the resultant curve. Compare the general wave shape thus found with that of the original wave form shown in Oscillogram 4.

Problem 4. Express the equation for the wave shape shown in Oscillogram 4 in terms of positive cosine components.

Wave Analysis (Second Graphical Method). Although the fundamental basis of the previous method of analysis is simple, there are a number of methods which require less time for numerical computation. One of these shorter methods follows.

Equation (1) may be written in the following form:

$$y = f(x) = A_0 + A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots$$
$$+ A_n \sin nx + B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x$$
$$+ \cdots + B_n \cos nx \qquad (20)$$

If q is a number equal to the order of the harmonic which is under

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investigation and $f(\pi/2q)$, $f(3\pi/2q)$, etc., are the values of $y = f(x)$ at $x = \pi/2q$, $x = 3\pi/2q$, etc., it can be shown that the following relations are true.[3]

$$2q(A_q - A_{3q} + A_{5q} - A_{7q} + \cdots) = f\left(\frac{\pi}{2q}\right) - f\left(\frac{3\pi}{2q}\right) + f\left(\frac{5\pi}{2q}\right) - \cdots - f\left[\frac{(4q-1)\pi}{2q}\right] \quad (21)$$

$$2q(B_q + B_{3q} + B_{5q} + \cdots) = f(0) - f\left(\frac{\pi}{q}\right) + f\left(\frac{2\pi}{q}\right) - f\left(\frac{3\pi}{q}\right) + \cdots - f\left[\frac{(2q-1)\pi}{q}\right] \quad (22)$$

When equations (21) and (22) are used, it must be remembered that the subscripts $3q$, $5q$, etc., represent the order of the harmonic obtained by multiplication of 3 times q, 5 times q, etc. Thus, if q is 3, B_{3q} would be B_9, B_{5q} would be B_{15}, etc.

Before proceeding to employ equations (21) and (22), it is necessary to estimate the maximum number of harmonics required in the analysis. The procedure is then to start with the highest harmonic and substitute the ordinates at the various angles indicated by the right members of equations (21) and (22). Since it is unlikely that all ordinates required will be given, it is usually necessary to plot the resultant wave in order that the required ordinates may be read from the curve. The necessity of having a graph of the curve will usually entail no extra work in practice because the method will usually be applied only when the resultant wave is obtained from an oscillogram similar to that illustrated in Oscillogram 4, page 174. After the harmonic coefficients are determined, A_0 is evaluated by substituting $x = 0$ in equation (20). Thus

$$f(0) = A_0 + B_1 + B_2 + B_3 + \cdots + B_n \quad (23)$$

$f(0)$ is read from the curve and, since everything except A_0 has been determined, A_0 can be calculated. As an example of the procedure, the wave employed in example 3 will be analyzed.

Example 5. Find the harmonic coefficients through the seventh harmonic for the wave given in Oscillogram 4, page 174, by employing equations (21), (22), and (23).

[3] See "Advanced Mathematics for Engineers," by Reddick and Miller, John Wiley & Sons, Inc., 2nd edition, 1947, p. 202.

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For the seventh harmonic, $q = 7$ and equation (21) is used as follows:

$$(2 \times 7)A_7 = f\left(\frac{\pi}{14}\right) - f\left(\frac{3\pi}{14}\right) + f\left(\frac{5\pi}{14}\right) - f\left(\frac{7\pi}{14}\right) + f\left(\frac{9\pi}{14}\right) - f\left(\frac{11\pi}{14}\right)$$
$$+ f\left(\frac{13\pi}{14}\right) - f\left(\frac{15\pi}{14}\right) + f\left(\frac{17\pi}{14}\right) - f\left(\frac{19\pi}{14}\right) + f\left(\frac{21\pi}{14}\right)$$
$$- f\left(\frac{23\pi}{14}\right) + f\left(\frac{25\pi}{14}\right) - f\left(\frac{27\pi}{14}\right)$$

Note that, since the seventh harmonic is the highest required, $A_{3q} = A_{21}$, A_{5q}, etc., are all zero.

$$\begin{aligned}
14A_7 &= f(12.86°) - f(38.57°) + f(64.29°) - f(90°) + f(115.7°) \\
&\quad - f(141.4°) + f(167.2°) - f(193°) + f(218.7°) - f(244.3°) \\
&\quad + f(270°) - f(296°) + f(321.5°) - f(347°) \\
&= 12.4 - 24.5 + 40.5 - 76.4 + 117.6 - 74.6 + 17.5 - (-12.4) \\
&\quad + (-24.5) - (-40.5) + (-76.4) - (-117.6) + (-74.6) \\
&\quad - (-17.5) \\
&= 25
\end{aligned}$$

$$A_7 = \frac{25}{14} = 1.79$$

$$\begin{aligned}
14B_7 &= f(0) - f(25.7°) + f(51.4°) - f(77.1°) + f(103°) - f(128.7°) \\
&\quad + f(154.3°) - f(180°) + f(205.5°) - f(231.3°) + f(257°) \\
&\quad - f(283°) + f(308.7°) - f(334.5°) \\
&= 0 - 20 + 32 - 56 + 101 - 107 + 41 - 0 + -20 + 32 - 56 \\
&\quad + 101 - 107 + 41 \\
&= -18
\end{aligned}$$

$$B_7 = -\frac{18}{14} = -1.286$$

Because the wave is symmetrical about the 180° point, even harmonics cannot exist. If, however, equations (21) and (22) are used to find the sixth harmonic, zero will be obtained.

Equations (21) and (22) are now used to calculate A_5 and B_5 as follows.

$$(2 \times 5)A_5 = f\left(\frac{\pi}{10}\right) - f\left(\frac{3\pi}{10}\right) + f\left(\frac{5\pi}{10}\right) - f\left(\frac{7\pi}{10}\right) + f\left(\frac{9\pi}{10}\right) - f\left(\frac{11\pi}{10}\right)$$
$$+ f\left(\frac{13\pi}{10}\right) - f\left(\frac{15\pi}{10}\right) + f\left(\frac{17\pi}{10}\right) - f\left(\frac{19\pi}{10}\right)$$

$$\begin{aligned}
10A_5 &= f(18°) - f(54°) + f(90°) - f(126°) + f(162°) - f(198°) \\
&\quad + f(234°) - f(270°) + f(306°) - f(342°) \\
&= 15 - 33 + 76.4 - 113 + 26 - (-15) + (-33) - (-76.4) \\
&\quad + (-113) - (-26) \\
&= 2(15 - 33 + 76.4 - 113 + 26) = 2(-28.6) = -57.2
\end{aligned}$$

$$A_5 = -5.72$$

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$$(2 \times 5)B_5 = f(0) - f\left(\frac{\pi}{5}\right) + f\left(\frac{2\pi}{5}\right) - f\left(\frac{3\pi}{5}\right) + f\left(\frac{4\pi}{5}\right) - f\left(\frac{5\pi}{5}\right) + f\left(\frac{6\pi}{5}\right) - f\left(\frac{7\pi}{5}\right) + f\left(\frac{8\pi}{5}\right) - f\left(\frac{9\pi}{5}\right)$$

$$\begin{aligned} 10B_5 &= f(0) - f(36°) + f(72°) - f(108°) + f(144°) - f(180°) \\ &\quad + f(216°) - f(252°) + f(288°) - f(324°) \\ &= 0 - 24 + 49 - 110 + 68 - 0 + (-24) - (-49) + (-110) \\ &\quad - (-68) \\ &= -34 \\ B_5 &= -3.4 \end{aligned}$$

Determination of A_3 and B_3:

$$(2 \times 3)A_3 = f\left(\frac{\pi}{6}\right) - f\left(\frac{3\pi}{6}\right) + f\left(\frac{5\pi}{6}\right) - f\left(\frac{7\pi}{6}\right) + f\left(\frac{9\pi}{6}\right) - f\left(\frac{11\pi}{6}\right)$$

$$\begin{aligned} 6A_3 &= f(30°) - f(90°) + f(150°) - f(210°) + f(270°) - f(330°) \\ &= 21.8 - 76.4 + 51.1 - (-21.8) + (-76.4) - (-51.1) \\ &= -7 \\ A_3 &= -1.167 \end{aligned}$$

$$\begin{aligned} 6B_3 &= f(0) - f\left(\frac{\pi}{3}\right) + f\left(\frac{2\pi}{3}\right) - f\left(\frac{3\pi}{3}\right) + f\left(\frac{4\pi}{3}\right) - f\left(\frac{5\pi}{3}\right) \\ &= f(0) - f(60°) + f(120°) - f(180°) + f(240°) - f(300°) \\ &= 0 - 37.4 + 118.5 - 0 + (-37.4) - (-118.5) = 162.2 \\ B_3 &= +27.03 \end{aligned}$$

For the fundamental, equations (21) and (22) become

$$(2 \times 1)(A_1 - A_3 + A_5 - A_7) = f\left(\frac{\pi}{2}\right) - f\left(\frac{3\pi}{2}\right)$$

$$2(A_1 - A_3 + A_5 - A_7) = f(90) - f(270) = 76.4 - (-76.4) = 152.8$$

Substituting the values of A_3, A_5, and A_7 found previously and solving for A_1 gives $A_1 = 82.74$.

In a similar way B_1 may be found as follows.

$$\begin{aligned} (2 \times 1)(B_1 + B_3 + B_5 + B_7) &= f(0) - f(\pi) = 0 \\ 2(B_1 + 27.03 - 3.4 - 1.286) &= 0 \\ B_1 &= -22.34 \end{aligned}$$

The foregoing method is easy to apply and entails less labor than the method employing analyzing tables. The accuracy, however, will vary with different wave shapes and will also be dependent upon the estimate of the number of harmonics required. It will be noted that the determination of the fundamental depends upon the values of the harmonics previously determined. It is therefore desirable to start with a high enough order of harmonic so that any higher-order components will be negligible so far as engineering accuracy is concerned. If only a single

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harmonic of some desired order is required, the method employing the analyzing tables may save time and be more accurate. The error in the method employing analyzing tables depends only upon the size of the intervals chosen and, obviously, approaches zero as the size of the interval is decreased and the number of them is increased. The determination of any one harmonic is independent of the determination of any other harmonics when analyzing tables are employed.

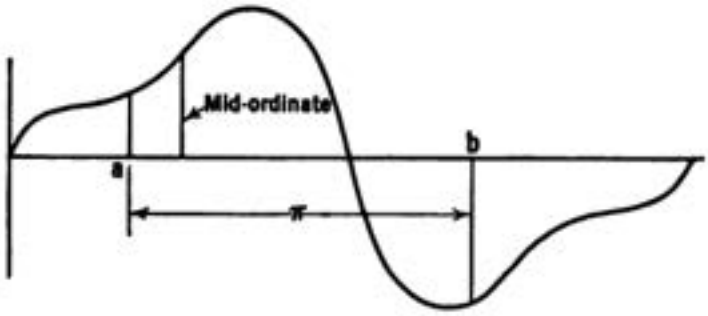

FIG. 13. Wave with unsymmetrical positive and negative loops.

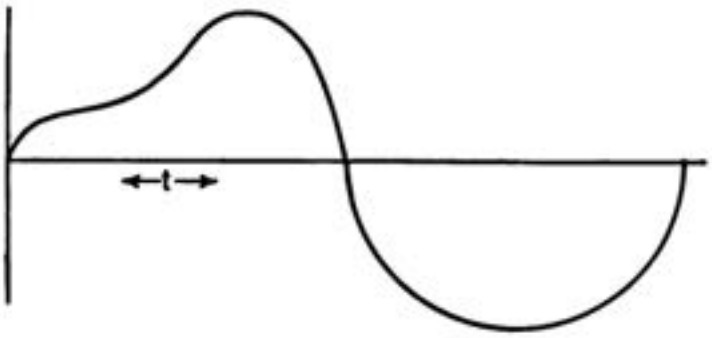

FIG. 14. Wave with unsymmetrical positive and negative loops.

Degrees of Symmetry of Non-Sinusoidal Waves. Non-sinusoidal waves may have symmetrical positive and negative loops, as shown in Fig. 8, or the loops may be unlike, as shown in Figs. 13 and 14. As indicated in the article on wave analysis (page 173), certain types of symmetry in a wave form will automatically eliminate the need for evaluating certain coefficients in the Fourier series which represents the wave.

When the variation from zero to 180° is repeated (except for sign) between 180° and 360°, the wave is said to possess half-wave symmetry. Mathematically a wave of this kind is described as having $[f(x + \pi) = -f(x)]$ symmetry. Expressed in another way, a wave has half-wave symmetry when any ordinate, such as b, Fig. 13, π radians distant from another ordinate, such as a, is equal in magnitude to that at point a but opposite in sign. Thus, the ordinate at any point a for

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a general wave is:

$$\begin{aligned} y_a = A_0 &+ C_1 \sin(\omega t + \alpha_1) + C_2 \sin(2\omega t + \alpha_2) \\ &+ C_3 \sin(3\omega t + \alpha_3) + C_4 \sin(4\omega t + \alpha_4) \\ &+ C_5 \sin(5\omega t + \alpha_5) + \cdots \end{aligned} \tag{24}$$

The ordinate π radians distant from a is found by adding π radians to ωt. If this angle $(\omega t + \pi)$ is substituted and if it is remembered that $(\omega t + \pi)$ for the fundamental corresponds to $n(\omega t + \pi)$ for the nth harmonic, the following results:

$$\begin{aligned} y_b = A_0 &+ C_1 \sin(\omega t + \alpha_1 + \pi) + C_2 \sin(2\omega t + \alpha_2 + 2\pi) \\ &+ C_3 \sin(3\omega t + \alpha_3 + 3\pi) + C_4 \sin(4\omega t + \alpha_4 + 4\pi) \\ &+ C_5 \sin(5\omega t + \alpha_5 + 5\pi) + \cdots \end{aligned} \tag{25}$$

Since the sine of any angle plus an even multiple of π radians is the same as the sine of the angle, and the sine of an angle plus any odd multiple of π radians is the same as the negative sine of the angle, equation (25) simplifies to:

$$\begin{aligned} y_b = A_0 &- C_1 \sin(\omega t + \alpha_1) + C_2 \sin(2\omega t + \alpha_2) \\ &- C_3 \sin(3\omega t + \alpha_3) + C_4 \sin(4\omega t + \alpha_4) \\ &- C_5 \sin(5\omega t + \alpha_5) + \cdots \end{aligned} \tag{26}$$

The ordinate y_b [equation (26)] would be exactly opposite to that of equation (24) if A_0 and all even harmonics in the wave were absent. Hence a wave is symmetrical with respect to the positive and negative loops if it contains no even harmonics and if A_0 is equal to zero. The converse of the foregoing statement is also true, that is, a wave which has $[f(\omega t) = -f(\omega t + \pi)]$ symmetry can contain neither even harmonics nor A_0. The effect of a second harmonic in destroying half-wave symmetry is shown graphically in Fig. 15. In analyzing waves possessing half-wave symmetry, the analysis need be carried through only ½ cycle or 180°.

A wave possessing half-wave symmetry as defined above may also be symmetrical about the midordinates of its positive and negative loops, namely, its 90° and 270° points. A wave of this kind is said to possess midordinate or quarter-wave symmetry, and the analysis need be carried through only ¼ cycle or 90°. The case where only the positive *or* negative loop is symmetrical about its midordinate is of relatively little importance. Thus the positive loop of the wave shown in Fig. 13 is not symmetrical about its midordinate, whereas that of Fig. 16 is symmetrical with respect to its midordinate. The wave will have the halves of its positive and negative loops symmetrical if its fundamental and all har-

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monics pass through zero values at the same time, and, further, if all even harmonics are absent. This fact is illustrated graphically in Fig. 17. The second harmonic, shown dotted, adds to the fundamental to the left of the midordinate of the positive loop and subtracts from it on the right-

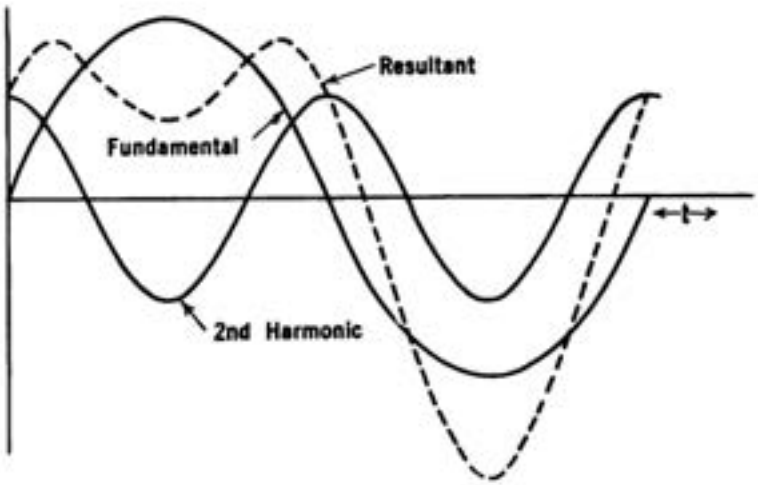

FIG. 15. Effect of second harmonic in destroying half-wave symmetry.

hand side. All the odd harmonics are symmetrical about the mid-ordinate a when they pass through zero at the same time as the fundamental. If the zero-ordinate point of the complex wave is chosen as a

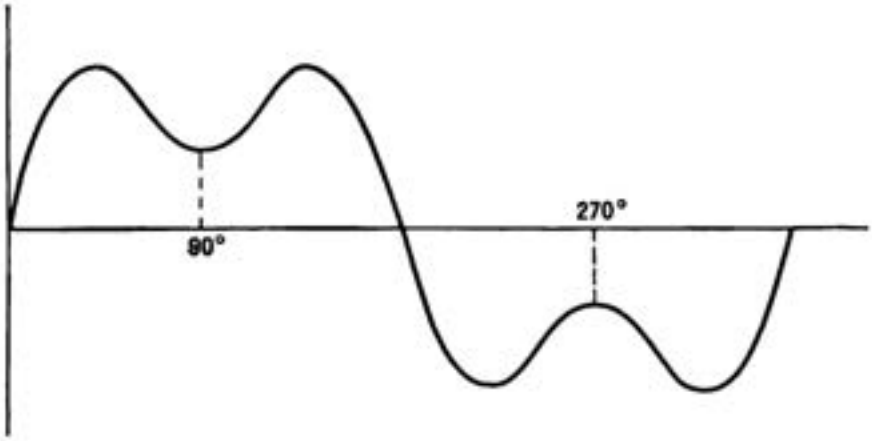

FIG. 16. Wave with positive and negative loops symmetrical about the midordinate (quarter-wave symmetry).

reference, it is plain that only odd sine terms can be present in the equation of a complex wave having quarter-wave symmetry.

Waves of Same Wave Shape. Waves are of the same wave shape if they contain the same harmonics, if the ratio of corresponding harmonics to their respective fundamentals is the same, and if the harmonics are spaced the same with respect to their fundamentals. Expressed

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in another way, for two waves of the same form the ratio of the magnitudes of corresponding harmonics must be constant, and, when the fundamentals are in phase, all the corresponding harmonics of the two waves must be in phase. The test is to note whether the ratio of corresponding harmonics is constant and then to shift one wave so that the fundamentals coincide. If the phase angles of corresponding harmonics in the two waves are then the same and if the first condition is also fulfilled, the waves are of the same wave shape or wave form.

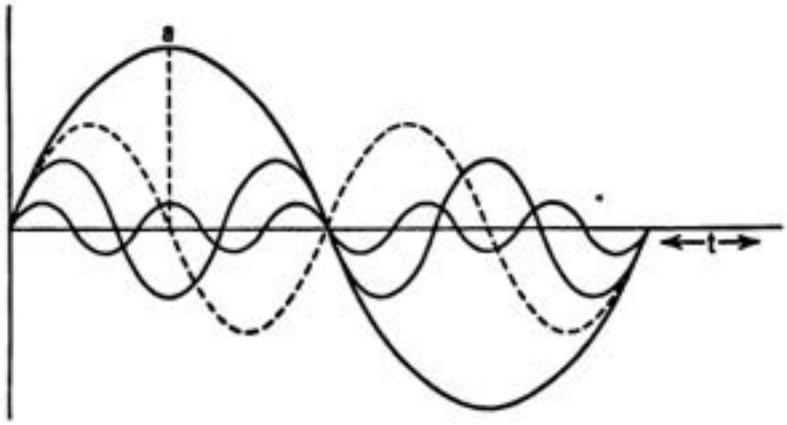

FIG. 17. Symmetry about the midordinate, a, is maintained if all odd harmonics are zero when the fundamental is zero. The second harmonic shown dotted will destroy this symmetry as will other even harmonics.

Example 6. Determine whether the following two waves are of the same shape:

$$e = 100 \sin (\omega t + 30°) - 50 \sin (3\omega t - 60°) + 25 \sin (5\omega t + 40°)$$

$$i = 10 \sin (\omega t - 60°) + 5 \sin (3\omega t - 150°) + 2.5 \cos (5\omega t - 140°)$$

Since all harmonics of the current wave are one-tenth of the corresponding harmonics in the voltage wave, the first requisite is fulfilled. Next, the fundamentals should be brought into phase by shifting the current wave forward 90° or the voltage wave backward 90°. The current wave will be shifted by adding 90° to the phase angle of its fundamental. Shifting the fundamental of a wave by $\alpha°$ corresponds to shifting the nth harmonic by $n\alpha°$. This may be verified by referring to Fig. 17. Suppose the reference axis is changed to the position marked a, thus shifting the wave ahead. This is a shift of 90°, or one quarter cycle for the fundamental. It is a shift of three quarter cycles for the third harmonic, or 270° and five quarter cycles for the fifth harmonic, or 450°. Hence, to maintain the same relation between the fundamental and all harmonics in the current waves, $3 \times 90°$ or 270° will be added to the third, and $5 \times 90°$ or 450° will be added to the fifth harmonic. Then:

$$\begin{aligned} i' &= 10 \sin (\omega t - 60° + 90°) + 5 \sin (3\omega t - 150° + 270°) \\ &\quad + 2.5 \cos (5\omega t - 140° + 450°) \\ &= 10 \sin (\omega t + 30°) + 5 \sin (3\omega t + 120°) + 2.5 \cos (5\omega t + 310°) \\ &= 10 \sin (\omega t + 30°) - 5 \sin (3\omega t - 60°) + 2.5 \sin (5\omega t + 40°) \end{aligned}$$

The corresponding harmonics of the current and voltage waves are hence in phase,

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and the two waves are of the same shape. Had either the third or fifth harmonic been out of phase with the corresponding harmonic in the voltage wave, the wave shapes would have been different.

The effect on wave shape of shifting a harmonic with respect to the fundamental can be understood through a study of Figs. 18, 19, and 20. In each figure the magnitudes of the fundamental and third harmonic are the same. As the third harmonic is shifted along the axis with respect to the fundamental, the wave form of the resultant is seen to change. This shifting of a harmonic with respect to the fundamental is sometimes spoken of as changing the phase of the harmonic with respect to the fundamental. This should not be construed to mean that there is a definite phase difference between a *vector* representing the fundamental and one representing the third harmonic. Vectors representing a fundamental and a higher harmonic cannot correctly be related on the same vector diagram without special interpretation.

Problem 5. Given the following equations for two wave forms of current:

$$i' = 10 \sin (\omega t + 30°) + 2 \sin 7\omega t$$
$$i'' = 35 \sin (\omega t - 10°) + 7 \sin (7\omega t + 80°)$$

Show that the wave form of the i' variation is like (or unlike) the wave form of the i'' variation. *Ans.:* Same form.

Effective Value of a Non-Sinusoidal Wave. In Chapter III the effective value of any wave was shown to be $\sqrt{\frac{1}{T}\int_0^T [f(t)]^2\,dt}$. Applying this expression to the general complex wave

$$i = I_0 + I_{m1} \sin \omega t + I_{m2} \sin (2\omega t + \alpha_2) + I_{m3} \sin (3\omega t + \alpha_3) + \cdots + I_{mn} \sin (n\omega t + \alpha_n)$$

gives

$$I = \left\{\frac{1}{T}\int_0^T [I_0 + I_{m1} \sin \omega t + I_{m2} \sin (2\omega t + \alpha_2) + I_{m3} \sin (3\omega t + \alpha_3) + \cdots + I_{mn} \sin (n\omega t + \alpha_n)]^2\,dt\right\}^{1/2}$$

$$= \sqrt{I_0^2 + \frac{I_{m1}^2 + I_{m2}^2 + I_{m3}^2 + I_{m4}^2 + \cdots + I_{mn}^2}{2}} \qquad (27)$$

Problem 6. Show by integration, including all steps, that the effective value of $v = V_{m1} \sin (\omega t + \alpha_1) + V_{m3} \sin (3\omega t + 30°)$ is $\sqrt{\frac{V_{m1}^2 + V_{m3}^2}{2}}$.

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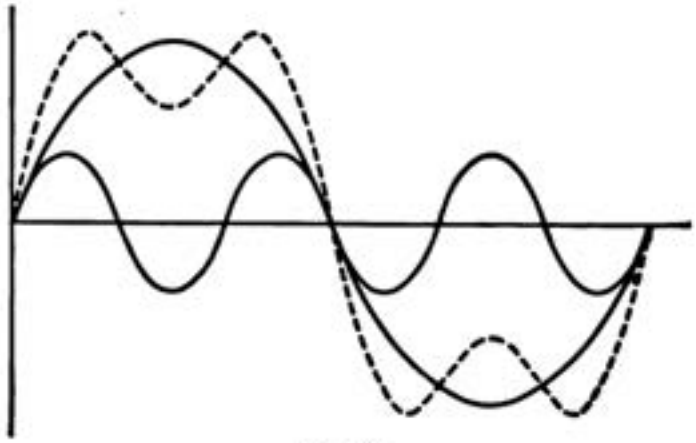

FIG. 18.

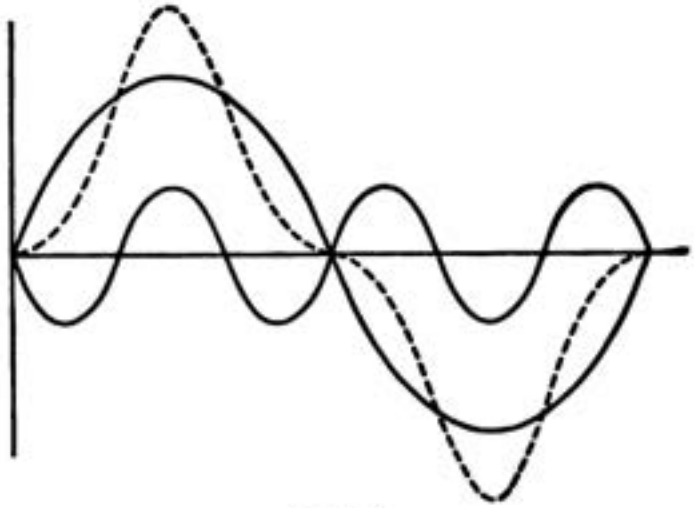

FIG. 19.

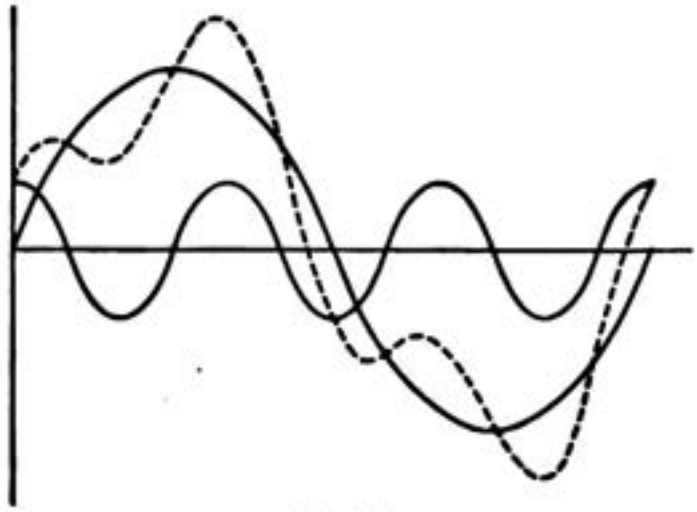

FIG. 20.

Figs. 18, 19, and 20 show the effect on wave shape of shifting a harmonic.

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Since

$$\frac{I_{m1}}{\sqrt{2}} = I_1, \qquad \frac{I_{m2}}{\sqrt{2}} = I_2, \qquad \text{etc.}$$

$$I = \sqrt{I_0^2 + I_1^2 + I_2^2 + I_3^2 + I_4^2 + \cdots + I_n^2} \tag{28}$$

Equation (27) is used when the maximum values of the harmonics are given, whereas equation (28) gives the equivalent expression if effective values of the harmonics are available. It is obvious that similar expressions hold for voltages.

Example 7. Find the effective value of the voltage wave used in example 6.

$$E = \sqrt{\frac{100^2 + 50^2 + 25^2}{2}} = 81 \text{ volts}$$

It should be noted that the effective value is the square root of the *sum* of the squares of the maximum values divided by 2, irrespective of the phase angles or signs of the harmonics. A similar statement is true when effective values of the harmonics are used in equation (28).

For one method of analysis in a-c machinery, known as the Blondell two-reaction method, it is necessary to have the effective value of the rectangular wave given by equation (15), page 171. For this wave, effective value equals $A_1\pi/4$.

Power Due to Non-Sinusoidal Voltages and Currents. The expression for average power in general was given as

$$P = \frac{1}{T}\int_0^T ei\,dt$$

When

$$e = E_{m1}\sin(\omega t + \alpha_1) + E_{m2}\sin(2\omega t + \alpha_2) + E_{m3}\sin(3\omega t + \alpha_3) + \cdots$$

and

$$i = I_{m1}\sin(\omega t + \alpha_1') + I_{m2}\sin(2\omega t + \alpha_2') + I_{m3}\sin(3\omega t + \alpha_3') + \cdots$$

$$\begin{aligned} P = \frac{1}{T}\int_0^T & [E_{m1}\sin(\omega t + \alpha_1) + E_{m2}\sin(2\omega t + \alpha_2) \\ & + E_{m3}\sin(3\omega t + \alpha_3) + \cdots]\ [I_{m1}\sin(\omega t + \alpha_1') \\ & + I_{m2}\sin(2\omega t + \alpha_2') + I_{m3}\sin(3\omega t + \alpha_3') + \cdots]\,dt \end{aligned} \tag{29}$$

Upon expansion, this yields products of terms of unlike frequencies and products of terms of like frequencies. As shown on page 164 the integral of the products of terms of unlike frequencies taken over a complete cycle of the lower frequency is zero. This leaves only the product of terms of like frequency, such as:

$$\frac{1}{T}\int_0^T A\sin(m\omega t + \alpha)\,B\sin(m\omega t + \alpha')\,dt$$

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which gives

$$\frac{AB}{2}\cos(\alpha - \alpha') \tag{30}$$

Thus equation (29) becomes

$$P = \frac{E_{m1}I_{m1}}{2}\cos(\alpha_1 - \alpha_1') + \frac{E_{m2}I_{m2}}{2}\cos(\alpha_2 - \alpha_2')$$
$$+ \frac{E_{m3}I_{m3}}{2}\cos(\alpha_3 - \alpha_3') + \cdots \tag{31}$$

Or, since

$$\frac{E_{m1}I_{m1}}{2} = \frac{E_{m1}}{\sqrt{2}}\frac{I_{m1}}{\sqrt{2}} = E_1I_1$$

$$P = E_1I_1\cos(\alpha_1 - \alpha_1') + E_2I_2\cos(\alpha_2 - \alpha_2')$$
$$+ E_3I_3\cos(\alpha_3 - \alpha_3') + \cdots \tag{32}$$

Average power when waves are non-sinusoidal is the algebraic sum of the powers represented by corresponding harmonics of voltage and current. No average power results from components of voltage and current of unlike frequency, provided that the time interval chosen is equal to an integral number of cycles of the lower-frequency variation. The foregoing statement can be proved either mathematically or graphically.

Example 8. Find the power represented by the following:

$$e = 100\sin(\omega t + 30°) - 50\sin(3\omega t + 60°) + 25\sin 5\omega t \text{ volts}$$
$$i = 20\sin(\omega t - 30°) + 15\sin(3\omega t + 30°) + 10\cos(5\omega t - 60°) \text{ amperes}$$
$$P = \frac{100 \times 20}{2}\cos[30° - (-30°)] + \frac{(-50)(15)}{2}\cos[60° - 30°]$$
$$+ \frac{25 \times 10}{2}\cos[-90° - (-60°)]$$
$$= 500 - 324.75 + 108.25$$
$$= 283.5 \text{ watts}$$

An alternative method of obtaining the power for the third-harmonic components follows.

$$e_3 = -50\sin(3\omega t + 60°) = +50\sin(3\omega t - 120°) \text{ volts}$$
$$i_3 = 15\sin(3\omega t + 30°) \text{ amperes}$$
$$P_3 = \frac{50 \times 15}{2}\cos(-120° - 30°) = 375\cos 150° = -324.75 \text{ watts}$$

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Problem 7. Find the power delivered by the following:

$$e = 100 \sin \omega t + 50 \sin (5\omega t - 80°) - 40 \cos (7\omega t + 30°) \text{ volts}$$

$$i = 30 \sin (\omega t + 60°) + 20 \sin (5\omega t - 50°) + 10 \sin (7\omega t + 60°) \text{ amperes}$$

Ans.: 1083 watts.

Volt-Amperes. Volt-amperes are determined by the product of the effective voltage and effective current.

Example 9. Find the volt-amperes for the waves in example 8.

$$Va = EI = \sqrt{\frac{100^2 + 50^2 + 25^2}{2}} \sqrt{\frac{20^2 + 15^2 + 10^2}{2}} = 81 \times 19.03$$

$$= 1541 \text{ volt-amperes}$$

In general,

$$\text{Volt-amperes} = \sqrt{\frac{E_{m1}^2 + E_{m2}^2 + E_{m3}^2 + \text{etc.}}{2}} \sqrt{\frac{I_{m1}^2 + I_{m2}^2 + I_{m3}^2 + \text{etc.}}{2}} \tag{33}$$

Power Factor. Power factor for non-sinusoidal waves is defined as the ratio of the power to the volt-amperes. Hence

$$\text{Power factor} = \frac{E_1 I_1 \cos (\alpha_1 - \alpha_1') + E_2 I_2 \cos (\alpha_2 - \alpha_2') + E_3 I_3 \cos (\alpha_3 - \alpha_3') + \text{etc.}}{\sqrt{E_1^2 + E_2^2 + E_3^2 + \text{etc.}} \sqrt{I_1^2 + I_2^2 + I_3^2 + \text{etc.}}} \tag{34}$$

Example 10. Find the power factor for the waves given in example 8.

Power from example 8 = 283.5 watts

Volt-amperes from example 9 = 1541

$$\text{Power factor} = \frac{283.5}{1541} = 0.1837$$

The conditions under which the power factor is unity when waves are non-sinusoidal are found from equation (34). To make the power factor 1, the numerator (power) should be as large as possible. Hence

$$\cos (\alpha_1 - \alpha_1') = \cos (\alpha_2 - \alpha_2') = \cos (\alpha_3 - \alpha_3') + \text{etc.} = 1$$

Then

$$\text{p.f.} = \frac{E_1 I_1 + E_2 I_2 + E_3 I_3 + \cdots}{\sqrt{(E_1^2 + E_2^2 + E_3^2 + \text{etc.})(I_1^2 + I_2^2 + I_3^2 + \text{etc.})}}$$

This expression can equal unity only if $E_1/I_1 = E_2/I_2 = E_3/I_3$.

To simplify the algebra, consider only the fundamental and one harmonic.

$$\frac{E_1 I_1 + E_2 I_2}{\sqrt{(E_1^2 + E_2^2)(I_1^2 + I_2^2)}} = 1$$

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$$E_1I_1 + E_2I_2 = \sqrt{E_1^2I_1^2 + E_1^2I_2^2 + E_2^2I_1^2 + E_2^2I_2^2}$$
$$E_1^2I_1^2 + 2E_1I_1E_2I_2 + E_2^2I_2^2 = E_1^2I_1^2 + E_1^2I_2^2 + E_2^2I_1^2 + E_2^2I_2^2$$
$$2E_1I_1E_2I_2 = E_1^2I_2^2 + E_2^2I_1^2$$

If $E_1/I_1 = E_2/I_2$, $E_1I_2 = E_2I_1$ and the above expression becomes $2E_2^2I_1^2 = 2E_2^2I_1^2$, under which conditions the premise is true. Hence, to have unity power factor, the voltage and current waves must be of the same wave shape and in phase. Even though the voltage and current waves pass through zero at the same instant, the power factor cannot be unity if any harmonic in one wave is absent in the other, or when its magnitude makes the wave shapes different.

Equivalent Sine Waves. Occasionally equivalent sine waves are used for certain calculations and comparisons. They must be used with discretion because calculations based upon them are usually in error by varying amounts. An equivalent sine wave of current or voltage is a sine wave the effective value of which is the same as the effective value of the non-sinusoidal wave which is being represented. When equivalent sine waves of corresponding non-sinusoidal voltages and currents are found, the phase angle between the equivalent sine waves is made such that the power and power factor are the same as those for the actual waves. Whether the equivalent angle of phase difference is one of lead or lag is determined by the angle between the fundamentals of the two waves. If the fundamental of current lags the fundamental of voltage, the equivalent sine wave of current must lag the equivalent sine wave of voltage. If the fundamentals are in phase and the power factor is not unity, the sign of the angle of equivalent phase difference is indeterminate.

Example 11. Find the equivalent sine waves for the current and voltage given in example 8.

$$\text{Effective voltage} = \sqrt{\frac{100^2 + 50^2 + 25^2}{2}} = 81 \text{ volts}$$

$$\text{Effective current} = \sqrt{\frac{20^2 + 15^2 + 10^2}{2}} = 19.03 \text{ amperes}$$

Power factor from example 10 = 0.1837

The angle of equivalent phase difference is $\cos^{-1} 0.1837 = 79.4°$. Since the fundamental of current lags the fundamental of voltage, the angle 79.4° is an angle of lag of current with respect to voltage for the equivalent sine waves. The equivalent sine waves of voltage and current, respectively, are:

$$e = \sqrt{2}\, 81 \sin \omega t \text{ volts}$$

$$i = \sqrt{2}\, 19.03 \sin (\omega t - 79.4°) \text{ amperes}$$

As indicated before, the use of equivalent sine waves in non-sinusoidal circuit analysis will generally lead to large errors, particularly in operations involving the addition

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or subtraction of the waves. Equivalent sine waves are sometimes used in specifying the deviation from a sine wave.

Problem 8. Find the equivalent sine waves for the waves given in Problem 7.
Ans.: 118.8 sin ωt volts; 37.4 sin (ωt + 60.8°) amperes.

Deviation Factor. Deviation factor is the ratio of the maximum difference between corresponding ordinates of an actual wave and an

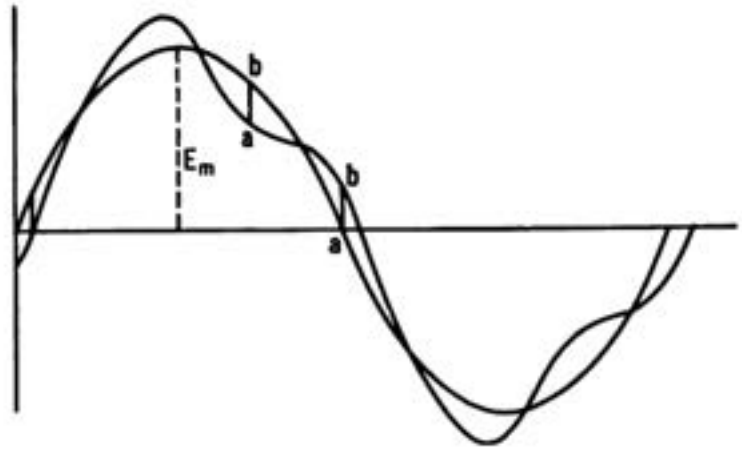

FIG. 21. Deviation of a distorted wave from an equivalent sine wave.

equivalent sine wave of the same length to the maximum ordinate of the equivalent sine wave when the two waves are superposed and shifted along the axis so as to make the maximum difference a minimum. For example, Fig. 21 shows a non-sinusoidal wave and an equivalent sine wave of the same period and length. These waves are shifted in such a way that the maximum difference between corresponding ordinates is as small as possible. In this particular case the maximum difference is ab. The ratio of ab to the maximum value E_m of the equivalent sine wave is the deviation factor. Deviation factor is sometimes used for specification purposes. A deviation factor of about 0.1 for commercial machines is usually allowable.

Series Circuit Analysis when Waves Are Non-Sinusoidal. The procedure is most readily understood from an example.

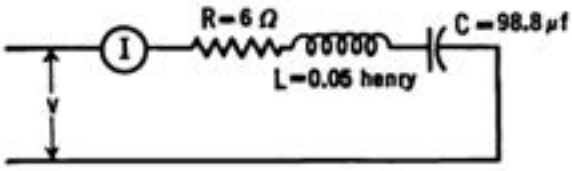

FIG. 22. See example 12.

Example 12. Given the circuit with the parameters shown in Fig. 22. When ω is 377 radians per second and the voltage v = 141.4 sin ωt + 70.7 sin (3ωt + 30°) −28.28 sin (5ωt − 20°) volts is impressed, find the current, I, that an ammeter would read. Also find the total power dissipated and the effective value of the voltage drop across the inductance. Also find the equation of the current wave.

Since the inductive and condensive reactances are different for different frequen-

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cies, each harmonic must be handled *separately*. Subscripts 1, 3, and 5 will designate the fundamental, third, and fifth harmonics, respectively. Either maximum or effective values may be used. If maximum values are used, maximum currents will result; when effective voltages are used, effective currents result. Whichever are used, the result can always be easily changed to give the other if desired. Since the effective values of the harmonic components of voltage in this particular case are more convenient numbers to handle, the solution will be negotiated through the use of effective values immediately.

Fundamental

$$V_1 = \frac{141.4}{\sqrt{2}} = 100 \text{ volts}$$

$$R_1 = 6 \text{ ohms}$$

$$X_{L1} = 377 \times 0.05 = 18.85 \text{ ohms}$$

$$X_{C1} = \frac{10^6}{377 \times 98.8} = 26.85 \text{ ohms}$$

$$\mathbf{Z}_1 = 6 + j18.85 - j26.85 = 6 - j8 \quad \text{or} \quad 10 \text{ ohms}$$

$$I_1 = \frac{V_1}{Z_1} = \frac{100}{10} = 10 \text{ amperes}$$

$$\mathbf{I}_1 \text{ leads } \mathbf{V}_1 \text{ by } \tan^{-1}\frac{8}{6} = 53.12°$$

$$P_1 = 10^2 \times 6 = 600 \text{ watts}$$

$$V_{L1} = I_1 X_{L1} = 10 \times 18.85 = 188.5 \text{ volts}$$

Third Harmonic

$$V_3 = \frac{70.7}{\sqrt{2}} = 50 \text{ volts}$$

$$R_3 = 6 \text{ ohms}$$

$$X_{L3} = 3X_{L1} = 3 \times 18.85 = 56.55 \text{ ohms}$$

$$X_{C3} = \frac{X_{C1}}{3} = \frac{26.85}{3} = 8.95 \text{ ohms}$$

$$\mathbf{Z}_3 = 6 + j56.55 - j8.95 = 6 + j47.6 \quad \text{or}$$
$$\sqrt{6^2 + 47.6^2} = 48.1 \text{ ohms}$$

$$I_3 = \frac{50}{48.1} = 1.04 \text{ amperes}$$

$$\mathbf{I}_3 \text{ lags } \mathbf{V}_3 \text{ by } \tan^{-1}\frac{47.6}{6} = 82.8°$$

$$P_3 = 1.04^2 \times 6 = 6.48 \text{ watts}$$

$$V_{L3} = 1.04 \times 56.55 = 58.9 \text{ volts}$$

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Fifth Harmonic

$$V_5 = \frac{28.28}{\sqrt{2}} = 20 \text{ volts}$$

$$R_5 = 6 \text{ ohms}$$

$$X_{L5} = 5X_{L1} = 5 \times 18.85 = 94.25 \text{ ohms}$$

$$X_{C5} = \frac{X_{C1}}{5} = \frac{26.85}{5} = 5.37 \text{ ohms}$$

$$\mathbf{Z}_5 = 6 + j94.25 - j5.37 = 6 + j88.88 \quad \text{or}$$
$$\sqrt{6^2 + 88.88^2} = 89 \text{ ohms}$$

$$I_5 = \frac{20}{89} = 0.225 \text{ ampere}$$

$$\mathbf{I}_5 \text{ lags } \mathbf{V}_5 \text{ by } \tan^{-1}\frac{88.88}{6} = 86.1°$$

$$P_5 = I_5^2 R_5 = 0.225^2 \times 6 = 0.304 \text{ watt}$$

$$V_{L5} = 0.225 \times 94.25 = 21.2 \text{ volts}$$

$$I_{\text{total}} = \sqrt{I_1^2 + I_3^2 + I_5^2} = \sqrt{10^2 + 1.04^2 + 0.225^2} = 10.05 \text{ amperes}$$

$$P_{\text{total}} = P_1 + P_3 + P_5 = 600 + 6.48 + 0.304 = 606.8 \text{ watts}$$

$$V_L = \sqrt{188.5^2 + 58.9^2 + 21.2^2} = \sqrt{39510} = 198.8 \text{ volts}$$

Since the fundamental of current leads the fundamental of voltage by 53.12°, the equation of the fundamental of current must be $\sqrt{2}\,10 \sin(\omega t + 53.12°)$. Similarly, for the third harmonic,

$$i_3 = \sqrt{2}\,1.04 \sin(3\omega t + 30° - 82.8°)$$

or

$$i_3 = \sqrt{2}\,1.04 \sin(3\omega t - 52.8°) \text{ amperes}$$

Also

$$i_5 = -\sqrt{2}\,0.225 \sin(5\omega t - 20° - 86.1°)$$
$$= -\sqrt{2}\,0.225 \sin(5\omega t - 106.1°) \text{ amperes}$$

The complete equation is:

$$i = 14.14 \sin(\omega t + 53.12°) + 1.47 \sin(3\omega t - 52.8°) - 0.318 \sin(5\omega t - 106.1°)$$
$$= 14.14 \sin(\omega t + 53.12°) + 1.47 \sin(3\omega t - 52.8°) + 0.318 \sin(5\omega t + 73.9°) \text{ amperes}$$

Parallel Circuit Analysis when Waves Are Non-Sinusoidal. This is not appreciably different from the preceding series-circuit problem.

Example 13. Given the circuit shown in Fig. 23, with the 60-cycle constants as shown. When a voltage $v = 141.4 \sin \omega t + 70.7 \sin(3\omega t + 30°) - 28.28 \sin(5\omega t - 20°)$ volts is impressed, find the ammeter value of the total current, I, the current in each branch, power dissipated by each branch, total power dissipated, and the equation of the resultant current. $\omega = 377$ radians per second.

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Fundamental

$$V_1 = \frac{141.4}{\sqrt{2}} = 100 \text{ volts magnitude}$$

$$\mathbf{V}_1 = 100 + j0 \text{ volts}$$

$$\mathbf{I}_{ab1} = \frac{100(5 + j15)}{(5 - j15)(5 + j15)} = 2 + j6 \quad \text{or} \quad 6.33 \text{ amperes}$$

$$\mathbf{I}_{cd1} = \frac{100}{10 + j2} = 9.62 - j1.925 \quad \text{or} \quad 9.82 \text{ amperes}$$

$$\mathbf{I}_{fe1} = \mathbf{I}_{ab1} + \mathbf{I}_{cd1} = 11.62 + j4.075 \quad \text{or} \quad 12.33 \text{ amperes}$$

$\mathbf{I}_{fe1}$ leads the fundamental of voltage by $\tan^{-1}\dfrac{4.075}{11.62} = 19.4°$

$$P_{ab1} = ei + e'i' = 100 \times 2 = 200 \text{ watts}$$

$$P_{cd1} = 100 \times 9.62 = 962 \text{ watts}$$

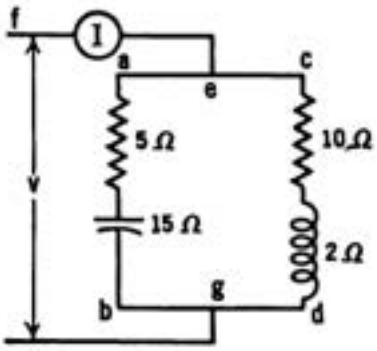

FIG. 23. Circuit with 60-cycle parameters.

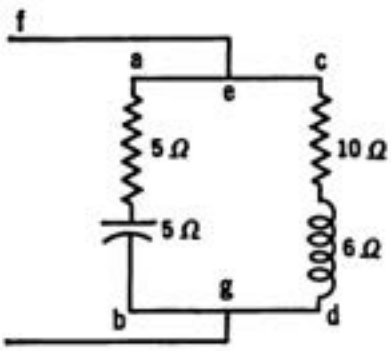

FIG. 24. Circuit of Fig. 23 showing parameters at 180 cycles.

Third Harmonic

The circuit with the parameters for the third harmonic is shown in Fig. 24. Only the reactances need be changed before proceeding as before.

$$V_3 = \frac{70.7}{\sqrt{2}} = 50 \text{ volts magnitude}$$

Take $\mathbf{V}_3$ along the reference axis for the third harmonic. (The most convenient reference axis should be chosen in any particular case in this type of analysis.)

$$\mathbf{V}_3 = 50 + j0 \text{ volts}$$

$$\mathbf{I}_{ab3} = \frac{50}{5 - j5} = 5 + j5 \quad \text{or} \quad 7.07 \text{ amperes}$$

$$\mathbf{I}_{cd3} = \frac{50}{10 + j6} = 3.68 - j2.21 \quad \text{or} \quad 4.3 \text{ amperes}$$

$$\mathbf{I}_{fe3} = 8.68 + j2.79 \quad \text{or} \quad 9.11 \text{ amperes}$$

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$$\mathbf{I}_{fe3} \text{ leads } \mathbf{V}_3 \text{ by } \tan^{-1}\frac{2.79}{8.68} = 17.85°$$

$$P_{ab3} = 50 \times 5 = 250 \text{ watts}$$

$$P_{cd3} = 50 \times 3.68 = 184 \text{ watts}$$

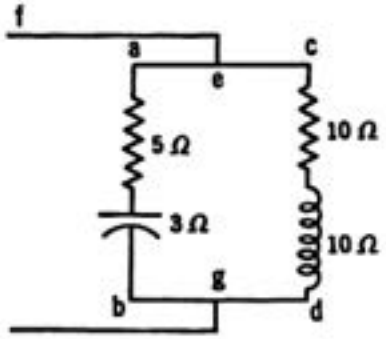

FIG. 25. Circuit of Fig. 23 showing parameters at 300 cycles.

Fifth Harmonic

The circuit with parameters for the fifth harmonic is shown in Fig. 25.

$$V_5 = \frac{28.28}{\sqrt{2}} = 20 \text{ volts}$$

Let $\mathbf{V}_5 = 20 + j0$ volts

$$\mathbf{I}_{ab5} = \frac{20}{5 - j3} = 2.94 + j1.763 \quad \text{or} \quad 3.43 \text{ amperes}$$

$$\mathbf{I}_{cd5} = \frac{20}{10 + j10} = 1 - j1 \quad \text{or} \quad 1.414 \text{ amperes}$$

$$\mathbf{I}_{fe5} = 3.94 + j0.763 \quad \text{or} \quad 4.01 \text{ amperes}$$

$$\mathbf{I}_{fe5} \text{ leads } \mathbf{V}_5 \text{ by } \tan^{-1}\frac{0.763}{3.94} = 10.95°$$

$$P_{ab5} = 20 \times 2.94 = 58.8 \text{ watts}$$

$$P_{cd5} = 20 \times 1 = 20.0 \text{ watts}$$

Ammeter value of total current $= \sqrt{12.33^2 + 9.11^2 + 4.01^2}$
$= 15.9$ amperes

Ammeter value of current in $ab = \sqrt{6.33^2 + 7.07^2 + 3.43^2}$
$= 10.1$ amperes

Ammeter value of current in $cd = \sqrt{9.82^2 + 4.3^2 + 1.414^2}$
$= 10.81$ amperes

$$P_{ab} = 200 + 250 + 58.8 = 508.8 \text{ watts}$$

$$P_{cd} = 962 + 184 + 20 = 1166 \text{ watts}$$

Total power dissipated $= 1674.8$ watts

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Since $\mathbf{I}_{fe1}$ leads $\mathbf{V}_1$ by 19.4°, the equation for the fundamental of the current wave must lead the voltage wave 141.4 sin ωt by 19.4°. Hence

$$i_1 = \sqrt{2}\,12.33 \sin(\omega t + 19.4°) \text{ amperes}$$

Similarly

$$i_3 = \sqrt{2}\,9.11 \sin(3\omega t + 30° + 17.85°)$$
$$= \sqrt{2}\,9.11 \sin(3\omega t + 47.85°) \text{ amperes}$$

and

$$i_5 = -\sqrt{2}\,4.01 \sin(5\omega t - 20° + 10.95°)$$
$$= \sqrt{2}\,4.01 \sin(5\omega t + 170.95°) \text{ amperes}$$

Therefore

$$i = i_1 + i_3 + i_5$$
$$= 17.45 \sin(\omega t + 19.4°) + 12.9 \sin(3\omega t + 47.85°)$$
$$+ 5.67 \sin(5\omega t + 171°) \text{ amperes}$$

Addition and Subtraction of Complex Waves. These operations are similar. Subtraction is performed by reversing the sign of the term to be subtracted and then adding. To illustrate, consider the bifurcated circuit shown in Fig. 26. Given

$$i_1 = 10 \sin(\omega t + 30°) - 5 \sin(3\omega t - 40°) \text{ amperes}$$

$$i_3 = 15 \sin(\omega t - 10°) + 10 \sin(3\omega t + 60°) \text{ amperes}$$

Find i_2.

From Kirchhoff's laws, $i_1 + i_2 = i_3$, or $i_2 = i_3 - i_1$.

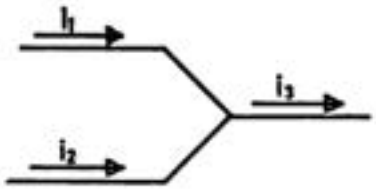

FIG. 26. Bifurcated line.

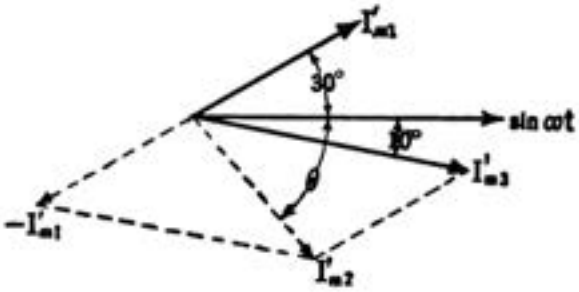

FIG. 27. Vector diagram for currents of fundamental frequency in Fig. 26.

Fundamental

Consider a wave whose equation is of the phase sin ωt as the reference. The solution will follow the vector diagram of Fig. 27. The number of primes on a symbol will indicate the order of the harmonic represented.

$$\mathbf{I}_{m1}' = 10(\cos 30° + j \sin 30°) = 8.66 + j5$$

$$\mathbf{I}_{m3}' = 15(\cos 10° - j \sin 10°) = 14.75 - j2.6$$

$$-\mathbf{I}_{m1}' = -8.66 - j5$$

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$$I_{m2}' = -I_{m1}' + I_{m3}' = 6.09 - j7.6 \quad \text{or} \quad 9.74 \text{ amperes}$$

$$\theta = \tan^{-1}\frac{-7.6}{6.09} = -51.3°$$

$$i_2' = 9.74 \sin(\omega t - 51.3°) \text{ amperes}$$

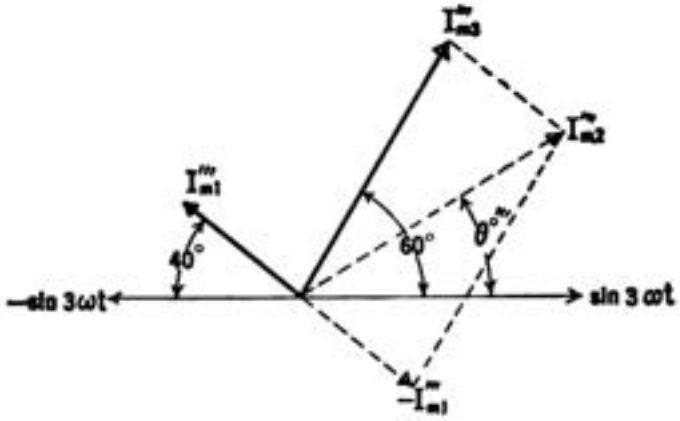

FIG. 28. Vector diagram for third harmonic currents in Fig. 26.

Third Harmonic

A wave of the phase of $\sin 3\omega t$ will be taken as the reference. Then the vector diagram representing the third-harmonic currents appears as shown in Fig. 28.

$$I_{m1}''' = 5(\cos 140° + j \sin 140°) = -3.83 + j3.214$$

$$I_{m3}''' = 10(\cos 60° + j \sin 60°) = 5 + j8.66$$

$$I_{m2}''' = I_{m3}''' - I_{m1}''' = 5 + j8.66 + 3.83 - j3.214 = 8.83 + j5.446$$
$$\text{or} \quad 10.37 \text{ amperes}$$

$$\theta''' = \tan^{-1}\frac{5.446}{8.83} = 31.6°$$

$$i_2''' = 10.37 \sin(3\omega t + 31.6°) \text{ amperes}$$

The complete solution is

$$i_2 = i_2' + i_2'''$$
$$= 9.74 \sin(\omega t - 51.3°) + 10.37 \sin(3\omega t + 31.6°) \text{ amperes}$$

Introduction of Harmonics Due to Variation in Circuit Parameters. Harmonics in a current wave may exist even though the voltage causing it is a pure sinusoid. For example, consider a very thin filament of wire which has a high temperature coefficient of resistivity. If the wire is sufficiently thin so that it will heat and cool during a cycle as

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the current varies from zero to a maximum, the resistance will vary during the cycle. At the maximum point a on the voltage wave, Fig. 29, the resistance will be higher than at point b. The current at a will, therefore, fall below the value that would permit it to be proportional to the voltage. The wave i_1 shows the current wave for a constant resistance, whereas the dotted wave i_2 shows how it will vary when the resistance increases for the higher values of current during a cycle.

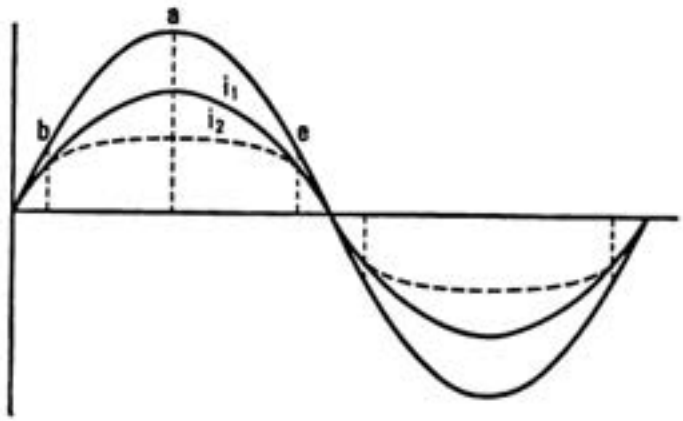

FIG. 29. Shape of i_2 wave is flatter than a sine wave owing to resistance increasing with current.

A very common example of harmonics in a current wave occurs when a sinusoidal voltage wave is impressed on an inductance coil with an iron core. As the current increases, the resulting operation on a higher part of the magnetization or saturation curve causes the inductance to become smaller. When the inductance becomes less, the inductive reactance is reduced and the current, therefore, rises more rapidly than it otherwise would. Thus the current wave becomes more peaked than a sinusoid. This is shown by Oscillogram 2, page 162, which was taken for an iron-core coil.

When the voltage on some device is to be reduced and it is desired to maintain the same wave form, a series resistance cannot be used if the current wave is not sinusoidal. The drop across the resistance will be non-sinusoidal, and this drop subtracted from an original sine wave of voltage will result in a non-sinusoidal wave across the device. In general, but not invariably, the subtraction of a non-sinusoidal voltage drop from a non-sinusoidal voltage will result in a non-sinusoidal wave of different shape from the original.

Modulated Waves. Modulated waves consist of a combination of waves of different frequencies and are, therefore, classified as complex or non-sinusoidal waves. The transmission of radio intelligence is usually accomplished by means of some combination of carrier and audio

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frequencies. Graphical representations of a carrier wave of relatively high frequency and of a modulating wave of relatively low frequency are shown in Fig. 30*a* and Fig. 30*b*, respectively. The carrier frequencies employed in the program broadcast band range from 540 to 1600 kc, and the modulating audio frequencies usefully employed at the transmitter range from about 30 to 10,000 cycles.

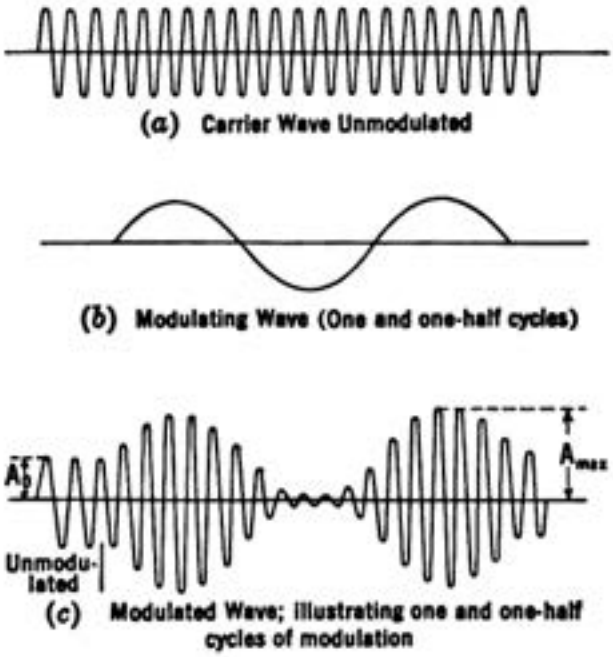

FIG. 30.

The carrier and modulating waves may be combined in a network at the transmitter in such a manner that useful variations in the resultant amplitude or frequency are obtained. Some of the basic principles involved may be understood by considering the case where the carrier frequency is generated by an ordinary type of alternator rather than by a vacuum tube oscillator. The carrier voltage will be represented by

$$e_c = A_0' \sin \omega t \tag{35}$$

where A_0' is the maximum magnitude of the carrier voltage and ω is the carrier angular velocity. Either A_0' or ω may be varied in accordance with the intelligence to be transmitted, thus producing amplitude or frequency modulation. In the case of the ordinary alternator, A_0' could be made to vary by changing the field current sinusoidally and the resultant wave would correspond generally to that shown in Fig. 30*c* or in Oscillogram 5.

Amplitude modulation may be investigated conveniently by letting A_0' of equation (35) take the form $(A_0 + E_m' \sin \omega_1 t)$, where E_m' is the

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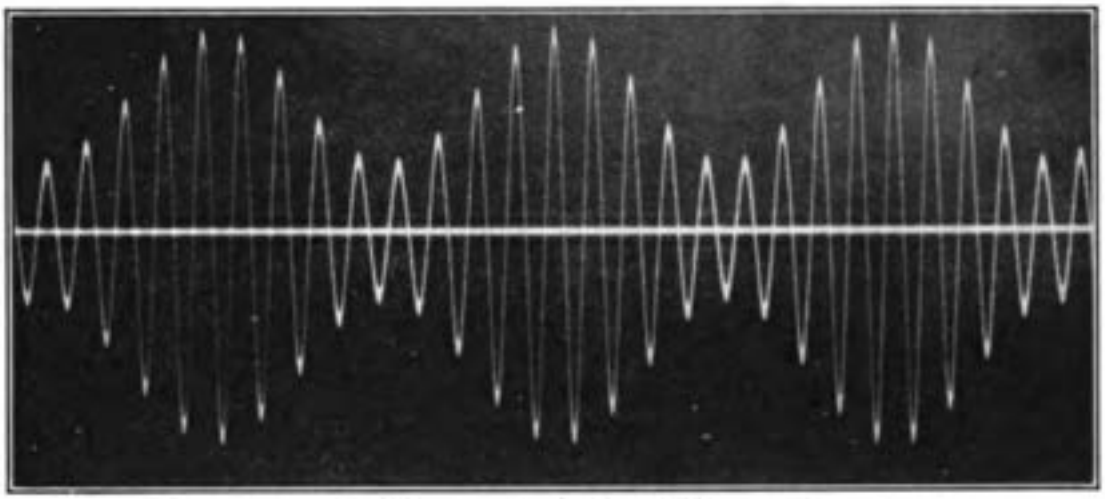
maximum amplitude of the modulating wave that is effectively superimposed on the carrier and ω_1 is the modulating angular velocity. E_m' is a measure of the degree of modulation (for a fixed value of A_0) and usually has values ranging from 50 to 100 per cent of A_0. Percentage modulation is defined as

$$\frac{E_m'}{A_0} \times 100 = \frac{A_{\max} - A_0}{A_0} \times 100$$

where the A's refer to the amplitudes shown in Fig. 30c.



OSCILLOGRAM 5. Photograph of a sinusoidally modulated wave.

In general, the equation of a sinusoidally modulated wave is:

$$\begin{aligned} e &= (A_0 + E_m' \sin \omega_1 t) \sin \omega t \\ &= A_0 \sin \omega t + E_m' \sin \omega_1 t \sin \omega t \end{aligned} \tag{36}$$

The product of two sine waves of different frequencies may be expressed in terms of the following two well-known trigonometric relations.

$$\cos (\omega t - \omega_1 t) = \cos \omega t \cos \omega_1 t + \sin \omega t \sin \omega_1 t \tag{37}$$

$$\cos (\omega t + \omega_1 t) = \cos \omega t \cos \omega_1 t - \sin \omega t \sin \omega_1 t \tag{38}$$

Subtracting equation (38) from (37) gives

$$\cos (\omega t - \omega_1 t) - \cos (\omega t + \omega_1 t) = 2 \sin \omega t \sin \omega_1 t \tag{39}$$

Substituting the value of $\sin \omega t \sin \omega_1 t$ from equation (39) in equation (36) gives

$$\begin{aligned} e &= A_0 \sin \omega t + \frac{E_m'}{2} \cos (\omega t - \omega_1 t) - \frac{E_m'}{2} \cos (\omega t + \omega_1 t) \\ &= A_0 \sin \omega t + \frac{E_m'}{2} \cos 2\pi (f - f_1)t - \frac{E_m'}{2} \cos 2\pi (f + f_1)t \end{aligned} \tag{40}$$

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Equation (40) consists of three terms. The first term, $A_0 \sin \omega t$, is of the same frequency as the original wave before modulation. This wave is called the carrier wave, and its frequency the carrier frequency. The second term, $(E_m'/2) \cos 2\pi (f - f_1)t$, has a frequency equal to $(f - f_1)$, the difference between the carrier frequency and the modulating frequency. This frequency $(f - f_1)$ is called the lower side-band frequency. The third term, $(E_m'/2) \cos 2\pi (f + f_1)t$, represents a frequency equal to $f + f_1$, the sum of the carrier and modulating frequencies. It is called the upper side-band frequency. Each of these three frequencies can be separated from the others in the resultant wave by the use of appropriate filters. If a carrier wave is modulated by a complex wave, each harmonic of the modulating wave gives rise to an upper and lower side-band frequency. Hence, in general, there are several different frequencies in each side band. The type of modulated wave presented above is primarily given as an example of non-sinusoidal waves. There are other types of modulated waves, but further discussion of them is beyond the scope of this text.

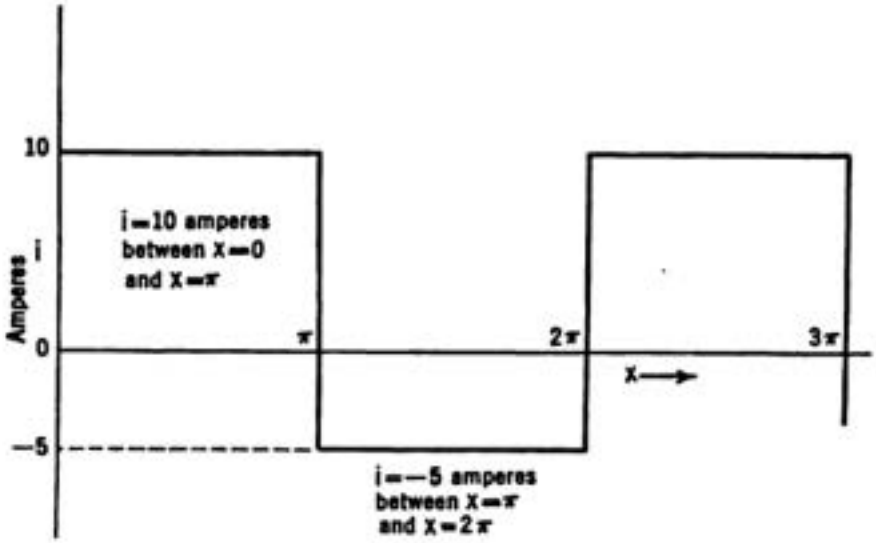

FIG. 31. See Problems 9 and 25.

PROBLEMS

9. (*a*) Employ the analytical method to determine the coefficients of the harmonics through the third harmonic for the wave shown in Fig. 31.

(*b*) Write the Fourier series in terms of sine components for the wave.

(*c*) Sketch the components, indicating the manner in which the components combine to approximate the original wave shape shown in Fig. 31.

10. (*a*) Employ the analytical method to determine the coefficients of the harmonics through the fifth harmonic for the wave shown in Fig. 32.

(*b*) Write the equation of the wave through the fifth harmonic.

(*c*) Sketch the components, indicating the manner in which the components combine to approximate the original wave shown in Fig. 32.

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11. A certain current wave has a height of 1 from 0° to 30°, then increases linearly in a positive direction to a value of 3 at 60°, after which it remains at a height of 3 until 120° is reached. It then decreases linearly to a value of zero at 150° and then remains at zero value until 360°. The cycle is then repeated. Find A_0, A_1, and B_1 of the Fourier series terms which represent this wave.

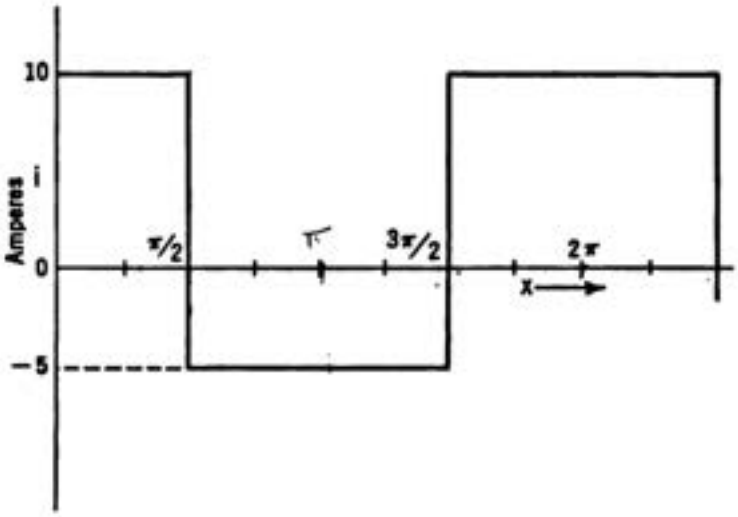

FIG. 32. See Problem 10.

12. A current wave is defined over one complete cycle by the following data:

x (in degrees)	i (in amperes)	x (in degrees)	i (in amperes)
0	−2.000	195	−3.613
15	+0.149	210	−5.000
30	+3.000	225	−6.364
45	+6.364	240	−7.660
60	+9.660	255	−8.634
75	+12.098	270	−9.000
90	+13.000	285	−8.634
105	+12.098	300	−7.660
120	+9.660	315	−6.364
135	+6.364	330	−5.000
150	+3.000	345	−3.613
165	+0.149	360	−2.000
180	−2.000	375	+0.149

(*a*) Employ the analyzing tables on pages 175 to 178, evaluate the Fourier series coefficients A_0, A_1, B_1, A_2, B_2, and A_3 of the above wave form, and write the Fourier series in equational form. (*Note*: Evaluations based on 15° intervals will be sufficiently accurate in this case since the actual Fourier series contains no terms beyond the A_3 term. Call any coefficient zero which is no greater in magnitude than the probable arithmetical error involved.)

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(*b*) Graph each of the components and combine these components to form the resultant wave. Check various values on the resultant graph against the original data.

13. Employ the method of equations (21) and (22) and evaluate the Fourier series coefficients through the third harmonic for the wave given in Problem 12.

14. Write the following equation in terms of three sine components only:

$$v = 4.0 \sin \omega t - 3.0 \cos \omega t - 7.66 \sin 2\omega t + 6.43 \cos 2\omega t$$
$$- 2 \sin 3\omega t - 1.5 \cos 3\omega t$$

15. Given an a-c wave form as defined by the following table of measured ordinates:

Ordinate No.	Degrees	Measured Ordinate	Ordinate No.	Degrees	Measured Ordinate
0	0	0.0	19	95	7.1
1	5	0.8	20	100	7.4
2	10	1.7	21	105	8.0
3	15	2.7	22	110	9.0
4	20	3.6	23	115	10.5
5	25	4.5	24	120	12.0
6	30	5.6	25	125	13.2
7	35	6.9	26	130	14.0
8	40	8.2	27	135	14.0
9	45	9.7	28	140	13.0
10	50	10.7	29	145	11.6
11	55	11.0	30	150	10.0
12	60	11.0	31	155	8.0
13	65	10.4	32	160	5.8
14	70	9.8	33	165	4.0
15	75	9.2	34	170	2.5
16	80	8.5	35	175	1.0
17	85	7.8	36	180	0.0
18	90	7.0			

Negative loop similar to positive loop.

(*a*) Graph the wave and analyze it by the Fourier series method for fundamental the third, the fifth, and the seventh harmonics by the use of analyzing tables.

(*b*) Write the equation of the wave in terms of its sine and cosine components.

(*c*) Write the equation of the wave in terms of sine components only.

(*d*) Synthesize the components graphically, and compare the resultant with the original wave.

16. Employ equations (21) and (22) instead of analyzing tables, and find the sine and cosine coefficients of the Fourier series to include the seventh harmonic for the wave in Problem 15. Express the resultant wave in terms of four sine components only.

17. Given an a-c wave form as defined by the measured ordinates shown on page 207.

Analyze the wave by using equations (21) and (22) for the first seven harmonics, and write the Fourier series equation for the wave.

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Degrees	Measured Ordinate	Degrees	Measured Ordinate
0	−0.6064	100	0.7848
10	0.1736	110	0.6767
20	0.9484	120	0.4966
30	1.4139	130	0.4200
40	1.4428	140	0.5669
50	1.149	150	0.8832
60	0.79	160	1.1420
70	0.5937	170	1.0880
80	0.6154	180	0.6064
90	0.737		

Negative loop similar to positive loop.

18. Show whether the following waves have symmetry with respect to the positive and negative loops:

$$e = 100 \sin (\omega t + 30°) - 50 \cos 3\omega t + 25 \sin (5\omega t + 150°) \text{ volts}$$

$$i = 20 \sin (\omega t + 40°) + 10 \sin (2\omega t + 30°) - 5 \sin (5\omega t - 50°) \text{ amperes}$$

19. Does either of the waves in Problem 18 possess symmetry about the mid-ordinate of the positive and negative loops? Why?

20. Are the following waves of the same wave form or shape? Give reason.

$$v = 100 \sin (\omega t + 70°) - 60 \sin (2\omega t - 30°) + 30 \sin (3\omega t - 60°)$$

$$i = 50 \cos (\omega t - 60°) + 30 \sin (2\omega t + 70°) - 15 \cos (3\omega t - 90°)$$

21. Are the following two waves of the same wave form? Give reason.

$$e = 100 \sin (\omega t - 20°) + 50 \sin (3\omega t + 60°) - 25 \cos (5\omega t - 30°) \text{ volts}$$

$$i = 20 \cos (\omega t - 60°) - 10 \sin (3\omega t + 15°) + 5 \sin (5\omega t - 70°) \text{ amperes}$$

22. Find the effective values of the voltage and current waves of Problem 18.

23. Find the effective value of:

$$v = 100 \sin (\omega t + 30°) - 40 \sin (2\omega t - 30°) + 40 \sin (2\omega t + 30°) + 20 \cos (5\omega t - 30°)$$

24. A complex wave has harmonics of the following effective values: fundamental 100 volts, third harmonic 70 volts, and fifth harmonic 50 volts. Find the voltmeter value of the complex wave.

25. The Fourier representation of the current variation shown in Fig. 31 is:

$$i = 2.5 + \frac{30}{\pi} \sin x + \frac{30}{3\pi} \sin 3x + \frac{30}{5\pi} \sin 5x + \frac{30}{7\pi} \sin 7x + \frac{30}{9\pi} \sin 9x + \cdots$$

Compare the effective value of the current as calculated by equation (27), page 188 (employing only the first six terms of the series given above), with the true effective value.

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26. The current flowing through a particular filter choke is: $i = 5 + 2 \sin x$ amperes, where x ($= 754t$) represents angular measure. Sketch the wave shape of this current variation.

(*a*) What are the maximum, minimum, and average values of current?

(*b*) Does the maximum value of the a-c component satisfy the relation: $I_{m(ac)} = 0.5\ (I_{max} - I_{min})$?

(*c*) What is the effective value of the current: $i = 5 + 2 \sin x$ amperes?

27. Assuming that a pulsating direct current is composed of a d-c component (I_{dc}) and a single-frequency a-c component, the general expression for the current variation is: $i = I_{dc} + I_{m(ac)} \sin x$.

(*a*) If only the average and effective values of the pulsating current were known, would it be possible to find the maximum value of the a-c component, $I_{m(ac)}$?

(*b*) The average value of $i = I_{dc} + I_{m(ac)} \sin x$ is 4 amperes, and the effective value is 5 amperes. Find $I_{m(ac)}$.

28. Considering only second harmonic distortion, the plate current of one class of amplifiers (with sinusoidally varying grid-cathode excitation) is given by the equation:

$$i = I_0 + I_{m1} \sin x - I_{m2} \cos 2x$$

where $I_0 = I_b + I_{m2}$, I_b being the steady plate current with no a-c grid excitation.

(*a*) Sketch the wave form of the current variation for $I_0 = 0.2$, $I_{m1} = 0.1$, and $I_{m2} = 0.01$ ampere. Indicate the value of I_b on the sketch.

(*b*) What are the maximum (I_{max}), minimum (I_{min}), and average values of the wave form sketched in (*a*)? Does the average value of current (I_0) satisfy the relation: $0.5(I_{max} + I_{min})$?

29. Refer to the plate current variation given in Problem 28, namely,

$$i = I_0 + I_{m1} \sin x - I_{m2} \cos 2x$$

(*a*) If it is known that the average value of plate current changes from the steady value $I_b = I_0 - I_{m2}$ (with no a-c grid excitation) to the average value I_0 with a-c grid excitation, show either graphically or analytically that:

$$I_{max} \text{ (with a-c grid excitation)} = I_b + I_{m1} + 2I_{m2}$$

$$I_{min} \text{ (with a-c grid excitation)} = I_b - I_{m1} + 2I_{m2}$$

$$I_{m1} = 0.5\ (I_{max} - I_{min})$$

$$I_{m2} = \frac{(I_{max} + I_{min}) - 2I_b}{4}$$

(*b*) Show that the ratio of I_{m2} to I_{m1} expressed in per cent is:

$$\frac{I_{m2}}{I_{m1}} \times 100 = \frac{0.5(I_{max} + I_{min}) - I_b}{(I_{max} - I_{min})} \times 100$$

Note: The above ratio is called the per cent second harmonic distortion, and, since the values of I_{max}, I_{min}, and I_b may be readily measured under the conditions of steady grid bias, the above relation is sometimes used to determine the per cent second harmonic distortion where unsymmetrical positive and negative peaks of plate current are encountered.

(*c*) Determine the per cent harmonic distortion from $(I_{m2}/I_{m1}) \times 100$ and from the equation given in (*b*) if $I_0 = 0.2$, $I_{m1} = 0.1$, and $I_{m2} = 0.01$ ampere. ($I_b = 0.2 - 0.01$ ampere.)

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30. Because of irregularities in the "straight" portion of the plate current-grid voltage characteristic of a vacuum tube, the equation for the plate current sometimes takes the general form

$$i = I_b + I_{m1} \sin x + I_{m3} \sin 3x$$

where I_b is the plate current corresponding to fixed values of grid-cathode and plate-cathode voltages. Find the maximum, the minimum, and the average values of i if $I_b = 0.2$, $I_{m1} = 0.07$, and $I_{m3} = 0.005$ ampere.

31. Calculate the power represented by the voltage and current in Problem 18.

32. Calculate the power represented by the current and voltage of Problem 21.

33. Calculate the power factor for the waves in Problem 18.

34. Determine the power factor for the waves in Problem 21.

35. Determine the equivalent sine waves for the voltage and current in Problem 18.

36. Find the deviation factor for the voltage

$$e = 100 \sin (\omega t - 25.36°) + 50 \sin (3\omega t + 58.92°)$$

37. A voltage $v = 100 \sin (\omega t + 30°) - 50 \sin (3\omega t + 60°) + 30 \cos 5\omega t$ volts is impressed on a resistance of 6 ohms in series with a capacitance of 88.4 μf and an inductance of 0.01061 henry. Find the ammeter value of the current, the power dissipated by the circuit, the power factor of the whole circuit, and the voltage drop across the capacitance if $\omega = 377$ radians per second.

38. A current of $i = 10 \sin (\omega t - 60°) + 5 \sin (2\omega t + 20°)$ amperes flows in a series circuit consisting of 8 ohms resistance, 10 ohms 60-cycle capacitive reactance, and 4 ohms 60-cycle inductive reactance. Find the equation of the impressed voltage wave. $\omega = 377$ radians per second.

39. A branch containing 5 ohms resistance in series with an inductance of 0.00796 henry is in parallel with another branch consisting of a resistance of 6 ohms in series with a 60-cycle capacitive reactance of 15 ohms. For a voltage of $e = 100 \sin (\omega t + 30°) - 50 \cos (3\omega t - 30°)$ volts impressed on the combination, find the equation of the current wave required by the combination. $\omega = 377$ radians per second.

40. Find the ammeter readings in each branch and the supply line to the circuit of Problem 39.

41. Determine the power dissipated in each branch of the circuit of Problem 39 and the total power taken by the whole circuit.

42. Calculate the power factor of the whole circuit in Problem 39 and the power factor of each branch.

43. The following two currents flow toward a certain junction:

$$i_1 = 20 \sin (\omega t + 30°) - 10 \sin (2\omega t - 30°) + 5 \sin (3\omega t - 40°) \text{ amperes}$$

$$i_2 = 15 \cos \omega t + 10 \cos (2\omega t - 60°) + 10 \cos (3\omega t + 50°) \text{ amperes}$$

Find the equation of the current leaving the junction. What is the ammeter or effective value of each of the three currents?

44. Subtract i_2 from i_1 in Problem 43, and find the equation of the resultant.

45. At 60 cycles a certain impedance, Z_1, consists of 4 ohms resistance, 6 ohms capacitive reactance, and 3 ohms inductive reactance in series. Another identical impedance, Z_2, is connected in parallel with Z_1. A third 60-cycle impedance (consisting of 1.5 ohms resistance and 2 ohms inductive reactance in series) is connected in series with the parallel combination of Z_1 and Z_2. If a voltage $v = 100 \sin 377t - 50 \sin 3(377t + 30°)$ volts is impressed on the entire series-parallel circuit, calcu-

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late: (*a*) the total rms current taken, (*b*) the rms current in each branch, (*c*) the equation of the current in branch Z_1, (*d*) the total power consumed, (*e*) the power factor of whole circuit.

46. The wave form given in Fig. 33 consists of a fundamental term $A_1 \sin x$ and one and only one other Fourier series term.

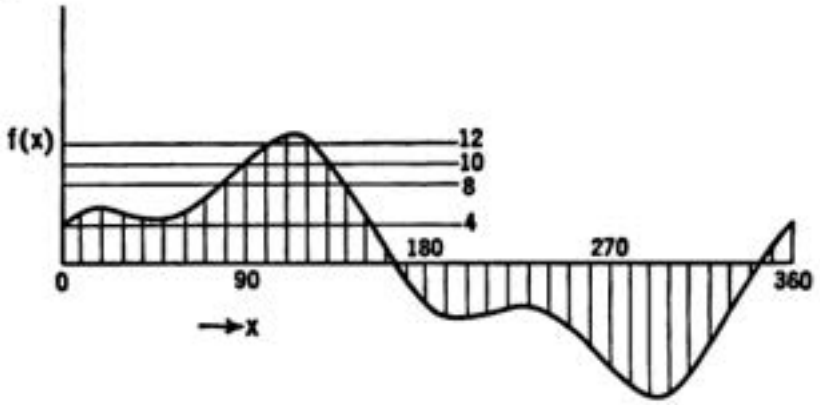

FIG. 33. See Problem 46.

(*a*) What are the numerical values of the coefficients of the two terms?

(*b*) Write the equation of the wave. *Note:* It is suggested that the problem be solved by inspection and checked by the second graphical method of analysis, given on pages 180–184.

47. A condenser having 20 $\mu\mu$f capacitance is connected in parallel with a coil having 20 microhenrys inductance and a series resistance as specified in (*a*) and (*b*) below. This parallel combination is energized with a pulse of current which is zero for $140° \lesseqgtr \omega t \lesseqgtr 40°$ during each cycle. The pulse reaches a maximum value of 100 milliamperes at $\omega t = 90°$ and

$$i(45°) = i(135°) = 18 \text{ milliamperes}$$
$$i(55°) = i(125°) = 49 \text{ milliamperes}$$
$$i(65°) = i(115°) = 73.5 \text{ milliamperes}$$
$$i(75°) = i(105°) = 90.5 \text{ milliamperes}$$
$$i(85°) = i(95°) = 99 \text{ milliamperes}$$

where $i(45°)$ means the value of i at $\omega t = 45°$.

Find the effective magnitude of the fundamental component of voltage developed across the parallel branches if $\omega = \frac{5}{3} \times 10^7$ radians per second. Compare this value of voltage with the third harmonic voltage developed across the parallel branches, recognizing the fact that the branches are tuned to the third harmonic. (*a*) Assume that $R = 10\Omega$ is the same for the fundamental and third harmonic. (*b*) Assume that $Q = \omega L/R$ is constant, R being 10Ω for the fundamental.

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CHAPTER VII

COUPLED CIRCUITS

Terminology. In electrical-engineering literature, the term "circuit" is used in a variety of ways. At times it is employed to designate a single branch of an electrical network; at other times it is used synonymously with the term "network" to mean a combination of two or more branches which are interrelated either electrically or magnetically, or both. In the present chapter the term "circuit" is employed to mean "any complete electrical loop around which Kirchhoff's emf law can be written."

Two circuits are said to be "coupled" when they are so related that energy interchanges can take place between them. More specifically, this means that a potential difference appears in either of the two circuits which are coupled, if and when the other is energized. The circuits involved may be coupled conductively, electromagnetically, or electrostatically. Various combinations of these principal modes of coupling may exist between circuits. However, the great majority of the circuits in actual practice are coupled either conductively or electromagnetically.

Coupled circuits interact upon one another, and in general the movement of electricity in any particular circuit is governed, not only by the circuit parameters of that circuit, but to some extent by the parameters of all circuits to which the circuit in question is coupled.

Conductively Coupled Circuits. Two circuits which are conductively coupled are shown in Fig. 1. In a circuit arrangement of this kind, circuit 1 may be viewed as the driving or primary circuit and circuit 2 as the receiving or secondary circuit. Z_{12}, the impedance of the branch which is common to both circuits, is called the mutual impedance between circuit 1 and circuit 2. The mutual impedance may consist, theoretically, of a pure resistance, a pure inductance, a pure capacitance, or some combination of these circuit elements.

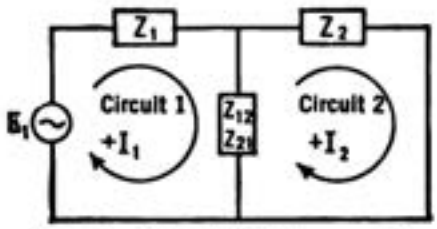

FIG. 1. Conductively coupled circuits.

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If the exciting voltage and circuit parameters of Fig. 1 are given, the currents, component voltages, and component powers can be evaluated by simple circuit analysis.

In general the "mesh current" method of solution[1] is particularly well suited to coupled circuit solutions. If this method of attack is employed, I_1 and I_2 are considered as the currents which flow around the complete loops of circuit 1 and circuit 2, respectively. The positive circuit directions assigned to I_1 and I_2 are, of course, arbitrary. If positive circuit directions are assigned to I_1 and I_2, as shown in Fig. 1, the actual current in the Z_{12} branch in the $+I_1$ direction is $I_1 - I_2$. The details of the "mesh current" method of solution as applied to Fig. 1 are given below. By definition:

$$\mathbf{Z}_{11} = \mathbf{Z}_1 + \mathbf{Z}_{12} \quad \text{(Impedance of circuit 1 to } I_1\text{)}$$

$$\mathbf{Z}_{22} = \mathbf{Z}_2 + \mathbf{Z}_{21} \quad \text{(Impedance of circuit 2 to } I_2\text{)}$$

If the circuit parameters are constant,

$$\mathbf{Z}_{12} = \mathbf{Z}_{21} \quad \text{(Mutual impedance between circuits 1 and 2)}$$

The application of Kirchhoff's emf law to circuits 1 and 2 of Fig. 1 results in:

$$\mathbf{Z}_{11}\mathbf{I}_1 - \mathbf{Z}_{12}\mathbf{I}_2 = \mathbf{E}_1 \tag{1}$$

$$-\mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 = 0 \tag{2}$$

Employing elementary determinants, the expressions for $\mathbf{I}_1$ and $\mathbf{I}_2$ become:

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{E}_1 & -\mathbf{Z}_{12} \\ 0 & \mathbf{Z}_{22} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{11} & -\mathbf{Z}_{12} \\ -\mathbf{Z}_{21} & \mathbf{Z}_{22} \end{vmatrix}} = \frac{\mathbf{E}_1\mathbf{Z}_{22}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}^2} \tag{3}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_{11} & \mathbf{E}_1 \\ -\mathbf{Z}_{21} & 0 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{11} & -\mathbf{Z}_{12} \\ -\mathbf{Z}_{21} & \mathbf{Z}_{22} \end{vmatrix}} = \frac{\mathbf{E}_1\mathbf{Z}_{21}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}^2} \tag{4}$$

The above method is generally applicable and may be extended to include any number of coupled circuits.

[1] In general circuit analysis many of the disagreeable details can be avoided by making use of this method. It is sometimes referred to as Maxwell's "cyclic current" method. See "A Treatise on Electricity and Magnetism," by Maxwell, Vol. 1, 3rd edition.

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Example 1. Let it be assumed that, in Fig. 1: $\mathbf{E}_1 = 100\,\underline{/0^\circ}$ volts, $\mathbf{Z}_1 = 3 + j4$ ohms, $\mathbf{Z}_{12} = 10 + j0$ ohms, and $\mathbf{Z}_2 = 4 - j8$ ohms. The impedance of the generator is considered to be negligibly small, or else its impedance is included in $\mathbf{Z}_1$.

$$\mathbf{Z}_{11} = (3 + j4) + (10 + j0) = 13 + j4 = 13.6\,\underline{/17.1^\circ} \text{ ohms}$$

$$\mathbf{Z}_{22} = (4 - j8) + (10 + j0) = 14 - j8 = 16.1\,\underline{/-29.7^\circ} \text{ ohms}$$

$$\mathbf{Z}_{11}\mathbf{Z}_{22} = 219\,\underline{/-12.6^\circ} = 214 - j47.8$$

$$\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}^2 = 114 - j47.8 = 123.7\,\underline{/-22.7^\circ}$$

$$\mathbf{I}_1 = \frac{(100\,\underline{/0^\circ})(16.1\,\underline{/-29.7^\circ})}{123.7\,\underline{/-22.7^\circ}} = 13.0\,\underline{/-7^\circ} \text{ amperes}$$

$$\mathbf{I}_2 = \frac{(100\,\underline{/0^\circ})(10\,\underline{/0^\circ})}{123.7\,\underline{/-22.7^\circ}} = 8.08\,\underline{/22.7^\circ} \text{ amperes}$$

The current in the $\mathbf{Z}_{12}$ branch in the direction of $\mathbf{I}_1$ is $\mathbf{I}_{12} = (\mathbf{I}_1 - \mathbf{I}_2)$.

$$\begin{aligned}\mathbf{I}_{12} &= 13.0\,(0.992 - j0.122) - 8.08\,(0.922 + j0.386)\\ &= (12.9 - j1.59) - (7.45 + j3.12)\\ &= 5.45 - j4.71 = 7.21\,\underline{/-40.8^\circ} \text{ amperes}\end{aligned}$$

The total power generated by the generator E_1 is:

$$\begin{aligned}P_{\text{gen}} &= E_1 I_1 \cos\theta \Big]_{\mathbf{I}_1}^{\mathbf{E}_1} = 100 \times 13.0 \cos(-7^\circ)\\ &= 1290 \text{ watts (approximately)}\end{aligned}$$

The total power absorbed by the network is:

$$\begin{aligned}I_1^2R_1 + I_2^2R_2 + I_{12}^2R_{12} &= 13.0^2 \times 3 + 8.08^2 \times 4 + 7.21^2 \times 10\\ &= 1288 \text{ watts (approximately)}\end{aligned}$$

Problem 1. Solve for $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_{12}$ in the above illustrative example by first reducing the coupled circuits to an equivalent series impedance. Draw the vector diagram of $\mathbf{E}_1$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_{12}$, $\mathbf{V}_{12}$, illustrating vectorially that $\mathbf{V}_{12} = \mathbf{E}_1 - \mathbf{I}_1\mathbf{Z}_1$.

Ans.: Given in the above illustrative example.

Mutual Impedance. Before proceeding with particular types of coupled circuits, we shall state some general definitions which will be useful later in this chapter and also in radio courses where the coefficient of coupling plays a far more prominent role than it does in a first course.

The mutual impedance between, say, circuits 1 and 2 of a general network is defined as the ratio of the voltage developed in circuit 2 per unit current in circuit 1 when all circuits except circuit 1 are open-circuited. This mutual impedance has already been employed in the foregoing section as $\mathbf{Z}_{21}$. If linear bilateral circuit elements are employed in the coupling of the two circuits, it should be plain that $\mathbf{Z}_{12}$,

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the ratio of the voltage developed in circuit 1 per unit current in circuit 2 with all circuits except circuit 2 open-circuited, is equal to $\mathbf{Z}_{21}$.

The definition given above for mutual impedance between two circuits can be generalized to apply to two pairs of terminals, $11'$ and $22'$, as shown in Fig. 2 where the network in the box may be any con-

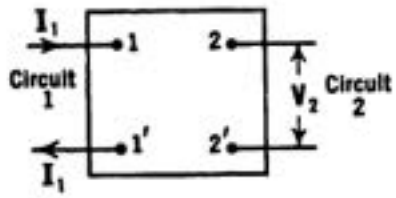

FIG. 2. Circuit 1 coupled to circuit 2 through an arbitrary network not shown.

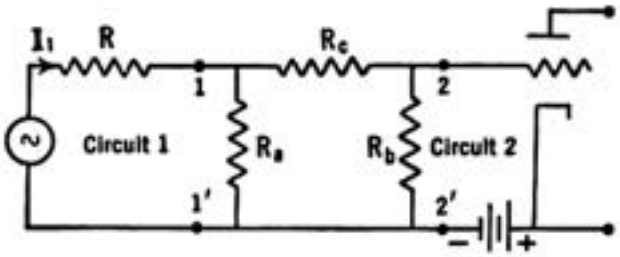

FIG. 3. Circuit 1 coupled to circuit 2 through a π set of resistances.

figuration of impedances. If, for example, the terminals $11'$ and $22'$ of Fig. 3 are selected, we would find upon measurement that

$$\mathbf{Z}_{21} = \frac{\mathbf{V}_2}{\mathbf{I}_1} = \frac{\mathbf{V}_b}{\mathbf{I}_1} = \frac{\dfrac{\mathbf{V}_a}{R_b + R_c} R_b}{\dfrac{\mathbf{V}_a(R_a + R_b + R_c)}{R_a(R_b + R_c)}} = \frac{R_a R_b}{R_a + R_b + R_c}$$

where $\mathbf{V}_b$ is the voltage developed across R_b (terminals $22'$) and $\mathbf{V}_a$ is the voltage drop across R_a. The same result would have been obtained had the π set of resistors $(R_a - R_b - R_c)$ been transformed to an equivalent $\mathbf{Y}$ set of resistors.

In many networks, particularly in the field of radio, the direct currents must be confined to specified paths and a-c energy is transferred from

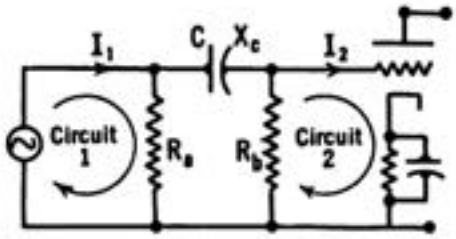

FIG. 4. Circuits coupled through R_a-C-R_b network.

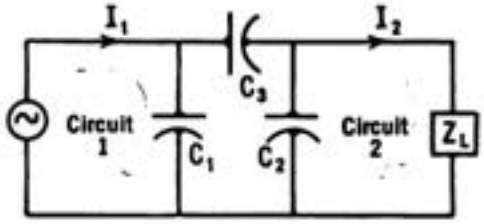

FIG. 5. Circuits coupled through C_1-C_2-C_3 network.

one circuit to another through the agency of an electric or magnetic field. In Fig. 4, for example, a-c energy may be transferred from circuit 1 to circuit 2 by way of the electric field existing between the plates of the coupling condenser, C.

A particular form of capacitive coupling is shown in Fig. 5. If the

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coupling reactance between circuit 1 and circuit 2 is defined as the voltage developed in circuit 2, namely, the voltage across C_2, per unit current in circuit 1, this coupling reactance is:

$$X_{\text{coupling}} = \frac{\dfrac{V_1}{X_2 + X_3} X_2}{\dfrac{V_1(X_1 + X_2 + X_3)}{X_1(X_2 + X_3)}} = \frac{X_1 X_2}{X_1 + X_2 + X_3}$$

where V_1 is the voltage across C_1 and the X's are the capacitive reactances of the respective condensers. The coupling capacitance between circuit 1 and circuit 2 (or vice versa) is:

$$C_{\text{coupling}} = \frac{1}{\omega X_{\text{coupling}}} = \frac{1}{\omega \dfrac{(1/\omega C_1)(1/\omega C_2)}{(1/\omega C_1) + (1/\omega C_2) + (1/\omega C_3)}}$$

$$= C_1 + C_2 + \frac{C_1 C_2}{C_3}$$

Problem 2. Show that the voltage developed across condenser C_1 per unit current flowing in circuit 2 of Fig. 5 is:

$$\frac{X_1 X_2}{X_1 + X_2 + X_3} = X_{\text{coupling}}$$

where $X_1 = 1/\omega C_1$, $X_2 = 1/\omega C_2$, and $X_3 = 1/\omega C_3$.

Problem 3. Consider R_a, R_b, and X_c of Fig. 4 to be a coupling device between circuit 1 and circuit 2. Show that the coupling impedance between the two circuits is:

$$\mathbf{Z}_{\text{coupling}} = \frac{(R_a^2 R_b + R_a R_b^2) + jR_a R_b X_c}{(R_a + R_b)^2 + X_c^2}$$

Note:

$$\mathbf{Z}_{\text{coupling}} = \frac{\mathbf{V}_b}{\mathbf{I}_1}$$

where $\mathbf{V}_b$ is the voltage developed across R_b by $\mathbf{I}_1$, or

$$\mathbf{Z}_{\text{coupling}} = \frac{\mathbf{V}_a}{\mathbf{I}_2}$$

where V_a is the voltage developed across R_a by $\mathbf{I}_2$.

Coefficient of Coupling. Given two pairs of terminals, $11'$ and $22'$, as shown in Fig. 2. The coefficient of coupling between circuit $11'$ and circuit $22'$ will be defined as:

$$\mathbf{k} = \frac{\mathbf{Z}_{12}}{\sqrt{\mathbf{Z}_{11'}\mathbf{Z}_{22'}}} = \frac{\mathbf{Z}_{21}}{\sqrt{\mathbf{Z}_{11'}\mathbf{Z}_{22'}}}$$

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where $\mathbf{Z}_{12}$ is the mutual impedance between circuits 2 and 1. $\mathbf{Z}_{21} = \mathbf{Z}_{12}$.

$\mathbf{Z}_{11'}$ is the impedance seen looking into terminals $11'$ with terminals $22'$ open-circuited.

$\mathbf{Z}_{22'}$ is the impedance seen looking into terminals $22'$ with terminals $11'$ open-circuited.

Example 2. Consider terminals $11'$ and $22'$ of Fig. 3. Let it be required to find the coefficient of coupling between circuits 1 and 2.

It has been shown that

$$\mathbf{Z}_{21} = \mathbf{Z}_{12} = \frac{R_a R_b}{R_a + R_b + R_c}$$

$$\mathbf{Z}_{11'} = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c}$$

$$\mathbf{Z}_{22'} = \frac{R_b(R_a + R_c)}{R_a + R_b + R_c}$$

$$\mathbf{k} = \frac{R_a R_b}{\sqrt{R_a(R_b + R_c)R_b(R_a + R_c)}}$$

If, for example, $R_c = 0$, the coefficient of coupling is unity. It should be noted that, with the general definition of coupling coefficient which has been given, $\mathbf{k}$ may be complex and greater than unity. In most cases, however, the coefficient of coupling is real and less than unity as in this example.

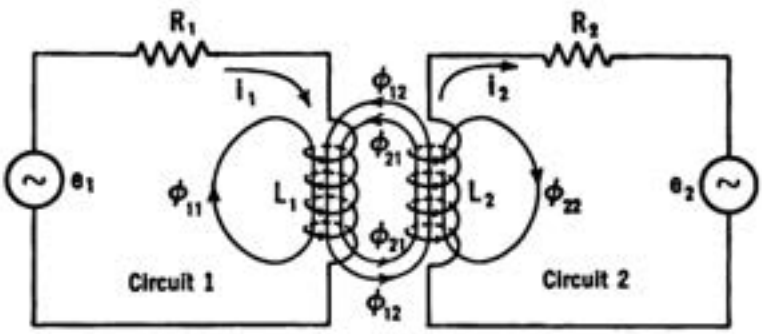

FIG. 6. Illustrating the four component fluxes ϕ_{11}, ϕ_{12}, ϕ_{22}, and ϕ_{21} into which the resultant magnetic field is separated for the purpose of analysis.

Magnetic Coupling. If a portion of the magnetic flux established by one circuit interlinks with a second circuit, the two circuits are coupled magnetically and energy may be transferred from one circuit to the other by way of the magnetic field which is common to the two circuits. The practical operation of many devices depends upon this type of coupling.

Separation of Magnetic Flux into Hypothetical Components. Magnetic coupling between two individual circuits is shown in Fig. 6. For the purpose of analysis, the total flux which is established by i_1, namely,

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ϕ_1, is divided into two components. One component of ϕ_1 is that part which links with circuit 1 but not with circuit 2, namely, ϕ_{11}. The second component of ϕ_1 is ϕ_{12}, that part which links with both circuit 2 and circuit 1. In a similar manner, the flux established by i_2 is separated into two components for the sake of detailed analysis.

By definition:

$$\phi_1 = \phi_{11} + \phi_{12} \tag{5}$$

and

$$\phi_2 = \phi_{22} + \phi_{21} \tag{6}$$

The four component fluxes are shown in Fig. 6, and a recapitulation of their definitions is given below:

ϕ_{11} the fractional part of ϕ_1 which links *only* with the turns of circuit 1. This is the leakage flux of circuit 1 with respect to circuit 2.

ϕ_{12} the fractional part of ϕ_1 which links with the turns of circuit 2. This is the mutual flux produced by circuit 1.

ϕ_{22} the fractional part of ϕ_2 which links *only* with the turns of circuit 2. This is the leakage flux of circuit 2 with respect to circuit 1.

ϕ_{21} the fractional part of ϕ_2 which links with the turns of circuit 1. This is the mutual flux produced by circuit 2.

It should be recognized that the actual flux established by i_1 or i_2 does not conform to the simple configurations shown in Fig. 6. For example, part of ϕ_{11} links with only a fraction of the total turns of circuit 1, and likewise a part of ϕ_{12} links with only a fractional part of the turns of circuit 2. ϕ_{11} is a hypothetical flux which, when linking with all the turns, N_1, produces the same total flux linkages as the true flux linkages in question. Similar concepts are held for the other component fluxes, and, when used quantitatively in this manner, they represent accurately the true condition of affairs as regards induced voltages.

Mutual Inductance. In order to describe the magnetic interaction between circuits or between portions of the same circuit, the circuit parameter M is introduced. It is called the coefficient of mutual inductance, or simply mutual inductance, and is dimensionally equivalent to the coefficient of self-inductance, L. The similarity between the concept of mutual inductance of (or between) two circuits and the concept of self-inductance may be shown in the following manner. Refer to Fig. 6. For the purpose at hand we shall define the self-

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inductance of circuit 1 as:

$$L_1 = \frac{N_1\phi_1}{i_1}\left[\text{flux linkages of circuit 1 per unit current in circuit 1}\right] \quad (7)$$

On the same basis of reckoning, the mutual inductance of circuit 1 with respect to circuit 2 is:

$$M_{21} = \frac{N_1\phi_{21}}{i_2}\left[\text{flux linkages of circuit 1 per unit current in circuit 2}\right] \quad (8)$$

Also the mutual inductance of circuit 2 with respect to circuit 1 is:

$$M_{12} = \frac{N_2\phi_{12}}{i_1}\left[\text{flux linkages of circuit 2 per unit current in circuit 1}\right] \quad (9)$$

If the ϕ/i characteristics in equations (7), (8), and (9) are not straight lines, then L_1, M_{21}, and M_{12} are variable circuit parameters and for certain types of analyses can best be written in the forms:

$$L_1 = N_1\frac{d\phi_1}{di_1} \quad (7a)$$

$$M_{21} = N_1\frac{d\phi_{21}}{di_2} \quad (8a)$$

$$M_{12} = N_2\frac{d\phi_{12}}{di_1} \quad (9a)$$

If, however, the flux is proportional to the current (i.e., permeability constant), both self-inductance and mutual inductance in equations (7), (8), and (9) are constant and as such are very useful circuit parameters in classical circuit theory.

Under the condition of constant permeability, the reluctance of the *mutual flux path* ($\mathcal{R}_{21}$ or $\mathcal{R}_{12}$) is a fixed quantity and $\mathcal{R}_{21} = \mathcal{R}_{12}$.

$$M_{21} = \frac{N_1\phi_{21}}{i_2} = \frac{KN_1N_2}{\mathcal{R}_{21}} \quad (10)$$

$$M_{12} = \frac{N_2\phi_{12}}{i_1} = \frac{KN_2N_1}{\mathcal{R}_{12}} \quad (11)$$

where K is a constant which depends for its value upon the units employed in evaluating $\phi = KNi/\mathcal{R}$. Therefore, if the permeability of the mutual flux path is constant, M_{21} and M_{12} are constant and $M_{21} = M_{12} = M$. This fact may also be proved in terms of the energies stored in the magnetic field when both circuits are energized.

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If the permeability of the mutual flux path is not constant, neither M_{21} nor M_{12} will be constant and the following method of representing mutually induced voltages in terms of M loses much of its effectiveness. Unless otherwise stated, absence of ferromagnetic material will be assumed, in which case $M_{21} = M_{12} = M$.

The units in which mutual inductance is expressed are identical with the units in which self-inductance is expressed, usually the henry or millihenry. If the flux linkages in equations (8) or (9) are expressed in weber-turns (10^8 maxwell-turns) and the current is expressed in amperes, M is given in henrys.

Problem 4. Refer to Fig. 6, page 216, and assume that the L_1 coil consists of 50 turns and that the L_2 coil consists of 500 turns.

(*a*) What is the mutual inductance between the two circuits (in millihenrys) if 5 amperes in circuit 1 establishes a total equivalent flux (ϕ_1) of 30,000 maxwells 27,500 maxwells of which link with the turns of the L_2 coil?

(*b*) What is the self-inductance of the L_1 coil?

Ans.: (*a*) $M_{12} = 27.5$ millihenrys; (*b*) $L_1 = 3$ millihenrys.

Mutual Reactance, X_M. It is evident that any change in i_2 of Fig. 6 will cause a corresponding change in ϕ_{21}. In accordance with Lenz's law, any time rate of change of ϕ_{21} will manifest itself in circuit 1 in the form of a generated or induced voltage the value of which is:

$$e_{12} = -N_1 \frac{d\phi_{21}}{dt} \quad \text{or} \quad v_{12} = N_1 \frac{d\phi_{21}}{dt} \tag{12}$$

where e_{12} is considered as a voltage rise or generated voltage and v_{12} is considered as a voltage drop.

Similarly any change in i_1 will manifest itself in circuit 2 as:

$$e_{21} = -N_2 \frac{d\phi_{12}}{dt} \quad \text{or} \quad v_{21} = N_2 \frac{d\phi_{12}}{dt} \tag{13}$$

It is through the agency of these mutually induced voltages that the phenomenon known as mutual inductance can be taken into account in circuit analysis.

The basic equations of voltage for the two circuits shown in Fig. 6 are:

$$R_1 i_1 + N_1 \frac{d\phi_1}{dt} + N_1 \frac{d\phi_{21}}{dt} = e_1 \tag{14}$$

and

$$R_2 i_2 + N_2 \frac{d\phi_2}{dt} + N_2 \frac{d\phi_{12}}{dt} = e_2 \tag{15}$$

If the permeability of the flux paths is assumed constant, the above

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equations can be written in more convenient forms, since:

$$N_1\phi_1 = L_1 i_1 \quad \therefore N_1 \frac{d\phi_1}{dt} = L_1 \frac{di_1}{dt} \tag{16}$$

$$N_1\phi_{21} = M_{21} i_2 \therefore N_1 \frac{d\phi_{21}}{dt} = M_{21} \frac{di_2}{dt} \tag{17}$$

$$N_2\phi_2 = L_2 i_2 \quad \therefore N_2 \frac{d\phi_2}{dt} = L_2 \frac{di_2}{dt} \tag{18}$$

$$N_2\phi_{12} = M_{12} i_1 \therefore N_2 \frac{d\phi_{12}}{dt} = M_{12} \frac{di_1}{dt} \tag{19}$$

Equations (14) and (15) may, therefore, be written in the following manner:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M_{21} \frac{di_2}{dt} = e_1 \tag{14a}$$

$$R_2 i_2 + L_2 \frac{di_2}{dt} + M_{12} \frac{di_1}{dt} = e_2 \tag{15a}$$

It will be observed that the effects of mutual inductance are entered into the basic voltage equations as voltage drops ($+M\, di/dt$). If, for example, $i_1 = I_{m1} \sin \omega t$, the voltage drop in circuit 2 due to mutual inductance is:

$$M_{12} \frac{di_1}{dt} = \omega M_{12} I_{m_1} \cos \omega t = X_{M_{12}} I_{m_1} \cos \omega t \tag{20}$$

In general, $\omega M = X_M$. It is called the mutual reactance and is an impedance function which expresses the ratio of the voltage of mutual inductance to the exciting current. It will be noted that the voltage of mutual inductance leads the exciting current by 90°. Hence the vector expression for the mutual reactance is:

$$\mathbf{X}_M = j\omega M = \omega M \underline{/90^\circ} \tag{21}$$

Circuit configurations in which M may possess either a positive or negative sign will be considered presently.

Problem 5. An inductance coil has a resistance of 10 ohms, a self-inductance of 1/37.7 henry, and a mutual inductance of 0.02 henry with respect to a neighboring coil. ($M_{12} = M_{21}$.) A voltage of 50 sin 377t volts is impressed across the terminals of the primary coil. Find the ohmic value of the mutual reactance and the effective value of the voltage across the open-circuited terminals of the neighboring coil.

Ans.: $X_M = 7.54$ ohms, $V_2 = 18.85$ volts.

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Problem 6. Let the effective values of the primary voltage and current of Problem 5 be known as V_1 and I_1, and draw a vector diagram illustrating $\mathbf{V}_1$, $\mathbf{I}_1$, $R_1\mathbf{I}_1$, $jX_{L1}\mathbf{I}_1$, $jX_M\mathbf{I}_1$, and $\mathbf{E}_{21}$. (*Note:* Considered as a generated voltage, $\mathbf{E}_{21}$ is 180° out of phase with $jX_M\mathbf{I}_1$, since the latter is a component voltage drop in circuit 2 in the same sense that $R\mathbf{I}_1$ and $jX_{L1}\mathbf{I}_1$ are component voltage drops in circuit 1.)

Ans.: $\mathbf{V}_1 = \frac{50}{\sqrt{2}}\underline{/0^\circ}$ volts, $\mathbf{I}_1 = 2.5\underline{/-45^\circ}$ amperes, $\mathbf{E}_{21} = 18.85\underline{/-135^\circ}$ volts.

Coefficient of Magnetic Coupling. The fractional part of ϕ_1 which links with N_2, ϕ_{12}/ϕ_1, and the fractional part of ϕ_2 which links with N_1, (ϕ_{21}/ϕ_2), are indices of the degree of coupling that exists between two windings. Where the windings are widely separated or are so situated in space that these fractions are small, the coupling is said to be loose. With closer proximity and proper orientation of the windings, ϕ_{12}/ϕ_1 and ϕ_{21}/ϕ_2 approach unity as a theoretical upper limit.

The coefficient of coupling between two windings which individually possess L_1 and L_2 units of self-inductance is defined as:

$$k_M = \sqrt{\left(\frac{\phi_{12}}{\phi_1}\right)\left(\frac{\phi_{21}}{\phi_2}\right)} = \sqrt{\frac{(M_{12}i_1/N_2)}{(L_1i_1/N_1)}\,\frac{(M_{21}i_2/N_1)}{(L_2i_2/N_2)}} = \sqrt{\left(\frac{M_{12}}{L_1}\right)\left(\frac{M_{21}}{L_2}\right)} \tag{22}$$

Under the condition of constant permeability, $\phi_{12}/\phi_1 = \phi_{21}/\phi_2$ and $M_{12} = M_{21} = M$. Therefore, if the permeability is constant,

$$k_M = \sqrt{\left(\frac{M}{L_1}\right)\left(\frac{M}{L_2}\right)} = \frac{M}{\sqrt{L_1L_2}} \tag{23}$$

Thus k_M is the geometric mean of the fractions (ϕ_{12}/ϕ_1) and (ϕ_{21}/ϕ_2) or between the fractions (M/L_1) and (M/L_2). Numerically the coefficient of coupling in practical installations may range from approximately 0.01 between certain types of radio circuits to as high as 0.98 or 0.99 between iron-core transformer windings.

Example 3. Let the number of turns of the two windings shown in Fig. 6 be $N_1 = 50$ and $N_2 = 500$. It will be assumed that 6000 maxwells link with the turns N_1, of circuit 1, per ampere of exciting current i_1, of which 5500 also link with N_2. Under the assumption of similar concentrated windings and of constant permeability of the flux paths, 60,000 maxwells will link with the turns N_2, of circuit 2, per ampere of exciting current i_2, and 55,000 of these flux lines will also link with N_1. The purpose of this numerical example is to specify the coefficient of coupling in terms of the fractions (ϕ_{12}/ϕ_1) and (ϕ_{21}/ϕ_2) and also in terms of the fractions (M_{12}/L_1) and (M_{21}/L_2). For 1 ampere of primary exciting current and for 1 ampere of secondary current:

$$\phi_1 = 6000 \text{ maxwells}$$
$$\phi_{12} = 5500 \text{ maxwells}$$
$$\phi_2 = 60{,}000 \text{ maxwells}$$
$$\phi_{21} = 55{,}000 \text{ maxwells}$$

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$$k_M = \sqrt{\left(\frac{\phi_{12}}{\phi_1}\right)\left(\frac{\phi_{21}}{\phi_2}\right)} = \sqrt{\left(\frac{55}{60}\right)\left(\frac{55}{60}\right)} = 0.917$$

$$L_1 = \frac{N_1\phi_1}{i_1} = \frac{50 \times 6000}{1} \times 10^{-8} = 0.003 \text{ henry}$$

$$M_{12} = \frac{N_2\phi_{12}}{i_1} = \frac{500 \times 5500}{1} \times 10^{-8} = 0.0275 \text{ henry}$$

$$L_2 = \frac{N_2\phi_2}{i_2} = \frac{500 \times 60{,}000}{1} \times 10^{-8} = 0.30 \text{ henry}$$

$$M_{21} = \frac{N_1\phi_{21}}{i_2} = \frac{50 \times 55{,}000}{1} \times 10^{-8} = 0.0275 \text{ henry}$$

$$k_M = \sqrt{\left(\frac{M_{12}}{L_1}\right)\left(\frac{M_{21}}{L_2}\right)} = \frac{M}{\sqrt{L_1L_2}} = \frac{0.0275}{\sqrt{0.003 \times 0.30}} = 0.917$$

Problem 7. The individual self-inductances of two windings are 0.094 henry and 0.0108 henry. The coefficient of coupling between the windings is 0.805. Find the mutual inductance of the two windings. *Ans.:* 0.0256 henry.

Problem 8. A winding of 1000 turns has a (ϕ_1/i_1) characteristic of 9400 maxwells per ampere and is coupled magnetically to a second winding of 338 turns. Assuming constant permeability of the flux paths and similar concentrated windings, find L_1, L_2, and M in henrys if the coefficient of coupling is 0.805.
Ans.: $L_1 = 0.094$ henry, $L_2 = 0.0108$ henry, $M = 0.0256$ henry.

Circuit Directions and the Sign of *M*. If only one circuit of an a-c network includes a generating device, the positive directions of the currents may be arbitrarily assigned if it is understood that the positive circuit direction given to the current through the generator arbitrarily defines the positive circuit direction of the generated voltage. When more than one generating device is present in an electrical network, the relative polarities and time phases of the generating devices with respect to the common branches must be taken into account in assigning the positive circuit directions of the currents in the coupled circuits.

In a given circuit or portion thereof the voltage of mutual inductance, $M\,di/dt$, may aid or oppose the voltage of self-inductance, $L\,di/dt$. If more than one circuit is involved, the currents are first given their positive circuit directions. When the positive circuit directions of the currents have been determined from the relative polarities of the several generating devices (if more than one generator exists), or when the positive circuit directions of the currents for a single generator have been arbitrarily assigned, the sign of M is considered positive if in a given winding the induced voltage of mutual inductance acts in the

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same direction as the induced voltage of self-inductance. If the induced voltage of mutual inductance opposes the induced voltage of self-inductance in a given winding, M is considered as a negative quantity.

In determining the sign of M, each particular case must be analyzed as to the relative positive circuit directions of the currents, the relative modes of winding of the coils involved, and the actual physical placement of one winding with respect to the other. It will be shown later that the sign of M between circuits which are not electrically connected and which are energized with a single generator in one circuit is wholly dependent upon the arbitrary positive circuit directions which are assigned to the currents in the separate circuits.

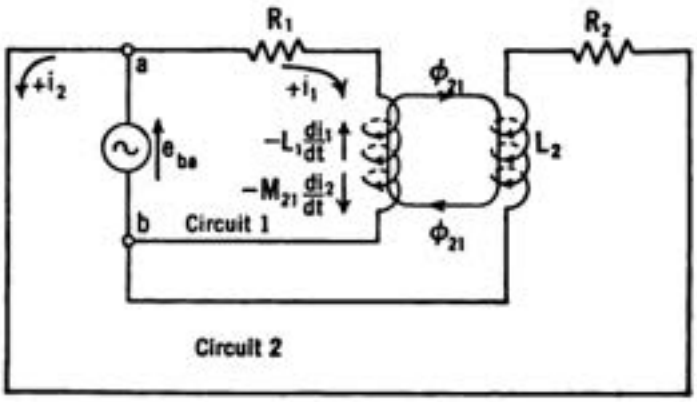

FIG. 7. Illustrating a particular case wherein the voltage of mutual inductance acts in circuit opposition to the voltage of self-inductance in a given coil.

Example 4. Consider the hypothetical arrangement of the two circuits shown in Fig. 7. If the clockwise direction around circuit 1 is taken as the positive circuit direction of i_1, the generator emf possesses a positive circuit direction from b to a through the generator. The latter direction fixes the positive circuit direction of i_2 as counter-clockwise around circuit 2.

By Lenz's law, the voltage of self-inductance in the L_1 coil *considered as an induced voltage* acts in a counter-clockwise direction around circuit 1 when di_1/dt is positive. If the positive circuit direction of i_2 and the modes of winding of the coils are considered, it is plain that voltage which is induced in the L_1 coil by the variation of ϕ_{21} is a clockwise direction around circuit 1 when di_2/dt or $d\phi_{21}/dt$ is positive.

Since $M\ di_2/dt$ acts oppositely to $L_1\ di_1/dt$ in circuit 1, M must be considered negative if L_1 is considered positive. Another way to determine the sign of M is to call M positive if the mmf's caused by the two currents combine to increase the total flux. If the mmf's oppose, the sign of M is negative. The general equation for voltage equilibrium in circuit 1 is:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + (-M)\frac{di_2}{dt} = e_{ba}$$

Problem 9. Show, by means of detailed and independent analysis, that the

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general equation for voltage equilibrium in circuit 2 of Fig. 7 is:

$$R_2 i_2 + L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} = e_{ba}$$

Instead of showing the actual modes of winding, a conventional method employing a dot-marked terminal, as shown in Fig. 8, is often used to yield the same information. This practice has been used for many years in the marking of iron-core instrument transformers, where the dots are known as polarity marks. The dots are placed so that a current entering the dot-marked terminal of any coil will produce a magnetomotive force and corresponding flux in the *same* direction around the magnetic circuit. Thus in Fig. 8 a current *entering* the dot-marked terminal of coil 1 causes a counter-clockwise flux in the magnetic circuit and a current *entering* the dot-marked terminal of coil 2 also causes a counter-clockwise flux in the same magnetic circuit. Hence the dots alone are sufficient to determine the relative modes

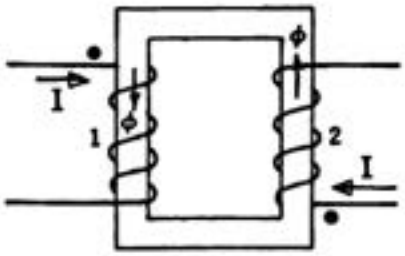

Fig. 8. Dot marks used to define relative polarities of two coils.

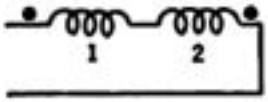

Fig. 9. Dot marks indicate $-M$.

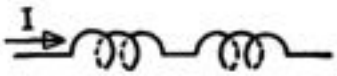

Fig. 10. Mode of winding and physical placement indicate $-M$.

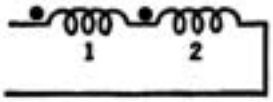

Fig. 11. Dot marks indicate $+M$.

of winding. The use of this convention is illustrated in Fig. 9. If a current entering the dot-marked terminal of coil 1 is assumed to produce a flux through the coils from left to right, this same current, since it is leaving the dot-marked terminal of coil 2, would cause a flux from right to left through the coils. Therefore, for the purpose of setting up an equation of voltage drops, M must be considered negative. Hence the relative modes of winding must be as shown in Fig. 10. If the coils of Fig. 9 were marked as shown in Fig. 11, a current entering the dot-marked terminal of coil 1 would also enter the dot-marked terminal of coil 2, the mmf's of the two coils would be additive, and the sign of M would be positive.

Mutual Inductance between Portions of the Same Circuit. Mutual inductance may be a significant factor in governing the flow of electricity in a single-series circuit where two or more portions of the circuit are coupled magnetically. A particular example is shown in Fig. 12. The

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arrangement consists of two magnetically coupled inductance coils connected in electrical series. Individually the coils possess L_a and L_b units of self-inductance together with R_a and R_b units of resistance, respectively.

If the coils are wound in the manner shown in Fig. 12, it is apparent that, in coil a, the voltage

$$-M_{ba}\frac{di}{dt} = -N_a\frac{d\phi_{ba}}{dt}$$

acts in the same circuit direction as the voltage $-L_a\,di/dt$. Likewise the voltage

$$-M_{ab}\frac{di}{dt} = -N_b\frac{d\phi_{ab}}{dt}$$

acts in the same circuit direction as $-L_b\,di/dt$. Hence M is positive.

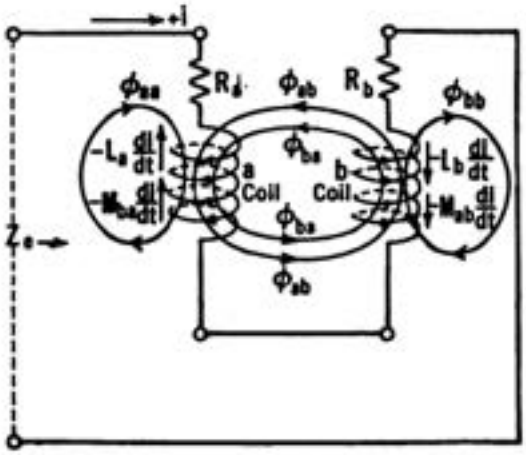

FIG. 12. Two inductance coils connected series-aiding.

Considered as voltage drops, the component voltages referred to above have circuit directions which agree with that of the applied voltage, v. Considered as voltage rises, the induced voltages are, of course, in circuit opposition to the applied voltage, v.

The facts involved can be stated in equation form as follows:

$$R_a i + L_a\frac{di}{dt} + M_{ba}\frac{di}{dt} + R_b i + L_b\frac{di}{dt} + M_{ab}\frac{di}{dt} = v \qquad (24)$$

If the mutual flux path is of constant permeability, the above equation reduces to:

$$(R_a + R_b)\,i + (L_a + L_b + 2M)\frac{di}{dt} = v \qquad (25)$$

If v varies sinusoidally with time and if all circuit parameters are con-

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stant, equation (25) may be written in terms of effective values as follows:

$$(R_a + R_b)\mathbf{I} + j\omega(L_a + L_b + 2M)\mathbf{I} = \mathbf{V} \tag{26}$$

It will be noted that M enters into the voltage equation in exactly the same manner as L. Hence ωM is a mutual reactance. The equivalent impedance of the series circuit shown in Fig. 12 follows directly from equation (26).

$$\mathbf{Z}_e = \frac{\mathbf{V}}{\mathbf{I}} = \sqrt{[R_a + R_b]^2 + [\omega(L_a + L_b + 2M)]^2} \angle \tan^{-1}\frac{\omega(L_a + L_b + 2M)}{(R_a + R_b)} \tag{27}$$

Equation (27) may also be written:

$$\mathbf{Z}_e = (R_a + R_b) + j\omega(L_a + L_b + 2M) = \mathbf{Z}_a + \mathbf{Z}_b + 2\mathbf{Z}_M \tag{27a}$$

where

$$\mathbf{Z}_a = R_a + j\omega L_a,\ \mathbf{Z}_b = R_b + j\omega L_b \quad \text{and} \quad \mathbf{Z}_M = 0 + j\omega M$$

If the two coils were connected together in the opposite sense, that is, with a polarity opposite to that shown in Fig. 12, the signs of the M terms in the above equations would be reversed.

Example 5. An inspection of equations (25), (26), and (27) will show that the equivalent inductance of the two coils connected in additive series is:

$$L_{e(\text{add})} = L_a + L_b + 2M$$

If the two coils are connected in subtractive series:

$$L_{e(\text{sub})} = L_a + L_b - 2M$$

The value of M may, therefore, be found experimentally by measuring $L_{e(\text{add})}$ and $L_{e(\text{sub})}$ since, from the above relations:

$$M = \frac{L_{e(\text{add})} - L_{e(\text{sub})}}{4}$$

Example 6. Let it be required to find the coefficient of coupling, the equivalent series-circuit impedance, and the magnitude of the current in a circuit arrangement similar to that shown in Fig. 12 if:

R_a = 1.0 ohm	M = +3 millihenrys
L_a = 4.0 millihenrys	ω = 1000 radians per second
R_b = 6.0 ohms	V = 40.5 volts, the applied voltage
L_b = 9.0 millihenrys	

(*a*) The coefficient of coupling is:

$$k = \frac{M}{\sqrt{L_a L_b}} = \frac{+3}{\sqrt{4 \times 9}} = +0.5$$

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(*b*) The equivalent series-circuit impedance is:

$$\begin{aligned} \mathbf{Z}_e &= (R_a + R_b) + j\omega(L_a + L_b + 2M) \\ &= (1 + 6) + j(1000)(0.004 + 0.009 + 0.006) \\ &= 7 + j19 = 20.25\underline{/69.8^\circ} \text{ ohms} \end{aligned}$$

(*c*) The series current is:

$$I = \frac{40.5}{20.25} = 2.0 \text{ amperes}$$

A vector diagram of $\mathbf{V}$, $\mathbf{I}$, $\mathbf{V}_a$, and $\mathbf{V}_b$ is shown in Fig. 13 together with the component voltages of $\mathbf{V}_a$ and $\mathbf{V}_b$.

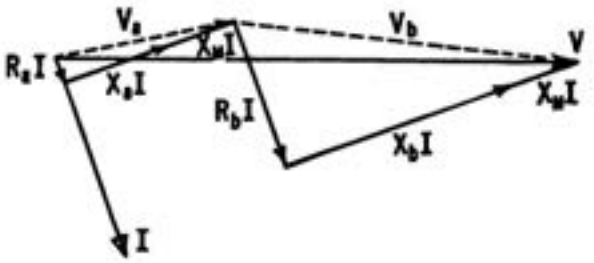

FIG. 13. Vector diagram of example 6.

Problem 10. Find the magnitude of the current in the above example if the two coils are connected in subtractive series, that is, $M = -3$ millihenrys. Draw a vector diagram illustrating the vector positions of $\mathbf{V}$, $\mathbf{I}$, $\mathbf{V}_a$, $\mathbf{V}_b$, and the various $R\mathbf{I}$ and $X\mathbf{I}$ component voltages. *Ans.*: $I = 4.09$ amperes.

Mutual Inductance between Parallel Branches. Reference to Fig. 14 will show that, in coil 1, $M_{21}\, di_2/dt$ acts in circuit opposition to $L_1\, di_1/dt$.

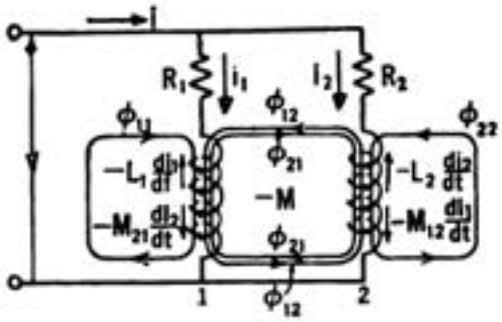

FIG. 14. Parallel arrangement of two inductance coils which are coupled magnetically. For the mode of winding shown and the assumed positive directions of currents as indicated, M is negative.

Similarly, in coil 2, $M_{12}\, di_1/dt$ acts in circuit opposition to $L_2\, di_2/dt$. In equation form:

$$R_1 i_1 + L_1 \frac{di_1}{dt} - M_{21} \frac{di_2}{dt} = v \tag{28}$$

$$R_2 i_2 + L_2 \frac{di_2}{dt} - M_{12} \frac{di_1}{dt} = v \tag{29}$$

It will be noted that the individual branch currents have been employed in the above equations.

If the circuit parameters are constant and a sinusoidal variation of v is assumed, the above equations may be written in terms of effective

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values as follows:

$$(R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2 = \mathbf{V} \tag{30}$$

$$(R_2 + j\omega L_2)\mathbf{I}_2 - j\omega M\mathbf{I}_1 = \mathbf{V} \tag{31}$$

Let

$$(R_1 + j\omega L_1) = \mathbf{Z}_1 \tag{32}$$

$$(R_2 + j\omega L_2) = \mathbf{Z}_2 \tag{33}$$

$$0 + j\omega M = \mathbf{Z}_M \tag{34}$$

With the above abbreviations, equations (30) and (31) reduce to:

$$\mathbf{Z}_1\mathbf{I}_1 - \mathbf{Z}_M\mathbf{I}_2 = \mathbf{V} \tag{35}$$

$$-\mathbf{Z}_M\mathbf{I}_1 + \mathbf{Z}_2\mathbf{I}_2 = \mathbf{V} \tag{36}$$

The individual branch currents, $\mathbf{I}_1$ and $\mathbf{I}_2$, may be found from the simultaneous solutions of equations (35) and (36).

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{V} & -\mathbf{Z}_M \\ \mathbf{V} & \mathbf{Z}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & -\mathbf{Z}_M \\ -\mathbf{Z}_M & \mathbf{Z}_2 \end{vmatrix}} = \frac{\mathbf{V}(\mathbf{Z}_2 + \mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2} \tag{37}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{V} \\ -\mathbf{Z}_M & \mathbf{V} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & -\mathbf{Z}_M \\ -\mathbf{Z}_M & \mathbf{Z}_2 \end{vmatrix}} = \frac{\mathbf{V}(\mathbf{Z}_1 + \mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2} \tag{38}$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = \frac{\mathbf{V}(\mathbf{Z}_1 + \mathbf{Z}_2 + 2\mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2} \tag{39}$$

The equivalent impedance of the two parallel branches shown in Fig. 14 for the case of negative M is:

$$\mathbf{Z}_e = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2}{\mathbf{Z}_1 + \mathbf{Z}_2 + 2\mathbf{Z}_M} \tag{40}$$

Example 7. In the circuit arrangement shown in Fig. 14 it will be assumed that:

$R_1 = 3.3$ ohms $\qquad L_2 = 0.0108$ henry

$L_1 = 0.094$ henry $\qquad M = -0.0256$ henry

$R_2 = 0.775$ ohm $\qquad \omega = 377$ radians per second

$$\mathbf{V} = 50\angle 0^\circ \text{ volts}$$

Let it be required to find $\mathbf{I}$, $\mathbf{I}_1$, $\mathbf{I}_2$, and the total power spent in the two parallel branches.

$$\mathbf{Z}_1 \text{ (individually)} = 3.3 + j35.4 = 35.5\angle 84.7^\circ \text{ ohms}$$

$$\mathbf{Z}_2 \text{ (individually)} = 0.775 + j4.07 = 4.17\angle 79.25^\circ \text{ ohms}$$

$$\mathbf{Z}_M = 0 + j\omega M = 0 + j9.65 = 9.65\angle 90^\circ \text{ ohms}$$

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Note: Z_M is herein considered as inherently positive since the appropriate negative signs have been introduced into equations (30) and (31).

$$Z_e = \frac{Z_1Z_2 - Z_M^2}{Z_1 + Z_2 + 2Z_M} = \frac{63.6\angle 140^\circ}{59.0\angle 86^\circ} = 1.078\angle 54^\circ \text{ ohms}$$

$$I = \frac{V}{Z_e} = \frac{50\angle 0^\circ}{1.078\angle 54^\circ} = 46.4\angle -54^\circ \text{ amperes}$$

$$I_1 = \frac{V(Z_2 + Z_M)}{Z_1Z_2 - Z_M^2} = \frac{(50\angle 0^\circ)(13.73\angle 86.8^\circ)}{63.6\angle 140^\circ}$$

$$I_1 = 10.8\angle -53.2^\circ \text{ amperes}$$

$$I_2 = \frac{V(Z_1 + Z_M)}{Z_1Z_2 - Z_M^2} = \frac{(50\angle 0^\circ)(45.1\angle 85.8^\circ)}{63.6\angle 140^\circ}$$

$$I_2 = 35.4\angle -54.2^\circ \text{ amperes}$$

$$P = VI\cos\theta\Big]_{I}^{V} = 50 \times 46.4 \times \cos 54^\circ = 1365 \text{ watts}$$

Check:

$$I = I_1 + I_2 = 10.8\angle -53.2^\circ + 35.4\angle -54.2^\circ$$

$$I = (6.46 - j8.65) + (20.8 - j28.8) = 27.26 - j37.45$$

$$I = 46.4\angle -54^\circ \text{ amperes}$$

$$P = I_1^2R_1 + I_2^2R_2 = 385 + 973 = 1358 \text{ watts}$$

Problem 11. Assume that the inductance coils in the above illustrative example are connected in parallel as shown in Fig. 14, except that the terminals of one coil are reversed from that shown in the figure. Show that, under these conditions:

$$Z_e = 3.095\angle 61.40^\circ \text{ ohms}$$

$$I = 16.16\angle -61.40^\circ \text{ amperes (V as reference)}$$

$$I_1 = 4.43\angle -222.1^\circ \text{ amperes}$$

$$I_2 = 20.4\angle -57.30^\circ \text{ amperes}$$

$$P = VI\cos\theta\Big]_{I}^{V} = 386 \text{ watts}$$

Draw the vector diagram of V, I, I_1, and I_2, and illustrate the manner in which the three component voltages in each branch combine vectorially to equal the applied voltage, V.

The Air-Core Transformer. In the conventional transformer arrangement shown schematically in Fig. 15, the individual circuits are not connected electrically. Circuit 1, energized by means of an alternating potential difference, is called the primary. Circuit 2 is called the

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secondary. As a result of the magnetic coupling between the circuits, circuit 2 has induced in it a voltage which is equal to:

$$-N_2 \frac{d\phi_{12}}{dt} = -M_{12} \frac{di_1}{dt} \tag{41}$$

The magnitude of the voltage induced in circuit 2 is proportional to the number of secondary turns, N_2, and is dependent upon the degree of coupling between the two windings.

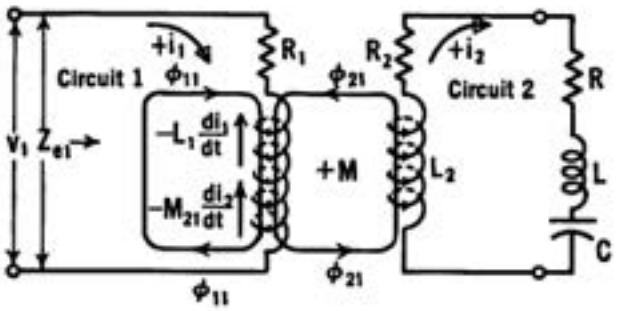

FIG. 15. Conventional air-core transformer arrangement.

The sign of M in the conventional transformer arrangement is dependent upon the arbitrary choice of the positive circuit direction of i_2. The majority of writers prefer to use the positive circuit direction of i_2 which allows them to employ the positive sign of M. For the relative modes of winding shown in Fig. 15, the positive clockwise direction of i_2 requires the use of $+M$, since under these conditions $M_{21}\, di_2/dt$ acts in the same circuit direction as $L_1\, di_1/dt$ in the primary winding. If the counter-clockwise direction around circuit 2 is taken as the positive circuit direction of i_2, then, of course, M must be considered negative. The resulting solutions will be identical in either case, except that all secondary currents and voltages will be reversed in sign. Experience with detailed solutions will convince the reader that the two different methods of attack yield identical physical results.

If the positive circuit directions are employed as indicated in Fig. 15, the mathematical analysis of the ordinary air-core transformer may be carried out as follows:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M_{21} \frac{di_2}{dt} = v_1 \tag{42}$$

$$(R_2 + R)i_2 + (L_2 + L)\frac{di_2}{dt} + \frac{\int i_2\, dt}{C_2} + M_{12}\frac{di_1}{dt} = 0 \tag{43}$$

If v_1 is assumed to have sinusoidal wave form and all circuit parameters are constant, the above equations may be written in terms of effective

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values as follows:

$$(R_1 + j\omega L_1)\mathbf{I}_1 + j\omega M\mathbf{I}_2 = \mathbf{V}_1 \tag{44}$$

$$(R_2 + j\omega L_2)\mathbf{I}_2 + \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]\mathbf{I}_2 + j\omega M\mathbf{I}_1 = 0 \tag{45}$$

For the sake of simplicity in writing, the following abbreviations are adopted:

$\mathbf{Z}_1 = (R_1 + j\omega L_1)$ (Individual primary winding impedance) (46)

$\mathbf{Z}_2 = (R_2 + j\omega L_2)$ (Individual secondary winding impedance) (47)

$\mathbf{Z}_M = (0 + j\omega M)$ (Mutual impedance assuming no core loss) (48)

$\mathbf{Z} = \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]$ (General expression for load impedance) (49)

Equations (44) and (45) become:

$$\mathbf{Z}_1\mathbf{I}_1 + \mathbf{Z}_M\mathbf{I}_2 = \mathbf{V}_1 \tag{44)-(50}$$

$$\mathbf{Z}_M\mathbf{I}_1 + (\mathbf{Z}_2 + \mathbf{Z})\mathbf{I}_2 = 0 \tag{45)-(51}$$

The simultaneous solutions of the above equations for $\mathbf{I}_1$ and $\mathbf{I}_2$ yield:

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{V}_1 & \mathbf{Z}_M \\ 0 & (\mathbf{Z}_2 + \mathbf{Z}) \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_M \\ \mathbf{Z}_M & (\mathbf{Z}_2 + \mathbf{Z}) \end{vmatrix}} = \frac{\mathbf{V}_1(\mathbf{Z}_2 + \mathbf{Z})}{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2} \tag{52}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{V}_1 \\ \mathbf{Z}_M & 0 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_M \\ \mathbf{Z}_M & (\mathbf{Z}_2 + \mathbf{Z}) \end{vmatrix}} = \frac{-\mathbf{V}_1\mathbf{Z}_M}{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2} \tag{53}$$

If $\mathbf{I}_1$ has been evaluated, it may, in certain cases, be more convenient to solve for $\mathbf{I}_2$ directly from equation (51).

$$\mathbf{I}_2 = \frac{-\mathbf{Z}_M\mathbf{I}_1}{(\mathbf{Z}_2 + \mathbf{Z})} \tag{54}$$

The secondary terminal voltage, or the voltage which appears across the load impedance, is:

$$\mathbf{V}_2 = \mathbf{Z}\mathbf{I}_2 = -\mathbf{Z}_M\mathbf{I}_1 - \mathbf{Z}_2\mathbf{I}_2 \tag{55}$$

Also:

$$\mathbf{V}_2 = \frac{-\mathbf{V}_1\mathbf{Z}_M\mathbf{Z}}{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2} \tag{56}$$

The above relations follow directly from equations (51) and (53). Equation (55) shows that the secondary circuit may be thought of

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as experiencing an induced voltage equal to $-\mathbf{Z}_M\mathbf{I}_1$, from which the internal secondary impedance drop, $\mathbf{Z}_2\mathbf{I}_2$, must be subtracted in order to obtain the secondary terminal voltage, $\mathbf{V}_2$.

Equivalent Impedance. The equivalent impedance of the transformer arrangement shown in Fig. 15 referred to the primary side is defined as the ratio of the applied voltage to the primary current. Thus:

$$\mathbf{Z}_{e1} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2}{(\mathbf{Z}_2 + \mathbf{Z})} \tag{57}$$

A more convenient form of the above equation is:

$$\mathbf{Z}_{e1} = \mathbf{Z}_1 - \frac{\mathbf{Z}_M^2}{(\mathbf{Z}_2 + \mathbf{Z})} = \mathbf{Z}_1 + \frac{\omega^2 M^2}{\mathbf{Z}_2 + \mathbf{Z}} \tag{58}$$

Equations (57) and (58) show that the air-core transformer, with respect to its primary terminals, is reducible to an equivalent series circuit.

Example 8 (for Z = 0). It will be assumed that, in Fig. 16*a*:

$R_1 = 3.3$ ohms $\qquad M = 0.0256$ henry

$L_1 = 0.094$ henry $\qquad \mathbf{Z} = 0$

$R_2 = 0.775$ ohm $\qquad \omega = 377$ radians per second

$L_2 = 0.0108$ henry $\qquad \mathbf{V}_1 = 50\angle 0°$ volts

$$\mathbf{Z}_1 = 3.3 + j35.4 = 35.5\angle 84.7° \text{ ohms}$$

$$\mathbf{Z}_2 = 0.775 + j4.07 = 4.14\angle 79.25° \text{ ohms}$$

$$\mathbf{Z}_M = 0 + j9.65 = 9.65\angle 90° \text{ ohms}$$

$$\mathbf{Z}_{e1} = \mathbf{Z}_1 - \frac{\mathbf{Z}_M^2}{\mathbf{Z}_2} = (3.3 + j35.4) + \frac{93.1\angle 0°}{4.14\angle 79.25°}$$

$$\mathbf{Z}_{e1} = (3.3 + j35.4) + (4.20 - j22.1) = 7.50 + j13.3 = 15.27\angle 60.55° \text{ ohms}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{\mathbf{Z}_{e1}} = \frac{50\angle 0°}{15.27\angle 60.55°} = 3.28\angle -60.55° \text{ amperes}$$

$$\mathbf{I}_2 = \frac{-\mathbf{I}_1\mathbf{Z}_M}{\mathbf{Z}_2} = \frac{(3.28\angle 119.45°)(9.65\angle 90°)}{4.14\angle 79.25°}$$

$$\mathbf{I}_2 = 7.66\angle 130.2° \text{ amperes}$$

The total power dissipated in the two circuits is:

$$P = V_1 I_1 \cos\theta \Big]_{\mathbf{I}_1}^{\mathbf{V}_1} = 50 \times 3.28 \times \cos(-60.55°) = 80.8 \text{ watts}$$

or

$$P = I_1^2 R_1 + I_2^2 R_2 = 3.28^2 \times 3.3 + 7.66^2 \times 0.775 = 81.0 \text{ watts}$$

The vector diagram of $\mathbf{V}_1$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $-\mathbf{Z}_M\mathbf{I}_1$ is shown in Fig. 16*b*. In the particular case shown in Fig. 16*b*, the voltage induced in circuit 2, $-\mathbf{Z}_M\mathbf{I}_1$, is balanced

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entirely by the internal secondary impedance drop, namely, $\mathbf{Z}_2\mathbf{I}_2$. If the counterclockwise direction around circuit 2 had been taken as the positive circuit direction, $\mathbf{I}_2$ and $\mathbf{Z}_M\mathbf{I}_1$ would appear on the vector diagram 180° from the positions shown in Fig. 16*b*.

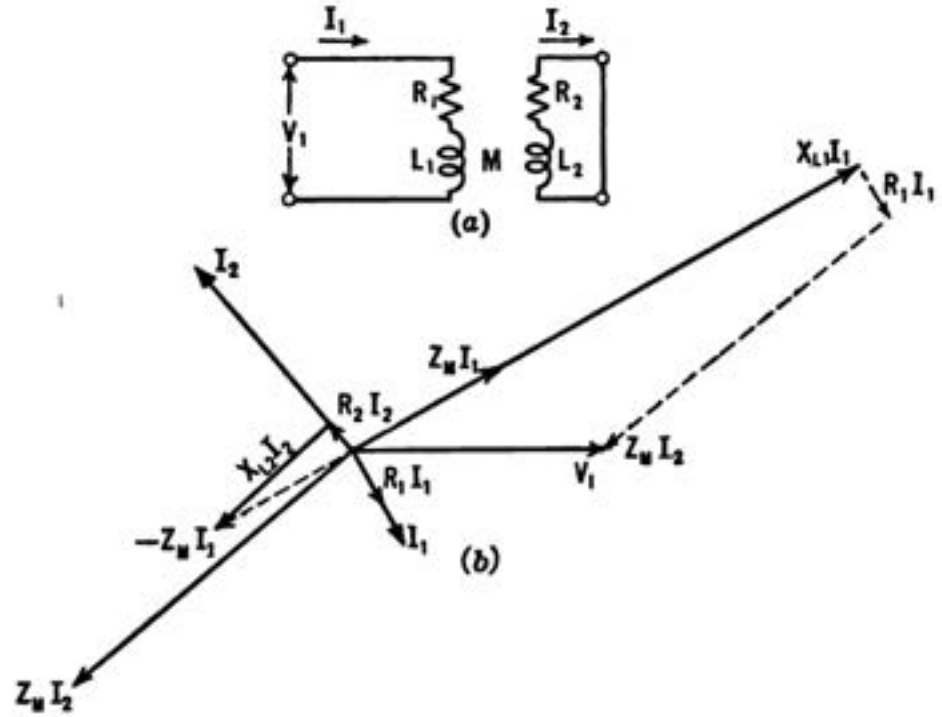

FIG. 16. Voltage and current relations in an air-core transformer the secondary of which is short-circuited. Note the manner in which $X_{L1}I_1$, R_1I_1, and Z_MI_2 combine vectorially to balance the applied voltage V_1.

Oscillogram 1 illustrates the instantaneous variations of v_1, i_1, and i_2 for the above numerical case. The salient features of the numerical solution are clearly shown. The primary current lags the applied voltage by approximately 60°, and the secondary current lags the primary current by approximately 170°. Within the limits of oscillographic accuracy, the maximum magnitudes of i_1 and i_2 agree with the results of the above numerical example.

Example 9 (for Z = 14.5 + *j*21.2 ohms). It will be assumed that in Fig. 17*a*:

$R_1 = 3.3$ ohms $\qquad M = 0.0256$ henry

$L_1 = 0.094$ henry $\qquad \mathbf{Z} = 14.5 + j21.2$ ohms

$R_2 = 0.775$ ohm $\qquad \omega = 377$ radians per second

$L_2 = 0.0108$ henry $\qquad \mathbf{V}_1 = 50\angle 0°$ volts

$$\mathbf{Z}_1 = 3.3 + j35.4 = 35.5\angle 84.7° \text{ ohms}$$

$$\mathbf{Z}_2 = 0.775 + j4.07 = 4.14\angle 79.25° \text{ ohms}$$

$$\mathbf{Z}_M = 0 + j9.65 = 9.65\angle 90° \text{ ohms}$$

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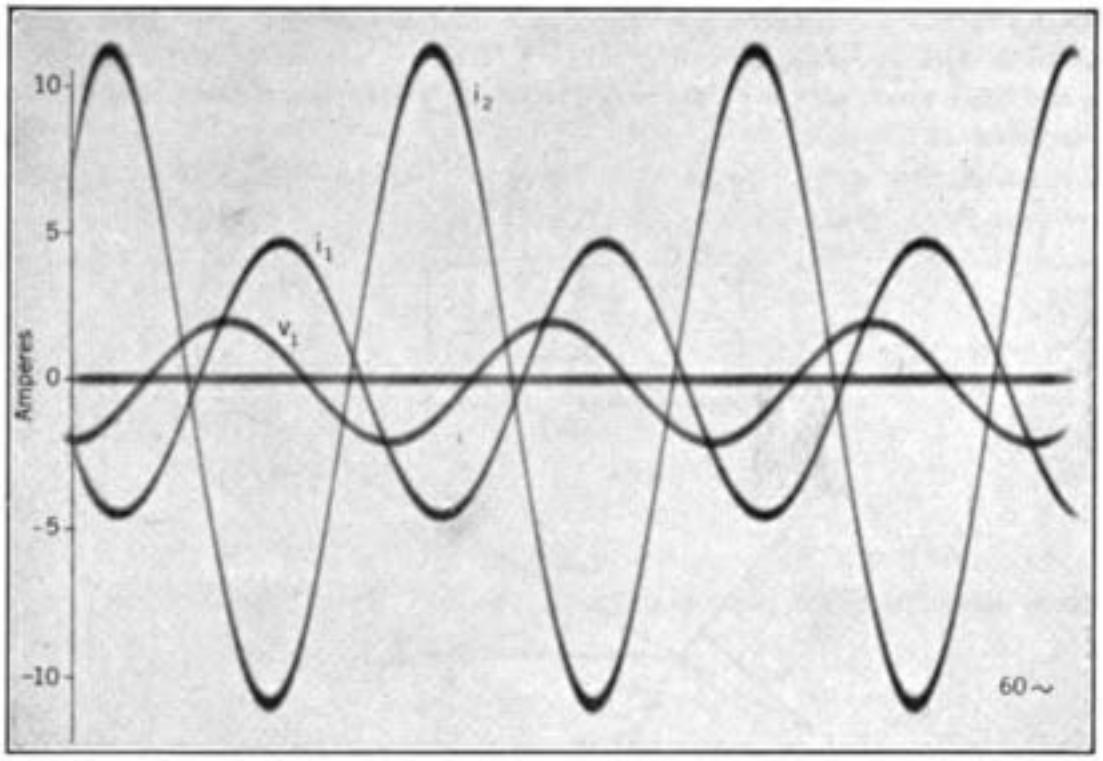

OSCILLOGRAM 1. Illustrating the time phase relations of primary and secondary currents of an air-core transformer with respect to the applied voltage wave. (For a short-circuited secondary. See Fig. 16a.) $v_1 = 70.7 \sin 377t$ volts.

$$\mathbf{Z}_{e1} = \mathbf{Z}_1 - \frac{\mathbf{Z}_M^2}{\mathbf{Z}_2 + \mathbf{Z}} = 35.5\underline{/84.7^\circ} + \frac{93.1\underline{/0^\circ}}{15.28 + j25.3}$$

$$\mathbf{Z}_{e1} = (3.3 + j35.4) + (1.63 - j2.7) = 4.93 + j32.7$$

$$\mathbf{Z}_{e1} = 33.0\underline{/81.4^\circ} \text{ ohms}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{\mathbf{Z}_{e1}} = \frac{50\underline{/0^\circ}}{33\underline{/81.4^\circ}} = 1.515\underline{/-81.4^\circ} \text{ amperes}$$

$$\mathbf{I}_2 = \frac{-\mathbf{I}_1\mathbf{Z}_M}{(\mathbf{Z}_2 + \mathbf{Z})} = \frac{(1.515\underline{/98.6^\circ})(9.65\underline{/90^\circ})}{29.6\underline{/58.9^\circ}}$$

$$\mathbf{I}_2 = 0.494\underline{/129.7^\circ} \text{ amperes}$$

$$\mathbf{V}_2 \text{ (terminal voltage)} = \mathbf{I}_2\mathbf{Z}$$

$$\mathbf{V}_2 = (0.494\underline{/129.7^\circ})(25.7\underline{/55.6^\circ}) = 12.7\underline{/185.3^\circ} \text{ volts}$$

The input power to the primary terminals is:

$$P_{\text{input}} = V_1 I_1 \cos\theta \Big]_{\mathbf{I}_1}^{\mathbf{V}_1} = 50 \times 1.515 \times \cos 81.4^\circ$$

$$= 50 \times 1.515 \times 0.1495 = 11.3 \text{ watts}$$

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The power delivered to the load is:

$$P_{\text{load}} = V_2 I_2 \cos\theta \Big]_{I_2}^{V_2} = 12.7 \times 0.494 \cos 55.6°$$

$$= 12.7 \times 0.494 \times 0.565 = 3.55 \text{ watts}$$

The efficiency of this particular air-core transformer working under the conditions stated above is 3.55/11.3 or 31.4 per cent.

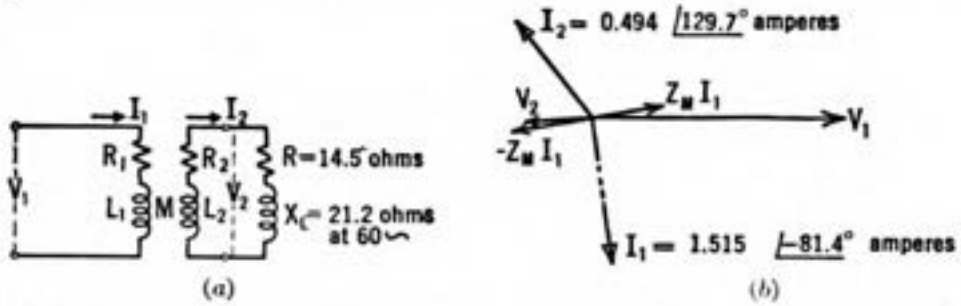
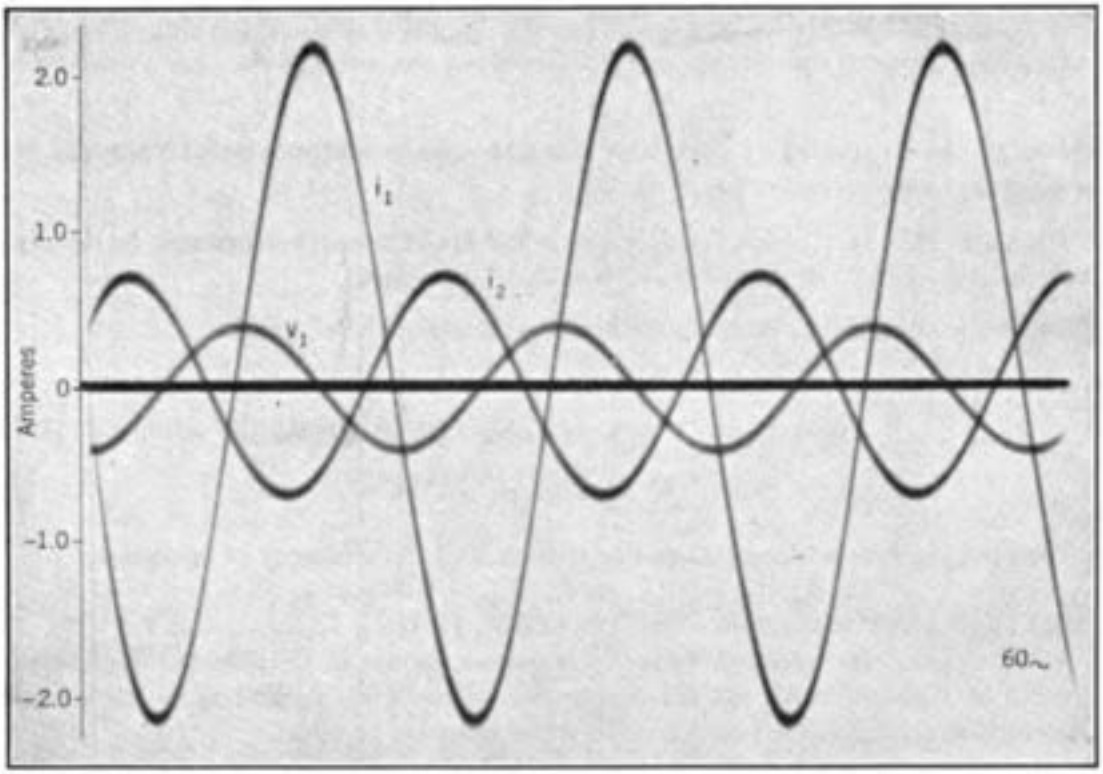


FIG. 17. Voltage and current relations in an air-core transformer the secondary of which is loaded as shown in (*a*).

Figure 17*b* is a vector diagram of $\mathbf{V}_1$, $\mathbf{I}_1$, $-\mathbf{Z}_M\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{V}_2$. Oscillogram 2 illustrates the variations of v_1, i_1, and i_2 for the particular case under discussion. The phase positions of the primary and secondary currents with respect to the applied voltage are shown in rectangular-coordinate form and agree with the calculated



OSCILLOGRAM 2. Illustrating the time-phase relations of primary and secondary currents of an air-core transformer with respect to the applied voltage wave. (For an inductive-type load placed across the secondary terminals of the transformer. See Fig. 17*a*.)
v_1 represents the applied voltage wave (effective value = 50 volts)
i_1 represents the primary current wave (effective value = 1.5 amperes)
i_2 represents the secondary current wave (effective value = 0.5 ampere)

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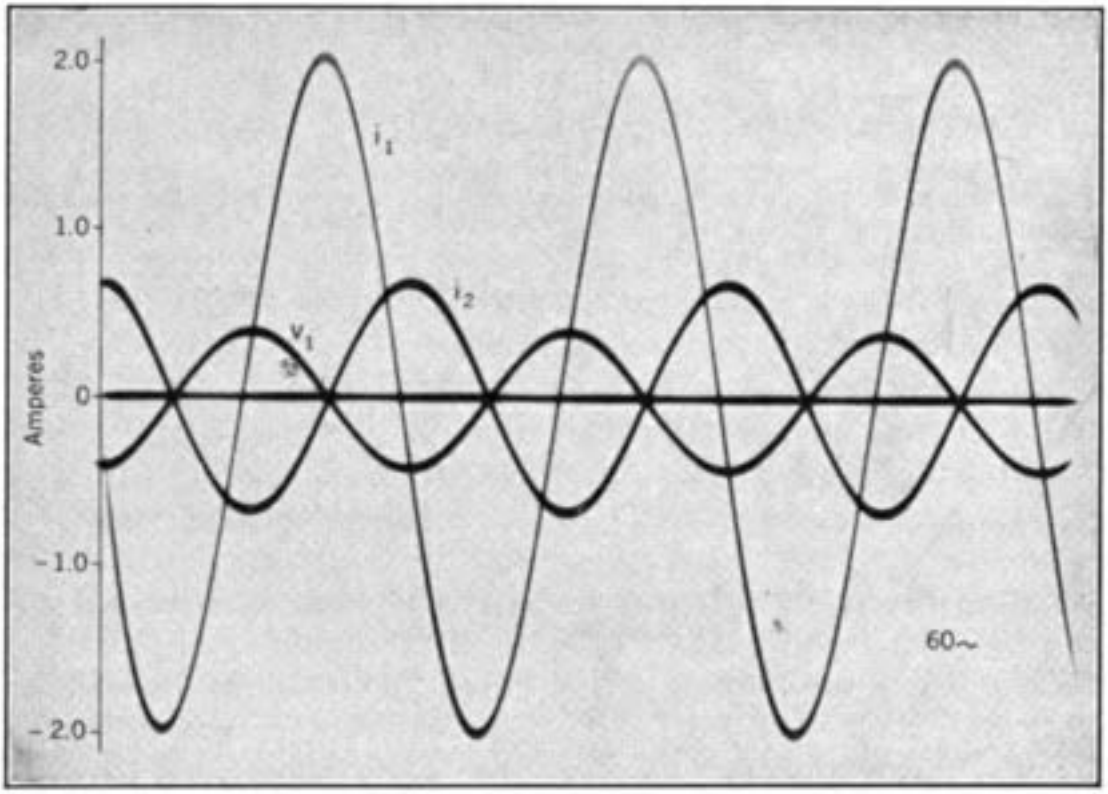

OSCILLOGRAM 3. Illustrating the time-phase relations of primary and secondary currents of an air-core transformer with respect to the applied voltage wave. (For a resistive-type load placed across the secondary terminals of the transformer. See Problem 12.)

values of these quantities. Likewise the wave shape and maximum values of the voltage and current waves are discernible.

Problem 12. Let the load impedance in the above numerical example be replaced with an impedance the value of which is $28.15\underline{/0^\circ}$ ohms.

(*a*) Show that, under this condition of operation,

$$\mathbf{Z}_{e1} = 35.5\underline{/79.5^\circ} \text{ ohms}$$

$$\mathbf{I}_1 = 1.409\underline{/-79.5^\circ} \text{ amperes} \quad (\mathbf{V}_1 \text{ as reference})$$

$$\mathbf{I}_2 = 0.465\underline{/182.4^\circ} \text{ amperes}$$

(*b*) Find the power input, the power output, and the efficiency of operation.

Ans.: $P_{in} = 12.8$ watts, $P_{out} = 6.08$ watts, efficiency $= 47.5\%$.

(*c*) Draw a vector diagram of $\mathbf{V}_1$, $\mathbf{I}_1$, $-\mathbf{I}_1\mathbf{Z}_M$, $\mathbf{I}_2$, $\mathbf{I}_2R_2$, $\mathbf{I}_2(j\omega L_2)$, and $\mathbf{V}_2$.

(*d*) Compare the results obtained with those shown in Oscillogram 3. Oscillogram 3 is a photographic record of the variations of v_1, i_1, and i_2 in the air-core transformer arrangement considered in this particular problem.

Transferred Impedance. One of the primary considerations in communication circuits is that of transferring maximum power from a low-power generating device to a receiver. It has been shown in Chapter V that maximum power is transferred (for a fixed generator

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voltage) when the impedance of the receiver (in complex form) is the conjugate of the impedance of the generator and associated transmission lines. That is, if $\mathbf{Z}_{gen} = R + jX$, then $\mathbf{Z}_{rec}$ should equal $R - jX$ for maximum power transfer. For impedance matches which will prevent reflection losses, $\mathbf{Z}_{gen} = \mathbf{Z}_{rec}$. (See Chapters XII and XIII.)

At audio frequencies, iron-core transformers may be used successfully for transforming voltage magnitudes and for matching impedances, but at radio frequencies air-core transformers are generally used. In iron-core transformers where the coefficient of coupling is relatively high and where $(\omega L_2)^2 \gg R_2'^2$, a resistance, R, placed across an N_2-turn secondary, *may* appear at the terminals of an N_1-turn primary as $(N_1/N_2)^2R$, approximately. The term "*may* appear" is used because several conditions must be fulfilled simultaneously before the $(N_1/N_2)^2$ factor can be used successfully, as will be shown presently.

Classical methods will be employed to show how an impedance placed across the secondary terminals of an air-core transformer appears at the primary terminals in modified form.[2]

Reference to equation (58) will show that the equivalent impedance of an air-core transformer referred to the primary side is:

$$\mathbf{Z}_{e1} = \mathbf{Z}_1 - \frac{\mathbf{Z}_M^2}{\mathbf{Z}_2'} = (R_1 + jX_1) + \frac{X_M^2}{(R_2' + jX_2')} \tag{59}$$

where $\mathbf{Z}_2' = (\mathbf{Z}_2 + \mathbf{Z})$, the total secondary impedance.

Since $\mathbf{Z}_M^2 = -\omega^2M^2$, and $\mathbf{Z}_2' = R_2' + j\omega L_2'$ (for a predominantly inductive secondary circuit), it follows that:

$$\mathbf{Z}_{e1} = (R_1 + j\omega L_1) + \left(\frac{\omega^2M^2}{R_2' + j\omega L_2'}\right) \tag{60}$$

Rationalizing equation (60) yields:

$$\mathbf{Z}_{e1} = \left[R_1 + \frac{\omega^2M^2R_2'}{R_2'^2 + \omega^2L_2'^2}\right] + j\omega\left[L_1 - \frac{\omega^2M^2L_2'}{R_2'^2 + \omega^2L_2'^2}\right] \tag{61}$$

It will be observed that R_2' appears at the primary terminals in modified form, namely, as:

$$\left[\frac{\omega^2M^2}{R_2'^2 + \omega^2L_2'^2}\right]R_2' = \frac{X_M^2}{Z_2'^2}R_2'$$

[2] It should be recognized that classical methods are applicable only where $M_{21} = M_{12}$ = a constant. Where iron-core transformers are involved, the $(N_1/N_2)^2$ factor is often used as an approximation, but since detailed analyses of iron-core transformers are usually considered in a-c machinery courses they will not be given here.

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If R_2' is very small compared with $\omega_2 L_2'$, if $L_2' = N_2\phi_2/i_2$, that is, if all of L_2' is concentrated in the secondary winding, and if $M = \sqrt{L_1 L_2'}$, then R_2' appears at the primary terminals as:

$$\left(\frac{N_1}{N_2}\right)^2 R_2' \quad \text{approximately}$$

Thus, if a high value of R_2' is to appear at the primary terminals at an apparently reduced value, N_1/N_2 must be made less than unity by the appropriate amount. The above transfer factor, $(N_1/N_2)^2$, can be theoretically approached only in the case of an ideal transformer the coefficient of coupling of which is unity. Even with unity coupling, R_2' is not actually transferred by the exact square of the turn ratio, N_1/N_2, as is sometimes supposed. In the iron-core transformer the conditions required to make $(N_1/N_2)^2$ the correct transfer factor are fulfilled to a degree which makes calculations fall well within engineering accuracy when this factor is applied. As a result, it is customary to use this factor in iron-core transformer practice.

Equation (61) reveals another interesting fact, namely, that the effective inductance at the primary terminals of a loaded transformer approaches zero only when $R_2'^2$ is negligibly small compared with $\omega^2 L_2'^2$ and when L_2' is entirely concentrated in the secondary winding. Under these conditions and if the coefficient of coupling is equal to unity,

$$\left[L_1 - \frac{\omega^2 M^2 L_2'}{\omega^2 L_2'^2}\right] = \left[L_1 - \frac{\omega^2 L_1 L_2'^2}{\omega^2 L_2'^2}\right] = 0$$

Example 10. Given an air-core (or constant-permeability) transformer, in which $N_1 = 500$ and $N_2 = 5000$. For the particular arrangement considered:

$$R_1 = 1.0 \text{ ohm} \qquad R_2 = 10 \text{ ohms}$$
$$L_1 = 0.03 \text{ henry} \qquad L_2 = 3.0 \text{ henrys}$$
$$M = 0.275 \text{ henry}$$
$$\mathbf{Z} = 90\underline{/0^\circ} \text{ ohms}$$

At 265.5 cycles per second, $\omega = 1667$ radians per second and

$$X_M = \omega M = 1667 \times 0.275 = 458.4 \text{ ohms}$$
$$X_M^2 = 458.4^2 = 210{,}000$$
$$\mathbf{Z}_2' = (10 + j5000) + (90 + j0) = 100 + j5000 \text{ ohms}$$
$$\mathbf{Z}_{e1} = (1 + j50) + \frac{210{,}000}{100 + j5000}$$
$$\mathbf{Z}_{e1} = (1 + j50) + (0.84 - j42) = 1.84 + j8.0 = 8.2\underline{/77^\circ} \text{ ohms}$$

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It will be noted that $\mathbf{Z}_2' = (100 + j5000)$ ohms appears at the primary terminals as $(0.84 - j42)$ ohms. This result emphasizes the wide discrepancy that may exist between ideal transformer operation and that actually obtained in an air-core transformer the coefficient of coupling of which is 0.917.

Under ideal conditions, the load impedance, $\mathbf{Z} = 90\angle 0°$ ohms, would appear at the primary terminals as

$$\left[\frac{N_1}{N_2}\right]^2 \times 90 = \frac{500^2}{5000^2} \times 90 = 0.90 \text{ ohm}$$

The ideal conditions referred to are: (1) perfect coupling, and (2) zero resistance in the transformer windings.

The reactive term in $\mathbf{Z}_{e1}$ may, of course, be neutralized with a series condenser in the primary circuit if a low resistive impedance at the primary circuit terminals is desired.

Problem 13. A generator which develops 10 volts (effective) at 265.5 cycles and which has an internal impedance of $2\angle 0°$ ohms is to be used to energize the 90-ohm load resistance of the above example in the two following ways:

(*a*) Directly. That is, with the generator terminals directly across the terminals of the 90-ohm load.

(*b*) Through the transformer of the above example and a primary series condenser the capacitive reactance of which is 8 ohms.

Find the power delivered to the 90-ohm load in (*a*) and in (*b*).

Ans.: (*a*) 1.063 watts; (*b*) 5.13 watts.

Primary Unity-Power-Factor Resonance. The inductive reactance of $\mathbf{Z}_{e1}$ caused by the introduction of a transformer may be neutralized in any one of several different ways. If, upon evaluation in a particular case, $\mathbf{Z}_{e1}$ possesses an inductive reactive component, suitable neutralizing capacitors may be placed in either the primary or the secondary circuits, and these capacitors may be arranged either in series or in parallel with the transformer windings. For the sake of analysis, let $\mathbf{Z}_{e1}$ be written in the form given in equation (61).

$$\mathbf{Z}_{e1} = \left[R_1 + \frac{\omega^2 M^2 R_2'}{R_2'^2 + \omega^2 L_2'^2}\right] + j\omega\left[L_1 - \frac{\omega^2 M^2 L_2'}{R_2'^2 + \omega^2 L_2'^2}\right] \tag{61}$$

R_2' is the total secondary circuit resistance. L_2' is the total secondary circuit self-inductance.

$$\mathbf{Z}_{e1} = R_{e1} + jX_{e1} \tag{62}$$

where

$$X_{e1} = \left[\omega L_1 - \frac{\omega^3 M^2 L_2'}{R_2'^2 + \omega^2 L_2'^2}\right] = \left[X_1 - \frac{X_M^2 X_2'}{R_2'^2 + X_2'^2}\right] \tag{63}$$

Series Primary Capacitor. Primary unity power factor can be obtained by introducing a capacitor in series with the primary, which has a capacitive reactance equal in magnitude to the inductive reactance

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represented in equation (63).

$$X_{C1(\text{series})} = \left[X_1 - \frac{X_M{}^2 X_2{}'}{R_2{}'^2 + X_2{}'^2}\right] \tag{64}$$

Parallel Primary Capacitor. A capacitor, placed in parallel with the primary terminals, can be used to produce primary unity power factor. It is simply necessary to make the susceptance (b_C) of the parallel capacitor equal in magnitude to the susceptance (b_L) of $\mathbf{Y}_{e1}$, where:

$$\mathbf{Y}_{e1} = \frac{1}{R_{e1} + jX_{e1}} = \frac{R_{e1}}{R_{e1}{}^2 + X_{e1}{}^2} - j\frac{X_{e1}}{R_{e1}{}^2 + X_{e1}{}^2} \tag{65}$$

The inductive susceptance of the uncompensated transformer looking into the primary terminals is given by the j component of the above equation. The capacitive susceptance of the parallel primary capacitor must, therefore, be equal to:

$$b_{C1(\text{parallel})} = \frac{X_{e1}}{R_{e1}{}^2 + X_{e1}{}^2} \tag{66}$$

Secondary Capacitors. Under the assumptions that have been made concerning equations (61), (62), and (63), $X_2{}'$ is an inductive reactance. The introduction of a capacitor in series with the secondary circuit or the introduction of a capacitor in parallel with the secondary load terminals will tend to neutralize the original inductive reactance and cause the net inductive $X_2{}'$ to be smaller in magnitude. If $R_2{}'^2$ is not too great, the lower value of $X_2{}'$ increases the magnitude of the subtractive term of equation (63), namely,

$$\left[\frac{X_M{}^2 X_2{}'}{R_2{}'^2 + X_2{}'^2}\right]$$

Provided $R_2{}'^2$ is sufficiently small in comparison with $X_2{}'^2$ to permit the required increase in the above expression, X_{e1} may be made equal to zero with the proper adjustment of the secondary capacitance. The correct value of secondary capacitance to employ in a particular case is not difficult to determine. However, the general algebraic expressions for the proper sizes of capacitors are of rather awkward algebraic form. In the circuits where this type of tuning is employed the desired effect is very often accomplished by means of a variable condenser which can be adjusted experimentally to the proper capacitance.

Adjustment of M. Assume that X_1 or $X_2{}'$ of equation (63) possesses

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a capacitive reactive component which is at least large enough to make

$$X_{e1} = \left[X_1 - \frac{X_M{}^2 X_2{}'}{R_2{}'^2 + X_2{}'^2}\right] = 0 \tag{67}$$

when the two windings are in their position of closest coupling. If now X_M is made smaller by decreasing the coefficient of coupling, X_{e1} will take on positive values, thus indicating a resulting inductive reactance. In general, the capacitive element employed would be adjusted to make X_{e1} slightly capacitive for the condition of maximum X_M. The primary current could thus be made to lead or lag the primary voltage by adjusting the degree of coupling between the two transformer windings.

Example 11. Let it be required to find the condenser of proper size to place in parallel with the primary terminals of Fig. 17*a* to produce primary unity power factor. The circuit parameters, and so forth, are given on page 233. For the case considered: $\mathbf{Z}_1 = 3.3 + j35.4$, $\mathbf{Z}_M = 0 + j9.65$, and $\mathbf{Z}_2{}' = (\mathbf{Z}_2 + \mathbf{Z}) = 15.28 + j25.27$ ohms at 60 cycles. Without the condenser:

$$\mathbf{Z}_{e1} = 4.93 + j32.7 \text{ ohms}$$

$$\mathbf{Y}_{e1} = \frac{4.93}{1094} - \frac{j32.7}{1094} = (0.0045 - j0.0299) \text{ mho}$$

Neglecting the resistance of the capacitor which is to be used:

$$b_{c1(\text{parallel})} = \frac{1}{X_{C1}} = 2\pi fC$$

$$C = \frac{0.0299}{377} = 79.3 \times 10^{-6} \text{ farad} = 79.3\ \mu\text{f}$$

Problem 14. Find the primary series capacitance to employ in the above example to produce primary unity power factor. *Ans.:* 81.1 μf.

Problem 15. Solve equation (63) for the value of $X_2{}'$ which will make $X_{e1} = 0$.

Ans.: $X_2{}' = \dfrac{X_M{}^2}{2X_1} \pm \sqrt{\dfrac{X_M{}^4}{4X_1{}^2} - R_2{}'^2}$.

Problem 16. Can a secondary series capacitance be employed in example 11 to produce primary unity power factor?

Ans.: No; $R_2{}'$ is too large for the specified values of X_1 and X_M.

Partial Resonance. In the coupled circuits of the type shown in Fig. 18, the two chief concerns are usually: (*a*) maximum value of I_2 (and of V_{C2}) for a given value of V_1; (*b*) sharply defined peak of I_2 for variable X_2, X_M, or ω.

In considering the salient features of these tuned coupled circuits, a slight modification in notation is desirable. Thus far we have dis-

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tinguished between the impedance of the primary winding ($\mathbf{Z}_1$), the impedance of the secondary winding ($\mathbf{Z}_2$), and the impedance of the load ($\mathbf{Z}$). It is plain from the development preceding equations (52) and (53), page 231, that no restrictions have been imposed on the nature of $\mathbf{Z}_1$. $\mathbf{Z}_1$ is simply the equivalent series-circuit impedance of the primary circuit. Similarly $\mathbf{Z}_2 + \mathbf{Z}$ is the equivalent series-circuit impedance of the secondary circuit. The equations in the remainder of this chapter will be simpler to write and easier to grasp if $\mathbf{Z}_1$ is under-

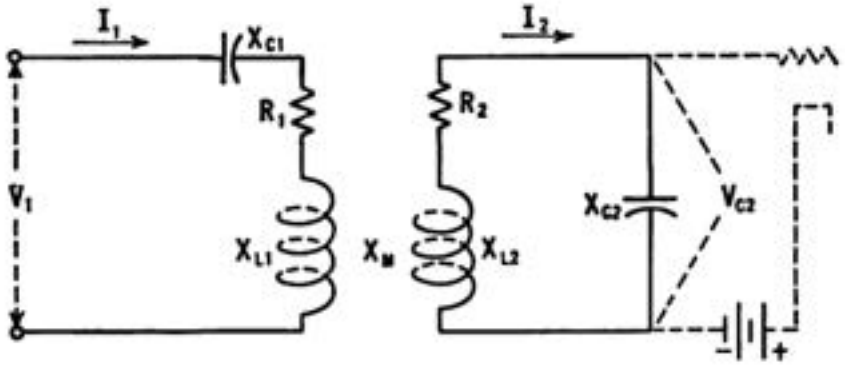

FIG. 18. A double-tuned circuit arrangement.

stood to be the total series impedance of the primary circuit and if $\mathbf{Z}_2$ is understood to be the total series impedance of the secondary circuit. Thus:

$$\mathbf{Z}_1 = R_1 + j(X_{L1} - X_{C1}) = R_1 + jX_1 \tag{68}$$

$$\mathbf{Z}_2 = R_2 + j(X_{L2} - X_{C2}) = R_2 + jX_2 \tag{69}$$

$$\mathbf{Z}_M = jX_M = j\omega M \quad \text{(as before)} \tag{70}$$

The equation for the secondary current $\mathbf{I}_2$ [as given in equation (53), page 231] becomes:

$$\mathbf{I}_2 = \frac{-\mathbf{V}_1\mathbf{Z}_M}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2} = \frac{-\mathbf{V}_1(jX_M)}{(R_1 + jX_1)(R_2 + jX_2) + X_M^2} \tag{71}$$

or

$$\mathbf{I}_2 = \frac{-\mathbf{V}_1X_M[(X_1R_2 + X_2R_1) + j(R_1R_2 - X_1X_2 + X_M^2)]}{(X_1R_2 + X_2R_1)^2 + (R_1R_2 - X_1X_2 + X_M^2)^2} \tag{72}$$

For simplicity in writing, let

$$a = X_1R_2 + X_2R_1 \quad \text{and} \quad b = R_1R_2 - X_1X_2 + X_M^2$$

Then:

$$\mathbf{I}_2 = \frac{-\mathbf{V}_1X_M(a + jb)}{a^2 + b^2} \tag{73}$$

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The magnitude of $\mathbf{I}_2$ is:

$$I_2 = V_1 X_M \sqrt{\frac{a^2 + b^2}{(a^2 + b^2)^2}} = \frac{V_1 X_M}{\sqrt{a^2 + b^2}} \tag{74}$$

or

$$I_2 = \frac{V_1 X_M}{\sqrt{X_1^2 R_2^2 + X_2^2 R_1^2 + R_1^2 R_2^2 + 2R_1 R_2 X_M^2 + X_1^2 X_2^2 - 2X_1 X_2 X_M^2 + X_M^4}} \tag{75}$$

In solving for I_2, where numerical values are involved, it is often more convenient to use equation (71) than equation (75). This is particularly true where X_1 or X_2 is equal to zero. Equation (75), however, is useful in determining maximum values of I_2 that can be obtained by varying any one of the parameters.

Partial resonance in coupled circuits is obtained when any one parameter is so varied as to cause maximum effective secondary current, I_2, under the condition of constant applied voltage, V_1.

From equation (75) it is evident that partial resonance can be obtained by adjusting any one of the five parameters: R_1, R_2, X_1, X_2, or X_M. (For fixed values of R_1, L_1, C_1, M, R_2, L_2, and C_2, partial resonance may be obtained by adjustment of the frequency.) Partial resonance will obviously be produced by adjusting any parameter which appears only in the positive terms of the denominator of equation (75) to zero. Hence partial resonance obtains, theoretically, when either R_1 or R_2 is equal to zero. Practically, neither R_1 nor R_2 can be zero and, as will be shown presently, the value of $R_1 R_2$ determines the optimum value of I_2 that can be obtained.

The values of X_1, X_2, or X_M which will produce partial resonance may, in general, be found by differentiating the expression for I_2 [as given in equation (75)] with respect to the proper X and equating dI_2/dX equal to zero. For example, the value of X_1 which will produce partial resonance may be determined by equating dI_2/dX_1 equal to zero and solving for X_1 in terms of the other parameters. Thus:

$$\frac{dI_2}{dX_1} = 0 = -V_1 X_M \tfrac{1}{2}[2X_1(R_2^2 + X_2^2) - 2X_2 X_M^2] \tag{76}$$

The only useful relationship which can be derived from the above is:

$$X_1(R_2^2 + X_2^2) = X_2 X_M^2 \tag{77}$$

The value of X_1 which will produce partial resonance is, therefore:

$$X_{1(\text{res})} = \frac{X_2 X_M^2}{R_2^2 + X_2^2} = \frac{X_2 X_M^2}{Z_2^2} \tag{78}$$

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Reference to equation (63), page 239, will show that the above value of X_1 is also the unity-power-factor-resonance value of X_1. In making this comparison it should be recognized that R_2 and X_2 of equation (78) mean the same as R_2' and X_2' of equation (63) because of the shift in notation which was made at the beginning of this section. In a similar manner, it may be shown that the value of X_2 for partial resonance is:

$$X_{2(\text{res})} = \frac{X_1 X_M^2}{R_1^2 + X_1^2} = \frac{X_1 X_M^2}{Z_1^2} \tag{79}$$

The interpretation of the above equation is that X_2 must have the value stated to produce maximum I_2. If $X_1 = 0$, then X_2 should be tuned to zero to produce maximum I_2 for a fixed value of X_M. If the primary circuit is not tuned to $X_{L1} - X_{C1} = 0$, then the secondary must be detuned to the value $X_1 X_M^2/Z_1^2$. Where sharpness of secondary tuning is of more importance than an optimum value of I_2, the primary is often purposely detuned to effect a pronounced peak in the I_2 versus X_{C2} graph. (See Problem 17, page 247.)

If X_1 and X_2 are both equal to zero (by virtue of $X_{L1} - X_{C1} = 0$ and $X_{L2} - X_{C2} = 0$), equation (75) reduces to

$$I_{2(\max)} = \frac{V_1 X_M}{R_1 R_2 + X_M^2} \tag{80}$$

If, now, X_M is varied by changing the coefficient of coupling between the coils, the optimum value of I_2 is obtained when

$$\frac{dI_{2(\max)}}{dX_M} = \frac{V_1(R_1 R_2 + X_M^2) - 2V_1 X_M^2}{(R_1 R_2 + X_M^2)^2} = 0 \tag{81}$$

or when

$$X_M = \omega M = \pm\sqrt{R_1 R_2} \quad \text{(called critical coupling)} \tag{82}$$

Under these conditions:

$$I_{2(\text{opt})} = \frac{V_1\sqrt{R_1 R_2}}{R_1 R_2 + R_1 R_2} = \frac{V_1}{2\sqrt{R_1 R_2}} \tag{83}$$

The relationships stated in equations (78), (79), (82), and (83) are of considerable importance in voltage amplification in radio circuits. Some of the essential features involved are illustrated numerically in the following examples and in graphical form in Figs. 19 and 20. For fixed values of the other parameters, there is a value of X_M or a coefficient of coupling which will produce maximum I_2 as shown in the graphs of Fig. 19. Frequency responses of coupled circuits for fixed values of R_1, L_1, C_1, M, R_2, L_2, and C_2 are shown in Fig. 20. Graphs of I_2 and V_{C2} versus X_{C2} are reserved for student exercises.

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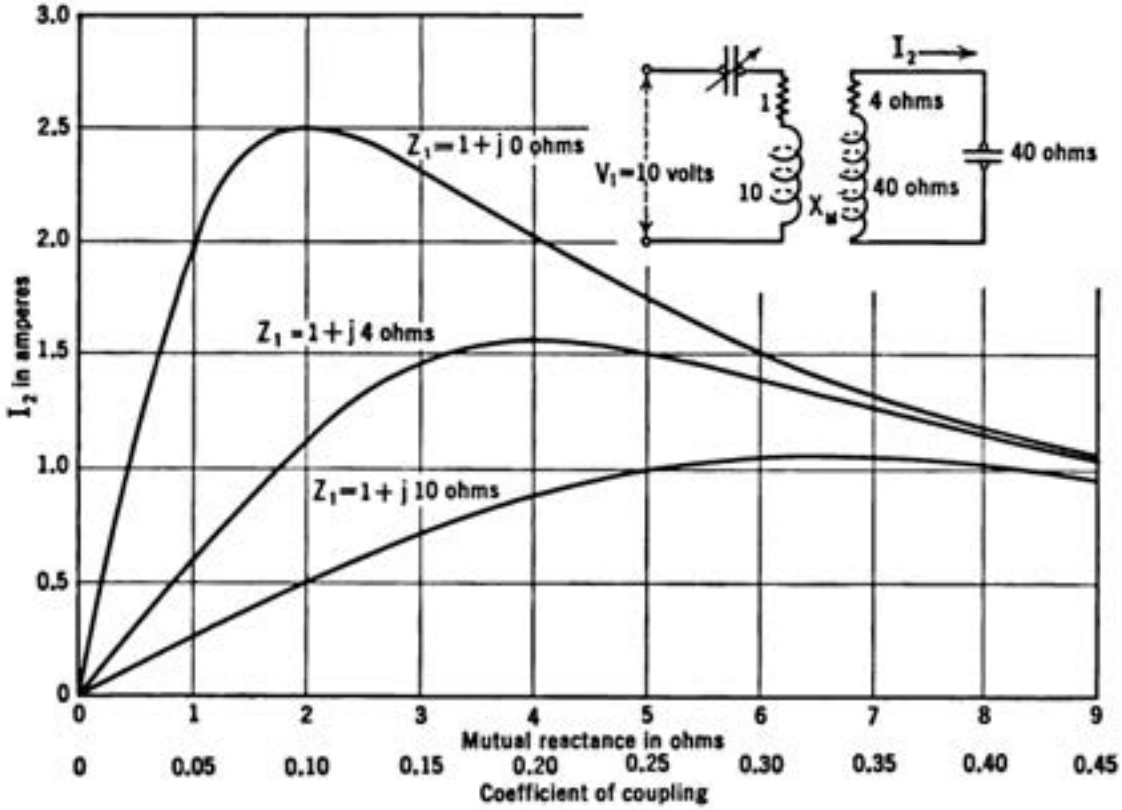

FIG. 19. Variation of secondary current with coefficient of coupling for different values of primary impedance. See example 12.

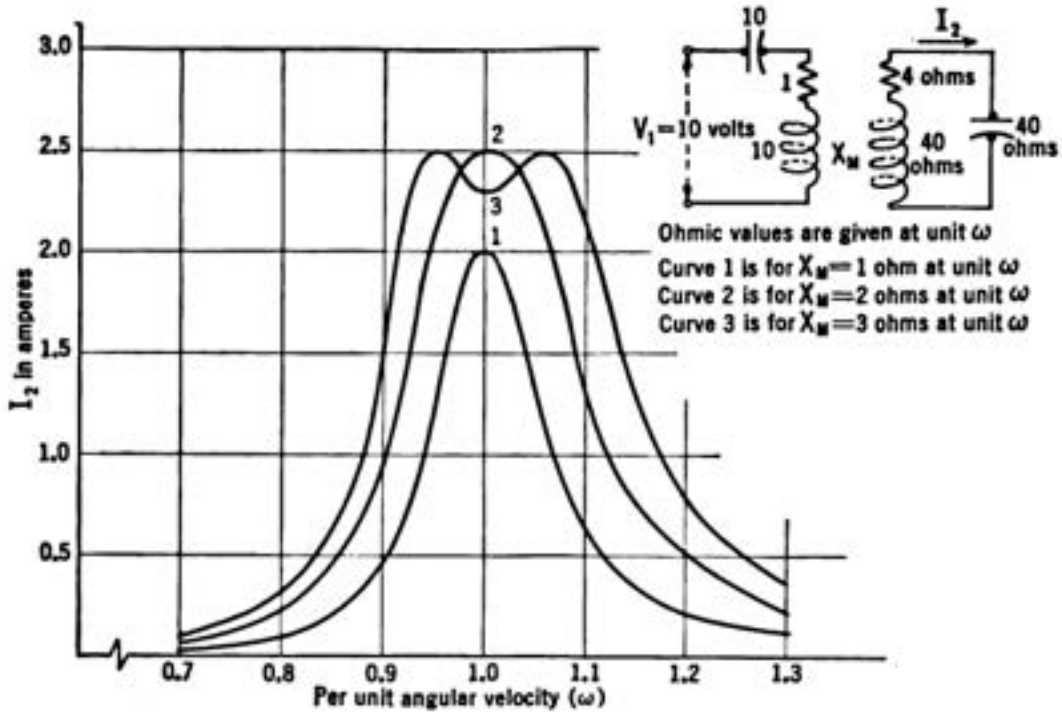

FIG. 20. Frequency responses of double-tuned circuits.

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Example 12. (*a*) Consider the coupled circuits shown in Fig. 19 under the following conditions:

$$\mathbf{Z}_1 = 1 + j10 \text{ ohms} \quad \mathbf{Z}_2 = 4 + j(40 - 40) \text{ ohms} \quad X_M \text{ variable}$$

In this case the primary is not tuned and the secondary is tuned, that is, $X_{C2} = X_{L2} = 40$ ohms at the frequency of the impressed voltage, V_1.

Solutions of equation (71) for $V_1 = 10$ volts and for various values of X_M will show the manner in which I_2 varies with the degree of coupling between the coils. The results of a series of such calculations are shown in the lower curve of Fig. 19. It will be observed that, for $\mathbf{Z}_1 = 1 + j10$ ohms, I_2 attains a maximum value at X_M equal to 6.5 ohms or at a coefficient of coupling of 0.325. Closer or looser coupling than 0.325 results in lesser values of I_2 and hence of $V_{C2} = I_2 X_{C2}$.

Calculations will show that in this case

$$V_{C2(\max)} = 1.063 \times 40 = 42.52 \text{ volts}$$

(*b*) The response of I_2 to variable X_M when the primary is partially tuned is shown in the middle graph of Fig. 19. In this case, 6 ohms of capacitive reactance is employed in the primary circuit and

$$\mathbf{Z}_1 = 1 + j4 \text{ ohms} \quad \mathbf{Z}_2 = 4 + j0 \text{ ohms} \quad X_M \text{ variable}$$

I_2 attains a maximum value at $X_M = 4.3$ ohms of 1.565 amperes. The maximum value of the secondary condenser voltage is:

$$V_{C2(\max)} = 1.565 \times 40 = 62.6 \text{ volts}$$

(*c*) The upper graph of Fig. 19 shows the response of I_2 to a variable X_M when both primary and secondary are tuned.

$$\mathbf{Z}_1 = 1 + j0 \text{ ohms} \quad \mathbf{Z}_2 = 4 + j0 \text{ ohms}, \quad X_M \text{ variable}$$

In accordance with equations (82) and (83), I_2 attains its optimum value of $V_1/2\sqrt{R_1R_2}$ at $X_M = \sqrt{R_1R_2}$.

$$I_{2(\text{opt})} = \frac{V_1}{2\sqrt{R_1R_2}} = \frac{10}{2 \times 2} = 2.5 \text{ amperes}$$

$$V_{C2(\text{opt})} = I_{2(\text{opt})} X_{C2} = 2.5 \times 40 = 100 \text{ volts}$$

The Q (or $\omega L/R$) of the coils in this case is equal to 10, and it will be observed that $V_{C2(\text{opt})}$ is equal to the driving voltage (10 volts) times the Q of the coils. That is, $V_{C2(\text{opt})} = V_1 Q = 10 \times 10 = 100$ volts. This fact is generally true where $X_{L2} = 4X_{L1}$, provided that both primary and secondary circuits are tuned to resonance and provided that the coupling reactance is adjusted to its critical value, namely, $\sqrt{R_1R_2}$. Under these conditions,

$$Q = \frac{X_{L1}}{R_1} = \frac{X_{L2}}{R_2} \quad \text{and} \quad R_1R_2 = \frac{X_{L1}X_{L2}}{Q^2}$$

$$I_{2(\text{opt})} = \frac{V_1}{2\sqrt{R_1R_2}} = \frac{V_1 Q}{\sqrt{4X_{L1}X_{L2}}} = \frac{V_1 Q}{X_{L2}}$$

$$V_{C2(\text{opt})} = I_{2(\text{opt})} X_{C2} = I_{2(\text{opt})} X_{L2} = V_1 Q$$

Thus it will be seen that the voltage developed across the secondary condenser of

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the coupled circuits shown in Fig. 18 may be equal to Q times the applied voltage. If, for example, the Q of the coils is 50, a voltage amplification of 50 can be obtained simply with the aid of the tuned coupled circuits. As indicated in Fig. 18, the voltage developed across the secondary condenser may be applied between the control grid and cathode of a vacuum tube in order to obtain further voltage amplification.

Example 13. The response of a coupled circuit to a constant driving voltage of variable frequency is shown in Fig. 20 for three different values of X_M. Since the critical coupling at unit angular velocity is 2 ohms, the graphs shown in Fig. 20 represent couplings which are less than, equal to, and greater than critical coupling.

In these graphs, unit angular velocity is called the angular velocity at which $X_{L1} - X_{C1} = 0$ and at which $X_{L2} - X_{C2} = 0$. At unit angular velocity,

$$\mathbf{Z}_1 = 1 + j(10 - 10), \quad \mathbf{Z}_2 = 4 + j(40 - 40) \quad X_M = 1, 2, \text{ or } 3 \text{ ohms}$$

At other values of ω, the X_L's and X_M vary directly as ω, and the X_C's vary inversely as ω.

For coupling less than critical coupling the maximum value of the secondary current is less than for critical coupling, and for couplings greater than critical coupling the current response is generally similar to the double-peaked curve shown in Fig. 20.

If a single pronounced peak of I_2 versus ω is desired, the coupling should not be greater than critical coupling, and the Q of the coils should be as high as practicable. If the Q of the coils is made higher than that used in Fig. 20, the peaks of the curves will be sharper and more clearly defined. Sharpness of tuning is particularly important in radio receiver circuits.

Problem 17. In the coupled circuits shown in Fig. 18, page 242:

$R_1 = 1.0$ ohm	$R_2 = 4.0$ ohms
$X_{L1} = 10$ ohms	$X_{L2} = 40$ ohms
$X_{C1} = 10$ ohms	X_{C2} is variable
$X_M = 2$ ohms	$V_1 = 10$ volts

Graph I_2 and V_{C2} versus X_{C2} between the limits of $X_{C2} = 20$ ohms and $X_{C2} = 60$ ohms.

Ans.: $I_{2(\max)} = 2.5$ amperes at $X_{C2} = 40$ ohms.
$V_{C2(\max)} = 102$ volts at $X_{C2} = 41.7$ ohms, approximately.

Note: The fact that circuits of this kind tune more sharply but to lesser peak values when one member is partially detuned may be shown by repeating the above problem using $\mathbf{Z}_1 = 1 + j4$ ohms rather than $\mathbf{Z}_1 = 1 + j0$.

Double-Tuned Circuit Analysis and Design in Terms of $f/f_0 - f_0/f$. The double-tuned circuit shown in Fig. 21*a* is widely used in radio engineering practice, and it is the purpose of this section to derive design equations which will specify the Q's of the circuits and the coefficient of coupling in terms of the band width and the degree of irregularity which can be tolerated in the response characteristic. The current

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generator ($g_m E_g$) in parallel with R_p is the plate circuit representation of a vacuum tube. $\left(di_b = \frac{\partial i_b}{\partial e_g} de_g + \frac{\partial i_b}{\partial e_b} de_b \text{ or } i_p = g_m e_g + \frac{e_p}{R_p}\cdot\right)$ See page 144.

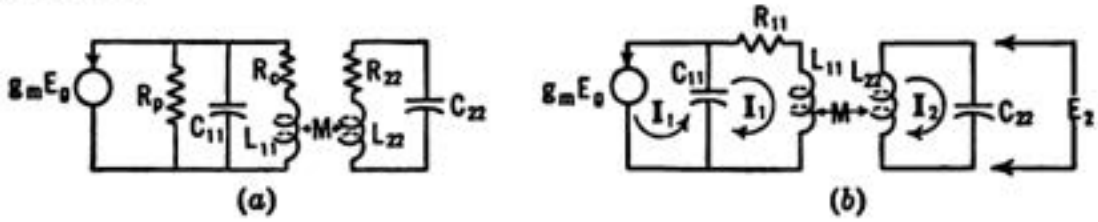

FIG. 21. The actual double-tuned circuit shown in (a) transforms readily to that shown in (b).

Wherever inductive and capacitive reactances are combined as shown in Fig. 22 the analysis is simplified considerably by letting

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = \frac{f}{f_0} - \frac{f_0}{f} = F \tag{84}$$

where $\omega_0 = 1/\sqrt{L_{11}C_{11}} = 1/\sqrt{L_{22}C_{22}}$ under the assumption that the primary and secondary circuits will be tuned to the same frequency.

It will be noted that F as defined above is the difference between two dimensionless quantities (f/f_0 and f_0/f) which individually characterize the variations of inductive and capacitive reactances relative to variations in frequency.

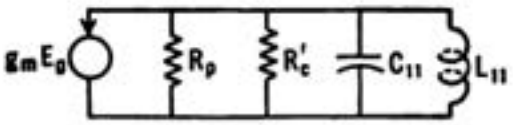

FIG. 22.

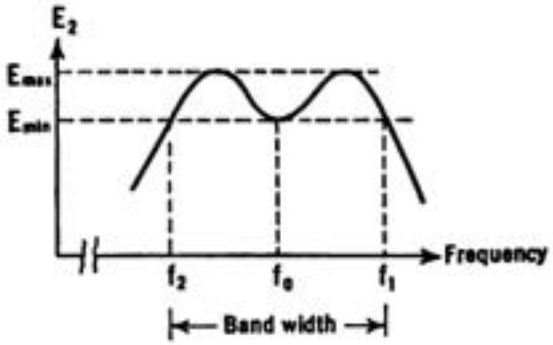

FIG. 23. Response curve of double-tuned circuit. $f_0 = \sqrt{f_1 f_2}$ is the center frequency.

As shown in Fig. 23, $f_1 - f_2$ will be called the band width and it will be assumed that $f_1 - f_2$ is small compared with f_0. For narrow-band responses of this kind, E_2 has a value of $E_{\min}$ within the pass band at

$$f_0 \approx \sqrt{f_1 f_2}$$

where f_1 and f_2 are the frequencies (other than f_0) at which the response, E_2, has values of $E_{\min}$. See Fig. 23. In this connection it will be

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noted that, if F_{min} symbolizes the value of F where $E_2 = E_{min}$, say at $f = f_1$, then

$$F_{min} = \frac{f_1}{f_0} - \frac{f_0}{f_1} \approx \frac{f_1}{f_0} - \frac{f_2}{f_0} \tag{85}$$

if $f_1 - f_2$ is small compared to f_0.

If we let $a = 1/Q_1$, $b = 1/Q_2$, and $k = M/\sqrt{L_{11}L_{22}}$:

$$\mathbf{Z}_{11}(\text{in Fig. 21}b) = R_{11} + j\left(\omega L_{11} - \frac{1}{\omega C_{11}}\right) = \omega_{01}L_{11}(a + jF_{11}) \tag{86}$$

$$\mathbf{Z}_{22} = \omega_{02}L_{22}(b + jF_{22}) \tag{87}$$

$$\mathbf{Z}_{12} = \mathbf{Z}_{21} = j\omega M = j\omega k\sqrt{L_{11}L_{22}} \tag{88}$$

Since $L_{11} = L_{22}$ in this case, we may write

$$\omega_{01} = \frac{1}{\sqrt{L_{11}C_{11}}} = \omega_{02} = \frac{1}{\sqrt{L_{22}C_{22}}} = \omega_0$$

where

$$\omega_0^2 = \frac{1}{\sqrt{L_{11}L_{22}C_{11}C_{22}}}$$

and

$$F_{11} = F_{22} = F$$

Employing the loop current method of analysis in Fig. 21b and treating g_mE_g as a known value of current, say I_t, circulating in the left-hand loop, we have

$$\left.\begin{aligned} \mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2 &= j\frac{g_mE_g}{\omega C_{11}} = j\frac{I_t}{\omega C_{11}} \\ \mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 &= 0 \end{aligned}\right\} \tag{89}$$

The output voltage is

$$\mathbf{E}_2 = -j\frac{1}{\omega C_{22}}\mathbf{I}_2 = \frac{\left(-j\dfrac{1}{\omega C_{22}}\right)\left(-j\dfrac{I_t}{\omega C_{11}}\right)(j\omega k\sqrt{L_{11}L_{22}})}{\omega_0^2L_{11}L_{22}[(ab - F^2) + j(a + b)F] + \omega^2k^2L_{11}L_{22}} \tag{90}$$

$$\mathbf{E}_2 = \frac{\dfrac{-jI_tk\sqrt{L_{11}L_{22}}}{\omega C_{11}C_{22}}}{\omega_0^2L_{11}L_{22}\left[\left(ab + \dfrac{\omega^2}{\omega_0^2}k^2 - F^2\right) + j(a + b)F\right]} \tag{91}$$

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Since we are interested particularly in the region shown in Fig. 23 where any ω is close to ω_0 if the per unit band width is small, we may set $\omega^2/\omega_0^2 \approx 1$ in equation (91) and obtain

$$\mathbf{E}_2 = \frac{-jI_t k}{\omega\sqrt{C_{11}C_{22}}[(k^2 + ab - F^2) + j(a + b)F]} \tag{92}$$

At $\omega = \omega_0$, the center angular frequency $F = 0$ and

$$\mathbf{E}_{20} = \mathbf{E}_0 = \frac{-jI_t k}{\omega_0\sqrt{C_{11}C_{22}}\,(k^2 + ab)} \tag{93}$$

Consider now the ratio of the magnitudes of E_2 and E_0 and let the ratio ω/ω_0 again be reckoned as unity. Under these conditions

$$\left(\frac{E_2}{E_0}\right)^2 = \frac{1}{1 + \dfrac{F^4 + (a^2 + b^2 - 2k^2)F^2}{(k^2 + ab)^2}} \tag{94}$$

or

$$\frac{E_2}{E_0} = \frac{1}{\sqrt{1 + \dfrac{F^4 + (a^2 + b^2 - 2k^2)F^2}{(k^2 + ab)^2}}} \tag{95}$$

From equation (95) it is plain that the shape of the E_2 curve (reckoned in per unit values relative to E_0) will be determined by the relative magnitudes of $a^2 + b^2$ and $2k^2$. If $a^2 + b^2 \gtreqless 2k^2$, then a single-peaked curve is obtained since, as F takes on values greater than 0 (f different from f_0), the E_2/E_0 curve will decrease continuously from its maximum value of unity, the value of E_2/E_0 when $F = 0$ or when $f = f_0$.

If, however, $a^2 + b^2 < 2k^2$, the denominator of equation (95) takes on a minimum value of E_2/E_0 takes on a maximum value where $f(F^2) = F^4 + (a^2 + b^2 - 2k^2)F^2$ is a minimum. This minimum may be found from

$$\frac{df(F^2)}{d(F^2)} = 2(F^2) - (2k^2 - a^2 - b^2) = 0$$

or where

$$F^2 = F_{\max}^2 = \frac{2k^2 - a^2 - b^2}{2} \tag{96}$$

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When plotted versus actual frequency, the response takes the form shown in Fig. 23 or, when plotted versus F, the form shown in Fig. 24.

We may write an expression for $(E_2/E_0)_{max} = E_{max}$ from equations (95) and (96), and, since E_{min} is taken as unity, we may write

$$\frac{E_{max}^2}{E_{min}^2} = \frac{1}{1 - \dfrac{(2k^2 - a^2 - b^2)^2}{4(k^2 + ab)^2}} \tag{97}$$

Let

$$\alpha^2 = 1 - \frac{E_{min}^2}{E_{max}^2} = \frac{(2k^2 - a^2 - b^2)^2}{4(k^2 + ab)^2} = \frac{F_{min}^4}{4(k^2 + ab)^2}, \tag{98}$$

where $F_{min}^2 = (2k^2 - a^2 - b^2)$. [See Fig. 24 and equation (95.)] It follows that

$$\alpha = \frac{F_{min}^2}{2(k^2 + ab)} \tag{99}$$

and

$$\frac{E_2}{E_0} = \frac{1}{\sqrt{1 + \dfrac{4\alpha^2 F^2 (F^2 - F_{min}^2)}{F_{min}^4}}} \tag{100}$$ [3]

$F_{min} = \sqrt{2k^2 - a^2 - b^2} = (f_1 - f_2)/f_0$ is the value of F at the edge of the pass band where $E_2/E_0 = 1 = E_{min}$.

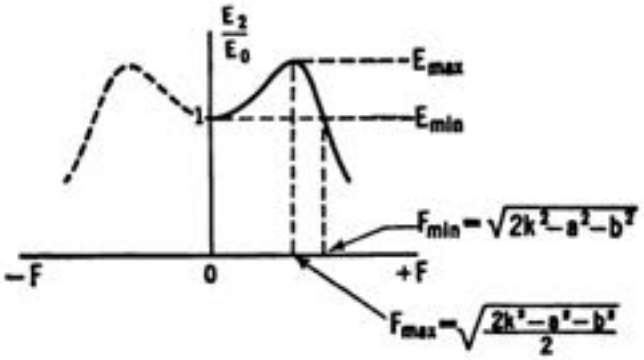

FIG. 24. A response curve, E_2/E_0 versus the variable F for $a^2 + b^2 < 2k^2$. $(F = f/f_0 - f_0/f.)$

Equation (100) is a convenient working equation since it includes α, a measure of the response irregularity which can be tolerated within the pass band $(f_1 - f_2)$, and F_{min}, a measure of the pass band width $(f_1 - f_2)$. From a design point of view, α and F_{min} would normally be specified (at least indirectly), and k, a, and b would then be so chosen

[3] These results are due to Dr. T. C. G. Wagner of the University of Maryland who has developed design formulas for double-, triple-, and quadruple-tuned circuits.

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that the specified values of α and $f_1 - f_2$ would be obtained in the final design. See Problem 43, page 261, and example 14 for applications.

Example 14. Let it be required to design a double-tuned circuit which will have a per unit band width $[(f_1 - f_2)/f_0]$ of 0.05 and a ratio of E_{max} to E_{min} equal to 1.25. If we make $a = b$ $(Q_1 = Q_2)$, we may readily show that:

$$a^2 = b^2 = \frac{F_{min}^2(1-\alpha)}{4\alpha} \quad \text{and} \quad k^2 = \frac{F_{min}^2(1+\alpha)}{4\alpha}$$

since $\alpha = F_{min}^2/2(k^2 + ab)$ and $F_{min}^2 = 2k^2 - a^2 - b^2$. In the particular case under discussion

$$F_{min}^2 = \left(\frac{f_1 - f_2}{f_0}\right)^2 = 0.05^2 = 0.0025 \text{ [see equation (85)]}$$

and

$$\alpha^2 = 1 - \frac{E_{min}^2}{E_{max}^2} = 1 - \frac{16}{25} \quad \text{or} \quad \alpha = 0.6$$

Thus

$$a^2 = b^2 = \frac{0.0025(0.4)}{2.4} = 0.000417 \quad \text{and} \quad Q_1 = Q_2 = 49$$

$$k^2 = \frac{0.0025(1.6)}{2.4} = 0.00167 \quad \text{and} \quad k = 0.041$$

Component Fluxes and Voltages in the Air-Core Transformer. Figure 25*a* shows diagrammatically the flux components in an air-core transformer. The current I_2 in the secondary produces an mmf which may be considered to cause two component fluxes: one the leakage flux ϕ_{22}, which links the turns of winding 2 only, and ϕ_{21}, which links both windings 2 and 1. The same conditions regarding the flux linkages as explained on page 217 for Fig. 6 apply to the present discussion, namely, that ϕ_{22} is a hypothetical component which, when linking all the turns of winding 2, produces the same total flux linkages as obtained from the true flux linkages in question. The current I_1 causes two component fluxes, ϕ_{12}, which links both windings, and ϕ_{11}, which links winding 1 only. Reference to example 9 on page 233 and application of Lenz's law will reveal in a general way the reason for the phase angle shown between $\mathbf{I}_1$ and $\mathbf{I}_2$ in the vector diagram (Fig. 25*b*). The component fluxes produced by $\mathbf{I}_1$ and $\mathbf{I}_2$ are also shown. It is plain from Fig. 25*a* that the resultant mutual flux is $\phi_M = \phi_{12} + \phi_{21}$. The total flux through winding 2 is $\phi_{2R} = \phi_M + \phi_{22} = \phi_2 + \phi_{12}$. Also the total flux through winding 1 is $\phi_{1R} = \phi_M + \phi_{11} = \phi_1 + \phi_{21}$. All these combinations are shown on the vector diagram. Equal numbers of turns on windings 1 and 2 are assumed.

Since $e = -N(d\phi/dt)$, the induced voltage due to a flux lags the flux by 90 degrees. Thus, on the vector diagram, $\mathbf{E}_{2R}$ is caused by ϕ_{2R}, $\mathbf{E}_{M2}$

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by ϕ_M, and $\mathbf{E}_{22}$ by ϕ_{22}. The resultant induced emf in winding 2 is therefore $\mathbf{E}_{2R}$. Because of the resistance R_2 of winding 2 the terminal voltage must be less than $\mathbf{E}_{2R}$ by the $\mathbf{I}_2R_2$ drop as shown. Hence $\mathbf{V}_2$ is the secondary terminal voltage. It is seen to be ahead of $\mathbf{I}_2$ by the secondary load power-factor angle.

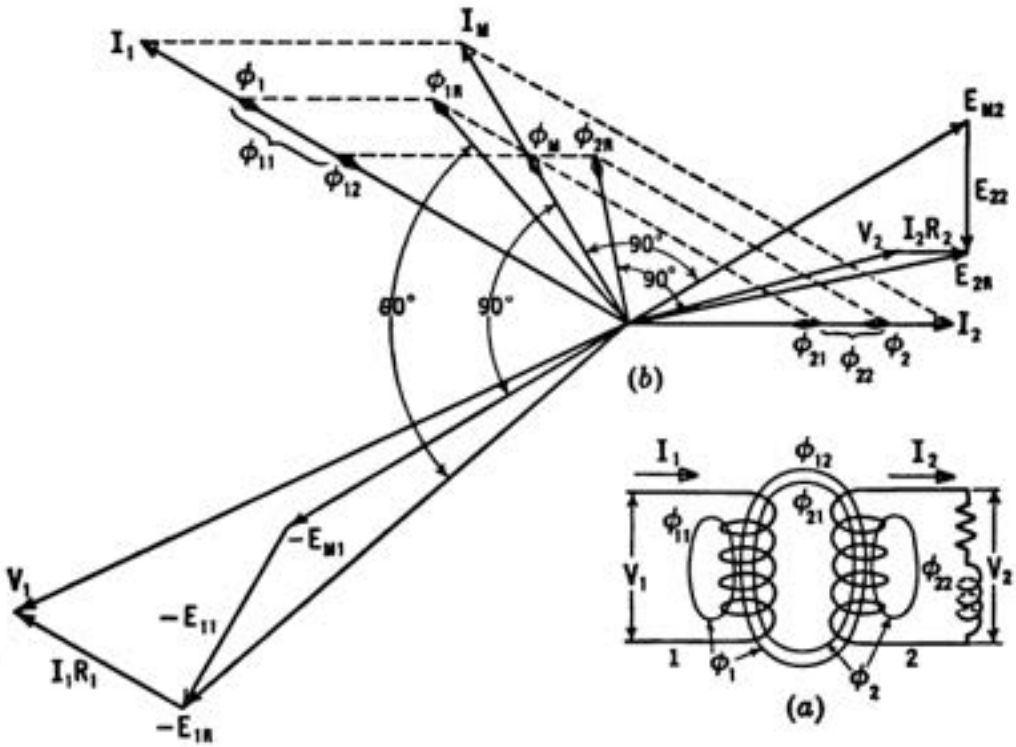

FIG. 25. Vector diagram of the air-core transformer.

The voltage drop impressed on winding 1 must be equal to the sum of all the drops through winding 1. Thus one component of the total drop must be the drop $-\mathbf{E}_{1R}$, which is equal and opposite to the induced voltage $\mathbf{E}_{1R}$ (not shown) in winding 1 caused by all the flux linking that winding. The remaining component drop is the $\mathbf{I}_1R_1$. Hence $\mathbf{V}_1 = \mathbf{I}_1R_1 + (-\mathbf{E}_{1R})$. The components of $-\mathbf{E}_{1R}$ are the voltage drops $-\mathbf{E}_{11}$ and $-\mathbf{E}_{M1}$, which overcome the induced voltages due to the primary leakage and mutual fluxes, respectively.

The leakage flux ϕ_{22} is (even for all practical purposes in iron-core transformers) proportional to the current $\mathbf{I}_2$. $\mathbf{E}_{22}$ is an induced voltage rise and is directly proportional to $\mathbf{I}_2$. The voltage $-\mathbf{E}_{22}$ is opposite to $\mathbf{E}_{22}$ and therefore leads the current by 90 degrees. It is thus in the direction of a reactance drop, and, since it is proportional to the current, a constant reactance may be multiplied by the current $\mathbf{I}_2$ to represent correctly the drop $-\mathbf{E}_{22}$. Such a reactance which may be used to replace the effect of the leakage flux is called a leakage reactance, and the

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corresponding drop a leakage reactance drop. The vector diagram which is commonly used is shown in Fig. 26. Only the flux ϕ_M in Fig. 25 is shown, and the drops $-\mathbf{E}_{22}$ and $-\mathbf{E}_{11}$ are replaced by their corresponding leakage reactance drops $\mathbf{I}_2\mathbf{X}_2$ and $\mathbf{I}_1\mathbf{X}_1$, respectively.

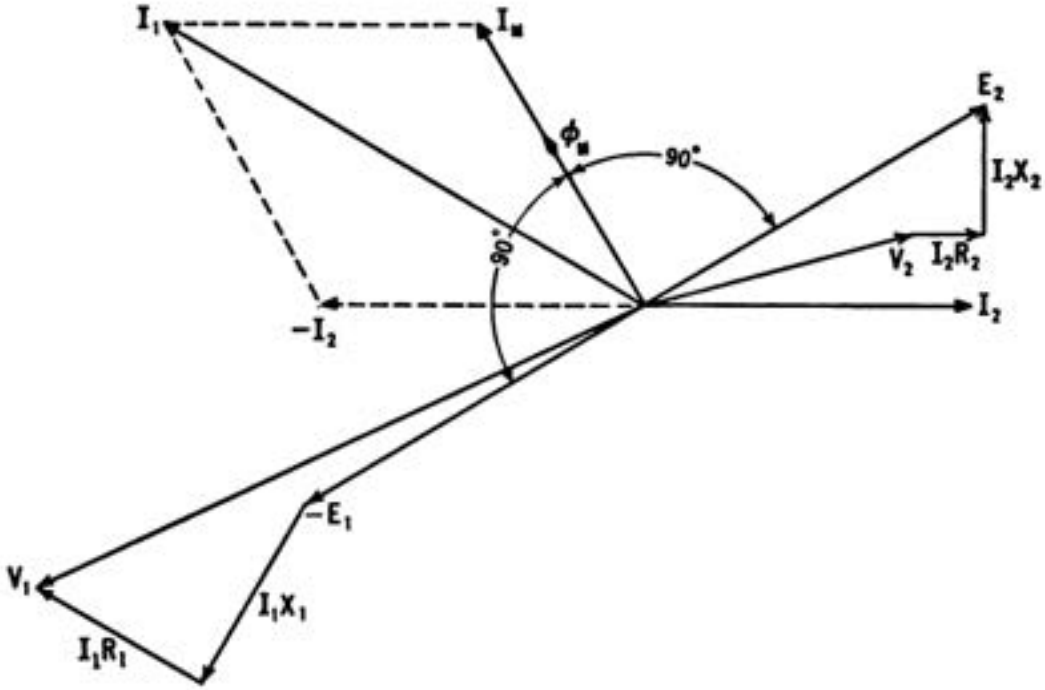

FIG. 26. Commonly used vector diagram for the air-core transformer shown in Fig. 25.

Leakage Reactance. Leakage reactance may be defined as $2\pi f$ times the leakage inductance. This may be shown as follows. By referring to Fig. 25*a*, leakage inductance

$$L_{S2} = \frac{N_2\phi_{22}}{I_2} \quad \text{or} \quad N_2\frac{d\phi_{22}}{di_2} \tag{101}$$

$$e_{22} = -N_2\frac{d\phi_{22}}{dt} \tag{102}$$

Dividing equation (102) by equation (101) gives

$$e_{22} = -L_{S2}\frac{di_2}{dt}$$

For sine waves

$$i_2 = I_{m2}\sin\omega t \tag{103}$$

and

$$e_{22} = -L_{S2}I_{m2}\omega\cos\omega t \tag{104}$$

Hence

$$E_{m22} = I_{m2}\omega L_{S2}$$

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Also

$$E_{22} = \frac{I_{m2}}{\sqrt{2}}\omega L_{S2} = I_2\omega L_{S2}$$

The magnitude of the leakage reactance drop has been defined equal to $E_{22} = I_2\omega L_{S2} = I_2X_2$. Therefore

$$X_2 = \omega L_{S2} \tag{105}$$

Since e_{22} in equation (104) is a voltage rise, the drop is $-e_{22} = L_{S2}\omega I_{m2} \cos \omega t$. Because this voltage drop is 90 degrees ahead of the current (equation 103), the complex expression for leakage reactance must be

$$X_2 = +j\omega L_{S2} \tag{106}$$

The Air-Core Autotransformer. Two inductance coils arranged as shown in Fig. 27 are called an autotransformer. If the driving voltage is applied to the terminals *ab* and the load connected across the terminals *ac*, the autotransformer functions as a step-up voltage device; whereas, if the driving voltage is applied to the terminals *ac* and the load connected to terminals *ab* or *bc*, it functions as a step-down voltage device. The mathematical analysis of the air-core autotransformer is reserved for student exercises. (See Problems 35, 36, and 37 at the end of this chapter.)

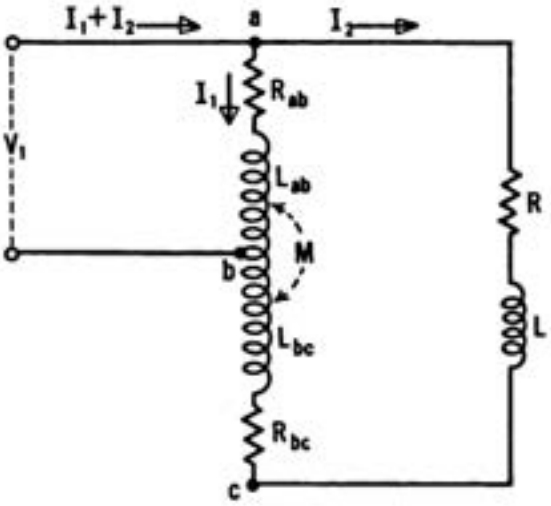

FIG. 27. Air-core autotransformer connected as a step-up voltage device.

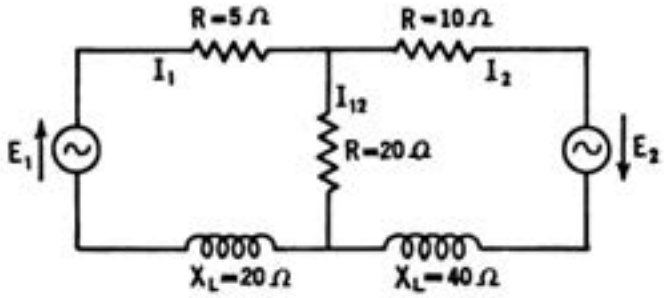

FIG. 28. See Problem 18.

PROBLEMS

18. In Fig. 28, $\mathbf{E}_1 = 100\angle 0°$ volts and $\mathbf{E}_2 = 100\angle +120°$ volts. The physical meaning of the foregoing statement is that the E_2 generator develops a maximum generated emf ($\sqrt{2} \times 100$ volts) in its arrow direction $\frac{1}{3}$ of a cycle or 120° before

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the E_1 generator develops its maximum generated emf in its arrow direction. Assuming that the resistances and reactances given in Fig. 28 include the generator impedances, find $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_{12}$.

19. In Fig. 2, page 214, it is found experimentally that $\mathbf{I}_1 = 1\underline{/90^\circ}$ ampere and $\mathbf{V}_{22'} = 4\underline{/0^\circ}$ volts (with terminals 22′ open-circuited) when $\mathbf{E}_1$ (the voltage applied to terminals 11′) is $6\underline{/0^\circ}$ volts. When a voltage of $6\underline{/0^\circ}$ volts is applied to terminals 22′ (with terminals 11′ open-circuited), $\mathbf{I}_2 = 1.5\underline{/90^\circ}$ amperes and $\mathbf{V}_{11'} = 6\underline{/0^\circ}$ volts.

(*a*) Find $\mathbf{Z}_{21}$ and $\mathbf{Z}_{12}$ from the above data.

(*b*) Find the coefficient of coupling between the two circuits.

(*c*) Draw a circuit configuration that might actually exist within the 11′2′2 box and that is consistent with the specified data.

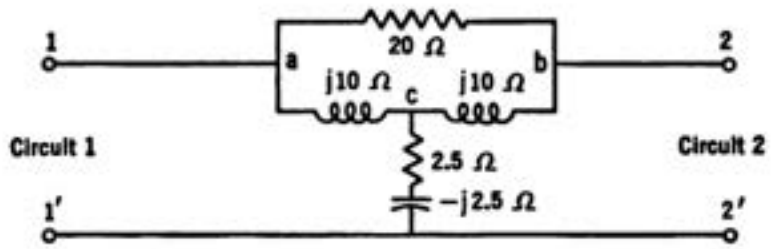

FIG. 29. See Problem 20.

20. Find the coefficient of coupling between circuits 1 and 2 in Fig. 29. *Hint:* Transform the *abc* delta to an equivalent wye, and then determine $\mathbf{Z}_{12}$ or $\mathbf{Z}_{21}$ of the equivalent circuit.

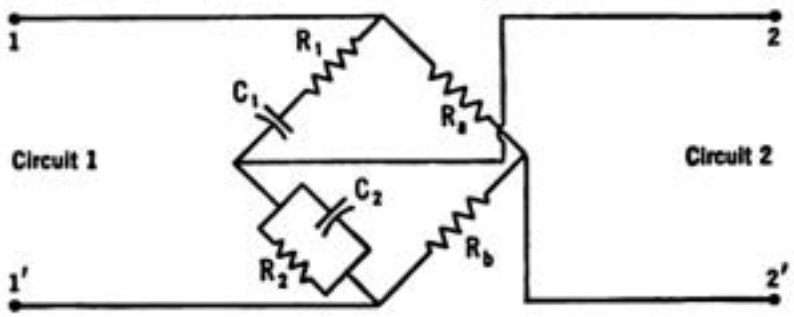

FIG. 30. See Problem 21.

21. Show that the coupling coefficient between circuits 1 and 2 in Fig. 30 is equal to zero if $\omega = 1/\sqrt{R_1R_2C_1C_2}$, $R_a = R_b$, $R_2 = 2R_1$, and $C_1 = 2C_2$.

22. Figures 31*a*, 31*b*, and 31*c* are the approximate equivalent circuits that are sometimes used in making voltage amplification calculations in resistance-coupled audio amplifiers. Show that the expressions given for E_2 in terms of μE_g are correct for each of the three configurations.

23. Two air-core inductance coils possess, individually, 60 and 30 millihenrys self-inductance, respectively. Measurements show that, if the two coils are connected in additive series as shown in Fig. 12, page 225, the equivalent self-inductance of the combination is 120 millihenrys.

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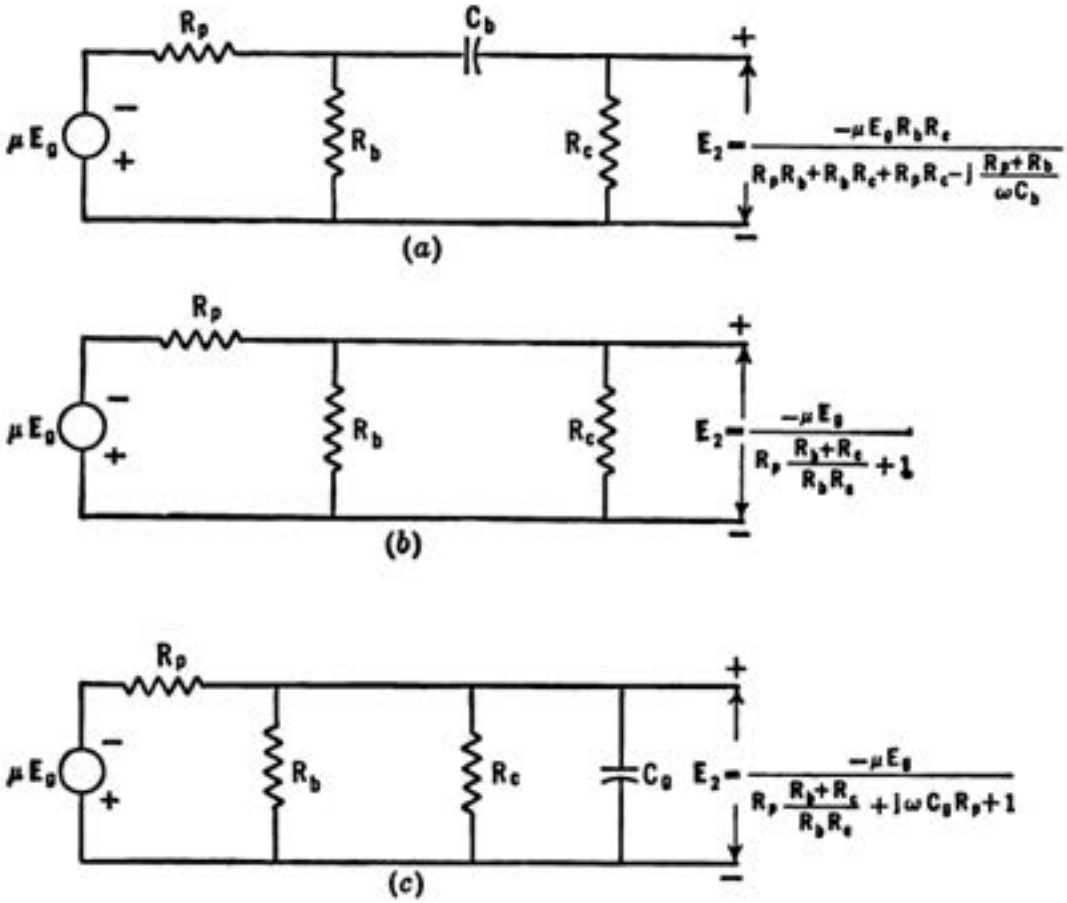

FIG. 31. Approximate equivalent circuits of resistance-capacitance coupled amplifiers. See Problem 22. (*a*) is for low-frequency range, (*b*) is for intermediate-frequency range, and (*c*) is for high-frequency range where the impedance of the blocking condenser C_b may be neglected.

(*a*) If the coils are connected in subtractive series, find the equivalent self-inductance of the combination.

(*b*) Find the coefficient of coupling between the coils.

24. Two inductance coils are connected in additive series. For 100 volts impressed on the combination, the current is 5 amperes and the power consumed is 200 watts. When the coils are reconnected in subtractive series and 100 volts are impressed, 8 amperes flow. Calculate the mutual inductance if the frequency for the above measurements is 69.5 cycles.

25. If the two coils in Problem 24 have equal resistances and the voltage drop across coil 1 is 36.05 volts for the additive series connection in Problem 24, (*a*) calculate L_1 and L_2 and the drop across coil 2 for this condition; (*b*) also calculate the coefficient of coupling.

26. The individual self-inductances of the two windings shown in Fig. 6, page 216, are 0.100 and 0.050 henry, respectively. The coefficient of coupling between the windings is 0.56. If the current in the 0.100-henry winding is a 60-cycle sinusoidal variation, the maximum magnitude of which is 10 amperes, find the effective value of voltage induced in the 0.050-henry winding as a result of the current variation in the 0.100-henry winding. Also find the magnitude of the rms induced voltage in the 0.1-henry winding.

27. In Fig. 32, $e_{ba} = 141.4 \sin 1131t$ volts and $e_{cd} = 70.7 \sin (1131t - 90°)$ volts.

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(*a*) Find I_{ba} and I_{cd}, assuming that Fig. 32 correctly represents the modes of winding as well as the physical placement of the two inductance coils. The internal impedances of the generators may be assumed to be negligibly small.

(*b*) Find the power generated by each generator.

(*c*) Draw a vector diagram of $\mathbf{E}_{ba}$, $\mathbf{I}_{ba}$, $\mathbf{I}_{ba}\mathbf{R}_1$, $\mathbf{I}_{ba}\mathbf{X}_{L1}$, $\mathbf{E}_{cd}$, $\mathbf{I}_{cd}$, $\mathbf{I}_{cd}\mathbf{R}_2$, $\mathbf{I}_{cd}\mathbf{X}_{L2}$, $\mathbf{I}_{cd}\mathbf{X}_M$, and $\mathbf{I}_{ba}\mathbf{X}_M$.

28. Branch 1 of two parallel branches consists of a resistance of 2 ohms in series with an inductive reactance of 3 ohms. Branch 2 consists of a resistance of 5 ohms in series with an inductive reactance of 12 ohms. The coefficient of coupling between the two inductances is 0.8, and the inductances are wound so that the mmf's

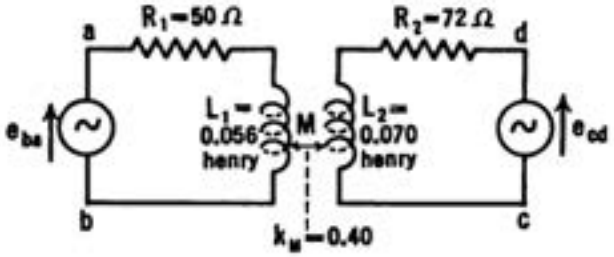

FIG. 32. See Problem 27.

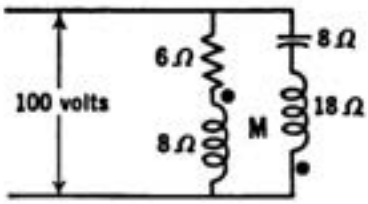

FIG. 33. See Problem 29.

due to I_1 and I_2 taken in the same direction from the junction are additive. If 100 volts are impressed on the two parallel branches, find I_1, I_2, the power supplied conductively to branch 2, the power supplied branch 2 electromagnetically, and the voltage drop across only the inductance of branch 2. What is the phase angle between the latter drop and the current in branch 2?

29. The coefficient of coupling for the coils in Fig. 33 is 0.5. Find the current in the resistance.

30. In the coupled circuits shown in Fig. 18, page 242,

$R_1 = 4.0$ ohms	$R_2 = 10$ ohms
$X_{L1} = 40$ ohms	$X_{L2} = 100$ ohms
$X_{C1} = 40$ ohms	$X_{C2} = 120$ ohms
$X_M = 50$ ohms	$V_1 = 100$ volts

Find I_2 and V_{C2}.

31. In the coupled circuits shown in Fig. 18, page 242,

$R_1 = 4$ ohms	$R_2 = 10$ ohms
$X_{L1} = 40$ ohms	$X_{L2} = 100$ ohms
$X_{C1} = 40$ ohms	$X_{C2} = 120$ ohms
$X_M = 50$ ohms	$V_1 = 100$ volts

Find the equivalent primary impedance, Z_{r1}, of the coupled circuits and the ohmic value of the secondary-circuit impedance referred to the primary terminals. How many ohms reactance does the secondary reflect into the primary, and is it inductive or capacitive?

32. Assume that an 83-μf capacitance is placed in series with the primary of Fig. 17*a*. Except for the insertion of the 83-μf capacitance into the primary circuit, the parameters are as given on page 233. Find the value of M which will produce unity-power-factor resonance.

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33. Show that the partial resonance which can be obtained by adjustment of the secondary reactance, X_2 (in coupled circuits of the kind shown in Fig. 18, page 242), occurs when $X_2 = X_1X_M^2/Z_1^2$. (See equation 79, page 244.)

34. Circuits 1 and 2 are inductively coupled. Circuit 1 consists of 2 ohms resistance in series with a coil of 16 ohms reactance and negligible resistance. Circuit 2 consists of 10 ohms resistance in series with an inductance coil of 100 ohms reactance and a condenser of 100 ohms.

(*a*) If the coefficient of coupling is 0.05, what is the drop across the condenser when 10 volts are applied to circuit 1?

(*b*) If a condenser is placed in series with circuit 1 so as to tune circuit 1 to resonance ($\omega L_1 = 1/\omega C_1$), what will be the drop across the condenser in circuit 2 for the same coefficient of coupling as before?

(*c*) If the coupling can be adjusted in part (*b*), what will be the greatest voltage drop across the secondary condenser?

35. Write the general differential equations for voltage equilibrium in the two circuits shown in Fig. 27, page 255, in terms of R_{ab}, L_{ab}, R_{bc}, L_{bc}, M, R, and L, and the branch currents i_1 and i_2. Note that this is essentially two parallel branches which are coupled.

36. Assuming that v_1 varies sinusoidally, write the general voltage equations for Fig. 27, page 255, in terms of the effective values of the branch currents, I_1 and I_2. Solve the equations thus found for $\mathbf{I}_1$ and $\mathbf{I}_2$. What circuit considered earlier in this chapter has similar equations for $\mathbf{I}_1$ and $\mathbf{I}_2$?

37. Assume that, in Fig. 27, page 255,

$R_{ab} = 4.0$ ohms	$M = 0.02$ henry
$L_{ab} = 0.07$ henry	$R = 10$ ohms
$R_{bc} = 0.5$ ohms	$L = 0.00$ henry
$L_{bc} = 0.01$ henry	$\omega = 377$ radians per second

If $\mathbf{V}_1 = 100\angle 0°$ volts, find $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_1 + \mathbf{I}_2$. Also calculate the total power supplied and that dissipated in each of circuits 1 and 2. Draw the complete vector diagram of the voltages and currents.

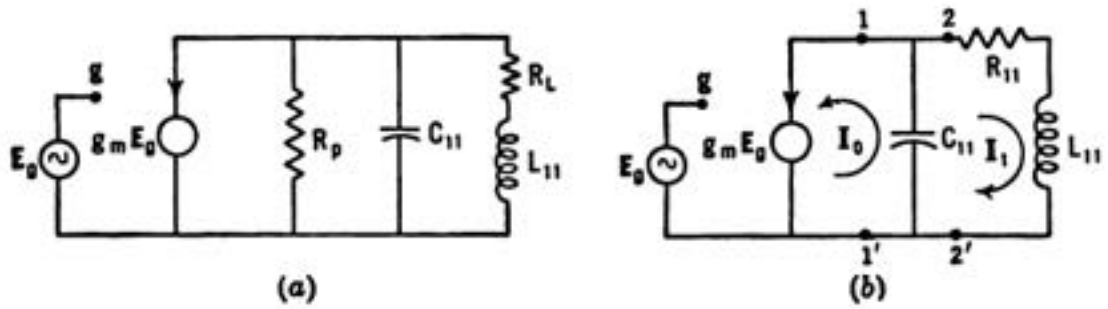

FIG. 34. See Problem 38.

38. Given the circuit arrangement shown in Fig. 34*a*, where the g_mE_g current generator in parallel with R_p is the equivalent a-c circuit of a pentode which has a voltage of E_g volts applied to its control grid.

(*a*) If $R_p = 750{,}000$ ohms, $R_L = 12$ ohms, $L_{11} = 382$ microhenrys, and C_{11} is adjusted to resonate the $L_{11}C_{11}$ parallel branches at 500 kc, find R_{11} of the equivalent circuit shown in Fig. 34*b*.

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(*b*) What is the Q of the coil itself, namely, $\omega_m L_{11}/R_L$, at 500 kc?

(*c*) What is the Q of the $C_{11}-R_{11}L_{11}$ parallel combination of Fig. 29*b* at 500 kc?

(*d*) Can I_1 in Fig. 34*b* be evaluated from the relation $\mathbf{Z}_{11}\mathbf{I}_1 = -(\mathbf{I}_0)\left(-j\dfrac{1}{\omega C_{11}}\right)$, where $\mathbf{Z}_{11} = R_{11} + j\left(\omega L_{11} - \dfrac{1}{\omega C_{11}}\right)$?

39. In Problem 38, it has been shown that the current generators of Fig. 34*b* and Fig. 35 can be replaced by equivalent voltage generators which have voltages of $-(\mathbf{I}_0)\left(-j\dfrac{1}{\omega C_{11}}\right)$.

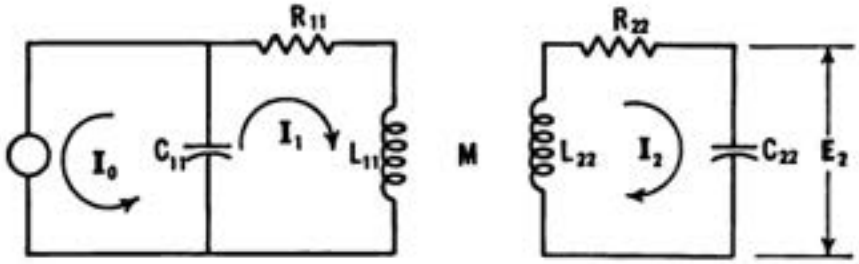

FIG. 35. See Problems 39, 40, 41, 42, and 43.

Show that the equivalent primary impedance (including the reflected impedance from the secondary) which the equivalent voltage generator in Fig. 35 sees is:

$$\mathbf{Z}_{11\,eq} = \frac{j\dfrac{\mathbf{I}_0}{\omega C_{11}}}{\mathbf{I}_1} = \mathbf{Z}_{11} + \frac{\omega^2 M^2}{\mathbf{Z}_{22}} = \omega_{m1}L_{11}\left[(a + jF_{11}) + \frac{\omega^2 k^2}{\omega_{m1}\omega_{m2}(b + jF_{22})}\right]$$

where

$$\omega_{m1} = \frac{1}{\sqrt{L_{11}C_{11}}} \qquad \omega_{m2} = \frac{1}{\sqrt{L_{22}C_{22}}}$$

$$\mathbf{Z}_{11} = R_{11} + j\left(\omega L_{11} - \frac{1}{\omega C_{11}}\right) \qquad \mathbf{Z}_{22} = R_{22} + j\left(\omega L_{22} - \frac{1}{\omega C_{22}}\right)$$

$$a = \frac{1}{Q_1} = \frac{\omega_{m1}L_{11}}{R_{11}} \qquad b = \frac{1}{Q_2} = \frac{\omega_{m2}L_{22}}{R_{22}}$$

$$F_{11} = \left(\frac{\omega}{\omega_{m1}} - \frac{\omega_{m1}}{\omega}\right) \qquad F_{22} = \left(\frac{\omega}{\omega_{m2}} - \frac{\omega_{m2}}{\omega}\right)$$

$$k = \frac{M}{\sqrt{L_{11}L_{22}}}$$

40. The results of Problem 39 are to be employed in the following exercises.

(*a*) Show that a voltmeter across L_{11} of Fig. 35 will read a maximum value when C_{11} is adjusted to $1/L_{11}\omega^2$ if loop 2 is open-circuited and that this voltage will be

$$\mathbf{V}_{L_{11}\max} = \frac{\mathbf{K}}{\omega_{m1}L_{11}a}$$

where $\mathbf{K} = [-(\mathbf{I}_0/\omega C_{11})]\,(\omega L_{11})$.

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(b) With C_{11} left at the value found above $(1/L_{11}\omega_{m1}^2)$, show that the voltmeter (which is across the L_{11} coil) will read a minimum value of

$$V_{L_{11}\min} = \frac{K}{\omega_{m1}L_{11}\left(a + \frac{k^2}{b}\right)}$$

when C_{22} is adjusted to $1/L_{22}\omega_{m1}^2$.

(c) Show that, if the experimental procedure outlined in (a) and (b) is followed, the coupling coefficient between the two coils is

$$k = \sqrt{ab\left(\frac{V_{L_{11}\max}}{V_{L_{11}\min}} - 1\right)}$$

41. In Fig. 35: $L_{11} = L_{22} = 500$ microhenrys; $C_{11} = C_{22} = 2000$ $\mu\mu$f; $M = 8.66$ microhenrys; $a = R_{11}/\omega_{m1}L_{11} = b = R_{22}/\omega_{m2}L_{22} = 0.01$.

(a) Find the magnitude of the voltage across the C_{22} condenser per milliampere of I_0 at $\omega = \omega_m = 1/\sqrt{L_{11}C_{11}}$ radians per second.

(b) Will the voltage found in part (a) be the maximum value of E_2 if the frequency is varied slightly about the value ω_m specified above?

42. (a) Make a sketch of $\dfrac{E_{C22}}{E_{C22(\omega=\omega_m)}}$ versus F for the circuit shown in Fig. 35 employing the circuit parameters specified in Problem 41. Calculate points for this sketch at

$$\omega = 1.010\omega_m \quad \text{or} \quad F = 2 \times 10^{-2}$$
$$\omega = 1.00707\omega_m \quad \text{or} \quad F = \sqrt{2} \times 10^{-2}$$
$$\omega = 1.005\omega_m \quad \text{or} \quad F = 10^{-2}$$
$$\omega = \omega_m \quad \text{or} \quad F = 0$$

using equation (95), namely:

$$\frac{E_{C22}}{E_{C22(\omega=\omega_m)}} = \frac{1}{\sqrt{1 + \dfrac{F^4 + (a^2 + b^2 - 2k^2)F^2}{(k^2 + ab)^2}}}$$

(b) Make a sketch of E_{C22} per milliampere of I_0 versus ω/ω_m employing the results of part (a). It may be assumed that the response curve is symmetrical about the center frequency ω_m.

43. Design a current-fed double-tuned circuit like that shown in Fig. 35 which has a per unit band width of 0.02 centered at $\omega_m = 10^6$ radians per second. Use $L_{11} = L_{22} = 500$ microhenrys. The permissible variation in the response curve over the pass band is 1.2516 decibels reckoned from $E_{\min}$ as reference. $(\alpha = 0.5)$

Note: Where $Q_1 = Q_2$, a design of this kind amounts simply to specifying some appropriate value for the Q's of the coils and then calculating the coefficient of coupling to employ between these coils to meet the conditions imposed. In this case, $F_{\min}^2/\alpha = 0.0004/0.5 = 2(k^2 + ab) = 2(k^2 + a^2)$. In a more general case, one of the Q's may be chosen almost arbitrarily. Then $F_{\min}^2/\alpha = 2(k^2 + ab)$ and $F_{\min}^2 = (2k^2 - a^2 - b^2)$ may be solved simultaneously for k and the other Q to meet the specified values of $F_{\min}$ and α.

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CHAPTER VIII

BALANCED POLYPHASE CIRCUITS

Generation of Polyphase Voltages. Polyphase voltages are generated in the same way as single-phase voltages. A polyphase system is simply several single-phase systems which are displaced in time phase from one another. The single-phase systems which form the polyphase systems are generally interconnected in some way.

In Fig. 1 is shown a single coil aa' on the armature of a two-pole machine. When the poles are in the position shown, the emf of conductor a of coil aa' is a maximum, and its direction is away from the reader. If a conductor is placed 120° from a at position b, it would

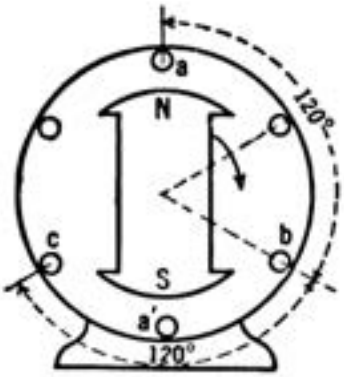

Fig. 1. Elementary three-phase generator.

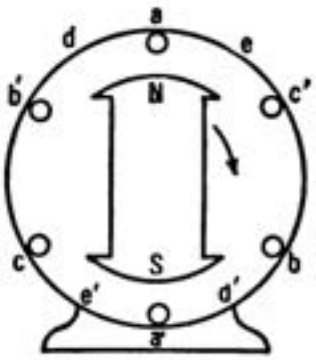

Fig. 2.

experience maximum emf in a direction away from the reader when the north pole axis was at b, or 120° later than when the pole axis was at a. In like manner, the maximum emf in the direction away from the reader for a conductor at c would occur 120° later than that at b, and 240° later than that at a. The placement of such conductors and the coils of which they are a part are shown in Fig. 2. Thus the coils aa', bb', and cc' would have emf's that are 120° out of time phase, as pictured in Fig. 3. This system is called three-phase because there are three waves of different time phase. In practice the space on the armature is completely covered with coils (except in single phase). For instance, the conductor of another coil could be placed in the slot to the right of conductor a in Fig. 2, and another to the left. The one to the right would have an emf which would lag that in a by the same angle that the one to the left would lead. The sum of the three emf's would give a resultant emf of the same phase as that in a. Conductors for phase a

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would cover the periphery from d to e and from d' to e'. The distance from d to e is called a phase belt. The emf of all the coils in series for the whole phase would have the same phase relation as the emf of the center conductor of the phase belt. For this reason only the center conductors of the phase belts will be considered. It is apparent that any number of phases could be developed through properly spacing the coils on the stator.

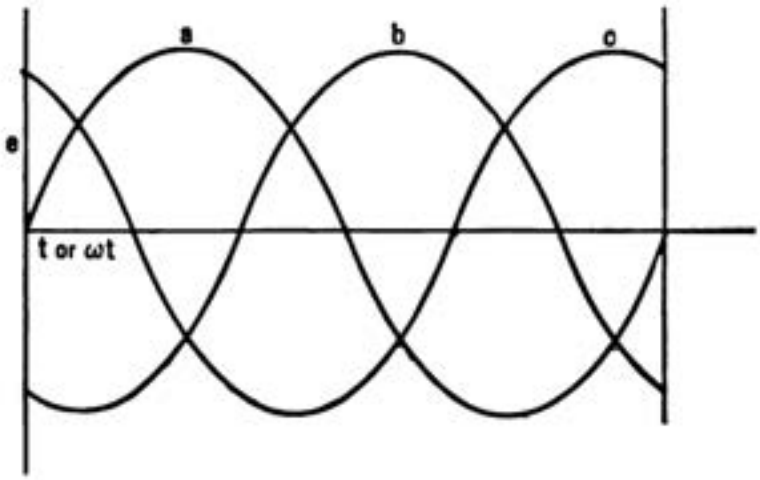

FIG. 3. Waves of emf generated by a three-phase generator.

In general, the electrical displacement between phases for a balanced n-phase system is $360/n$ electrical degrees. Three-phase systems are the most common, although for certain special applications a greater number of phases is used. For instance, practically all mercury-arc rectifiers for power purposes are either six- or twelve-phase. Most rotary converters are six-phase. Practically all modern generators are three-phase. Three-phase is also invariably used for transmitting large amounts of power. In general, three-phase apparatus is more efficient, uses less material for a given capacity, and costs less than single-phase apparatus. It will be shown later that, for a fixed amount of power to be transmitted a fixed distance at a fixed line loss with a fixed voltage between conductors, three-phase is more economical in the use of copper than any other number of phases.

In the development of the three-phase voltages in Fig. 3, clockwise rotation of the field structure of the alternator in Fig. 2 was assumed. This assumption made the emf of phase b lag that of a by 120°. Also, the emf of phase c lagged that of phase b by 120°. In other words, the order in which the emf's of phases a, b, and c came to their corresponding maximum values was abc. This is called the phase order or sequence abc. If the rotation of the field structure in Fig. 2 is reversed, the order in which the phases would attain their corresponding maximum voltages

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would be reversed. The phase sequence would be *acb*. This means that the emf of phase *c* would then lag that of phase *a* by 120° instead of by 240° as in the first case. In general, the phase sequence of the voltages applied to a load is fixed by the order in which the three-phase lines are connected. Interchanging any pair of lines reverses the phase sequence. For three-phase induction motors the effect of reversing the sequence is to reverse the direction of rotation. For three-phase unbalanced loads the effect is, in general, to cause a completely different set of values for line currents; hence when calculating such systems it is essential that phase sequence be specified or confusion may arise.

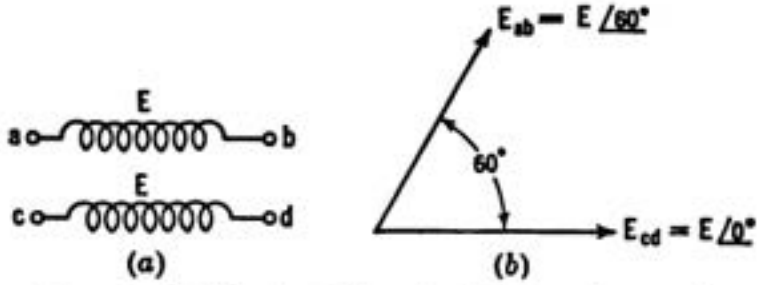

FIG. 4. Coils having induced emf's shown in part (*b*).

Vector Diagrams and Double-Subscript Notation. When drawing vector diagrams of polyphase circuits it is imperative that directions in which the circuit is being traced be noted and recorded. For example, let us assume that the two coils shown in Fig. 4*a* possess induced voltages or emf's that are 60° out of phase and that the coils are to be connected in additive series, that is, in such a manner that the emf's add at a 60°

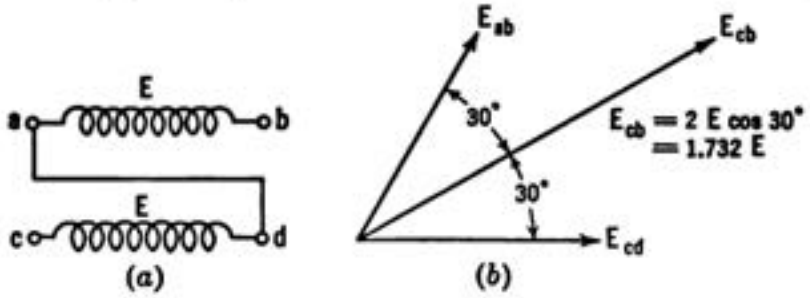

FIG. 5. Resultant emf shown in (*b*) for connection of coils shown in (*a*).

angle. From the information given it would be impossible to know whether terminal *a* should be connected to terminal *c* or terminal *d*. But if it were stated that the emf from *a* to *b* is 60° out of phase with that from *c* to *d* as shown in Fig. 4*b*, the way to connect the coils would be definitely fixed. Under such conditions, double-subscript notation is very convenient.

The order in which the subscripts are written denotes the direction in which the circuit is being traced. Thus the emf from *a* to *b* in Fig. 4*a*

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may be designated as $\mathbf{E}_{ab}$ and that from c to d as $\mathbf{E}_{cd}$. (See Fig. 4*b*.) If d is connected to a as shown in Fig. 5*a*, the emf from c to b is determined by adding all the emf's *in the directions encountered* as the circuit is traced from c to b. Hence $\mathbf{E}_{cb} = \mathbf{E}_{cd} + \mathbf{E}_{ab}$ as shown in Fig. 5*b*. This procedure will be further illustrated in succeeding articles.

Problem 1. In Fig. 4*a*, connect terminal b to terminal c and compare the resultant voltage $\mathbf{E}_{ad}$ with voltage $\mathbf{E}_{cb}$ of Fig. 5*b*.

Ans.: $\mathbf{E}_{ad} = \mathbf{E}_{cb}$.

Problem 2. (*a*) Connect terminal d to terminal b in Fig. 4*a* and find the voltage $\mathbf{E}_{ca}$ if $E = 120$ volts. $\mathbf{E}_{ab}$ and $\mathbf{E}_{cd}$ have the same vector relation as shown in Fig. 4*b*.

Ans.: $\mathbf{E}_{ca} = 120 \angle -60°$ volts.

(*b*) With terminal d connected to terminal b as above, find $\mathbf{E}_{ac}$.

Ans.: $\mathbf{E}_{ac} = 120 \angle 120°$ volts.

A vector diagram is simply a means of representing certain electrical quantities that are related by a circuit. A vector diagram therefore must always be drawn in conjunction with a circuit. Sometimes circuits may be visualized instead of actually drawn, but without a definite picture of the circuit represented a vector diagram means nothing and cannot be intelligently drawn. It should be clearly recognized, however, that a circuit vector diagram of voltages and currents represents time-phase relations and not space relations of the circuit. This means that the space configuration of a circuit diagram is in no way indicative of the time-phase relations of the voltages or currents.

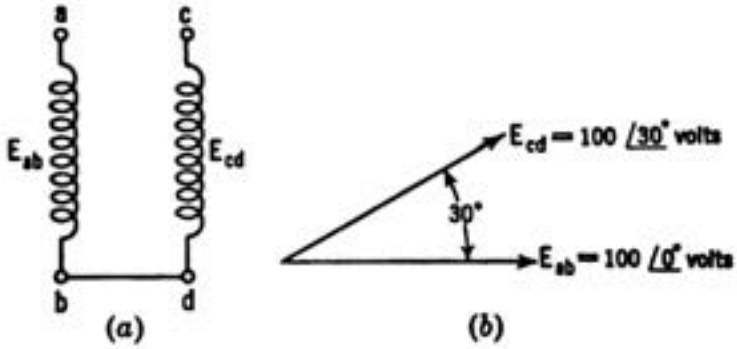

FIG. 6. See Problem 3.

Problem 3. Find the magnitude and vector position of voltage $\mathbf{E}_{ca}$ in Fig. 6*a* if $\mathbf{E}_{ab}$ and $\mathbf{E}_{cd}$ are displaced from each other by 30° in time phase as shown in Fig. 6*b*.

Ans.: $\mathbf{E}_{ca} = 51.76 \angle 105°$ volts.

Two- and Four-Phase Systems. A two-phase system is an electrical system in which the voltages of the phases are 90° out of time phase. A two-phase system is pictured by the drum and Gramme ring windings in Figs. 7 and 8. From the position of the coils on the armature in Fig. 8 it can be seen that the emf's of the four coils are related in time

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phase as shown in Fig. 9. If the zero terminals of coils $a0$ and $c0$ are connected, the emf from a to c is $\mathbf{E}_{a0} + \mathbf{E}_{0c}$. This operation is shown in Fig. 10. Likewise, when the zeros of coils $b0$ and $d0$ are connected

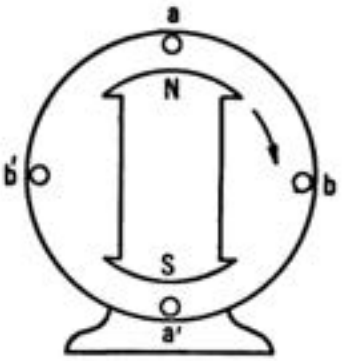

FIG. 7. Elementary drum-type two-phase generator.

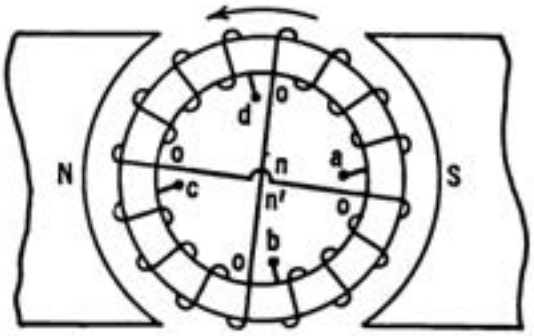

FIG. 8. Elementary Gramme ring-type two-phase generator.

$\mathbf{E}_{bd} = \mathbf{E}_{b0} + \mathbf{E}_{0d}$. This is also shown in Fig. 10. The emf's $\mathbf{E}_{ac}$ and $\mathbf{E}_{bd}$ are 90° apart in time phase, and the system shown in Fig. 8 constitutes a two-phase system. A two-phase system is the equivalent of two separate single-phase systems that are separated 90° in time phase.

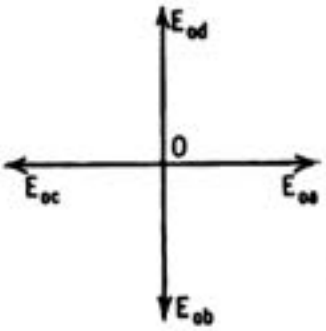

FIG. 9. Emf's of coils on generator in Fig. 8.

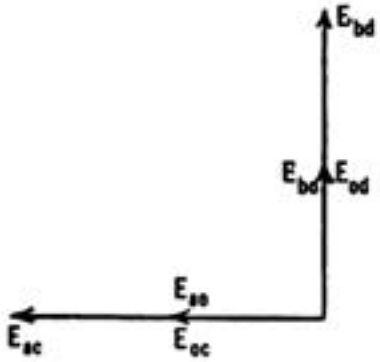

FIG. 10. Resultant emf's of two coils in series connected as shown in Fig. 8.

A four-phase and a two-phase system differ only in internal connections. Thus if connection is made between the two windings at n and n', the system would be called a four-phase system. The vector diagram of phase or coil voltages is shown in Fig. 9. Since there now is an electrical connection between the two groups of coils that constituted the two-phase system, there will be emf's between terminals d and a and also between b and c, as may be seen by studying the diagrammatic representation of the coils shown in Fig. 11. This connection is called a four-phase star. The voltages E_{da}, E_{ab}, E_{bc}, and E_{cd} are called the line voltages, while voltages E_{0a}, E_{0b}, E_{0c}, and E_{0d} are called the phase voltages, or voltages to neutral. From the circuit it is evident that

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$\mathbf{E}_{da} = \mathbf{E}_{d0} + \mathbf{E}_{0a}$. This combination and similar ones for all the line voltages are shown in Fig. 12. Another method of showing the same thing is illustrated in Fig. 13. Thus, in the four-phase star, line voltage is the $\sqrt{2}$ times phase voltage and it is either 45° or 135° out of phase with the phase voltage, depending upon which voltages are considered.

FIG. 11. Diagrammatic representation of Fig. 8 when n and n' are connected to form point o.

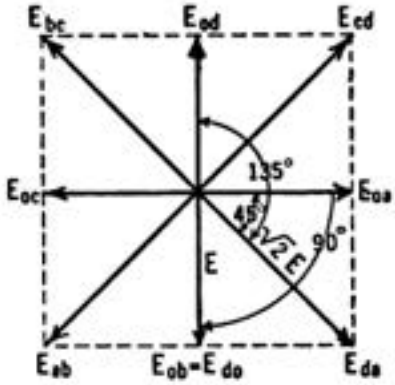

FIG. 12. Voltages of the four-phase star shown in Fig. 11.

Since $\mathbf{E}_{0a} + \mathbf{E}_{0b} + \mathbf{E}_{0c} + \mathbf{E}_{0d} = 0$, it would be possible to connect the four coils shown in Figs. 8 and 11 so that their voltages add in this way and no current would flow in the series circuit of the coils. This connection, shown in Fig. 14, is called a mesh connection, and in this case it would be known as a four-phase mesh. The line connections

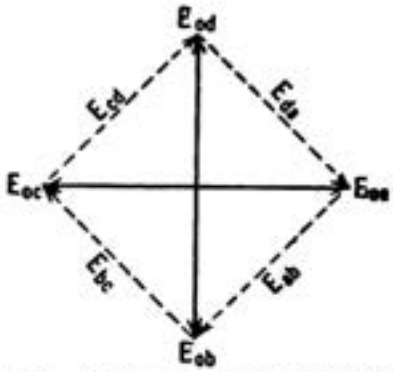

FIG. 13. Alternative representation of Fig. 12.

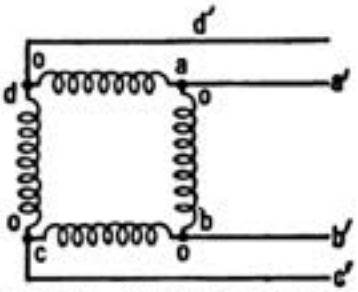

FIG. 14. Four-phase mesh.

are made at points a, b, c, and d. The vector diagram of the emf's for this system is shown in Fig. 15. For balanced loads the currents in adjacent phases are 90° out of phase as shown in Fig. 16. The aa' line current is $\mathbf{I}_{aa'} = \mathbf{I}_{da} + \mathbf{I}_{ba}$, as shown in Fig. 16. Thus line current of a balanced four-phase mesh is the $\sqrt{2}$ times phase current and is either 45° or 135° out of phase with the phase currents, according to which are being considered. Note that what was true about line and phase voltages in the star is true about line and phase currents in the mesh.

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Inspection of the star system shows that line and phase currents must be identical, and the same thing is true regarding line and phase voltages in the mesh.

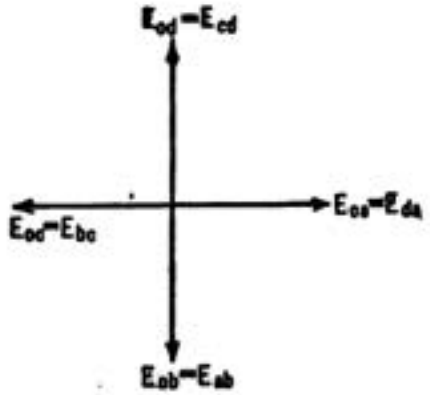

FIG. 15. Vector diagram of emf's of the four-phase mesh shown in Fig. 14.

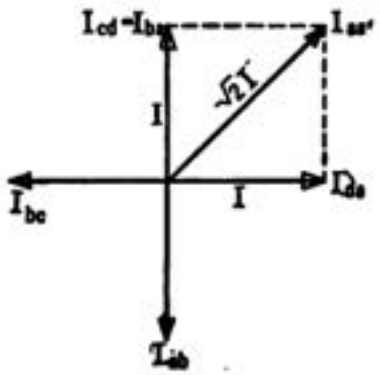

FIG. 16. Vector diagram of currents of the four-phase mesh shown in Fig. 14 under conditions of balanced load.

Sometimes a two-phase system is used with only three wires. When this is done, one wire is common to both phases. The circuit diagram of Fig. 8 when connected for such use is shown in Fig. 17, and the vector diagram is shown in Fig. 18. It will be noted that this is essentially half of the four-phase system shown in Fig. 11 when line wires are connected to points 0, *d*, and *c*.

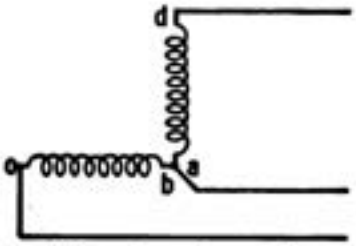

FIG. 17. Two-phase three-wire system.

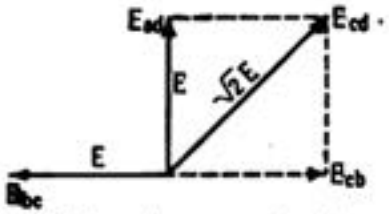

FIG. 18. Vector diagram of voltages for Fig. 17.

Three-Phase, Four-Wire Systems of Generated Emf's. The generation of three-phase was explained at the beginning of this chapter. If six wires were connected to terminals *a*, *a'*, *b*, *b'*, *c*, and *c'* of Fig. 2, the system might be called a six-wire, three-phase system. Such a generator could be loaded with three independent single-phase loads. Though such a system is not used, one that is widely used may be derived from it by making a common connection between terminals *a'*, *b'*, and *c'*. Four wires are all that would then be necessary, three for terminals *a*, *b*, and *c*, and one for the common connection *a'b'c'*. Such a system, called a four-wire, three-phase system, is shown diagrammatically in Fig. 19. This system is now extensively used for a-c networks and is rapidly displacing the formerly much used d-c networks in the down-

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town areas of large cities. The common wire connecting to n is called the neutral. Lighting loads are placed from line to neutral; motor and other three-phase power loads are connected between the three lines a, b, and c. The generated voltage waves of this system are shown in Fig. 3, and the vector diagram that portrays the same thing is shown in Fig. 20. The three voltages shown are called phase voltages or line-

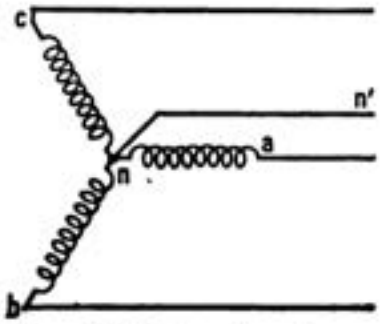

FIG. 19. Three-phase four-wire system.

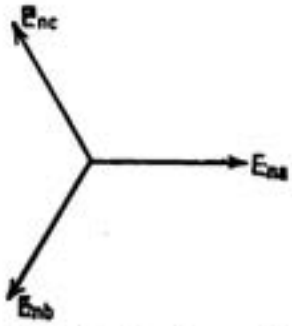

FIG. 20. Line-to-neutral voltages of Fig. 19.

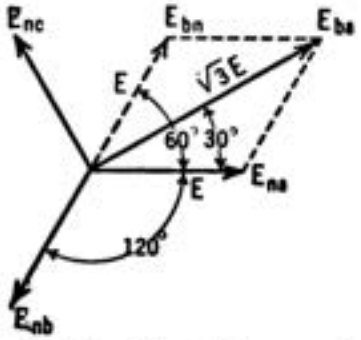

FIG. 21. Line voltage equals phase voltage times $\sqrt{3}$ in the wye connection.

to-neutral voltages. They are sometimes called the wye voltages of the system, and the connection of Fig. 19 is called a wye connection. The voltages between terminals a, b, and c are called the line or terminal voltages. Under balanced conditions they are definitely related to the phase voltages, as the following shows:

$$\mathbf{E}_{ba} = \mathbf{E}_{bn} + \mathbf{E}_{na}$$

This combination is shown in Fig. 21 where the magnitude of the phase

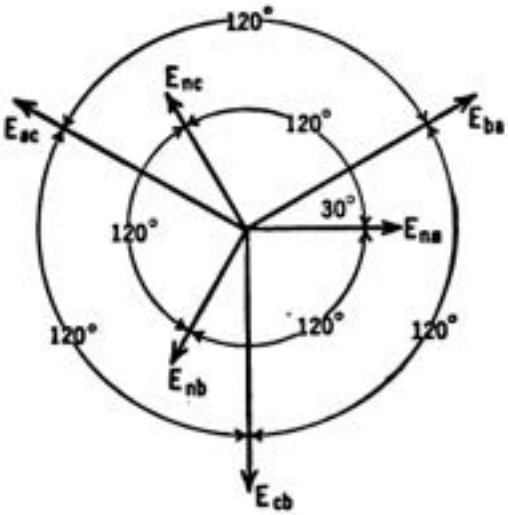

FIG. 22. Line and phase voltages of the wye connection (Fig. 19).

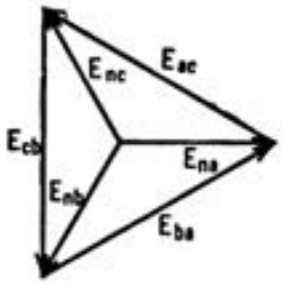

FIG. 23. Alternative representation of Fig. 22.

voltage is considered as E. Hence line voltage in the balanced three-phase star or wye connection is the $\sqrt{3}$ times the phase voltage and makes an angle with the component phase voltages of either 30° or 150°, depending upon which are considered. The complete vector diagram showing all line voltages is given in Fig. 22. Figure 23 shows the same

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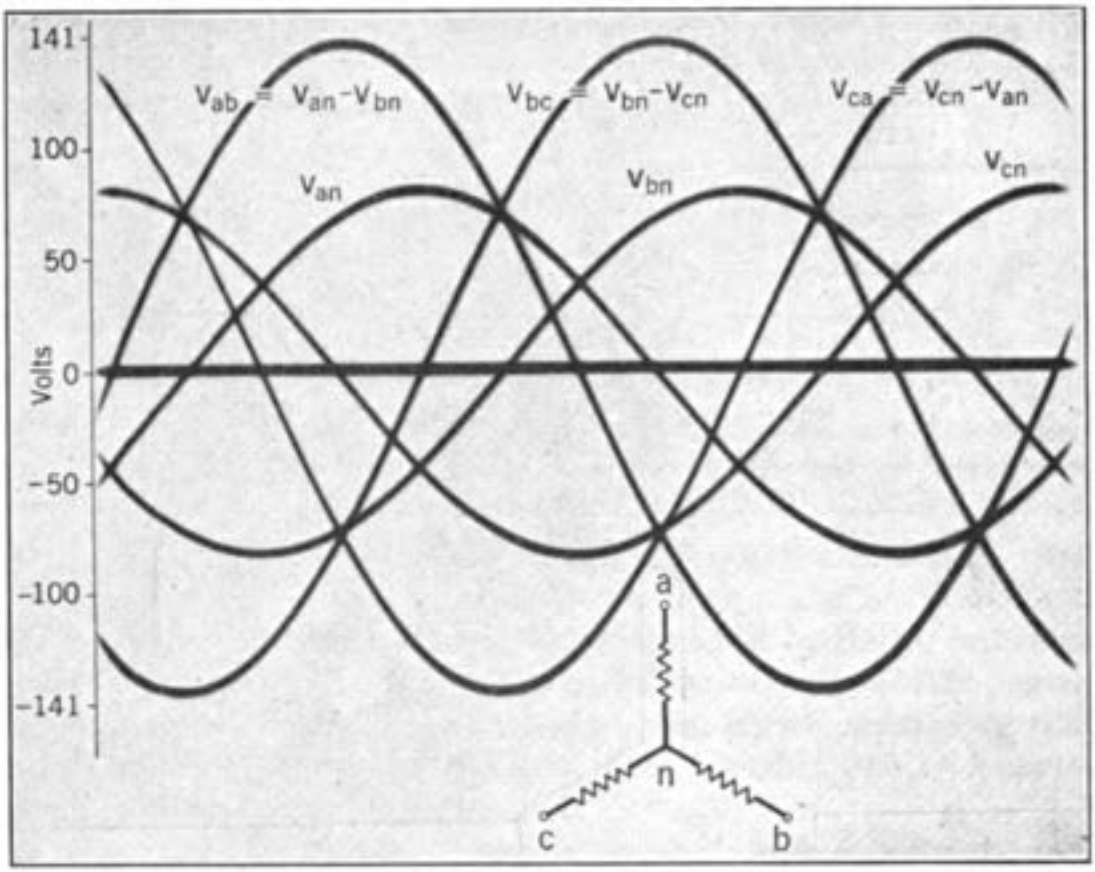
system in terms of a polar vector diagram of phase voltages and a funicular diagram of line voltages. Oscillogram 1 shows these relationships as obtained from an actual load.



OSCILLOGRAM 1. Illustrating the 30° angular displacement between the phase voltages and the systematically labeled line-to-line voltages in a balanced, three-phase, wye-connected load. Effective value of each line-to-line voltage is 100 volts.

When the system is balanced, the currents in the three phases are all equal in magnitude and differ by 120° in time phase, as shown in Fig. 24. The phase of currents with respect to the wye voltages is defined by the circuit parameters in any particular case. An inspection of Fig. 19 shows that line and phase currents are identical. The current in the neutral wire is obtained through the application of Kirchhoff's current law. Thus

$$\mathbf{I}_{n'n} = \mathbf{I}_{na} + \mathbf{I}_{nb} + \mathbf{I}_{nc}$$

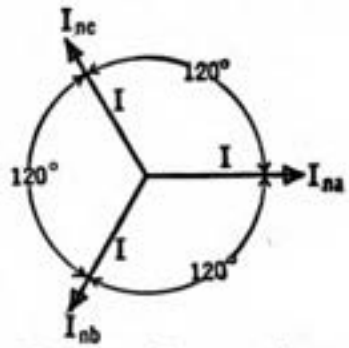


FIG. 24. Currents in a balanced-wye system.

If the system is balanced, $\mathbf{I}_{na}$, $\mathbf{I}_{nb}$, and $\mathbf{I}_{nc}$ are equal in magnitude and displaced from one another in time phase by 120° as shown in Fig. 24. Under these conditions it is apparent that the current in the neutral is zero since $\mathbf{I}_{na} + \mathbf{I}_{nb} + \mathbf{I}_{nc} = 0$.

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Problem 4. (*a*) Draw a polar (or single-origin) vector diagram which will represent the same phase voltages and the same line voltages as shown in Oscillogram 1 using $\mathbf{V}_{bn}$ as reference. Specify the effective magnitude of the phase voltages, the sequence of the phase voltages, and the sequence of the line voltages.

Ans.: V/phase = 57.7 volts.
Phase voltage sequence: *an-bn-cn.*
Line voltage sequence: *ab-bc-ca.*

(*b*) Draw a polar (or single-origin) vector diagram which will represent the same phase voltages as shown in Oscillogram 1, namely $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$, together with the line voltages $\mathbf{V}_{ba}$, $\mathbf{V}_{cb}$, and $\mathbf{V}_{ac}$, using $\mathbf{V}_{cn}$ as reference. Specify the sequence of these line voltages.

Ans.: Line voltage sequence: *ba-cb-ac.*

Three-Phase, Three-Wire Systems. The usual three-phase system consists of only three wires. In this event loads are not placed between the lines and neutral, and the neutral wire is therefore not brought out. The balanced relations discussed in the previous article are obviously unaffected by omitting the neutral wire and therefore apply to the three-phase, three-wire system.

The Delta Connection. If only three wires are used, the three-phase system may be connected in mesh similar to the four-phase system previously considered. Since

$$\mathbf{E}_{na} + \mathbf{E}_{nb} + \mathbf{E}_{nc} = 0$$

for the three-phase system, the three coils shown in Fig. 19 can be connected as shown in Fig. 25, and no current of fundamental frequency

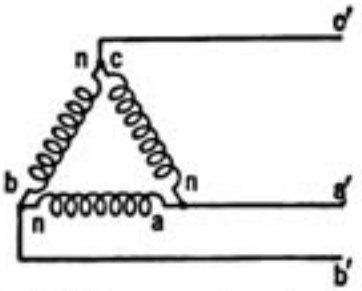

FIG. 25. Delta connection of the coils shown in Fig. 19.

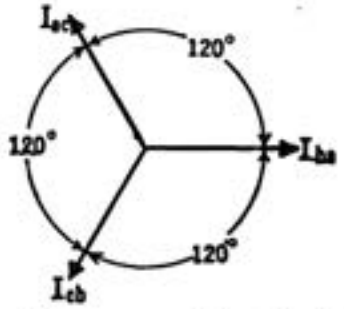

FIG. 26. Phase currents for the balanced delta of Fig. 25.

will flow around the series circuit of the three coils. This three-phase mesh connection is called a delta connection. It will be noted that star and mesh are general terms applicable to any number of phases, but wye and delta are special cases of the star and mesh when three-phase is considered. Inspection of Fig. 25 shows that phase voltages and line voltages are identical but that line and phase currents are different. The vector diagram of phase currents for a balanced load is shown in Fig. 26. Line currents are found through the application of Kirchhoff's

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current law. Thus

$$\mathbf{I}_{aa'} = \mathbf{I}_{ba} + \mathbf{I}_{ca}$$

This operation is carried out in Fig. 27. For a balanced system, line current is the $\sqrt{3}$ times phase current in magnitude and is out of phase with the component-phase currents by either 30° or 150°, depending

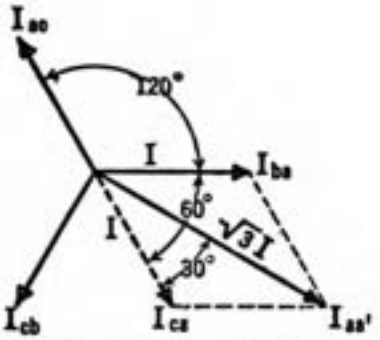


FIG. 27. Combination of phase currents gives line current for Fig. 25.

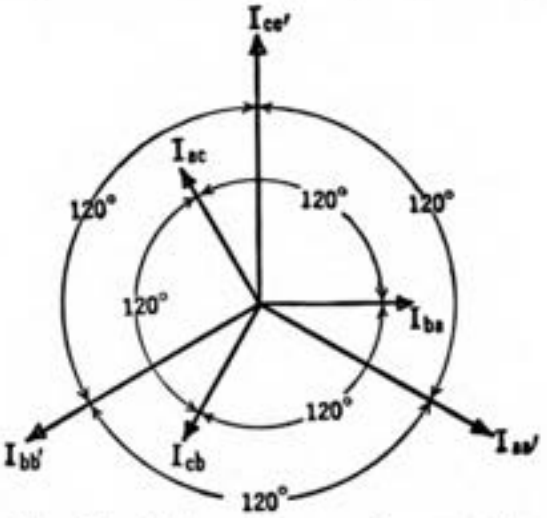
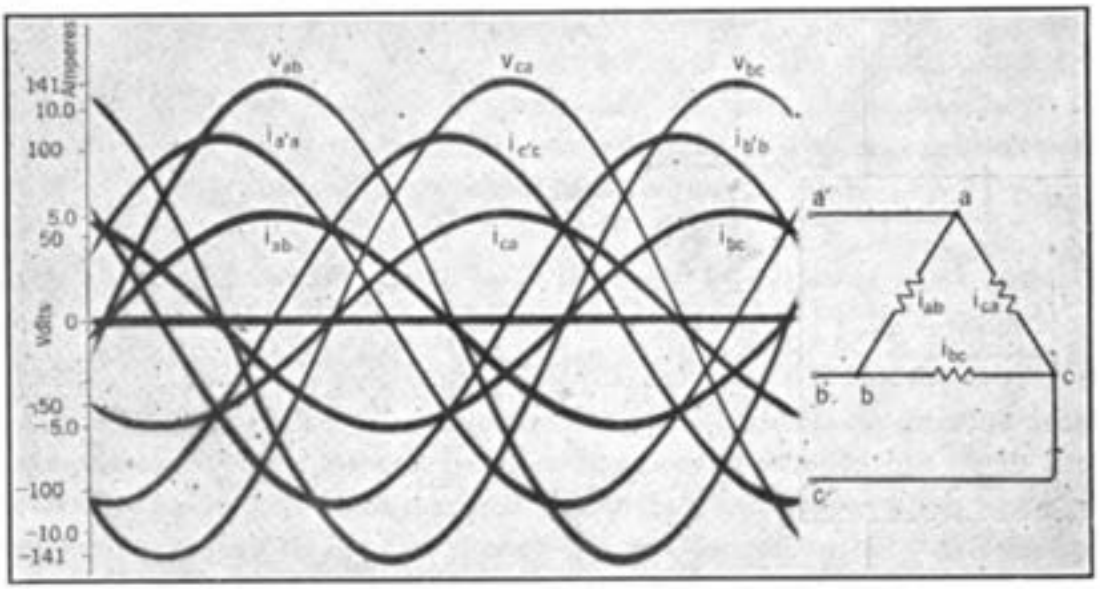


FIG. 28. Vector diagram of currents for a balanced delta is shown in Fig. 25.

upon which are considered. The complete vector diagram of currents for the three-phase balanced delta connection is shown in Fig. 28. Oscillogram 2 shows the relations discussed above as obtained from an actual load labeled as in the accompanying circuit diagram.



OSCILLOGRAM 2. Oscillographic study of a balanced, delta-connected, unity-power-factor load. The line-to-line voltages (or phase voltages) together with the phase currents and line currents are illustrated.

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It should be understood that all the vectors on a vector diagram like that shown in Fig. 28 may be reversed, that is, changed individually through 180°, and, if a reversal in the order of subscripts accompanies this change, the resulting vector diagram will represent the same thing as does Fig. 28. As applied to the circuit shown on Oscillogram 2, for example, it is immaterial whether $\mathbf{I}_{ab}$ is considered to flow in the direction of $\mathbf{V}_{ab}$ or whether $\mathbf{I}_{ba}$ is considered to flow in the direction of $\mathbf{V}_{ba}$. Those who prefer to consider line voltages *ao*, *ca*, and *bc* rather than line voltages *ba*, *ac*, and *cb* will label a circuit diagram like that shown on Oscillogram 2, whereas those who prefer to consider line voltages *ba*, *ac*, and *cb* will employ $\mathbf{I}_{ba}$, $\mathbf{I}_{ac}$, and $\mathbf{I}_{cb}$ as the delta-phase currents.

Problem 5. Refer to Oscillogram 2. Draw a complete vector diagram of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{ca}$, $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, $\mathbf{I}_{ca}$, $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ employing $\mathbf{V}_{bc}$ as reference. From the scaled ordinates given on Oscillogram 2, determine the effective values of line (or phase) voltage, phase current, and line current.

Ans.: $V = 100$ volts; $I_p = 3.5$ amperes; $I_l = 6$ amperes.

The *n*-Phase Star and Mesh. The circuit and vector diagrams of two adjacent phases of an *n*-phase star system are shown in Figs. 29

FIG. 29. Two adjacent phases of an *n*-phase star.

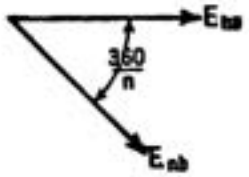

FIG. 30. Line-to-neutral voltages of adjacent phases of an *n*-phase star (Fig. 29).

and 30, respectively. The line voltage $\mathbf{E}_{ab}$ is $\mathbf{E}_{an} + \mathbf{E}_{nb}$. Remembering that the angle of phase difference between voltages of adjacent phases is $360°/n$, and calling the magnitude of phase voltage E_p, the general calculation of the line voltage can be understood from the vector relations shown in Fig. 31. Hence the line voltage is

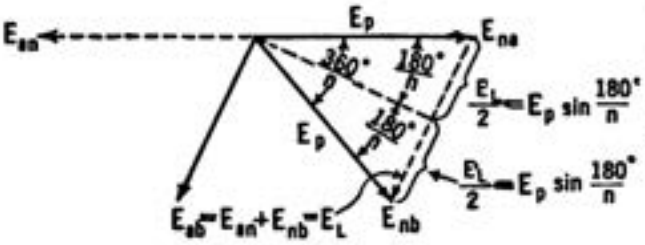

FIG. 31. Combination of line-to-neutral voltages to give line-to-line voltages in an *n*-phase star.

$$E_L = 2E_p \sin\frac{180°}{n} \quad (1)$$

From the circuit of Fig. 29 it is evident that line current and phase current are identical. Hence

$$I_L = I_p$$

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From the circuit and vector diagrams shown for part of an n-phase mesh system in Fig. 32, the use of previously outlined principles will show that

$$E_L = E_p$$

and

$$I_L = 2I_p \sin \frac{180°}{n} \tag{2}$$

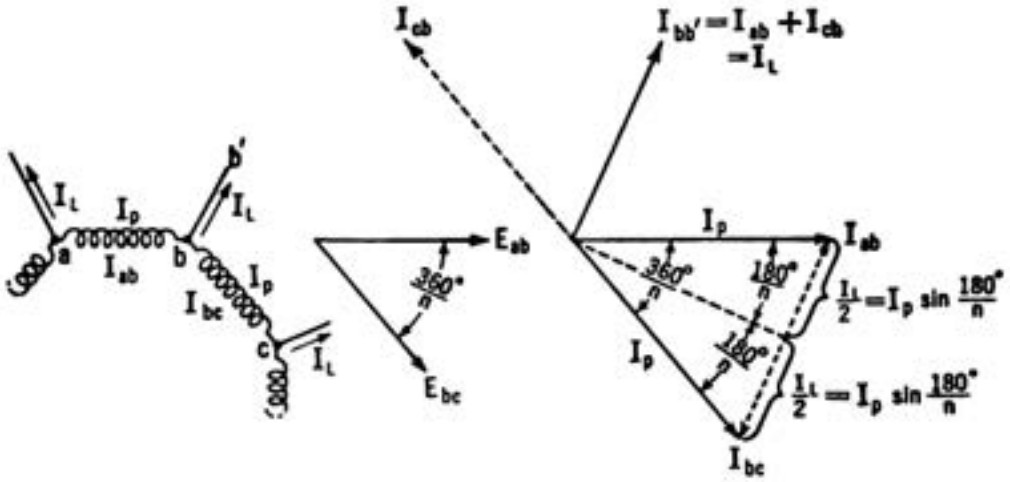

FIG. 32. Circuit diagram of adjacent phases and corresponding vector diagrams for an n-phase mesh.

Example 1. The line currents issuing from a balanced four-phase, mesh-connected generator (like that shown in Fig. 14, page 267) are known to be 70.7 amperes in magnitude. Let it be required to find the magnitude of the phase currents employing the general relationship stated in equation (2).

$$I_p = \frac{70.7}{2 \sin \frac{180°}{4}} = \frac{70.7}{2 \sin 45°} = \frac{70.7}{1.414} = 50 \text{ amperes}$$

Problem 6. Find the magnitude of the line currents issuing from a balanced six-phase, mesh-connected generator if the phase currents are known to be 100 amperes in magnitude. Illustrate solution by means of a vector diagram.

Ans.: $I_L = I_p = 100$ amperes.

Problem 7. Find the voltage between *adjacent* lines of a balanced twelve-phase, star-connected system if the phase voltages are 50 volts in magnitude. Illustrate solution by means of a vector diagram.

Ans.: 25.88 volts.

Problem 8. Find the voltage between *alternate* lines of a balanced six-phase, star-connected system if the phase voltages are 132.8 volts in magnitude.

Ans.: 230 volts.

Balanced Wye Loads. When three identical impedances are connected to a common point, n, Fig. 33, they constitute a balanced wye load. If balanced three-phase voltages are impressed on such a load, it would seem that all impedances should have equal voltage drops

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across them and that the ratio and phase of line and phase voltages should be the same as those discussed for the wye-connected generators. Application of Kirchhoff's laws as discussed in the next chapter shows that this is true. Hence the voltage drop V_p across each impedance in terms of the line voltage is

$$V_p = \frac{V_L}{\sqrt{3}}$$

The current, power, etc., may then be found in accordance with single-phase circuit analysis. As a general rule, all balanced three-phase circuits are calculated on a *per phase* basis in exactly the same manner as the corresponding calculations are made for any single-phase circuit. If this procedure is followed it is important that *per phase* values of V and I are not confused with line voltages and line currents even though line currents in a wye connection are the same as the phase currents, and the line voltages in a delta connection are the same as the phase voltages. As a general rule, all balanced three-phase circuits are calculated per phase just as the calculations were made for single-phase circuits.

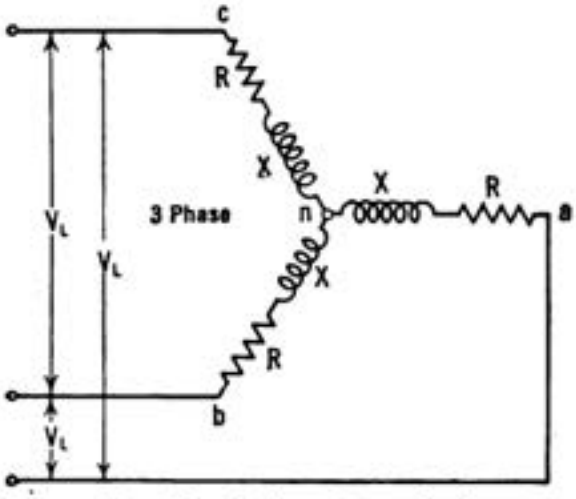

FIG. 33. Balanced wye load.

Example 2. Given the line voltages V_L in Fig. 33 as 220 volts balanced three-phase, and R and X of each phase 6 ohms resistance and 8 ohms inductive reactance. Find the line current, power per phase, and total power.

$$V_p = \frac{V_L}{\sqrt{3}} = \frac{220}{\sqrt{3}} = 127 \text{ volts}$$

$$I_L = I_p = \frac{127}{\sqrt{6^2 + 8^2}} = \frac{127}{10} = 12.7 \text{ amperes}$$

Power per phase $= I_p^2 R_p = 12.7^2 \times 6 = 968$ watts
Total power $= 3 \times 968 = 2904$ watts

The example given could have been worked by means of complex numbers. Since there was no need for the vector expressions of voltages and currents, it was simpler to use magnitudes only. When it is necessary to combine the line current due to some particular load with that from another load, the vector expressions or their equivalents are required. To illustrate the vector method of handling the above example, assume

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the phase sequence V_{ba}, V_{cb}, V_{ac}. This means that $\mathbf{V}_{cb}$ lags $\mathbf{V}_{ba}$ by 120°. It would be possible to use any line voltage or any phase voltage as a reference. The vector diagram of a similar set of voltages to those required here is shown in Fig. 22 where E is used instead of V. The phase voltage of phase na will be taken as the reference (sometimes called the standard phase). Thus:

$$\mathbf{V}_{na} = 127 + j0 \text{ volts}$$
$$\mathbf{V}_{nb} = 127 \underline{/-120^\circ} = 127 (\cos 120^\circ - j \sin 120^\circ) = -63.5 - j110 \text{ volts}$$
$$\mathbf{V}_{nc} = 127 \underline{/120^\circ} = -63.5 + j110 \text{ volts}$$

If the vector expressions for line voltages are desired, they may be obtained by the following procedure.

$$\mathbf{V}_{ba} = \mathbf{V}_{bn} + \mathbf{V}_{na} = 63.5 + j110 + 127 + j0 = 190.5 + j110 \text{ volts, etc.}$$
$$\mathbf{I}_{na} = \frac{\mathbf{V}_{na}}{\mathbf{Z}_{na}} = \frac{127 + j0}{6 + j8} = 7.62 - j10.16 = 12.7 \underline{/-53.13^\circ} \text{ amperes}$$
$$\mathbf{I}_{nb} = \frac{\mathbf{V}_{nb}}{\mathbf{Z}_{nb}} = \frac{-63.5 - j110}{6 + j8} = \frac{127 \underline{/-120^\circ}}{10 \underline{/53.13^\circ}} = 12.7 \underline{/-173.13^\circ} \text{ amperes}$$
$$\mathbf{I}_{nc} = \frac{\mathbf{V}_{nc}}{\mathbf{Z}_{nc}} = \frac{127 \underline{/120^\circ}}{10 \underline{/53.13^\circ}} = 12.7 \underline{/66.87^\circ} \text{ amperes}$$
$$P_{na} = vi + v'i' = 127 \times 7.62 = 968 \text{ watts}$$

or

$$P_{nb} = 127 \times 12.7 \cos (120^\circ - 173.13^\circ) = 968 \text{ watts}$$

The vector diagram of the voltages and currents for this load as drawn from the vector solution is shown in Fig. 34.

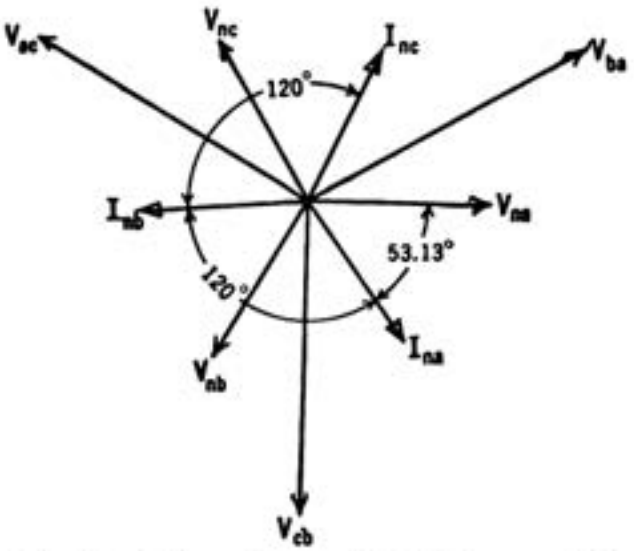

FIG. 34. Vector diagram of load in example 2.

Balanced Delta Loads. Three identical impedances connected as shown in Fig. 35 constitute a balanced delta load. The voltage drop across each impedance is known when the line voltage is given. Hence

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the phase currents may be determined directly as V_p/Z_p. The magnitudes of the line currents are simply phase currents multiplied by $\sqrt{3}$.

Example 3. Reconnect the impedances given in example 2 in delta, and calculate phase current, line current, phase power, and total power. ($R = 6$ ohms and $X = 8$ ohms per phase.)

$$V_L = V_p = 220 \text{ volts}$$

$$I_p = \frac{220}{\sqrt{6^2 + 8^2}} = 22 \text{ amperes}$$

$$I_L = \sqrt{3} \times 22 = 38.1 \text{ amperes}$$

Power per phase $= 22^2 \times 6 = 2904$ watts.

Total power $= 2904 \times 3 = 8712$ watts.

Alternative vector solution using sequence $\mathbf{V}_{ba}$, $\mathbf{V}_{cb}$, $\mathbf{V}_{ac}$. Use $\mathbf{V}_{ba}$ as the reference voltage.

$$\mathbf{V}_{ba} = 220 \angle 0^\circ \text{ volts}$$

$$\mathbf{V}_{cb} = 220 \angle -120^\circ \text{ volts}$$

$$\mathbf{V}_{ac} = 220 \angle 120^\circ \text{ volts}$$

$$\mathbf{I}_{ba} = \frac{220 \angle 0^\circ}{10 \angle 53.13^\circ} = 22 \angle -53.13^\circ = 13.2 - j17.6 \text{ amperes}$$

$$\mathbf{I}_{cb} = \frac{220 \angle -120^\circ}{10 \angle 53.13^\circ} = 22 \angle -173.13^\circ = -21.85 - j2.63 \text{ amperes}$$

$$\mathbf{I}_{ac} = \frac{220 \angle 120^\circ}{10 \angle 53.13^\circ} = 22 \angle 66.87^\circ = 8.65 + j20.2 \text{ amperes}$$

$$P_{ba} = 220 \times 22 \cos 53.13^\circ = 2904 \text{ watts}$$

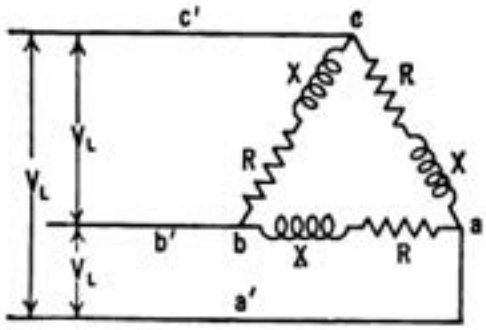

Fig. 35. Balanced delta load.

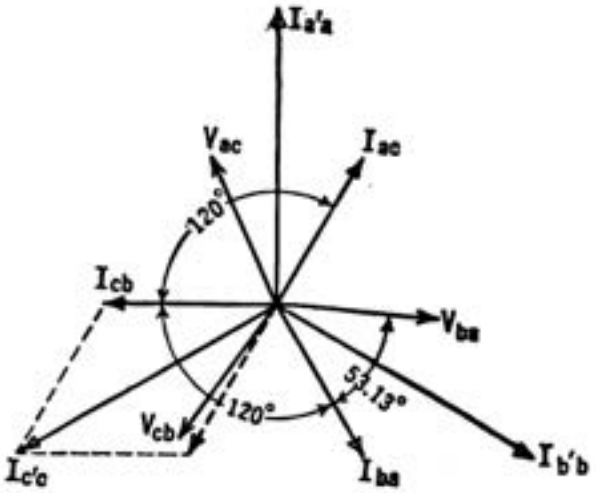

Fig. 36. Vector diagram for load of example 3.

Total power $= 3 \times 2904 = 8712$ watts.

$$\mathbf{I}_{c'c} = \mathbf{I}_{cb} + \mathbf{I}_{ca} = -30.5 - j22.8 = 38.1 \angle -143.13^\circ \text{ amperes}$$

$$\mathbf{I}_{b'b} = \mathbf{I}_{bc} + \mathbf{I}_{ba} = +35.05 - j15 = 38.1 \angle -23.13^\circ \text{ amperes}$$

$$\mathbf{I}_{a'a} = \mathbf{I}_{ab} + \mathbf{I}_{ac} = -4.55 + j37.8 = 38.1 \angle 96.87^\circ \text{ amperes}$$

The vector diagram of this delta load as drawn from the vector solution is shown in Fig. 36.

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Three-Origin Vector Diagram of a Balanced Three-Phase System. Figure 37 shows a polar vector diagram of a three-phase balanced unity-power-factor wye load. Figure 38 shows a vector diagram of a

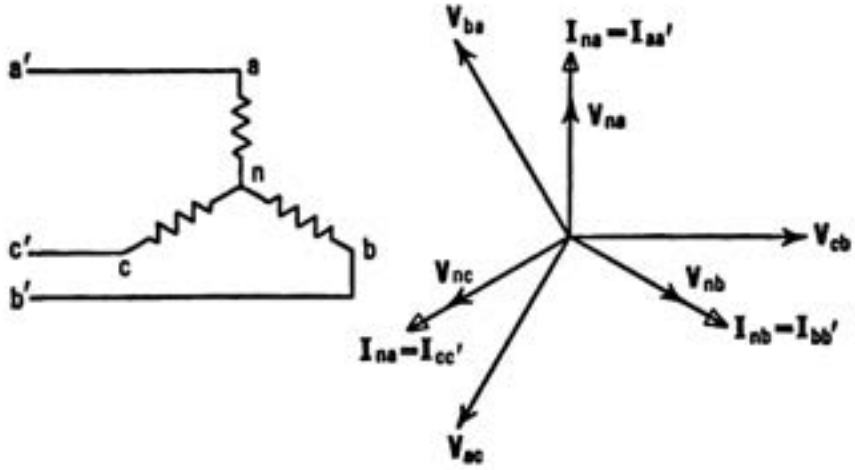

FIG. 37. Polar vector diagram of unity-power-factor, balanced wye-connected load.

balanced unity-power-factor delta load. A comparison of these two diagrams will show that the phase relation between *line* currents and *line* voltages is identical for both loads. Therefore a single vector diagram can be used to represent the relations between line currents and line

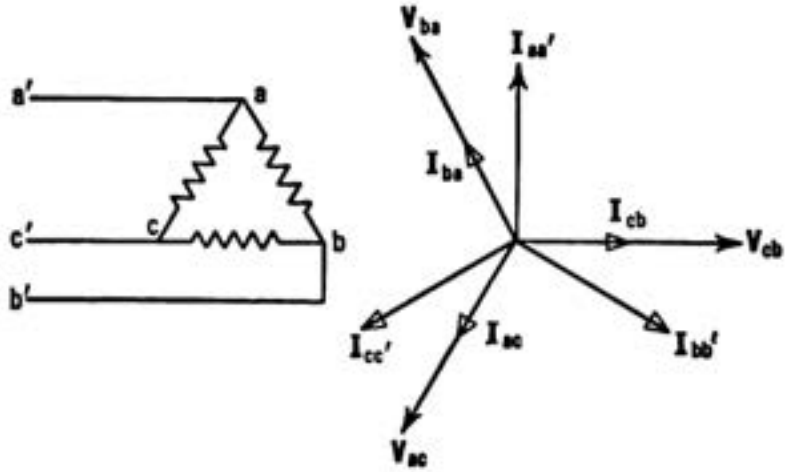

FIG. 38. Polar vector diagram of unity-power-factor, balanced delta-connected load.

voltages for a balanced three-phase load whether the load is wye- or delta-connected. In other words, it is not necessary to know which connection is used in order to represent properly the phase relations of line voltages and currents. This fact makes it convenient in many cases to use a three-origin vector diagram which is explained as follows.

If it is remembered that a vector can be translated without changing its value, the line voltages for the above loads may be arranged to form

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a closed triangle, as shown in Fig. 39. Also the line currents may be drawn from the corners of the triangle so formed as indicated. The three corners comprise the three origins; hence the name of the diagram. It will be observed that, at unity power factor, line current $I_{aa'}$ bisects the angle at origin a made by the line voltages at that point. A similar situation obtains for the other line currents. The bisectors of these angles may therefore be called the unity-power-factor positions of the line currents for a balanced three-phase load regardless of delta or wye connection. If a load having a power-factor angle of θ is to be represented, it is necessary only to let the three line currents swing from their unity-power-factor positions by the angle θ. That this is true is evident from a study of the changes in Figs. 37 and 38 when a load having a power-factor angle θ is represented. To show how the three-origin diagram might be used to represent a three-phase load, study the following example.

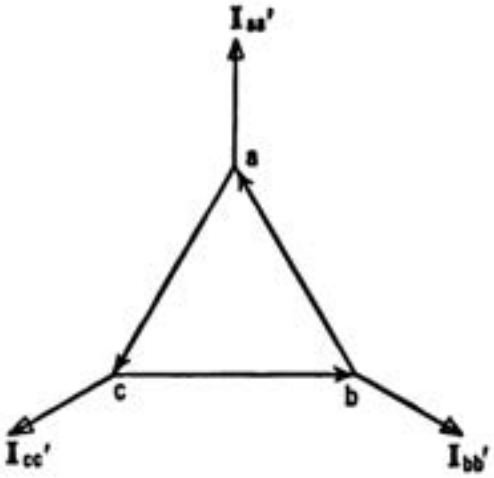

FIG. 39. Three-origin vector diagram of line voltages (cb-ac-ba) and line currents ($I_{bb'}$, $I_{cc'}$, $I_{aa'}$).

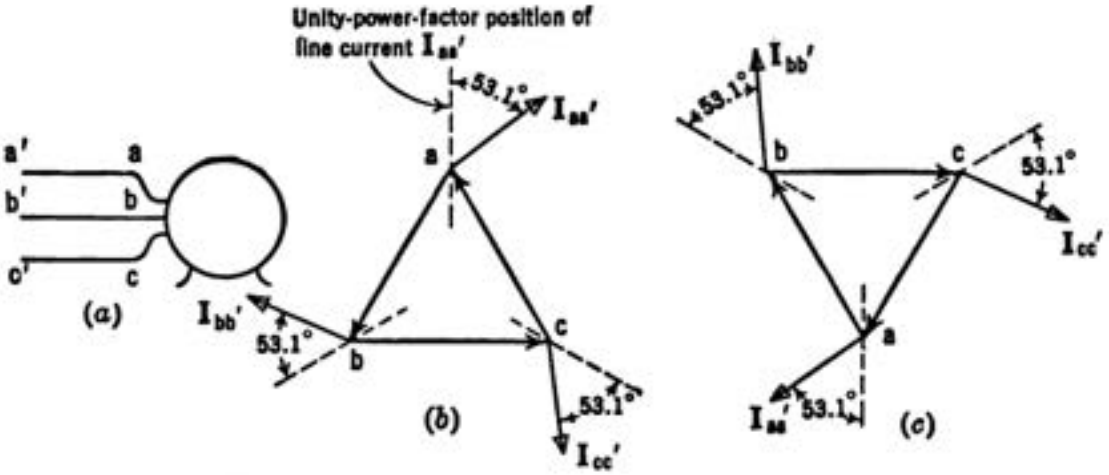

FIG. 40. Three-origin vector diagrams for both sequences of line voltages.

Example 4. A balanced three-phase, 0.6 p.f. lagging load takes 10 kva at 200 volts. Show the vector diagram of the line voltages and currents.

The load is represented by the circle, and the lines are labeled a, b, and c, as shown in Fig. 40. Assume V_{bc} as a reference, and complete the line voltage triangle as shown in (b) or (c) according to the sequence desired. The bisectors of the angles are shown dotted and are the unity-power-factor positions of the respective currents leaving points a, b, and c. The actual power-factor angle for the load is $\cos^{-1} 0.6 =$

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53.1°, and the currents are therefore drawn lagging their unity-power-factor positions by this angle, as shown. Had the load operated at a leading power factor, the currents would have swung ahead of their unity-power-factor positions by 53.1°.

The above type of diagram lends itself to a simple visualization of line voltages and currents for a balanced three-phase load and contributes to an easy understanding of operating conditions in individual transformers for certain types of connections when supplying balanced loads. They may also be used to effect the proper combination of line currents from several balanced three-phase loads independent of whether the loads themselves are delta- or wye-connected. It should be recognized from this discussion that, as far as phase relations between line currents and line voltages are concerned, one is at liberty to assume a delta- or wye-connected load even though the actual type of connection is known or unknown. Also, if convenient, the directions of the currents shown in Fig. 40 may be reversed and so labeled.

Power Calculations in Balanced Systems. The determination of power in balanced polyphase systems is based upon calculations per phase. If the voltage per phase is V_p, the phase current I_p, and the angle between them θ_p, the power per phase is:

$$P_p = V_p I_p \cos\theta_p \tag{3}$$

The power for all phases of an n-phase system is:

$$P_t = nP_p = nV_p I_p \cos\theta_p \tag{4}$$

The universality of three-phase warrants the development of equation (4) to give power in terms of line current I_L and line voltage V_L. Consider the wye connection. Then

$$\begin{aligned} P_t &= 3V_p I_p \cos\theta_p = 3\frac{V_L}{\sqrt{3}} I_L \cos\theta_p \\ &= \sqrt{3} V_L I_L \cos\theta_p \end{aligned} \tag{5}$$

For the delta connection:

$$\begin{aligned} P_t &= 3V_p I_p \cos\theta_p = 3V_L \frac{I_L}{\sqrt{3}} \cos\theta_p \\ &= \sqrt{3} V_L I_L \cos\theta_p \end{aligned} \tag{6}$$

The equations for power in terms of line voltages and line currents for *balanced three-phase* loads whether delta- or wye-connected are identical and equal to $\sqrt{3}V_L I_L \cos\theta_p$. In this expression, $\sqrt{3}V_L I_L \cos\theta_p$, for balanced three-phase power, it must be remembered that θ_p is the angle between *phase voltage and phase current* and *not* between line voltage and line current.

Problem 9. Three-phase line voltages of 2300 volts magnitude are impressed on a balanced wye-connected load which consists of 100 ohms resistance per phase

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in series with 173.2 ohms inductive reactance per phase. Find the line current and the total power taken by the three-phase load. Calculate P_t as $3I_p^2R_p$, as $3V_pI_p \cos \theta_p$, and as $\sqrt{3}V_LI_L \cos \theta_p$.

Ans.: $I_L = I_p = 6.64$ amperes, $P_t = 13.22$ kw.

Problem 10. Repeat Problem 9, assuming that the three impedances are connected in delta (rather than in wye) across the same line voltages.

Ans.: $I_L = 19.92$ amperes, $P_t = 39.66$ kw.

Volt-Amperes. The volt-amperes of a *balanced* three-phase system are defined as the sum of the volt-amperes of the separate phases or three times the number of volt-amperes per phase. Hence

$$\text{va}_t = 3\text{va}_p = 3V_pI_p$$

In terms of line voltage and line current, volt-amperes are:

For delta:
$$3V_L\frac{I_L}{\sqrt{3}} = \sqrt{3}V_LI_L \tag{7}$$

For wye:
$$3\frac{V_L}{\sqrt{3}}I_L = \sqrt{3}V_LI_L \tag{8}$$

For an n-phase system under balanced conditions the total volt-amperes are n times the volt-amperes per phase.

Reactive Volt-Amperes. The reactive volt-amperes for a balanced three-phase system are defined as the sum of the reactive volt-amperes for each phase, or three times the reactive volt-amperes per phase. In terms of line voltage and line current the reactive volt-amperes or reactive power is:

For wye:
$$P_X = 3V_pI_p \sin \theta_p = 3\frac{V_L}{\sqrt{3}}I_L \sin \theta_p$$
$$= \sqrt{3}V_LI_L \sin \theta_p \tag{9}$$

For delta:
$$P_X = 3V_pI_p \sin \theta_p = 3V_L\frac{I_L}{\sqrt{3}} \sin \theta_p$$
$$= \sqrt{3}V_LI_L \sin \theta_p \tag{10}$$

Summarizing for either *balanced* delta or wye, the totals for the systems are:

$$P = \sqrt{3}V_LI_L \cos \theta_p \tag{11}$$

$$\text{va} = \sqrt{3}V_LI_L \tag{12}$$

$$P_X = \sqrt{3}V_LI_L \sin \theta_p \tag{13}$$

The sine of the angle between phase voltage and phase current ($\sin \theta_p$) is called the *reactive factor* of a *balanced* system.

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Problem 11. Three-phase line voltages of 440 volts are impressed on a balanced delta-connected load which consists of 8 ohms resistance in series with 6 ohms inductive reactance per phase.

(*a*) Find the volt-amperes per phase, the reactive volt-amperes per phase, and the reactive factor of each phase.

Ans.: $va_p = 19{,}360$, $rva_p = 11{,}616$, r.f. $= 0.6$.

(*b*) Find the total volt-amperes of the system, the total reactive volt-amperes of the system, and the reactive factor of the system.

Ans.: $va_t = 58{,}080$, $rva_t = 34{,}848$, r.f. $= 0.6$.

Power Factor. The power factor of a balanced three-phase system, when the wave forms of voltage and current are sinusoidal, is defined as the cosine of the angle between *phase voltage* and *phase current* independent of whether the connection is delta or wye. It should be noted that the volt-amperes of equation (12) are equal to $\sqrt{P^2 + P_X{}^2}$. Thus

$$\begin{aligned} \text{va} &= \sqrt{(\sqrt{3}V_L I_L \cos\theta_p)^2 + (\sqrt{3}V_L I_L \sin\theta_p)^2} \\ &= \sqrt{3}V_L I_L \sqrt{\cos^2\theta_p + \sin^2\theta_p} = \sqrt{3}V_L I_L \end{aligned} \tag{14}$$

From equation (11),

$$\text{p.f.} = \cos\theta_p = \frac{P}{\sqrt{3}V_L I_L} \tag{15}$$

From equation (13),

$$\text{r.f.} = \sin\theta_p = \frac{P_X}{\sqrt{3}V_L I_L} \tag{16}$$

From equations (15) and (14),

$$\text{p.f.} = \frac{P}{\sqrt{P^2 + P_X{}^2}} \tag{17}$$

From equations (16) and (14),

$$\text{r.f.} = \frac{P_X}{\sqrt{P^2 + P_X{}^2}} \tag{18}$$

Example 5. A 5-horsepower, 220-volt, three-phase motor has an efficiency of 85 per cent and operates at 86 per cent power factor. Find the line current.

$$\text{Power input} = \sqrt{3}V_L I_L \text{ p.f.} = \frac{5 \times 746}{0.85} = 4390 \text{ watts}$$

$$I_L = \frac{4390}{\sqrt{3}\,220 \times 0.86} = 13.4 \text{ amperes}$$

Balanced Three-Phase Loads in Parallel. The combination of a number of balanced loads which are in parallel may be effected through changing all loads to equivalent delta loads and then combining the impedances of corresponding phases according to the law governing parallel circuits. Also all loads may be changed to equivalent wye loads

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and the impedances of corresponding phases paralleled. In addition to these methods, the power of the several loads may be added arithmetically and the reactive volt-amperes may be added algebraically. The total volt-amperes will then be obtained as $\sqrt{P^2 + P_X{}^2}$.

Example 6. A 3-phase motor takes 10 kva at 0.6 power factor lagging from a source of 220 volts. It is in parallel with a balanced delta load having 16 ohms resistance and 12 ohms capacitive reactance in series in each phase. Find the total volt-amperes, power, line current, and power factor of the combination.

Solution a. Assume motor to be Y-connected.

$$\text{Motor line current} = \text{phase current} = \frac{10{,}000}{\sqrt{3}\,220} = 26.25 \text{ amperes}$$

$$\text{Equivalent impedance per phase of motor} = \frac{220}{\sqrt{3}\,26.25} = 4.84 \text{ ohms}$$

$$R = 4.84 \cos\theta = 4.84 \times 0.6 = 2.904 \text{ ohms}$$

$$X = 4.84 \sin\theta = 4.84 \times 0.8 = 3.872 \text{ ohms}$$

$$\text{Equivalent wye of delta load } \mathbf{Z}_p = \frac{16 - j12}{3} = 5.33 - j4 \text{ ohms}$$

$$\mathbf{Z}_0 = \frac{(5.33 - j4)(2.904 + j3.872)}{5.33 - j4 + 2.904 + j3.872} = 3.91 \underline{/17.17^\circ} \text{ ohms}$$

$$I_0 = \frac{220}{\sqrt{3}\,3.91} = 32.5 \text{ amperes}$$

$$\text{va} = \sqrt{3}\,220 \times 32.5 = 12{,}370 \text{ volt-amperes}$$

$$\text{p.f.}_0 = \cos 17.1^\circ = 0.955$$

$$P = 12{,}370 \times 0.955 = 11{,}810 \text{ watts}$$

Solution b. Assume motor to be delta-connected.

$$\text{Motor line current} = \frac{10{,}000}{\sqrt{3}\,220} = 26.25 \text{ amperes}$$

$$\text{Motor phase current} = \frac{26.25}{\sqrt{3}} = 15.14 \text{ amperes}$$

$$\text{Equivalent impedance per phase of motor} = \frac{220}{15.14} = 14.52 \text{ ohms}$$

$$R = 14.52 \cos\theta = 14.52 \times 0.6 = 8.712 \text{ ohms}$$

$$X = 14.52 \times 0.8 = 11.616 \text{ ohms}$$

The delta-phase impedance of $16 - j12$ is now paralleled with the motor phase impedance $8.712 + j11.616 = 14.52 \underline{/53.1^\circ}$.

$$\mathbf{Z}_0 = \frac{(16 - j12)14.52 \underline{/53.1^\circ}}{16 - j12 + 8.712 + j11.616} = 11.73 \underline{/17.17^\circ} \text{ ohms}$$

$$I_0 = \sqrt{3}\,\frac{220}{11.73} = 32.5 \text{ amperes}$$

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The rest of the requirements are now obtained as in solution *a*.

Solution c. For the delta load, phase current is $220/\sqrt{16^2 + 12^2} = 11$ amperes.

$$P = 11^2 \times 16 \times 3 = 5810 \text{ watts for all phases}$$
$$P_X = 11^2 \times 12 \times 3 = 4350 \text{ vars for all phases (capacitive)}$$

For the motor

$$P = 10 \times 0.6 = 6 \text{ kw}$$
$$P_X = 10 \times 0.8 = 8 \text{ kilovars (inductive)}$$
$$\text{Summation of power} = 5.81 + 6 = 11.81 \text{ kw}$$
$$\text{Summation of kilovars} = 4.35 - 8 = -3.65 \text{ kilovars}$$
$$\text{kva}_0 = \sqrt{11.81^2 + 3.65^2} = 12.37$$
$$I_0 = \frac{12{,}370}{\sqrt{3}\,220} = 32.5 \text{ amperes}$$
$$\text{p.f.}_0 = \frac{11.81}{12.37} = 0.955$$

Of the three solutions, that which is most convenient for the quantities given should be employed.

Single-Phase and Balanced Three-Phase Power. A comparison of the variation with respect to time of instantaneous single-phase and three-phase power brings out certain fundamental differences. As shown in Chapter II, single-phase power follows a double-frequency sine law with respect to time plus a constant. The instantaneous power for each of three phases, when currents and voltages are sine waves, of a balanced three-phase system is given by the following general equations.

$$p_a = V_m I_m \sin \omega t \sin (\omega t - \theta)$$
$$p_b = V_m I_m \sin (\omega t - 120°) \sin (\omega t - 120° - \theta)$$
$$p_c = V_m I_m \sin (\omega t - 240°) \sin (\omega t - 240° - \theta)$$

The total three-phase power is

$$\begin{aligned} p_3 = P_a + P_b + P_c = V_m I_m [&\sin \omega t \sin (\omega t - \theta) \\ &+ \sin (\omega t - 120°) \sin (\omega t - 120° - \theta) \\ &+ \sin (\omega t - 240°) \sin (\omega t - 240° - \theta)] \end{aligned}$$
$$p_3 = 1.5 V_m I_m \cos \theta \tag{19}$$

For single-phase, say phase *a*,

$$\begin{aligned} p_1 &= V_m I_m \sin \omega t \sin (\omega t - \theta) \\ &= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} \cos (2\omega t - \theta) \end{aligned} \tag{20}$$

Equation (19) shows the instantaneous value of three-phase power to be independent of time. In other words, balanced three-phase power

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under steady-state conditions is constant from instant to instant. In contrast, equation (20) for single-phase power shows it to follow a double-frequency variation with respect to time. This comparison is graphically illustrated in Fig. 41.

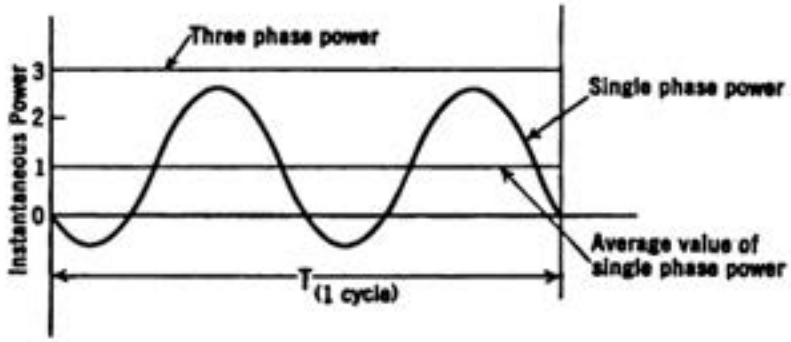

FIG. 41. Comparison of variations of single- and balanced three-phase power.

Power Measurement in Balanced Systems. A wattmeter gives a reading proportional to the product of the current through its current coil, the voltage across its potential coil, and the cosine of the angle between this voltage and current. Since the total power in a three-phase circuit is the sum of the powers of the separate phases, the total power

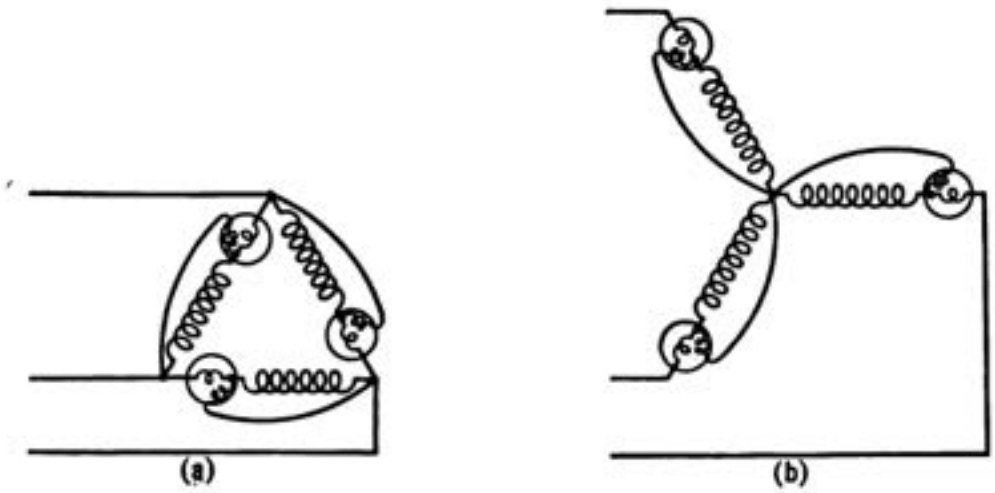

FIG. 42. A wattmeter in each phase may be used to measure three-phase power.

could be measured by placing a wattmeter in each phase, as shown in Fig. 42. It is not generally feasible to break into the phases of a delta-connected load. Therefore the method shown in part (*a*) of Fig. 42 is not applicable. For the wye load shown in part (*b*), it is necessary to connect to the neutral point. This point is not always accessible. Hence another method making use of only two wattmeters is generally employed in making three-phase power measurements. This con-

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nection is shown in Fig. 43. To show that two such wattmeters may be used to measure power, the readings of each will be established and their sum compared with equation (11), which has been shown to be correct for balanced three-phase power. It is important to take the direction of the voltage through the circuit the same as that taken for current when establishing wattmeter readings. Thus if the current coil of W_a, Fig. 43, is considered carrying current I_{an}, the potential across the voltage coil should be taken from a through the circuit, which in

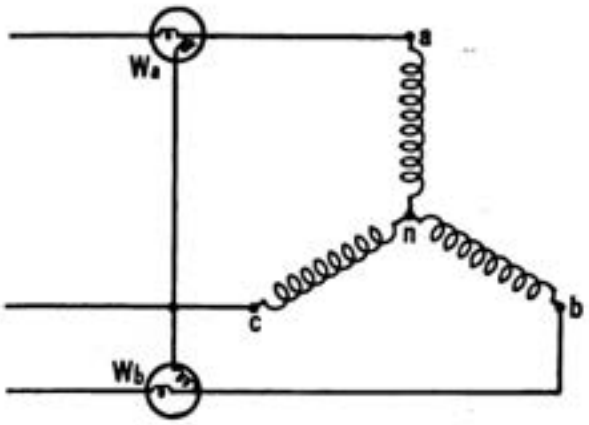

FIG. 43. Connection of two wattmeters to measure three-phase power.

this particular case is V_{ac}. Figure 44 shows the vector diagram of the voltages and currents for a balanced system like that of Fig. 43. From this figure the power represented by the currents and voltages of each wattmeter is:

$$W_a = V_{ac}I_{an} \cos (\theta - 30°) \tag{21}$$

$$W_b = V_{bc}I_{bn} \cos (\theta + 30°) \tag{22}$$

In equations (21) and (22) the subscripts serve only to assist in seeing which voltages and currents were used. Since the load is balanced, $V_{ac} = V_{bc}$, $I_{an} = I_{bn}$ and only magnitudes are involved. Dropping the subscripts gives

$$W_a = VI \cos (\theta - 30°) \tag{23}$$

$$W_b = VI \cos (\theta + 30°) \tag{24}$$

$$\begin{aligned} W_a + W_b &= VI \cos (\theta - 30°) + VI \cos (\theta + 30°) \\ &= VI [\cos \theta \cos 30° + \sin \theta \sin 30° + \cos \theta \cos 30° - \sin \theta \sin 30°] \\ &= \sqrt{3}VI \cos \theta \end{aligned} \tag{25}$$

Hence $W_a + W_b$ correctly measures the power in a balanced three-phase system of any power factor. As will be shown later, the algebraic sum of the readings of two wattmeters will give the correct value for power under any conditions of unbalance, wave form, or power factor.

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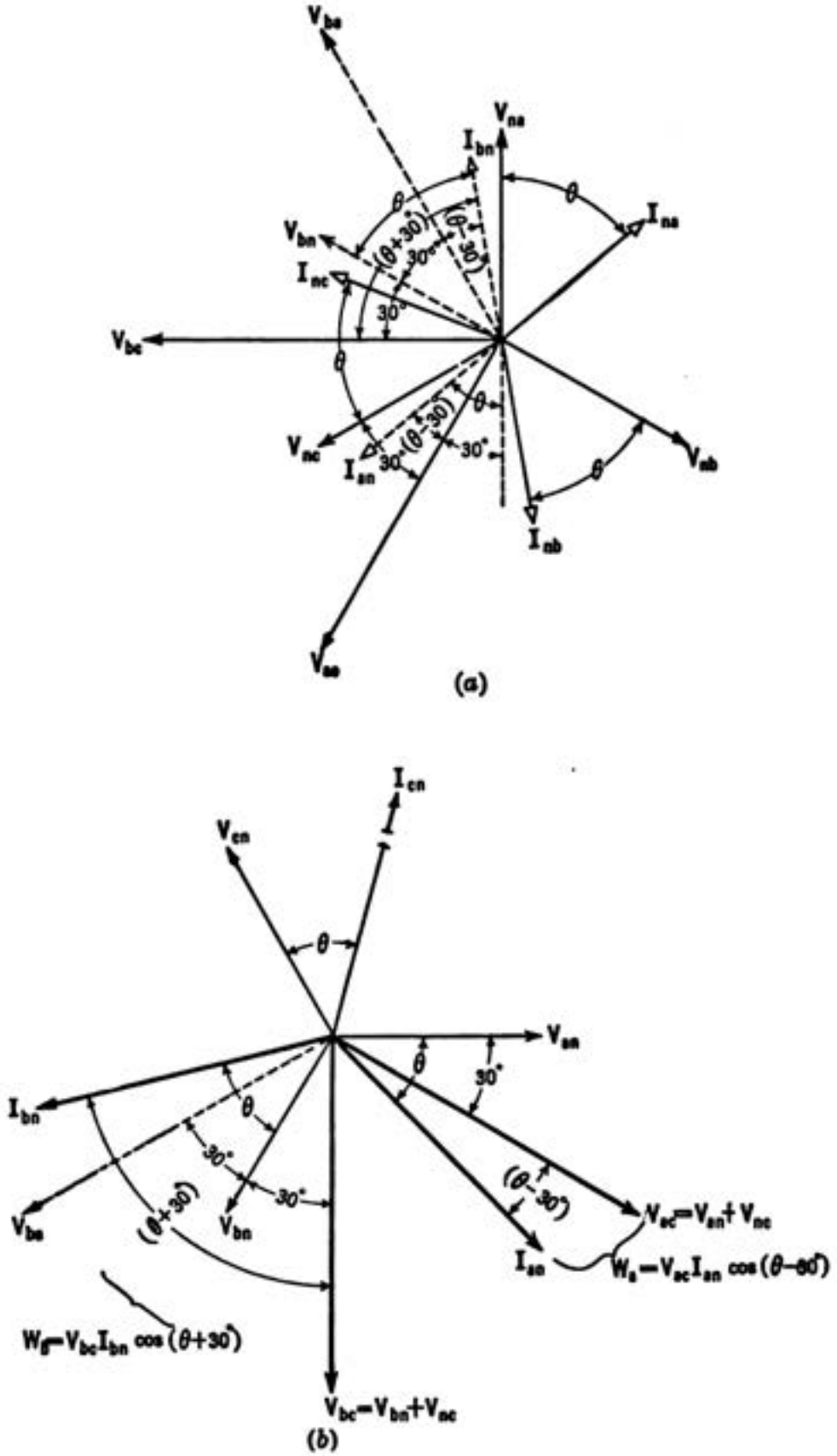

FIG. 44. Alternative ways of drawing the vector diagrams for a power-factor angle θ of the system shown in Fig. 43.

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For each value of θ (i.e., for each power factor) there is a definite ratio of W_a/W_b. If the ratio of the smaller to the larger reading is always taken and plotted against the corresponding $\cos\theta$ (i.e., power factor), a curve called the watt ratio curve results. This curve is shown in Fig. 45. Reference to the vector diagram of Fig. 44 and the curve of Fig. 45 shows that at 0.5 power factor one wattmeter reads zero. For the case under discussion 0.5 lagging power factor makes W_b read zero, while 0.5 leading power factor makes W_a read zero. When the power factor is zero, each wattmeter has the same deflection but the readings are of opposite signs. The foregoing facts are easily deducible from the vector diagram shown in Fig. 44 and also follow from equations (23) and (24). It is essential in the two-wattmeter method that the proper sign be given the wattmeter readings and that the sum be taken algebraically.

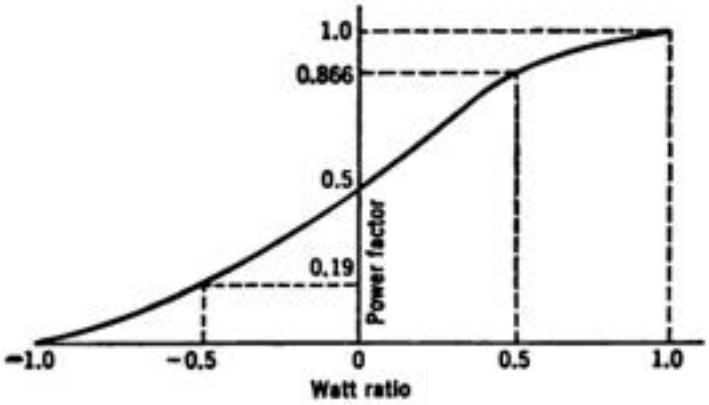

FIG. 45. Watt ratio curve for two-wattmeter method of measuring power (applicable only to balanced loads).

There are several ways to determine whether a wattmeter reading should be taken positive or negative. One of the best methods follows. Refer to Fig. 43. Open line a. Then all power must be transferred to the load over lines b and c. If wattmeter b is connected so that it reads "up scale," it will then be known to have this deflection when the power it reads is going to the load. Next reconnect line a and open line b. Then connect W_a so that it reads up scale. Now close line b. If at any time after this either wattmeter needle goes backward against the down-scale stop, power through this wattmeter channel is being transferred to the generator and this power must be of opposite sign to that registered by the other. Either the potential or current coil will have to be reversed to secure an up-scale reading. The foregoing test is applicable under any conditions of loading, although it may not always be feasible because of the necessity for opening the lines.

A second test applicable only when the load is practically *balanced*

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is to disconnect from the common potential point c of Fig. 43 the potential coil of the wattmeter which has the smaller reading and connect it to the line containing the current coil of the other wattmeter. If the needle goes against the down-scale stop, the wattmeter reading was negative. The foregoing is best explained through a consideration of the circuit diagram of Fig. 43 and the corresponding vector diagram of Fig. 44. As previously shown, W_a reads the power represented by V_{ac} and I_{an} while W_b reads that due to V_{bc} and I_{bn}. Since the angle $(\theta + 30°)$ between V_{bc} and I_{bn} is larger than the angle $(\theta - 30°)$ between V_{ac} and I_{an} for the load represented by Fig. 44, wattmeter W_b will have the smaller deflection. If the potential coil of W_b is now removed from line c in Fig. 43 and connected to line a, the meter will deflect because of the potential V_{ba} and current I_{bn}. The angle between V_{ba} and I_{bn} is seen to be $(\theta - 30°)$ or the same as that between the voltage and current for wattmeter W_a. W_a and W_b will then read alike. Furthermore, since W_b was connected to read up scale when the angle between its voltage and current was less than 90°, it will continue to read up scale when it receives the potential V_{ba}. If, however, the power factor was below 0.5, the angle $(\theta + 30°)$ on Fig. 44 would be more than 90°. If the wattmeter W_b were made to read up scale under such conditions, it would reverse its deflection when given the potential V_{ba} as outlined above since it would then be subjected to a voltage and current of $(\theta - 30°)$, which is less than 90° out of phase. When the potential coil connection of W_b is moved from line c to a in Fig. 43, this wattmeter receives a potential of V_{ba}, while that for W_a (taken similarly from the line containing the current coil) is V_{ac}. These potentials are in the same order or direction around the diagram. Hence the potential coils are said to be connected in the same cyclic order about the circuit, and under these conditions both wattmeters would be expected to show the same deflection. This was found to be true in the above analysis.

Example 7. In a circuit like that shown in Fig. 43, W_a reads 800 and W_b reads 400 watts. When the potential coil of W_b is disconnected at c and connected at a, the needle goes against the down-scale stop.

Solution. The test indicates that W_b is reading -400 watts. Hence:

$$P = W_a + W_b = 800 + (-400) = 400 \text{ watts}$$

$$\text{Watt ratio} = \frac{W_b}{W_a} = \frac{-400}{800} = -0.5$$

From a watt ratio curve like that shown on page 288, the power factor may be determined directly as 0.19.

The power factor, $\cos\theta$, could also have been calculated from a simultaneous

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solution of equations (23) and (24) since:

$$\cos\theta = \cos\left(\tan^{-1}\frac{\sqrt{3}\,(W_a - W_b)}{W_a + W_b}\right)$$

This relation is made apparent in the next article.

Reactive Volt-Amperes. The reactive volt-amperes in a balanced three-phase circuit may be expressed by:

$$P_X = \sqrt{3}\,(W_a - W_b) \tag{26}$$

This may be shown as follows:

$$\begin{aligned}\sqrt{3}\,(W_a - W_b) &= \sqrt{3}\,[VI\cos(\theta - 30°) - VI\cos(\theta + 30°)]\\ &= \sqrt{3}VI\,[\cos\theta\cos 30° + \sin\theta\sin 30° - \cos\theta\cos 30°\\ &\quad + \sin\theta\sin 30°]\\ &= \sqrt{3}VI\sin\theta\end{aligned}$$

This is the same as equation (13) for reactive power given on page 281. Since the ratio of the reactive volt-amperes, $\sqrt{3}V_LI_L\sin\theta$, to the power, $\sqrt{3}V_LI_L\cos\theta$, is the $\tan\theta$, it follows from equations (25) and (26) that

$$\tan\theta = \frac{\sqrt{3}\,(W_a - W_b)}{W_a + W_b} \tag{27}$$

where θ is the power-factor angle.

Example 8. The power factor in the preceding example could have been easily calculated by means of the relation stated in equation (26). Thus:

$$P_X = \sqrt{3}\,(W_a - W_b) = \sqrt{3}\,[800 - (-400)] = 2078 \text{ vars}$$

$$(P = W_a + W_b = 800 - 400 = 400 \text{ watts})$$

$$\text{va} = \sqrt{P^2 + P_X^2} = \sqrt{400^2 + 2078^2} = 2114 \text{ volt-amperes}$$

$$\text{p.f.} = \frac{P}{\text{va}} = \frac{400}{2114} = 0.19$$

Three-Phase, Four-Wire Systems. If a three-phase, four-wire system is balanced, the fourth wire or neutral will carry no current. The system is the same as when the neutral is omitted, in which case it is the same as a balanced three-phase, three-wire system. It can therefore be metered as previously shown for the three-wire system. Another method is given later. Under any other conditions three meters or their equivalent are necessary. Unbalanced systems are considered in the next chapter.

Delta Systems. The measurement of power in a three-phase system was discussed with reference to a wye-circuit diagram and the corre-

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sponding vector diagram. When it is remembered that a delta system can always be replaced by an equivalent wye system, the preceding discussion will be seen to apply to the delta system. Furthermore only line voltages and line currents were involved in the discussion of the two-wattmeter method of measuring power, and there is no difference between these quantities for the delta and wye systems.

Oscillograms 3 and 4, which were obtained from a delta system as shown and labeled in Fig. 46, may be profitably studied.

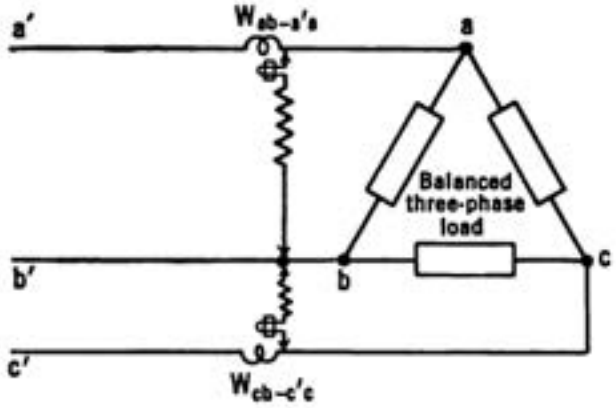

FIG. 46. Circuit arrangement for which Oscillograms 3 and 4 were taken.

Problem 12. Refer to Oscillogram 3. (*a*) If the line-to-line voltages have instantaneous maximum values of 155.5 volts and the delta-line currents have instantaneous maximum values of 14.14 amperes, find the average power readings of the wattmeters $W_{ab-a'a}$ and $W_{cb-c'c}$.

(*b*) Draw a vector diagram indicating all currents and voltages shown on Oscillogram 3. Use $\mathbf{V}_{ab}$ as reference, and include the delta-phase currents $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, and $\mathbf{I}_{ca}$ which are not shown on the oscillogram but which combine to form the delta-line currents $\mathbf{I}_{a'a}$ and $\mathbf{I}_{c'c}$.

Ans.: (*a*) $W_{ab-a'a} = W_{cb-c'c} = 952.6$ watts.

(*b*) *ab-bc-ca* sequence of line-to-line voltages; $\mathbf{I}_{ab}$ in time phase with $\mathbf{V}_{ab}$; $\mathbf{I}_{a'a}$ lags $\mathbf{V}_{ab}$ by 30°; $\mathbf{I}_{c'c}$ leads $\mathbf{V}_{cb}$ by 30°.

General *n*-Wire Balanced System. The total power taken by a balanced *n*-phase system is *n* times the power per phase. A single wattmeter connected to measure the product of the current, potential, and the cosine of the angle between the current and potential may be used to obtain the power of a balanced *n*-phase system. The wattmeter reading obtained is multiplied by *n*. If it is not possible to break into a phase of a mesh-connected load or to obtain the neutral of a star-connected one, power may still be measured with a single wattmeter. For the *n*-phase system, *n* equal resistances may be connected in star and then to the lines. A neutral is thus established, and power is measured as though the neutral wire of a star system were available.

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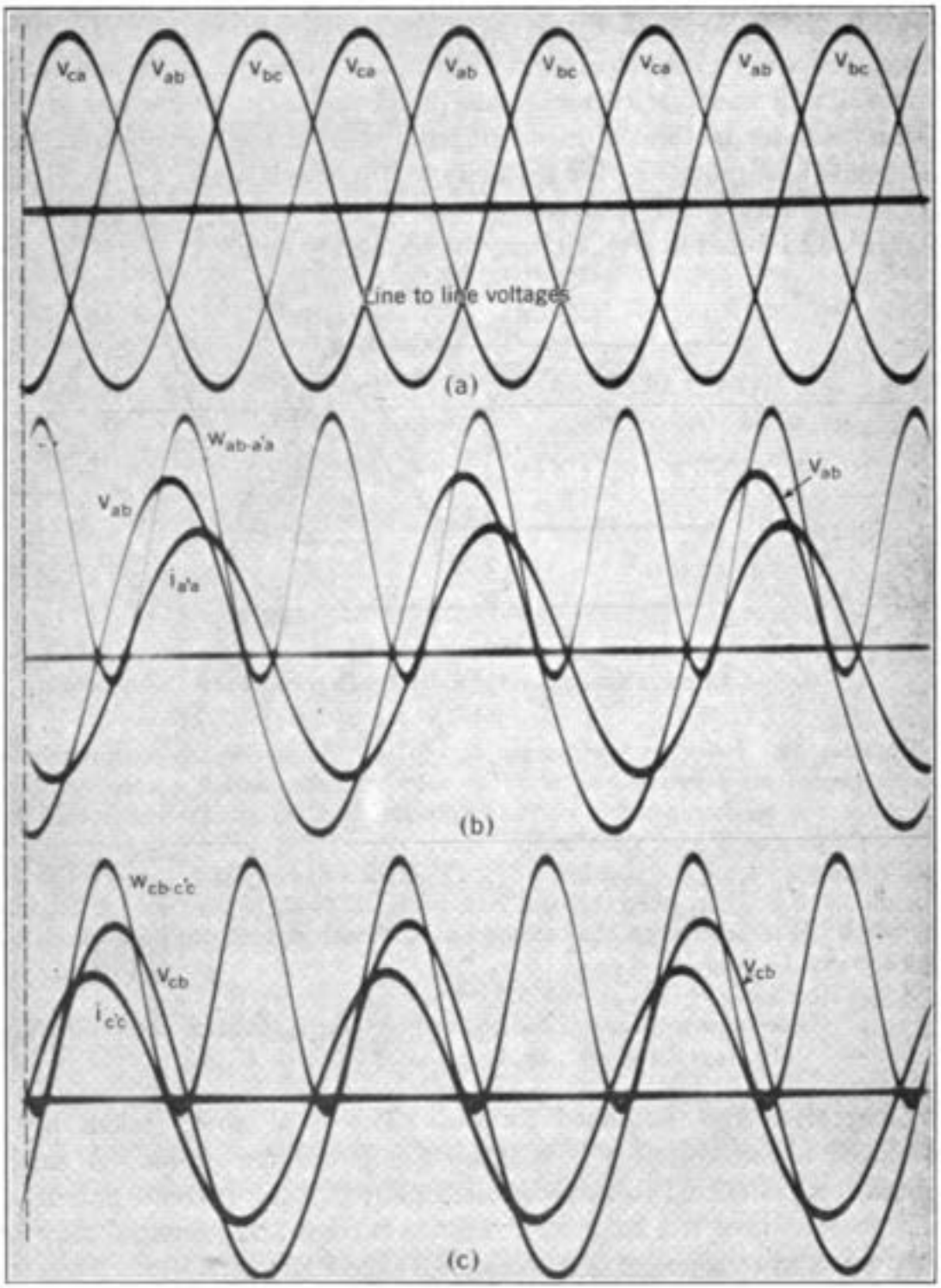

OSCILLOGRAM 3. Oscillographic representation of all voltages and currents involved in the two-wattmeter method of measuring balanced three-phase power at unity power factor. In (*a*) the sequence of line-to-line voltages is shown. v_{ca} is the voltage not used. In (*b*) $w_{ab-a'a}$ is a graph of the instantaneous driving torque of the wattmeter element which is operated by v_{ab} and $i_{a'a}$. In (*c*) $w_{cb-c'c}$ is a graph of the instantaneous driving torque of the wattmeter element which is operated by v_{cb} and $i_{c'c}$.

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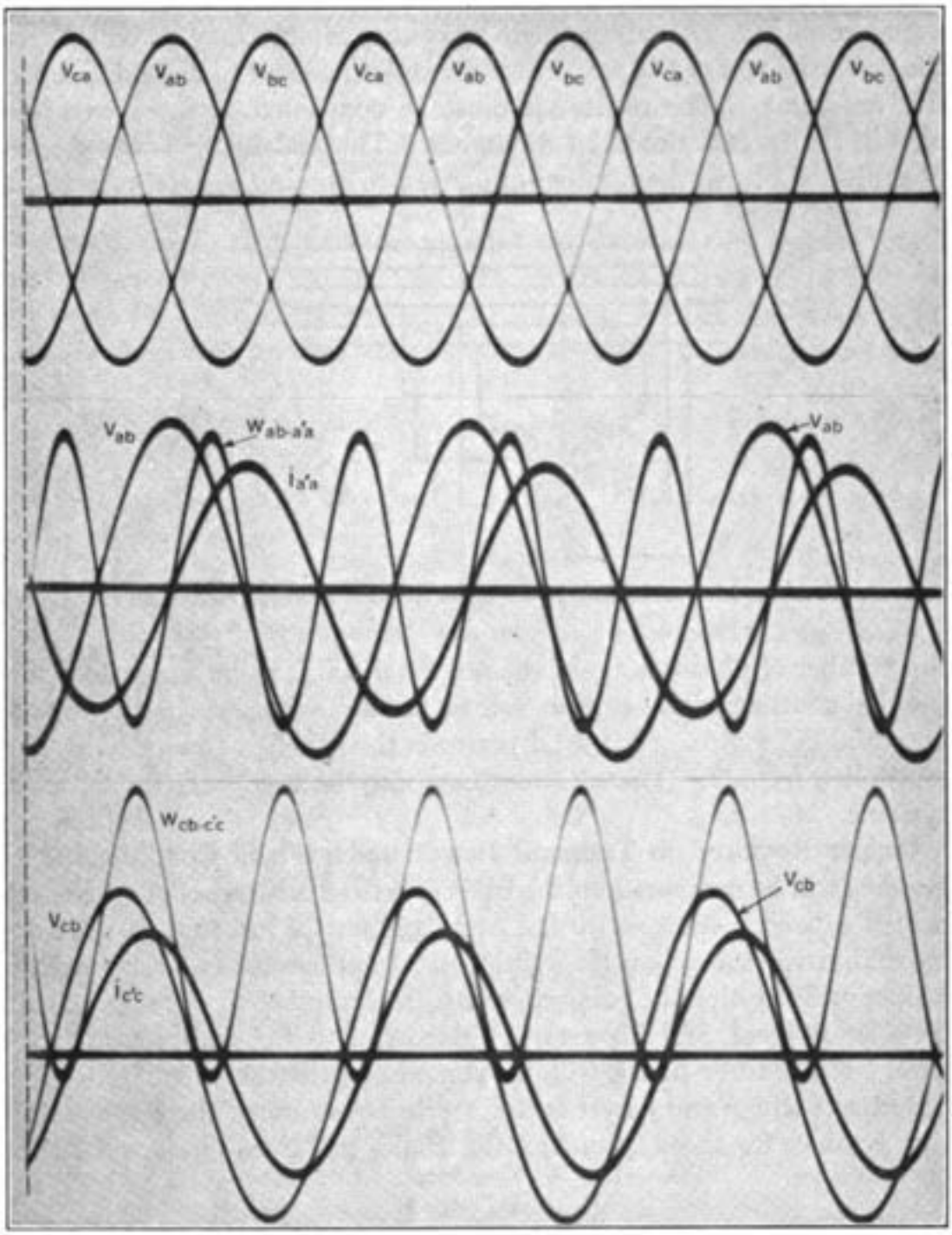

Oscillogram 4. Oscillographic representation of all voltages and currents involved in the two-wattmeter method of measuring balanced three-phase power at 0.5 p.f. lag, the condition under which one wattmeter reads zero. In the upper oscillogram, the sequence of line-to-line voltages is shown. The voltage v_{ca} is the voltage not used in the two-wattmeter method in this case. (In the center oscillogram, $w_{ab-a'a}$ is a graph of the instantaneous driving torque of the wattmeter element which is operated by v_{ab} and $i_{a'a}$. In the lower oscillogram, $w_{cb-c'c}$ is a graph of the instantaneous driving torque of the wattmeter element which is operated by v_{cb} and $i_{c'c}$.

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The method is shown in Fig. 47. If the number of phases is even, as, for example, in Fig. 47, only a single resistance is necessary provided that the potential coil of the wattmeter can be connected at the midpoint of this resistance. The resistance must be connected between two lines having the largest potential difference. The wattmeter reading must be multiplied by n, the number of phases, to obtain the total power. If

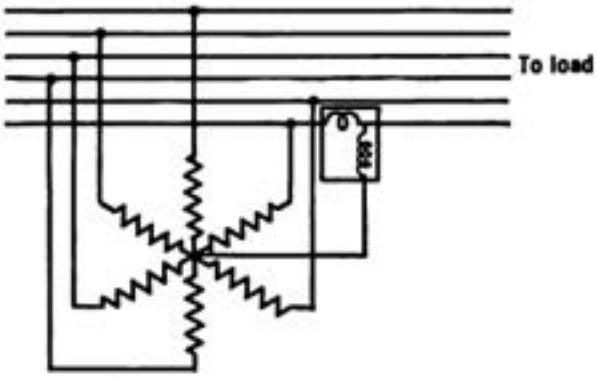

FIG. 47. A method for measuring power to an n-phase balanced load (load not shown).

the number of phases is even, the potential coil may be connected from the line containing the current coil to the line which yields the highest potential difference. The total power is then the wattmeter indication multiplied by $n/2$. These connections may be used only for balanced systems.

Copper Required to Transmit Power under Fixed Conditions. All systems will be compared on the basis of a fixed amount of power transmitted a fixed distance with the same amount of loss and at the same maximum voltage between conductors. In all cases the total weight of copper will be directly proportional to the number of wires, since the distance is fixed, and inversely proportional to the resistance of each wire. First, three-phase will be compared with single-phase. Since the same voltage and power factor are to be assumed, the same respective symbols for these quantities for single- and three-phase will suffice.

$$P_1 = VI_1 \cos\theta$$

$$P_3 = \sqrt{3}VI_3 \cos\theta$$

Since

$$P_1 = P_3$$

$$VI_1 \cos\theta = \sqrt{3}VI_3 \cos\theta$$

$$I_1 = \sqrt{3}I_3$$

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Also $$I_1^2R_1 \times 2 = I_3^2R_3 \times 3$$

or $$\frac{R_1}{R_3} = \frac{3I_3^2}{2I_1^2} = \frac{3I_3^2}{3I_3^2 \times 2} = \frac{1}{2}$$

$$\frac{\text{Copper three-phase}}{\text{Copper single-phase}} = \frac{\text{No. of wires three-phase}}{\text{No. of wires single-phase}} \times \frac{R_1}{R_3} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$$

The above shows that the same amount of power may be transmitted a fixed distance with a fixed line loss with only three-fourths of the amount of copper that would be required for single-phase, or one-third more copper is required for single-phase than would be necessary for three-phase.

Comparison of Three-Phase with Four-Phase.

$$P_3 = \sqrt{3}VI_3 \cos\theta$$

$$P_4 = 4\frac{V}{2}I_4 \cos\theta$$

(*Note:* V is highest voltage between any pair of wires.) Therefore

$$\sqrt{3}VI_3 \cos\theta = 4\frac{V}{2}I_4 \cos\theta$$

$$\sqrt{3}I_3 = \frac{4}{2}I_4$$

$$\frac{I_3}{I_4} = \frac{2}{\sqrt{3}}$$

$$3I_3^2R_3 = 4I_4^2R_4$$

$$\frac{R_4}{R_3} = \frac{3I_3^2}{4I_4^2} = \frac{3}{4} \times \frac{4}{3} = 1$$

$$\frac{\text{Copper three-phase}}{\text{Copper four-phase}} = \frac{3}{4} \times \frac{1}{1} = \frac{3}{4}$$

This is the same relation as shown for single-phase. If other systems are compared with three-phase in this manner, it will be found that three-phase is more economical in the use of copper than any other number of phases.

When a fixed amount of power is transmitted a fixed distance with a fixed loss for the same voltage to neutral, there is no difference between any of the systems. Consider three-phase and single-phase. The voltage to neutral single-phase is half the voltage between lines. This

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point is called the neutral, since the potential from either line to it is the same.

$$P_3 = P_1$$

$$3V_nI_3 \cos\theta = 2V_nI_1 \cos\theta$$

$$\frac{I_3}{I_1} = \frac{2}{3}$$

$$3I_3^2R_3 = 2I_1^2R_1$$

$$\frac{R_1}{R_3} = \frac{3I_3^2}{2I_1^2} = \frac{3}{2} \times \frac{4}{9} = \frac{2}{3}$$

$$\frac{\text{Copper three-phase}}{\text{Copper single-phase}} = \frac{3}{2} \times \frac{2}{3} = 1 \quad \text{(for same voltage to neutral)}$$

Comparison of Three-Phase with n-Phase for the Same Voltage to Neutral.

$$P_3 = P_n$$

$$3V_nI_3 \cos\theta = nV_nI_n \cos\theta$$

$$\frac{I_3}{I_n} = \frac{n}{3}$$

$$3I_3^2R_3 = nI_n^2R_n$$

$$\frac{R_n}{R_3} = \frac{3}{n}\frac{I_3^2}{I_n^2} = \frac{3}{n}\frac{n^2}{3^2} = \frac{n}{3}$$

$$\frac{\text{Copper three-phase}}{\text{Copper } n\text{-phase}} = \frac{3}{n}\frac{n}{3} = 1 \quad \text{(for same voltage to neutral)}$$

There is no difference in the amount of copper required between any of the systems if the voltage to neutral is fixed and if the same amount of power is transmitted a fixed distance at a fixed line loss.

Two-phase transmission was not considered in the above comparisons. When it is recognized that two-phase is the same as two independent single-phase systems, it is evident that two-phase, four-wire transmission requires the same amount of copper as single-phase. There are twice as many wires, but each is only one-half of the cross section of those necessary for single-phase.

Problem 13. Refer to Fig. 48. Find the ratio of the copper required for two-phase, three-wire transmission to that required for three-phase, three-wire transmission under the following conditions, all imposed simultaneously.

(*a*) A fixed amount of power transmitted.

(*b*) The same distance.

(*c*) With the same total line loss.

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(*d*) With the same highest line voltage between any pair of lines in the two systems.
(*e*) With the same current density in the three two-phase conductors.

Hint:

From condition (*a*): $P_{2t} = 2V_{p2}I_2 \cos\theta = P_{3t} = 3V_{p3}I_3 \cos\theta$

From condition (*d*): $I_2 = \frac{\sqrt{3}}{\sqrt{2}} I_3$

From condition (*c*): $2I_2^2R_2 + (\sqrt{2}I_2)^2R_{2'} = 3I_3^2R_3$

From condition (*e*): Area of $R_{2'}$ wire $= \sqrt{2} \times$ area of R_2 wire

From condition (*b*): $R_{2'} = \frac{R_2}{\sqrt{2}}$ *Ans.:* 1.94.

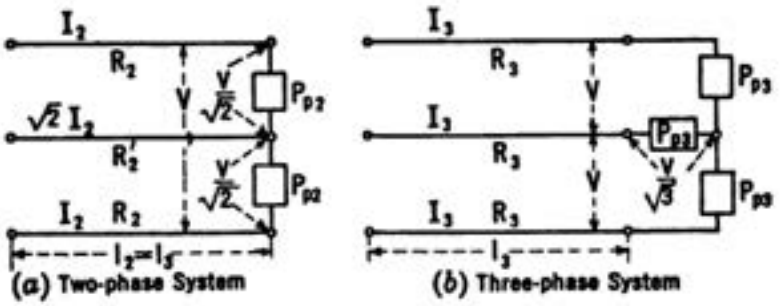

FIG. 48. See Problem 13.

Harmonics in the Wye System. An emf generated in a conductor will be sinusoidal only when the flux cutting the conductor varies according to a sine law. In a-c generators it is rather difficult, if not entirely impossible, to obtain an exact sine wave of distribution of the field flux. The slots and teeth change the reluctance of the path for the flux and cause ripples in the flux wave. Even if the distribution of the field flux were sinusoidal at no load, the distribution would be altered as the load came on, owing to the effect of the armature reaction of the current in the armature. The result is to induce in each phase an emf wave that is somewhat distorted from a true sine wave. In modern machines this distortion is relatively small. Through certain arrangements of the inductors on the armature and through certain ways of connecting them, some of the harmonics in the wave are reduced or are made to cancel entirely. When iron-core transformers are connected in wye, or any other way for that matter, the exciting current cannot be sinusoidal even though the impressed voltage is a

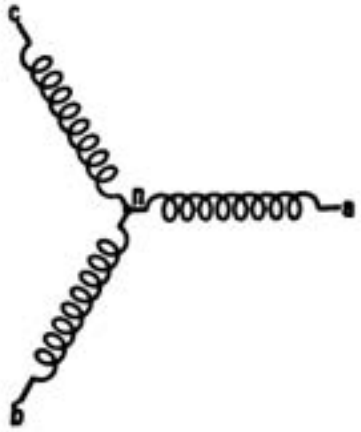

FIG. 49. Diagrammatic sketch of a wye-connected generator.

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perfect sine wave. This is due to the varying reluctance of the magnetic circuit with the consequent requirement of more ampere-turns to produce a given change in flux when the core operates at the higher flux densities. It therefore becomes of some importance to consider the effects of certain harmonics of currents and voltages in the phases of a three-phase system in affecting the line voltage of the system.

Assume that the emf induced in phase a of the wye-connected generator diagrammatically shown in Fig. 49 is

$$e_{na} = E_{m1} \sin \omega t + E_{m3} \sin (3\omega t + \alpha_3) + E_{m5} \sin (5\omega t + \alpha_5) + E_{m7} \sin (7\omega t + \alpha_7) \tag{28}$$

The sequence e_{na}, e_{nb}, e_{nc} will be used. Hence the fundamental of emf in phase nb will lag that in na by 120°, while that in phase nc will lag phase na by 240°. As usual, a shift of one degree for the fundamental will be a shift of n degrees for the nth harmonic. Then

$$\begin{aligned} e_{nb} &= E_{m1} \sin (\omega t - 120°) + E_{m3} \sin (3\omega t + \alpha_3 - 360°) \\ &\quad + E_{m5} \sin (5\omega t + \alpha_5 - 600°) + E_{m7} \sin (7\omega t + \alpha_7 - 840°) \\ &= E_{m1} \sin (\omega t - 120°) + E_{m3} \sin (3\omega t + \alpha_3) \\ &\quad + E_{m5} \sin (5\omega t + \alpha_5 - 240°) + E_{m7} \sin (7\omega t + \alpha_7 - 120°) \end{aligned} \tag{29}$$

$$\begin{aligned} e_{nc} &= E_{m1} \sin (\omega t - 240°) + E_{m3} \sin (3\omega t + \alpha_3) \\ &\quad + E_{m5} \sin (5\omega t + \alpha_5 - 120°) + E_{m7} \sin (7\omega t + \alpha_7 - 240°) \end{aligned} \tag{30}$$

The equations of the phase voltages show that all third harmonics are in phase. Also the phase sequence for the fifth harmonic is reversed from that of the fundamental. The sequence of the seventh is the same

TABLE I

DISPLACEMENT BETWEEN VARIOUS HARMONICS IN THE PHASES OF FIG. 49

Displacement in electrical degrees

Harmonic	1	3	5	7	9	11	13
Phase A	0	0	0	0	0	0	0
Phase B	120	0	240	120	0	240	120
Phase C	240	0	120	240	0	120	240

as that for the fundamental. In general it will be found that the fundamental and all harmonics obtained by adding a multiple of 6 to the fundamental will have the same sequence. These are first, seventh, thirteenth, nineteenth, twenty-fifth, and so on. In like manner, the fifths, elevenths, seventeenths, twenty-thirds, etc., have like sequences

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but opposite to that of the fundamentals. Also the third, ninth, and all multiples of the third will be found to be in phase. These results are tabulated in Table I. The relation between the fundamentals and third harmonics in each phase for $\alpha_3 = 0$ in equations (28), (29), and (30) is shown in Fig. 50.

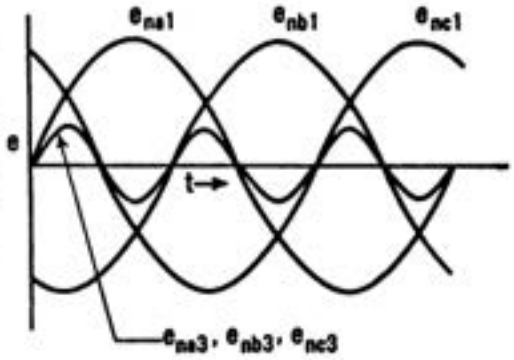

FIG. 50. Fundamental and third harmonic voltages.

The line voltage of the wye may be found by summing up the potentials encountered in passing through the circuit between the line terminals in question. With reference to Fig. 49,

$$e_{ba} = e_{bn} + e_{na}$$

Each harmonic must be handled separately. The combination of e_{bn} and e_{na} is shown by vector diagrams in Fig. 51. For the fundamental, e_{ba} is 30° ahead of e_{na}. Since $e_{na_1} = E_{m1} \sin \omega t$, $e_{ba_1} = \sqrt{3}E_{m1} \sin (\omega t + 30°)$. For the third harmonic, $e_{ba_3} = 0$. For the fifth, e_{ba_5}

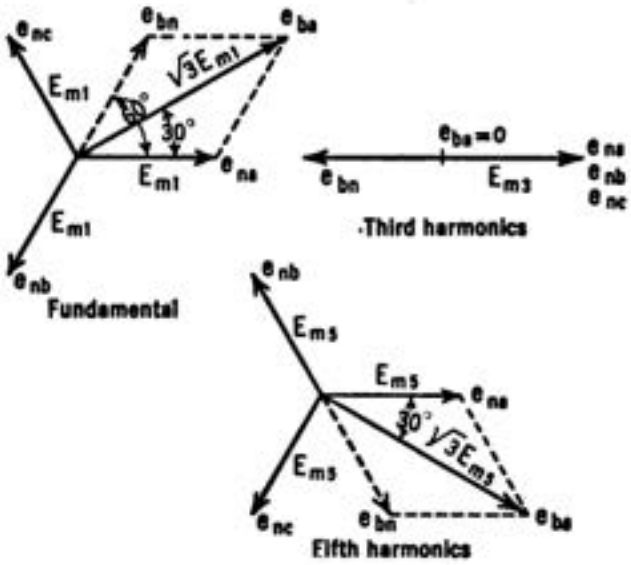

FIG. 51. Line voltages in Fig. 49 are found for each harmonic separately.

lags e_{na_5} by 30°. Hence $e_{ba_5} = \sqrt{3}E_{m5} \sin (5\omega t + \alpha_5 - 30°)$. The seventh harmonic vector diagram is similar to that for the fundamental. The complete equation for the line voltage e_{ba} is

$$e_{ba} = \sqrt{3}E_{m1} \sin (\omega t + 30°) + \sqrt{3}E_{m5} \sin (5\omega t + \alpha_5 - 30°) + \sqrt{3}E_{m7} \sin (7\omega t + \alpha_7 + 30°) \tag{31}$$

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Similarly,

$$e_{ac} = \sqrt{3}E_{m1} \sin (\omega t + 150°) + \sqrt{3}E_{m5} \sin (5\omega t + \alpha_5 - 150°)$$
$$+ \sqrt{3}E_{m7} \sin (7\omega t + \alpha_7 + 150°) \tag{32}$$
$$e_{cb} = \sqrt{3}E_{m1} \sin (\omega t - 90°) + \sqrt{3}E_{m5} \sin (5\omega t + \alpha_5 + 90°)$$
$$+ \sqrt{3}E_{m7} \sin (7\omega t + \alpha_7 - 90°) \tag{33}$$

The vector diagram of the third-harmonic voltages shows that the third harmonics in the two phases between any pair of terminals are in opposition and cancel. The third harmonics cannot contribute anything to line voltage, although they do contribute toward the total voltage between one terminal and neutral. The rms magnitude of the voltage to neutral in the example just considered is

$$E_{na} = \sqrt{\frac{E_{m1}^2 + E_{m3}^2 + E_{m5}^2 + E_{m7}^2}{2}}$$

The rms magnitude of the voltage between terminals is

$$E_{ba} = \sqrt{3}\sqrt{\frac{E_{m1}^2 + E_{m5}^2 + E_{m7}^2}{2}}$$

The ratio of line and phase voltage of a wye connection can be the $\sqrt{3}$ only when there is no third harmonic or its multiples in the wave of phase voltage.

Consider next the harmonics in the current waves for the wye. Kirchhoff's current law applied to the wye connection without a neutral wire connected states that

$$i_{na} + i_{nb} + i_{nc} = 0$$

Under balanced conditions this equation can be fulfilled only when the three currents are equal in magnitude and 120° apart in time phase, or when the magnitudes of each current are equal to zero. Since the third harmonics and their multiples are the only ones that are not 120° apart, each of them must be zero to fulfil the conditions imposed by Kirchhoff's current law. The vector diagrams for the harmonics of current appear exactly as those for phase voltages in Fig. 51. If, in each phase, e is replaced by i, the diagrams will represent currents. If the third harmonics of current do exist, there must be a neutral connection. This neutral or fourth wire furnishes the return path for the third harmonics of each phase. Since all third harmonics, in accordance with the diagram in Fig. 51, would have to be in phase, their arithmetic sum would flow in the neutral. A third-harmonic pressure or voltage may

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exist in each phase, but, unless a path through the neutral is provided, the three pressures do not have a closed circuit upon which they can act and, therefore, no third-harmonic current can flow. In a balanced wye-connected circuit without neutral connection, therefore, all harmonics except the third and its multiples can exist. In a four-wire, three-phase circuit (neutral wire connected) all harmonics in the current wave can exist.

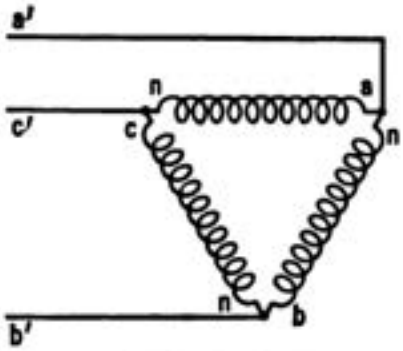

FIG. 52. Coils of Fig. 49 reconnected in delta.

Harmonics in the Delta System. If three coils having induced voltages as given by e_{na}, e_{nb}, and e_{nc} in the previous article are connected in delta, those voltages that do not add to zero around the loop will cause a circulating current to flow. Under any circumstances, in the delta of Fig. 52, the sum of the three terminal voltages taken in the same direction around the delta must be zero. Expressed algebraically,

$$v_{ca} + v_{ab} + v_{bc} = 0 \tag{34}$$

Because the sum of the generated emf's, $e_{na} + e_{nb} + e_{nc}$, is equal to zero for all except triple-frequency voltages and its multiples, no circulatory current of other than triple frequency and its multiples can exist. Hence there will be no impedance drops at no load, and the generated voltages for all except the third harmonic and its multiples will appear across the terminals. For the third harmonics and its multiples the situation is different. Since the third-harmonic generated voltages of all phases of a three-phase system were shown to be equal and in phase,

$$e_{na_3} + e_{nb_3} + e_{nc_3} = 3E_{m3} \sin (3\omega t + \alpha_3)$$

will cause a current to circulate in the delta. This current multiplied by the impedance of the loop will be equal to the resultant third-harmonic voltage $3E_{m3} \sin (3\omega t + \alpha_3)$. Since the terminal voltage is equal to the generated voltage minus the internal drop, there will be no third-harmonic voltage between terminals in the delta if the phase emf's and impedances are balanced. In this way equation (34) is fulfilled for the third-harmonic voltages.

There is no third harmonic in the terminal voltage of the wye; neither is the wye connection subject to a third-harmonic circulating current. In the wye the third-harmonic voltages between terminals do not appear, as the result of their being in opposition between two terminals and neutralizing. In the delta, the third-harmonic voltage does not appear in the terminal voltage because it is short-circuited by the mesh connec-

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tion and is consumed in the form of internal impedance drop. The equations of the terminal voltages of the delta generator or transformer at no load are the same as the generated voltages of each phase with the third-harmonic voltage and its multiples omitted. Thus

$$v_{ca} = E_{m1} \sin \omega t + E_{m5} \sin (5\omega t + \alpha_5) + E_{m7} \sin (7\omega t + \alpha_7) \quad (35)$$

$$v_{ab} = E_{m1} \sin (\omega t - 120°) + E_{m5} \sin (5\omega t + \alpha_5 - 240°) + E_{m7} \sin (7\omega t + \alpha_7 - 120°) \quad (36)$$

$$v_{bc} = E_{m1} \sin (\omega t - 240°) + E_{m5} \sin (5\omega t + \alpha_5 - 120°) + E_{m7} \sin (7\omega t + \alpha_7 - 240°) \quad (37)$$

Compare equations (35), (36), and (37) with equations (28), (29), and (30).

All harmonics of current are possible in the phases of the delta, since it is simply a closed series loop. Thus for phase *ca*, Fig. 52, we may have

$$i_{ca} = I_{m1} \sin \omega t + I_{m3} \sin (3\omega t + \alpha_3) + I_{m5} \sin (5\omega t + \alpha_5) + I_{m7} \sin (7\omega t + \alpha_7) \quad (38)$$

If the sequence is such that phase *ab* lags *ca* by 120°, the currents in the other phases are found by displacing the fundamentals by the usual 120° and the *n*th harmonic by *n* times this angle. Thus

$$\begin{aligned} i_{ab} &= I_{m1} \sin (\omega t - 120°) + I_{m3} \sin (3\omega t + \alpha_3 - 360°) \\ &\quad + I_{m5} \sin (5\omega t + \alpha_5 - 600°) + I_{m7} \sin (7\omega t + \alpha_7 - 840°) \\ &= I_{m1} \sin (\omega t - 120°) + I_{m3} \sin (3\omega t + \alpha_3) \\ &\quad + I_{m5} \sin (5\omega t + \alpha_5 - 240°) + I_{m7} \sin (7\omega t + \alpha_7 - 120°) \end{aligned} \quad (39)$$

$$i_{bc} = I_{m1} \sin (\omega t - 240°) + I_{m3} \sin (3\omega t + \alpha_3) + I_{m5} \sin (5\omega t + \alpha_5 - 120°) + I_{m7} \sin (7\omega t + \alpha_7 - 240°) \quad (40)$$

The line currents are obtained in terms of phase current as indicated below.

$$i_{a'a} = i_{ac} + i_{ab}$$

$$i_{b'b} = i_{ba} + i_{bc}$$

$$i_{c'c} = i_{ca} + i_{cb}$$

These operations are performed similarly to those illustrated in the vector diagrams of Fig. 51 for voltages. The results are

$$i_{a'a} = \sqrt{3} I_{m1} \sin (\omega t - 150°) + \sqrt{3} I_{m5} \sin (5\omega t + \alpha_5 + 150°) + \sqrt{3} I_{m7} \sin (7\omega t + \alpha_7 - 150°) \quad (41)$$

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$$i_{b'b} = \sqrt{3}I_{m1} \sin (\omega t + 90°) + \sqrt{3}I_{m5} \sin (5\omega t + \alpha_5 - 90°) + \sqrt{3}I_{m7} \sin (7\omega t + \alpha_7 + 90°) \quad (42)$$

$$i_{c'c} = \sqrt{3}I_{m1} \sin (\omega t - 30°) + \sqrt{3}I_{m5} \sin (5\omega t + \alpha_5 + 30°) + \sqrt{3}I_{m7} \sin (7\omega t + \alpha_7 - 30°) \quad (43)$$

Equations (41), (42), and (43) show that no third-harmonic currents can exist in the lines of a delta. The third-harmonic current in one phase coming to a line connection exactly equals the third-harmonic current in the other phase leaving the junction. This leaves no third-harmonic current to flow in the line connection.

The magnitude of the phase current is

$$I_p = \sqrt{\frac{I_{m1}^2 + I_{m3}^2 + I_{m5}^2 + I_{m7}^2}{2}}$$

The magnitude of the line current is

$$I_L = \sqrt{\frac{(\sqrt{3}I_{m1})^2 + (\sqrt{3}I_{m5})^2 + (\sqrt{3}I_{m7})^2}{2}} = \sqrt{3}\sqrt{\frac{I_{m1}^2 + I_{m5}^2 + I_{m7}^2}{2}}$$

The ratio of line to phase current can be the $\sqrt{3}$ only when no third-harmonic currents exist.

Example 9. Only fundamentals and third harmonics are assumed to exist in the voltages of a wye connection like that shown in Fig. 49. Voltmeter readings as follows are obtained: $V_{na} = 150$, $V_{ba} = 220$. Calculate the magnitude of the third-harmonic voltage.

Solution. Since V_{ba} contains only fundamental voltage, the fundamental to neutral is $220/\sqrt{3} = 127$.

$$V_{na} = \sqrt{V_1^2 + V_3^2} \quad \text{or} \quad V_3 = \sqrt{150^2 - 127^2} = 79.9$$

The possibility of a third-harmonic circulating current in a delta makes this connection for a-c generators somewhat less desirable than the wye, although there are several other more important factors that make wye connection for generators predominate. Although the third-harmonic current is undesirable in the delta generator it *is* desirable in transformers, since there it acts as a component of the magnetizing current for the core which is essential if a sine wave of flux and induced voltage is to be obtained. Some high-voltage transformers which are connected wye on both primary and secondary have a third winding which is delta-connected to allow a third-harmonic circulating cur-

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rent to flow, thus supplying the transformers with the necessary triple-frequency component of magnetizing current. A delta-connected winding of this kind is called a tertiary winding.

PROBLEMS

14. What is the phase voltage and also the voltage between adjacent lines of a six-phase star connection if the greatest voltage between any pair of lines is 156 volts?

15. The voltage between adjacent lines of a twelve-phase star is 100 volts. Find the voltage to neutral, the voltage between alternate lines, and the greatest voltage between any pair of lines.

16. Find the phase current in a six-phase mesh if the line current is 10 amperes; also for a twelve-phase mesh for the same line current.

17. Given six coils each having an induced voltage of 63.5 volts. Adjacent coil voltages are 60° apart. In how many ways may you connect these coils to form a balanced three-phase wye system of voltages if all coils must be used for each system and if the magnitude of the line voltages of each system must be different? What are the line voltages for each wye system?

18. A generator has six coils, adjacent coils being displaced 30 electrical degrees. If each coil voltage is 114 volts, show how to connect them and calculate the line or terminal voltage for three-phase star. Repeat for three-phase mesh. Repeat for two-phase, where line voltage is taken as the phase voltage.

19. A generator has six coils, adjacent coils being displaced 30 electrical degrees. If all coils are used to form a three-phase mesh, what must be the emf of each coil to yield balanced three-phase voltages of 230 volts each? If all coils are connected for three-phase star, what must be the emf of each coil to give an emf between lines of 230 volts?

20. Draw vector diagrams which represent the currents and voltages shown in Oscillograms 3 and 4, pages 292 and 293, and label them in accordance with the labeling on the oscillogram.

21. Three-phase line voltages of 230 volts are impressed on a balanced wye load having 16 ohms resistance and 12 ohms reactance in series in each phase. Find the line current and total power. If the three impedances are reconnected in delta and placed across the same line voltages, what are the line and phase currents and the total power?

22. A current of 10 amperes flows in the lines to a twelve-phase mesh-connected load having 5 ohms resistance and 8 ohms capacitive reactance in series in each phase. What is the voltage between alternate lines on the load? Draw the vector diagram of the voltages and phase currents of two adjacent phases, and also show the line current from the junction of these two phases.

23. A balanced wye load consists of 3 ohms resistance and 4 ohms capacitive reactance in series per phase. Balanced three-phase voltages of 100 volts each are impressed across the lines at the load. If the load is connected to a generator through three lines of equal impedance, each line containing a resistance of 1 ohm and an inductive reactance of 4 ohms, find the voltage at the generator terminals.

24. A balanced wye load having 8 ohms resistance and 6 ohms inductive reactance in series in each phase is supplied through lines each having 1 ohm resistance and 2 ohms inductive reactance. If the sending-end voltage between lines is 250 volts, what will be the voltage between lines at the load?

25. A balanced delta load contains a resistance of 12 ohms and a capacitive re-

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actance of 16 ohms in series in each phase. If the balanced impressed line voltages on the load are 115 volts each, calculate the line and phase currents.

26. A balanced delta load having 18 ohms resistance and 24 ohms capacitive reactance in series in each phase is supplied through lines each having 1 ohm resistance and 2 ohms inductive reactance. If the line-to-line voltage at the sending end is 250 volts, find the line-to-line voltage at the load terminals. Also find the total power consumed by the load.

27. A balanced wye inductive load takes 5.4 kw at 0.6 power factor at a line voltage of 200 volts. It is in parallel with a pure resistive balanced wye load taking 5 kw. Find the resultant line current supplied the combination.

28. The total power supplied two balanced three-phase loads in parallel is 12 kw at 0.8 power factor lagging. One of the loads takes 10 kva at 0.8 power-factor lead. The second load is a delta-connected balanced load. Find the resistance and reactance per phase of the delta load if the line voltage is 230 volts. If the unknown load were wye-connected, what would be the resistance and reactance per phase?

29. Each phase of a delta load has 6 ohms resistance and 9 ohms capacitive reactance in series. Each phase of a wye load has 8 ohms resistance and 6 ohms inductive reactance in series. The two loads are connected in parallel across three-phase line voltages of 100 volts. Calculate the resultant line current, the total power consumed, and the power factor of the combination.

30. A three-phase, 5-hp, 220-volt induction motor (balanced load) has an efficiency of 86 per cent and operates at 86.6 per cent lagging power factor. It is paralleled with a three-phase resistance furnace consisting of three 36-ohm resistances connected in delta. Find the kilovolt-amperes demanded by the combination, the power factor, and the line current.

31. A three-phase generator supplies balanced voltages of 230 volts each at its terminals when it carries a load which requires 10 amperes. If the power factor at the generator terminals is 0.8 leading, calculate the voltage at the load if the load is connected through lines each having 1 ohm resistance and 5 ohms inductive reactance.

32. A balanced three-phase load requires 10 kva at 0.5 lagging power factor. Find the kva size of a condenser bank which may be paralleled with the load to bring the power factor of the combination to 0.866 lag, and also to 0.866 lead.

33. If the line voltage for Problem 32 is 230 volts and the frequency 60 cycles, find the capacitance in microfarads of condensers required in each phase of the condenser bank if they are delta-connected. What capacitance is required if they are wye-connected?

34. Three 15 $\underline{/60^\circ}$-ohm load impedances are connected in delta and supplied by lines, each line containing 1 ohm resistance and 1 ohm inductive reactance. If the line voltages on the supply side of the line impedances are balanced three-phase of 115 volts each, find the voltage across the load impedances. Also calculate the power loss in the supply lines and the power dissipated by the load itself.

35. If the current through each of the load impedances in Problem 34 is 20 amperes, find the required voltage on the supply side of the line impedances.

36. The motor M in Fig. 53 has 2300 volts balanced three-phase voltages impressed at its terminals and takes 120 kva at 0.6 leading power factor. Calculate the line volts, power input, and the power factor at a, b, c.

37. If the motor in Fig. 53 is removed from the circuit and balanced three-phase line voltages of 2300 volts each are impressed at a, b, and c, how many volts will appear between lines at the motor end of the line?

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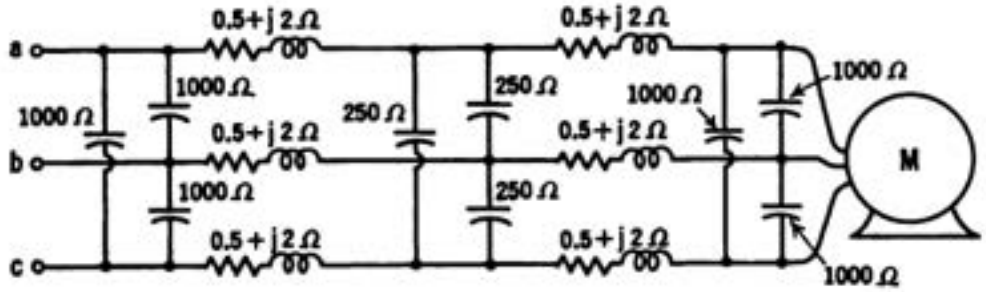

FIG. 53. See Problems 36 and 37.

38. A three-phase resonant shunt is connected to three-phase, 2300-volt lines to furnish a low impedance for a certain frequency so as to reduce the inductive interference with a telephone line. The shunt consists of three 10-kva, 60-cycle, 2300-volt capacitors connected in delta. In series with each line terminal from the delta is an inductance of 2.5 millihenrys. At what frequency does this three-phase combination resonate, that is, offer minimum impedance? Assume that resistances of condensers and inductances are negligible.

39. (*a*) Three coils each having 36 ohms resistance and 100 millihenrys inductance are connected in delta. Find the microfarad capacitance of each condenser which may be placed in each of the three lines from the delta to produce resonance (unity p.f.) of the system as a whole for a frequency of 800 cycles. This is a type of resonant shunt sometimes connected to power lines to reduce inductive interference with telephone circuits.

(*b*) Assume that the capacitors calculated for each line in (*a*) are removed and connected in delta. Find how many henrys of inductance would be required in each line from this delta to bring the power factor of the combination to unity at 800 cycles.

40. Find the readings of W_a and W_b in Fig. 54 for the sequence V_{na}, V_{nc}, V_{nb}. Find the power dissipated in each phase.

41. A balanced three-phase load takes 5 kw and 20 reactive kva. Find the readings of two wattmeters properly connected to measure the total power.

42. In Fig. 54 find the reading of W_R. Also calculate the total reactive volt-amperes taken by the load. What is the ratio of the total reactive volt-amperes taken to the reading of W_R?

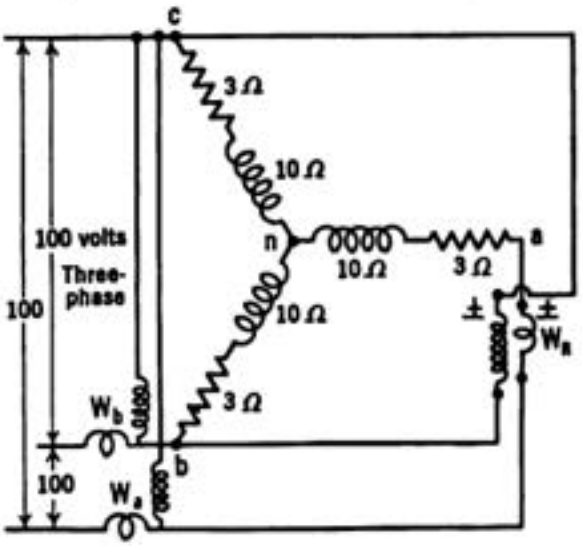

FIG. 54. See Problems 40, 42, and 43.

43. Prove that the ratio of the reading of W_R of Fig. 54 to the total reactive volt-amperes obtained in Problem 42 will obtain for all balanced loads when the impressed voltages are sinusoidal balanced three-phase.

44. (*a*) Calculate analytically the power-factor angle for a balanced three-phase circuit in which two wattmeters properly connected to measure three-phase power read +1000 and +800 watts, respectively.

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(*b*) Also calculate the angle if the meters read +1000 and −800 watts, respectively.

45. Two wattmeters measuring power to a balanced three-phase load read 1200 and −400 watts, respectively. How many volt-amperes does the load take? At what power factor?

46. Each phase of a balanced twelve-phase star-connected load consists of 3 ohms resistance and 4 ohms inductive reactance in series. Balanced twelve-phase line voltages of 51.76 volts between adjacent lines are applied to the load. Calculate the line current, power factor, and total power consumed by the load.

47. The voltage induced in phase *na* of a 3-phase wye-connected generator is

$$e_{na} = 127 \sin \omega t + 50 \sin (3\omega t - 30°) + 30 \sin (5\omega t + 40°)$$

If the sequence is e_{na}, e_{nb}, e_{nc}, find the equation with respect to time of the line voltage e_{ab}. *Note:* Phase voltages of polyphase generators differ only in phase angle.

48. If the phases of the generator in Problem 47 are reconnected in delta, what will be the equation with respect to time of the line voltage across phase *na*?

49. A wye-connected generator has a generated voltage per phase which contains only the fundamental, third, fifth, and seventh harmonics. The line voltage as measured by a voltmeter is 230 volts; the voltage to neutral is 160 volts. Calculate the magnitude of the third harmonic in the generated voltage.

50. The induced emf of a delta generator with one corner of the delta open as shown in Fig. 55 contains only odd harmonics up to the seventh. A voltmeter across *ac* reads 2500 volts, and, across *bb′* when negligible current flows, 1800 volts. Find the reading of a voltmeter connected from *a* to *b′*.

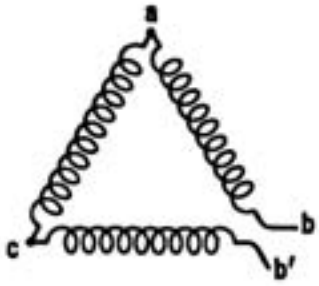

Fig. 55. See Problems 50 and 51.

51. The induced phase voltage of a delta generator with one corner open as shown in Fig. 55 contains odd harmonics up to the seventh. A voltmeter connected from *a* to *b′* reads 2500 volts, and from *a* to *c* it reads 2200 volts when negligible current flows. What should it read from *b* to *b′*?

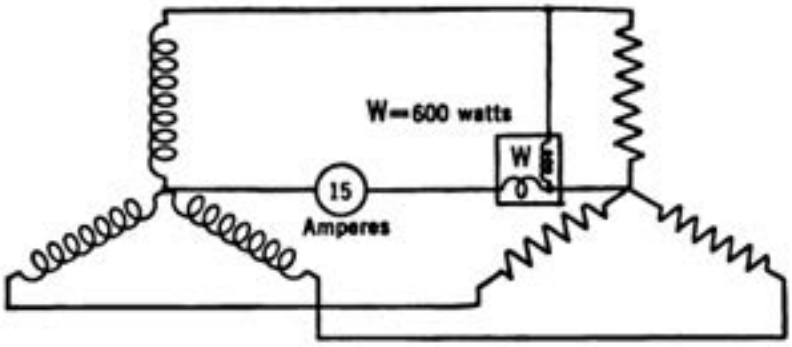

Fig. 56. See Problem 52.

52. Figure 56 shows a generator connected to a balanced pure resistance load. An ammeter in the neutral reads 15 amperes, and the wattmeter shown reads 600 watts. A voltmeter shows a balanced line voltage of 230 volts. Find the line currents to the load and the voltage from line to neutral at the load, assuming that the generated voltage contains only fundamental and third harmonic components.

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CHAPTER IX

UNBALANCED POLYPHASE CIRCUITS

Unbalanced Loads. The previous chapter developed the method of calculating the currents in the various branches of balanced polyphase loads when the impedances and impressed voltages are known. In the present chapter, methods of calculating the various branch currents will be developed when known voltages are impressed upon unbalanced loads. Any polyphase load in which the impedance in one or more phases differs from those of other phases is said to be unbalanced. Even though the load impedances of the various phases are identical, one of the methods of calculating unbalanced loads must be employed if the voltages impressed on the load are unequal and differ in phase by angles which are not equal. Some of the simpler types of unbalanced loads which are solvable by rather simple direct methods will be considered first.

Unbalanced Delta Loads. If the three-phase line voltages across the terminals of an unbalanced delta load are fixed, the voltage drop across each phase impedance is known. The currents in each phase can therefore be determined directly. The line currents can be found by adding vectorially the two component currents coming toward or flowing away from the line terminal in question as was done in series-parallel circuit analysis. The following example will illustrate the procedure.

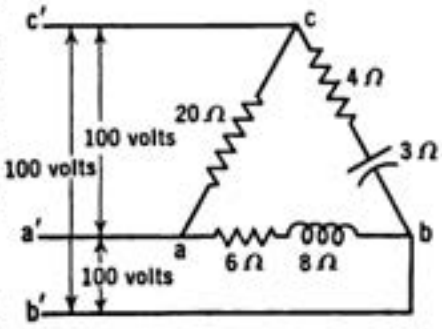

FIG. 1. Unbalanced delta load. See example 1.

Example 1. Given the unbalanced delta load shown in Fig. 1. Calculate all currents for the three-phase balanced voltages shown on the figure, if the voltage sequence is *ab–ca–bc*.

Since the voltages shown are assumed to be maintained at the terminals *a*, *b*, and *c*, the complex expressions for the phase voltages may be established. Take some phase voltage as a reference, say $\mathbf{V}_{ab}$ for this example. Therefore,

$$\mathbf{V}_{ab} = 100 + j0$$

$$\mathbf{V}_{bc} = 100 \underline{/120^\circ} = -50 + j86.6$$

$$\mathbf{V}_{ca} = 100 \underline{/-120^\circ} = -50 - j86.6 \text{ volts}$$

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Then

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{100 + j0}{6 + j8} = 6 - j8 = 10 \underline{/-53.1^\circ} \text{ amperes}$$

$$\mathbf{I}_{bc} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_{bc}} = \frac{-50 + j86.6}{4 - j3} = -18.39 + j7.856 = 20 \underline{/156.9^\circ} \text{ amperes}$$

$$\mathbf{I}_{ca} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_{ca}} = \frac{-50 - j86.6}{20 + j0} = -2.5 - j4.33 = 5 \underline{/-120^\circ} \text{ amperes}$$

The line currents are:

$$\begin{aligned}\mathbf{I}_{a'a} &= \mathbf{I}_{ab} + \mathbf{I}_{ac} = 6 - j8 + 2.5 + j4.33 = 8.5 - j3.67 \\ &= 9.26 \underline{/-23.4^\circ} \text{ amperes} \\ \mathbf{I}_{b'b} &= \mathbf{I}_{ba} + \mathbf{I}_{bc} = -6 + j8 - 18.39 + j7.856 \\ &= -24.39 + j15.856 = 29 \underline{/146.9^\circ} \text{ amperes} \\ \mathbf{I}_{c'c} &= \mathbf{I}_{ca} + \mathbf{I}_{cb} = -2.5 - j4.33 + 18.39 - j7.856 \\ &= 15.89 - j12.186 = 20 \underline{/-37.3^\circ} \text{ amperes}\end{aligned}$$

Unbalanced Wye Loads. If the load voltages at the terminals a, b, and c of an unbalanced wye load like that shown in Fig. 2 can be assumed to remain constant at their specified values, then the phase currents of an equivalent delta which replaces the wye can be found directly as shown in example 1. The line currents to this equivalent delta are obviously the currents in the phases of the wye load.

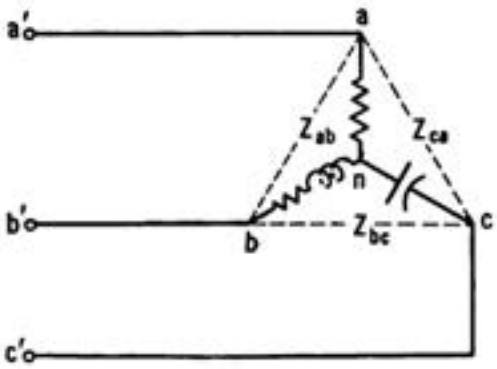

FIG. 2. Conversion from a wye-connected load to an equivalent delta-connected load.

Example 2. A balanced set of three-phase voltages is connected to an unbalanced set of wye-connected impedances as shown in Fig. 2. The following values are assumed to be known:

$$\mathbf{V}_{ab} = 212 \underline{/90^\circ} \text{ volts} \qquad \mathbf{Z}_{an} = 10 + j0 \text{ ohms}$$

$$\mathbf{V}_{bc} = 212 \underline{/-150^\circ} \text{ volts} \qquad \mathbf{Z}_{bn} = 10 + j10 \text{ ohms}$$

$$\mathbf{V}_{ca} = 212 \underline{/-30^\circ} \text{ volts} \qquad \mathbf{Z}_{cn} = 0 - j20 \text{ ohms}$$

The line currents $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ are to be determined by the wye to delta conversion method. (See Chapter V, page 134, for the general theory involved in making wye to delta conversions.)

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In Fig. 2 the equivalent delta impedances may be expressed in terms of the wye impedances as follows:

$$Z_{ab} = \frac{(Z_{an}Z_{bn} + Z_{bn}Z_{cn} + Z_{cn}Z_{an})}{Z_{nc}} = \frac{S}{Z_{nc}}$$

$$Z_{bc} = \frac{S}{Z_{an}} \quad \text{and} \quad Z_{ca} = \frac{S}{Z_{bn}}$$

Numerically, the equivalent delta impedances are:

$$Z_{ab} = \frac{300 - j300}{0 - j20} = (15 + j15) = 21.2 \angle 45^\circ \text{ ohms}$$

$$Z_{bc} = \frac{300 - j300}{10 - j0} = (30 - j30) = 42.4 \angle -45^\circ \text{ ohms}$$

$$Z_{ca} = \frac{300 - j300}{10 + j10} = (0 - j30) = 30.0 \angle -90^\circ \text{ ohms}$$

The load currents in the equivalent delta are:

$$I_{ab} = \frac{V_{ab}}{Z_{ab}} = \frac{212 \angle 90^\circ}{21.2 \angle 45^\circ} = 10 \angle 45^\circ \text{ amperes}$$

$$I_{bc} = \frac{V_{bc}}{Z_{bc}} = \frac{212 \angle -150^\circ}{42.4 \angle -45^\circ} = 5.0 \angle -105^\circ \text{ amperes}$$

$$I_{ca} = \frac{V_{ca}}{Z_{ca}} = \frac{212 \angle -30^\circ}{30 \angle -90^\circ} = 7.07 \angle 60^\circ \text{ amperes}$$

The actual line and load currents are:

$$\begin{aligned} I_{a'a} &= I_{ab} - I_{ca} \\ &= 10 \angle 45^\circ - 7.07 \angle 60^\circ = 3.66 \angle 15^\circ \text{ amperes} \\ I_{b'b} &= I_{bc} - I_{ab} \\ &= 5 \angle -105^\circ - 10 \angle 45^\circ = 14.56 \angle -125.1^\circ \text{ amperes} \\ I_{c'c} &= I_{ca} - I_{bc} \\ &= 7.07 \angle 60^\circ - 5 \angle -105^\circ = 11.98 \angle 66.2^\circ \text{ amperes} \end{aligned}$$

As a single check on the above arithmetic let the calculated value of $[I_{a'a}Z_{an} - I_{b'b}Z_{bn}]$ be compared with the originally specified value of V_{ab}, which was $212 \angle 90^\circ$ volts.

$$\begin{aligned} [I_{a'a}Z_{an} - I_{b'b}Z_{bn}] &= (35.4 + j9.48) - (35.35 - j202.6) \\ &= (0.05 + j212.1) \text{ volts} \qquad (\textit{Check}) \end{aligned}$$

The conversion of a wye to its equivalent delta along with the solution of the delta as illustrated in the above example will usually require an equal or greater amount of work than the direct solution of the wye employing two simultaneous equations obtained by the application of Kirchhoff's laws.

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Vector diagrams of the voltages and currents involved in the foregoing example are given in Fig. 3.

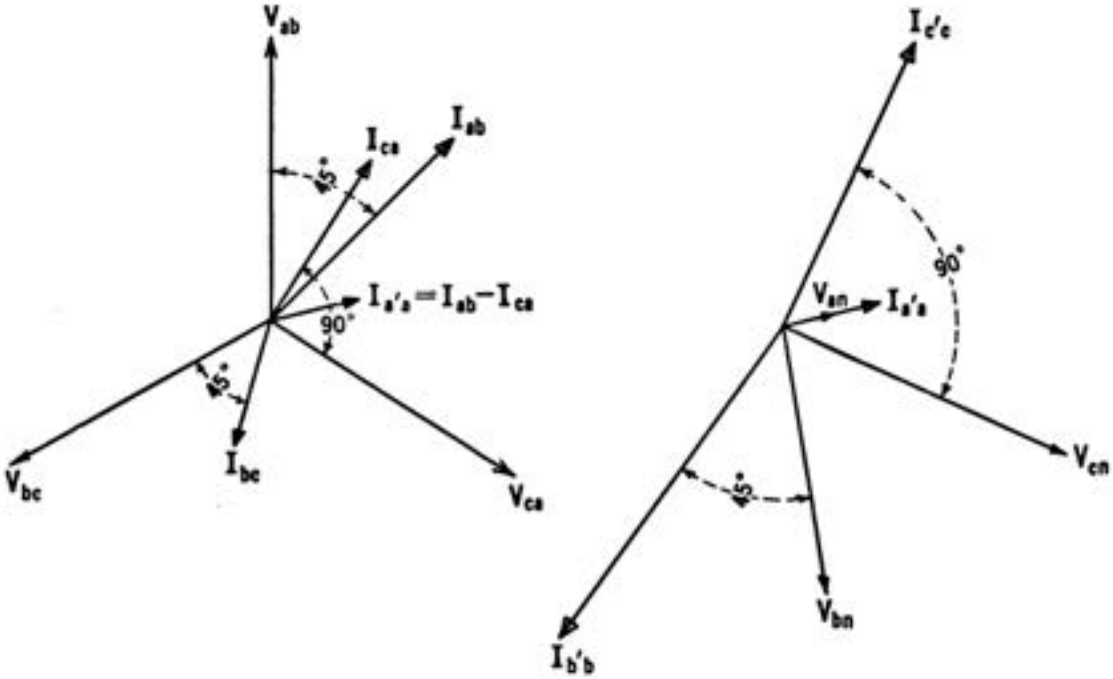

FIG. 3. Vector diagrams for example 2.

Problem 1. Determine the values of $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ in example 2.

Ans.: $\mathbf{V}_{an} = 36.6 \underline{/15^\circ}$; $\mathbf{V}_{bn} = 205.6 \underline{/-80.1^\circ}$; $\mathbf{V}_{cn} = 239.6 \underline{/-23.8^\circ}$ volts.

Problem 2. Determine the power dissipated in each of the three phases (*an*, *bn*, and *cn*) of example 2.

Ans.: $P_{an} = 134$; $P_{bn} = 2120$; $P_{cn} = 0$ watts.

Problem 3. Find the magnitudes of $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ in Fig. 2 if $\mathbf{V}_{ab} = 212 \underline{/90^\circ}$, $\mathbf{V}_{bc} = 212 \underline{/-30^\circ}$, and $\mathbf{V}_{ca} = 212 \underline{/-150^\circ}$ volts. As in example 2, $\mathbf{Z}_{an} = (10 + j0)$, $\mathbf{Z}_{bn} = (10 + j10)$, and $\mathbf{Z}_{cn} = (0 - j20)$ ohms.

Ans.: $I_{a'a} = 13.65$; $I_{b'b} = 6.20$; $I_{c'c} = 7.54$ amperes.

Combined Delta and Wye Loads. Delta-connected loads are sometimes operated in conjunction with wye-connected loads as shown in Fig. 4. If the three-phase, line-to-line voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ remain sensibly constant irrespective of load conditions, a relatively simple solution may be effected by first converting the wye load to an equivalent delta load. The two parallel deltas may then be combined to form a single equivalent delta-connected load and the equivalent delta currents calculated directly as

$$\mathbf{I}_{ab(eq)} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab(eq)}} \quad \mathbf{I}_{bc(eq)} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_{bc(eq)}} \quad \mathbf{I}_{ca(eq)} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_{ca(eq)}}$$

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Incidentally, only $[B - (J - 1)]$ *independent* emf equations can be written for a particular network.

A few simple rules aid greatly in writing the necessary and sufficient equations to solve a network. In establishing current or junction equations, the first junction should be chosen which will include the least possible number of unknown currents. Then continue setting up junction equations, being careful to include in each succeeding equation as few unknown currents as possible which have not been previously incorporated in an equation. This procedure is continued as long as at least one unknown current not previously used may be incorporated in an equation. It is desirable to place some mark on the current as it is used in the network to indicate that it has been employed in an equation.

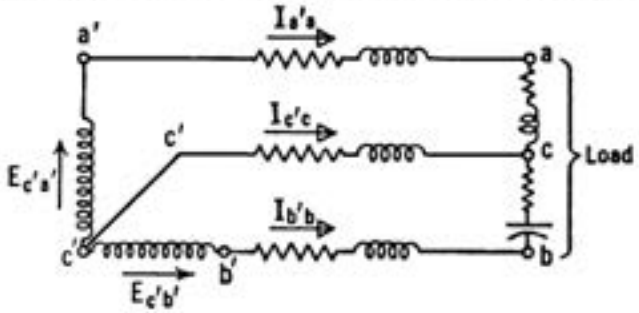

FIG. 5. A simple three-branch network.

A similar procedure should be followed when establishing loop or emf equations. The simplest loops should be chosen and as few additional unknown currents as possible should be included in each succeeding emf equation. When no unknown currents not previously used remain, the necessary and sufficient loop equations will have been established. If the above rules are followed, no more and no less than the necessary and sufficient current and emf equations can be established.

It must be remembered that, basically, Kirchhoff's laws apply only to instantaneous currents and voltages. But it has been shown that sinusoidal components of currents and voltages can be manipulated in terms of their complex or vector expressions and that the results obtained are equivalent to those obtained when instantaneous values are employed. Kirchhoff's laws may, therefore, be written in terms of complex currents and voltages provided that the laws are applied individually to each harmonic component which is present in the particular wave forms under investigation.

In networks that contain not more than three or four branches it is entirely feasible to derive general expressions for each individual branch current. These expressions can then be employed to effect individual current solutions. The process will be illustrated. General expressions for the branch currents, $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ of Fig. 5 will be derived.

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Since there are but two junctions in the network shown in Fig. 5, Kirchhoff's current law can be applied independently only once. Kirchhoff's emf law can be applied to each of the three closed paths, but only two of the resulting equations will be independent. The three independent equations which will be employed in the proposed derivation are:

$$\mathbf{I}_{a'a} + \mathbf{I}_{b'b} + \mathbf{I}_{c'c} = 0 \tag{1}$$

$$(\mathbf{Z}_{c'a'} + \mathbf{Z}_{a'a} + \mathbf{Z}_{ac})\mathbf{I}_{a'a} - \mathbf{Z}_{c'c}\mathbf{I}_{c'c} = \mathbf{E}_{c'a'} \tag{2}$$

$$(\mathbf{Z}_{c'b'} + \mathbf{Z}_{b'b} + \mathbf{Z}_{bc})\mathbf{I}_{b'b} - \mathbf{Z}_{c'c}\mathbf{I}_{c'c} = \mathbf{E}_{c'b'} \tag{3}$$

Equations (1), (2), and (3) may be solved simultaneously for $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ in any one of several different ways. One of the simplest and most straightforward methods of effecting solutions of simultaneous circuit equations is the determinant method. When not more than three-row, three-column determinant theory is involved the determinant method of solution is extremely compact and concise. Any simple reorganization of the basic equations which will reduce the order of the determinant involved should not be overlooked. In general, the order of the determinant can be reduced by systematically eliminating certain currents from the voltage equations. The number of simultaneous equations is thereby reduced and the solution simplified. For example, any one of the three currents can be eliminated from equations (2) and (3) by substituting for it its value in terms of the other two currents. Thus $\mathbf{I}_{c'c}$ can be eliminated by substituting for it its value $(-\mathbf{I}_{a'a} - \mathbf{I}_{b'b})$. The device of eliminating certain currents by means of substituting current equation relations into the voltage equations is an expedient which can be employed frequently. It may be employed when labeling the diagram as illustrated on page 325.

Upon the elimination of $\mathbf{I}_{c'c}$ from equations (2) and (3) the following relations are obtained:

$$(\mathbf{Z}_{c'a'} + \mathbf{Z}_{a'a} + \mathbf{Z}_{ac} + \mathbf{Z}_{c'c})\mathbf{I}_{a'a} + \mathbf{Z}_{c'c}\mathbf{I}_{b'b} = \mathbf{E}_{c'a'} \tag{4}$$

$$\mathbf{Z}_{c'c}\mathbf{I}_{a'a} + (\mathbf{Z}_{c'b'} + \mathbf{Z}_{b'b} + \mathbf{Z}_{bc} + \mathbf{Z}_{c'c})\mathbf{I}_{b'b} = \mathbf{E}_{c'b'} \tag{5}$$

For the sake of simplicity in writing, the following abbreviations will be adopted:

$$(\mathbf{Z}_{c'a'} + \mathbf{Z}_{a'a} + \mathbf{Z}_{ac} + \mathbf{Z}_{c'c}) = \mathbf{Z}_1 \tag{6}$$

$$(\mathbf{Z}_{c'b'} + \mathbf{Z}_{b'b} + \mathbf{Z}_{bc} + \mathbf{Z}_{c'c}) = \mathbf{Z}_2 \tag{7}$$

$$\mathbf{Z}_{c'c} = \mathbf{Z}_c \tag{8}$$

Equations (4) and (5) then become:

$$\mathbf{Z}_1\mathbf{I}_{a'a} + \mathbf{Z}_c\mathbf{I}_{b'b} = \mathbf{E}_{c'a'} \tag{9}$$

$$\mathbf{Z}_c\mathbf{I}_{a'a} + \mathbf{Z}_2\mathbf{I}_{b'b} = \mathbf{E}_{c'b'} \tag{10}$$

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Simultaneous solutions of the above equations for $\mathbf{I}_{a'a}$ and $\mathbf{I}_{b'b}$ yield:

$$\mathbf{I}_{a'a} = \frac{\begin{vmatrix} \mathbf{E}_{c'a'} & \mathbf{Z}_c \\ \mathbf{E}_{c'b'} & \mathbf{Z}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_c \\ \mathbf{Z}_c & \mathbf{Z}_2 \end{vmatrix}} = \frac{\mathbf{E}_{c'a'}\mathbf{Z}_2 - \mathbf{E}_{c'b'}\mathbf{Z}_c}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_c^2} \tag{11}$$

$$\mathbf{I}_{b'b} = \frac{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{E}_{c'a'} \\ \mathbf{Z}_c & \mathbf{E}_{c'b'} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_c \\ \mathbf{Z}_c & \mathbf{Z}_2 \end{vmatrix}} = \frac{\mathbf{E}_{c'b'}\mathbf{Z}_1 - \mathbf{E}_{c'a'}\mathbf{Z}_c}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_c^2} \tag{12}$$

$\mathbf{I}_{c'c}$ can be evaluated in terms of $\mathbf{I}_{a'a}$ and $\mathbf{I}_{b'b}$.

$$\mathbf{I}_{c'c} = -\mathbf{I}_{a'a} - \mathbf{I}_{b'b} \tag{13}$$

After the branch currents have been determined, the component voltages and powers follow directly. For example, the load voltages are:

$$\mathbf{V}_{ac} = \mathbf{Z}_{ac}\mathbf{I}_{a'a} \tag{14}$$

$$\mathbf{V}_{bc} = \mathbf{Z}_{bc}\mathbf{I}_{b'b} \tag{15}$$

The powers generated by the individual alternator phases are:

$$W_{c'a'(\text{gen.})} = E_{c'a'}I_{a'a}\cos\theta\Big]_{\mathbf{I}_{a'a}}^{\mathbf{E}_{c'a'}} \tag{16}$$

$$W_{c'b'(\text{gen.})} = E_{c'b'}I_{b'b}\cos\theta\Big]_{\mathbf{I}_{b'b}}^{\mathbf{E}_{c'b'}} \tag{17}$$

General expressions may also be written for the generator impedance voltage drops, line drops, etc.

It should be noted that the above analysis is of a fairly general character. Although the individual impedances are given particular form in Fig. 5, no limitations as to the nature of the individual impedances have been imposed. It is understood of course, that the actual circuit parameters involved in an analysis of this kind must be constant. No limitations other than sinusoidal wave form have been imposed upon the nature of the driving voltages $\mathbf{E}_{c'a'}$ and $\mathbf{E}_{c'b'}$. These voltages might represent any two generated voltages of sinusoidal wave form because at this stage of the analysis the voltages in question have not been restricted as to magnitude or relative time phase. In fact, the voltages $\mathbf{E}_{c'a'}$ and $\mathbf{E}_{c'b'}$ could be d-c driving voltages and the derived expressions would hold equally well. Therefore the generalized derivations contained in equations (11) to (17) inclusive apply to any network of the type depicted in Fig. 5.

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Example 3. It will be assumed that the generated voltages in Fig. 5 are: $\mathbf{E}_{c'a'} = 100\,\underline{/0^\circ}$ volts and $\mathbf{E}_{c'b'} = 100\,\underline{/90^\circ}$ volts. These voltages represent a two-phase system of generated emf's, the phase sequence of which is $c'b' - c'a'$. The particular impedances that are to be considered will be assumed to have the following numerical values:

$$\mathbf{Z}_{c'a'} = \mathbf{Z}_{c'b'} = (1 + j3) \text{ ohms}$$
$$\mathbf{Z}_{a'a} = \mathbf{Z}_{b'b} = (1 + j1) \text{ ohms}$$
$$\mathbf{Z}_{c'c} = (2 + j2) \text{ ohms}$$
$$\mathbf{Z}_{ac} = (12 + j21.2) \text{ ohms}$$
$$\mathbf{Z}_{bc} = (15 - j16.6) \text{ ohms}$$

Branch currents, load voltages, and generated powers are to be determined. Employing the abbreviations adopted in equations (6), (7), and (8):

$$\mathbf{Z}_1 = 16 + j27.2 = 31.5\,\underline{/59.5^\circ} \text{ ohms}$$
$$\mathbf{Z}_2 = 19 - j10.6 = 21.7\,\underline{/-29.2^\circ} \text{ ohms}$$
$$\mathbf{Z}_c = 2.0 + j2.0 = 2.83\,\underline{/45^\circ} \text{ ohms}$$

The common denominator of equations (11) and (12) for this particular numerical example is:

$$\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_c^2 = (684\,\underline{/30.3^\circ}) - (8\,\underline{/90^\circ}) = 680\,\underline{/29.7^\circ}$$

$$\mathbf{I}_{a'a} = \frac{(100\,\underline{/0^\circ})\,(21.7\,\underline{/-29.2^\circ}) - (100\,\underline{/90^\circ})\,(2.83\,\underline{/45^\circ})}{680\,\underline{/29.7^\circ}}$$

$$= \frac{2450\,\underline{/-31.0^\circ}}{680\,\underline{/29.7^\circ}} = 3.60\,\underline{/-60.7^\circ}$$

$$= (1.76 - j3.14) \text{ amperes}$$

$$\mathbf{I}_{b'b} = \frac{(100\,\underline{/90^\circ})\,(31.5\,\underline{/59.5^\circ}) - (100\,\underline{/0^\circ})\,(2.83\,\underline{/45^\circ})}{680\,\underline{/29.7^\circ}}$$

$$= \frac{3230\,\underline{/154.4^\circ}}{680\,\underline{/29.7^\circ}} = 4.75\,\underline{/124.7^\circ}$$

$$= (-2.70 + j3.90) \text{ amperes}$$

$$\mathbf{I}_{c'c} = -\mathbf{I}_{a'a} - \mathbf{I}_{b'b}$$
$$= -(1.76 - j3.14) - (-2.70 + j3.90)$$
$$= 0.94 - j0.76 = 1.21\,\underline{/-39.0^\circ} \text{ amperes}$$

$$\mathbf{V}_{ac} = \mathbf{I}_{a'a}\mathbf{Z}_{ac} = (3.60\,\underline{/-60.7^\circ})\,(24.4\,\underline{/60.5^\circ})$$
$$= 87.8\,\underline{/-0.2^\circ} \text{ volts}$$

$$\mathbf{V}_{bc} = \mathbf{I}_{b'b}\mathbf{Z}_{bc} = (4.75\,\underline{/124.7^\circ})\,(22.4\,\underline{/-47.9^\circ})$$
$$= 106.4\,\underline{/76.8^\circ} \text{ volts}$$

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Figure 6 is a vector diagram of the voltages and currents.

$$W_{c'a'(\text{gen.})} = \left[ei + e'i' \right]_{I_{a'a}}^{E_{c'a'}}$$
$$= 100 \times 1.76 = 176 \text{ watts}$$
$$W_{c'b'(\text{gen.})} = \left[ei + e'i' \right]_{I_{b'b}}^{E_{c'b'}}$$
$$= 100 \times 3.90 = 390 \text{ watts}$$

The total power generated is:

$$176 + 390 = 566 \text{ watts}$$

The total power absorbed or spent in the entire system is:

$$(3.60^2 \times 14) + (4.75^2 \times 17) + (1.21^2 \times 2) = 567 \text{ watts}$$

Obviously, the small discrepancy between the generated power and the absorbed power is the result of the approximate manner in which the calculations were performed.

Problem 4. Work through all details of the above numerical example for the opposite phase sequence of voltages. Let

$$E_{c'a'} = 100 \underline{/90^\circ} \text{ volts and } E_{c'b'} = 100 \underline{/0^\circ} \text{ volts}$$

Ans.: $P_{\text{total}} = 566$ watts.

$$I_{a'a} = 2.79 \underline{/33.4^\circ},\ I_{b'b} = 4.54 \underline{/24.7^\circ},\ I_{c'c} = 7.32 \underline{/-152^\circ} \text{ amperes}$$

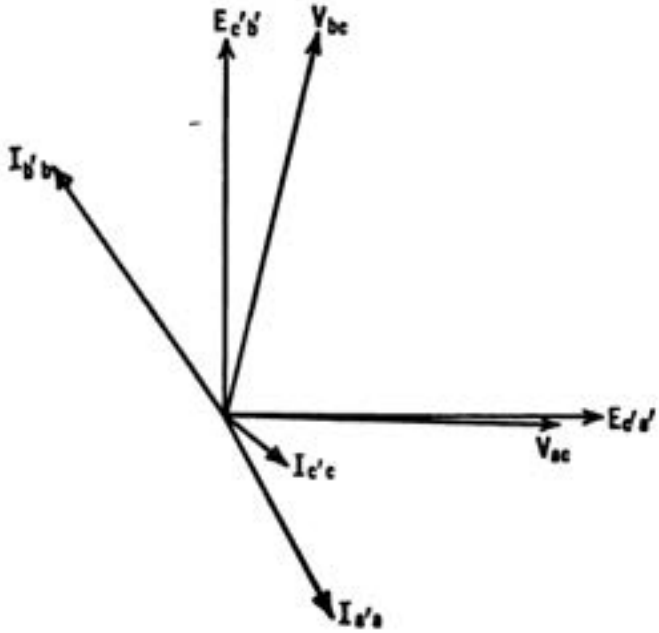

FIG. 6. A vector diagram of the voltages and currents involved in example 3.

Positive Circuit Directions. A great deal of needless confusion exists in the minds of many students relative to the correct positive circuit directions of the quantities involved in polyphase circuit analysis. The basic principles concerning circuit direction have been presented in the earlier chapters. (See pages 52–53, 222–223, and 264.) These princi-

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ples are, of course, entirely applicable to polyphase circuits as well as to single-phase circuits.

In general, all generated emf's in polyphase systems have specified *relative* polarities and angular positions with respect to one another. This information must be known either directly or indirectly if the circuit investigation is to proceed. For example, if a three-phase alternator is connected in wye it may be assumed that the individual phases are connected subtractively at a common junction as shown in Fig. 7. It is only by means of subtractive polarities that a three-phase, wye-connected machine can give balanced line-to-line voltages. Unless otherwise specified, the individual phase generated emf's of a three-phase machine may be assumed to be 120° apart in time phase. The foregoing facts are sufficient for a specification of the positive circuit directions in the network shown in Fig. 7.

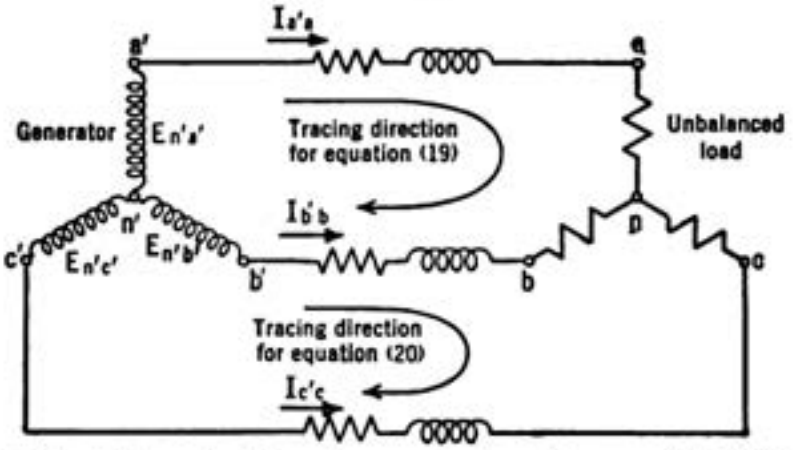

FIG. 7. A three-wire three-phase network. (See pages 319–320.)

A positive circuit direction may be arbitrarily assigned to any one generated emf. For example, if the a phase generated emf in Fig. 7 is considered, either $\mathbf{E}_{n'a'}$ or $\mathbf{E}_{a'n'}$ may be taken as positive. One of these having been selected as positive, the positive circuit directions of the other systematically labeled emf's are fixed because of the relatively fixed polarities that the generated emf's bear toward one another. If $\mathbf{E}_{n'a'}$ is taken as positive then $\mathbf{E}_{n'b'}$ is positive because, in tracing from n' to a' to b' to n', the two voltages must be in circuit opposition. In a similar manner $\mathbf{E}_{n'c'}$ must be taken as positive if $\mathbf{E}_{n'a'}$ is originally selected as positive. Thus either of the two following systems of generated voltages may be employed in analyzing the network shown in Fig. 7.

(1) $$\mathbf{E}_{n'a'}, \mathbf{E}_{n'b'}, \mathbf{E}_{n'c'}$$

or

(2) $$\mathbf{E}_{a'n'}, \mathbf{E}_{b'n'}, \mathbf{E}_{c'n'}$$

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In the simple case shown in Fig. 7, one and only one source of emf is present in each branch. The obvious procedure is to assign to the individual branch currents positive circuit directions which agree with those assigned to the generated voltages. It should be noted, however, that this obvious procedure is not necessary. The positive circuit directions of the currents can be arbitrarily assigned — some in the direction of their generated voltages and others against. Due consideration must be given to the assigned positive directions of the currents and voltages, particularly during the establishment of Kirchhoff's current and emf equations. If all quantities are entered into these equations with their proper signs then the current solutions obtained will be those for the currents in the arbitrarily assigned directions. To obtain the expression for current in the opposite direction through any particular branch it is simply necessary to reverse the sign of the branch current originally found.

If two or more emf's in any one branch are connected in additive series then the positive circuit direction assigned to the current in this branch will probably agree with that which has been assigned as the positive circuit direction of generated voltage. If a branch contains two generated voltages which are connected in subtractive series it is plain that the positive circuit direction of current will agree with the positive circuit direction of one emf and disagree with the other.

Two general methods are employed in writing the basic Kirchhoff equations of a given network. One is to affix arrows on the circuit diagram which designate the assumed positive circuit directions of the currents and voltages; the other is to employ double-subscript notation which automatically states the positive circuit directions of the currents or voltages that are being considered. The two schemes are entirely equivalent, and each has its particular field of usefulness.

The Wye-Wye System with No Neutral Connection. The three-phase, wye-connected generator shown in Fig. 7 is connected to a three-phase, wye-connected load through line impedances $\mathbf{Z}_{a'a}$, $\mathbf{Z}_{b'b}$, and $\mathbf{Z}_{c'c}$. The Kirchhoff current and voltage equations for this particular network can be written in terms of double-subscript notation as follows:

$$\mathbf{I}_{a'a} + \mathbf{I}_{b'b} + \mathbf{I}_{c'c} = 0 \tag{18}$$

$$(\mathbf{Z}_{n'a'} + \mathbf{Z}_{a'a} + \mathbf{Z}_{an})\mathbf{I}_{a'a} - (\mathbf{Z}_{n'b'} + \mathbf{Z}_{b'b} + \mathbf{Z}_{bn})\mathbf{I}_{b'b} = \mathbf{E}_{n'a'} - \mathbf{E}_{n'b'} \tag{19}$$

$$(\mathbf{Z}_{n'b'} + \mathbf{Z}_{b'b} + \mathbf{Z}_{bn})\mathbf{I}_{b'b} - (\mathbf{Z}_{n'c'} + \mathbf{Z}_{c'c} + \mathbf{Z}_{cn})\mathbf{I}_{c'c} = \mathbf{E}_{n'b'} - \mathbf{E}_{n'c'} \tag{20}$$

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The tracing directions employed in writing equations (19) and (20) are indicated in Fig. 7. Equations (18), (19), and (20) may be solved simultaneously for $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ in a manner almost identical with that employed in solving equations (1), (2), and (3) of the present chapter.

In manipulating equations (19) and (20) either analytically or numerically, it is convenient to adopt the following abbreviations:

$$\mathbf{E}_{n'a'} - \mathbf{E}_{n'b'} = \mathbf{E}_{b'a'} \qquad \text{(See Fig. 7)} \tag{21}$$

$$\mathbf{E}_{n'b'} - \mathbf{E}_{n'c'} = \mathbf{E}_{c'b'} \tag{22}$$

$$\mathbf{Z}_{n'a'} + \mathbf{Z}_{a'a} + \mathbf{Z}_{an} = \mathbf{Z}_a \tag{23}$$

$$\mathbf{Z}_{n'b'} + \mathbf{Z}_{b'b} + \mathbf{Z}_{bn} = \mathbf{Z}_b \tag{24}$$

$$\mathbf{Z}_{n'c'} + \mathbf{Z}_{c'c} + \mathbf{Z}_{cn} = \mathbf{Z}_c \tag{25}$$

If the above abbreviations are adopted and if $\mathbf{I}_{c'c}$ is eliminated from equation (20), equations (19) and (20) reduce directly to

$$\mathbf{Z}_a\mathbf{I}_{a'a} - \mathbf{Z}_b\mathbf{I}_{b'b} = \mathbf{E}_{b'a'} \tag{26}$$

$$\mathbf{Z}_c\mathbf{I}_{a'a} + (\mathbf{Z}_b + \mathbf{Z}_c)\mathbf{I}_{b'b} = \mathbf{E}_{c'b'} \tag{27}$$

Problem 5. Solve equations (26) and (27) explicitly for $\mathbf{I}_{a'a}$ and $\mathbf{I}_{b'b}$ in terms of the **E**'s and **Z**'s, employing determinants.

Ans.: $$\mathbf{I}_{a'a} = \frac{\mathbf{E}_{b'a'}(\mathbf{Z}_b + \mathbf{Z}_c) + \mathbf{E}_{c'b'}\mathbf{Z}_b}{\mathbf{Z}_a(\mathbf{Z}_b + \mathbf{Z}_c) + \mathbf{Z}_b\mathbf{Z}_c} = \frac{\mathbf{E}_{b'a'}(\mathbf{Z}_b + \mathbf{Z}_c) + \mathbf{E}_{c'b'}\mathbf{Z}_b}{\mathbf{D}}$$

$$\mathbf{I}_{b'b} = \frac{\mathbf{E}_{c'b'}\mathbf{Z}_a - \mathbf{E}_{b'a'}\mathbf{Z}_c}{\mathbf{D}}, \quad (\mathbf{I}_{c'c} = -\mathbf{I}_{a'a} - \mathbf{I}_{b'b}).$$

Single-Subscript Notation. In symmetrically arranged networks of the kind shown in Figs. 7 and 8, single-subscript notation may be used to advantage and is actually preferred by some circuit analyzers over double-subscript notation in those cases where no ambiguity can possibly arise. The scheme usually employed is to consider the positive circuit directions of both the *E*'s and *I*'s as being *out* from the generator thus giving load voltage drops ($\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$) as shown in Fig. 8. If arrows are placed on the circuit diagram as in Fig. 8, the positive circuit directions of the **E**'s, **I**'s, and **V**'s are clearly defined. In terms of single-subscript notation, the basic equations for the network shown in Fig. 8 are:

$$\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0 \tag{28}$$

$$(\mathbf{Z}_g + \mathbf{Z}_1)\mathbf{I}_a - (\mathbf{Z}_g + \mathbf{Z}_2)\mathbf{I}_b = \mathbf{E}_a - \mathbf{E}_b \tag{29}$$

$$(\mathbf{Z}_g + \mathbf{Z}_2)\mathbf{I}_b - (\mathbf{Z}_g + \mathbf{Z}_3)\mathbf{I}_c = \mathbf{E}_b - \mathbf{E}_c \tag{30}$$

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The relation

$$(\mathbf{Z}_g + \mathbf{Z}_3)\mathbf{I}_c - (\mathbf{Z}_g + \mathbf{Z}_1)\mathbf{I}_a = \mathbf{E}_c - \mathbf{E}_a \tag{31}$$

might, of course, be used in place of either equation (29) or (30).

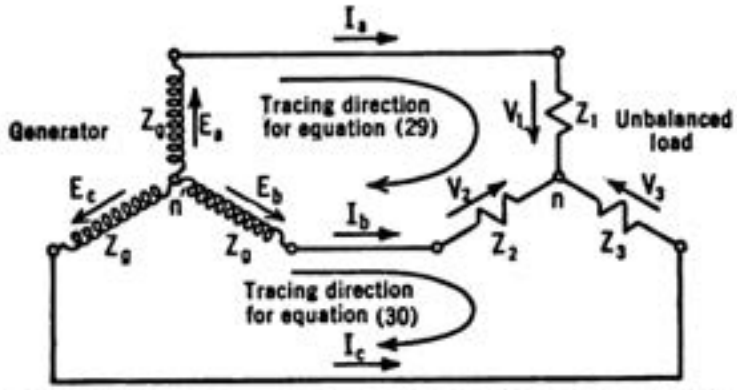

FIG. 8. A wye-wye circuit arrangement in which the positive circuit directions are indicated by means of arrows.

If the E's and Z's are known either directly or indirectly it is possible to solve equations (28), (29), and (30) simultaneously for $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$. It should be noted that the omission of the connecting line impedances in the network shown in Fig. 8 places no serious limitation on the generality of the solution because all impedances from n' to n in any given branch may be added in complex form to give a single impedance.

If $\mathbf{I}_c$ is eliminated from equation (30), the basic relations stated in equations (28), (29), and (30) reduce to:

$$(\mathbf{Z}_g + \mathbf{Z}_1)\mathbf{I}_a - (\mathbf{Z}_g + \mathbf{Z}_2)\mathbf{I}_b = \mathbf{E}_a - \mathbf{E}_b \tag{29}$$

$$(\mathbf{Z}_g + \mathbf{Z}_3)\mathbf{I}_a + (2\mathbf{Z}_g + \mathbf{Z}_2 + \mathbf{Z}_3)\mathbf{I}_b = \mathbf{E}_b - \mathbf{E}_c \tag{32}$$

The common denominator of the solutions for $\mathbf{I}_a$ and $\mathbf{I}_b$ in determinant form is:

$$\mathbf{D} = \begin{vmatrix} (\mathbf{Z}_g + \mathbf{Z}_1) & -(\mathbf{Z}_g + \mathbf{Z}_2) \\ (\mathbf{Z}_g + \mathbf{Z}_3) & (2\mathbf{Z}_g + \mathbf{Z}_2 + \mathbf{Z}_3) \end{vmatrix} \tag{33}$$

After $\mathbf{D}$ has been defined, the solutions for the three branch currents can be written as follows:

$$\mathbf{I}_a = \frac{\begin{vmatrix} (\mathbf{E}_a - \mathbf{E}_b) & -(\mathbf{Z}_g + \mathbf{Z}_2) \\ (\mathbf{E}_b - \mathbf{E}_c) & (2\mathbf{Z}_g + \mathbf{Z}_2 + \mathbf{Z}_3) \end{vmatrix}}{\mathbf{D}} \tag{34}$$

$$\mathbf{I}_b = \frac{\begin{vmatrix} (\mathbf{Z}_g + \mathbf{Z}_1) & (\mathbf{E}_a - \mathbf{E}_b) \\ (\mathbf{Z}_g + \mathbf{Z}_3) & (\mathbf{E}_b - \mathbf{E}_c \end{vmatrix}}{\mathbf{D}} \tag{35}$$

$$\mathbf{I}_c = -\mathbf{I}_a - \mathbf{I}_b \tag{36}$$

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After the branch currents have been evaluated in any particular case, the component voltages and powers in the three-wire, three-phase system can be determined in terms of elementary circuit theory.

Example 4. In Fig. 8 it will be assumed that:

$$\mathbf{E}_a = 1000 + j0 = 1000 \angle 0^\circ \text{ volts}$$

$$\mathbf{E}_b = -500 - j866 = 1000 \angle -120^\circ \text{ volts}$$

$$\mathbf{E}_c = -500 + j866 = 1000 \angle -240^\circ \text{ volts}$$

$$\mathbf{Z}_1 = 20 + j20 = 28.28 \angle 45^\circ \text{ ohms}$$

$$\mathbf{Z}_2 = 50 + j0 = 50.0 \angle 0^\circ \text{ ohms}$$

$$\mathbf{Z}_3 = 30 + j52 = 60.0 \angle 60^\circ \text{ ohms}$$

$$\mathbf{Z}_g = 2 + j8 = 8.25 \angle 76^\circ \text{ ohms}$$

The following quantities are to be evaluated: $\mathbf{I}_a$, $\mathbf{I}_b$, $\mathbf{I}_c$, $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $(\mathbf{V}_1 - \mathbf{V}_2)$, $(\mathbf{V}_2 - \mathbf{V}_3)$, and $(\mathbf{V}_3 - \mathbf{V}_1)$.

Certain voltage and impedance combinations will first be evaluated in order to simplify the numerical expressions which are to follow.

$$(\mathbf{E}_a - \mathbf{E}_b) = 1500 + j866 = 1732 \angle 30^\circ \text{ volts}$$

$$(\mathbf{E}_b - \mathbf{E}_c) = 0 - j1732 = 1732 \angle -90^\circ \text{ volts}$$

$$(\mathbf{Z}_g + \mathbf{Z}_1) = 22 + j28 = 35.6 \angle 51.8^\circ \text{ ohms}$$

$$(\mathbf{Z}_g + \mathbf{Z}_2) = 52 + j8 = 52.6 \angle 8.8^\circ \text{ ohms}$$

$$(\mathbf{Z}_g + \mathbf{Z}_3) = 32 + j60 = 68.0 \angle 61.9^\circ \text{ ohms}$$

$$(2\mathbf{Z}_g + \mathbf{Z}_2 + \mathbf{Z}_3) = 84 + j68 = 108.0 \angle 39.0^\circ \text{ ohms}$$

By substituting the numerical values of the above impedances in equation (33):

$$\mathbf{D} = \begin{vmatrix} (35.6 \angle 51.8^\circ) & -(52.6 \angle 8.8^\circ) \\ (68.0 \angle 61.9^\circ) & (108 \angle 39.0^\circ) \end{vmatrix}$$

$$= 1130 + j7225 = 7310 \angle 81.1^\circ \text{ ohms}^2$$

The values of $\mathbf{I}_a$ and $\mathbf{I}_b$, determined with the aid of equations (34) and (35), are:

$$\mathbf{I}_a = \frac{\begin{vmatrix} (1732 \angle 30^\circ) & -(52.6 \angle 8.8^\circ) \\ (1732 \angle -90^\circ) & (108 \angle 39.0^\circ) \end{vmatrix}}{7310 \angle 81.1^\circ}$$

$$= \frac{81{,}110 + j84{,}650}{7310 \angle 81.1^\circ}$$

$$= \frac{117{,}000 \angle 46.2^\circ}{7310 \angle 81.1^\circ} = 16.0 \angle -34.9^\circ \text{ amperes}$$

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$$\mathbf{I}_b = \frac{\begin{vmatrix} (35.6\ \underline{/51.8^\circ}) & (1732\ \underline{/30^\circ}) \\ (68.0\ \underline{/61.9^\circ}) & (1732\ \underline{/-90^\circ}) \end{vmatrix}}{7310\ \underline{/81.1^\circ}}$$

$$= \frac{52{,}440 - j155{,}900}{7310\ \underline{/81.1^\circ}}$$

$$= \frac{164{,}500\ \underline{/-71.4^\circ}}{7310\ \underline{/81.1^\circ}} = 22.5\ \underline{/-152.5^\circ} \text{ amperes}$$

The value of $\mathbf{I}_c$ may be determined in terms of $\mathbf{I}_a$ and $\mathbf{I}_b$.

$$\begin{aligned} \mathbf{I}_c &= -\mathbf{I}_a - \mathbf{I}_b \\ &= (-16\ \underline{/-34.9^\circ}) - (22.5\ \underline{/-152.5^\circ}) \\ &= -(13.12 - j9.15) - (-19.98 - 10.39) \\ &= 6.86 + j19.54 = 20.7\ \underline{/70.8^\circ} \end{aligned}$$

A check on a large part of the foregoing arithmetic can be obtained by testing the balance of the voltage equation which was not employed directly in the current solutions. The voltage equation referred to is:

$$(\mathbf{Z}_g + \mathbf{Z}_3)\mathbf{I}_c - (\mathbf{Z}_g + \mathbf{Z}_1)\mathbf{I}_a = \mathbf{E}_c - \mathbf{E}_a$$

Numerically, the above equation becomes:

$$(32 + j60)\ (6.86 + j19.54) - (22 + j28)\ (13.12 - j9.15) = -1500 + j866$$

or

$$-1497 + j870 = -1500 + j866 \qquad \textit{(Check)}$$

The phase load voltages are:

$$\begin{aligned} \mathbf{V}_1 &= \mathbf{Z}_1\mathbf{I}_a = (28.28\ \underline{/45^\circ})\ (16.0\ \underline{/-34.9^\circ}) \\ &= 452.5\ \underline{/10.1^\circ} = (445 + j79.3) \text{ volts} \\ \mathbf{V}_2 &= \mathbf{Z}_2\mathbf{I}_b = (50.0\ \underline{/0^\circ})\ (22.5\ \underline{/-152.5^\circ}) \\ &= 1125\ \underline{/-152.5^\circ} = (-998 - j520) \text{ volts} \\ \mathbf{V}_3 &= \mathbf{Z}_3\mathbf{I}_c = (60.0\ \underline{/60^\circ})\ (20.7\ \underline{/70.8^\circ}) \\ &= 1242\ \underline{/130.8^\circ} = (-811 + j941) \text{ volts} \end{aligned}$$

The line-to-line load voltages are:

$$\begin{aligned} \mathbf{V}_1 - \mathbf{V}_2 &= (445 + j79.3) - (-998 - j520) \\ &= 1443 + j599 = 1562\ \underline{/22.6^\circ} \text{ volts} \\ \mathbf{V}_2 - \mathbf{V}_3 &= (-998 - j520) - (-811 + j941) \\ &= -187 - j1461 = 1474\ \underline{/-97.3^\circ} \text{ volts} \\ \mathbf{V}_3 - \mathbf{V}_1 &= (-811 + j941) - (445 + j79.3) \\ &= -1256 + j862 = 1522\ \underline{/145.5^\circ} \text{ volts} \end{aligned}$$

As a check on a large portion of the load voltage calculations let the following relation be checked for balance:

$$\mathbf{V}_1 - \mathbf{V}_2 = (\mathbf{E}_a - \mathbf{I}_a\mathbf{Z}_g) - (\mathbf{E}_b - \mathbf{I}_b\mathbf{Z}_g)$$

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Numerically the above relation becomes:

$$1443 + j599 \approx (1000 + j0 - 99 - j87) - (-500 - j866 - 43 + j181)$$
$$\approx 1444 + j598 \qquad \textit{(Check)}$$

Vector diagrams of the principal voltages and currents involved in the above example are shown in Fig. 9.

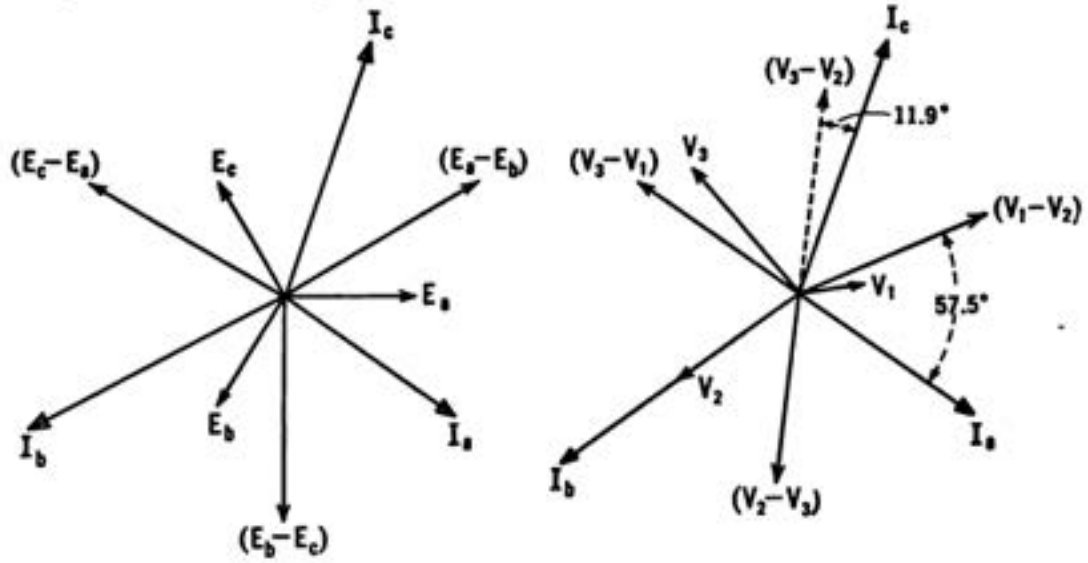

FIG. 9. Vector diagrams for example 4 and Problem 6.

Exercise for Comparing Solutions by Single- and Double-Subscript Notations in a Polyphase Network. Rewrite the above network solution in terms of double-subscript notation employing the labels given in Fig. 7. For example,

$\mathbf{E}_a$, $\mathbf{E}_b$, and $\mathbf{E}_c$ of Fig. 8 are, respectively, $\mathbf{E}_{n'a'}$, $\mathbf{E}_{n'b'}$, and $\mathbf{E}_{n'c'}$ of Fig. 7.

$\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ of Fig. 8 are, respectively, $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ of Fig. 7.

$(\mathbf{E}_a-\mathbf{E}_b)$ of Fig. 8 is $\mathbf{E}_{b'a'}$ of Fig. 7, etc.

Problem 6. Two wattmeters, W_a and W_c, are to be connected into the three-phase circuit shown in Fig. 8 in the following manner:

W_a	W_c
Current coil carries I_a	Current coil carries I_c
Potential circuit across $(\mathbf{V}_1-\mathbf{V}_2)$	Potential circuit across $(\mathbf{V}_3-\mathbf{V}_2)$

(*a*) Draw the circuit diagram showing the location of each wattmeter.

(*b*) Calculate the readings of W_a and W_c from the current and voltage solutions of the above illustrative example.

(*c*) Compare the sum of the two wattmeter readings with $(I_a^2R_1 + I_b^2R_2 + I_c^2R_3)$, the total power absorbed by the load, recognizing that slide-rule calculations were employed in example 4.

Ans: (*b*) W_a reads 13,430 watts, W_c reads 29,900 watts.
(*c*) $\sum$ load I^2R's = 43,390 watts.

Through an application of Kirchhoff's current law, when labeling a circuit diagram it is possible to eliminate the necessity of writing the usual current equations to effect a solution. The only equations required are the ordinary loop equations. Consider the circuit shown in

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Fig. 10. The currents in the $\mathbf{Z}_1$ and $\mathbf{Z}_2$ branches are labeled $\mathbf{I}_a$ and $\mathbf{I}_b$ respectively as shown. Since there is only one remaining branch at the junction of $\mathbf{Z}_1$ and $\mathbf{Z}_2$, the current in it may be written on the diagram

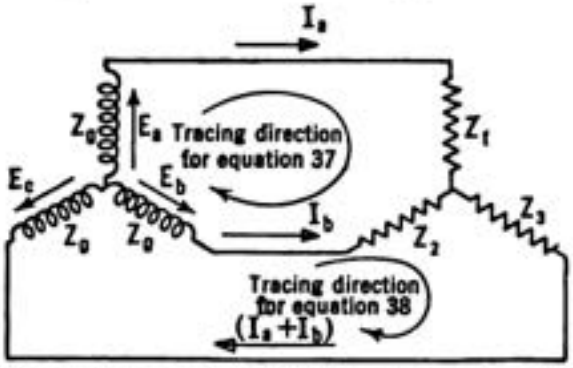

Fig. 10. A method of labeling which eliminates the writing of junction equations.

in terms of the currents $\mathbf{I}_a$ and $\mathbf{I}_b$, already assumed, as $(\mathbf{I}_a + \mathbf{I}_b)$. This fact follows from Kirchhoff's current law. There are now only two unknown currents, and two independent equations will suffice. An attempt at setting up a junction equation will show its futility. The only equations necessary are therefore two independent loop equations as follows:

$$(\mathbf{Z}_g + \mathbf{Z}_1)\mathbf{I}_a - (\mathbf{Z}_g + \mathbf{Z}_2)\mathbf{I}_b = \mathbf{E}_a - \mathbf{E}_b \tag{37}$$

and

$$(\mathbf{Z}_g + \mathbf{Z}_2)\mathbf{I}_b + (\mathbf{Z}_g + \mathbf{Z}_3)(\mathbf{I}_a + \mathbf{I}_b) = \mathbf{E}_b - \mathbf{E}_c$$

or if rearranged

$$(\mathbf{Z}_g + \mathbf{Z}_3)\mathbf{I}_a + (2\mathbf{Z}_g + \mathbf{Z}_2 + \mathbf{Z}_3)\mathbf{I}_b = \mathbf{E}_b - \mathbf{E}_c \tag{38}$$

Equations (37) and (38) are seen to be the same as equations (29) and (32) respectively. In a network consisting of many branches application of the above principle usually greatly reduces the number of simultaneous equations to be solved, and therefore makes the task of solving for the unknowns much less cumbersome.

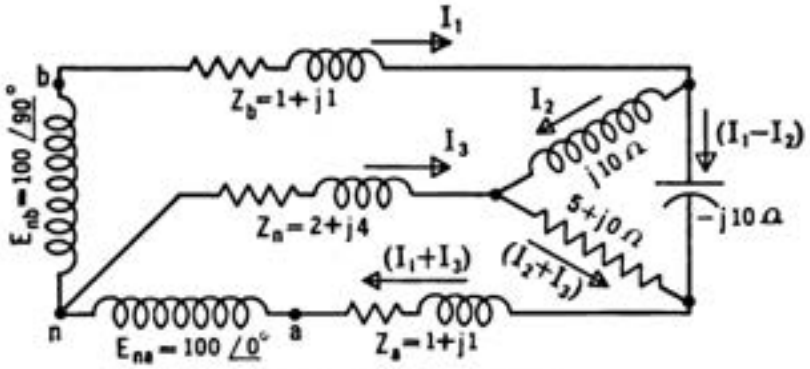

Fig. 11. See Problem 7.

Problem 7. Study the details of the labeling in Fig. 11, set up the necessary equations, and solve for $\mathbf{I}_1$.

Ans.: $-5.38 - j2.9$ amperes.

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The Wye-Wye System with Neutral Connection. Four-wire, three-phase systems similar to the one shown in Fig. 12 are sometimes employed in the transmission and distribution of electrical energy. The connection of the point n' of the wye-connected generator (or transformer bank) to the point n of the wye-connected load distinguishes Fig. 12 from the three-wire, three-phase systems shown in Figs. 7 and 8.

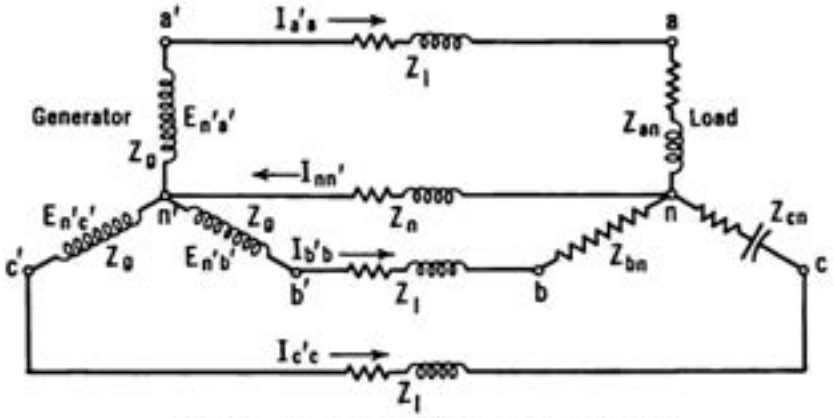

FIG. 12. A four-wire three-phase system.

In general, the details involved in solving for $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, $\mathbf{I}_{c'c}$, and $\mathbf{I}_{nn'}$ of Fig. 12 are similar to those which have been presented for the wye-wye system without neutral connection. If the wye-wye system of Fig. 12 is solved straightforwardly by the determinant method, three-row, three-column matrices are encountered, and a considerable amount of labor is involved in effecting a complete solution in a perfectly general case. Due to the inherent symmetry of the basic voltage equations, however, several simplifications may be made. If, for example, Kirchhoff's emf law is applied to loops $n'a'ann'$, $n'b'bnn'$, and $n'c'cnn'$, it is plain that

$$\mathbf{I}_{a'a} = \frac{\mathbf{E}_{n'a'} - \mathbf{I}_{nn'}\mathbf{Z}_n}{(\mathbf{Z}_g + \mathbf{Z}_l + \mathbf{Z}_{an})}; \quad \mathbf{I}_{b'b} = \frac{\mathbf{E}_{n'b'} - \mathbf{I}_{nn'}\mathbf{Z}_n}{(\mathbf{Z}_g + \mathbf{Z}_l + \mathbf{Z}_{bn})};$$

$$\mathbf{I}_{c'c} = \frac{\mathbf{E}_{n'c'} - \mathbf{I}_{nn'}\mathbf{Z}_n}{(\mathbf{Z}_g + \mathbf{Z}_l + \mathbf{Z}_{cn})}$$

Since

$$\mathbf{I}_{a'a} + \mathbf{I}_{b'b} + \mathbf{I}_{c'c} = \mathbf{I}_{nn'} \tag{39}$$

it follows that

$$\frac{\mathbf{E}_{n'a'} - \mathbf{I}_{nn'}\mathbf{Z}_n}{\mathbf{Z}_a} + \frac{\mathbf{E}_{n'b'} - \mathbf{I}_{nn'}\mathbf{Z}_n}{\mathbf{Z}_b} + \frac{\mathbf{E}_{n'c'} - \mathbf{I}_{nn'}\mathbf{Z}_n}{\mathbf{Z}_c} = \mathbf{I}_{nn'} \tag{40}$$

where, for simplicity in writing,

$$\mathbf{Z}_g + \mathbf{Z}_l + \mathbf{Z}_{an} = \mathbf{Z}_a \tag{41}$$

$$\mathbf{Z}_g + \mathbf{Z}_l + \mathbf{Z}_{bn} = \mathbf{Z}_b \tag{42}$$

$$\mathbf{Z}_g + \mathbf{Z}_l + \mathbf{Z}_{cn} = \mathbf{Z}_c \tag{43}$$

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The remaining details are reserved for student analysis. (See Problem 8 below and Problem 26 at the close of the chapter.)

Problem 8. Solve equation (40) explicitly for $I_{nn'}$ and state in words how to find $I_{a'a}$, $I_{b'b}$, and $I_{c'c}$ after $I_{nn'}$ has been evaluated.

$$Ans.:\ I_{nn'} = \frac{E_{n'a'}Z_bZ_c + E_{n'b'}Z_cZ_a + E_{n'c'}Z_aZ_b}{Z_aZ_bZ_c + Z_n(Z_bZ_c + Z_cZ_a + Z_aZ_b)}.$$

The Wye-Delta System. A three-phase, wye-connected generator is shown connected to a delta-type load in Fig. 13. A study of Fig. 13 will show that there are six branch currents, $I_{a'a}$, $I_{b'b}$, $I_{c'c}$, I_{ab}, I_{bc}, and

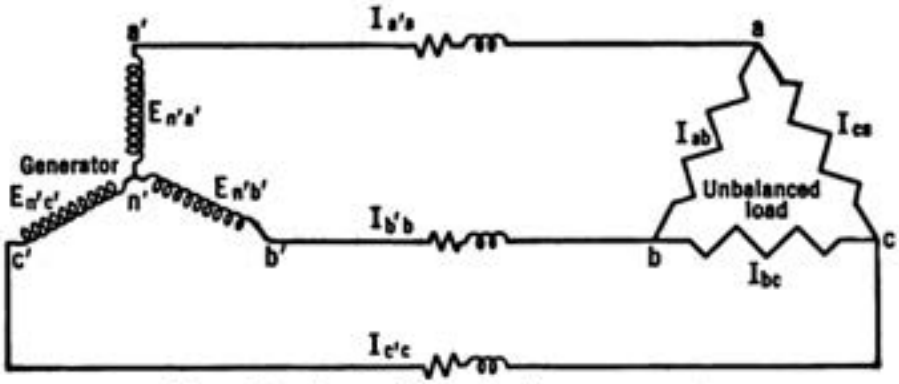

FIG. 13. A wye-delta circuit arrangement.

I_{ca}, in this network. Although these six currents might be determined by the conventional method (which in this case involves the establishment of three current equations and three emf equations) the problem is simplified if the delta-connected load is first converted to an equivalent wye-connected load. This type of conversion is given below.

Delta-to-Wye Conversion. A particular wye-delta system is shown in Fig. 14*a* where all of the impedances, including those of the connecting lines, are indicated on the diagram. The theory involved in making the conversion from the delta-connected load of Fig. 14*a* to the wye-connected load of Fig. 14*b* is given in Chapter V, pages 145–149.

As applied to Fig. 14, the generated phase voltages are assumed to be balanced at 1350 volts each and are assigned the following vector positions for purposes of illustration:

$$E_{n'a'} = 1350 + j0 \text{ volts} \tag{44}$$

$$E_{n'b'} = -675 - j1170 \text{ volts} \tag{45}$$

$$E_{n'c'} = -675 + j1170 \text{ volts} \tag{46}$$

The equivalent wye impedances are:

$$Z_{an} = \frac{Z_{ab}Z_{ca}}{(Z_{ab} + Z_{bc} + Z_{ca})} = \frac{(40 + j60)\ (50 - j20)}{190 + j40}$$
$$= (18.5 + j7.69) \text{ ohms} \tag{47}$$

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$$\mathbf{Z}_{bn} = \frac{\mathbf{Z}_{bc}\mathbf{Z}_{ab}}{(\mathbf{Z}_{ab} + \mathbf{Z}_{bc} + \mathbf{Z}_{ca})} = \frac{(100 + j0)\ (40 + j60)}{190 + j40}$$
$$= (26.53 + j26.0) \text{ ohms} \tag{48}$$

$$\mathbf{Z}_{cn} = \frac{\mathbf{Z}_{ca}\mathbf{Z}_{bc}}{(\mathbf{Z}_{ab} + \mathbf{Z}_{bc} + \mathbf{Z}_{ca})} = \frac{(50 - j20)\ (100 + j0)}{190 + j40}$$
$$= (23.1 - j15.37) \text{ ohms} \tag{49}$$

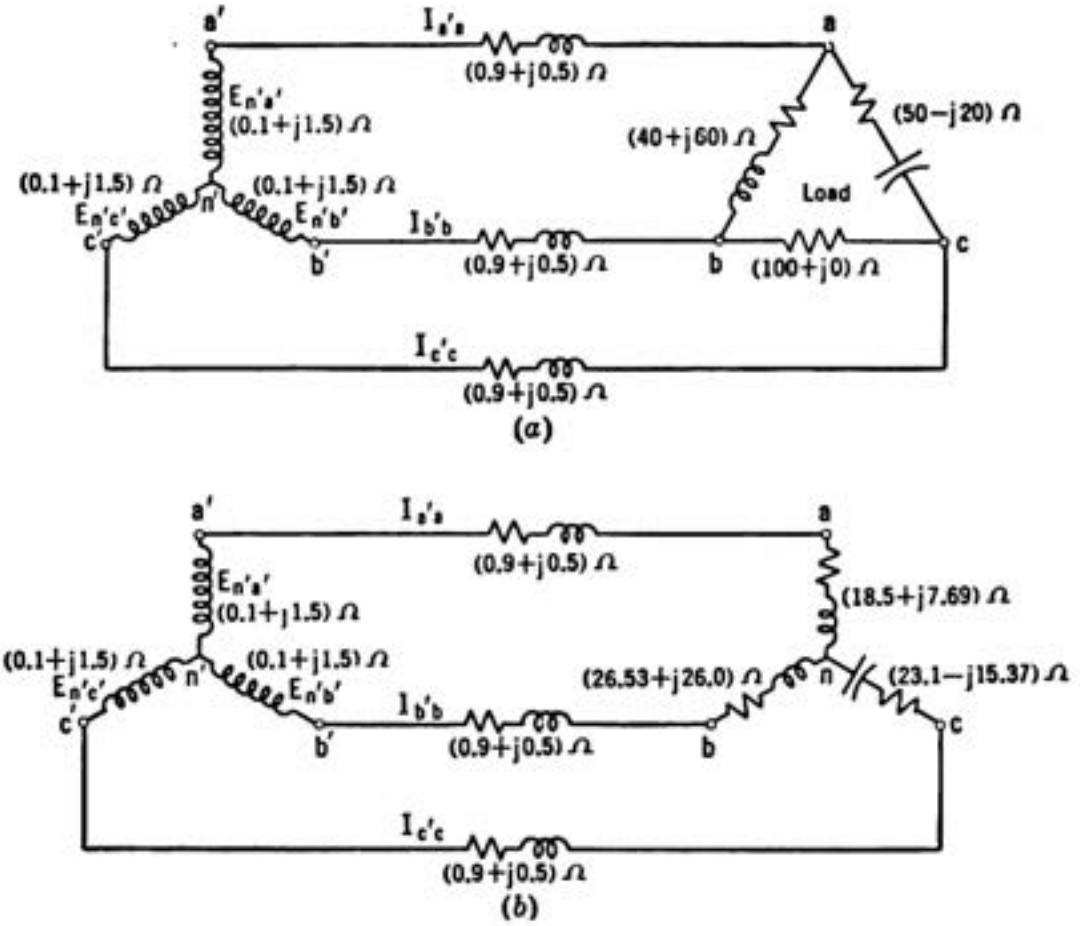

Fig. 14. Illustrating the conversion of a delta-connected load to an equivalent wye-connected load.

The equivalent wye impedances thus determined replace the original delta-connected impedances, and the circuit arrangement reduces to that shown in Fig. 14*b*. If the numerical values of the impedances and generated voltages are employed, the basic equations of the network shown in Fig. 14*b* take the following forms:

$$\mathbf{I}_{a'a} + \mathbf{I}_{b'b} + \mathbf{I}_{c'c} = 0 \tag{50}$$

$$(19.5 + j9.69)\mathbf{I}_{a'a} - (27.53 + j28.0)\mathbf{I}_{b'b} = 2025 + j1170 \tag{51}$$

$$(27.53 + j28.0)\mathbf{I}_{b'b} - (24.1 - j13.37)\mathbf{I}_{c'c} = 0 - j2340 \tag{52}$$

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The three simultaneous equations stated above can be reduced to two simultaneous equations by the simple expedient of eliminating one current from either equation (51) or (52). By substituting for $\mathbf{I}_{c'c}$ in equation (52) its value $(-\mathbf{I}_{a'a} - \mathbf{I}_{b'b})$, equation (52) reduces to

$$(24.1 - j13.37)\mathbf{I}_{a'a} + (51.63 + j14.63)\mathbf{I}_{b'b} = 0 - j2340 \qquad (53)$$

The two simultaneous equations in $\mathbf{I}_{a'a}$ and $\mathbf{I}_{b'b}$, namely, equations (51) and (53), may be written in polar form as follows:

$$(21.8\angle 26.4^\circ)\mathbf{I}_{a'a} - (39.3\angle 45.5^\circ)\mathbf{I}_{b'b} = 2340\angle 30^\circ \qquad (54)$$

$$(27.6\angle -29.0^\circ)\mathbf{I}_{a'a} + (53.6\angle 15.8^\circ)\mathbf{I}_{b'b} = 2340\angle -90^\circ \qquad (55)$$

Assuming that $\mathbf{I}_{a'a}$ and $\mathbf{I}_{b'b}$ are to be evaluated with the aid of determinants, the common denominator of the two current expressions takes the form shown below:

$$\mathbf{D} = \begin{vmatrix} (21.8\angle 26.4^\circ) & -(39.3\angle 45.5^\circ) \\ (27.6\angle -29.0^\circ & (53.6\angle 15.8^\circ) \end{vmatrix}$$
$$= 2200\angle 29.9^\circ \text{ ohms}^2 \qquad (56)$$

$$\mathbf{I}_{a'a} = \frac{\begin{vmatrix} (2340\angle 30^\circ) & -(39.3\angle 45.5^\circ) \\ (2340\angle -90^\circ) & (53.6\angle 15.8^\circ) \end{vmatrix}}{2200\angle 29.9^\circ}$$
$$= \frac{155{,}300\angle 9.5^\circ}{2200\angle 29.9^\circ} = 70.6\angle -20.4^\circ \text{ amperes} \qquad (57)$$

$$\mathbf{I}_{b'b} = \frac{\begin{vmatrix} (21.8\angle 26.4^\circ) & (2340\angle 30^\circ) \\ (27.6\angle -29.0^\circ) & (2340\angle -90^\circ) \end{vmatrix}}{2200\angle 29.9^\circ}$$
$$= \frac{62{,}800\angle -131.8^\circ}{2200\angle 29.9^\circ} = 28.5\angle -161.7^\circ \text{ amperes} \qquad (58)$$

$$\mathbf{I}_{c'c} = -(70.6\angle -20.4^\circ) - (28.5\angle -161.7^\circ)$$
$$= -(66.2 - j24.6) - (-27.1 - j8.94)$$
$$= (-39.1 + j33.54) = 51.5\angle 139.4^\circ \text{ amperes} \qquad (59)$$

After the line currents have been determined, the load potential drops, the delta-phase currents, and the various component powers can be evaluated in terms of elementary circuit theory.

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Problem 9. Draw a vector diagram illustrating $\mathbf{E}_{n'a'}$, $\mathbf{E}_{n'b'}$, $\mathbf{E}_{n'c'}$, $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, $\mathbf{I}_{c'c}$, $\mathbf{E}_{b'a'}$, $\mathbf{E}_{a'c'}$, and $\mathbf{E}_{c'b'}$ of the above network solution. Calculate the total power generated by the three-phase generator of Fig. 14*b*.

Ans.: Total power generated = 184 kw (approx.)

Problem 10. Determine $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{ca}$, $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, and $\mathbf{I}_{ca}$ of Fig. 14*a* making use of the calculated values given in the text material. Draw a vector diagram of these load voltages and currents and calculate the total power dissipated in the delta-connected load shown in Fig. 14(*a*).

Ans.: $\mathbf{V}_{ab} = 1901 + j995 = 2145 \angle 27.65°$ volts.

$\mathbf{V}_{bc} = -97 - j2322 = 2328 \angle -92.4°$ volts.

$\mathbf{V}_{ca} = -1804 + j1327 = 2240 \angle 143.6°$ volts.

$\mathbf{I}_{ab} = 26.1 - j14.25 = 29.7 \angle -28.7°$ amperes.

$\mathbf{I}_{bc} = -0.975 - j23.26 = 23.28 \angle -92.4°$ amperes.

$\mathbf{I}_{ca} = -40.2 + j10.47 = 41.55 \angle 165.4°$ amperes.

Total power dissipated = 176 kw (approx.)

Problem 11. Compare total power dissipated in the delta load of Fig. 14*a*, namely, 176 kw, plus the total I^2R loss in lines and generator windings with the total power generated, namely, 184 kw. (Recognize the fact that slide-rule calculations have been employed throughout the above network solution.)

Ans.: Line and generator I^2R loss = 8.45 kw (approx.)

Phase-Sequence Effects. The direction of rotation of polyphase induction motors is dependent upon the phase sequence of the applied voltages. Also, the two wattmeters in the two-wattmeter method of measuring three-phase power interchange their readings when subjected to a reversal of phase sequence even though the system is balanced. But the magnitudes of the various currents and component voltages in balanced systems are not affected by a reversal of phase sequence.

In an unbalanced polyphase system, a reversal of voltage phase sequence will, in general, cause certain branch currents to change in magnitude as well as in time-phase position, although the total watts and vars generated remain the same. (See example following.)

Unless otherwise stated, the term " phase sequence " refers to voltage phase sequence. It should be recognized that, in unbalanced systems, the line currents and phase currents have their own phase sequence which may or may not be the same as the voltage sequence.

Example 5. The effects of reversal of voltage sequence upon the magnitudes of the currents in the wye-connected load of Fig. 2 are illustrated by the results of example 2 and of Problem 3.

For the *ab–ca–bc* voltage sequence of example 2, page 310:

$$I_{a'a} = 3.66,\ I_{b'b} = 14.56, \text{ and } I_{c'c} = 11.98 \text{ amperes}$$

For the *ab–bc–ca* voltage sequence of Problem 3, page 311:

$$I_{a'a} = 13.65,\ I_{b'b} = 6.20, \text{ and } I_{c'c} = 7.54 \text{ amperes}$$

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Methods of Checking Voltage Phase Sequence. Sometimes in practice it becomes desirable and even necessary to know the phase sequence of a particular polyphase system. There are two general methods for checking voltage phase sequence; one based on direction of rotation of induction motors; the other, on unbalanced polyphase circuit phenomena.

Method One. Small polyphase induction motors which have previously been checked against a known phase sequence can be employed to test the phase sequence of a given system. In two- and three-phase systems, only two different phase sequences are possible, and consequently the direction in which the motor rotates can be used as an indicator of phase sequence. The principle of operation involves rotating magnetic field theory which rightfully belongs in the domain of a-c machinery.

Method Two. In general, any unbalanced set of load impedances can be employed as a voltage phase sequence checker. The different effects produced by changes in phase sequence can be determined theoretically, and when an effect peculiar to one sequence is noted in the actual installation, that effect can be used to designate the phase sequence of the system.

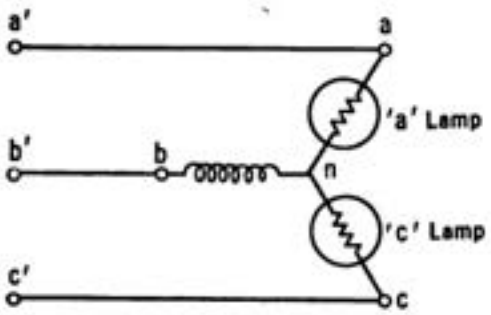

FIG. 15. A two-lamp method for checking phase sequence in three-phase systems. Lamp 'a' is brighter for *ab-bc-ca* sequence, lamp 'c' is brighter for *ab-ca-bc* sequence.

One of the most common devices for checking phase sequence in three-phase systems is the unbalanced circuit arrangement shown in Fig. 15. The three line wires, the voltage phase sequence of which is to be tested, are arbitrarily labeled. The free end of one lamp is connected to the line marked *a*. The other lamp is connected to line *c*, and the inductance coil is connected to line *b* as shown in Fig. 15. *If lamp 'a' is brighter than lamp 'c,' the phase sequence of the line-to-line voltages is ab–bc–ca. If lamp 'c' is brighter than lamp 'a,' the phase sequence is ab–ca–bc.*

The foregoing statements are based upon the results of theoretical analyses, the details of which are outlined below. Assuming that the lamps are similar, their brightnesses will depend upon the voltages $Z_{an}I_{an}$ and $Z_{cn}I_{cn}$. These voltages may be determined by the Kirchhoff equation method as shown below.

$$\mathbf{I}_{an} + \mathbf{I}_{bn} + \mathbf{I}_{cn} = 0 \tag{60}$$

$$\mathbf{Z}_{an}\mathbf{I}_{an} - \mathbf{Z}_{bn}\mathbf{I}_{bn} = \mathbf{V}_{ab} \tag{61}$$

$$\mathbf{Z}_{bn}\mathbf{I}_{bn} - \mathbf{Z}_{cn}\mathbf{I}_{cn} = \mathbf{V}_{bc} \tag{62}$$

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Upon the elimination of $\mathbf{I}_{cn}$ from equation (62), there results:

$$\mathbf{Z}_{cn}\mathbf{I}_{an} + (\mathbf{Z}_{bn} + \mathbf{Z}_{cn})\mathbf{I}_{bn} = \mathbf{V}_{bc} \tag{63}$$

Equations (61) and (63) can now be solved by inspection for $\mathbf{I}_{an}$ and the result multiplied by $\mathbf{Z}_{an}$. The voltage across the a lamp is:

$$\mathbf{Z}_{an}\mathbf{I}_{an} = \mathbf{Z}_{an}\left[\frac{\mathbf{V}_{ab}(\mathbf{Z}_{bn} + \mathbf{Z}_{cn}) + \mathbf{V}_{bc}\mathbf{Z}_{bn}}{\mathbf{Z}_{an}(\mathbf{Z}_{bn} + \mathbf{Z}_{cn}) + \mathbf{Z}_{cn}\mathbf{Z}_{bn}}\right] \tag{64}$$

The voltage across the c lamp is:

$$\mathbf{Z}_{cn}\mathbf{I}_{cn} = \mathbf{V}_{ca} + \mathbf{Z}_{an}\mathbf{I}_{an} \tag{65}$$

Example 6. For the sake of illustrating the effect of reversal of phase sequence upon the magnitudes of $\mathbf{Z}_{an}\mathbf{I}_{an}$ and $\mathbf{Z}_{cn}\mathbf{I}_{cn}$, a numerical case will be considered. The lamps $\mathbf{Z}_{an}$ and $\mathbf{Z}_{cn}$ of Fig. 15 will be assumed to be pure resistances each of 100 ohms magnitude. $\mathbf{Z}_{bn}$ will be assumed equal to $100\ \underline{/90^\circ}$ ohms, that is, a hypothetically pure inductance. The magnitude of the line-to-line voltages will be taken as 100 volts each and will first be assigned the following vector positions:

$$\mathbf{V}_{ab} = 100\ \underline{/0^\circ} \text{ volts}$$
$$\mathbf{V}_{bc} = 100\ \underline{/-120^\circ} \text{ volts}$$
$$\mathbf{V}_{ca} = 100\ \underline{/-240^\circ} \text{ volts}$$

Under these conditions:

$$\mathbf{Z}_{an}\mathbf{I}_{an} = 100\ \underline{/0^\circ}\left[\frac{(100\ \underline{/0^\circ})\ (141.4\ \underline{/45^\circ}) + (100\ \underline{/-120^\circ})\ (100\ \underline{/90^\circ})}{22{,}380\ \underline{/63.45^\circ}}\right]$$
$$= 86.4\ \underline{/-48.45^\circ} \text{ volts} \tag{66}$$
$$\mathbf{Z}_{cn}\mathbf{I}_{cn} = (100\ \underline{/-240^\circ}) + (86.3\ \underline{/-48.45^\circ})$$
$$= 23.2\ \underline{/71.55^\circ} \text{ volts} \tag{67}$$

The a lamp is therefore brighter than the c lamp for phase sequence ab–bc–ca.

Now let the line-to-line voltages be assigned vector positions which represent a reversal of phase sequence, namely,

$$\mathbf{V}_{ab} = 100\ \underline{/0^\circ} \text{ volts}$$
$$\mathbf{V}_{bc} = 100\ \underline{/-240^\circ} \text{ volts}$$
$$\mathbf{V}_{ca} = 100\ \underline{/-120^\circ} \text{ volts}$$

For ab–ca–bc phase sequence:

$$\mathbf{Z}_{an}\mathbf{I}_{an} = 100\ \underline{/0^\circ}\left[\frac{(100\ \underline{/0^\circ})\ (141.1\ \underline{/45^\circ}) + (100\ \underline{/-240^\circ})\ (100\ \underline{/90^\circ})}{22{,}380\ \underline{/63.45^\circ}}\right]$$
$$= 23.2\ \underline{/11.55^\circ} \text{ volts} \tag{68}$$
$$\mathbf{Z}_{cn}\mathbf{I}_{cn} = 100\ \underline{/-120^\circ} + 23.2\ \underline{/11.55^\circ}$$
$$= 86.4\ \underline{/-108.45^\circ} \text{ volts} \tag{69}$$

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The c lamp is therefore brighter than the a lamp for phase sequence ab–ca–bc. The above numerical results would be somewhat different if the resistance of the inductance coil had been considered. However, if the ratio (X_L/R) of the coil is relatively high, the difference between the lamp voltages is easily discernible.

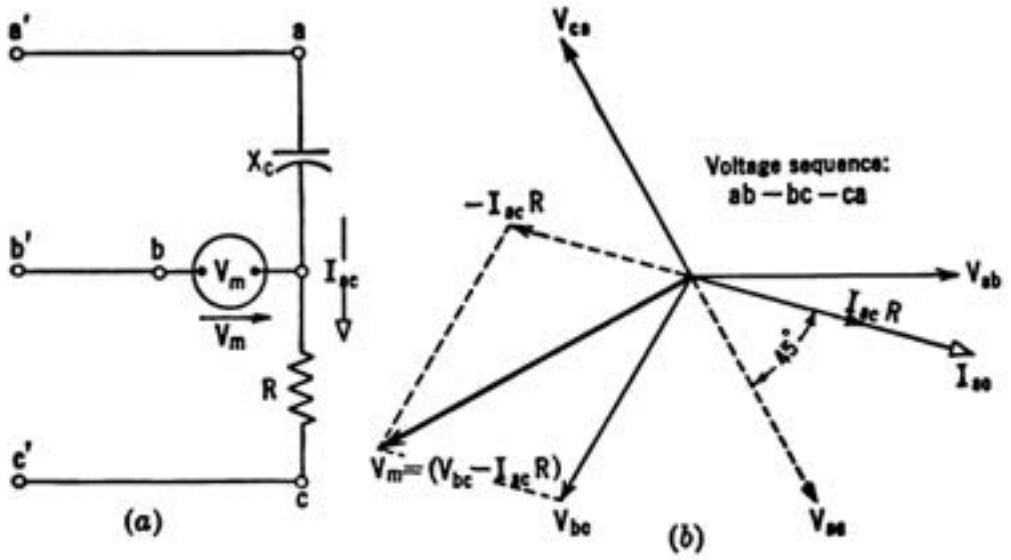

FIG. 16. A voltmeter method of checking phase sequence in three-phase systems. See example 7 and Problems 12 and 13.

Example 7. Another convenient form of voltage sequence checker is shown in Fig. 16a. It consists of a condenser (X_C), a resistor (R), and a voltmeter (Vm). The voltmeter (whose current consumption is negligibly small compared with the current through X_C and R) is connected between the line labeled b and the junction between X_C and R. X_C and R are connected in series across the voltage $\mathbf{V}_{ac}$ (or $\mathbf{V}_{ca}$) with the condenser connected to the a line and the resistor to the c line. If $X_C = 100$ ohms, $R = 100$ ohms, and $V_{ab} = V_{bc} = V_{ca} = 141.4$ volts:

$$\mathbf{I}_{ac} = \frac{141.4 \angle -60^\circ}{141.4 \angle -45^\circ} = 1 \angle -15^\circ \text{ amperes} \begin{cases} \text{for sequence } ab\text{–}bc\text{–}ca \text{ as shown} \\ \text{in Fig. 16}b. \end{cases}$$

$$\mathbf{V}_{bc} = \mathbf{Vm} + \mathbf{I}_{ac}R \quad \text{or} \quad \mathbf{Vm} = \mathbf{V}_{bc} - \mathbf{I}_{ac}R$$

$$\mathbf{Vm} = (141.4 \angle -120^\circ) - (1 \angle -15^\circ)(100 \angle 0^\circ)$$

$$= -167.3 - j96.6 = 193 \angle -150^\circ \text{ volts}$$

The above result shows that the voltmeter (Vm) reads above the line voltage (in the ratio of 193 to 141 in this case) for voltage sequence ab–bc–ca. The same general result is obtained with any combination of X_C and R provided X_C is roughly equal in ohmic value to R or greater in ohmic value than R.

Problem 12. Show by means of a qualitative vector diagram that the voltmeter (Vm) of Fig. 16a reads below line voltage for voltage sequence ab–ca–bc.

Problem 13. What is the magnitude of the voltmeter reading in Fig. 16a if $X_C = 100$ ohms, $R = 100$ ohms, and $V_{ab} = V_{bc} = V_{ca} = 141.4$ volts if the voltage sequence is ab–ca–bc?

Ans.: 51.8 volts.

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The Three-Wattmeter Method of Measuring Three-Phase Power. The total power delivered to a three-phase, wye-connected load with

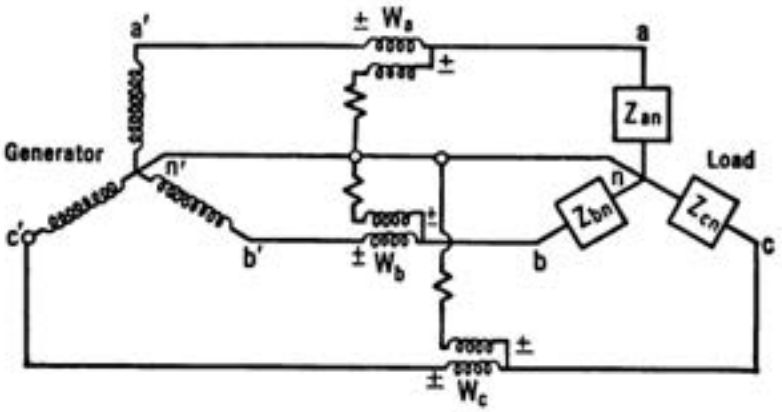

FIG. 17. The three-wattmeter method of measuring four-wire three-phase power.

neutral connection can obviously be measured with three wattmeters connected as shown in Fig. 17. W_a measures the *an* phase power, W_b measures the *bn* phase power, and W_c measures the *cn* phase power. The sum of the three wattmeter readings therefore equals the total power consumed by the load. It is plain that if each individual phase of the wye-connected load is dissipative in character all the wattmeters shown in Fig. 17 will indicate positive power.

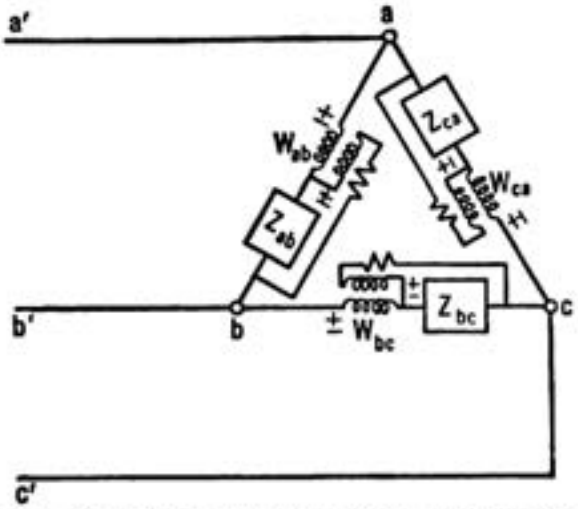

FIG. 18. The three-wattmeter method of measuring individual phase powers in a delta-connected load.

The total power absorbed by an unbalanced delta-connected load can be measured with the aid of three wattmeters as shown in Fig. 18. Individual phase powers are measured by the wattmeters. This method of measuring power would not, in general, be used unless the individual phase powers were desired.

The Two-Wattmeter Method of Measuring Three-Wire, Three-Phase Power. Except for inherent meter losses and errors, the three wattmeters connected as shown in Fig. 19 will measure accurately the power consumed by the three-phase load *abc*. A general proof of the foregoing statement will be given, and then certain important deductions will be made therefrom.

The total average power delivered to the three-phase load shown in

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Fig. 19 over a time interval T is:

$$P_{abc} = \frac{1}{T}\int_0^T (v_{an}i_{a'a} + v_{bn}i_{b'b} + v_{cn}i_{c'c})\,dt \tag{70}$$

The total average power measured by the three wattmeters shown in Fig. 19 is:

$$P_{\text{meters}} = \frac{1}{T}\int_0^T (v_{a0}i_{a'a} + v_{b0}i_{b'b} + v_{c0}i_{c'c})\,dt \tag{71}$$

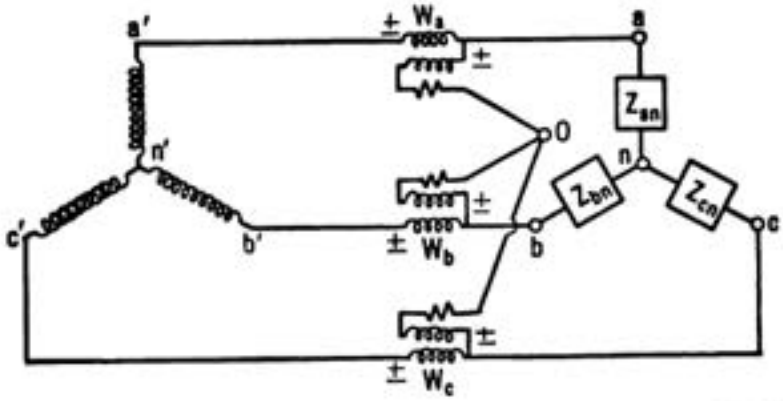

FIG. 19. A three-wattmeter method of measuring three-phase power which is independent of the potential and hence of the physical position of the point O.

Under any condition it is plain that

$$v_{a0} = v_{an} - v_{0n} \tag{72}$$

$$v_{b0} = v_{bn} - v_{0n} \tag{73}$$

$$v_{c0} = v_{cn} - v_{0n} \tag{74}$$

Equation (71) may therefore be written as:

$$P_{\text{meters}} = \frac{1}{T}\int_0^T (v_{an}i_{a'a} + v_{bn}i_{b'b} + v_{cn}i_{c'c})\,dt - \frac{1}{T}\int_0^T v_{0n}(i_{a'a} + i_{b'b} + i_{c'c})\,dt \tag{75}$$

Since $(i_{a'a} + i_{b'b} + i_{c'c}) = 0$, it follows that

$$P_{\text{meters}} = \frac{1}{T}\int_0^T (v_{an}i_{a'a} + v_{bn}i_{b'b} + v_{cn}i_{c'c})\,dt \tag{76}$$

It is thus shown that the three wattmeters in Fig. 19 measure the load power irrespective of voltage or current balance, of wave form, and of the potential of the point O. The last fact is highly significant. It indicates that the wattmeter potential coils need not have equal resistances when employed as shown in Fig. 19. It also indicates that the

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point O can be placed on any one of the three lines, thereby reducing one wattmeter reading to zero. Although the proof was based on a wye-connected load, the entire proof holds equally well for delta-connected loads. A simple way of extending the proof to cover delta loads is to recognize the fact that any delta load can be reduced to an equivalent wye-connected load. (See Chapter V, pages 131–134.)

The practical significance of placing point O on one of the three lines is that only two wattmeters are required to measure the total three-phase power. This expedient is widely utilized in measuring three-wire, three-phase power because it possesses no inherent limitations as regards balance or wave form.

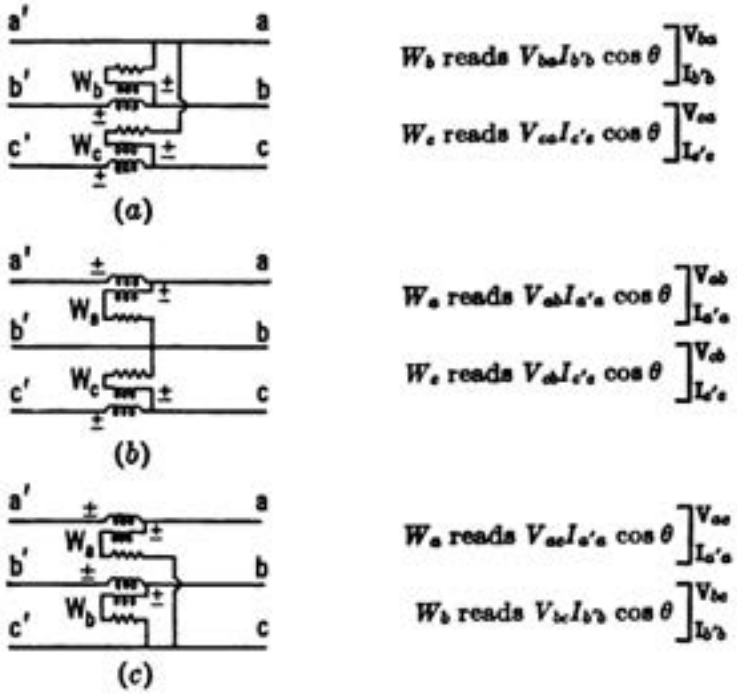

FIG. 20. Different circuit positions that the two wattmeters employed to measure three-phase power can take.

The two wattmeters used to measure three-phase power may be placed in the circuit as shown in Fig. 20*a*, *b*, or *c*. The three combinations are obtained by placing the point O of Fig. 19 on lines *a*, *b*, and *c*, respectively.

For the relative polarities of the wattmeter coils shown in Figs. 19 and 20 the instruments will read up-scale if positive power is being metered. Under the condition of sinusoidal wave form of current and voltage, positive power is indicated if the current through the current coil in the $\pm$ direction is less than 90° out of phase with the voltage which is across the potential circuit in the $\pm$ direction. If one of the meters reads down-scale when connected as shown in Fig. 20, the rela-

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tive polarity of the coils is changed to obtain up-scale reading and this reading is reckoned as negative power in finding the algebraic sum of the wattmeter readings. Other methods of checking for negative wattmeter readings will be considered presently.

Example 8. In Fig. 21, *abc* represents a balanced three-phase system of voltages. The magnitude of each voltage is 200 volts, and the phase sequence is *ab–ca–bc*.

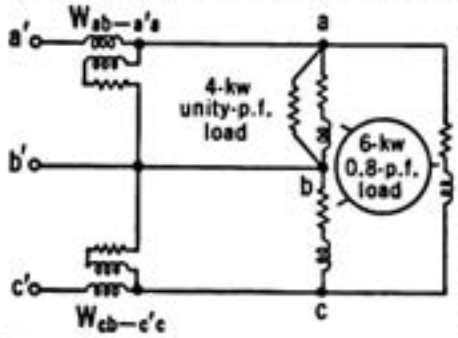

FIG. 21. A particular unbalanced three-phase load.

A balanced, 0.8-power-factor, induction motor load of 6 kw is connected across *abc* and a 4-kw, unity-power-factor load is connected across *ab* as shown in the diagram.

Let it be required to find the individual readings of the wattmeters, $W_{ab-a'a}$ and $W_{cb-c'c}$, which are connected to measure the total load power. The subscripts designate the voltage and current which are operative in a given meter in producing positive up-scale deflection. Obviously, the meter will read down-scale, thus indicating negative power if the operative voltage and current are separated by more than 90° in time phase.

Let $\mathbf{V}_{ab}$ be selected as reference. Then:

$$\mathbf{V}_{ab} = 200\,\underline{/0^\circ},\ \mathbf{V}_{bc} = 200\,\underline{/-240^\circ},\quad \text{and}\quad \mathbf{V}_{ca} = 200\,\underline{/-120^\circ}\ \text{volts}$$

The current in each phase of the induction motor is:

$$I_\phi = \frac{2000}{200 \times 0.8} = 12.5 \text{ amperes}$$

and these phase currents lag the applied phase voltages by $\cos^{-1} 0.8$ or 36.9°. The unity-power-factor load current is, of course, in phase with $\mathbf{V}_{ab}$. Therefore

$$\begin{aligned}
\mathbf{I}_{ab} &= \frac{4000}{200}\,\underline{/0^\circ} + 12.5\,\underline{/-36.9^\circ} \\
&= (20 + j0) + (10 - j7.5) \\
&= (30 - j7.5) \text{ amperes} \\
\mathbf{I}_{bc} &= 12.5\,\underline{/-240^\circ - 36.9^\circ} = 12.5\,\underline{/83.1^\circ} \\
&= (1.5 + j12.4) \text{ amperes} \\
\mathbf{I}_{ca} &= 12.5\,\underline{/-120^\circ - 36.9^\circ} = 12.5\,\underline{/-156.9^\circ} \\
&= (-11.5 - j4.90) \text{ amperes}
\end{aligned}$$

The line currents are:

$$\begin{aligned}
\mathbf{I}_{a'a} &= (30 - j7.5) - (-11.5 - j4.90) \\
&= 41.5 - j2.60 = 41.6\,\underline{/-3.58^\circ} \text{ amperes} \\
\mathbf{I}_{b'b} &= (1.5 + j12.4) - (30 - j7.5) \\
&= -28.5 + j19.9 = 34.7\,\underline{/145^\circ} \text{ amperes} \\
\mathbf{I}_{c'c} &= (-11.5 - j4.90) - (1.5 + j12.4) \\
&= -13.0 - j17.3 = 21.7\,\underline{/-127^\circ} \text{ amperes}
\end{aligned}$$

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A vector diagram of the voltages and currents is shown in Fig. 22. Since the magnitudes and relative time-phase positions of the line-to-line voltages and the line currents are known, the wattmeter readings can be determined.

$$W_{ab-a'a} = V_{ab}I_{a'a}\cos\theta \Big]_{I_{a'a}}^{V_{ab}}$$

$$= 200 \times 41.6 \cos 3.58° = 8300 \text{ watts}$$

$$W_{cb-c'c} = V_{cb}I_{c'c}\cos\theta \Big]_{I_{c'c}}^{V_{cb}}$$

$$= 200 \times 21.7 \cos 67° = 1700 \text{ watts}$$

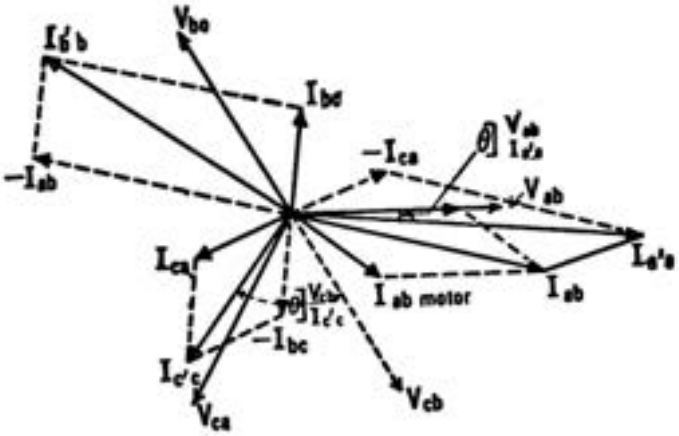

FIG. 22. Vector diagram of voltages and currents in a particular unbalanced three-phase circuit. (See Fig. 21.)

The other wattmeter combinations which will correctly measure the three-phase power are:

(1) $W_{ac-a'a}$ together with $W_{bc-b'b}$,
(2) $W_{ba-b'b}$ together with $W_{ca-c'c}$.

In the present example:

$$W_{ac-a'a} = V_{ac}I_{a'a}\cos\theta \Big]_{I_{a'a}}^{V_{ac}}$$

$$= 200 \times 41.6 \times \cos 63.58° = 3705 \text{ watts}$$

$$W_{bc-b'b} = V_{bc}I_{b'b}\cos\theta \Big]_{I_{b'b}}^{V_{bc}}$$

$$= 200 \times 34.7 \times \cos 25° = 6295 \text{ watts}$$

Problem 14. Calculate the readings of $W_{ba-b'b}$ and $W_{ca-c'c}$ in the above example and compare the sum of the wattmeter readings thus found with the total connected load.

Ans.: $W_{ba-b'b} = 5685$, $W_{ca-c'c} = 4315$ watts.

The Use of $n - 1$ Wattmeters to Measure n-Wire Power. In general, $n - 1$ wattmeter elements can be employed to measure n-wire power. The wattmeter elements may take the form of individual wattmeters, in which case the total power is equal to the algebraic sum of the wattmeter readings; or all movable members may be connected to a common

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shaft in which case the total power is indicated directly on one scale. The latter type of instrument is called a polyphase wattmeter.

The fact that $n - 1$ wattmeters can measure the power in an n-wire system can be established in several different ways. If the student recognizes that any mesh-connected load is reducible to an equivalent star-connected load, he should have little difficulty in devising his own proof.

Methods of Checking for Positive or Negative Readings of Wattmeters. In determining the algebraic sum of the wattmeter readings in the $n - 1$ wattmeter method of measuring n-wire power, due regard must be given to the sign of each wattmeter reading. Only the two-wattmeter method of measuring three-phase power will be considered at this point.

In general, the two wattmeters employed to measure three-phase power read differently. (Equal readings occur only at balanced unity-power-factor loads and at singular conditions of unbalance.) Various schemes are in use for determining the correct sign of up-scale readings of the wattmeters.

If the power being measured is the power taken by a dissipative type load, then the algebraic sum of the two wattmeter readings is always positive. Under these conditions the higher reading must be positive and only the lower reading requires checking. In the following discussion it will be assumed that the lower-reading meter is reading up-scale but that the conditions of balance and power factor are such as to render the sign of the reading doubtful.

One of the most direct and satisfactory methods of checking signs which is applicable to loads of any degree of unbalance is that explained on page 288, Chapter VIII. However, the method is not always practicable because of the difficulty in opening lines. Another method of checking the sign of the lower reading is to interchange the two wattmeters, leaving the common potential connection intact. It is assumed that the current coils are reconnected without change in polarity. If both meters read up-scale in their new positions, both the original readings were positive. If one meter reads down-scale in its new position, the original lower reading was negative.

Various other methods have been suggested for checking the sign of the lower reading, but practically all of them are predicated upon a certain degree of balance, either of the voltages or currents or both. The limitations of any method of checking should be well known before any attempt is made to employ it in practice. The theory underlying the problem of checking wattmeter signs is similar to the theory of checking the correctness of polyphase watthour meter connections. (See Chapter X.)

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Reactive Volt-Amperes in Unbalanced Four-Wire, Three-Phase Systems. The reactive volt-amperes of each individual phase of the load shown in Fig. 23 can be measured with three reactive volt-ampere meters. (For a description of a reactive volt-ampere meter and the

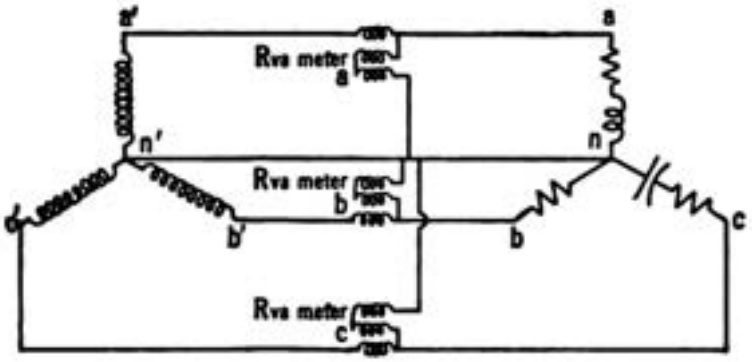

FIG. 23. Measurement of total reactive volt-amperes in a four-wire, three-phase system with three reactive volt-ampere meters.

measurement of single-phase vars see pages 374–376.) Sinusoidal wave forms of currents and voltages are assumed since the term "reactive volt-amperes" as well as any measurements of that quantity are ambiguous when other than sinusoidal wave forms are encountered.

In Fig. 23:

$$\text{Meter } a \text{ reads } V_{an}I_{an} \sin \theta \Big]_{I_{an}}^{V_{an}} \text{ vars}$$

$$\text{Meter } b \text{ reads } V_{bn}I_{bn} \sin \theta \Big]_{I_{bn}}^{V_{bn}} \text{ vars}$$

$$\text{Meter } c \text{ reads } V_{cn}I_{cn} \sin \theta \Big]_{I_{cn}}^{V_{cn}} \text{ vars}$$

The algebraic sum of the above readings is of practical importance. Assume the phase angle to be positive if the current lags the voltage and negative if the current leads the voltage. These conventions are merely matters of definition. (See page 54.) A meter properly connected to give up-scale readings for lagging-current reactive volt-amperes will read down-scale when subjected to leading-current reactive volt-amperes. If then in a particular case a meter reads down-scale, the relative polarities of the current and potential circuits of the meter are reversed. The resulting up-scale reading is considered as negative reactive volt-amperes in finding the total reactive volt-amperes of the system. With negative reactive volt-amperes defined as it is, the total vars of a system may, of course, be negative.

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Example 9. In Fig. 23 let

$$\mathbf{V}_{an} = 100 \angle 0^\circ \text{ volts} \qquad \mathbf{Z}_{an} = 25 \angle 45^\circ \text{ ohms}$$
$$\mathbf{V}_{bn} = 100 \angle -120^\circ \text{ volts} \qquad \mathbf{Z}_{bn} = 50 \angle 0^\circ \text{ ohms}$$
$$\mathbf{V}_{cn} = 100 \angle -240^\circ \text{ volts} \qquad \mathbf{Z}_{cn} = 20 \angle -60^\circ \text{ ohms}$$

The individual readings of the three reactive volt-ampere meters and the algebraic sum of the readings are to be determined.

$$\mathbf{I}_{an} = \frac{100 \angle 0^\circ}{25 \angle 45^\circ} = 4.0 \angle -45^\circ \text{ amperes}$$

$$\mathbf{I}_{bn} = \frac{100 \angle -120^\circ}{50 \angle 0^\circ} = 2.0 \angle -120^\circ \text{ amperes}$$

$$\mathbf{I}_{cn} = \frac{100 \angle -240^\circ}{20 \angle -60^\circ} = 5.0 \angle 180^\circ \text{ amperes}$$

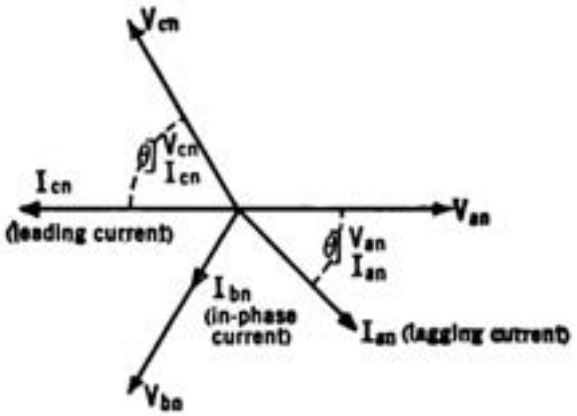

FIG. 24. Vector diagram of the phase voltages and phase currents of the four-wire, three-phase load shown in Fig. 23 for a particular set of load impedances.

The relative vector positions of the phase voltages and phase currents which actuate the meters are shown in Fig. 24.

Reactive volt-ampere meter a reads

$$(100 \times 4 \times 0.707) = 283 \text{ vars}$$

Reactive volt-ampere meter b reads

$$(100 \times 2 \times 0.0) = 0 \text{ var}$$

Reactive volt-ampere meter c reads

$$(100 \times 5 \times -0.866) = -433 \text{ vars}$$

The algebraic sum of the meter readings or the "total" number of vars is -150.

If wattmeters were to replace the reactive volt-ampere meters shown in Fig. 23, their readings would be as shown below:

$$W_a = 100 \times 4 \times 0.707 = 283 \text{ watts}$$
$$W_b = 100 \times 2 \times 1.000 = 200 \text{ watts}$$
$$W_c = 100 \times 5 \times 0.500 = 250 \text{ watts}$$

The total number of watts is 733.

Power Factor in Unbalanced Three-Phase Systems. Power factor in a single-phase system or in a balanced polyphase system has a definite physical significance. It is the ratio of the phase watts to the phase volt-amperes. Under conditions of sinusoidal wave form, power factor is equivalent to the cosine of the time-phase angular displacement between phase voltage and phase current.

In an unbalanced polyphase system each phase has its own particular

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power factor. The result is that the term "power factor" as applied to the combined unbalanced polyphase system can have only such meaning as is given to it by definition. The average of the individual phase power factors is a good general indication of the ratio of total watts to total volt-amperes in certain cases where the phase loads are all inductive or all capacitive. Where both capacitive and inductive phase loads are encountered, the compensating effect of capacitive reactive volt-amperes and inductive reactive volt-amperes is not taken into account. Another serious limitation to "average" power factor concept is that the individual phase power factors are not easily determined in many practical installations. "Average" power factor is generally not considered when specifying the power factor of an unbalanced polyphase system.

One recognized definition[1] called vector power factor of an unbalanced polyphase systems is:

$$\text{Vector p.f.} = \frac{\sum VI \cos\theta}{\sqrt{(\sum VI \sin\theta)^2 + (\sum VI \cos\theta)^2}} \tag{77}$$

$$\sum VI \cos\theta = V_aI_a \cos\theta_a + V_bI_b \cos\theta_b + V_cI_c \cos\theta_c + \cdots \tag{78}$$

$$\sum VI \sin\theta = V_aI_a \sin\theta_a + V_bI_b \sin\theta_b + V_cI_c \sin\theta_c + \cdots \tag{79}$$

The subscripts employed in the above equations refer to individual phase values. For example, θ_a is the angular displacement between phase voltage and phase current in the a phase of the system. $\sum VI \cos\theta$ is the total power consumed by the polyphase load, the power factor of which is under investigation. $\sum VI \sin\theta$ is the algebraic sum of the individual phase reactive volt-amperes. In evaluating $\sum VI \sin\theta$ in any particular case due regard must be given to the sign of each component.

It is evident that the denominator of equation (77) can be evaluated as if it were the magnitude of a resultant vector, the right-angle components of which are $(\sum VI \cos\theta)$ and $(\sum VI \sin\theta)$. This fact is illustrated graphically in Fig. 25 for the particular three-phase system discussed on pages 340–341. Considering watts and vars as the right-angle components which go to form "vector volt-amperes" it is plain that:

$$\sum \mathbf{VI} = \sqrt{(\sum VI \sin\theta)^2 + (\sum VI \cos\theta)^2} \angle \beta \tag{80}$$

or

$$\sum \mathbf{VI} = V_aI_a \angle \theta_a + V_bI_b \angle \theta_b + V_cI_c \angle \theta_c \tag{81}$$

[1] See American Standard Definitions of Electrical Terms, Approved American Standards Association, August 12, 1941.

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Power factor, as defined by equation (77), can now be written in any one of several different ways.

$$\text{Vector p.f.} = \cos \tan^{-1} \frac{(\sum VI \sin \theta)}{(\sum VI \cos \theta)} = \cos \beta \tag{82}$$

or

$$\text{Vector p.f.} = \frac{\sum VI \cos \theta}{\text{magnitude of } \sum \mathbf{VI}} \tag{83}$$

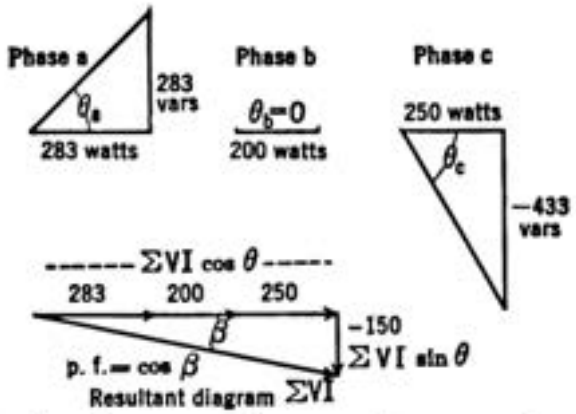

FIG. 25. Illustrating the concept of vector volt-amperes in a particular case.

Example 10. The "average" power factor of the unbalanced load described on pages 340–341 is to be compared with the power factor as defined by equations (77), (82), or (83). The circuit arrangement is shown in Fig. 23, and the previously determined values are indicated below.

$\mathbf{V}_{an} = 100 \angle 0°$ volts	$\mathbf{I}_{an} = 4.0 \angle -45°$ amperes
$\mathbf{V}_{bn} = 100 \angle -120°$ volts	$\mathbf{I}_{bn} = 2.0 \angle -120°$ amperes
$\mathbf{V}_{cn} = 100 \angle -240°$ volts	$\mathbf{I}_{cn} = 5.0 \angle 180°$ amperes
a-phase vars = 283	a-phase watts = 283
b-phase vars = 000	b-phase watts = 200
c-phase vars = −433	c-phase watts = 250
$\sum VI \sin \theta = -150$ vars	$\sum VI \cos \theta = 733$ watts

The individual phase power factors are:

$$\text{P.f.}_a = 0.707 \text{ (result of lagging current)}$$
$$\text{P.f.}_b = 1.000 \text{ (result of in-phase current)}$$
$$\text{P.f.}_c = 0.500 \text{ (result of leading current)}$$

The arithmetical average of the above phase power factors is:

$$\text{P.f.}_{av} = \frac{2.207}{3} = 0.736$$

The power factor of the unbalanced load as defined by equation (77) is:

$$\text{Vector p.f.} = \frac{733}{\sqrt{(-150)^2 + (733)^2}} = \frac{733}{748} = 0.98$$

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Inasmuch as the latter determination of power factor recognizes the compensating effect of "leading" and "lagging" reactive volt-amperes it is somewhat more significant than the "average" power factor.

Measurement of $\sum VI \sin\theta$ in a Three-Wire, Three-Phase Circuit. Power factors in three-wire, three-phase systems are very often measured in terms of $\sum VI \cos\theta$ and $\sum VI \sin\theta$. $\sum VI \cos\theta$ can be measured with the aid of either two or three wattmeters as shown in previous articles. It may be shown that $\sum VI \sin\theta$ can also be measured in a three-wire, three-phase system with either two or three reactive volt-ampere meters. Only the two-meter method of measuring $\sum VI \sin\theta$ will be considered.

The two meters shown in Fig. 26 are assumed to be reactive volt-ampere meters which are capable of reading $VI \sin\theta \Big]_{I}^{V}$. (See pages 374–376 for a general discussion of the reactive volt-ampere meter as a measuring instrument.)

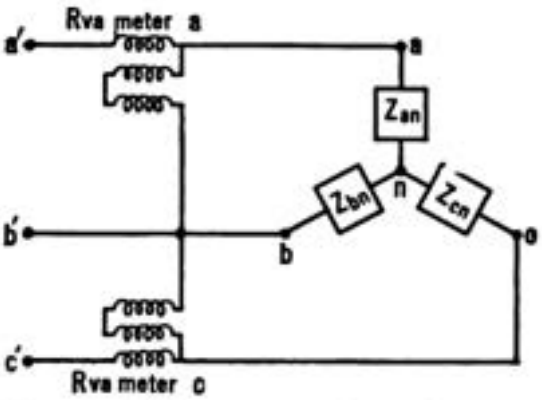

Fig. 26. The two reactive volt-ampere meter method of measuring $\Sigma VI \sin\theta$ in a three-wire, three-phase system.

The meters shown in Fig. 26 are connected into the circuit in a manner which is exactly like two wattmeters in the two-wattmeter method of measuring three-phase power. It will be shown presently that, when they are connected in this fashion, the algebraic sum of the two reactive volt-ampere meter readings is equal to $\sum VI \sin\theta$ of the three-phase circuit. $\sum VI \sin\theta$ for a polyphase system has been defined in equation (79) of the present chapter.

Connected as shown in Fig. 26:

$$\text{Reactive volt-ampere meter } a \text{ reads } \left\{ V_{ab} I_{a'a} \sin\theta \Big]_{I_{a'a}}^{V_{ab}} \right\}$$

$$\text{Reactive volt-ampere meter } c \text{ reads } \left\{ V_{cb} I_{c'c} \sin\theta \Big]_{I_{c'c}}^{V_{cb}} \right\}$$

For the sake of analysis, the above readings will be expressed temporarily in terms of the complex components of the voltages and currents. In Chapter IV it was shown that under the conditions of sinusoidal wave form:

$$VI \sin\theta \Big]_{I}^{V} = v'i - vi' \tag{84}$$

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where

$$\mathbf{V} = v + jv' \quad \text{and} \quad \mathbf{I} = i + ji'$$

Reference to Fig. 26 will show that $\mathbf{I}_{a'a} = \mathbf{I}_{an}$ and that $\mathbf{I}_{c'c} = \mathbf{I}_{cn}$. Also $\mathbf{V}_{ab} = \mathbf{V}_{an} - \mathbf{V}_{bn}$ and $\mathbf{V}_{cb} = \mathbf{V}_{cn} - \mathbf{V}_{bn}$.

$$\begin{aligned} V_{ab}I_{a'a} \sin \theta \Big]_{\mathbf{I}_{a'a}}^{\mathbf{V}_{ab}} &= V_{ab}I_{an} \sin \theta \Big]_{\mathbf{I}_{an}}^{\mathbf{V}_{ab}} \\ &= (v'_{ab}i_{an} - v_{ab}i'_{an}) \\ &= (v'_{an}i_{an} - v'_{bn}i_{an} - v_{an}i'_{an} + v_{bn}i'_{an}) \\ &= (v'_{an}i_{an} - v_{an}i'_{an}) + (v_{bn}i'_{an} - v'_{bn}i_{an}) \end{aligned} \tag{85}$$

$$\begin{aligned} V_{cb}I_{c'c} \sin \theta \Big]_{\mathbf{I}_{c'c}}^{\mathbf{V}_{cb}} &= V_{cb}I_{cn} \sin \theta \Big]_{\mathbf{I}_{cn}}^{\mathbf{V}_{cb}} \\ &= (v'_{cb}i_{cn} - v_{cb}i'_{cn}) \\ &= (v'_{cn}i_{cn} - v'_{bn}i_{cn} - v_{cn}i'_{cn} + v_{bn}i'_{cn}) \\ &= (v'_{cn}i_{cn} - v_{cn}i'_{cn}) + (v_{bn}i'_{cn} - v'_{bn}i_{cn}) \end{aligned} \tag{86}$$

It will be noticed that $(v_{bn}i'_{an} - v'_{bn}i_{an})$ of equation (85) and $(v_{bn}i'_{cn} - v'_{bn}i_{cn})$ of equation (86) can be added so as to yield

$$v_{bn}(i'_{an} + i'_{cn}) - v'_{bn}(i_{an} + i_{cn}) = (v'_{bn}i_{bn} - v_{bn}i'_{bn}) \tag{87}$$

Therefore the sum of equations (85) and (86) reduces to

$$(v'_{an}i_{an} - v_{an}i'_{an}) + (v'_{bn}i_{bn} - v_{bn}i'_{bn}) + (v'_{cn}i_{cn} - v_{cn}i'_{cn})$$

which in turn is easily recognized as the total reactive volt-amperes of the three-phase load or $\sum VI \sin \theta$.

No restrictions as to the balance of either voltage or current have been imposed upon the foregoing derivation. Two reactive volt-ampere meters connected into a three-wire, three-phase circuit as shown in Fig. 26 will, therefore, measure $\sum VI \sin \theta$ regardless of the condition of balance. Although the generality is rather difficult to incorporate into the derivation, the algebraic sum of the readings will be equal to $\sum VI \sin \theta$ whenever the reactive volt-amperes are restricted to those cases where both voltage and current wave forms are sinusoidal, provided the reactive volt-ampere meters are connected into the three-wire, three-phase line in a manner similar to the wattmeters shown in Fig. 20*a*, *b*, or *c*.

Example 11. In Fig. 27, *abc* represents an unbalanced three-phase system of voltages, the phase sequence of which is *ab–bc–ca*. In magnitude:

$$V_{ab} = 200, \; V_{bc} = 141.4 \quad \text{and} \quad V_{ca} = 141.4 \text{ volts}$$

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If $\mathbf{V}_{ab}$ is assumed to occupy the reference axis position, then:

$$\mathbf{V}_{ab} = 200\,\underline{/0^\circ},\ \mathbf{V}_{bc} = 141.4\,\underline{/-135^\circ},\ \mathbf{V}_{ca} = 141.4\,\underline{/-225^\circ}\ \text{volts}$$

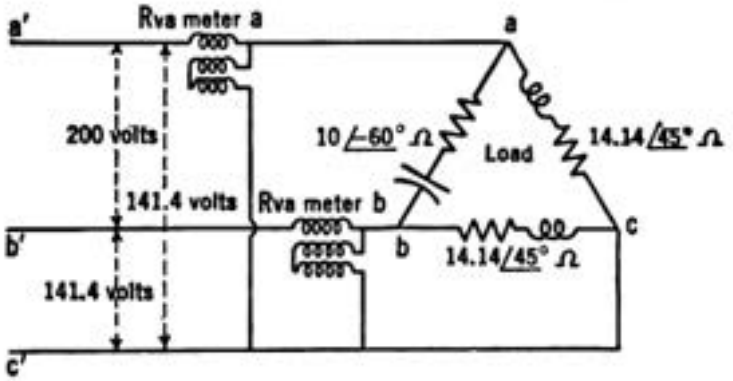

FIG. 27. A particular unbalanced three-phase load.

It will be assumed that the load impedances have the values shown on the circuit diagram, namely,

$$\mathbf{Z}_{ab} = 10\,\underline{/-60^\circ}\ \text{ohms}$$
$$\mathbf{Z}_{bc} = 14.14\,\underline{/45^\circ}\ \text{ohms}$$
$$\mathbf{Z}_{ca} = 14.14\,\underline{/45^\circ}\ \text{ohms}$$

Assuming that the line-to-line voltages remain fixed at the values given above, the delta-phase currents are:

$$\mathbf{I}_{ab} = \frac{200\,\underline{/0^\circ}}{10\,\underline{/-60^\circ}} = 20\,\underline{/60^\circ}\ \text{amperes}$$
$$\mathbf{I}_{bc} = \frac{141.4\,\underline{/-135^\circ}}{14.14\,\underline{/45^\circ}} = 10\,\underline{/180^\circ}\ \text{amperes}$$
$$\mathbf{I}_{ca} = \frac{141.4\,\underline{/-225^\circ}}{14.14\,\underline{/45^\circ}} = 10\,\underline{/90^\circ}\ \text{amperes}$$

From which

$$\mathbf{I}_{a'a} = \mathbf{I}_{ab} - \mathbf{I}_{ca} = 10 + j7.32 = 12.4\,\underline{/36.2^\circ}\ \text{amperes}$$
$$\mathbf{I}_{b'b} = \mathbf{I}_{bc} - \mathbf{I}_{ab} = -20 - j17.32 = 26.45\,\underline{/-139.1^\circ}\ \text{amperes}$$
$$\mathbf{I}_{c'c} = \mathbf{I}_{ca} - \mathbf{I}_{bc} = 10 + j10 = 14.14\,\underline{/45^\circ}\ \text{amperes}$$

The voltages and currents are represented graphically in Fig. 28.

The meters shown in Fig. 27 are assumed to be reactive volt-ampere meters, and the present example concerns itself with the predetermination of their readings.

Reactive volt-ampere meter a reads:

$$V_{ac}I_{a'a}\sin\theta\Big]_{\mathbf{I}_{a'a}}^{\mathbf{V}_{ac}} = 141.4 \times 12.4 \times \sin -81.2^\circ = -1732\ \text{vars}$$

Reactive volt-ampere meter b reads:

$$V_{bc}I_{b'b}\sin\theta\Big]_{\mathbf{I}_{b'b}}^{\mathbf{V}_{bc}} = 141.4 \times 26.45 \sin 4.1^\circ = 268\ \text{vars}$$

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The algebraic sum of the meter readings is:

$$-1732 + 268 = -1464 \text{ vars}$$

The actual value of $\sum VI \sin \theta$ as determined from the individual phase voltages and currents is:

$$\sum VI \sin \theta = -(200 \times 20 \times 0.866) + (141.4 \times 10 \times 0.707) + (141.4 \times 10 \times 0.707) = -1464 \text{ vars}$$

Problem 15. If the reactive volt-ampere meters shown in Fig. 27 are placed so that the current coils carry $I_{a'a}$ and $I_{c'c}$, what will be the individual meter readings in vars? It is assumed that the potential circuits of the meters are connected in such a manner that the algebraic sum of the readings will be equal to $\sum VI \sin \theta$.

Ans.: Meter a reads -1464 vars; meter c reads zero.

Problem 16. What is the power factor of the unbalanced load shown in Fig. 27 as determined from $\sum VI \sin \theta$ and $\sum VI \cos \theta$?

Ans.: 0.939.

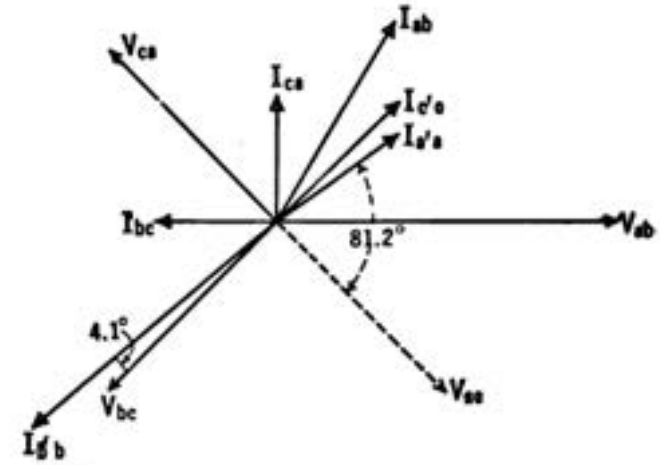

FIG. 28. Vector voltages and currents in the three-phase circuit shown in Fig. 27.

Vector Relations as Found from Experimentally Determined Magnitudes of Current and Voltage. Time-phase relations in unbalanced polyphase systems may in some instances be determined from ammeter and voltmeter readings. If, for example, the magnitudes of the three line-to-line voltages of a three-wire system are known, the relative time-phase angular displacements of these voltages can be determined from a graphical layout of the kind shown in Fig. 29. Unique determinations of the phase angles require a knowledge of the time-phase sequence of the voltages in question.

Three-Phase, Line-to-Line Voltages. The basic relation involved in a graphical layout of line-to-line voltages is:

$$\mathbf{V}_{ab} + \mathbf{V}_{bc} + \mathbf{V}_{ca} = 0 \tag{88}$$

Therefore the vector voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ form a closed triangle and their relative angular positions may be determined either trigonometri-

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cally or graphically from a closed triangle of the kind shown in Fig. 29. Applying the law of cosines to one of the triangles shown in Fig. 29:

$$\alpha = \cos^{-1} \frac{V_{ab}^2 + V_{bc}^2 - V_{ca}^2}{2V_{ab}V_{bc}} \quad (89)$$

$$\beta = \cos^{-1} \frac{V_{ab}^2 + V_{ca}^2 - V_{bc}^2}{2V_{ab}V_{ca}} \quad (90)$$

The actual vector positions of the three voltages with respect to a common reference may be written as follows if the sequence is known to be *ab–bc–ca*:

$$\mathbf{V}_{ab} = V_{ab}\underline{/0^\circ} \quad (91)$$

$$\mathbf{V}_{bc} = V_{bc}\underline{/-(180 - \alpha)^\circ} \quad (92)$$

$$\mathbf{V}_{ca} = V_{ca}\underline{/-(180 + \beta)^\circ} \quad (93)$$

Study of Fig. 29 will show that if the sequence is *ab–ca–bc*, then $\mathbf{V}_{bc} = V_{bc}\underline{/(180 - \alpha)^\circ}$ and $\mathbf{V}_{ca} = V_{ca}\underline{/(180 + \beta)^\circ}$ with respect to $\mathbf{V}_{ab}$ as reference.

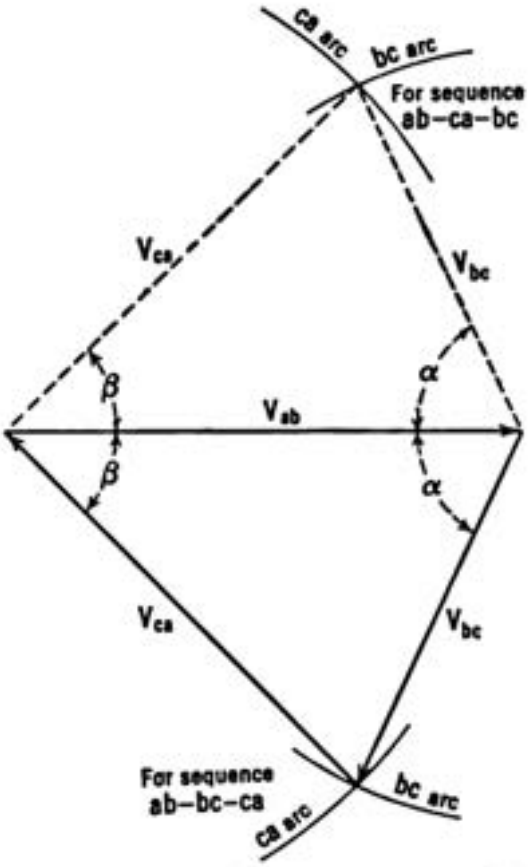

FIG. 29. Graphical construction for determining phase relation between three measured line voltages.

Details of the Graphical Layout. One vector voltage may conveniently be placed along the $+x$-axis position and used as reference. For the case shown in Fig. 29, $\mathbf{V}_{ab}$ is chosen as reference. Then, (1) an arc, equal (to scale) to V_{bc}, is swung from the head of $\mathbf{V}_{ab}$; (2) an arc, equal (to scale) to V_{ca}, is swung from the tail of $\mathbf{V}_{ab}$. The two arcs thus described will, in general, intersect at two points as shown in Fig. 29 and the actual time-phase sequence of the voltages will determine which of the points should be used. If the sequence is *ab–bc–ca*, and if the arcs are swung in accordance with the directions given above, the lower intersection in Fig. 29 should be used because it properly locates $\mathbf{V}_{bc}$ and $\mathbf{V}_{ca}$ with respect to $\mathbf{V}_{ab}$. A vector drawn from the head of $\mathbf{V}_{ab}$ to the point in question represents $\mathbf{V}_{bc}$. A vector drawn from the point in question to the tail of $\mathbf{V}_{ab}$ represents $\mathbf{V}_{ca}$.

Note: Other schemes such as reversing the sense of the vectors, laying off $\mathbf{V}_{ca}$ from the head of $\mathbf{V}_{ab}$, and the like may be employed and if properly interpreted the correct results can be obtained. The scheme given above is preferred here since later it will permit the associations

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of $\mathbf{V}_{an}$ and $\mathbf{V}_{ab}$, $\mathbf{V}_{bn}$ and $\mathbf{V}_{bc}$, and $\mathbf{V}_{cn}$ and $\mathbf{V}_{ca}$ at common origins. Orderly notational schemes of this kind are helpful. (See Fig. 30*b*).

Problem 17. The line-to-line voltages of a three-phase system are $V_{ab} = 160$, $V_{ca} = 155.5$, and $V_{bc} = 120.8$ volts. Find the vector expressions for the line-to-line voltages with respect to $\mathbf{V}_{ab}$ as reference, if it is known that the sequence of voltages is *ab–ca–bc*.

Ans.: $\mathbf{V}_{ab} = 160\,\underline{/0°}$, $\mathbf{V}_{ca} = 155.5\,\underline{/-135°}$, $\mathbf{V}_{bc} = 120.8\,\underline{/114.45°}$ volts.

The Six Voltages of a Wye-Connected Load (or Generator). If the magnitudes of the six major potential differences of a wye-connected load (or generator) are known, their vector relations can be determined. The line-to-line voltages are laid out graphically as outlined above and then the phase voltages are properly located with respect to the line voltages. The vector positions of the phase voltages are determined as outlined below.

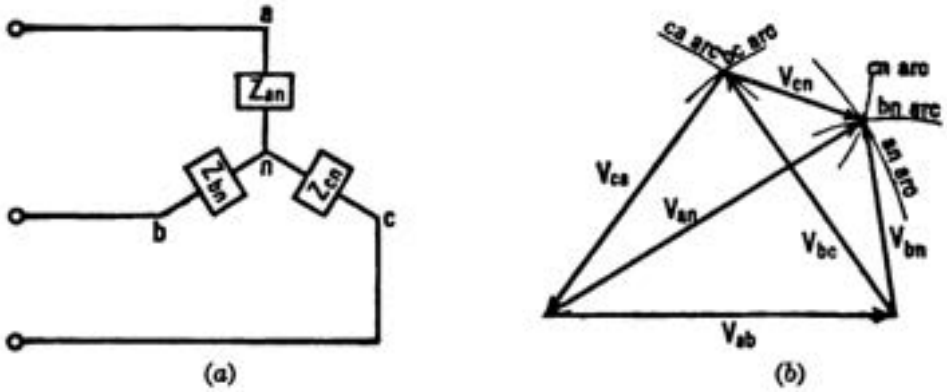

FIG. 30. Graphical method for determining the vector relations of voltages in a wye-connected load.

The wye-connected load shown in Fig. 30*a* will be considered, and it will be assumed that the closed triangle of line-to-line voltages has been constructed. It is evident from the circuit diagram that

$$\left.\begin{aligned} \mathbf{V}_{an} - \mathbf{V}_{bn} &= \mathbf{V}_{ab} \\ \mathbf{V}_{bn} - \mathbf{V}_{cn} &= \mathbf{V}_{bc} \\ \mathbf{V}_{cn} - \mathbf{V}_{an} &= \mathbf{V}_{ca} \end{aligned}\right\} \tag{94}$$

irrespective of balance. These facts immediately suggest a graphical method of laying out $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$. From the tail of $\mathbf{V}_{ab}$ an arc equal to V_{an} is described; from the tail of $\mathbf{V}_{bc}$ an arc equal to V_{bn} is described; and from the tail of $\mathbf{V}_{ca}$ an arc equal to V_{cn} is described. The three arcs should intersect at a point which may be either within or without the closed triangle of line-to-line voltages. The degree of precision with which the arcs coincide at a single point in any particular case is an indication of the exactness with which the voltages were measured, provided sinusoidal wave forms of voltage are involved.

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In order for the relations stated in (94) to be satisfied it is plain that the phase voltages must be given vector polarities as shown in Fig. 30*b*. Figure 30*b* thus becomes a three-origin vector diagram. $\mathbf{V}_{ab}$ and $\mathbf{V}_{an}$ have their tails at one origin; $\mathbf{V}_{bc}$ and $\mathbf{V}_{bn}$ have their tails at a second origin; and $\mathbf{V}_{ca}$ and $\mathbf{V}_{cn}$ have their tails at a third origin.

The line-to-line voltages can be located with respect to one another and then each phase voltage can be properly located with respect to one or more line voltages. The actual phase angles involved in any particular case may be determined by either graphic or trigonometric methods. The accuracy of graphic methods can usually be made comparable to the accuracy with which six voltages of a dynamic system can be measured experimentally.

The Six Currents of a Delta-Connected Load (or Generator). The method described above for locating the relative vector positions of voltages in wye-connected loads may be employed to find the relative vector relations of currents in delta-connected loads. It should be noted that ammeter readings of the six currents together with voltmeter readings of the three line-to-line voltages in a delta system do not yield sufficient information from which a complete vector diagram can be constructed. The time phase of one current with respect to one voltage must be known before the voltage vectors and current vectors can be properly associated.

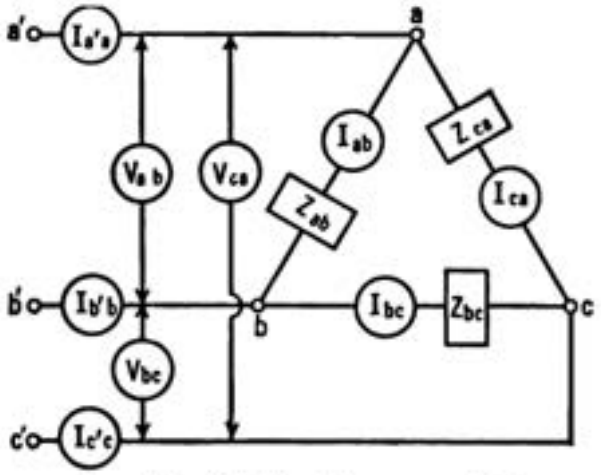

FIG. 31. Delta load for example 12.

Example 12. The following information regarding the delta-connected load shown in Fig. 31 is available:

(1) $V_{ab} = 200$, $V_{bc} = 141.4$, and $V_{ca} = 141.4$ volts. The voltage phase sequence is *ab–bc–ca*.

(2) $I_{ab} = 7.07$, $I_{bc} = 10$, $I_{ca} = 10$, $I_{a'a} = 15.8$, $I_{b'b} = 7.07$, and $I_{c'c} = 14.14$ amperes.

(3) $\mathbf{Z}_{ab}$ is inductive in character, having a ratio of X to R equal to unity.

(4) All load impedances are of a dissipative nature.

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The complex expressions for all currents and voltages are to be determined with respect to $\mathbf{V}_{ab}$ as a reference.

A rough graphical layout of the line-to-line voltages will show that: $\mathbf{V}_{ab} = 200 \underline{/0°}$, $\mathbf{V}_{bc} = 141.4 \underline{/-135°}$, and $\mathbf{V}_{ca} = 141.4 \underline{/135°}$ volts.

Regardless of the degree of unbalance:

$$\mathbf{I}_{ab} - \mathbf{I}_{ca} = \mathbf{I}_{a'a}$$
$$\mathbf{I}_{bc} - \mathbf{I}_{ab} = \mathbf{I}_{b'b}$$
$$\mathbf{I}_{ca} - \mathbf{I}_{bc} = \mathbf{I}_{c'c}$$

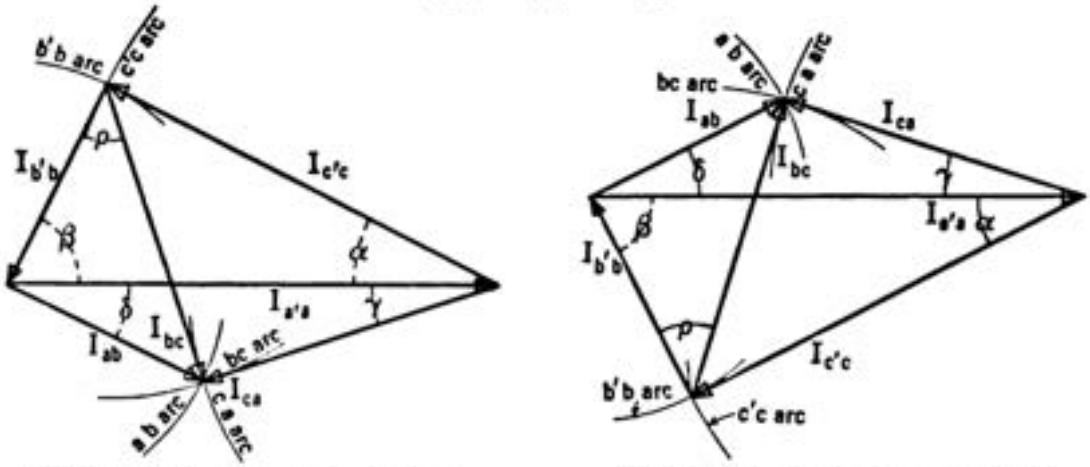
Hence:

$$\mathbf{I}_{a'a} + \mathbf{I}_{b'b} + \mathbf{I}_{c'c} = 0$$

The line currents of Fig. 31 may, therefore, be laid out to form a closed triangle as shown in Fig. 32. After the closed triangle of line currents has been formed, angles α and β in Fig. 32 may be determined, and hence the vector positions of $\mathbf{I}_{c'c}$ and $\mathbf{I}_{b'b}$ with respect to $\mathbf{I}_{a'a}$ as reference.

Next, arcs equal (to scale) to I_{ab}, I_{bc}, and I_{ca} are described about the tails of vectors $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ respectively, if $\mathbf{I}_{ab}$ and $\mathbf{I}_{a'a}$ are to have a common origin, $\mathbf{I}_{bc}$ and $\mathbf{I}_{b'b}$ are to have a common origin, and $\mathbf{I}_{ca}$ and $\mathbf{I}_{c'c}$ are to have a common origin. In the scheme employed in Fig. 32 $\mathbf{I}_{c'c}$ must follow $\mathbf{I}_{a'a}$ if

$$(\mathbf{I}_{ab} - \mathbf{I}_{ca}) = \mathbf{I}_{a'a}$$



(*a*) For $a'a$–$b'b$–$c'c$ sequence of line currents.

(*b*) For $a'a$–$c'c$–$b'b$ sequence of line currents.

FIG. 32. Graphical method for determining the vector relations of currents in a delta-connected load. See example 12.

as it should. A study of the basic current relations involved will show why the positive directions of $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, and $\mathbf{I}_{ca}$ have been drawn as shown in Fig. 32. Regardless of the scheme employed the basic current relations must be satisfied and if the graphical layout satisfies Kirchhoff's current law in all respects then a correct interpretation of the graphical layout will yield the relative vector positions of the vector currents.

After the line and phase currents have been laid out graphically the angles α, β, γ, δ, and ρ in Fig. 32 may be determined. For the particular case under discussion, graphical solutions yield the following:

$\alpha = 26.5°$ $\quad\delta = 26.5°$ $\quad\rho = 45.0°$

$\beta = 63.5°$ $\quad\gamma = 18.5°$

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The relative vector positions may be determined directly from the angles.

For $a'a$–$b'b$–$c'c$ Current sequence		For $a'a$–$c'c$–$b'b$ Current sequence	
$\mathbf{I}_{a'a} = 15.8\ \underline{/0^\circ}$	amperes	$\mathbf{I}_{a'a} = 15.8\ \underline{/0^\circ}$	amperes
$\mathbf{I}_{b'b} = 7.07\ \underline{/-116.5^\circ}$	"	$\mathbf{I}_{b'b} = 7.07\ \underline{/116.5^\circ}$	"
$\mathbf{I}_{c'c} = 14.14\ \underline{/153.5^\circ}$	"	$\mathbf{I}_{c'c} = 14.14\ \underline{/-153.5^\circ}$	"
$\mathbf{I}_{ab} = 7.07\ \underline{/-26.5^\circ}$	"	$\mathbf{I}_{ab} = 7.07\ \underline{/26.5^\circ}$	"
$\mathbf{I}_{bc} = 10.0\ \underline{/-71.5^\circ}$	"	$\mathbf{I}_{bc} = 10.0\ \underline{/71.5^\circ}$	"
$\mathbf{I}_{ca} = 10.0\ \underline{/-161.5^\circ}$	"	$\mathbf{I}_{ca} = 10.0\ \underline{/161.5^\circ}$	"

The above vector positions are, of course, only relative to $\mathbf{I}_{a'a}$. Some added information is required before the vector currents can be specified with respect to the line-to-line voltages of the system. The necessary information is contained in the original statement of the problem. $\mathbf{Z}_{ab}$ is such as to cause $\mathbf{I}_{ab}$ to lag $\mathbf{V}_{ab}$ by 45°. Therefore

$$\mathbf{I}_{ab} = 7.07\ \underline{/-45^\circ} \text{ amperes (with } \mathbf{V}_{ab} \text{ as reference)}$$

But the vector position of $\mathbf{I}_{ab}$ in the above $a'a$–$b'b$–$c'c$ sequence table is:

$$\mathbf{I}_{ab} = 7.07\ \underline{/-26.5^\circ} \text{ (with } \mathbf{I}_{aa'} \text{ as reference)}$$

The vector position of $\mathbf{I}_{ab}$ (as well as that of all the other currents in the $a'a$–$b'b$–$c'c$ sequence table) must, therefore, be rotated through $-(45^\circ - 26.5^\circ) = -18.5^\circ$ if the voltages and currents are to be properly related in a single diagram with $\mathbf{V}_{ab}$ as reference.

In the $a'a$–$c'c$–$b'b$ sequence table, $\mathbf{I}_{ab}$ appears as a current which is 26.5° ahead of $\mathbf{I}_{a'a}$. Since circuit conditions require that $\mathbf{I}_{ab}$ be 45° behind $\mathbf{V}_{ab}$, all currents in the $a'a$–$c'c$–$b'b$ sequence table must be rotated through -71.5° in order for the currents to be properly located with respect to $\mathbf{V}_{ab}$ as reference.

The vector values referred to $\mathbf{V}_{ab}$ as reference are:

For $a'a$–$b'b$–$c'c$ Current sequence		For $a'a$–$c'c$–$b'b$ Current sequence	
$\mathbf{V}_{ab} = 200\ \underline{/0^\circ}$	volts	$\mathbf{V}_{ab} = 200\ \underline{/0^\circ}$	volts
$\mathbf{V}_{bc} = 141.4\ \underline{/-135^\circ}$	"	$\mathbf{V}_{bc} = 141.4\ \underline{/-135^\circ}$	"
$\mathbf{V}_{ca} = 141.4\ \underline{/135^\circ}$	"	$\mathbf{V}_{ca} = 141.4\ \underline{/135^\circ}$	"
$\mathbf{I}_{a'a} = 15.8\ \underline{/-18.5^\circ}$	amperes	$\mathbf{I}_{a'a} = 15.8\ \underline{/-71.5^\circ}$	amperes
$\mathbf{I}_{b'b} = 7.07\ \underline{/-135^\circ}$	"	$\mathbf{I}_{b'b} = 7.07\ \underline{/45^\circ}$	"
$\mathbf{I}_{c'c} = 14.14\ \underline{/135^\circ}$	"	$\mathbf{I}_{c'c} = 14.14\ \underline{/135^\circ}$	"
$\mathbf{I}_{ab} = 7.07\ \underline{/-45^\circ}$	"	$\mathbf{I}_{ab} = 7.07\ \underline{/-45^\circ}$	"
$\mathbf{I}_{bc} = 10.0\ \underline{/-90^\circ}$	"	$\mathbf{I}_{bc} = 10.0\ \underline{/0^\circ}$	"
$\mathbf{I}_{ca} = 10.0\ \underline{/180^\circ}$	"	$\mathbf{I}_{ca} = 10.0\ \underline{/90^\circ}$	"

Vector diagrams for both sequences of line currents are shown in Fig. 33. At this stage of the analysis it becomes evident that for the magnitudes involved in this particular problem the line current sequence cannot be $a'a$–$c'c$–$b'b$. Reference to

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Fig. 33*b* will disclose the reason. The fact that the current I_{bc} is 135° removed from its driving voltage V_{bc} violates the assumed condition that each phase of the delta-connected load is dissipative in character. Therefore the actual state of affairs is represented by the $a'a$–$b'b$–$c'c$ diagram. The process of arriving at the correct solution by the elimination of "impossible" solutions is sometimes employed in practice when a minimum amount of information is available.

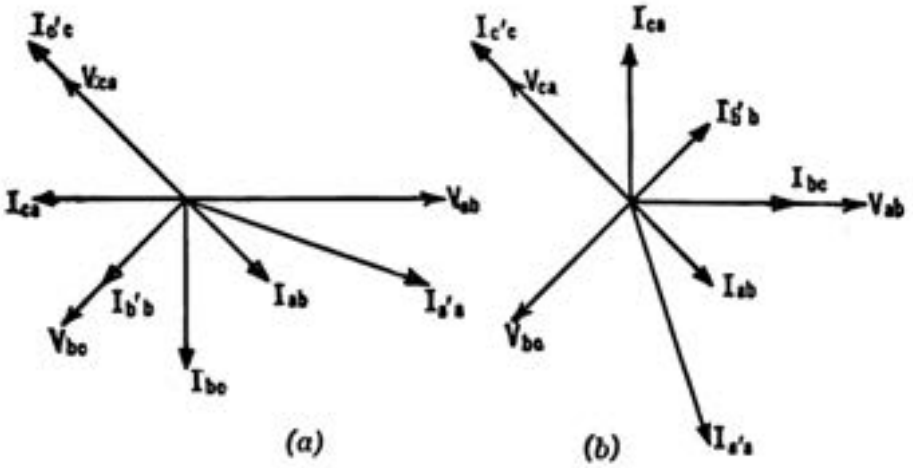

FIG. 33. Vector diagrams of voltages and currents in a particular delta load. (*a*) is for $a'a$–$b'b$–$c'c$ sequence of line currents and (*b*) is for $a'a$–$c'c$–$b'b$ sequence of line currents. See example 12.

The vector diagram of the correct solution in the present case is Fig. 33*a*. It is evident that the phases *bc* and *ca* are both capacitive in character since the phase currents lead their respective voltages by approximately 45°. The power spent in each individual phase of the delta-connected load can easily be calculated after the correct vector relations have been determined.

The above example illustrates the general method employed in finding vector relations in unbalanced three-phase circuits from ammeter and voltmeter readings. In general, vector relations cannot be uniquely determined from ammeter and voltmeter readings. Unique determination requires a knowledge of the phase sequence of the voltages and one or two other specific facts concerning the unbalanced system which is under investigation. The determination of vector relations from a minimum amount of information finds an important field of application in checking the correctness of polyphase watthour meter connections.

Vector Relations Determined from Oscillographic Records. Vector expressions for related currents and voltages of sinusoidal wave form may be determined directly from an oscillogram of the simultaneous wave variations. As far as an unbalanced polyphase circuit is concerned, the method is of academic interest only. The many details attendant upon securing nine or ten simultaneous oscillographic records cause the method to be of very little practical importance.

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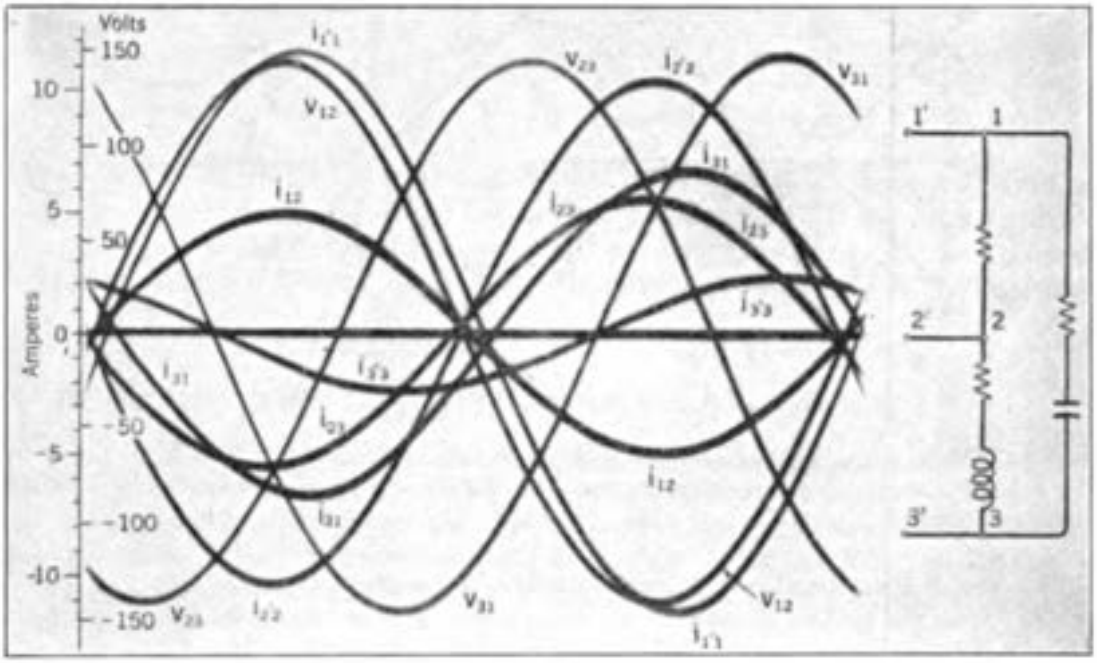
Oscillogram 1 illustrates the simultaneous variations in the line-to-line voltages and the line and phase currents in a particular unbalanced delta-connected load. This particular oscillogram furnishes the basis for an instructive type of problem, namely, given the experimentally determined values of voltage and current; find the circuit parameters. (See Problem 43, page 363.)



OSCILLOGRAM 1. Photographic records of voltages and currents in an unbalanced three-phase circuit. (For use in connection with Problem 43, page 363.) Effective values of voltages and currents as read with voltmeters and ammeters are: $V_{12} = 100$, $V_{23} = 101$, and $V_{31} = 103.5$ volts. $I_{12} = 3.55$, $I_{23} = 3.97$, $I_{31} = 4.95$, $I_{1'1} = 8.35$, $I_{2'2} = 7.40$, and $I_{3'3} = 1.79$ amperes.

Maxwell's Cyclic-Current Method of Solution. The neatness and compactness of the *cyclic-* or *loop-current* method of solving circuit problems make it ideally adapted to general circuit analysis.[2] The chief difference between the *loop-current* method and the *branch-current* method is that in the former a current is assumed to flow around a complete loop rather than from junction to junction as in the branch-current method. Any actual branch in a network will carry at least one (and sometimes more than one) loop current. In general:

$$\mathbf{I}_{\text{branch}} = \sum \mathbf{I}_{\text{loop}} \tag{95}$$

where the algebraic sum of the loop currents flowing in a specified branch is indicated by the summation sign.

[2] The loop-current method of solution is particularly well adapted to those cases involving mutual coupling between different parts of the network and has been considered briefly in Chapter VII, pages 211–212.

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In the application of the loop-current method, loop currents are arbitrarily assigned to complete paths (around which Kirchhoff's emf law can be applied) until all branches of the network are traversed by at least one loop current. For example, in Fig. 34 only two loop currents

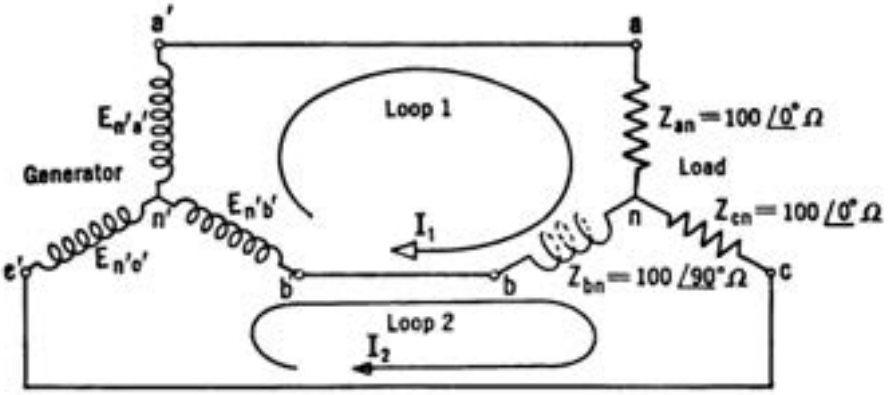

FIG. 34. Loop-current method of labeling. See example 13.

($\mathbf{I}_1$ and $\mathbf{I}_2$) are necessary to traverse all branches, and the actual branch currents in this case are:

$$\text{Branch current } \mathbf{I}_{n'a'an} = \text{Loop current } \mathbf{I}_1$$

$$\text{Branch current } \mathbf{I}_{n'b'bn} = \text{Loop current } \mathbf{I}_2 - \text{Loop current } \mathbf{I}_1$$

$$\text{Branch current } \mathbf{I}_{n'c'cn} = -(\text{Loop current } \mathbf{I}_2)$$

Where three loop currents are employed, as in Fig. 36, the basic voltage equations take the following general form:

$$\left.\begin{aligned} \mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2 + \mathbf{Z}_{13}\mathbf{I}_3 &= \mathbf{E}_1 \quad (\text{or } \mathbf{V}_1) \\ \mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 + \mathbf{Z}_{23}\mathbf{I}_3 &= \mathbf{E}_2 \quad (\text{or } \mathbf{V}_2) \\ \mathbf{Z}_{31}\mathbf{I}_1 + \mathbf{Z}_{32}\mathbf{I}_2 + \mathbf{Z}_{33}\mathbf{I}_3 &= \mathbf{E}_3 \quad (\text{or } \mathbf{V}_3) \end{aligned}\right\} \tag{96}$$

The above equations are simply general compact statements of Kirchhoff's emf law and since they may be contracted or expanded to accommodate any number of loop currents the method possesses a generality not found in the branch-current method. Furthermore, the notation in (96) is so systematically arranged that the necessity of even writing the emf equations may be eliminated as soon as the meanings of $\mathbf{Z}_{11}$, $\mathbf{Z}_{12}$, and the like are clearly understood. The determinant forms of the solutions for all the loop currents may be written directly from an inspection of the network which is under consideration.

$\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ in (96) are the resultant emf's (or voltage drops if generators are not considered), acting in the tracing direction, in the respective loops. The tracing direction around any loop is usually taken to coincide with the positive direction which has been arbitrarily assigned to the loop current.

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$\mathbf{Z}_{11}$ is the impedance of loop 1 to current $\mathbf{I}_1$, $\mathbf{Z}_{22}$ is the impedance of loop 2 to current $\mathbf{I}_2$, etc. $\mathbf{Z}_{12}$ is the impedance of loop 1 through which $\mathbf{I}_2$ flows and $\mathbf{Z}_{21}$ is the impedance of loop 2 through which $\mathbf{I}_1$ flows. If the circuit parameters are constant, $\mathbf{Z}_{12} = \mathbf{Z}_{21}$. Corresponding meanings are attached to $\mathbf{Z}_{13}$, $\mathbf{Z}_{23}$, etc.

The matter of signs is easily taken care of if the self-impedance voltage drops in (96), namely $\mathbf{Z}_{11}\mathbf{I}_1$, $\mathbf{Z}_{22}\mathbf{I}_2$, and $\mathbf{Z}_{33}\mathbf{I}_3$, are taken as positive (by virtue of tracing the loops in the arrow directions of $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$). Then $\mathbf{Z}_{12}\mathbf{I}_2$ is positive if both $\mathbf{I}_1$ and $\mathbf{I}_2$ traverse $\mathbf{Z}_{12}$ in the same direction. If the assigned directions of $\mathbf{I}_1$ and $\mathbf{I}_2$ are opposite in the mutual impedance, $\mathbf{Z}_{12}$, then $\mathbf{Z}_{12}\mathbf{I}_2$ is considered as negative to account for the physical fact that the $\mathbf{Z}_{12}\mathbf{I}_2$ voltage drop acts in opposition to the $\mathbf{Z}_{11}\mathbf{I}_1$ voltage drop in loop 1. Probably the best way of learning how the loop-current method *goes* is to study a few particular cases.

Example 13. Let it be required to find the branch currents $\mathbf{I}_{an}$, $\mathbf{I}_{bn}$, and $\mathbf{I}_{cn}$ of Fig. 34 by the loop-current method if

$$\mathbf{E}_{n'a'} = 57.7\,\underline{/-30^\circ},\ \mathbf{E}_{n'b'} = 57.7\,\underline{/-150^\circ},\quad \text{and}\quad \mathbf{E}_{n'c'} = 57.7\,\underline{/90^\circ}\ \text{volts}$$

Since only two loop currents are required to traverse all the branch, (96) reduces to

$$\mathbf{Z}_{11}\mathbf{I}_1 - \mathbf{Z}_{12}\mathbf{I}_2 = \mathbf{E}_1 = \mathbf{E}_{b'n'} + \mathbf{E}_{n'a'} = 100\,\underline{/0^\circ}\ \text{volts}$$
$$-\mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 = \mathbf{E}_2 = \mathbf{E}_{c'n'} + \mathbf{E}_{n'b'} = 100\,\underline{/-120^\circ}\ \text{volts}$$

where the minus signs account for the opposite directions of $\mathbf{I}_1$ and $\mathbf{I}_2$ through $\mathbf{Z}_{n'b'bn}$. If the generator impedances of Fig. 34 are neglected:

$$\mathbf{Z}_{11} = 100\,\underline{/0^\circ} + 100\,\underline{/90^\circ} = 141.4\,\underline{/45^\circ}\ \text{ohms}$$
$$\mathbf{Z}_{22} = 100\,\underline{/90^\circ} + 100\,\underline{/0^\circ} = 141.4\,\underline{/45^\circ}\ \text{ohms}$$

Without regard for sign, which has been taken care of in the above voltage equations

$$\mathbf{Z}_{12} = \mathbf{Z}_{21} = 100\,\underline{/90^\circ}\ \text{ohms}$$

The voltage equations may be solved directly for $\mathbf{I}_1$ and $\mathbf{I}_2$ as shown below:

$$\mathbf{I}_1 = \mathbf{I}_{an} = \frac{\begin{vmatrix} 100\,\underline{/0^\circ} & -100\,\underline{/90^\circ} \\ 100\,\underline{/-120^\circ} & 141.4\,\underline{/45^\circ} \end{vmatrix}}{\begin{vmatrix} 141.4\,\underline{/45^\circ} & -100\,\underline{/90^\circ} \\ -100\,\underline{/90^\circ} & 141.4\,\underline{/45^\circ} \end{vmatrix}} = \frac{19320\,\underline{/15^\circ}}{22380\,\underline{/63.45^\circ}} = 0.864\,\underline{/-48.45^\circ}\ \text{ampere}$$

$$\mathbf{I}_2 = \mathbf{I}_{nc} = \frac{\begin{vmatrix} 141.4\,\underline{/45^\circ} & 100\,\underline{/0^\circ} \\ -100\,\underline{/90^\circ} & 100\,\underline{/-120^\circ} \end{vmatrix}}{22380\,\underline{/63.45^\circ}} = \frac{5185\,\underline{/-45^\circ}}{22380\,\underline{/63.45^\circ}} = 0.232\,\underline{/-108.45^\circ}\ \text{ampere}$$

$$\mathbf{I}_{cn} = -\mathbf{I}_{nc} = 0.232\,\underline{/71.55^\circ},\quad \text{and}\quad \mathbf{I}_{bn} = \mathbf{I}_2 - \mathbf{I}_1$$

Example 14. In Fig. 35 are shown three load impedances $\mathbf{Z}_{an}$, $\mathbf{Z}_{bn}$, and $\mathbf{Z}_{cn}$ which are energized by $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, (and, of course, $\mathbf{V}_{ca}$). The *an* coil is assumed to be coupled

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magnetically to the *cn* coil and, as shown in Fig. 35, the coefficient of coupling between the coils is assumed to be $\sqrt{3}/6$. If the network is to be analyzed by the loop-current method employing $\mathbf{I}_1$ and $\mathbf{I}_2$ in the directions shown:

$$\omega M_{ac} = \omega M_{ca} = \frac{\sqrt{3}}{6}\sqrt{\omega L_{an} \times \omega L_{cn}} = \frac{\sqrt{3}}{6}\sqrt{1 \times 3} = 0.5 \text{ ohm}$$

The positive sign of M is used here because the coils magnetize along a common axis in the same direction if wound as shown and if positive values of $\mathbf{I}_1$ and $\mathbf{I}_2$ are present. (See page 222.) Assume $\mathbf{V}_{ab} = 100 \angle 0°$ volts and $\mathbf{V}_{bc} = 100 \angle -120°$ volts.

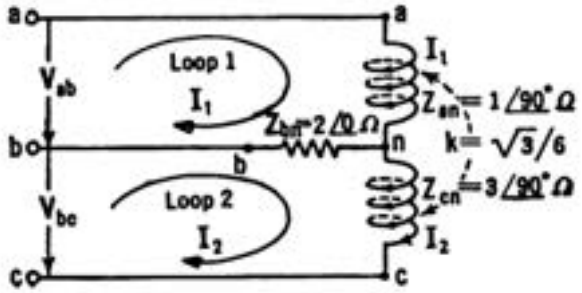

FIG. 35. See example 14.

For the network shown in Fig. 35, the basic voltage equations become

$$\mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2 = \mathbf{V}_{ab} = 100 \angle 0° \text{ volts}$$
$$\mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 = \mathbf{V}_{bc} = 100 \angle -120° \text{ volts}$$
$$\mathbf{Z}_{11} = (2 + j1),\ \mathbf{Z}_{22} = (2 + j3), \quad \text{and} \quad \mathbf{Z}_{12} = \mathbf{Z}_{21} = (-2 + j0.5) \text{ ohms}$$

Note: The minus sign in $\mathbf{Z}_{12}$ accounts for the fact that $\mathbf{I}_2$ flows through $\mathbf{Z}_{bn}$ opposite to $\mathbf{I}_1$ and $+j0.5$ in $\mathbf{Z}_{12}$ accounts for the fact that the $(j\omega M\mathbf{I}_2)$ voltage drop acts in the same direction in loop 1 as the $(j\omega L\mathbf{I}_1)$ voltage drop.

$$\mathbf{I}_1 = \frac{\begin{vmatrix} (100 + j0) & (-2 + j0.5) \\ (-50 - j86.6) & (2 + j3) \end{vmatrix}}{\begin{vmatrix} (2 + j1) & (-2 + j0.5) \\ (-2 + j0.5) & (2 + j3) \end{vmatrix}} = \frac{56.7 + j152}{-2.75 + j10} = 12.68 - j9.15$$
$$= 15.6 \angle -35.8° \text{ amperes}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} (2 + j1) & (100 + j0) \\ (-2 + j0.5) & (-50 - j86.6) \end{vmatrix}}{(-2.75 + j10)} = \frac{186.6 - j273}{-2.75 + j10} = -30.15 - j10.36$$
$$= 31.8 \angle -161° \text{ amperes}$$

The branch currents follow directly from $\mathbf{I}_1$ and $\mathbf{I}_2$ as shown in example 13.

Example 15. The network shown in Fig. 36 represents two generators operating in parallel. An accidental ground on the line leading out from terminal *c* is assumed to exist as shown and the problem is that of determining the short-circuit current $\mathbf{I}_{npc}$ or loop current $\mathbf{I}_3$ in Fig. 36.

A study of Fig. 36 will show that the self-impedances of loops 1, 2, and 3 are, respectively:

$$\mathbf{Z}_{11} = (7.28 + j18) = 19.4 \angle 68° \text{ ohms}$$
$$\mathbf{Z}_{22} = (7.28 + j18) = 19.4 \angle 68° \text{ ohms}$$
$$\mathbf{Z}_{33} = (4.04 + j7.0) = 8.08 \angle 60° \text{ ohms}$$

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Next, the mutual impedances will be obtained from an inspection of Fig. 36 and minus signs will be affixed to those mutual impedances that carry loop currents of opposite directions. (Actually these impedances are not negative quantities but when associated with oppositely directed currents they produce negative voltage drops as shown in examples 13 and 14.) On this basis all the currents in (96) may be reckoned as positive and

$$\mathbf{Z}_{12} = \mathbf{Z}_{21} = -(3.64 + j9.0) = -9.7\,\underline{/68^\circ} \text{ ohms}$$

$$\mathbf{Z}_{23} = \mathbf{Z}_{32} = -(0.50 + j3.0) = -3.04\,\underline{/80.5^\circ} \text{ ohms}$$

$$\mathbf{Z}_{13} = \mathbf{Z}_{31} = 0 \text{ (Since loops 1 and 3 have no common path.)}$$

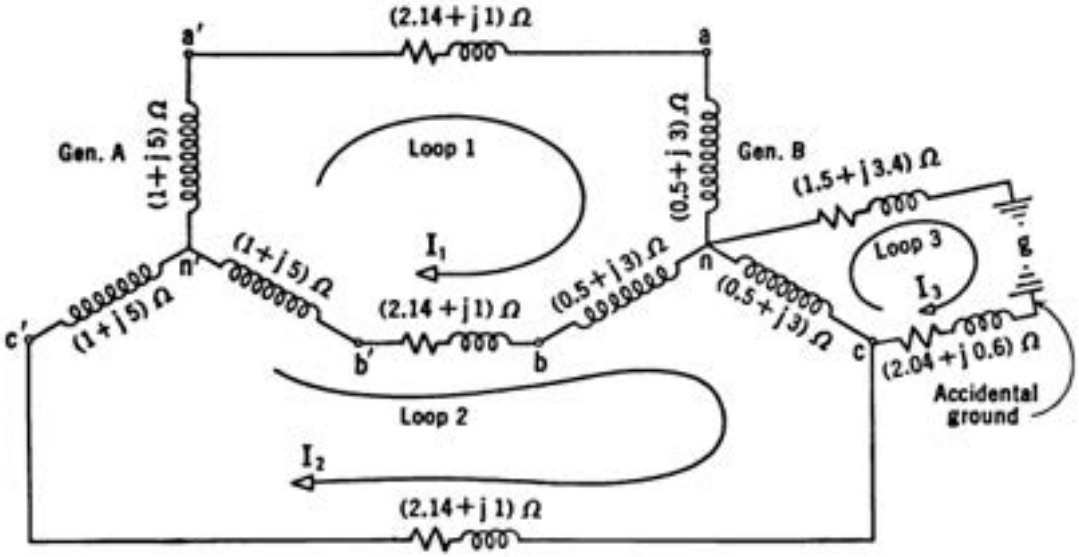

FIG. 36. See example 15.

For the particular example under investigation it will be assumed that the individual phase voltages of the generators have the following values:

$$\mathbf{E}_{n'a'} = \mathbf{E}_{na} = 4000\,\underline{/0^\circ} \text{ volts}$$

$$\mathbf{E}_{n'b'} = \mathbf{E}_{nb} = 4000\,\underline{/-120^\circ} \text{ volts}$$

$$\mathbf{E}_{n'c'} = \mathbf{E}_{nc} = 4000\,\underline{/-240^\circ} \text{ volts}$$

The resultant voltages which exist in the three loops of Fig. 36 are:

$$\mathbf{E}_1 = \mathbf{E}_{n'a'} - \mathbf{E}_{na} + \mathbf{E}_{nb} - \mathbf{E}_{n'b'} = 0$$

$$\mathbf{E}_2 = \mathbf{E}_{n'b'} - \mathbf{E}_{nb} + \mathbf{E}_{nc} - \mathbf{E}_{n'c'} = 0$$

$$\mathbf{E}_3 = -\mathbf{E}_{nc}$$

$$= -4000\,\underline{/-240^\circ} = 4000\,\underline{/-60^\circ} \text{ volts}$$

The equations for voltage equilibrium in the three meshes of Fig. 36 are:

$$(19.4\,\underline{/68^\circ})\mathbf{I}_1 - (9.7\,\underline{/68^\circ})\mathbf{I}_2 + 0 = 0$$

$$-(9.7\,\underline{/68^\circ})\mathbf{I}_1 + (19.4\,\underline{/68^\circ})\mathbf{I}_2 - (3.04\,\underline{/80.5^\circ})\mathbf{I}_3 = 0$$

$$0 - (3.04\,\underline{/80.5^\circ})\mathbf{I}_2 + (8.08\,\underline{/60^\circ})\mathbf{I}_3 = 4000\,\underline{/-60^\circ}$$

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The above equations will be solved simultaneously for $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ with the aid of elementary determinant theory. The common denominator of each current solution is:

$$\mathbf{D} = \begin{vmatrix} (19.4\angle 68^\circ) & -(9.7\angle 68^\circ) & 0 \\ -(9.7\angle 68^\circ) & (19.4\angle 68^\circ) & -(3.04\angle 80.5^\circ) \\ 0 & -(3.04\angle 80.5^\circ) & (8.08\angle 60^\circ) \end{vmatrix}$$

$$\mathbf{D} = [-2920 - j837] - [(-117.8 - j135.4) + (-733 - j210)]$$
$$= (-2068 - j492) = 2122\angle 193.4^\circ \text{ ohms}^3$$

The desired current in the present instance is $\mathbf{I}_{a'c'}$ or $\mathbf{I}_3$.

$$\mathbf{I}_3 = \frac{\begin{vmatrix} (19.4\angle 68^\circ) & -(9.7\angle 68^\circ) & 0 \\ -(9.7\angle 68^\circ) & (19.4\angle 68^\circ) & 0 \\ 0 & -(3.04\angle 80.5^\circ) & (4000\angle -60^\circ) \end{vmatrix}}{2122\angle 193.4^\circ}$$

$$\mathbf{I}_3 = \frac{1{,}131{,}000\angle 76^\circ}{2122\angle 193.4^\circ} = 533\angle -117.4^\circ \text{ amperes}$$

Problem 18. Find the magnitudes of $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ in Fig. 36 utilizing the calculations of example 15 in so far as they are helpful.

Ans.: $I_{a'a} = 55.6$, $I_{b'b} = 55.6$; and $I_{c'c} = 111.2$ amperes.

PROBLEMS

19. An unbalanced delta system labeled *abc* at the corners consists of $\mathbf{Z}_{ab} = 10\angle -60^\circ$ $\mathbf{Z}_{bc} = 5\angle 0^\circ$, and $\mathbf{Z}_{ca} = 10\angle 60^\circ$ ohms. If $\mathbf{V}_{cb} = 100\angle 0^\circ$ and the voltage sequence is *cb-ba-ac*, find the vector expressions for the currents entering the terminals *a*, *b*, and *c*. The three-phase supply voltages are balanced. Also solve for the opposite sequence.

20. An unbalanced load labeled *abc* at the corners consists of $\mathbf{Z}_{ab} = 5\angle 40^\circ$, $\mathbf{Z}_{bc} = 10\angle -30^\circ$, and $\mathbf{Z}_{ca} = 8\angle 45^\circ$ ohms. Three-phase balanced line voltages of 115 volts each are applied. If the sequence is *cb-ac-ba*, calculate the complex expressions for the line currents leaving terminals *a*, *b*, and *c* for $\mathbf{V}_{cb} = 115\angle 0^\circ$ volts.

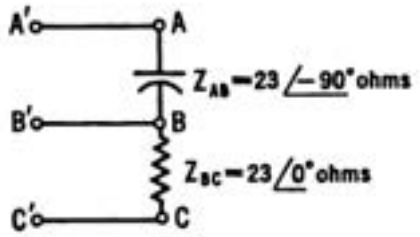

FIG. 37. See Problem 21.

21. Refer to Fig. 37. V_{AB} and V_{CB} represent a balanced two-phase system of voltage drops, the magnitude of each being 115 volts. The voltage phase sequence is *AB-CB*. V_{AB} is to be used as reference. Find $\mathbf{I}_{AB}$, $\mathbf{I}_{CB}$, $\mathbf{I}_{BB'}$ and draw a vector diagram of the voltages and currents.

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22. A wye-connected set of impedances consists of $\mathbf{Z}_{an} = 5\angle 0°$, $\mathbf{Z}_{bn} = 5\angle 60°$, and $\mathbf{Z}_{cn} = 5\angle -60°$ ohms. Find the equivalent delta-connected impedances $\mathbf{Z}_{ab}$, $\mathbf{Z}_{bc}$, and $\mathbf{Z}_{ca}$ which can be used to replace the wye-connected set of impedances.

23. Refer to Fig. 38. The terminals $a'b'c'$ represent a balanced three-phase system of voltages the sequence of which is $b'c'$-$a'b'$-$c'a'$. The magnitude of each line-to-line voltage is 230 volts. Find the readings of ammeters placed in the $a'a$, $b'b$, and $c'c$ lines.

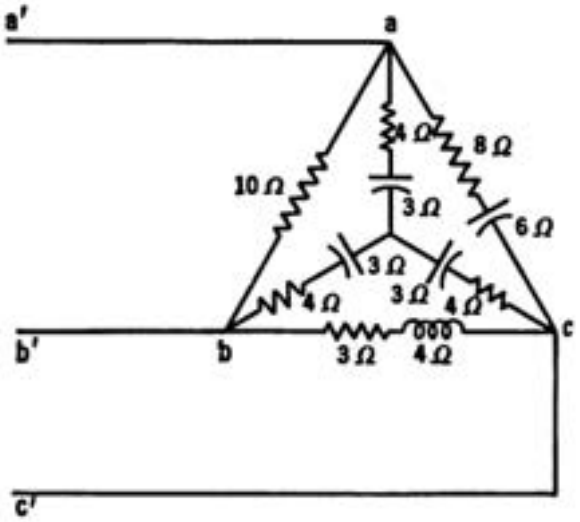

FIG. 38. See Problem 23.

24. Assume that the windings $c'a'$ and $b'c'$ of Fig. 5, page 313, are the secondaries of an open-delta bank of transformers in which $\mathbf{E}_{b'c'} = 115\angle 0°$ and $\mathbf{E}_{c'a'} = 115\angle -120°$ volts. *Note:* An open-delta connection is simply the result of connecting together additively two generated or induced emf's, the time-phase displacement between which is practically equal to 120°.

(*a*) Solve for the three line currents, $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$, if

$$\mathbf{Z}_{b'c'} = \mathbf{Z}_{c'a'} = \mathbf{Z}_{a'a} = \mathbf{Z}_{b'b} = \mathbf{Z}_{c'c} = 1 + j1 \text{ ohms}$$
$$\mathbf{Z}_{ac} = 10 + j10 \text{ ohms}$$
$$\mathbf{Z}_{bc} = 10 - j10 \text{ ohms}$$

(*b*) Find the load voltages $\mathbf{V}_{ac}$ and $\mathbf{V}_{cb}$.

(*c*) Compare the sum total of the power developed by the secondaries with the total dissipative power of the network.

25. What is the total generated power in Problem 24 if $\mathbf{E}_{c'a'} = 115\angle 0°$ and $\mathbf{E}_{b'c'} = 115\angle -120°$ volts?

26. In Fig. 7, page 318, it will be assumed that the generated voltages are:

$\mathbf{E}_{n'a'} = 100\angle 0°$, $\mathbf{E}_{n'b'} = 100\angle -120°$, $\mathbf{E}_{n'c'} = 100\angle -240°$ volts and that

$$\mathbf{Z}_{n'a'an} = (2 - j1) \text{ ohms}$$
$$\mathbf{Z}_{n'b'bn} = (1 - j3) \text{ ohms}$$
$$\mathbf{Z}_{n'c'cn} = (3 + j4) \text{ ohms}$$

Find the line currents $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$. Draw a vector diagram of the line-to-line voltages and the line currents.

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27. Refer to Fig. 12, page 326. Let it be assumed that the following quantities are known.

$$\mathbf{E}_{n'a'} = 1000 + j0 = 1000\underline{/0^\circ} \text{ volts}$$
$$\mathbf{E}_{n'b'} = -500 - j866 = 1000\underline{/-120^\circ} \text{ volts}$$
$$\mathbf{E}_{n'c'} = -500 + j866 = 1000\underline{/120^\circ} \text{ volts}$$
$$\mathbf{Z}_{an} = 20 - j20 = 28.28\underline{/-45^\circ} \text{ ohms}$$
$$\mathbf{Z}_{bn} = 50 + j0 = 50.0\underline{/0^\circ} \text{ ohms}$$
$$\mathbf{Z}_{cn} = 30 + j52 = 60.0\underline{/60^\circ} \text{ ohms}$$
$$\mathbf{Z}_g = 2 + j8 = 8.25\underline{/76^\circ} \text{ ohms}$$
$$\mathbf{Z}_l = 1 + j1 = 1.41\underline{/45^\circ} \text{ ohms}$$
$$\mathbf{Z}_n = 2.5 + j1 = 2.70\underline{/21.8^\circ} \text{ ohms}$$

Write the expressions for $\mathbf{I}_{aa'}$, $\mathbf{I}_{bb'}$, and $\mathbf{I}_{cc'}$, employing determinants and the numerical values of the **E**'s and **Z**'s specified above. Use loop currents $\mathbf{I}_1 = \mathbf{I}_{a'a}$, $\mathbf{I}_2 = \mathbf{I}_{b'b}$, and $\mathbf{I}_3 = \mathbf{I}_{c'c}$ all returning through line nn'. (Results may be left in the form of the ratio of two matrices.)

28. A delta-connected set of impedances consists of $\mathbf{Z}_{ab} = 5\underline{/0^\circ}$, $\mathbf{Z}_{bc} = 5\underline{/60^\circ}$, and $\mathbf{Z}_{ca} = 5\underline{/-60^\circ}$ ohms. Find the equivalent wye-connected impedances $\mathbf{Z}_{an}$, $\mathbf{Z}_{bn}$, and $\mathbf{Z}_{cn}$ which can be employed to replace the above delta-connected impedances.

29. Refer to Fig. 13, page 327. Assume that the generator is capable of maintaining a balanced three-phase system of voltages $\mathbf{E}_{b'a'}$, $\mathbf{E}_{a'c'}$, $\mathbf{E}_{c'b'}$, the sequence of which is $b'a'$-$a'c'$-$c'b'$. The magnitude of each line voltage is 100 volts. $\mathbf{Z}_{a'a} = \mathbf{Z}_{b'b} = \mathbf{Z}_{c'c} = 0.5 + j0.5$ ohm. $\mathbf{Z}_{ab} = 10\underline{/0^\circ}$, $\mathbf{Z}_{bc} = 10\underline{/60^\circ}$, and $\mathbf{Z}_{ca} = 10\underline{/-60^\circ}$ ohms. Find $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, and $\mathbf{I}_{ca}$ with respect to $\mathbf{V}_{a'b'}$ as a reference.

30. Explain, by means of qualitative vector diagrams, the operation of a three-phase-sequence indicator that employs an inductance coil in place of the condenser shown in Fig. 16*a*, page 333. Does the voltmeter read above or below line voltage for sequence *ab-ca-bc*?

31. Devise some scheme for checking the phase sequence of two-phase voltages.

32. Find the reading of a wattmeter which has its current coil in the $A'A$ line and its potential coil across the voltage V_{AC} in Problem 21 and Fig. 37.

33. Refer to Fig. 18, page 334. $V_{ab} = 200$, $\mathbf{V}_{bc} = 141.4$, and $\mathbf{V}_{ca} = 141.4$ volts. Sequence *ab-bc-ca*. $\mathbf{Z}_{ab} = \mathbf{Z}_{bc} = \mathbf{Z}_{ca} = (8 - j6)$ ohms. Find the reading of each of the wattmeters. Find reading of a wattmeter with its current coil in line *a* and potential coil from *a* to *b*; also one with current coil in line *c* and potential coil from *c* to *b*.

34. (*a*) If a wattmeter W_a has its current coil in line *a* and its potential coil from line *a* to *c* of Fig. 1, page 308, what will it read for a sequence V_{ab}-V_{ca}-V_{bc}? If another wattmeter W_b has its current coil in line *b* and its potential coil connected from line *b* to *c*, what will it read?

(*b*) If W_a and W_b were varmeters what would they read?

35. (*a*) Find readings of wattmeters W_a and W_b with their current coils in lines *a* and *b*, respectively, supplying the load of Problem 19 if the potential coils are properly connected so that the sum of the readings will give the total power consumed by the load.

(*b*) Find readings if W_a and W_b are varmeters.

36. Refer to Fig. 39. $\mathbf{V}_{a'b'}$, $\mathbf{V}_{b'c'}$, and $\mathbf{V}_{c'a'}$ represent a balanced three-phase system of voltage drops, the magnitude of each being 200 volts. The voltage

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sequence is $a'b'$-$b'c'$-$c'a'$. Two balanced three-phase loads indicated by the circles are connected to the terminals abc as shown in Fig. 39. In addition to the two balanced loads, a single-phase, 4-kw, unity-power-factor load is placed across the bc terminals as indicated.

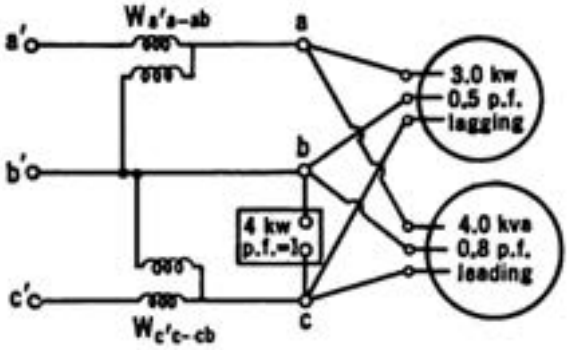

FIG. 39. See Problem 36.

(a) Find the reading of $W_{a'a-ab}$ and $W_{c'c-cb}$.

(b) If reactive volt-ampere meters replaced $W_{a'a-ab}$ and $W_{c'c-cb}$, find their respective readings.

(c) Find the combined vector power factor of the composite load.

37. In Fig. 26, page 344, it will be assumed that $\mathbf{V}_{a'b'}$, $\mathbf{V}_{b'c'}$, and $\mathbf{V}_{c'a'}$ represent a balanced three-phase system of voltages the sequence of which is $a'b'$-$c'a'$-$b'c'$. $\mathbf{Z}_{an} = 10\angle 0°$, $\mathbf{Z}_{bn} = 10\angle -60°$, and $\mathbf{Z}_{cn} = 10\angle 90°$ ohms. Assume line-to-line voltage of 100 volts.

(a) Find the readings of the two reactive volt-ampere meters shown in Fig. 26.

(b) Find the readings of wattmeters placed at similar positions in the circuit, namely, at the $a'a$-ab and the $c'c$-cb positions.

(c) Find the vector power factor of the unbalanced load as recognized by the A.I.E.E.

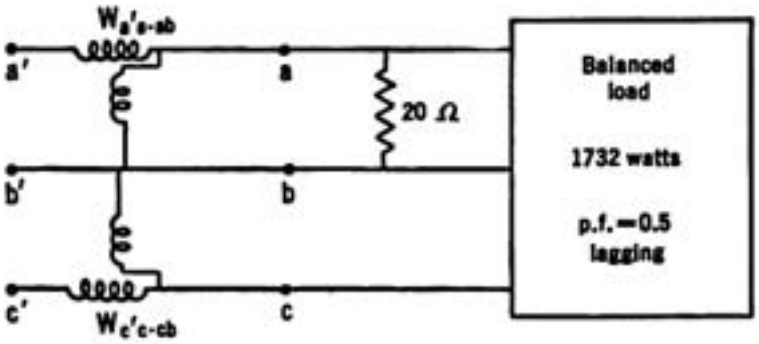

FIG. 40. See Problem 40.

38. In Fig. 40, $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{ca}$ are balanced three-phase voltages each having a magnitude of 200 volts and a phase sequence of ab-bc-ca. Determine the readings of the two wattmeters shown in the figure.

39. In Fig. 41, $\mathbf{E}_{n'a'}\mathbf{E}_{n'b'}\mathbf{E}_{n'c'}$ are balanced three-phase voltages with magnitudes of 115.4 volts and a phase sequence of $n'a'$-$n'b'$-$n'c'$. Find the following quantities and express all complex quantities with reference to $\mathbf{V}_{ab}$.

(a) $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{ca}$.

(b) $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, $\mathbf{I}_{ca}$.

(c) $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, $\mathbf{I}_{c'c}$.

(d) The sum of the readings of the wattmeters W_a, W_b, W_c when they are connected as shown.

(e) The individual readings of wattmeters W_a, W_b, W_c if the common point O is connected to line $b'b$.

40. The line-to-line voltages of a three-phase system are $V_{ab} = 200$, $V_{bc} = 150$, and $V_{ca} = 120$ volts. Write the polar expressions for $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ with respect to $\mathbf{V}_{ab}$ as reference for both phase sequences.

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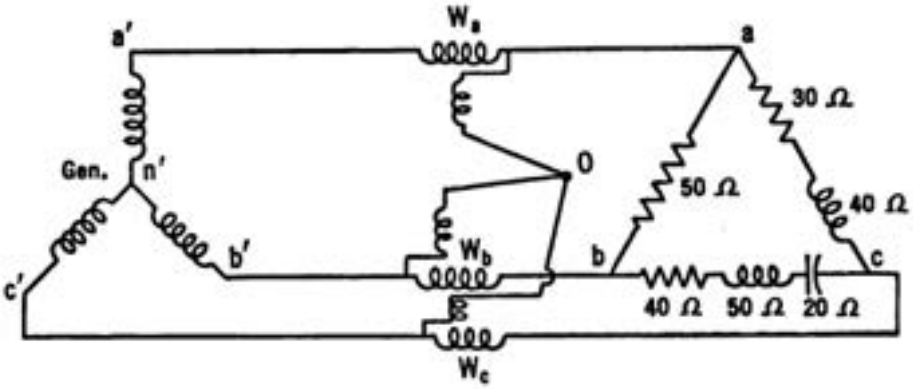

FIG. 41. See Problem 39.

41. Refer to Fig. 30*a*, page 349. In a particular case it is assumed that $V_{ab} = 140$, $V_{bc} = 120$, $V_{ca} = 150$, $V_{an} = 200$, $V_{bn} = 80$, and $V_{cn} = 104.2$ volts. The sequence of the line-to-line voltages is *ab-ca-bc*.

(*a*) Determine the vector expression for all voltages with respect to $\mathbf{V}_{ab}$ as a reference.

(*b*) Draw a single-origin vector diagram of the six voltages.

42. Refer to Fig. 31, page 350. Experiment shows that, in a particular case, the ammeters read as follows: $I_{a'a} = 20$, $I_{b'b} = 14$, $I_{c'c} = 15$, $I_{ab} = 12$, $I_{bc} = 2$, and $I_{ca} = 15$ amperes. Assume that the sequence of the line currents is known to be $a'a$-$c'c$-$b'b$. Write the polar expressions for $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, $\mathbf{I}_{c'c}$, $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, and $\mathbf{I}_{ca}$ with $\mathbf{I}_{ab}$ as reference.

43. Refer to Oscillogram 1, page 354. Write the complex or vector expressions for $\mathbf{V}_{12}$, $\mathbf{V}_{23}$, and $\mathbf{V}_{31}$ with $\mathbf{V}_{23}$ as a reference.

44. Calculate the line currents in Problem 26 by Maxwell's cyclic or the loop-current method.

45. Refer to example 15, pages 357–359, including Fig. 36. Solve for $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ by the loop-current method, neglecting the resistive components of *all* branch impedances for a voltage sequence E_{na}-E_{nc}-E_{nb}. (Results may be left in the form of the ratio of two matrices.)

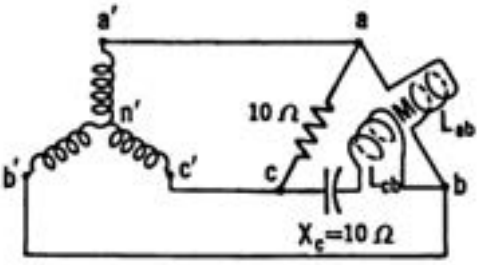

FIG. 42. See Problems 46 and 47.

46. In Fig. 42, $L_{nb} = L_{cb} = 0.01$ henry and the coefficient of coupling is 0.5. Assume no resistances or inductances except as indicated on the figure. The sequence of the balanced driving voltages is $n'a'$-$n'b'$-$n'c'$, and $\mathbf{E}_{n'a'} = 57.7 \angle 90°$ volts. For $\omega = 1000$ radians per second calculate the line and phase currents for the load. Use Maxwell's cyclic-current method.

47. Set up the determinant form of the solution for $I_{aa'}$ in Problem 46 if 3 ohms pure resistance is inserted in each line to the load and the same sequence and reference as specified in Problem 46 are employed. For uniformity in checking results, use loop currents as follows:

$$\text{Loop current } I_1 = I_{a'acc'}$$

$$\text{Loop current } I_2 = \mathbf{I}_{c'cbb'}$$

$$\text{Loop current } I_3 = I_{a'n'b'ba}$$

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CHAPTER X

ALTERNATING-CURRENT MEASUREMENTS

The methods employed in measuring alternating electrical quantities such as those discussed in the previous chapters are many and varied. In the present chapter only a few of the basic methods will be given in somewhat simplified form so that the student may gain some ideas regarding the functioning of many of the instruments he is likely to use. For a more extensive treatment of measurements and measuring instruments the reader is referred to more complete works on these subjects.[1]

The Electromagnetic Oscillograph. Alternating current and voltage measurements can be made under certain conditions with the aid of a simple permanent-magnet-type galvanometer. If a very light-weight ribbon, arranged as shown in Fig. 1, carries an alternating current, it is plain that the attached mirror experiences a tendency to oscillate about its vertical axis. The instantaneous turning effort exerted on the mirror is approximately proportional to the instantaneous current flowing in the ribbon, since the current-carrying conductors are located in a strong magnetic field which is not appreciably disturbed by the feeble magnetic field established by the current in the ribbon. This results in an angular deflection of the galvanometer mirror about its vertical axis which is approximately proportional to the instantaneous current flowing in the ribbon.

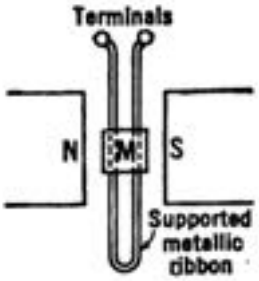

FIG. 1. Schematic arrangement of the principal working parts of an electromagnetic oscillograph galvanometer.

During the actual operation of an electromagnetic oscillograph, a beam of light is played upon the galvanometer mirror and its reflection is either viewed or recorded photographically. Obviously, for visual observation of the phenomena a succession of reflected light beams must be presented to the observer. This is accomplished in a variety of ways with the aid of auxiliary mirrors, which are synchronized with the alternating variations under observation. For photo-

[1] For example: Jansky, "Electrical Meters," McGraw-Hill Book Co., New York; Stubbings, "Commercial A. C. Measurements," Chapman & Hall, London; Hund, "High Frequency Measurements," McGraw-Hill Book Co., New York; Hague, "A. C. Bridges," Sir Isaac Pitman & Sons, London; Campbell and Childs, "Measurement of Inductance, Capacitance, and Frequency," D. Van Nostrand Co., New York.

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graphic recording it is necessary simply to drive a photographic film or paper past the deflecting light beam in such a manner that the direction of film travel is at right angles to the direction of light-beam travel or deflection. In some types an oscillating mirror arrangement with a stationary photographic film is employed.

The manner in which oscillograph galvanometers of permanent-magnet type are connected into an electric circuit to measure alternating current and voltage is shown in Fig. 2. The similarity of operation between the oscillograph galvanometers shown in Fig. 2 and ordinary d-c measuring instruments of the permanent-magnet type is apparent.

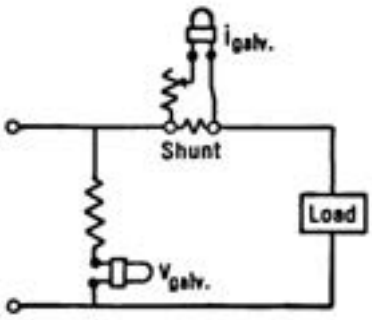

FIG. 2. Circuit placement of voltage and current galvanometers when the load impedance is relatively high.

An oscillographic record of an alternating electrical quantity is a most informative type of measurement. Under favorable conditions, wave shape, frequency, and time phase with respect to other quantities which are recorded simultaneously can be read directly from an oscillogram. The oscillograms shown in the previous chapters were taken with an oscillating mirror type instrument as described above. As a measuring instrument the electromagnetic oscillograph possesses some very serious limitations. The frequency that can be recorded faithfully must be well below the natural period of the moving element. The oscillograph galvanometer, as normally constructed, cannot record frequencies much higher than 5000 or 6000 cycles per second with any degree of fidelity. Moreover the galvanometer requires an appreciable current for full-scale deflection and this is often a serious limitation in certain low-power or low-current capacity circuits.

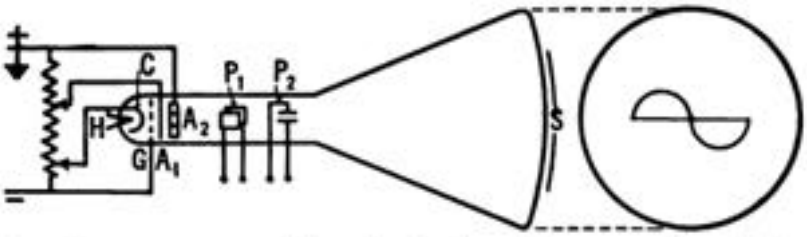

FIG. 3. Schematic arrangement of the principal working parts of a cathode-ray tube of the hot-cathode variety.

The Cathode-Ray Oscillograph. A schematic diagram of a cathode-ray tube of the hot-cathode type is shown in Fig. 3. A device of this kind can be used to visualize or photograph alternating currents and voltages and, when used in this manner, it possesses neither of the two

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serious limitations which were mentioned in connection with the electromagnetic oscillograph. Under many operating conditions it can be arranged to draw practically no power from the circuit which is under investigation and it will respond up to several hundred megacycles per second, hence its extreme usefulness as a measuring device in high-frequency communication circuits. Except in certain types of frequency measurements, the accuracy obtained with a cathode-ray oscillograph is not as high as that obtained by most other measuring instruments.

The physical principles involved in the operation of a cathode-ray tube as an oscillograph are listed below.

(*a*) The cathode or electron emitter, marked C in Fig. 3, is heated by means of the heater marked H. The energy supply circuit of heater H is not shown.

(*b*) The grid, G, is given a negative potential with respect to the cathode, as shown in Fig. 3. Its chief function is to control the supply of electrons which go to make up the electron beam.

(*c*) The first anode, A_1, being at a positive potential with respect to the cathode, accelerates some of the emitted electrons in the forward direction through the tube and starts them on their way toward the fluorescent screen S. The second anode, A_2, further accelerates the electrons which have passed through the opening in A_1. The second anode also aids in focusing this stream of electrons to a fine beam which appears on the screen as a brilliantly lighted spot.

(*d*) Let it be assumed that an alternating potential difference is to be observed and that the magnitude is such that this potential difference can be applied directly to the pair of plates marked P_2. Since the electron beam passes between the plates, the beam will be deflected in a vertical plane approximately in proportion to the instantaneous potential difference which is to be observed. Without the proper excitation on the P_1 plates to give the electron beam horizontal travel, the alternating potential difference applied to the P_2 plates appears on the screen merely as a vertical light trace.

(*e*) An important feature of the cathode-ray oscillograph, particularly from a circuit viewpoint, is the "sweep circuit" which is employed to give the electron beam its horizontal motion. If a linear time axis is desired, a potential difference which is of sawtooth wave form is applied to the pair of plates marked P_1 in Fig. 3. With a potential difference of sawtooth wave form applied to the P_1 plates, the electron beam moves horizontally from one side of the screen to the other with practically uniform velocity and then returns to its starting position in a relatively small period of time. (See Fig. 7, page 556.) The details connected

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with sawtooth wave-form production constitute what is essentially a problem in transient phenomena. One such set of details is presented on pages 554–557.

(*f*) With the P_1 plates deflecting the electron beam horizontally and the P_2 plates deflecting it vertically, the alternating potential difference which is under investigation appears as a graph in rectangular coordinates on the fluorescent screen. The assumed uniform time axis is provided by the sweep circuit voltage and the vertical displacement by the instantaneous values of the potential difference being observed. Obviously the sweep circuit frequency must be synchronized in some manner with the frequency which is under test if a stationary trace is to be obtained. Various different methods have been devised for interlocking the two frequencies.

Since the electron beam behaves as if it were composed of negatively charged particles of electricity, the beam may be deflected either electromagnetically or electrostatically. Various types of measurements, other than the one described above, can therefore be made with a cathode-ray tube. The cathode-ray oscillograph has become an indispensable instrument in connection with modern radio transmitters. It is used chiefly in measuring percentage modulation and in monitoring, although it is also employed in measuring frequency, checking grid excitation, detecting parasitic oscillations, etc.

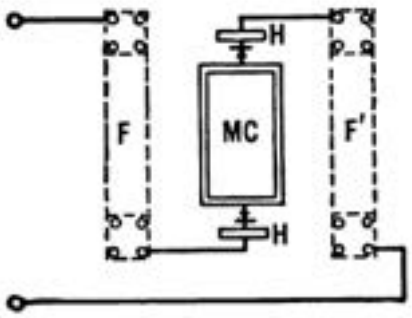

Fig. 4. Simple electrodynamometer in which the fixed coil FF' and the movable coil MC carry the same current.

Electrodynamometer Ammeters. The principle of operation of an electrodynamometer instrument has been referred to in Chapter III, page 43. The air-core fixed coils F and F' and the movable coil MC, shown in Fig. 4, are connected in series through the hair springs H. Since the current in the MC coil is the same as the current that produces the FF' field, the force and consequent torque on the MC coil in any given position are proportional to the square of the current. Expressed in the form of an equation:

$$\text{Instantaneous torque} = k\phi_{FF'}i_{MC} = k(k_1 i_{MC})i_{MC}$$
$$= k_2 i_{MC}^2 \qquad (1)$$

With a radial field produced by FF', the moving element deflects proportional to the average of the instantaneous torques because the coil has sufficient inertia to prevent it from responding to the instan-

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taneous variations. Hence, the coil assumes a position corresponding to the average square of the instantaneous current. If a suitable square root scale is placed under the pointer, the meter will show the $\sqrt{\text{average } i_{MC}^2}$ or effective value of the current through the coils.

In practice a completely radial field is not attained, and the resulting deviation of the actual deflection from the theoretical which was previously outlined is compensated for by suitably modifying the scale.

In general, the plane of the movable coil is arranged to move through an angle of approximately 90° from a position about 45° from the plane of a fixed coil, through the position where the planes are perpendicular, to the point where they are again about 45° apart as illustrated in Fig. 5. This construction results in a large deflection per ampere in the middle of the scale and smaller movements per ampere at each end. Hence, the scales are relatively open in the middle and somewhat cramped at each end. As a result, an a-c instrument of this kind should not be used in the lower one-third of the scale range if the greatest accuracy is desired.

The fact that the entire current in a simple electrodynamometer ammeter must flow through the movable coil, and hence through the small hair springs, precludes its use for over one or two amperes. For currents larger than this, a shunted electrodynamometer instrument may be used. The field FF' and the movable coil are connected in parallel as shown in Fig. 6 so that only a small portion of the total current may

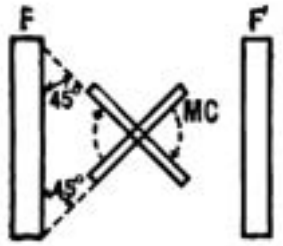

Fig. 5. Relative positions taken by the movable coil MC and the fixed coils FF' in an electrodynamometer.

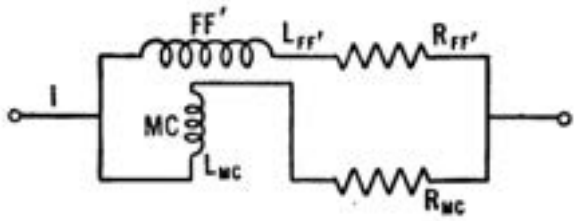

Fig. 6. Diagrammatic sketch of a shunted electrodynamometer.

pass through the movable coil. The currents in the two coils will be proportional and in phase if $L_{MC}/R_{MC} = L_{FF'}/R_{FF'}$, and the operation will then follow the laws previously described. Obviously, this arrangement may be used for very large currents even though the current in the movable coil is limited to small amounts.

Electrodynamometer Voltmeters. The electrodynamometer voltmeter is built similarly to the simple electrodynamometer ammeter described in the preceding article, except that a high resistance is placed

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in series with the coils. The coils thus receive a current proportional to the voltage across the meter, and therefore the meter reads rms volts.

Iron-Vane Ammeters and Voltmeters. Iron-vane type of meters, especially ammeters, are less expensive and, consequently, somewhat more common for frequencies in the vicinity of 60 cycles than are electrodynamometer instruments. Most of them employ the same general principle, which will be presented with reference to Fig. 7.

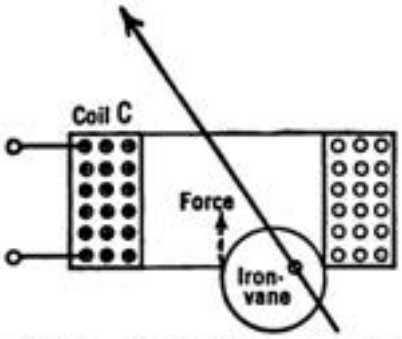

FIG. 7. Elementary type of iron-vane meter.

The force tending to turn the iron vane is proportional to the product of the flux established by the coil C and the magnetism induced in the magnetic vane. If, at all times, the flux density in the magnetic vane is maintained sufficiently low by proper design, the flux produced by the coil and the corresponding flux induced in the iron vane will be proportional to the current in the coil. The force and corresponding torque on the iron vane in any given position will then be proportional to the square of the current, and the meter thus operates similarly to the electrodynamometer type previously discussed. Iron-vane type meters are inherently adapted to measuring larger currents than the electrodynamometer instruments because the movable element carries no part of the circuit current.

Induction Ammeters and Voltmeters. Throughout the various stages of their development, induction-type meters have taken several radically different forms. The actual driving or moving element is generally a metal disc or cylinder which is not conductively connected to the electric circuit under investigation. Eddy currents are induced in the disc or cylinder by mutual induction. These currents, acting in conjunction with a magnetic field of suitable time phase and space position, produce the required driving torque in the movable element.

In induction ammeters and voltmeters, rotation of the moving element is usually restrained by a spiral spring so that the pointer attached to the moving element indicates effective amperage or voltage. In order for the device to register proportional to average i^2, it is necessary to approximate the conditions described in the above article on electrodynamometer measurements. Actually, the driving torque of the induction meter is approximately proportional to a product which is similar in nature to the product $(\phi_{FF} \cdot i_{MC})$ of the electrodynamometer. The relations involved in the induction-type meter are not simple because of the indirect manner in which the disc eddy currents are produced and because iron cores are employed in the construction of the instru-

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ments. Explanations can be presented either in terms of "shifting" magnetic fields or in terms of "disc eddy currents." Inasmuch as the "disc eddy current" theory involves several interesting circuit concepts, this form of explanation will be employed here.

From Fig. 8, it is evident that the circuit current flowing through the F coil can be employed to produce a magnetic flux, say ϕ_F, which is very nearly in time phase with the current that is to be measured. Since an iron core is used, ϕ_F will lag the circuit current by a small angle, as shown in Fig. 9. This principle is portrayed in Fig. 6, page 399, where I_M would produce the flux corresponding to ϕ_F.

The time rate of change of ϕ_F, namely $d\phi_F/dt$, through the metal disc will produce eddy currents in the disc which tend to prevent the change of flux. But these eddy currents, acting in conjunction with ϕ_F, will not establish a net driving torque in the moving element. A detailed study of Figs. 8 and 9 will show that neither the space position nor the time phase of the eddy currents produced in the disc by the

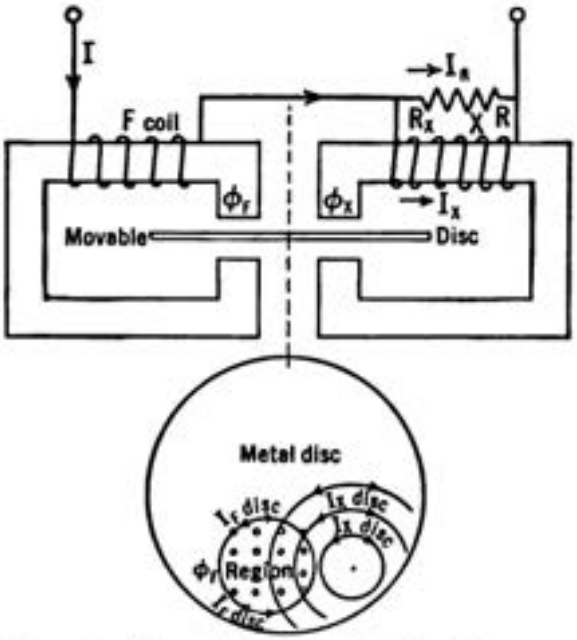

FIG. 8. Illustrating the principle of operation of a modern induction-type ammeter.

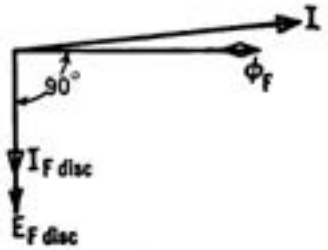

FIG. 9. Relative time phase positions of I, ϕ_F, and $E_{F\,disc}$ in the induction-type ammeter shown in Fig. 8.

variation of ϕ_F, namely $\mathbf{I}_{F\,disc}$, acting in conjunction with ϕ_F is proper to yield a net driving torque. The eddy currents in question react magnetically along the same axis in space as ϕ_F. Hence, no driving torque could result from the interaction of ϕ_F and the eddy currents due to $d\phi_F/dt$ even if the time-phase difference between these two quantities were other than 90°. The 90° time-phase displacement between ϕ_F and $\mathbf{I}_{F\,disc}$, as shown in Fig. 9, can be explained as follows.

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The voltage induced in an elemental path of the disc by $d\phi_F/dt$, namely $\mathbf{E}_{F\ \text{disc}}$, is 90° behind ϕ_F. Since the ratio of R to L of the elemental path is very high, the eddy currents, $\mathbf{I}_{F\ \text{disc}}$, are practically in time phase with $\mathbf{E}_{F\ \text{disc}}$ and hence 90° behind ϕ_F. It will be remembered that the product of two sinusoidally varying quantities of like frequency which are 90° apart in time phase yields an average value equal to zero when considered over an integral number of cycles.

Furthermore, since the magnetic axes of the field ϕ_F and the field of the eddy currents, $I_{F\ \text{disc}}$, coincide, no torque could be produced regardless of the time phase. However, the induction meter has another field ϕ_X which will also induce a voltage in the disc, which, in turn, causes a current $I_{X\ \text{disc}}$ to flow. Inspection of Fig. 8 will show that the magnetic axis of $I_{X\ \text{disc}}$ does not coincide with that of the field ϕ_F and that these currents also flow within the range of the field ϕ_F. If $I_{X\ \text{disc}}$ and ϕ_F are not in time quadrature, torque will be developed. The required phase relation is obtained through the circuit arrangement shown in Fig. 8. That this circuit accomplishes the result may be seen by a study of the vector diagram of Fig. 10. In this diagram V_{RX} is the potential drop across the divided circuit consisting of the X coil and a non-inductive resistance R. A current I_R in phase with V_{RX} will flow through the resistance while the current I_X through the X coil, which is highly inductive, will lag V_{RX} by nearly 90°. The sum of these two currents is I, the current flowing in the F coil. The current I_X in the X coil produces a field ϕ_X which lags I_X by a very small angle, as previously explained. The flux ϕ_X induces an emf in the disc $E_{X\ \text{disc}}$, which lags the flux producing it by 90°. This emf causes a current in the disc which is practically in phase with $E_{X\ \text{disc}}$ because of the very low inductance of the disc compared to its resistance. The result is a current $I_{X\ \text{disc}}$ which, by proper design, is nearly in time-phase opposition (180°) or in time phase with ϕ_F. The matter of time phase or time-phase opposition of quantities in the disc due to the two separate coils is dependent upon the mode of making the series connection of the F coil and the X coil. Therefore, a torque is developed proportional to $\phi_F i_{X\ \text{disc}}$, since the phase angle between the two quantities in a given meter is practically independent of the cur-

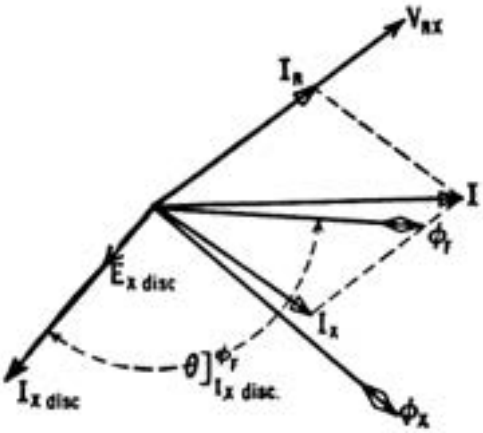

FIG. 10. Vector diagram of the principal quantities involved in the operation of an induction-type ammeter of the kind shown in Fig. 8.

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rents. Since on the straight-line portion of the magnetization curve and at constant frequency ϕ_P is $k_1 i$ and $i_{X\ \text{disc}}$ is $k_2 i$, the instantaneous torque developed is proportional to i^2, as in the electrodynamometer type of instrument. Hence, with a suitable scale, effective values are indicated.

The induction voltmeter operates on the same principles as the ammeter, the difference being in the use of a high series resistance to permit connecting the instrument across the source of potential. Induction-type instruments are suitable only for low frequencies and relatively small frequency ranges.

Other Classes of Meters. In addition to those described above, there are several other types of meters. Thermal-type instruments depend upon the expansion of a stretched wire, when it is heated by the current, for their indication. Another thermal type utilizes the heat produced by the current to raise the temperature of the hot junction of a thermocouple, and the thermocouple is connected to a d-c meter. Since the thermocouple voltage is approximately proportional to the heating, and this in turn is proportional to the average square of the alternating current, the meter indicates effective values if a suitable scale is used. Such instruments are mostly used for high frequencies.

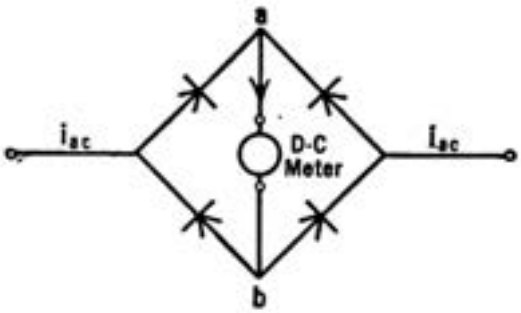

Fig. 11. Schematic arrangement of a rectifier-type meter.

Rectifier-type meters are in fairly common use at the present time, especially for measurement of very small currents. The alternating current is rectified by means of copper oxide rectifiers arranged as shown in Fig. 11, and the rectified current is measured by a d-c instrument. Since the alternating current is measured in terms of its average value, the meter must be calibrated for each wave form for which it is used.

Instrument Transformers. Large values of alternating current are usually reduced by means of iron-core transformers and then measured with 5-ampere meters. Such transformers not only reduce the value of the current to a magnitude which can be easily measured but also serve to insulate the meter from the circuit in which the current measurement is desired. Employment of instrument transformers eliminates the two chief disadvantages of shunts. Shunts used with a-c meters, in addition to permitting the meter to be raised to line potential, are rather inconvenient to apply since a meter will give different indications with shunts having the same impedance but different ratios of inductance to resistance.

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High values of alternating potentials are usually measured by reducing the potential with iron-core potential instrument transformers to values which can be read on the 150-volt range of ordinary voltmeters.

Power Measurements. An electrodynamometer-type instrument is usually employed in making power measurements in low-frequency circuits. The circuit current flows through the fixed coils FF' of Fig. 12, and a current which is approximately proportional to the instantaneous circuit voltage is allowed to pass through the movable

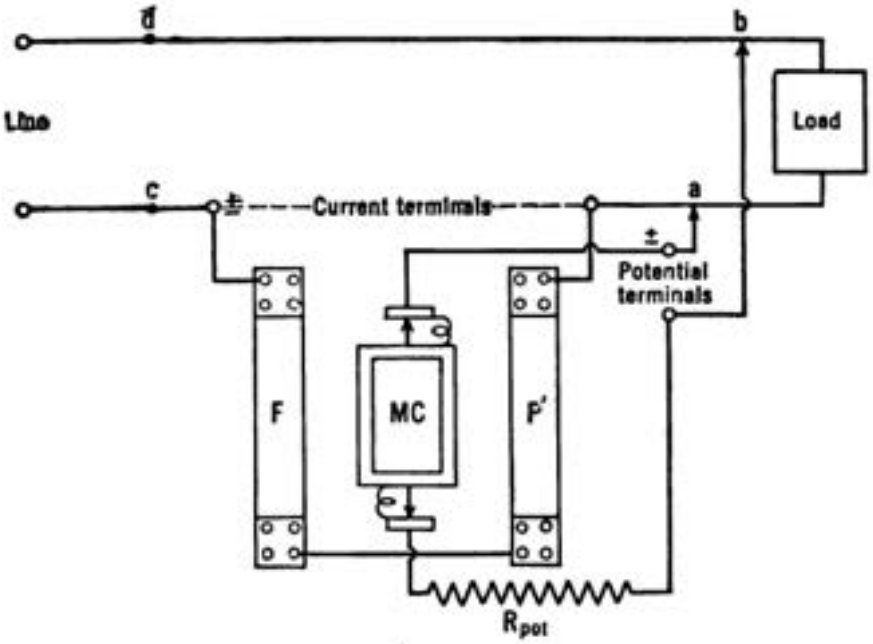

FIG. 12. The electrodynamometer-type wattmeter.

coil. As a result the movable element exerts an instantaneous driving torque which is approximately equal to the product of the instantaneous circuit current and voltage. Except in a power oscillograph galvanometer, the movable element cannot respond to the instantaneous variations in driving torque. The ordinary wattmeter indicates a value which is proportional to the average circuit power.

$$\text{Average torque} = \frac{K}{T}\int_0^T i_{mc}\phi_{FF'}\,dt$$

$$= \frac{K_1}{T}\int_0^T vi\,dt = K_1P \qquad (2)$$

In Chapters II and III it was shown that, if $v = V_m \sin \omega t$ and $i = I_m \sin (\omega t - \theta)$, then:

$$P = \frac{V_m I_m}{2}\cos\theta = VI\cos\theta \qquad (3)$$

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The manner in which the coils are connected into a circuit is shown in Fig. 12.

Unless compensated in some manner, the wattmeter registers at least a portion of its own loss. In order to determine the actual circuit power, the meter loss, which is also recorded, must be subtracted from the meter reading. Under normal working conditions the correction factor involved is of the order of 5 watts.

If the potential circuit of the wattmeter is connected across the "load" side of the current coils, as shown in Fig. 12, it is evident that the meter will read (VI_{pot}) watts even when the load circuit is open. (It is assumed that $\mathbf{I}_{pot}$ is in phase with $\mathbf{V}$.) Therefore, the correction factor is:

$$VI_{pot} = V\frac{V}{R_{pot}} = \frac{V^2}{R_{pot}} \text{ watts} \tag{4}$$

The above value should be subtracted from the meter reading to obtain the true watts taken by the load *provided the coils are connected as shown in Fig. 12.* Inasmuch as R_{pot} can usually be determined from the lid data of a wattmeter and V is generally known, the above correction factor is easily evaluated in any particular case.

If the potential terminals of the wattmeter shown in Fig. 12 are connected on the "line" side of the current coils, as for example at points c and d, the wattmeter measures the loss in its current coil. The loss involved can be computed if the load current and the resistance of the current coils are known. It is customary, however, to connect the potential terminals of a wattmeter on the "load" side of the current coil, that is, at points a and b in Fig. 12.

The Induction Wattmeter. An induction-type wattmeter is sometimes employed on switchboards and the like. The principle of operation is somewhat similar to that which has been given for induction-type ammeters and voltmeters. More specifically, the principle of operation agrees with that of the induction watthour meter, a detailed discussion of which may be found on pages 377–382.

Measurement of Reactive Volt-Amperes. In Chapter III it was shown that the reactive power or, more precisely, the reactive volt-amperes of a circuit are:

$$VI \sin \theta \Big]_{\mathbf{I}}^{\mathbf{V}}$$

if both voltage and current follow sinusoidal variations with respect to time. In general, any wattmeter which measures

$$VI \cos \theta \Big]_{\mathbf{I}}^{\mathbf{V}}$$

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by means of separate voltage and current circuits can be employed to measure reactive volt-amperes simply by shifting the time phase of one of the torque-producing quantities 90° from its wattmeter position. Under these conditions:

$$VI \cos (\theta \mp 90°) = \pm VI \sin \theta$$

When an electrodynamometer instrument is employed to measure reactive volt-amperes, an inductive reactance of the appropriate number of ohms is placed in series with the movable coil rather than the resistance shown in Fig. 12. The inherent resistance of the inductance coil causes the current which it carries to lag V by something less than 90°. The actual current through the movable coil can be shifted

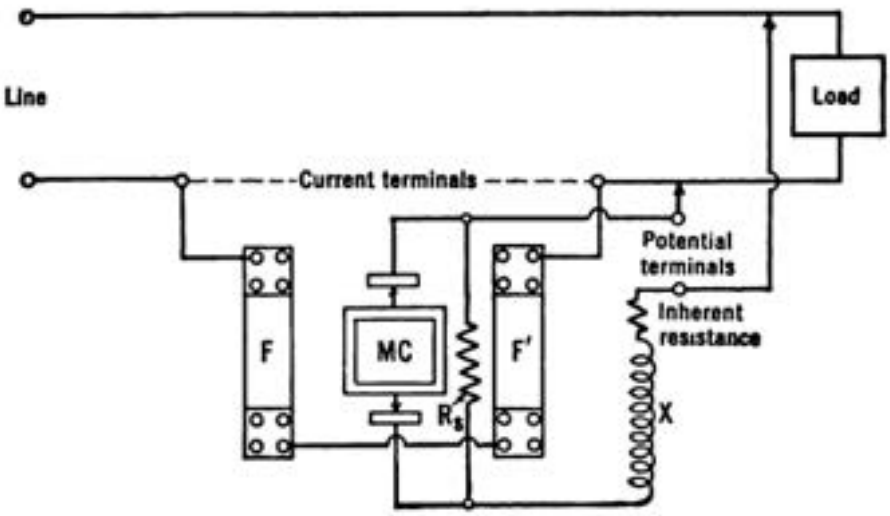

FIG. 13. Reactive volt-ampere meter of the electrodynamometer type.

to a time-phase position which is exactly 90° behind V by shunting the movable coil with a non-inductive resistance as shown in Fig. 13. If the current flowing through the movable coil is 90° behind the circuit voltage and if both the circuit current and circuit voltage are of sinusoidal wave form, the average torque developed by the instrument shown in Fig. 13 is:

$$\text{Average torque} = \frac{K}{T}\int_0^T i_{MC}\phi_{PP'}\, dt$$

$$= \frac{K_1}{T}\int_0^T V_m \sin (\omega t - 90°) I_m \sin (\omega t - \theta)\, dt$$

$$= K_1 \frac{V_m I_m}{2} \sin \theta = K_1 VI \sin \theta \Big]_{\mathrm{I}}^{\mathrm{V}} \qquad (5)$$

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The meter can therefore be calibrated to read reactive power or reactive volt-amperes directly. If **V** and **I** are in phase, the reactive volt-ampere meter reads zero. If reactive volt-amperes caused by a lagging current are considered positive, then θ in equation (5) is positive if **V** leads **I**. With proper adjustment of the relative polarities of the coils, the meter may be made to read "up scale" for either positive or negative reactive volt-amperes. Either "positive" or "negative" values may be read if the meter is designed to read zero at the mid-scale position.

It should be recognized that the quantity "reactive volt-amperes" differs materially in physical significance from the quantity "real power," even though the algebraic expressions for the two are quite similar under sinusoidal conditions. From the discussions contained in Chapters II and III it should be evident that "real power" has a definite physical significance regardless of the wave form of v and i. The quantity "reactive volt-amperes," however, is merely a hypothetical component of the total volt-amperes, which under sinusoidal conditions assumes an algebraic form that is convenient to manipulate in connection with volt-ampere calculations. It has been shown that, under sinusoidal conditions, "reactive volt-amperes" can be measured as a discrete quantity, and this quantity is useful when used in connection with "real power" measurements in finding the total volt-amperes and the power factor of the circuit. But when other than sinusoidal conditions are encountered the quantity "reactive volt-amperes" becomes rather ambiguous.

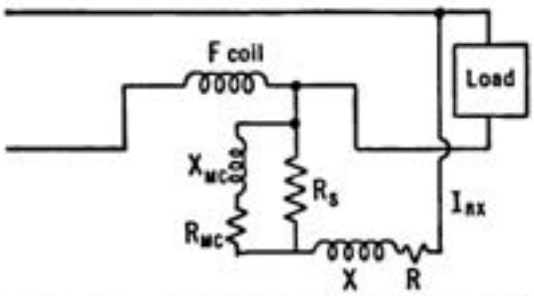

Fig. 14. Circuit diagram of a reactive volt-ampere meter. See Problem 1.

Because of the circuit adjustment necessary to secure the correct 90° phase position of $\mathbf{I}_{MC}$ with respect to **V** in Fig. 13, it is plain that this phase position is dependent upon the frequency. The meter, properly calibrated for one frequency, should not be used at a different frequency without first making the proper adjustment in the shunting resistance R_S.

Problem 1. In Fig. 14: $X = 4000$ ohms at 60 cycles and $X/R = 4$; $X_{MC} = 4$ ohms at 60 cycles and $X_{MC}/R_{MC} = 2$. Find the value of R_S which will cause $\mathbf{I}_{MC}$ to lag **V** by 90°. It may be assumed that R_S, X_{MC}, and R_{MC} are negligibly small in defining either the magnitude or phase position of $\mathbf{I}_{RX}$.

Ans.: 14.1 ohms.

Power-Factor Measurement. Power factor may be determined from a meter designed for the purpose and called a power-factor meter

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although it is generally more accurate to determine power factor from voltage, current, and power measurements. The latter method is the one usually preferred in experimental or test measurements whereas the power-factor meter is commonly used in central stations as a criterion of machine operation. For the details of construction and operation of power-factor meters, the reader is referred to works devoted to meters.[2]

The Induction Watthour Meter. The most common form of a-c meter for measuring and recording energy consumption is the induction type watthour meter. It has a current winding and a potential winding similar to those indicated in Fig. 15*a*. In general the potential winding of a watthour meter is connected on the "line" side of the current winding as shown in Fig. 15*a*. The moving element usually takes the form of an aluminum disc in which torque-producing eddy currents are established. The speed at which this disc rotates should be directly proportional to the power absorbed by the load. Under these conditions the number of revolutions made by the disc in a given period of time is a direct measure of the energy consumed during this period of time. A detailed analysis of the relation existing between disc speed and power consumed by the load is given below.

Torque Production. Assume that the circuit current, I, flowing through the winding on the two current lugs of Fig. 15*a* establishes a

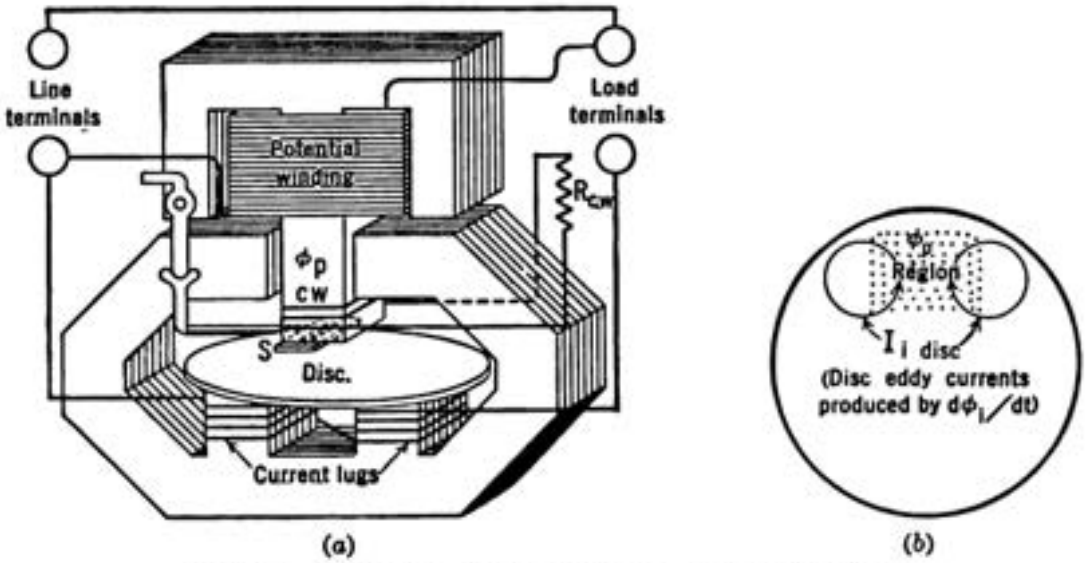

FIG. 15. Single-phase induction-type watthour meter.

flux, ϕ_i, the polarity of which is "up" through the disc over one of the current lugs when it is "down" through the disc over the other current

[2] Jansky, "Electrical Meters," McGraw-Hill Book Co., New York.

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lug. Eddy currents, $\mathbf{I}_{i\ \text{disc}}$, are established in the disc by $d\phi_i/dt$, and these eddy currents lag ϕ_i by practically 90°. (See "Induction Ammeters and Voltmeters," pages 369–372.) The eddy currents, $\mathbf{I}_{i\ \text{disc}}$, are unsymmetrically located with respect to the flux established by the potential lug, namely ϕ_p, hence $\mathbf{I}_{i\ \text{disc}}$ and ϕ_p are not in space phase. (See Fig. 15*b*.) Hence, $\mathbf{I}_{i\ \text{disc}}$ acting in conjunction with ϕ_p can produce an average net driving torque, provided these two quantities are not in time quadrature.

The potential coil establishes a flux, ϕ_p, which lags the current in the potential winding, $\mathbf{I}_p$, by a small angle because of the iron core. (See pages 399 and 400.) But the current in the potential winding lags the line voltage by approximately 80 or 85°, owing to the high ratio of reactance to resistance of the potential winding. The time-phase relations of the various quantities are illustrated in Fig. 16*a* for the case of a unity-power-factor load. The average driving torque developed under these conditions is:

$$\text{Average torque} = K\phi_p I_{i\ \text{disc}} \cos\theta \Big]_{I_{i\ \text{disc}}}^{\phi_p}$$

It will be noted in Fig. 16*a* that, although **V** and **I** are in time phase, ϕ_p and $\mathbf{I}_{i\ \text{disc}}$ are not in time phase, owing principally to the fact that $\mathbf{I}_p$ does not lag behind **V** by exactly 90°. If the meter were to be used only at unity power factor, the restraining torque could be properly adjusted to make the meter read correctly at unity power factor in spite of the angular difference between ϕ_p and $\mathbf{I}_{i\ \text{disc}}$. However, any departure from unity power factor would not be accompanied by a proportional decrease in driving torque which, of course, should be the case if the meter is to register correctly at all power factors.

Power-Factor Error. Let it be assumed that the angle between ϕ_p and $\mathbf{I}_{i\ \text{disc}}$ in Fig. 16*a* in a particular case is 10°; also that the power factor of the circuit which is under investigation drops to 0.866 lag and again to 0.866 lead. With no power-factor error, the meter should in each case develop a driving torque equal to 86.6 per cent of the driving torque developed at unity power factor. It is assumed in making the above statement that the magnitudes of **V** and **I**, and therefore ϕ_p and $\mathbf{I}_{i\ \text{disc}}$, remain unchanged. The ratio of the driving torque developed at 0.866 power factor (*I* lagging) to the driving torque developed at unity power factor is:

$$\frac{\text{Av. torque (0.866 p.f. lag)}}{\text{Av. torque (unity p.f.)}} = \frac{K\phi_p I_{i\ \text{disc}} \cos 40^\circ}{K\phi_p I_{i\ \text{disc}} \cos 10^\circ} = 0.779$$

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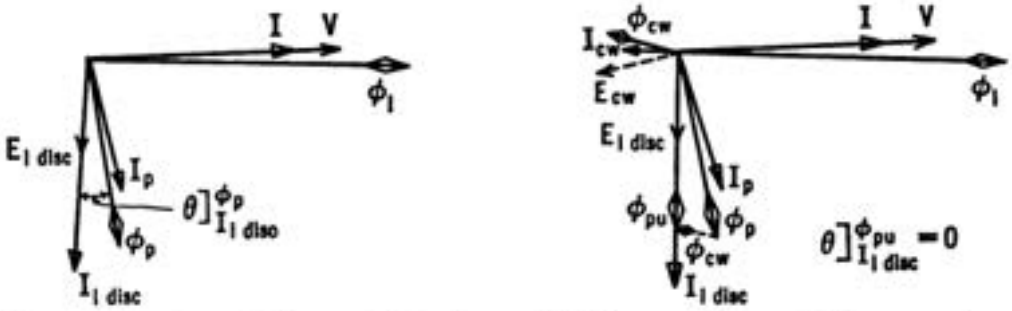
The ratio of the driving torque developed at 0.866 power factor (I leading) to unity-power-factor-driving torque is:

$$\frac{\text{Av. torque (0.866 p.f. lead)}}{\text{Av. torque (unity p.f.)}} = \frac{K\phi_p I_{i\,\text{disc}} \cos 20^\circ}{K\phi_p I_{i\,\text{disc}} \cos 10^\circ} = 0.954$$

Under the conditions of operation that have been assumed, it is evident that the meter will register too low for lagging loads and too high for leading loads. A little consideration will show that the angular dis-



(a) No compensating winding and at unity power factor.

(b) Compensating winding properly adjusted, at unity power factor. Note that as the angle between V and I changes a similar change in angle between ϕ_p and $I_{i\,\text{disc}}$ occurs.

FIG. 16. Vector diagrams of the major quantities involved in the operation of the induction watthour meter.

placement between ϕ_p and $\mathbf{I}_{i\,\text{disc}}$ should be identical with the angular displacement between $\mathbf{V}$ and $\mathbf{I}$ if accurate results at all power factors are desired. The obvious method of accomplishing this is to shift ϕ_p back until at unity power factor it coincides with the unity-power-factor position of $\mathbf{I}_{i\,\text{disc}}$, as shown in Fig. 16*b*.

Power-Factor Error Correction. Strictly speaking, ϕ_p in Fig. 16*a* is not shifted to correct for the inherent power-factor error of the induction watthour meter. But the useful flux which emanates from the end of the potential lug, ϕ_{pu}, is actually shifted with respect to the flux established by the potential winding, ϕ_p, by means of a compensating winding, marked *cw* in Fig. 15*a*. The phenomena involved can be explained in terms of coupled circuit theory, since the compensating winding is simply an *RL* circuit which is magnetically coupled to the potential winding. The addition of the compensating winding actually shifts $\mathbf{I}_p$ slightly ahead of the phase position shown in Fig. 16*a*. [See equation (61), page 239.] But the compensating winding establishes a component flux which is very nearly in time phase with the current in the compensating winding. A detailed analysis will show that the component of flux produced by the compensating winding, namely ϕ_{cw},

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possesses a time-phase displacement with respect to ϕ_p such that when combined with ϕ_p the result is ϕ_{pu}, a flux which lags ϕ_p. The amount of lag depends largely upon the circuit parameters of the compensating winding. The phase relations involved at unity power factor for a correctly adjusted meter are shown in Fig. 16*b*. ϕ_{pu} may be brought into its correct time phase by the proper adjustment of R_{cw}, L_{cw}, or M_{p-cw}. In practice the adjustment of R_{cw} is often employed, although in certain instances the proper phase relations are established by simply moving the compensating winding with respect to the potential winding, thus varying the mutual induction between these two windings.

It has been shown by means of an example on page 378 that the natural tendency of an uncompensated meter is to underregister for lagging-current loads. Hence, if, after a compensating winding has been installed, the meter still underregisters at lagging (current) power factors, the indication is that the effect produced by the compensating winding is not sufficiently great. One remedy is to reduce R_{cw}, thereby increasing the component flux, ϕ_{cw}. If, however, the meter overregisters at lagging (current) power factors after the compensating winding has been installed, the indication is that the effect is too great and R_{cw} should be increased.

Light Load Adjustment. In order to overcome the effects of friction the watthour meter is provided with a friction-compensating device. A small torque which is more or less independent of the load current is required to overcome frictional effects. An adjustable torque of this nature may be conveniently developed in an induction watthour meter by placing a "shading" ring directly under a portion of the potential lug. The "shading" ring referred to is shown in Fig. 15*a*, and is marked *S* for purposes of identification. The principle of operation of this friction-compensating device is that of a "shaded-pole" motor. The flux, ϕ_{pu}, which emanates from the lower end of the potential lug, is, of course, alternating in nature. When the flux ϕ_{pu} tends to increase from zero to a maximum value in one direction, the "shading" ring has induced in it a current, the mmf of which tends to prevent the establishment of $\phi_{pu\ max}$ within the region which is encompassed by the "shading" ring. When the flux tends to recede from a maximum value to zero, the action of the "shading" ring is to oppose the change of flux, thereby tending to maintain the flux within the region encompassed by the ring. The net effect of the "shading" ring is to produce a "shifting" magnetic field which, cutting across the aluminum disc, develops the torque required to overcome frictional effects. The direction of the driving torque produced is in the direction of the "shift" of the magnetic field, that is, *from* the unshaded portion *to* the shaded

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portion. The position of the "shading" ring is, in general, adjustable, thereby permitting an adjustment in the amount of auxiliary torque developed. If the compensation is too great, the meter may "creep" at no load. If the position of the "shading" ring is such as to develop a negative auxiliary torque, the meter may creep backwards at no load.

Full Load Adjustment. The restraining torque of the induction watthour meter is developed by eddy currents produced in the aluminum disc owing to the rotation of the disc with respect to one or more permanent magnets. These permanent magnets are sometimes called "drag" magnets. They are not shown in Fig. 15*a*. The proper amount of restraining torque can be obtained by adjusting the radial positions of the "drag" magnets with respect to the disc, or by adjusting a magnetic shunt which, if provided, is placed near the pole tips of the magnets. Since the restraining torque is proportional to the speed of the disc and the driving torque is proportional to $VI \cos \theta \Big]_{I}^{V}$, it follows that the speed of the disc is proportional to the power passing the meter. In equation form:

$$\text{Restraining torque} = K \times \text{rpm}$$

$$\text{Driving torque} = K_1 VI \cos \theta \Big]_{I}^{V}$$

Therefore:

$$\text{rpm (of disc)} = K_2 VI \cos \theta \Big]_{I}^{V} = K_2 \times \text{power}$$

The revolutions made by the disc in a specified period of time, say T_h hours, are:

$$\text{Revolutions (of disc)} = \int_0^{T_h} K_2 p \, dt$$

where p is the instantaneous power passing the meter.

Calibration. Since the accuracy of the watthour meters which are in use directly influences the gross revenue of the power companies, it is important that these meters be kept in adjustment. The accepted method of testing the watthour meter is to compare its registration with the registration of a "rotating standard." A rotating standard is simply a well-calibrated watthour meter with a graduated device for determining or counting accurately the number of revolutions made by its disc in a particular length of time. The range and flexibility of the restraining torque adjustment are desirable features of the induction watthour meter. They greatly simplify the problem of calibration

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and adjustment. (For the details connected with watthour meter calibration see "Handbook for Electrical Metermen," published by the Edison Electric Institute.)

Measurement of Polyphase Energy. Except for the type of recording instruments employed, polyphase energy measurements in a given circuit are similar to power measurements in the same circuit. Three-wire three-phase energy measurements are based on the two-wattmeter method of measuring power. Instead of two meters which deflect proportional to the product of the voltage, current, and cosine of the angle between them, two meters are used, the rotating discs of which run at speeds proportional to the product of voltage, current, and cosine of the angle between them. The number of revolutions of such a meter depends upon the time that power is used, and hence records the energy consumed. Three-phase meters, generally called polyphase watthour meters, have two meters or elements mounted on the same shaft. The rotations caused by the two elements are therefore automatically added algebraically. Two such meters are shown in Fig. 17. The earlier type had two discs on the same shaft with each element operating on its individual disc. Most of the present polyphase meters have only a single disc with both elements operating on the same disc. A polyphase watthour meter is connected in a circuit exactly like the two wattmeters shown in Fig. 43, page 286. The operation will be explained with the aid of the connection and vector diagrams shown in Fig. 18 which may be seen to be only slightly modified forms of those in Figs. 43 and 44, page 287, which were used in discussing the two-wattmeter measurement of power.

It will be assumed that the upper element of the watthour meter shown in Fig. 17 is connected as indicated by W_U in Fig. 18. The lower element connection is assumed to correspond to W_L. Also assume a balanced load connected to the lines. The vector diagram of Fig. 18 is drawn for such a load. Since the upper element receives the voltage $\mathbf{V}_{ac}$ and the current $\mathbf{I}_{an}$ it tends to rotate at a speed proportional to $VI \cos(\theta - 30°)$. Because the lower element operates from $\mathbf{V}_{bc}$ and $\mathbf{I}_{bn}$ it tends to rotate at a speed proportional to $VI \cos(\theta + 30°)$. In time dt the upper element tends to make a number of revolutions proportional to $VI \cos(\theta - 30°)\, dt$ while the corresponding tendency of the lower element is proportional to $VI \cos(\theta + 30°)\, dt$. Since both elements act on the same shaft the total number of revolutions made by the meter in time dt is proportional to the algebraic sum:

$$\begin{aligned} &VI \cos(\theta - 30°)\, dt + VI \cos(\theta + 30°)\, dt \\ &= [VI \cos(\theta - 30°) + VI \cos(\theta + 30°)]\, dt \\ &= \sqrt{3} VI \cos\theta\, dt \end{aligned}$$

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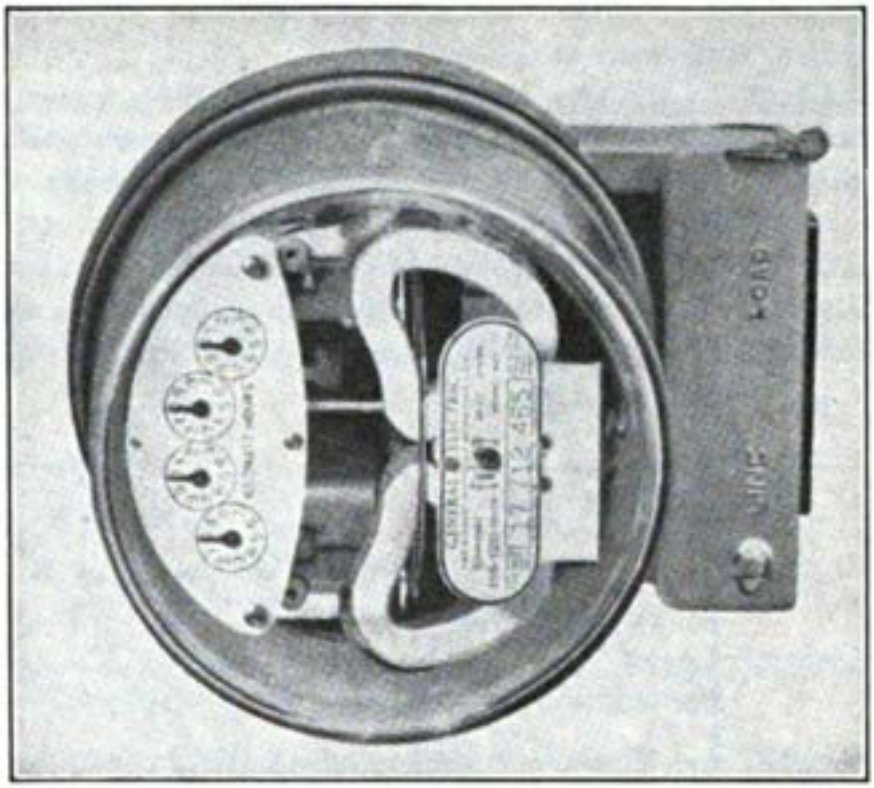

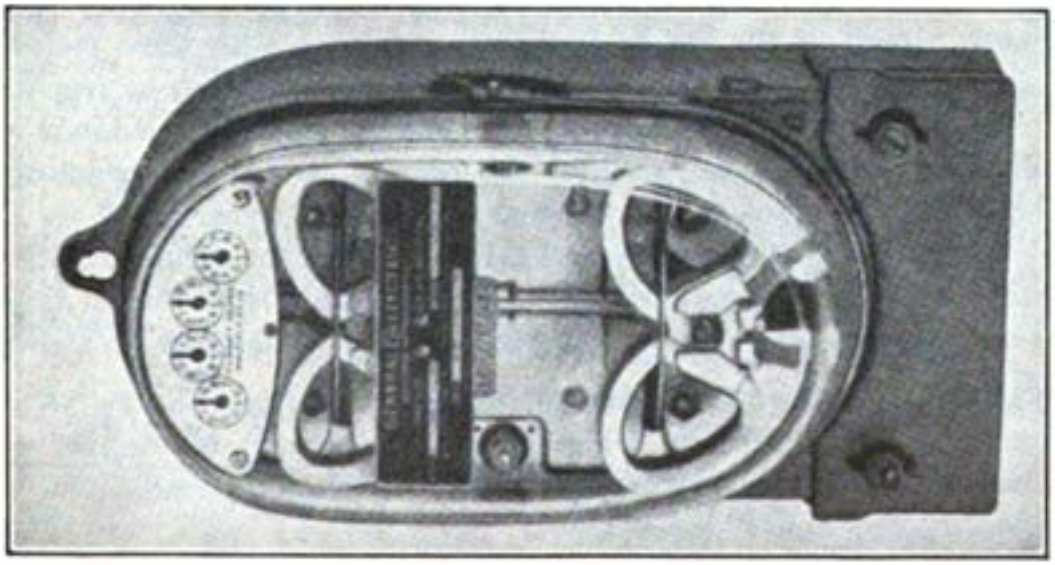

FIG. 17. Two-element two-disc and two-element single-disc watthour meters.

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The total number of revolutions made in the time T_1 is proportional to $\int_0^{T_1} \sqrt{3}VI \cos\theta \, dt$. This will be recognized as the total energy delivered or consumed in the time T_1.

In Chapter IX it was shown that two wattmeters connected like those shown in Fig. 20, page 336, may be used to measure power under any conditions of power factor or unbalanced load. Since each element of the watthour meter measures the same quantities as each of the wattmeters, and in addition takes into account the common factor time, the watthour meter, when properly adjusted and connected, will also record correctly the energy under any condition of unbalance or power factor

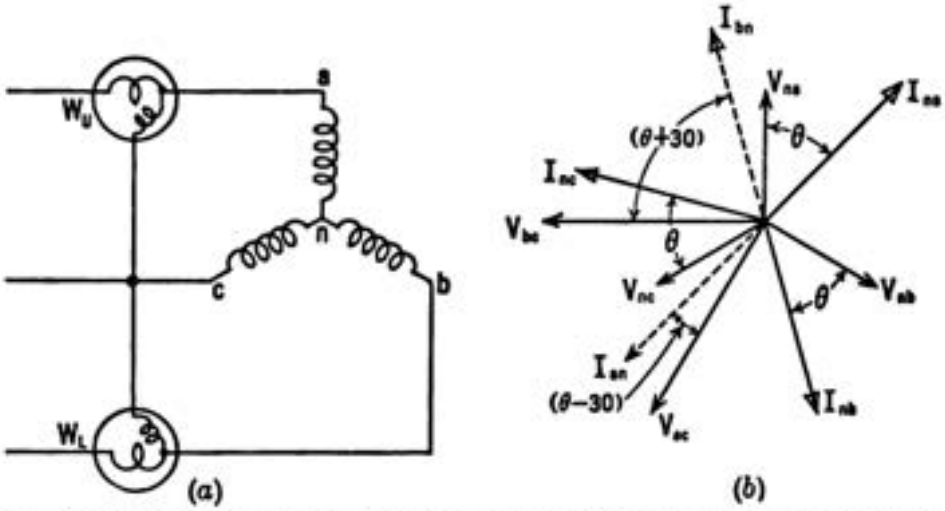

FIG. 18. Connection and vector diagrams for explaining how each element of a two-element watthour operates in measuring energy to a balanced three-phase load.

except for those inaccuracies which are introduced by saturation of the magnetic circuit of the meter, friction, etc. Even with these factors considered, ordinary watthour meters can be easily made to register within 0.5 per cent of the correct value under practically all kinds of conditions. An accuracy within 1 per cent is generally satisfactory, and several state laws allow as much as ± 2 per cent error.

To illustrate the quantities recorded by the meter for some particular unbalanced load, assume a single-phase resistance load between lines a and b of Fig. 19a. The corresponding vector diagram is shown in Fig. 19b. The upper element W_U reads $\left[V_{ac}I_{ab} \cos \theta_{\mathbf{V}_{ac}}^{\mathbf{I}_{ab}}\right] T_1$. The lower element W_L reads $\left[V_{bc}I_{ba} \cos \theta_{\mathbf{V}_{bc}}^{\mathbf{I}_{ba}}\right] T_1$. The total reading is

$$\begin{aligned} W_U + W_L &= \left[V_{ac}I_{ab} \cos \theta_{\mathbf{V}_{ab}}^{\mathbf{I}_{ac}} + V_{bc}I_{ba} \cos \theta_{\mathbf{V}_{bc}}^{\mathbf{I}_{ba}}\right] T_1 \\ &= [VI \cos 60^\circ + VI \cos 60^\circ] T_1 \\ &= VIT_1 \end{aligned}$$

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where V and I are line voltage and current respectively and T_1 is the time of operation under the specified conditions. This is known to be the correct expression for the energy due to the single-phase unity-power-factor load considered.

Problem 2. An additional resistance equal to R_{ba} in Fig. 19*a* is connected between points *c* and *b*. Find the expression for the energy measured by each element of the watthour meter. Compare the sum of your answers with what you know to be correct.

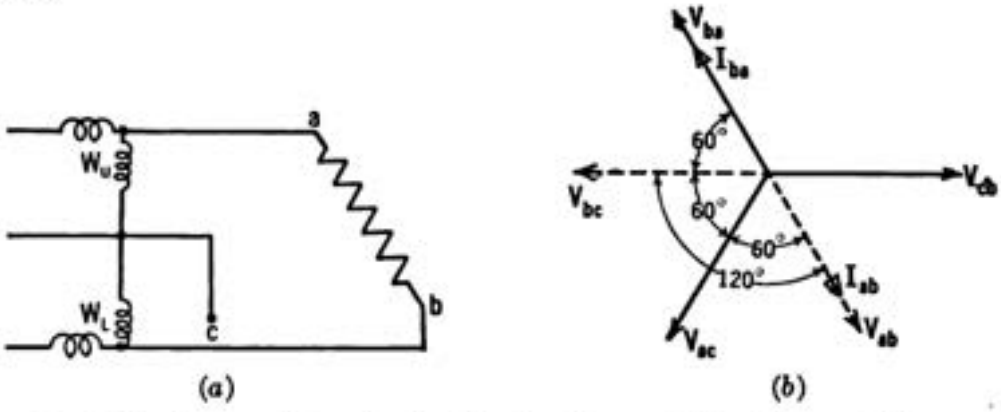

FIG. 19. Single-phase resistance load and vector diagram illustrating quantities measured by upper and lower elements of a two-element watthour meter.

Connections. Polyphase watthour meters must be correctly connected into the circuit if the readings are to be accurate. For instance, the upper element referred to in the circuit of Fig. 18*a*, must be connected so that the current coil carries the current I_{an} when the potential coil has the potential V_{ac}. Similarly, the lower element must have the current I_{bn} and the potential V_{bc}. Of course, both current and voltage may be reversed without affecting the result.

To aid in connecting the meter properly, polarity marks as shown in Fig. 20 could be used. Correct connection can then be made by connecting line $a'a$ so that the current $I_{a'a}$ enters at the ± mark on the current coil and connecting the potential coil so that the potential from *a* to *c* is the same as that from ± on the potential coil to its unmarked terminal. To fulfill the latter condition, the ± on the potential coil must be connected to line $a'a$. Actually, the coils of most polyphase watthour meters are connected at the factory to the terminals on the meter such that these ± marks referred to are on the side of the meter marked "line," as shown in Fig. 21. In the newer

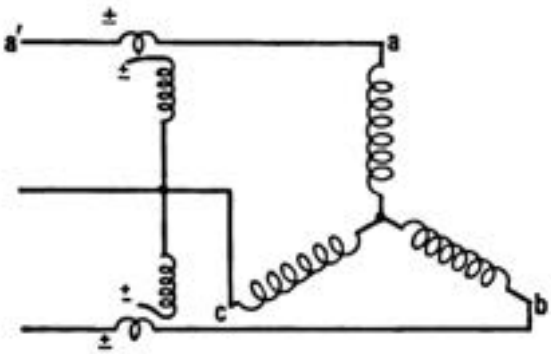

FIG. 20.

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meters these line terminals are brought out at the base of the meter. The polarity markings may then be, and usually are, omitted.

The usual connection of a three-phase watthour meter is then connected into a circuit as shown in Fig. 22. It will be noted that Fig. 22 differs from the two-wattmeter connection in Fig. 43, page 286, only in the connection of the potential coils to the line side of the current coils. This arrangement prevents the meter from recording the loss in its own

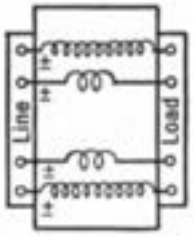

Fig. 21. Standard terminal connections of vertical two-element meter where line and load connections are on the sides of the meter. (Polarity marks are omitted in practice.)

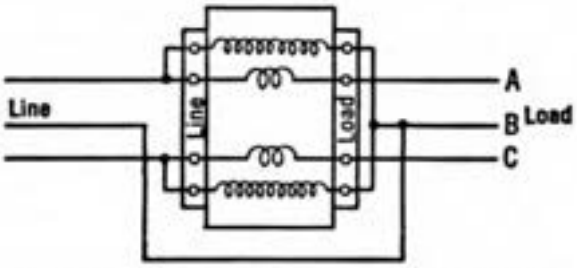

Fig. 22. Connection of two-element meter in a three-phase circuit.

potential coils. Otherwise, the meter would continue to run and record the loss in the potential coils even when there was no load on the system. Under load conditions the energy lost due to the resistance loss in the current coils will be recorded, but this is relatively very small and it is zero at no load on the system.

Connections Employing Current and Potential Transformers. If current and potential transformers are employed, the same currents and voltages reduced or otherwise changed in magnitude only must be applied to the coils of the watthour meter. If through the use of a transformer the current coil obtains only one-tenth of the actual line current at the same phase angle, it is apparent that the meter will register one-tenth as much as it would if it were connected directly into the line. Hence the reading of the meter would be multiplied by 10 to obtain the correct amount of energy consumed. A similar relation exists for the potential circuits. Thus, if 10 to 1 current and potential transformers are used, the meter reading will have to be multiplied by 100.

A meter connected with potential and current transformers is shown in Fig. 23. The crosses represent polarity marks and may be used to trace the directions of voltages and currents. Thus, if a voltage causes a current to enter at a polarity mark on the primary, the induced voltage in the secondary may be considered to cause a current to flow away from the transformer at a polarity mark on the secondary. Similarly, a current entering at a polarity mark on the primary of a current transformer can be considered as leaving the transformer in the same phase

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position but changed in magnitude at the polarity mark on the secondary. If these directions are followed, Fig. 23 will be found to be electrically the same as Fig. 22.

In general, the connections as shown in Fig. 23 are not recommended because it is desirable to ground the secondaries of current and potential transformers and the case of the meter. If the secondaries are

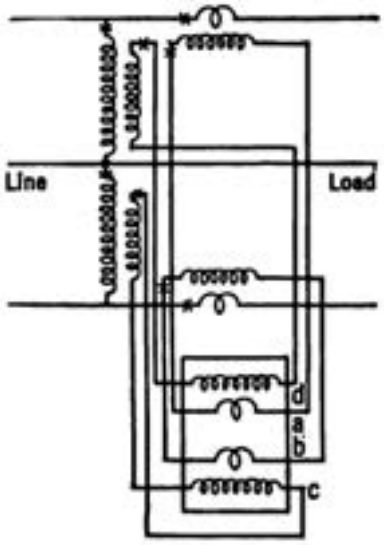

Fig. 23. Two-element meter connected with current and potential transformers in a three-phase circuit. (Not recommended.)

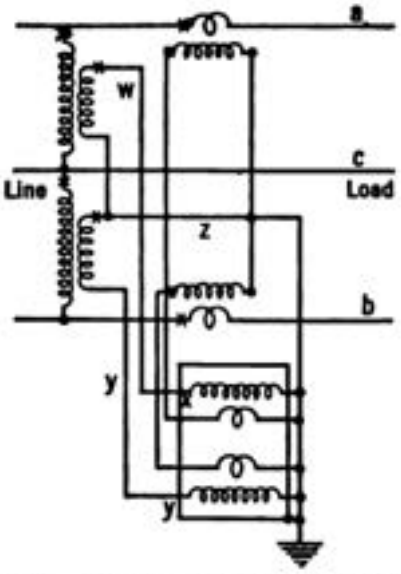

Fig. 24. Recommended connection of a two-element meter with current and potential transformers in a three-phase circuit.

grounded at points a, b, c, and d of Fig. 23 along with the meter case, the connections become equivalent to those shown in Fig. 24. Electrically this circuit is the same as far as the meter is concerned as Fig. 23. Figure 24 gives the connection usually employed because it is less hazardous to life, under abnormal conditions, than that shown in Fig. 23.

Fig. 25. Reversed connection to current coil.

Occasionally incorrect connections are made when watthour meters are used in conjunction with current and potential transformers. In the case of some faulty connections highly erroneous measurements may be obtained. For example, assume that the current to the upper element was reversed by reversing connections on the primary as indicated in Fig. 25. If the single-phase load shown in Fig. 19a were being metered, the upper element would then read $\left[V_{ac}I_{ba} \cos \theta^{I_{ba}}_{V_{ac}}\right] T_1$

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instead of $\left[V_{ac}I_{ab}\cos\theta^{\mathbf{I}_{ab}}_{\mathbf{V}_{ac}}\right]T_1$ as before. The lower element would be unaffected. Hence the meter as a whole would read:

$$\left[V_{ac}I_{ba}\cos\theta^{\mathbf{I}_{ba}}_{\mathbf{V}_{ac}}\right]T_1 + \left[V_{bc}I_{ba}\cos\theta^{\mathbf{I}_{ba}}_{\mathbf{V}_{bc}}\right]T_1$$
$$= [VI\cos 120^\circ + VI\cos 60^\circ]T_1$$
$$= VIT_1[-0.5 + 0.5] = 0$$

For balanced three-phase loads the upper and lower elements would be subtractive above 0.5 power factor and additive below 0.5 power factor, exactly the reverse of that which is correct.

Problem 3. If the above incorrect connection were made when the meter was connected to the load specified for Problem 2, page 385, determine the reading of each element of the meter and the total error in the registration of the energy consumed.
Ans.: $-0.5VIT_1$, $1.5VIT_1$, 50 per cent registration.

Methods of Checking Correctness of Connections. If the coils in a meter are correctly connected to the meter terminals at the factory, if the polarity markings on all transformers are correct, and if the polarity markings are correctly interpreted, it should be possible to connect the meter into a circuit correctly without the use of additional equipment such as voltmeters, ammeters, etc. Because of the possibility of some one condition mentioned not being fulfilled, it is desirable to know some methods of checking the correctness of the connections.

First a check should be made to determine whether the upper element of the meter obtains its potential from the line containing its own current coil to the line not containing a current coil. The corresponding check should be made on the lower element. All the following checks assume that this check has been made. This connection check having been completed, there are quite a large number of checks which might be used. No one check is always practicable.

Open-Line Check. If the load is such as to permit breaking one line at a time, line *a*, Fig. 24, may be opened. Then all the energy must be transmitted single-phase to the load between wires *c* and *b*. If the coils of the active element are correctly connected, the meter must, under these conditions, run forward if the load is dissipative in character. Next, line *a* is reconnected, and line *b*, the line containing the current coil of the lower element, is opened. Now all the energy transferred to the load is transmitted single-phase over lines *a* and *c*, and the upper element must record all this energy. The meter should therefore run forward. Then all lines are reconnected and the meter will be known to be correctly connected. This test is equally applicable to direct meter-

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ing and metering with current and potential transformers. Furthermore, this test is applicable under any conditions of unbalance or power factor. Unfortunately, the open-line check cannot always be applied.

Open Potential Coil Check. If the load is nearly balanced and the power factor is above 0.5, both elements of the meter should produce forward rotation. This is known either from a study of a three-phase vector diagram or from a study of the watt ratio curve on page 288. Each element can be checked for forward rotation when operating alone by de-energizing the other element. This is most readily accomplished by opening the potential connection to the element to be de-energized. If the power factor is known to be below 0.5, the check is still applicable provided it is correctly interpreted. Under 0.5 power factor one element must produce a torque opposing forward rotation of the meter. Therefore, when the potential connection to one element is opened, the meter should run forward, and when reconnected and the other opened the meter disc should rotate backward, or vice versa. When all connections are made, the meter disc should invariably run forward. This check is equally applicable to installations where the meter is used alone or in conjunction with current and potential transformers.

Akerman Check. This check is easy to apply, but it has a number of limitations. First, the load should be approximately balanced. For instance, a difference between the largest and smallest current of more than 25 per cent of the smallest makes the check questionable. Next, the leads must be traced to determine whether the upper element of the meter obtains its potential from the line containing its own current coil to the line not containing a current coil. The corresponding check should be made on the lower element. The above connections must be made before proceeding with the remainder of the check. Then the potential transformers should be checked to see that the voltage between leads w and y, Fig. 24, is approximately the same as the secondary potential transformer voltage. If these conditions are fulfilled, opening the common potential connection z will permit the disc to continue forward rotation irrespective of the power factor. Backward rotation indicates an incorrect connection. This can be shown through use of the vector diagram of a balanced three-phase system when it is recognized that each potential coil of the meter receives one-half of the line voltage ab when the common potential lead z is opened. This check has the advantage of being independent of power factor.

Frequency Measurement. One of the common methods of measuring frequency utilizes the resonant-reed-type frequency meter. These meters contain electromagnets that are excited by a current, the frequency of which is to be measured. The magnetic fields established

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by the electromagnets are trained upon a number of reeds which, by virtue of difference in mass, type of suspension, or restraint, possess different natural periods of vibration. With reeds of proper natural periods, frequencies over a particular range are indicated by the mechanical vibration of the calibrated reeds. The tips of the reeds are generally colored white, and the vibrating reeds appear as white lines opposite a calibrated scale which indicates the frequency directly. Meters of this kind are arranged so that the meter terminals are connected across the line voltage either directly or through potential transformers.

Another type of frequency meter is similar to utilizing two induction voltmeter elements on a single disc, so arranged as to produce opposite torques. One element is connected in series with a resistance and the other in series with an inductive reactance. As the frequency increases, the current is reduced in the inductive element, thereby reducing its torque and causing the disc to rotate to a new position under the influence of the resistance element. The opposite effect is produced by a decrease in frequency. If a suitably calibrated scale is employed, the meter can be made to indicate frequency in cycles.

Since frequency in alternators is dependent upon speed and poles, devices which indicate speed can be employed to measure frequency. In addition to the foregoing there are several other much used methods of measuring frequency.

The above methods of measuring frequency are primarily applicable to power frequencies. For the measurement of high frequencies where great accuracy is not required, a wave or frequency meter shown diagrammatically in Fig. 26 is employed. The *RLC* circuit equipped with some indicating device is loosely coupled to the circuit the frequency of which is to be measured. Maximum current is first obtained by proper adjustment of *L* or *C*. Theoretically, the resonant frequency can be calculated from the circuit parameters which yield maximum current, but usually the device is calibrated against some form of standard frequency meter. The frequency which is to be measured is then determined by reference to a calibration chart for the particular values of *L* and *C* employed to give maximum current in the *RLC* loop. The operation of the device depends upon its being sufficiently loosely coupled so that no appreciable reaction in the test circuit results from the current in the *RLC* loop.

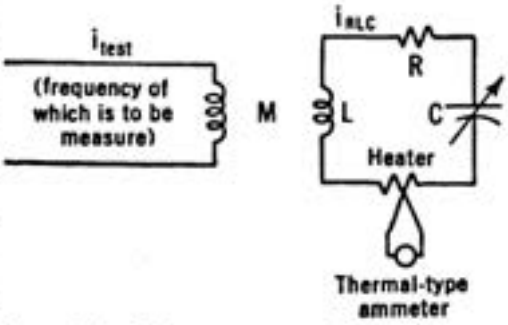

Fig. 26. Wavemeter method of measuring high frequencies.

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PROBLEMS

4. Assume that the actual current flowing in a particular circuit is composed of a d-c component, I_{dc}, and an additive a-c component such that the resultant current is:

$$i = I_{dc} + I_{ac\ max} \sin \omega t.$$

A permanent-magnet, moving-coil d-c ammeter and an ordinary electrodynamometer-type ammeter are placed in series in the circuit.

(*a*) Which meter reads the larger number of amperes?

(*b*) Which reading is the more useful if the current in question is employed for electroplating purposes?

(*c*) Which is more useful if the current is employed for heating purposes?

5. The lid data of a particular wattmeter are:

Current range, 7.5/15
Voltage range, 100/200 volts
Watt range, 120/240/480 watts
Resistance of 100-volt potential circuit, 2500 ohms
Resistance of 200-volt potential circuit, 5000 ohms

(*a*) Is the instrument a unity-power-factor or low-power-factor wattmeter?

(*b*) If the number of scale divisions is 240, find the multiplying factor to be employed with each of the four possible combinations of current and potential ranges.

(*c*) If the meter reads 160 scale divisions when the highest current range and highest potential range are in use, convert the reading to watts. If the actual circuit voltage is 220 volts, find the "corrected" reading of the wattmeter in watts, assuming that the potential circuit is connected on the "load" side of the current coils as shown in Fig. 12, page 373.

6. The 50-volt potential circuit of the wattmeter shown in Fig. 12, page 373, has a resistance of 2056.5 ohms and an inductance (concentrated largely in the movable coil) of 3.4 millihenrys. The frequency is 60 cycles.

(*a*) By what angle does the current in the potential circuit lag the load voltage?

(*b*) Qualitatively what effect does the inductance have on the accuracy of the instrument, if it is assumed that the instrument has been calibrated on direct current? Consider both magnitude and phase-angle discrepancies.

(*c*) By how many parts in a million does the inductance affect the magnitude of the potential circuit current, and does this magnitude discrepancy cause the instrument to read high or low?

(*d*) By how many parts in a million does the phase-angle discrepancy caused by the inductance affect the accuracy at a load power factor of unity, and does this phase-angle discrepancy cause the instrument to read high or low?

(*e*) Are the magnitude and phase-angle discrepancies cumulative or subtractive for load power factors of less than 0.9998 (current lagging)?

7. The induction instrument shown in Fig. 27 can be employed to measure reactive volt-amperes. Explain in detail the physical operation of the instrument, and draw a qualitative vector diagram showing $\mathbf{V}$, $\mathbf{I}_p$, $\boldsymbol{\phi}_p$, $\mathbf{I}$, $\mathbf{I}_R$, $\mathbf{I}_X$, $\mathbf{E}_{X\ disc}$, and $\mathbf{I}_{X\ disc}$. $\mathbf{E}_{X\ disc}$ and $\mathbf{I}_{X\ disc}$ are the voltage and current which are induced in the disc as a result of $d\phi_X/dt$. For the sake of uniformity and definiteness in drawing the vector diagram, the following assumptions may be made: (*a*) $X_p/R_p = 4$; (*b*) $\boldsymbol{\phi}_p$ in phase with $\mathbf{I}_p$; (*c*) $X_X/R_X = 5$; and (*d*) $\boldsymbol{\phi}_X$ in phase with $\mathbf{I}_X$. (See "In-

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duction Ammeters and Voltmeters," page 369, for a discussion of the physical principles involved in induction-type meters.)

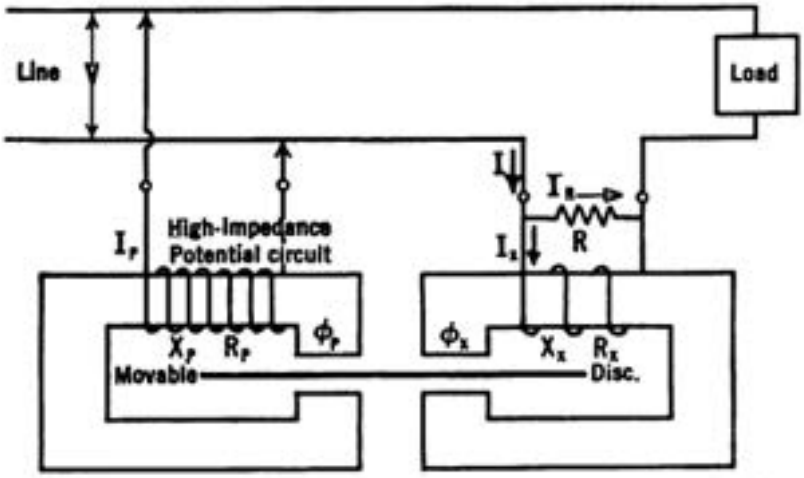

FIG. 27. Induction-type reactive volt-ampere meter. (See Problem 7.)

8. Figure 28 shows a commonly used connection to the delta secondary of a transformer bank for supplying both power and lighting service from a single transformer bank. For the metering scheme shown, what percentage of the energy used is recorded in the following cases:

(*a*) A unity-power-factor load from *a* to *d*?

(*b*) A 0.5 lagging power-factor load from *a* to *d*?

(*c*) A 0.866 lagging power-factor load from *a* to *d*?

(*d*) A 0.866 leading power-factor load from *a* to *d*?

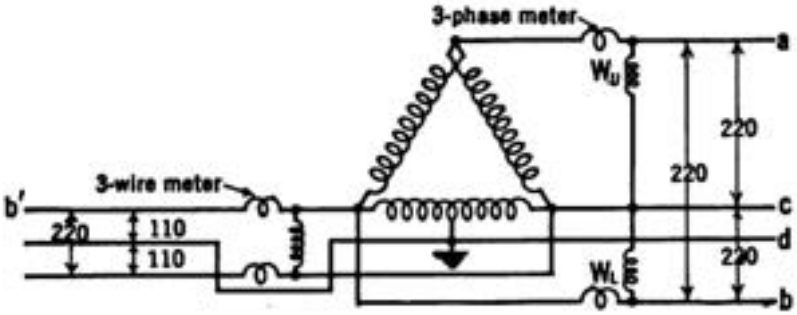

FIG. 28. See Problems 8 and 9.

(*e*) What is the effect on the meter of a load from *a* to *d* of less than 0.5 lagging power factor?

(*f*) Any load from *c* to *d*?

(*g*) Any load from *b* to *d*?

(*h*) A unity-power-factor load from *a* to *b'*. *Note:* A 3-wire meter reads the equivalent of the product of the current in its current coil by one-half the voltage across its voltage coil by the cosine of the angle between the current and voltage.

9. Devise a metering system that could be used to record the energy taken from the transformer secondary in Fig. 28 for any condition of loading between any lines of the single-phase, three-wire and the three-phase, three-wire systems.

10. Show by means of a vector diagram that the Akerman check will fail for one sequence of voltages and not the other, if a 20 per cent lagging power-factor load is connected from *a* to *b* in Fig. 24 and at the same time a like load is connected from *c* to *b*.

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CHAPTER XI

DETERMINATION OF CIRCUIT PARAMETERS

Although the circuit parameters R, L, M, and C may be reasonably constant under certain conditions, their effective magnitudes are often influenced by temperature, frequency, potential gradient, current density, magnetic density, etc. At low frequencies the parameters can usually be considered as lumped, and certain effects, such as internal distributed capacitance, dielectric loss, and capacitance to ground, are of minor importance and are generally neglected. Circuit parameters should, therefore, be measured under conditions which are as nearly identical to actual operating conditions as possible.

Generally speaking, there are three distinct methods of determining circuit parameters:

1. Calculations based on physical dimensions for particular values of current, magnetic flux, and dielectric flux densities.
2. Calculations based on experimentally determined values of voltage, current, power, and frequency.
3. Substitution methods whereby the unknown parameter is compared with a known standard.

Adequate treatments of these general methods cannot possibly be encompassed in this book. A few examples which involve elementary circuit theory are given below.

Determination of Z by the V, I, P Method. Except for the sign of the phase angle, the complete expression for the impedance of a particular branch can be evaluated directly from experimentally determined values of V, I, and P as shown in Fig. 1. If "corrected" values of V, I, and P are employed

$$\mathbf{Z}_e = \frac{V}{I} \angle \cos^{-1} \frac{P}{VI} = \frac{V}{I} \angle \theta \tag{1}$$

The impedance thus determined is the effective or equivalent series circuit impedance between the terminals across which the potential difference is measured. When other than sinusoidal wave form is encountered, the equivalent impedance possesses very little physical significance except that it specifies the ratio of V to I for the particular wave forms involved.

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Because of the dynamical nature of the measured quantities, simultaneous readings of the measuring instruments are generally taken. This practice requires that certain corrections be made in at least some of the meter readings in order to compensate for the circuit disturbances which result from the insertion of the meters. Also due consideration must be given to the relative physical placement of the measuring instruments if a minimum amount of circuit disturbance is to be realized.

For the case shown in Fig. 1 it is assumed that Z is small as compared with the impedance of the voltmeter and the impedance of the potential circuit of the wattmeter. If Z is relatively very small, then the fact that the ammeter includes in its reading the voltmeter current and the wattmeter potential circuit current may not be a significant discrepancy. If Z is even approximately comparable in magnitude to the impedance of the potential circuits, then a serious error will be introduced into the ammeter reading if the arrangement shown in Fig. 1 is employed. Under these conditions either a suitable correction should be made in the ammeter reading or the ammeter should be moved to the load side of the potential circuits. If the ammeter is moved to the load side of the potential circuit, both the voltmeter and wattmeter readings will be in error because of the extraneous voltage drop across the ammeter. If the errors thus introduced are significant, suitable corrections must be made in the voltmeter and wattmeter readings.

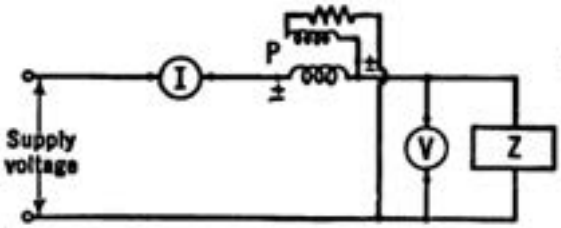

FIG. 1. Wattmeter P reads the power loss in both the potential circuit of the wattmeter and the voltmeter.

In making either voltage or current corrections, due consideration must be given to the vector relations of the quantities involved. If the wattmeter potential circuit and the voltmeter are connected on the load side of an uncompensated wattmeter current coil, as shown in Fig. 1, the V^2/R loss in the potential circuits should be subtracted directly from the wattmeter reading to obtain the correct power absorbed by the load.

Problem 1. Let it be assumed that in Fig. 1 the ammeter reads 2.02 amperes, the voltmeter reads 114 volts, and the wattmeter reads 37.5 watts. The resistance of the potential winding of the wattmeter is 3500 ohms; the resistance of the voltmeter is 2000 ohms. The load is known to be inductive. (*a*) Find the value of Z, neglecting all meter disturbances. (*b*) Find the value of Z after correcting the wattmeter reading for the loss in the potential winding of the wattmeter and for the loss in the voltmeter. (*c*) Find the value of Z after correcting both the wattmeter reading and the ammeter reading.

Ans.: (*a*) $56.4\,\underline{/80.6°}$, (*b*) $56.4\,\underline{/83.2°}$, (*c*) $56.7\,\underline{/83.15°}$ ohms.

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Problem 2. (*a*) If the resistance of the wattmeter potential circuit of Fig. 1 is R_p and the resistance of the voltmeter is R_v, show that

$$\text{True power, } P = W - \frac{V^2}{R}$$

where $R = \dfrac{R_pR_v}{R_p + R_v}$ and W is the actual reading of the wattmeter.

(*b*) If W is the actual reading of the wattmeter, I the actual reading of the ammeter, and I_s the current through the load impedance, show that

$$I_s = \sqrt{I^2 + \frac{V^2}{R^2} - \frac{2W}{R}}$$

where, as before, $R = R_pR_v/(R_p + R_v)$.

Hint: The in-phase component of $I_s = (W/V) - (V/R)$.

$$\text{The quadrature component of } I_s = \sqrt{I^2 - \frac{W^2}{V^2}}$$

These statements carry with them the assumption that the inductive reactances of both the wattmeter potential circuit and the voltmeter are negligibly small as compared with the resistances.

Effective Resistance. The effective a-c resistance of a circuit or circuit element is defined as:

$$R_e = \frac{P}{I^2} \tag{2}$$

For example, the effective resistance of the load impedance shown in Fig. 1 can be determined directly from the corrected readings of the ammeter and wattmeter. R_e as found from equation (2) may, in any particular case, differ materially from the ohmic or d-c resistance.

Owing to electromagnetic or electrostatic coupling with other circuits or owing to the proximity of magnetic or dielectric material, the effective a-c resistance is greater than the ohmic or d-c resistance. Also, skin effect in the conducting wire itself tends to make R_e greater than the ohmic resistance. Reference to Fig. 2 will disclose one of the reasons for skin effect. The internal regions of the wire exhibit a higher impedance to the flow of current than the outer regions. The result is that the current densities are greater near the surface than near the center of the conductor. For this reason it is customary to use tubular conductors if the frequency is very high, since the current naturally concentrates in a thin outside layer. At power frequencies, skin effect is not an appreciable factor except in conductors of large cross-sectional area. In conductors smaller than about 250,000 circular mils cross-

Fig. 2. Illustrating internal flux linkages.

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sectional area the effective resistance at power frequencies is practically identical in magnitude to the ohmic resistance, provided skin effect is the only factor which contributes to make R_e greater than R_{ohmic}. For a 250,000-circular-mil stranded copper conductor the increase in resistance due to skin effect above the ohmic resistance is about 0.9 per cent.

Another effect is present at high frequencies in parallel cylindrical wires which tends to make R_e greater than R_{ohmic}. The currents concentrate on the parts of the cylinders which are nearest to one another, thus increasing the current densities in these particular parts of the conductors. The phenomenon involved is usually described as "proximity effect." However, the manner in which skin and proximity effects vary with conductor size and shape and with frequency is rather complicated. General quantitative predetermination of these effects involves certain types of functions which are not generally understood by undergraduates.[1]

In addition to the increased current density near the surface of a conductor owing to the higher impedance of the inner paths, the relative values of self-inductance and distributed self-capacitance of the conductor can materially influence the magnitude of skin effect at the higher frequencies. The self-resonating effect of uncoupled air-core inductance coils at high frequencies and the way in which this phenomenon contributes to make R_e greater than R_{ohmic} will be better appreciated after pages 401–403 have been studied. In self-resonating coils R_e may become thousands of times larger than R_{ohmic} and, if the coil in question is not coupled to other circuits, the increase in apparent resistance is chargeable to one form of skin effect.

Equation (61), Chapter VII, on page 239, shows that the resistance of any coupled secondary circuit appears at the terminals of the primary circuit as a modified *additive* resistive component. This additive resistive component is one of the factors which makes R_e of the primary circuit greater than R_{ohmic}. With air-core coupling between the circuits, R_e may be sensibly constant within certain ranges of frequency and as such it may be manipulated as a constant circuit parameter. For iron-core coupling the core loss due to the mutual flux is, in certain cases, practically constant. Therefore, the effect of core loss due to the mutual flux cannot be represented by the loss in an additive series resistance. The usual equivalent circuit representation of a constant core loss of this type is a fictitious resistive branch which is placed across some potential difference of practically constant magnitude.

[1] Hund, "High-Frequency Measurements," pp. 263–267, 1st ed.; MacLachlan, "Bessel Functions for Engineers," pp. 140–141; Starr, "Electric Circuits and Wave Filters," pp. 104–105.

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The effective resistance of an iron-clad coil is generally several times larger than the ohmic resistance of the coil, even at power frequencies, and the core loss in the adjacent iron is the chief contributing factor. Since the core loss is a complicated function of the magnetic flux established in the iron, the effective resistance of an iron-clad coil is a rather intricate function of the applied voltage.

An understanding of what is actually meant by the term "effective resistance" and an appreciation of the various factors which cause R_e to be greater than R_{ohmic} are important concepts in a-c circuit theory as well as in a-c machinery theory.

Effective Reactance. The effective or equivalent series circuit reactance of a network or circuit element is defined as:

$$X_e = \sqrt{Z_e^2 - R_e^2} \tag{3}$$

Z_e and R_e having been determined from physical measurements of voltage, current, and power, the magnitude of X_e may be evaluated directly from equation (3). Owing to the fact that R_e is greater than R_{ohmic}, X_e is smaller than the "true" reactance. (See page 400 for a definition of "true" self-inductance, L_{true}, which combined with ω forms what is sometimes called "true" reactance.)

It has been shown in Chapter VII that the inductive reactance of a coupled secondary circuit causes the effective reactance of the primary winding to be less than the individual reactance of the primary winding. For the coupled circuit arrangement shown in Fig. 3:

$$R_e = R_1 + \frac{X_M^2}{Z_2^2} R_2 \tag{4}$$

and

$$X_e = X_1 - \frac{X_M^2}{Z_2^2} X_2 \tag{5}$$

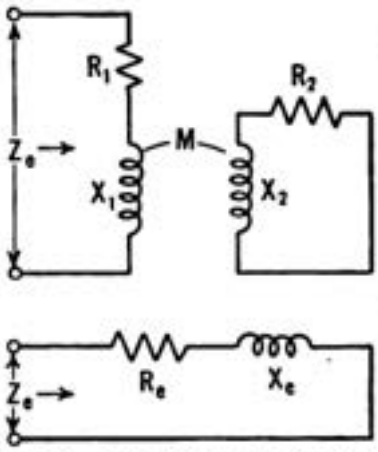

FIG. 3. Reduction of air-core coupled circuits to an equivalent series circuit.

The above relations follow directly from equation (61) of Chapter VII, page 239. For the particular air-core case considered in Fig. 3 the equivalent reactance of the primary, X_e, has a definite physical significance and can be defined in terms of conceivable flux linkages. But where magnetic core losses or dielectric losses cause R_e to be greater than R_{ohmic}, X_e loses much of its physical significance. X_e is simply a hypothetical reactance which, if multiplied by the current, forms an IX_e drop such that:

$$\sqrt{(IX_e)^2 + (IR_e)^2} = (IZ_e) = V_{\text{applied}} \tag{6}$$

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Example 1. Let it be assumed that, in Fig. 4, the ammeter reads 2.60 amperes, the voltmeter reads 110 volts, and the wattmeter reads 85.0 watts. The impedance of the wattmeter potential will be considered as being 4000 ohms pure resistance and the impedance of the voltmeter as 6000 ohms pure resistance. The values of Z_e, R_e, and X_e are to be determined.

The power supplied to the terminals of the iron-clad inductance coil is:

$$P = 85.0 - \left[\frac{110^2}{4000} + \frac{110^2}{6000}\right] = 79.96, \text{ say } 80 \text{ watts}$$

$$I_{coil} = \sqrt{I^2 + \frac{V^2}{R^2} - \frac{2W}{R}} \quad \text{(See Problem 2, page 395.)}$$

$$= \sqrt{2.6^2 + \frac{12{,}100}{2400^2} - \frac{170}{2400}} = 2.59 \text{ amperes}$$

It will be noted that, whereas the magnitude of the current taken by the voltmeter and potential circuit of the wattmeter is approximately 46 milliamperes, the actual discrepancy introduced into the ammeter is only about 10 milliamperes in this particular case.

The desired values can now be calculated from the corrected meter readings.

$$Z_e = \frac{110}{2.59} \Big/ \cos^{-1} \frac{80}{110 \times 2.59} = 42.5 \angle 73.7^\circ \text{ ohms}$$

$$R_e = \frac{80}{2.59^2} = 11.9 \text{ ohms}$$

$$X_e = \sqrt{42.5^2 - 11.9^2} = 40.8 \text{ ohms}$$

If the effective inductance, L_e, is defined as X_e/ω, it is only necessary to know the angular velocity of the supply system in order to be able to evaluate L_e.

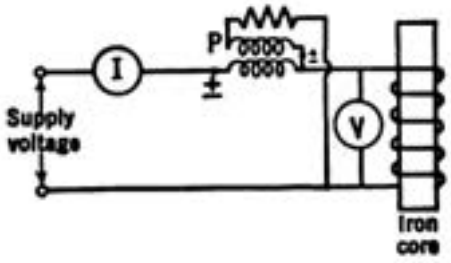

FIG. 4. Measurement of Z_e, R_e and X_e by the voltmeter-ammeter-wattmeter method.

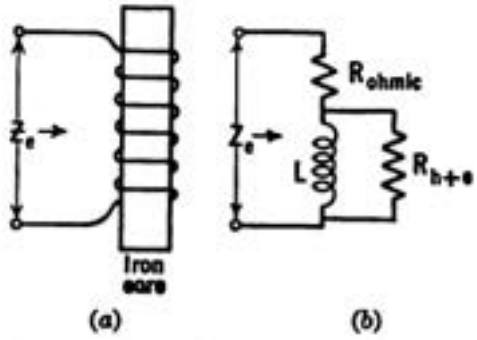

FIG. 5. Approximate equivalent circuit of an iron-core inductance coil.

Vector Diagram of Iron-Clad Circuit Quantities from Voltmeter, Ammeter, and Wattmeter Readings. Several important concepts in the field of a-c machinery are dependent upon the equivalent circuit theory of iron-clad circuits. The iron-clad circuit shown in Fig. 5*a* can be represented quite faithfully by the equivalent electric circuit

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shown in Fig. 5*b*. The R_{h+e} branch is a resistive branch which is theoretically connected in parallel with the pure inductance of the coil in order to provide a path wherein a number of watts equal to the actual core loss can be hypothetically spent. The subscripts $(h + e)$ used in connection with R_{h+e} refer to "hysteresis" and "eddy-current" components of the core loss. The fact that the core loss in question is a function of the flux established by the inductance coil, together with the fact that the flux established by the inductance coil is to some extent dependent upon the magnitude of IR_{ohmic}, dictates the circuit position of the hypothetical R_{h+e} branch.

If the analysis is to be based entirely on experimental readings of V, I, and P, neither R_{h+e} nor L can be evaluated accurately. An approximate procedure is employed which is entirely consistent with the accuracy of the experimental readings and the accuracy with which Fig. 5*b* actually simulates iron-clad circuit phenomena. In the approximate procedure referred to, the applied voltage is thought of as being balanced by iR_{ohmic} and $N\,d\phi/dt$, in which case:

$$iR_{\text{ohmic}} + N\frac{d\phi}{dt} = v \tag{7}$$

If v is of sinusoidal wave form and i is assumed to be of sinusoidal wave form, then:

$$\mathbf{I}R_{\text{ohmic}} + \mathbf{V}_L = \mathbf{V} \tag{8}$$

In this equation $\mathbf{V}_L$ is the effective vector value of the $N\,d\phi/dt$ voltage, V_L is the voltage corresponding to $-E_{1R}$ on the vector diagram shown in Fig. 25, page 253.

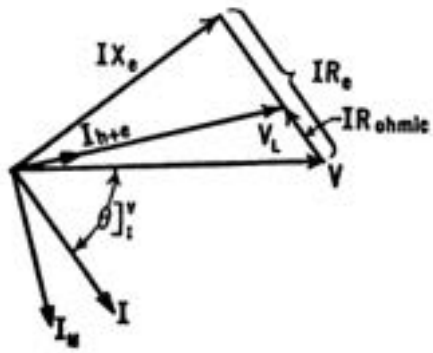

FIG. 6. Vector diagram of iron-clad inductance-coil quantities.

On the basis of sinusoidal wave forms of v and i, the vector position of $\mathbf{I}$ is determined as

$$\theta = \cos^{-1}\frac{P}{VI} \tag{9}$$

Assuming that the d-c or ohmic resistance of the coil is known, the value of $\mathbf{V}_L$ is, approximately:

$$\mathbf{V}_L = \mathbf{V} - \mathbf{I}R_{\text{ohmic}} \tag{10}$$

The vector relations of the above quantities are illustrated in Fig. 6. The hypothetical separation of the actual current I into its two right-angle components is a desirable step in certain types of analyses. The method employed in evaluating the component currents is indicated by the circuit arrangement shown in Fig. 5*b*.

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The actual core loss is the true watts delivered to the iron-clad circuit minus I^2R_{ohmic}. Frequently I^2R_{ohmic} is a negligibly small quantity.

$$I_{h+e} = \frac{\text{Core loss}}{V_L} \tag{11}$$

$$I_M = \sqrt{I^2 - I_{h+e}^2} \tag{12}$$

$\mathbf{I}_{h+e}$ and $\mathbf{I}_M$ are the right-angle components of $\mathbf{I}$. $\mathbf{I}_{h+e}$ is in phase with $\mathbf{V}_L$ and $\mathbf{I}_M$ is 90° behind $\mathbf{V}_L$, as shown in Fig. 6.

The present method of iron-core circuit analysis illustrates how the applied voltage can be broken up into two distinct sets of component voltages. Either:

$$\mathbf{V} = \mathbf{I}R_{\text{ohmic}} + \mathbf{V}_L \tag{13}$$

or

$$\mathbf{V} = \mathbf{I}R_e + \mathbf{I}X_e \tag{14}$$

Reference to the vector diagram shown in Fig. 6 will show that R_e is greater than R_{ohmic} and that L_e is less than L_{true}. In this latter connection:

$$L_e = \frac{IX_e}{\omega I} \quad \text{and} \quad L_{\text{true}} = \frac{V_L}{\omega I_M}$$

Example 2. It will be assumed that, in Fig. 4, the "corrected" ammeter reading is 1.25 amperes, the "corrected" wattmeter reading is 28 watts, the voltmeter reading is 100 volts, and R_{ohmic} is 0.2 ohm. The frequency is 60 cycles.

The method of constructing a vector diagram for these particular conditions is shown below.

Let

$$\mathbf{V} = 100 \angle 0° \text{ volts}$$

$$\theta \Big]_{\mathbf{I}}^{\mathbf{V}} = \cos^{-1} \frac{28}{100 \times 1.25} = 77°$$

$$\mathbf{I} = 1.25 \angle -77° \text{ amperes}$$

$$\begin{aligned} \mathbf{V}_L &= 100 \angle 0° - (1.25 \angle -77°)(0.2) \\ &= (100 + j0) - (0.056 - j0.244) \\ &= 99.94 + j0.244 = 99.95 \angle 0.13° \text{ volts} \end{aligned}$$

$$\text{Core loss} = 28 - (1.25^2 \times 0.2) = 27.69 \text{ watts}$$

$$\mathbf{I}_{h+e} = \frac{27.69}{99.95} = 0.277 \text{ ampere}$$

$$I_M = \sqrt{1.25^2 - 0.277^2} = 1.22 \text{ amperes}$$

The magnitudes of $\mathbf{I}_{h+e}$ and $\mathbf{I}_M$ having been determined as outlined above, these quantities are assigned "in-phase" and "quadrature" positions with respect to $\mathbf{V}_L$ as shown in Fig. 6. With the aid of these concepts, the physical phenomena involved in the actual composite electric and magnetic arrangement shown in Fig. 4 are interpreted in terms of elementary electric circuit theory.

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Problem 3. Find the numerical values of R_e, L_e, and L_{true} of the iron-clad circuit of the above illustrative example. If the coil consists of 200 turns, find the maximum value of the magnetic flux in the iron core.

Ans.: $R_e = 17.95\ \Omega$, $L_e = 0.2065$ henry; $L_{true} = 0.217$ henry.

Mutual Inductance Measured in Terms of Self-Inductance. Self-inductance can be measured either by the voltmeter-ammeter-wattmeter method or by some bridge method. If the terminals of the two coils shown in Figs. 7*a* and 7*b* are accessible, either general method may be employed in measuring $L_{e\,add}$ and $L_{e\,sub}$. Reference to Fig. 7 parts (*a*) and (*b*) respectively will show that the following equations are applicable.

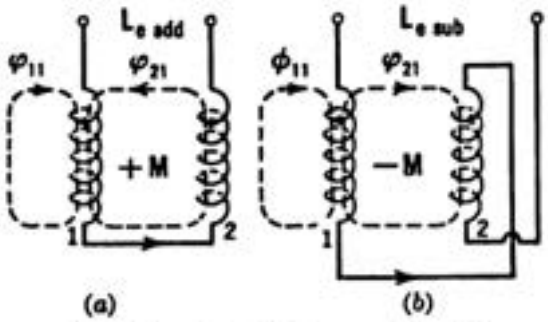

FIG. 7. $M = \frac{1}{4}(L_{e\,add} - L_{e\,sub})$

$$L_{e\,add} = L_1 + L_2 + 2M \quad (15)$$

$$L_{e\,sub} = L_1 + L_2 - 2M \quad (16)$$

from which:

$$M = \frac{1}{4}\,(L_{e\,add} - L_{e\,sub}) \quad (17)$$

Thus M can be determined directly from two equivalent self-inductance measurements, provided $M_{add} = M_{sub} = M$.

Inherent Difficulties Encountered in Making Measurements. It is often practically impossible to measure circuit parameters by means of the voltmeter-ammeter-wattmeter method. In low-power circuit elements the allowable current may be too small to operate successfully a normal wattmeter or ammeter. In other cases the circuit may be so radically disturbed as a result of the introduction of meters that the measurements made are useless in so far as original circuit parameter determination is concerned. Vacuum-tube devices which are sometimes employed to measure voltage, current, and power draw very little or no power from the circuit which is under investigation, and for this reason they are useful adjuncts to normal metering equipment in certain cases.

Bridge methods are generally to be preferred in making measurements involving low-power circuit elements. This is particularly true if the effective values of the parameters are to be determined at frequencies ranging from 100 to 2500 cycles. Certain difficulties attend the measurement of any circuit parameter at the higher frequencies.

Effect of Distributed Capacitance in Inductance Coil Measurements. The individual turns of an inductance coil possess capacitive effects one to the other. Under conditions of dynamic operation, the turns are maintained at differences of potential with respect to each other, and the distributed capacitive effects which are present are represented

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in approximate fashion by the discrete condensers shown in Fig. 8*a*. As a first approximation the distributed self-capacitance of a coil can be represented by a parallel condenser, as illustrated in Fig. 8*b*. In this case the resistance R is practically equal to the straight-wire effective a-c resistance of the wire which forms the inductance coil. It will be shown presently that the effective resistance of the coil under certain conditions may differ materially from the straight-wire effective resistance R.

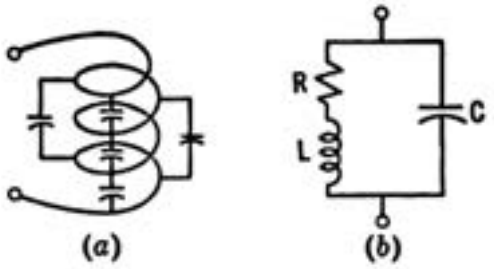

FIG. 8. Illustrating the manner in which the distributed self-capacitance of an inductance coil can be represented approximately by discrete condensers.

If the dielectric loss is an appreciable factor, then the C branch of Fig. 8*b* should contain a resistance of suitable magnitude to account for this loss. If all stray dielectric and magnetic losses are assumed to be negligibly small, the equivalent impedance of the two parallel branches of Fig. 8*b* is:

$$\mathbf{Z}_e = \frac{\mathbf{Z}_{RL}\mathbf{Z}_C}{\mathbf{Z}_{RL} + \mathbf{Z}_C} = \frac{(R + j\omega L)\left(-j\dfrac{1}{\omega C}\right)}{R + j\left(\omega L - \dfrac{1}{\omega C}\right)} = \frac{\omega L - jR}{\omega CR + j(\omega^2 CL - 1)}$$

$$= \left[\frac{R}{\omega^2 C^2 R^2 + (\omega^2 CL - 1)^2}\right] + j\omega\left[\frac{L(1 - \omega^2 CL) - CR^2}{\omega^2 C^2 R^2 + (\omega^2 CL - 1)^2}\right] \quad (18)$$

or

$$\mathbf{Z}_e = R_e + j\omega L_e \quad \text{(equivalent series circuit parameters)} \quad (19)$$

where

$$R_e = \frac{R}{\omega^2 C^2 R^2 + (\omega^2 CL - 1)^2} \quad (20)$$

$$L_e = \frac{L(1 - \omega^2 CL) - CR^2}{\omega^2 C^2 R^2 + (\omega^2 CL - 1)^2} \quad (21)$$

The equivalent resistance of the coil as given by equation (20) is a rather complicated function of frequency since R is in itself an intricate function of frequency. At low frequencies R does not differ materially from R_{ohmic} if the cross-sectional area of the conductor is not greater than about 250,000 circular mils. Also for coils of usual design at low values of ω, $(\omega^2 C^2 R^2)$ is negligibly small as compared with unity and $(\omega^2 CL)$ is generally considerably less than unity. Therefore, at very low frequencies, R_e is only slightly larger than R_{ohmic}. But, as the

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frequency increases to a point where R varies appreciably with ω and $(\omega^2C^2R^2)$ becomes a significant factor, the exact variation of R_e cannot be conveniently anticipated in terms of elementary functions. Situations of this kind are responsible for the wide differences that exist between low-frequency and high-frequency measurements.

Reference to equations (20) and (21) will show that, whereas R_e is always positive, L_e may under certain conditions vanish or become negative. Physically, a negative value of L_e implies that the inductance coil is actually functioning as a condenser owing to its distributed capacitance. Although advantage may sometimes be taken of this phenomenon in the design of certain selective circuits at radio frequencies, distributed capacitance is usually detrimental.

The point at which the inductance coil ceases to operate as an inductance coil occurs when

$$L(1 - \omega^2CL) = CR^2 \tag{22}$$

or when

$$\omega = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{L}} \tag{23}$$

It should be remembered that the above equation is based upon the approximate representation shown in Fig. 8*b*. Although the relations established are only approximate, they serve to portray certain important physical concepts.

Elementary Bridge Theory. Alternating-current bridges which take the same general form as the familiar d-c Wheatstone bridge are very widely used to measure inductance, capacitance, and frequency. The inductance measurements referred to may be either those of self-inductance or mutual inductance.

A simple form of a-c bridge is shown in Fig. 9*a*. Many of the bridges employed in practice contain two resistance arms and two impedance arms. For best results the two resistance arms are accurately calibrated resistances which are wound so as to reduce self-inductance and self-capacitance effects. To obtain greatest flexibility the resistance arms are adjustable, but for certain operating conditions they may take the form of fixed resistances. The other two arms of the bridge shown in Fig. 9*a* are the impedance arms: one, the standard impedance $(R_s + jX_s)$; the other, the unknown impedance $(R_x + jX_x)$ which is to be measured.

Consider the arrangement shown in Fig. 9*a*. After adjustments of R_c, R_d, R_s, and X_s have been made so that no potential difference exists

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between points c and d, then:

$$\mathbf{I}_c R_c = \mathbf{I}_d R_d \tag{24}$$

and

$$\mathbf{I}_c(R_x + jX_x) = \mathbf{I}_d(R_s + jX_s) \tag{25}$$

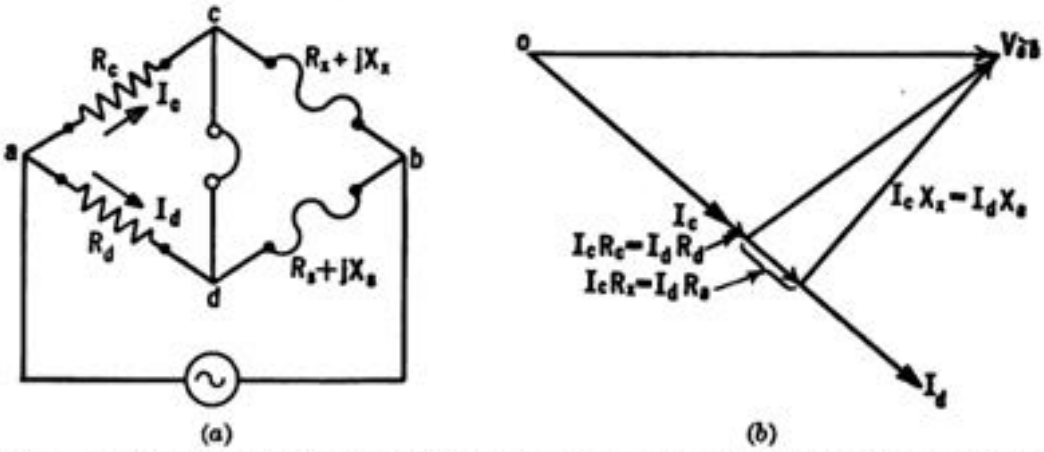

FIG. 9. Similar-angle bridge with the vector diagram of a particular balanced condition.

Dividing equation (25) by equation (24) and rearranging:

$$R_d R_x + jR_d X_x = R_c R_s + jR_c X_s \tag{26}$$

whence

$$R_x = \frac{R_c}{R_d} R_s \tag{27}$$

and

$$X_x = \frac{R_c}{R_d} X_s \tag{28}$$

Equations (26) and (28) show that, if the unknown impedance is inductive in character, the standard impedance must also be inductive in character. A vector diagram of the two branch currents and the various component voltages for a particular case is contained in Fig. 9*b*. If the unknown impedance is capacitive in character, then the standard impedance must also be capacitive in character provided that the arms are arranged as shown in Fig. 9*a*. Bridges of this general class are called similar-angle bridges.

If the reactive element of the unknown impedance is opposite in nature to the reactive element of the standard impedance, then the bridge arms must be arranged as shown in Fig. 10. Bridges of this general class are called opposite-angle bridges.

Problem 4. With bridge arms arranged as shown in Fig. 10 and with the adjustments made so that no potential difference exists between points c and d, show that

$$R_x = \frac{R_c R_d R_s}{R_s^2 + X_s^2} \quad \text{and} \quad X_x = -\frac{R_c R_d X_s}{R_s^2 + X_s^2}$$

Draw the vector diagram for a particular balanced condition. Make all necessary assumptions regarding the relative magnitudes of the various parameters.

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The elementary forms of bridges shown in Fig. 9a and Fig. 10 are very useful in making measurements of L, C, and M at frequencies less than about 2000 cycles provided that these circuit parameters are practically independent of the frequency employed in making the measurements. Special forms of bridges are required if parameters vary appreciably with frequency, or if the values of the parameters are desired at frequencies higher than about 2000 or 2500 cycles.[2] With more elaborate forms of the bridge, measurements can be made at frequencies up to about 5,000,000 cycles.

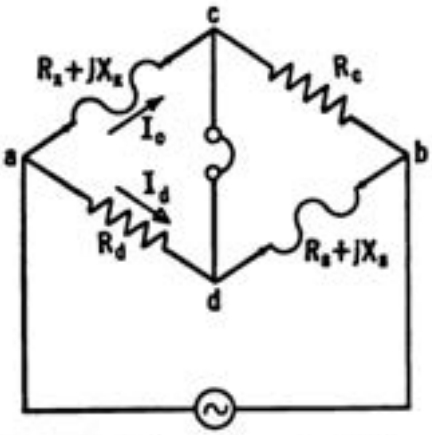

Fig. 10. Opposite-angle bridge. (See Problem 4.)

The Owen's Bridge. The circuit arrangement shown in Fig. 11 is known as the Owen's bridge. It is a versatile measuring device inasmuch as inductance can be measured in terms of capacitance and resistance; or capacitance can be measured in terms of inductance and resistance. Also, the equivalent series circuit resistance of the inductance coil under investigation or the equivalent series circuit resistance of a high-loss condenser placed at the C_2 position in Fig. 11 can be measured.

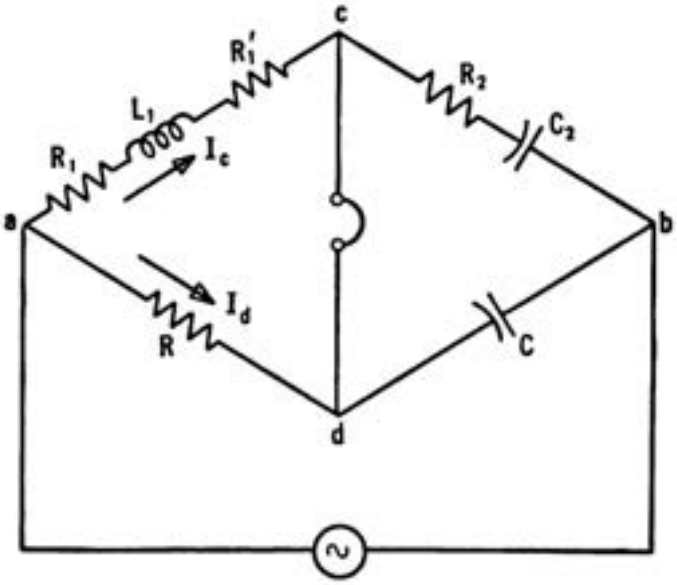

Fig. 11. Owen's bridge.

In general, however, the latter measurement cannot be made to a satisfactory degree of accuracy, so the bridge finds its greatest field of usefulness in the field of inductance measurements. It should be

[2] See "Electrical Engineers' Handbook: Electric Communication and Electronics," fourth edition, edited by Pender and McIlwain, pp. 11–24 ff., John Wiley & Sons, Inc., 1950.

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recognized that the unshielded arrangement shown in Fig. 11 is merely a diagrammatic sketch of an Owen's bridge.

The arm *ac* consists of an inductance which has R_1 units resistance and an additional adjustable resistance of R_1' units. If the condenser C in arm *bd* is an unknown capacitance to be measured, then L_1 may take the form of an adjustable self-inductance of known magnitude. If L_1 is an unknown self-inductance, then C in arm *bd* usually takes the form of an adjustable capacitance of known magnitude. Since the theory of the bridge as outlined below presupposes that arm *bd* is purely capacitive, the condenser C should be a low-loss condenser.

The condenser C_2 in arm *cb* may be fixed and need not necessarily be a low-loss condenser. Either the resistance R_2 in arm *bc* or the resistance R in arm *ad*, or both, are adjustable and, if the other parameters are within certain limits as regards magnitude, complete balance may be obtained by proper adjustment of R_1' and R or of R_1' and R_2. From the general equations given below it is evident that various other methods of balancing may be employed.

When the circuit parameters are so adjusted that point *c* is at the same potential as point *d*, it is plain that

$$\mathbf{I}_c\left[(R_1 + R_1') + j\omega L_1\right] = \mathbf{I}_d R \tag{29}$$

and

$$\mathbf{I}_c\left(R_2 - j\frac{1}{\omega C_2}\right) = \mathbf{I}_d\left(-j\frac{1}{\omega C}\right) \tag{30}$$

Dividing equation (29) by equation (30) and making two or three rather simple algebraic transformations, the following relation is obtained:

$$(R_1 + R_1') + j\omega L_1 = \frac{RC}{C_2} + j\omega CRR_2 \tag{31}$$

Therefore the two sets of conditions that exist when the bridge is balanced are:

(1) $$R_1 + R_1' = \frac{RC}{C_2} \tag{32}$$

(2) $$L_1 = CRR_2 \tag{33}$$

The condition stated in equation (32) can be obtained by proper adjustments of R_1' and/or R. The condition stated in equation (33) can then be obtained by proper adjustment of R_2. In practice a true balance can be obtained by making successive adjustments of the resistances referred to above. At this point it might be well for the

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beginner to realize that balancing the more elaborate forms of a-c bridges is an art which requires laboratory technique of a high order. All too often the amateur confuses positions of "false" balance with the position of "true" balance.

Equations (32) and (33) indicate that balance is obtained in an Owen's bridge irrespective of the frequency of the oscillator. This is true only when the circuit parameters are absolutely constant since equations (32) and (33) have been derived employing this assumption. When the parameters themselves are appreciably influenced by frequency, and they are at the higher frequencies, then balance is not independent of frequency. The fact that the effective values of the various parameters are influenced by frequency makes the measurement of them more difficult at the higher frequencies.

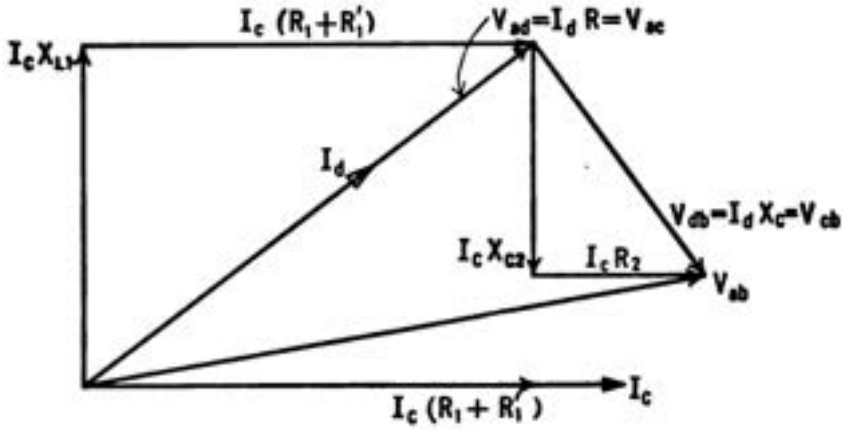

FIG. 12. Vector diagram of a balanced Owen's bridge.

Example 3. In Fig. 11 let R_1 and L_1 be unknown. The circuit elements, R, R_1', and R_2 are presumably adjustable, and it is known that C and C_2 are low-loss condensers the capacitances of which are 0.192 μf and 0.370 μf respectively. At an oscillator frequency of 1000 cycles, the bridge balances at

$$R = 1600 \text{ ohms}, \; R_1' = 799.4 \text{ ohms}, \text{ and } R_2 = 326 \text{ ohms}$$

The values of R_1 and L_1 follow directly from equations (32) and (33).

$$R_1 = \frac{RC}{C_2} - R_1' = \frac{1600 \times 0.192}{0.370} - 799.4 = 30.6 \text{ ohms}$$

$$L_1 = CRR_2 = 0.192 \times 10^{-6} \times 1600 \times 326 = 0.100 \text{ henry}$$

A vector diagram of the above balanced condition is shown in Fig. 12.

PROBLEMS

5. When 150 volts are impressed across a pair of terminals, 10 amperes flow and 700 watts are consumed. What are the impedance, resistance, and reactance, looking into the pair of terminals?

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6. In the circuit arrangement shown in Fig. 13, R is a known fixed resistance of 15 ohms. The readings of the three voltmeters are: $V = 120$ volts, $V_R = 60$ volts, and $V_Z = 80$ volts. Neglecting the circuit disturbance caused by the voltmeters, find Z and draw a vector diagram illustrating the various quantities. What is the total power absorbed by the RZ series branch?

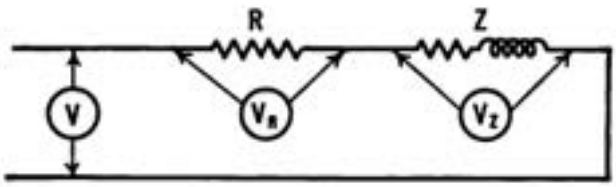

FIG. 13. See Problem 6.

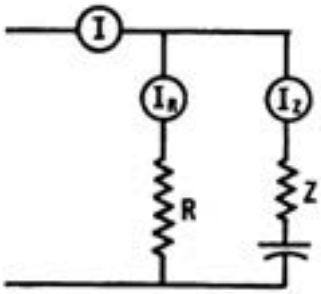

FIG. 14. See Problem 7.

7. In the circuit arrangement shown in Fig. 14, R is a known fixed resistance of 30 ohms. The readings of the three ammeters are: $I = 8$ amperes, $I_R = 4.0$ amperes, and $I_Z = 6.0$ amperes. Neglecting the circuit disturbance caused by the ammeters, find Z and draw a vector diagram illustrating the various quantities. What is the power absorbed by the Z branch?

8. Coils 1 and 2 are connected in additive series. When 100 volts at 60 cycles are connected across coil 1 with the remaining terminal of coil 2 free of any connection, the current input is 10 amperes and the power 500 watts. The voltage across the whole of the two coils in series is then 300 volts. When 100 volts at 60 cycles are impressed across coil 2 only, the current is 1 ampere and power taken is 30 watts. Calculate L_1, L_2, R_1, R_2, M, and the coefficient of coupling.

9. When 100 volts are impressed at the terminals of Fig. 7*a* the current is 10 amperes, total power consumed is 400 watts, and power consumed by coil 2 is 300 watts, while the drop across coil 1 is 40 volts. When the coils are connected as in Fig. 7*b* an impressed voltage of 100 volts at the terminals causes 20 amperes to flow. Calculate the mutual inductance, coefficient of coupling of the two coils if the frequency is 60 cycles, and the drop across coil 2 for the subtractive connection.

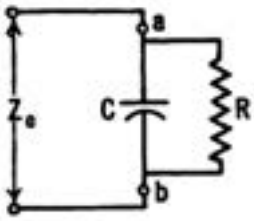

FIG. 15. See Problems 10 and 11.

10. Referring to Fig. 15, show that the equivalent series circuit resistance and capacitance are, respectively, equal to:

$$R\left(\frac{1}{1 + \omega^2C^2R^2}\right) \quad \text{and} \quad C\left(1 + \frac{1}{\omega^2C^2R^2}\right)$$

Assume that C is a fixed capacitance of $10\mu f$ and that ω is constant at 5000 radians per second. Plot the equivalent series circuit resistance, R_e, and the equivalent series circuit capacitance, C_e, with respect to R as R is varied from 5 to 100 ohms.

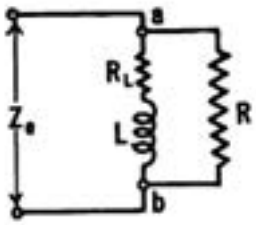

FIG. 16. See Problem 12.

11. What are the values of the equivalent series circuit resistance R_e and the equivalent series circuit capacitance C_e for $R = 5$ ohms in Problem 10? also for $R = 100$ ohms?

12. An inductance coil possessing L units of self-inductance is assumed to have R_L units of resistance. This coil is paralleled with a purely resistive circuit element

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of R units resistance as shown in Fig. 16. Find the general expressions for the equivalent series circuit resistance and the equivalent series circuit inductance of the parallel combination. Graph R_e and L_e versus R (between limits of $R = 0$ and $R = 100$ ohms) for the particular case of $f = 500$ cycles per second, $L = 0.056$ henry, and $R_L = 25$ ohms.

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CHAPTER XII

TRANSMISSION LINE CALCULATIONS

Line Constituents. A transmission line consists of the equivalent of two or more electrical conductors for the purpose of transmitting electric energy. For single-phase transmission the line may consist of a single conductor with a ground return or of two ordinary wires. For three-phase transmission, three wires are generally used although in some installations a neutral wire or its equivalent is employed. The wires of a transmission line are separated by some dielectric as air for overhead transmission, or by other insulating materials as in cables. Since the two conductors are separated by a dielectric, they form a condenser, the capacitance of which is uniformly distributed along the wires. When a difference of potential is applied between the wires, charging current flows. This effect could be simulated by a large number of condensers connected between the two wires as shown in Fig. 1. V_s denotes the sending-end voltage, and V_r represents receiver-end voltage. A representation of this kind is approximate because it shows the shunted capacitance lumped at certain points instead of being uniformly distributed. Within reasonable limits of accuracy it is permissible to make line calculations on the basis of lumped capacitance. Under the conditions of relatively low voltage and relatively short distances the shunted capacitance can even be neglected without seriously affecting the accuracy.

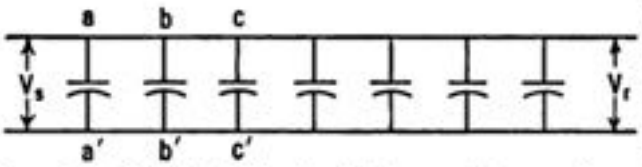

FIG. 1. Distributed shunted capacitance of a transmission line simulated by a large number of shunted condensers.

In addition to shunted capacitance the line has series resistance and inductance or inductive reactance. Thus the sections between con-

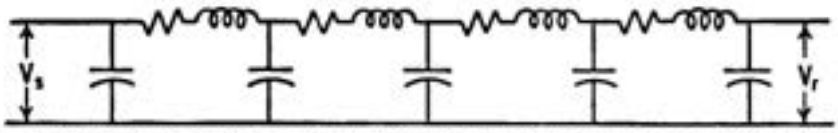

FIG. 2. Modification of Fig. 1 to account for series resistance and inductance of a transmission line.

densers, like ab and $a'b'$, bc and $b'c'$, etc., form loops through which flux will be set up by the mmf of the current flowing in the wires. These sections also have resistance. Hence, to account for these parameters, Fig. 1 should be modified to appear as shown in Fig. 2. Strictly speak-

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ing, each condenser should be shunted by a non-inductive resistance to account for any leakage of current from conductor to conductor because of imperfect insulation, moisture content of the air, and other factors. On a clear dry day the leakage is so small that it may usually be neglected. The greater the number of sections, like those shown in Fig. 2, into which the line is divided, the more nearly it will simulate the actual line which has uniformly distributed parameters. If more than two or three shunted condensers are used, it is just about as simple to calculate the line by assuming uniformly distributed parameters instead of concentrated quantities. Three of the usual arrangements of concentrated parameters will be considered.

The T Line. The T representation of a line is shown in Fig. 3. When all the shunted capacitance, C, of the line is concentrated in one condenser and half of the total series impedance, Z, is placed in each arm as indicated in Fig. 3, the circuit is known as the nominal T line. It is called nominal because the representation is only approximately correct. When the circuit parameters indicated in Fig. 3 are multiplied by certain hyperbolic correction factors,[1] the T thus formed represents the line exactly between terminals (V_s and V_r) and it then becomes the exact equivalent T. Calling Y the admittance due to the shunted capacitance C and using the quantities as labeled in Fig. 3, the determination of V_s in terms of the receiver voltage and current is made as follows.

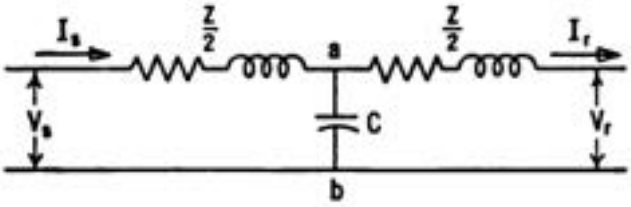

FIG. 3. T representation of a transmission line.

$$\mathbf{V}_{ab} = \mathbf{V}_r + \mathbf{I}_r\frac{\mathbf{Z}}{2}$$

$$\mathbf{I}_{ab} = \mathbf{V}_{ab}\mathbf{Y}$$

$$\mathbf{I}_s = \mathbf{I}_r + \mathbf{I}_{ab} = \mathbf{I}_r + \mathbf{Y}\left(\mathbf{V}_r + \mathbf{I}_r\frac{\mathbf{Z}}{2}\right) \tag{1}$$

$$\begin{aligned}\mathbf{V}_s &= \mathbf{V}_{ab} + \mathbf{I}_s\frac{\mathbf{Z}}{2}\\ &= \left(\mathbf{V}_r + \mathbf{I}_r\frac{\mathbf{Z}}{2}\right) + \left[\mathbf{I}_r + \mathbf{Y}\left(\mathbf{V}_r + \mathbf{I}_r\frac{\mathbf{Z}}{2}\right)\right]\frac{\mathbf{Z}}{2}\\ &= \mathbf{V}_r\left(1 + \frac{\mathbf{YZ}}{2}\right) + \mathbf{I}_r\left(\mathbf{Z} + \frac{\mathbf{YZ}^2}{4}\right)\end{aligned} \tag{2}$$

[1] See "Hyperbolic Functions Applied to Electrical Engineering," by A. E. Kennelly or "Electric Circuits Theory and Applications," by O. G. C. Dahl.

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From equation (1),

$$\mathbf{I}_s = \mathbf{I}_r\left(1 + \frac{\mathbf{YZ}}{2}\right) + \mathbf{YV}_r \tag{3}$$

Equations (2) and (3) give the sending-end voltage and current in complex form. As indicated, all quantities in the equations must be expressed in vector form. The receiver current must be properly related in complex form to the receiver voltage. The power factor of the load determines the angle between $\mathbf{V}_r$ and $\mathbf{I}_r$. $\mathbf{V}_s$ and $\mathbf{I}_s$ being in complex form, power input may be determined in the usual way. The vector diagram of the T circuit of Fig. 3 is shown in Fig. 4. This diagram follows the above equations for calculating $\mathbf{V}_s$ and $\mathbf{I}_s$.

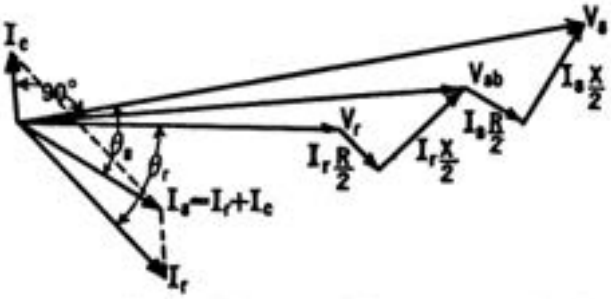

FIG. 4. Vector diagram of T representation in FIG. 3.

Problem 1. A 60-cycle, 3-phase line 200 miles long has a shunted capacitance to neutral per mile of 146×10^{-4} μf, an inductive reactance of 0.78 ohm per wire per mile, and a resistance of 0.42 ohm per wire per mile. The receiver voltage is 100,000 volts between lines. Use the nominal T line, and find the sending voltage and current for an 0.8 power-factor lagging load requiring 75 amperes per line at the receiver. *Ans.* 64,600 $\underline{/7.4°}$ volts, 62.3 $\underline{/24°}$ amperes.

The π Line. If one-half of the total line capacitance is concentrated at each end of the line and all the series resistance and reactance are concentrated at the center as shown in Fig. 5, the resultant configuration portrays the nominal π representation of a transmission line. Like the T line it is possible to alter the parameters by applying hyperbolic correction factors to obtain a π circuit which yields the correct relations between terminals. A π circuit thus corrected is called an exact equivalent π.

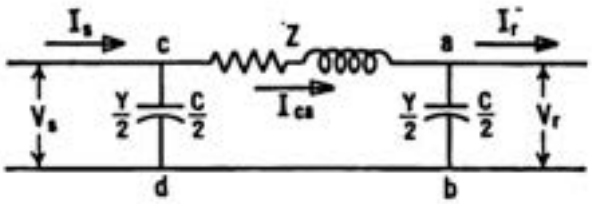

FIG. 5. π representation of a transmission line.

The π circuit is easily solved through a procedure similar to that employed for the T circuit.

$$\mathbf{I}_{ab} = \mathbf{V}_r\frac{\mathbf{Y}}{2}$$

$$\mathbf{I}_{ca} = \mathbf{I}_r + \mathbf{I}_{ab} = \mathbf{I}_r + \mathbf{V}_r\frac{\mathbf{Y}}{2}$$

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$$\mathbf{V}_s = \mathbf{V}_r + \mathbf{I}_{ca}\mathbf{Z} = \mathbf{V}_r + \left(\mathbf{I}_r + \mathbf{V}_r\frac{\mathbf{Y}}{2}\right)\mathbf{Z}$$

$$= \mathbf{V}_r\left(1 + \frac{\mathbf{ZY}}{2}\right) + \mathbf{I}_r\mathbf{Z} \tag{4}$$

$$\mathbf{I}_s = \mathbf{I}_{ca} + \mathbf{I}_{cd}$$

$$\mathbf{I}_{cd} = \mathbf{V}_s\frac{\mathbf{Y}}{2} = \left[\mathbf{V}_r\left(1 + \frac{\mathbf{ZY}}{2}\right) + \mathbf{I}_r\mathbf{Z}\right]\frac{\mathbf{Y}}{2}$$

$$\mathbf{I}_s = \mathbf{I}_r + \mathbf{V}_r\frac{\mathbf{Y}}{2} + \left[\mathbf{V}_r\left(1 + \frac{\mathbf{ZY}}{2}\right) + \mathbf{I}_r\mathbf{Z}\right]\frac{\mathbf{Y}}{2}$$

$$= \mathbf{I}_r\left(1 + \frac{\mathbf{ZY}}{2}\right) + \mathbf{V}_r\mathbf{Y}\left(1 + \frac{\mathbf{ZY}}{4}\right) \tag{5}$$

Equations (4) and (5) are the solution of the π representation of a transmission line. The vector diagram of the π circuit is shown in Fig. 6, and the above calculations follow this diagram.

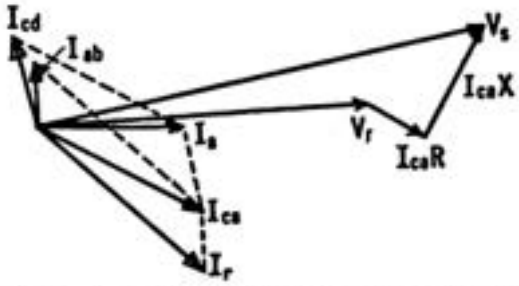

FIG. 6. Vector diagram of π line in Fig. 5.

Problem 2. Use the nominal π-line representation and solve Problem 1.
Ans.: 65,300 $\angle 7.4°$ volts, 59.75 $\angle 22.2°$ amperes.

The Steinmetz Representation of the Transmission Line. Another representation of the transmission line suggested by C. P. Steinmetz which yields approximately correct results is shown in Fig. 7. In the

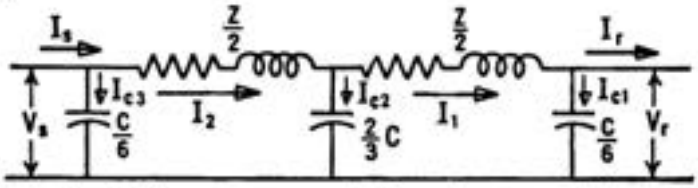

FIG. 7. Steinmetz representation of a transmission line.

figure, Z represents the total series impedance and C the total shunted capacitance. The student can work out the details of the solution by following the methods employed for the T and π lines. This circuit and the solution are slightly more cumbersome, but the results are generally

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somewhat closer to the theoretically correct values than those obtained from the use of the nominal T or π sections. The calculations must follow the vector diagram shown in Fig. 8.

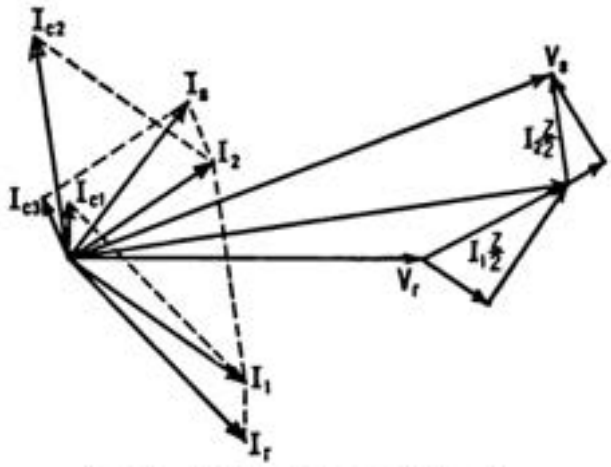

Fig. 8. Vector diagram of Fig. 7.

Problem 3. Derive the equations for the sending-end voltage and current in terms of the receiver quantities for the Steinmetz representation of a transmission line.

$$\textit{Ans.:}\ \mathbf{V}_s = \left(1 + \frac{\mathbf{ZY}}{2} + \frac{\mathbf{Z}^2\mathbf{Y}^2}{36}\right)\mathbf{V}_r + \mathbf{Z}\left(1 + \frac{\mathbf{ZY}}{6}\right)\mathbf{I}_r$$

$$\mathbf{I}_s = \left(1 + \frac{5\mathbf{ZY}}{36} + \frac{\mathbf{Z}^2\mathbf{Y}^2}{216}\right)\mathbf{YV}_r + \left(1 + \frac{\mathbf{ZY}}{2} + \frac{\mathbf{Z}^2\mathbf{Y}^2}{36}\right)\mathbf{I}_r$$

Problem 4. Solve Problem 1 according to the Steinmetz representation of the line.

Ans.: 64,900 $\angle 7.3°$ volts, 60.9 $\angle 22.9°$ amperes.

Exact Solution of a Long Line with Uniformly Distributed Parameters. In the line shown in Fig. 9 let the series impedance per mile be **Z**, the shunted admittance per mile **Y**, and the length of the line considered l.

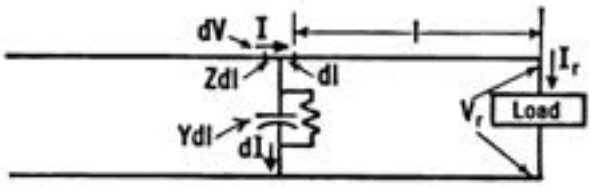

Fig. 9. Circuit used for deriving the exact solution of a transmission line.

The elementary voltage drop in the element dl is the current **I** in the element times the impedance **Z** dl of the element. Considering only the space variation of **V** and **I**,

$$d\mathbf{V} = \mathbf{IZ}\,dl \tag{6}$$

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The current leaving the line over the length dl is the voltage $\mathbf{V}$ times the shunted admittance $\mathbf{Y}\,dl$ for the element. Thus

$$d\mathbf{I} = \mathbf{VY}\,dl \tag{7}$$

Equations (6) and (7) are solved for $\mathbf{V}$ and $\mathbf{I}$ as follows:

$$\frac{d\mathbf{V}}{dl} = \mathbf{IZ} \tag{8}$$

$$\frac{d\mathbf{I}}{dl} = \mathbf{VY} \tag{9}$$

Differentiating equation (8) with respect to l gives

$$\frac{d^2\mathbf{V}}{dl^2} = \mathbf{Z}\frac{d\mathbf{I}}{dl} \tag{10}$$

Substituting equation (9) in equation (10),

$$\frac{d^2\mathbf{V}}{dl^2} = \mathbf{ZYV} \tag{11}$$

This is a linear differential equation of the second order, the solution of which can be shown[2] to be of the form

$$\mathbf{V} = C_1\epsilon^{\mathbf{m}_1 l} + C_2\epsilon^{\mathbf{m}_2 l}$$

where C_1 and C_2 are constants of integration and m_1 and m_2 are roots of the auxiliary equation, namely,

$$\begin{aligned} \mathbf{m}^2 &= \mathbf{ZY} \\ \mathbf{m} &= +\sqrt{\mathbf{ZY}} \quad \text{or} \quad -\sqrt{\mathbf{ZY}} \end{aligned} \tag{12}$$

The two roots $\mathbf{m}_1$ and $\mathbf{m}_2$ are respectively $+\sqrt{\mathbf{ZY}}$ and $-\sqrt{\mathbf{ZY}}$. Therefore

$$\begin{aligned} \mathbf{V} &= C_1\epsilon^{\mathbf{m}_1 l} + C_2\epsilon^{\mathbf{m}_2 l} \\ &= C_1\epsilon^{\sqrt{\mathbf{ZY}}l} + C_2\epsilon^{-\sqrt{\mathbf{ZY}}l} \end{aligned} \tag{13}$$

From equation (8),

$$\mathbf{I} = \frac{1}{\mathbf{Z}}\frac{d\mathbf{V}}{dl} \tag{14}$$

[2] See any book on differential equations, such as "Differential Equations," by D. A. Murray, p. 63.

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Differentiating equation (13) and substituting the result in (14) gives

$$\mathbf{I} = C_1 \frac{\sqrt{\mathbf{ZY}}}{\mathbf{Z}} \epsilon^{\sqrt{\mathbf{ZY}}l} - \frac{C_2\sqrt{\mathbf{ZY}}}{\mathbf{Z}} \epsilon^{-\sqrt{\mathbf{ZY}}l}$$

$$= C_1\sqrt{\mathbf{Y}/\mathbf{Z}}\, \epsilon^{\sqrt{\mathbf{ZY}}l} - C_2\sqrt{\mathbf{Y}/\mathbf{Z}}\, \epsilon^{-\sqrt{\mathbf{ZY}}l} \tag{15}$$

The constants of integration C_1 and C_2 in equations (13) and (15) can be evaluated from known boundary conditions. In this case the boundary conditions at the receiver are assumed to be known. Thus in Fig. 9 when

$$l = 0 \tag{16}$$

$$\mathbf{I} = \mathbf{I}_r \tag{17}$$

and

$$\mathbf{V} = \mathbf{V}_r \tag{18}$$

Substituting equations (16), (17), and (18) in equations (13) and (15),

$$\mathbf{V}_r = C_1 + C_2 \tag{19}$$

$$\mathbf{I}_r = C_1\sqrt{\mathbf{Y}/\mathbf{Z}} - C_2\sqrt{\mathbf{Y}/\mathbf{Z}} \tag{20}$$

Equations (19) and (20) are now solved simultaneously for C_1 and C_2. Multiplying equation (19) by $\sqrt{\mathbf{Y}/\mathbf{Z}}$ gives

$$\mathbf{V}_r\sqrt{\mathbf{Y}/\mathbf{Z}} = C_1\sqrt{\mathbf{Y}/\mathbf{Z}} + C_2\sqrt{\mathbf{Y}/\mathbf{Z}} \tag{21}$$

Adding equations (20) and (21),

$$\mathbf{I}_r + \sqrt{\mathbf{Y}/\mathbf{Z}}\,\mathbf{V}_r = 2C_1\sqrt{\mathbf{Y}/\mathbf{Z}}$$

$$C_1 = \frac{\mathbf{V}_r + \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2} \tag{22}$$

Subtracting equation (20) from equation (21),

$$\mathbf{V}_r\sqrt{\mathbf{Y}/\mathbf{Z}} - \mathbf{I}_r = 2C_2\sqrt{\mathbf{Y}/\mathbf{Z}}$$

$$C_2 = \frac{\mathbf{V}_r - \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2} \tag{23}$$

It is apparent that C_1 and C_2 in the above equations are complex coefficients and might have been written in bold-face type. The expressions for voltage and current at any distance l from the receiver are

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obtained by substituting equations (22) and (23) in equations (13) and (15). Then

$$\mathbf{V} = \left(\frac{\mathbf{V}_r + \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} + \left(\frac{\mathbf{V}_r - \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\right)\epsilon^{-\sqrt{\mathbf{ZY}}l} \tag{24}$$

$$\mathbf{I} = \left(\frac{\mathbf{I}_r + \mathbf{V}_r\sqrt{\mathbf{Y}/\mathbf{Z}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} + \left(\frac{\mathbf{I}_r - \mathbf{V}_r\sqrt{\mathbf{Y}/\mathbf{Z}}}{2}\right)\epsilon^{-\sqrt{\mathbf{ZY}}l} \tag{25}$$

Equations (24) and (25) may be used as the working equations for the exact solution of long lines. Under certain conditions it is convenient to have equations (24) and (25) expressed in terms of hyperbolic functions. This is done as follows.

From equation (24),

$$\mathbf{V} = \frac{\mathbf{V}_r}{2}\epsilon^{\sqrt{\mathbf{ZY}}l} + \frac{\mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\epsilon^{\sqrt{\mathbf{ZY}}l} + \frac{\mathbf{V}_r}{2}\epsilon^{-\sqrt{\mathbf{ZY}}l} - \frac{\mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\epsilon^{-\sqrt{\mathbf{ZY}}l}$$

$$= \mathbf{V}_r\left(\frac{\epsilon^{\sqrt{\mathbf{ZY}}l} + \epsilon^{-\sqrt{\mathbf{ZY}}l}}{2}\right) + \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}\left(\frac{\epsilon^{\sqrt{\mathbf{ZY}}l} - \epsilon^{-\sqrt{\mathbf{ZY}}l}}{2}\right)$$

Since the analytic definition of

$$\sinh\theta = \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2}$$

and

$$\cosh\theta = \frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2}$$

$$\mathbf{V} = \mathbf{V}_r\cosh\sqrt{\mathbf{ZY}}l + \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}\sinh\sqrt{\mathbf{ZY}}l \tag{26}$$

Similarly

$$\mathbf{I} = \mathbf{I}_r\cosh\sqrt{\mathbf{ZY}}l + \mathbf{V}_r\sqrt{\mathbf{Y}/\mathbf{Z}}\sinh\sqrt{\mathbf{ZY}}l \tag{27}$$

Equations (26) and (27) are particularly convenient to use if tables of complex hyperbolic functions are available; otherwise, equations (24) and (25) may be more convenient.[3]

Physical Interpretation of Equations for Exact Solution. Equations (24) and (25) may be modified somewhat to make their physical significance more apparent. Since $\sqrt{\mathbf{ZY}}$ is a complex expression, we may substitute an expression such as $(\alpha + j\beta)$ for it. Also, letting

[3] See "Tables" or "Charts of Complex Hyperbolic Functions," by A. E. Kennelly, Harvard University Press.

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$Z_0 = \sqrt{Z/Y}$ and $Y_0 = \sqrt{Y/Z}$, equations (24) and (25) may be written:

$$V = \left(\frac{V_r + I_r Z_0}{2}\right)\epsilon^{(\alpha+j\beta)l} + \left(\frac{V_r - I_r Z_0}{2}\right)\epsilon^{-(\alpha+j\beta)l} \tag{28}$$

$$I = \left(\frac{I_r + V_r Y_0}{2}\right)\epsilon^{(\alpha+j\beta)l} + \left(\frac{I_r - V_r Y_0}{2}\right)\epsilon^{-(\alpha+j\beta)l} \tag{29}$$

Recognizing that $\epsilon^{(\alpha+j\beta)l} = \epsilon^{\alpha l}\epsilon^{j\beta l}$ and that $\epsilon^{-(\alpha+j\beta)l} = \epsilon^{-\alpha l}\epsilon^{-j\beta l}$, we may write equations (28) and (29) as follows:

$$V = \left(\frac{V_r + I_r Z_0}{2}\right)\epsilon^{\alpha l}\epsilon^{j\beta l} + \left(\frac{V_r - I_r Z_0}{2}\right)\epsilon^{-\alpha l}\epsilon^{-j\beta l} \tag{30}$$

$$I = \left(\frac{I_r + V_r Y_0}{2}\right)\epsilon^{\alpha l}\epsilon^{j\beta l} + \left(\frac{I_r - V_r Y_0}{2}\right)\epsilon^{-\alpha l}\epsilon^{-j\beta l} \tag{31}$$

The quantity $\sqrt{ZY} = (\alpha + j\beta)$ is called the propagation constant. It determines how the wave is propagated with reference to change in magnitude and phase along the line. Equation (30) consists of two parts. The first, $\left(\frac{V_r + I_r Z_0}{2}\right)\epsilon^{\alpha l}\,\epsilon^{j\beta l}$, represents a quantity that increases in magnitude ($\epsilon^{\alpha l}$ increases) as we go from the receiver to the sending end or it becomes smaller as we proceed from the sending to the receiver end. This term must therefore represent a voltage wave which is being propagated from the sending to the receiver end. Hence it is called the direct wave or direct component. The direct component is analogous to a wave started in a body of water. As the wave leaves the source it becomes smaller and smaller. The second part of equation (30) is $\left(\frac{V_r - I_r Z_0}{2}\right)\epsilon^{-\alpha l}\,\epsilon^{-j\beta l}$. As we proceed from the load to the generator this component becomes smaller, since l increases and $\epsilon^{-\alpha l}$ decreases. Hence this wave must be originating at the receiver, and it is therefore called the reflected wave. It is analogous to the phenomena in a body of water as a wave strikes a bank. A reflection occurs, and a wave is then seen traveling away from the bank with gradually diminishing magnitude. Since, for a given distance of travel, α determines the magnitude of the wave, it is a measure of how much the wave is increased or decreased in magnitude, or, in other words, attenuated. For this reason it is called the attenuation constant or factor. The attenuation factor is the real part of the propagation constant. The factors $\epsilon^{j\beta l}$ and $\epsilon^{-j\beta l}$ will be recognized as operators which produce opposite rotations. The operator $\epsilon^{j\beta l}$ causes the direct component to advance in phase from its position at the load as we proceed from the

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receiving to the sending end, while $\epsilon^{-j\beta l}$ causes the reflected wave to fall behind its phase position at the receiver. Since β determines the change in phase for a given distance l along the line, it is called the phase constant. It is sometimes called the wave length constant because it determines the distance along the line over which a complete wave is subtended. This will be explained in more detail later. The loci of the variation of the direct and reflected waves can be represented as spirals, as shown in Fig. 10. The sum *od* of the direct and reflected waves of voltage at any point along the line such as at βl gives the resultant voltage at that point. When βl is 90°, the direct component of voltage *oa* is opposite to the reflected component *ob*. The resultant *oc*, which is the voltage of the line at this 90° or quarter-wave-length point, may be very small because of the cancellation effect of the two waves. A generator producing a low voltage, if connected at this point, could subtend a comparatively high voltage at the receiver. This is essentially a resonance phenomenon and is called quarter-wave resonance. As βl increases from this 90° point the voltage of the line increases until βl becomes 180°. Here the direct and reflected waves add. This phenomenon is called half-wave resonance. As βl increases to 270° the direct and reflected waves are again opposite (as at quarter-wave) and we then have three-quarter-wave resonance.

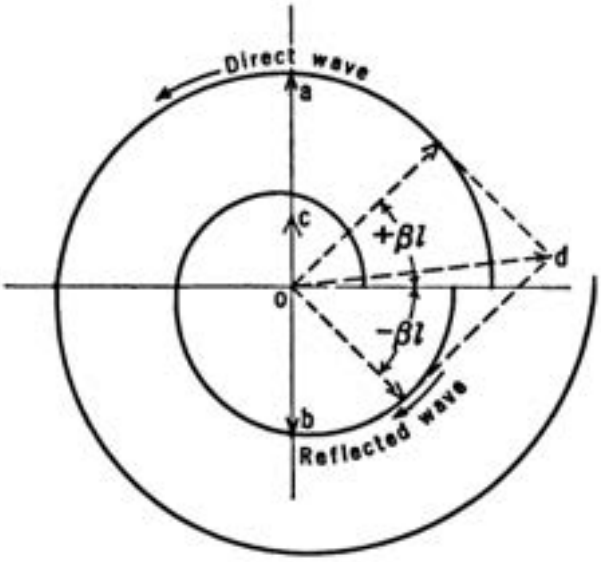

FIG. 10. Variation of direct and reflected waves of voltage with respect to the line angle βl for a particular case.

Surge Impedance. Inspection of equation (28) makes it apparent that dimensionally $I_r Z_0$ must be a voltage. Hence Z_0 must be an impedance. Further evidence of this fact is obtained when it is remembered that $Z_0 = \sqrt{Z/Y}$. The reciprocal of Y is dimensionally an impedance, and the $\sqrt{Z/Y}$ then becomes $\sqrt{\text{impedance}^2}$ which is an impedance. Hence the quantity $Z_0 = \sqrt{Z/Y}$ is called the surge impedance of the line. The reciprocal, $\sqrt{Y/Z}$, is called surge admittance. The surge impedance is the impedance offered to the propagation of a wave along a line. In effect it is the impedance an advancing wave of voltage or current encounters as it travels along the line.

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Terminal Reflections. The receiver voltage $\mathbf{V}_r$ is $\mathbf{I}_r\mathbf{Z}_r$ where $\mathbf{Z}_r$ is the impedance of the load. If $\mathbf{Z}_r$ is made equal to $\mathbf{Z}_0$ the receiver voltage $\mathbf{V}_r$ would equal $\mathbf{I}_r\mathbf{Z}_0$. Then the reflected wave in equation (30) is zero and the equation of the voltage along the line becomes:

$$\mathbf{V} = \left(\frac{\mathbf{V}_r + \mathbf{V}_r}{2}\right)\epsilon^{\alpha l}\epsilon^{j\beta l} = \mathbf{V}_r\epsilon^{\alpha l}\epsilon^{j\beta l} \tag{32}$$

This variation is exponential in character, and no terminal reflections exist. The voltage, V, increases exponentially in magnitude as we proceed from the receiving end to the sending end. Simultaneously with the increase in magnitude there is a uniform advance in phase of $\mathbf{V}$ with respect to the load voltage $\mathbf{V}_r$. The wave encounters the same impedance (surge impedance) at the load as it did while advancing along the line. This termination makes the line behave as if it were infinitely long. Hence a line terminated in its surge impedance is sometimes called an infinite line. In communication work, terminating a line in an impedance equal to the surge impedance is sometimes called matching.

If a long line is short-circuited at the receiver $V_r = 0$ and equation (30) becomes

$$\mathbf{V} = \frac{\mathbf{I}_r\mathbf{Z}_0}{2}\epsilon^{\alpha l}\epsilon^{j\beta l} - \frac{\mathbf{I}_r\mathbf{Z}_0}{2}\epsilon^{-\alpha l}\epsilon^{-j\beta l} \tag{33}$$

Where l is 0,

$$\mathbf{V}_{l=0} = \frac{\mathbf{I}_r\mathbf{Z}_0}{2} - \frac{\mathbf{I}_r\mathbf{Z}_0}{2}$$

$$= \text{direct wave} - \text{reflected wave}$$

Thus it may be said that the voltage is reflected with a change in sign. The current wave under the same conditions becomes

$$\mathbf{I}_{l=0} = \frac{\mathbf{I}_r}{2} + \frac{\mathbf{I}_r}{2}$$

$$= \text{direct wave} + \text{reflected wave}$$

It follows, then, that the current wave is reflected with the same sign or the direct and reflected waves of current add arithmetically at the receiver.

If the line is open-circuited at the receiver, $I_r = 0$. Imposing this condition on equations (30) and (31) shows the voltage wave to be reflected with the same sign and the current with a change in sign.

Velocity of Propagation. In the foregoing equations, distance along the line, namely l, has been considered the independent variable. The

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other independent variable, time, has been tacitly taken into account by the use of vector quantities. In the evaluation of the velocity of propagation the interrelation of time-phase and space-phase effects must be recognized.

It is evident from the use of β in the foregoing equations that this quantity determines the phase shift of **V** or **I** per unit length of line, and as such it represents a number of radians per unit length of line. The length of line required to effect a complete cycle or 2π radians of phase shift is

$$\lambda = \frac{2\pi}{\beta} \text{ units} \tag{34}$$

where λ and β are expressed in any consistent set of units. To simplify visualizing the phenomenon, consider only the voltage wave.

Since λ is the distance for a phase shift of 2π radians, it is the distance along the line (see Fig. 11) from one zero value say at a on the voltage wave to a corresponding zero value at b, 2π radians or 360° from the first zero point. The distance λ thus represents the length of line over which a complete space wave or cycle of voltage is subtended, and in consequence λ is called the wave length of the propagated wave. As time

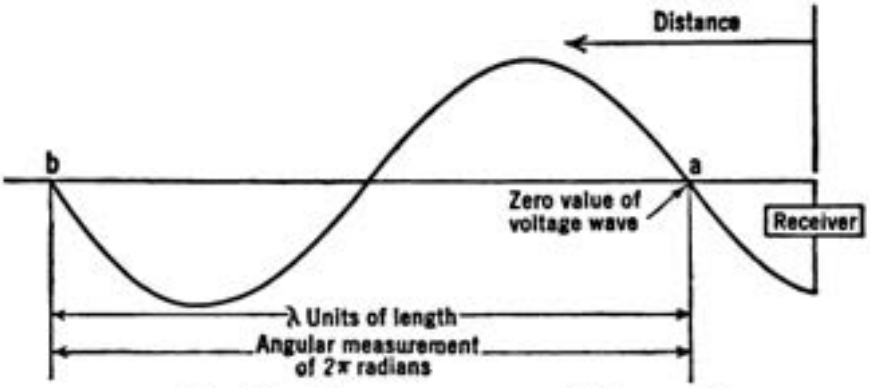

FIG. 11. A space wave or cycle from a to b.

elapses, the alternating voltage at point a will rise to a positive maximum, decrease to zero, then increase to a negative maximum, thence to zero. In this length of time, point b on the wave will have arrived at a in Fig. 11. In other words, during this length of time, the time for one cycle, $1/f$, all points on the wave will have traveled a distance of λ. The velocity of travel or propagation must then be $\frac{\lambda}{1/f}$, or λf units of length per second. Hence the velocity of propagation is

$$v = \lambda f = \frac{2\pi}{\beta} f \tag{35}$$

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Equation (34) shows that the wave length for any line is determined by the quantity β. Hence β is often termed the wave-length constant, and it may be evaluated in terms of the circuit parameters from the original substitution, namely, $\sqrt{\mathbf{ZY}} = \alpha + j\beta$. Since $\mathbf{Z} = R + jX$ and $\mathbf{Y} = g - jb$, it follows that

$$\alpha + j\beta = \sqrt{(R + jX)\,(g - jb)} \tag{36}$$

$$\alpha^2 + j2\alpha\beta - \beta^2 = Rg - jRb + jgX + bX$$

$$\alpha^2 - \beta^2 = Rg + bX \tag{37}$$

$$2\alpha\beta = gX - Rb \tag{38}$$

Solving equations (37) and (38) simultaneously for β gives

$$\beta = \sqrt{\frac{\pm ZY - (Rg + bX)}{2}} \tag{39}$$

The preceding derivation shows that all terms in equation (39) are expressed algebraically and not in complex form. All the quantities are per unit values, that is, per centimeter, per mile, etc.

It is interesting to find the velocity of propagation under the conditions of zero series resistance and a negligibly small value of g. Imposing these conditions gives

$$\beta = \sqrt{\frac{\pm bX - bX}{2}} = \sqrt{-bX}$$

The two signs before ZY in equation (39) and before bX above resulted from the solution of a quadratic equation. As often occurs, one of such solutions has no physical reality. If the plus sign were used in the algebraic manipulation above, β would be zero, which would in turn give an infinite velocity of propagation. Obviously, this is impossible. When making arithmetic computations the proper sign to employ is that which will give a physically possible and plausible result. Had the equations been based on $g + jb$, it would have been necessary to use the plus sign before the ZY and bX above. Since b is the shunted susceptance due to the line capacitance, it must carry a negative sign upon substitution of a numerical value for it in accordance with the conventions employed in this book. Substituting the value of β above in equation (35) gives

$$v = \frac{2\pi f}{\sqrt{-bX}} = \frac{2\pi f}{\sqrt{2\pi fC \times 2\pi fL}} = \frac{1}{\sqrt{LC}} \text{ (for } r = 0 \text{ and } g = 0) \tag{40}$$

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In equation (40) v is in miles per second if L is expressed in henrys per mile and C in farads per mile. If one further assumption is made in equation (40), namely, that the inductance due to the flux within the conductor is negligible, the velocity will be the same as that of light. This is illustrated by example 2, pages 430–434.

Example 1. An open-wire telephone line has a resistance of 10.26 ohms, an inductance of 0.00366 henry, and a capacitance of 0.00822 μf per loop mile (one mile of outgoing plus one mile of return conductor). Calculate the velocity of propagation for a 200-cycle and also for a 2000-cycle frequency, assuming that the values of R, L, and C are the same at both frequencies. Assume $g = 0$ in both cases.

At 200 cycles

$$X = 2\pi\, 200 \times 0.00366 = 4.6 \text{ ohms per loop mile}$$
$$Z = \sqrt{10.26^2 + 4.6^2} = 11.22 \text{ ohms per loop mile}$$
$$b = -2\pi fC = -2\pi\, 200 \times 0.00822 \times 10^{-6} = -10.32 \times 10^{-6} \text{ mho per loop mile}$$
$$Y = 10.32 \times 10^{-6} \text{ mho per loop mile}$$
$$\beta = \sqrt{\frac{\pm 11.22 \times 10.32 \times 10^{-6} - (-10.32 \times 10^{-6} \times 4.6)}{2}}$$
$$= \sqrt{\frac{163.5 \times 10^{-6}}{2}} = 9.05 \times 10^{-3}$$
$$v = \frac{2\pi f}{\beta} = \frac{2\pi\, 200}{9.05 \times 10^{-3}} = 139 \times 10^{3} = 139{,}000 \text{ miles per second}$$

At 2000 cycles

$$X = 2\pi\, 2000 \times 0.00366 = 46 \text{ ohms per loop mile}$$
$$Z = \sqrt{10.26^2 + 46^2} = 47.1 \text{ ohms per loop mile}$$
$$b = -2\pi fC = -2\pi\, 2000 \times 0.00822 \times 10^{-6}$$
$$= -103.2 \times 10^{-6} \text{ mho per loop mile}$$
$$Y = 103.2 \times 10^{-6} \text{ mho per loop mile}$$
$$\beta = \sqrt{\frac{\pm 47.1 \times 103.2 \times 10^{-6} - (-103.2 \times 10^{-6} \times 46)}{2}}$$
$$= \sqrt{\frac{9610 \times 10^{-6}}{2}} = 69.3 \times 10^{-3}$$
$$v = \frac{2\pi f}{\beta} = \frac{2\pi\, 2000}{69.3 \times 10^{-3}} = 181{,}400 \text{ miles per second}$$

If parameters per mile to ground or neutral were used, Z would be halved, Y and b doubled and β would be the same.

Confusion sometimes arises as to what the velocity of propagation refers to physically. The velocity of propagation of a voltage or current wave is the velocity at which the impulse or pressure travels. It is not the velocity of current flow. The velocity of current flow for normal current densities in copper is very low, although the velocity of the pressure wave is high. The phenomenon is somewhat analogous to the application of pressure at one end of a long pipe filled with water. The

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pressure appears at the far end of the pipe very soon after it is applied at the near end. However, the actual rate of flow of water in the pipe may be very low, especially if only a comparatively small stream is permitted to emerge at the far end.

Determination of Transmission Line Parameters. 1. *Inductance.* The inductance per wire is used in transmission line calculations. It may be derived as follows. Consider two parallel conductors as shown in

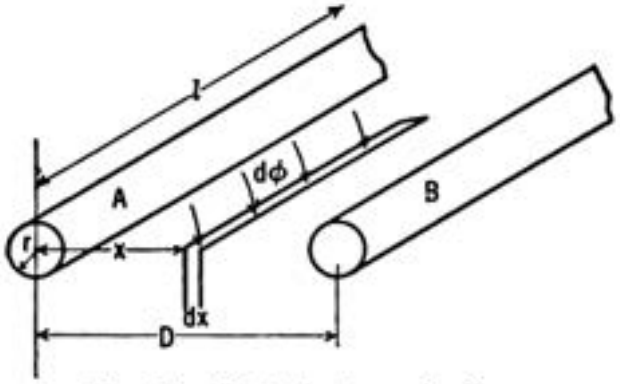

FIG. 12. Part of a two-wire line.

Fig. 12, each having a radius r and separated a distance D between centers. The fundamental equation of inductance when permeability is constant is

$$L = \frac{N\phi}{I} 10^{-9} \text{ henry}$$

where I is in abamperes and ϕ is in maxwells. The field intensity based on the law of Biot-Savart, at a distance of x centimeters from a long straight conductor carrying a current is $2I/x$ gilberts per centimeter, which in air is numerically equal to the flux density. Referring now to the open-wire line shown in Fig. 12,

$$d\phi = \left(\frac{2I}{x}\right)(l\,dx)$$

The total flux that exists outside of conductor A which causes an inductive effect on conductor A is

$$\phi_1 = \int_r^D \frac{2I}{x} l\,dx = 2lI \log_e \frac{D}{r}$$

$$L_1 = \frac{N\phi_1}{I} 10^{-9} = 2l \log_e \frac{D}{r} 10^{-9}$$

$$= 2 \times 2.3026 l \log_{10} \frac{D}{r} 10^{-9} \text{ henry} \tag{41}$$

where l is expressed in centimeters.

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The flux included from $x = (D - r)$ to $x = (D + r)$ has some effect in inducing a net emf in the two conductors connected in series to form a coil. The effect is due to this flux linking all of conductor A and only part of conductor B. Integrating between the limits $x = r$ and $x = D$ includes the full effectiveness of the flux from $x = (D - r)$ to $x = D$ in causing the inductance. This balances the partial effectiveness of the flux from $x = D$ to $x = (D + r)$ which is neglected in taking the limits from r to D. The flux from $x = (D + r)$ to $x = \infty$ links both conductors and therefore produces equal and opposing emf's around the loop. Hence it has no net inductive effect. Equation (41) gives the inductance of conductor A due to all the flux on the outside of conductor A which is effective. The flux within the conductor causes some inductance which may be calculated as follows.

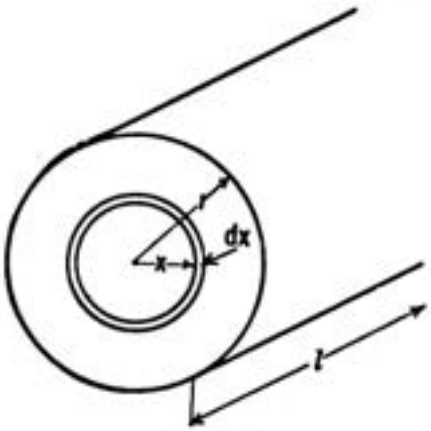

Fig. 13.

Assume that the current in conductor A is uniformly distributed across the cross-section. Let I' be the current per unit area. Refer to the cross-section of conductor A shown in Fig. 13. The total current responsible for the mmf causing flux through the element dx is $\pi x^2 I'$.

$$\text{mmf} = 4\pi NI = 4\pi\ (\pi x^2 I')$$

If the permeability of the conductor material is unity, the reluctance of the flux path formed by the element dx and a length of conductor l is

$$\mathcal{R} = \frac{2\pi x}{l\,dx} \text{ cgs units when } x \text{ is in centimeters}$$

$$d\phi = \frac{4\pi^2 x^2 I'}{\dfrac{2\pi x}{(l\,dx)}} = 2\pi x I' l\,dx \text{ maxwells}$$

The flux $d\phi$ links only the fibers of the conductor from the center to a distance x or $K\pi x^2$ fibers. To obtain the flux which links the whole conductor that produces the same effect as the actual flux which links $K\pi x^2$ fibers, it is only necessary to find the flux linking $K\pi r^2$ fibers (the entire conductor), which is equivalent to the flux $d\phi$ linking $K\pi x^2$ fibers. Calling the flux in question $d\phi_e$, we have for equivalent linkages

$$d\phi_e K\pi r^2 = d\phi K\pi x^2$$

or

$$d\phi_e = \frac{x^2}{r^2} d\phi$$

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Hence

$$d\phi_e = \frac{x^2}{r^2}(2\pi x I' l\, dx)$$

$$\phi_e = \int_0^r \frac{2\pi x^3 I' l\, dx}{r^2} = \left(\frac{2\pi I' l}{r^2}\right)\left(\frac{r^4}{4}\right)$$

$$= \frac{\pi r^2 I' l}{2}$$

But $\pi r^2 I'$ is the total current I. Therefore

$$\phi_e = \frac{Il}{2}$$

The inductance due to this flux is

$$L_2 = \frac{N\phi_e}{I} 10^{-9} = \frac{1 \times Il\, 10^{-9}}{2I} = \frac{l\, 10^{-9}}{2} \text{ henry} \tag{42}$$

The total inductance of conductor A is the sum of equations (41) and (42).

$$L = L_1 + L_2 = \left[\frac{l}{2} + 4.6052 l \log_{10}\frac{D}{r}\right] 10^{-9} \text{ henry} \tag{43}$$

The inductance per mile is

$$L_{\text{mile}} = 0.5 \times 5280 \times 30.48\, 10^{-9}$$
$$+ 4.6052 \times 5280 \times 30.48 \times 10^{-9} \log_{10}\frac{D}{r}$$
$$= 0.805\, 10^{-4} + 0.741 \times 10^{-3} \log_{10}\frac{D}{r} \text{ henry} \tag{44}$$

Equation (44) is the working equation. Usually the reactance is desired. It is found by multiplying the values obtained from equation (44) by the angular velocity $2\pi f$.

2. *Capacitance between Wires and to Ground.* The defining equation for capacitance is $C = Q/V$. The defining equation for difference of potential V is

$$V = \frac{W}{Q} = \frac{\text{work}}{\text{charge}}$$

The difference of electrostatic potential between two conductors is the work done in carrying a unit charge from the surface of one conductor to the other. Work is the product of force and the distance through which the force acts. By definition, if all quantities are expressed in the cgs electrostatic system of units, the force on a unit charge is numerically

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equal to the electrostatic field intensity. The electrostatic field intensity at point p, Fig. 14, at a perpendicular distance of r centimeters from a long straight wire is found as follows.

Let all quantities be expressed in the cgs electrostatic system of units, and let σ be the charge per unit length of wire. From Coulomb's law $f = QQ'/d^2$ dynes. Hence the force on a unit charge at point p due to a length of conductor dl is

$$df' = \frac{1 \times \sigma\, dl}{\rho^2}$$

where ρ is the distance in centimeters from p to dl. As θ varies between minus and plus 90° (on the basis of an infinite length of wire), it is

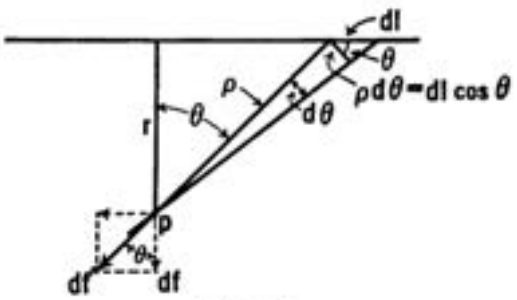

FIG. 14.

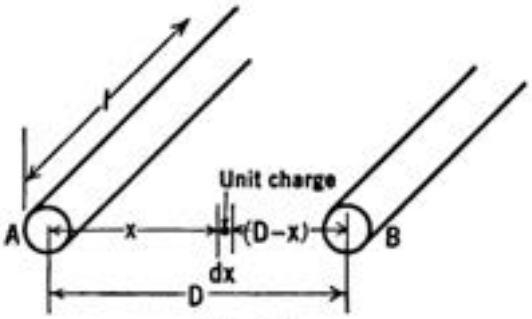

FIG. 15.

apparent that all the components of df' parallel to the wire add to zero. Therefore only the components perpendicular to the conductor need be added to obtain the resultant force on the unit charge.

$$\begin{aligned} df &= df' \cos\theta \\ &= \frac{\sigma\, dl}{\rho^2}\cos\theta = \frac{\sigma\rho\, d\theta}{\rho^2} = \frac{\sigma\, d\theta}{\rho} \\ &= \frac{\sigma\, d\theta}{r/\cos\theta} = \frac{\sigma}{r}\cos\theta\, d\theta \\ f &= \int_{-\pi/2}^{\pi/2} \frac{\sigma}{r}\cos\theta\, d\theta = \frac{2\sigma}{r}\text{ dynes} \end{aligned} \tag{45}$$

The force on the unit charge in Fig. 15 is due to the effect of conductor A (say + charge) and that of conductor B (negative charge if A is positive).

$$\begin{aligned} f_A &= \frac{2\sigma}{x} \\ f_B &= \frac{2\sigma}{D - x} \\ f &= f_A + f_B = \frac{2\sigma}{x} + \frac{2\sigma}{D - x} \end{aligned} \tag{46}$$

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$$dW = f\,dx = \left(\frac{2\sigma}{x} + \frac{2\sigma}{D - x}\right) dx$$

$$W = V = \int_r^{D-r} \left(\frac{2\sigma}{x} + \frac{2\sigma}{D - x}\right) dx = 4\sigma \log_e \frac{D - r}{r} \tag{47}$$

The charge on the line for a length l is σl. Therefore

$$C = \frac{Q}{V} = \frac{\sigma l}{4\sigma \log_e \dfrac{D - r}{r}} = \frac{l}{4 \log_e \dfrac{D - r}{r}} \text{ cgs esu} \tag{48}$$

where r now represents the radius of the conductor and is not the same as in the derivation of equation (45). All quantities in equation (48) are in the cgs or absolute electrostatic system of units, giving C in esu or statfarads.

Equation (48) gives the capacitance between two parallel wires. The capacitance to ground or neutral is usually desired in the calculation of transmission lines. Since the plane of neutral potential is symmetrically located between positive and negative charges (assuming a uniform dielectric such as air), the potential between one wire and neutral,[4] or what is also ground potential, is one-half of the potential

[4] The preceding and following equations of capacitance are only approximately correct because they are based on several assumptions which are only partially fulfilled. First, the charge on the conductor is assumed uniform. This assumption requires in part that the conductors be removed an infinite distance from all charged bodies and that the conductors are circular in shape. Under such conditions the distribution of the electrostatic field is pictured in Fig. 16. Equipotential surfaces are those in which all electrostatic lines of force enter and leave perpendicularly. One equipotential surface XX' is shown in Fig. 16. This surface is at a distance halfway between the two conductors and is therefore at a potential midway between the positively and negatively charged conductors. Such a surface is said to be at zero potential, and it is sometimes called the neutral plane between conductors, or simply the neutral. If the earth is considered a conductor and to be at zero potential, it may be assumed to be the same as the equipotential plane XX'. Hence the potential and capacitance to earth or ground may be taken the same as that to the equipotential surface XX' in Fig. 16 provided $D/2$ is relatively small compared with the physical height of the conductor above actual ground. Even though all the above assumptions are not completely fulfilled, the equations given yield results which are sufficiently accurate for most work concerning transmission lines. For more accurate derivations of capacitance the reader is referred to works on electrostatics and electrodynamics.

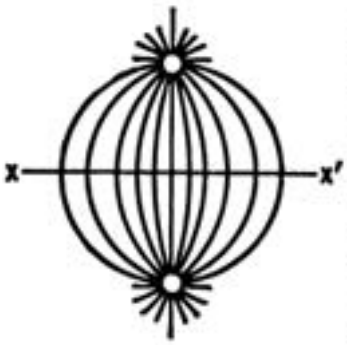

Fig. 16. Equipotential line XX' is at a potential midway between the positively charged top and negatively charged lower conductors.

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[given in equation (47)] between wires. Hence

$$V_g = \frac{1}{2}\left(4\sigma \log_e \frac{D-r}{r}\right) = 2\sigma \log_e \left(\frac{D-r}{r}\right)$$

and

$$C_g = \frac{\sigma l}{2\sigma \log_e \dfrac{D-r}{r}} = \frac{l}{2 \log_e \dfrac{D-r}{r}} \text{esu} \tag{49}$$

Expressed in farads per mile, equations (48) and (49) for the capacitance between conductors and between one conductor and ground become:

$$C_{\text{farads per mile}} = \frac{1940 \times 10^{-11}}{\log_{10} \dfrac{D-r}{r}} \tag{50}$$

$$C_{g\ \text{farads per mile}} = \frac{3880 \times 10^{-11}}{\log_{10} \dfrac{D-r}{r}} \tag{51}$$

Equations (50) and (51) are the working equations. As long as D and r are expressed in the same units, the actual units are immaterial.

Equations (44), (50), and (51) form the basis of tables wherein values of L or C may be immediately determined when the size of wire and spacings are known. Samples of tables where the quantities are expressed in units per thousand feet are shown in Tables I and II.[5]

When equations (44), (50), and (51) are applied to three-phase transmission the distance D is that for equilateral spacing, as shown in Fig. 17. These equations are often applied to plane spacings, as shown

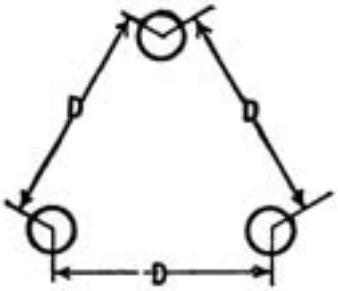

FIG. 17. Equilateral spacing of a transmission line.

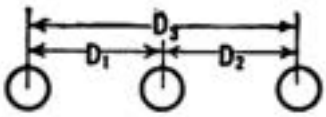

FIG. 18. Plane spacing of a transmission line.

in Fig. 18, in which cases D is taken as the geometric mean distance, that is, $D = \sqrt[3]{D_1 D_2 D_3}$. The results thus obtained are sufficiently accurate for most computations.

[5] Reprinted by permission from "Electrical Engineers' Handbook: Electric Power," fourth edition, edited by Pender and Del Mar, pp. 14–39 and 14–34, John Wiley & Sons, Inc., 1949.

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TABLE I

SELF-INDUCTANCE OF SOLID NON-MAGNETIC WIRES*

Millihenrys per 1000 FEET of each wire of a single-phase or of a symmetrical three-phase line

Size of Wire, cir mils or A.W.G.	Diam. of Wire, inches	Inches between Wires, center to center							
		1	3	6	9	12	18	24	30
1,000,000	1.0000	0.05750	0.1245	0.1667	0.1915	0.2090	0.2337	0.2512	0.2648
750,000	0.8660	0.06627	0.1332	0.1755	0.2002	0.2178	0.2425	0.2600	0.2736
500,000	0.7071	0.07863	0.1456	0.1879	0.2126	0.2301	0.2548	0.2724	0.2860
350,000	0.5916	0.08950	0.1565	0.1987	0.2235	0.2410	0.2657	0.2832	0.2968
250,000	0.5000	0.09976	0.1667	0.2090	0.2337	0.2512	0.2760	0.2935	0.3071
0000	0.4600	0.1048	0.1718	0.2141	0.2388	0.2563	0.2810	0.2986	0.3122
000	0.4096	0.1119	0.1789	0.2211	0.2459	0.2634	0.2881	0.3057	0.3193
00	0.3648	0.1190	0.1860	0.2282	0.2529	0.2705	0.2952	0.3127	0.3263
0	0.3249	0.1260	0.1930	0.2353	0.2600	0.2775	0.3022	0.3198	0.3334
1	0.2893	0.1331	0.2001	0.2423	0.2671	0.2846	0.3093	0.3269	0.3405
2	0.2576	0.1402	0.2072	0.2494	0.2741	0.2917	0.3164	0.3339	0.3475
4	0.2043	0.1543	0.2213	0.2635	0.2883	0.3058	0.3305	0.3481	0.3617
6	0.1620	0.1685	0.2354	0.2777	0.3024	0.3199	0.3447	0.3622	0.3758
8	0.1285	0.1826	0.2496	0.2918	0.3165	0.3341	0.3588	0.3763	0.3899
10	0.1019	0.1967	0.2637	0.3060	0.3307	0.3482	0.3729	0.3905	0.4041
12	0.08081	0.2109	0.2778	0.3201	0.3448	0.3623	0.3871	0.4046	0.4182
14	0.06408	0.2250	0.2920	0.3342	0.3590	0.3765	0.4012	0.4187	0.4323
16	0.05082	0.2391	0.3061	0.3484	0.3731	0.3906	0.4153	0.4329	0.4465

Size of Wire, cir mils or A.W.G.	Feet between Wires, center to center								
	3	4	5	6	8	10	15	20	25
1,000,000	0.2760	0.2935	0.3071	0.3182	0.3358	0.3494	0.3741	0.3916	0.4052
750,000	0.2847	0.3023	0.3159	0.3270	0.3445	0.3581	0.3828	0.4004	0.4140
500,000	0.2971	0.3146	0.3282	0.3393	0.3569	0.3705	0.3952	0.4127	0.4263
350,000	0.3080	0.3255	0.3391	0.3502	0.3678	0.3814	0.4061	0.4236	0.4372
250,000	0.3182	0.3358	0.3494	0.3605	0.3780	0.3916	0.4163	0.4339	0.4475
0000	0.3233	0.3408	0.3544	0.3656	0.3831	0.3967	0.4214	0.4390	0.4526
000	0.3304	0.3479	0.3615	0.3726	0.3902	0.4038	0.4285	0.4460	0.4596
00	0.3374	0.3550	0.3686	0.3797	0.3972	0.4108	0.4356	0.4531	0.4667
0	0.3445	0.3620	0.3756	0.3867	0.4043	0.4179	0.4426	0.4601	0.4737
1	0.3516	0.3691	0.3827	0.3938	0.4114	0.4250	0.4497	0.4672	0.4808
2	0.3586	0.3762	0.3898	0.4009	0.4184	0.4320	0.4568	0.4743	0.4879
4	0.3728	0.3903	0.4039	0.4150	0.4326	0.4462	0.4709	0.4884	0.5020
6	0.3869	0.4045	0.4181	0.4292	0.4467	0.4603	0.4850	0.5026	0.5162
8	0.4011	0.4186	0.4322	0.4433	0.4608	0.4744	0.4992	0.5167	0.5303
10	0.4152	0.4327	0.4463	0.4574	0.4750	0.4886	0.5133	0.5308	0.5444
12	0.4293	0.4469	0.4605	0.4716	0.4891	0.5027	0.5274	0.5450	0.5586
14	0.4435	0.4610	0.4746	0.4857	0.5033	0.5169	0.5416	0.5591	0.5727
16	0.4576	0.4751	0.4887	0.4998	0.5174	0.5310	0.5557	0.5732	0.5868

* The inductances given in this table also apply, with a practically negligible error (about 1 per cent), to ordinary stranded wires of the same cross-section.

Example 2. Exact Solution of a Transmission Line. A 60-cycle transmission line 200 miles long consists of three No. 0000 solid conductors with 10-ft equilateral spacing. Calculate the sending voltage when the receiver voltage is 110 kv between lines and when the line is supplying a balanced load of 18,000 kw at 0.8 power-factor lag. Also calculate the sending-end current and the efficiency of the line at 25° C. Assume that the conductance to ground is negligible.

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TABLE II

CAPACITANCE TO NEUTRAL* OF SMOOTH ROUND WIRES

Microfarads per 1000 FEET of each wire of a single-phase or of a symmetrical three-phase line

Size of Wire, A.W.G.	Diam. of Wire, inches	Inches between Wires, center to center							
		1	3	6	9	12	18	24	30
0000	0.4600	0.01199	0.006608	0.005192	0.004618	0.004282	0.003884	0.003643	0.003477
000	0.4096	0.01099	0.006317	0.005013	0.004477	0.004161	0.003783	0.003555	0.003396
00	0.3648	0.01016	0.006055	0.004847	0.004344	0.004045	0.003688	0.003470	0.003319
0	0.3249	0.009458	0.005812	0.004692	0.004218	0.003936	0.003597	0.003390	0.003245
1	0.2893	0.008855	0.005587	0.004546	0.004100	0.003833	0.003511	0.003313	0.003174
2	0.2576	0.008332	0.005381	0.004408	0.003988	0.003735	0.003428	0.003239	0.003107
4	0.2043	0.007455	0.005010	0.004157	0.003781	0.003553	0.003274	0.003102	0.002980
6	0.1620	0.006753	0.004688	0.003933	0.003595	0.003388	0.003134	0.002975	0.002863
8	0.1285	0.006177	0.004406	0.003732	0.003426	0.003238	0.003005	0.002859	0.002755
10	0.1019	0.005693	0.004155	0.003551	0.003273	0.003100	0.002886	0.002751	0.002655
12	0.08081	0.005277	0.003931	0.003386	0.003132	0.002974	0.002776	0.002651	0.002562
14	0.06408	0.004921	0.003730	0.003235	0.003003	0.002858	0.002675	0.002558	0.002475

Size of Wire, A.W.G.	Feet between Wires, center to center								
	3	4	5	6	8	10	15	20	25
0000	0.003351	0.003171	0.003043	0.002947	0.002806	0.002706	0.002542	0.002436	0.002361
000	0.003276	0.003103	0.002981	0.002889	0.002753	0.002657	0.002498	0.002396	0.002323
00	0.003204	0.003039	0.002922	0.002833	0.002702	0.002610	0.002456	0.002358	0.002287
0	0.003135	0.002977	0.002864	0.002779	0.002653	0.002564	0.002416	0.002320	0.002251
1	0.003069	0.002917	0.002809	0.002727	0.002606	0.002520	0.002376	0.002284	0.002217
2	0.003006	0.002860	0.002756	0.002677	0.002560	0.002477	0.002338	0.002249	0.002184
4	0.002887	0.002752	0.002656	0.002582	0.002474	0.002396	0.002266	0.002182	0.002121
6	0.002777	0.002652	0.002563	0.002494	0.002392	0.002319	0.002197	0.002118	0.002061
8	0.002676	0.002559	0.002476	0.002412	0.002317	0.002248	0.002133	0.002059	0.002004
10	0.002581	0.002473	0.002395	0.002335	0.002245	0.002181	0.002073	0.002002	0.001951
12	0.002493	0.002392	0.002319	0.002262	0.002178	0.902118	0.002016	0.001949	0.001900
14	0.002411	0.002316	0.002247	0.002194	0.002115	0.002058	0.001961	0.001898	0.001852

* The capacitance *between* wires equals one-half the values given in this table.

All calculations will be made per phase to neutral or ground.

$$V_n = \frac{110{,}000}{\sqrt{3}} = 63{,}500 \text{ volts}$$

$$I_r = \frac{18{,}000{,}000}{\sqrt{3}\ 110{,}000 \times 0.8} = 118 \text{ amperes}$$

From wire tables the 60-cycle resistance per mile of No. 0000 at 25° C is 0.271 ohm. The diameter of No. 0000 wire is 460 mils.

$$\begin{aligned} L_{\text{mile}} &= 0.805 \times 10^{-4} + 0.741 \times 10^{-3} \log_{10} \frac{120}{0.23} \\ &= 0.805 \times 10^{-4} + 0.741 \times 10^{-3} \times 2.718 \\ &= 0.805 \times 10^{-4} + 2.01 \times 10^{-3} \\ &= 20.9 \times 10^{-4} \text{ henry per mile} \end{aligned}$$

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Reactance per mile $= 2\pi\, 60 \times 20.9 \times 10^{-4} = 0.788$ ohm

$$b = \frac{-1}{X_c} = -2\pi f C$$

$$C_{\text{mile}} = \frac{3880 \times 10^{-11}}{\log_{10} \dfrac{120 - 0.23}{0.23}} = 1430 \times 10^{-11} \text{ farad}$$

Susceptance per mile $= -2\pi\, 60 \times 1430 \times 10^{-11} = -0.538 \times 10^{-5}$ mho

$$\mathbf{Y} = g - jb = +j0.538 \times 10^{-5} = 0.538 \times 10^{-5} \angle 90^\circ \text{ mho}$$

$$\mathbf{Z} = r + jX = 0.271 + j0.788 = 0.834 \angle 71.05^\circ \text{ ohms}$$

$$\sqrt{\mathbf{ZY}} = \sqrt{0.834 \angle 71.05^\circ\, 0.538 \times 10^{-5} \angle 90^\circ} = 2.12 \times 10^{-3} \angle 80.5^\circ$$

$$\sqrt{\mathbf{Z/Y}} = \sqrt{\frac{0.834 \angle 71.05^\circ}{0.538 \times 10^{-5} \angle 90^\circ}} = 3.94 \times 10^{2} \angle -9.48^\circ \text{ ohms}$$

$$\sqrt{\mathbf{Y/Z}} = 0.254 \times 10^{-2} \angle 9.48^\circ \text{ mhos}$$

For $l = 200$ miles,

$$\epsilon^{\sqrt{\mathbf{ZY}}l} = \epsilon^{0.424 \angle 80.5^\circ} = \epsilon^{0.07 + j0.418} = \epsilon^{0.07}\epsilon^{j23.9^\circ}$$

$$\mathbf{V}_r = 63{,}500 + j0 \text{ volts}$$

$$\mathbf{I}_r = 118 \angle -36.9^\circ \text{ amperes}$$

$$\left(\frac{\mathbf{V}_r + \mathbf{I}_r\sqrt{\mathbf{Z/Y}}}{2}\right) = \left(\frac{63{,}500 + 118 \angle -36.9^\circ \times 3.94 \times 10^{2} \angle -9.48^\circ}{2}\right)$$

$$= 47{,}800 - j16{,}800 \text{ volts}$$

$$\left(\frac{\mathbf{V}_r + \mathbf{I}_r\sqrt{\mathbf{Z/Y}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} = (47{,}800 - j16{,}800)\epsilon^{0.07}\epsilon^{j23.9^\circ}$$

$$= (51{,}300 - j18{,}050)\epsilon^{j23.9^\circ}$$

$$= 54{,}400 \angle -19.4^\circ \angle 23.9^\circ$$

$$= 54{,}400 \angle 4.5^\circ \text{ volts}$$

$$\left(\frac{\mathbf{V}_r - \mathbf{I}_r\sqrt{\mathbf{Z/Y}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} = (15{,}700 + j16{,}800)\epsilon^{-0.07}\epsilon^{-j23.9^\circ}$$

$$= (14{,}610 + j15{,}630)\epsilon^{-j23.9^\circ}$$

$$= 21{,}400 \angle 46.9^\circ \angle -23.9^\circ$$

$$= 21{,}400 \angle 23^\circ \text{ volts}$$

$$\mathbf{V}_s = \left(\frac{\mathbf{V}_r + \mathbf{I}_r\sqrt{\mathbf{Z/Y}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} + \left(\frac{\mathbf{V}_r - \mathbf{I}_r\sqrt{\mathbf{Z/Y}}}{2}\right)\epsilon^{-\sqrt{\mathbf{ZY}}l}$$

$$= 54{,}400 \angle 4.5^\circ + 21{,}400 \angle 23^\circ$$

$$= 54{,}200 + j4270 + 19{,}680 + j8355$$

$$= 73{,}880 + j12{,}625 = 74{,}970 \angle 9.7^\circ \text{ volts}$$

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The current at the sending end could be calculated in a similar way. However, for illustrative purposes it will be calculated from equation (27).

$$\mathbf{I}_s = \mathbf{I}_r \cosh \sqrt{\mathbf{ZY}}l + \mathbf{V}_r \sqrt{\mathbf{Y}/\mathbf{Z}} \sinh \sqrt{\mathbf{ZY}}l$$

The following relations are convenient to use when dealing with hyperbolic functions of complex angles:

$$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$
$$\sinh jx = j \sin x$$
$$\cosh jx = \cos x$$

$$\sqrt{\mathbf{ZY}}l = 2.12 \times 10^{-3} \underline{/80.5^\circ} \times 200 = 0.424 \underline{/80.5^\circ} = 0.07 + j0.418$$

$$\begin{aligned}\cosh (0.07 + j0.418) &= \cosh 0.07 \cosh j0.418 + \sinh 0.07 \sinh j0.418 \\ &= \cosh 0.07 \cos 0.418 + j \sinh 0.07 \sin 0.418 \\ &= \cosh 0.07 \cos 23.9^\circ + j \sinh 0.07 \sin 23.9^\circ \\ &= 1.00245 \times 0.9143 + j0.07 \times 0.4051 \\ &= 0.915 + j0.02835\end{aligned}$$

$$\begin{aligned}\sinh (0.07 + j0.418) &= \sinh 0.07 \cosh j0.418 + \cosh 0.07 \sinh j0.418 \\ &= \sinh 0.07 \cos 23.9^\circ + j \cosh 0.07 \sin 23.9^\circ \\ &= 0.07 \times 0.9143 + j1.00245 \times 0.4051 \\ &= 0.0639 + j0.406\end{aligned}$$

$$\mathbf{V}_r \sqrt{\mathbf{Y}/\mathbf{Z}} = 63{,}500 \times 0.254 \times 10^{-2} \underline{/9.48^\circ} = 161.30 \underline{/9.48^\circ} \text{ amperes}$$

$$\begin{aligned}\mathbf{V}_r \sqrt{\mathbf{Y}/\mathbf{Z}} \sinh \sqrt{\mathbf{ZY}}l &= 161.3 \underline{/9.48^\circ} (0.0639 + j0.406) \\ &= -0.66 + j66.3 \text{ amperes}\end{aligned}$$

$$\mathbf{I}_r \cosh \sqrt{\mathbf{ZY}}l = (118 \underline{/-36.9^\circ})(0.915 + j0.0284) = 88.4 - j62.1 \text{ amperes}$$

$$\begin{aligned}\mathbf{I}_s &= 88.4 - j62.1 - 0.66 + j66.3 \\ &= 87.8 + j4.2 = 87.9 \underline{/2.8^\circ} \text{ amperes}\end{aligned}$$

As a check on the sending voltage, V_s will be calculated by the hyperbolic equation

$$\mathbf{V}_r \cosh \sqrt{\mathbf{ZY}}l + \mathbf{I}_r \sqrt{\mathbf{Z}/\mathbf{Y}} \sinh \sqrt{\mathbf{ZY}}l$$

$$\begin{aligned}\mathbf{V}_s &= 63{,}500 \times 0.915 \underline{/1.75^\circ} + (118 \underline{/-36.9^\circ} \times 3.94 \times 10^2 \underline{/-9.48^\circ})(0.0639 + j0.406) \\ &= 58{,}100 + j1770 + 15{,}700 + j10{,}880 \\ &= 73{,}800 + j12{,}652 = 74{,}850 \underline{/9.7^\circ} \text{ volts}\end{aligned}$$

$$\begin{aligned}P_s &= vi + v'i' = 73{,}800 \times 87.8 + 12{,}652 \times 4.2 \\ &= 6{,}490{,}000 + 53{,}100 \\ &= 6{,}543{,}000 \text{ watts per phase}\end{aligned}$$

$$\text{Efficiency} = \frac{6000}{6543} = 0.917$$

If tables of complex hyperbolic functions are available, the hyperbolic solution is greatly simplified.

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Calculation of Velocity of Propagation. From equations (35) and (39),

$$v = \frac{2\pi f}{\beta}$$

$$\beta = \sqrt{\frac{\pm ZY - (Rg + bX)}{2}}$$

$$ZY = [2.12 \times 10^{-3}]^2 = 4.5 \times 10^{-6}$$

$$Rg = 0$$

$$bX = -0.538 \times 10^{-5} \times 0.788 = -0.424 \times 10^{-5}$$

$$\beta = \sqrt{\frac{\pm 4.5 \times 10^{-6} + 4.24 \times 10^{-6}}{2}} = 2.09 \times 10^{-3}$$

$$v = \frac{377}{2.09 \times 10^{-3}} = 180{,}300 \text{ miles per second}$$

If the resistance and the inductance due to the flux within the conductor are neglected, the velocity from equation (40) is

$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2.01 \times 10^{-3} \times 1430 \times 10^{-11}}}$$

$$= 186{,}400 \text{ miles per second, or the velocity of light}$$

PROBLEMS

5. Solve Problem 1, page 412, by the exact method of calculating transmission lines.

6. Points A and B are 150 miles apart and are connected by a parallel-wire line having parameters as follows:

Effective resistance per loop mile at 1000 cycles, 60 ohms
Effective inductance per loop mile at 1000 cycles, 0.0042 henry
Effective capacitance per loop mile at 1000 cycles, 0.00755 μf
Shunted conductance is negligible.

The line is assumed to be terminated at point B with an impedance equal to its surge impedance. Find the voltage, current, and power received at point B when 50 volts at 1000 cycles are impressed at A. (A loop mile consists of one mile of outgoing plus one mile of return conductor.) Use V_A as reference.

7. Calculate by means of the formula the inductance in henrys per mile of No. 0000 wire with an equilateral spacing of 6 feet.

8. Calculate the capacitance per mile between wires and between one wire and neutral or ground for the line in Problem 7.

9. A 3-phase 60-cycle transmission line is 150 miles long and consists of three No. 0000 wires spaced at corners of an equilateral triangle which are 15 feet apart. The line is to deliver 138,000 line-to-line volts and 45,000 kw total power at 0.8 p.f. lagging at the receiver. Calculate the required sending-end voltage, current, power factor, and efficiency of transmission if the nominal T line is used. See bottom of page 431 for resistance of No. 0000 wire. Use $V_{\text{line-to-neutral}}$ as reference.

10. Work Problem 9 if the nominal π line is employed.

11. Work Problem 9 if the Steinmetz three-condenser method of representing the line is used.

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12. Work Problem 9 if the exact method of calculating long lines is employed.

13. Calculate the velocity of propagation of the wave in Problem 12.

14. (*a*) If 138,000 line-to-line volts were maintained at the sending end of the line in Problem 9, what would be the receiver-end voltage with the receiver end open? Employ the exact method of solution. (*b*) What is the magnitude of the direct wave at the receiver? (*c*) of the reflected wave?

15. What is the velocity of propagation of the wave in Problem 6?

16. What is the attenuation in decibels per mile of the transmission line described in Problem 6?

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CHAPTER XIII

ELECTRIC WAVE FILTERS

The frequency characteristics of certain types of networks can be employed to separate waves of different frequencies. The separation may be effected primarily for the purpose of selecting a desired band of frequencies or for the purpose of rejecting an undesired band. Selected bands are called pass or transmission bands, and rejected bands are called stop or attenuation bands. Any network which possesses definite properties of frequency discrimination and which is capable of separating electric waves of different frequencies is called an *electric wave filter* or, simply, a *filter*.

Selective Properties of Circuit Elements and Elementary Circuits. Single reactive circuit elements are sometimes employed to pass or reject frequency bands when only broad discrimination is to be made. Thus *blocking* condensers in many vacuum tube circuits discriminate very satisfactorily between waves of zero frequency (direct current) and high-frequency waves. Inductance coils can be employed to pass direct current and practically eliminate frequencies which are of the order of 1000 kilocycles.

High-Frequency Line Drain. A high-frequency disturbance can be drained from a low-frequency, two-wire line with a condenser arrangement similar to that shown in Fig. 1*a*. The condensers constitute a

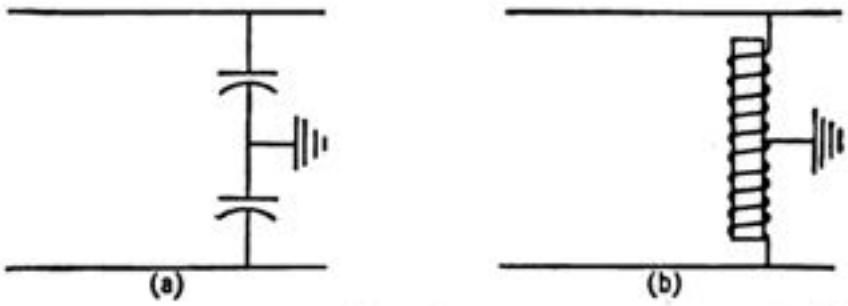

FIG. 1. Devices for draining induced disturbances from two-wire lines.

relatively high impedance to the low-frequency line voltage, both line-to-line and line-to-ground. At the same time a relatively low line-to-ground impedance is presented to the high-frequency variation which in the present case is assumed to be the result of an induced disturbance.

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Low-Frequency Line Drain. A method sometimes used to drain a low-frequency induced disturbance from a two-wire line is shown diagrammatically in Fig. 1*b*. The drain coil is ironclad and offers a relatively high impedance to current which tends to flow from line-to-line. If, however, both lines are raised simultaneously above (or below) ground potential by induction, the currents which flow from the lines to ground magnetize the core in opposite directions. With respect to the induced currents, the two halves of the coil are in series opposition with the result that the impedances offered to these currents to ground are relatively very low. The device can be used to drain charges from telephone lines which are electrostatically induced from neighboring power lines.

Typical Smoothing Network. A very common form of filter is the elementary π-section shown in Fig. 2. This particular type of filter section is widely used to give d-c output from rectified a-c wave forms.

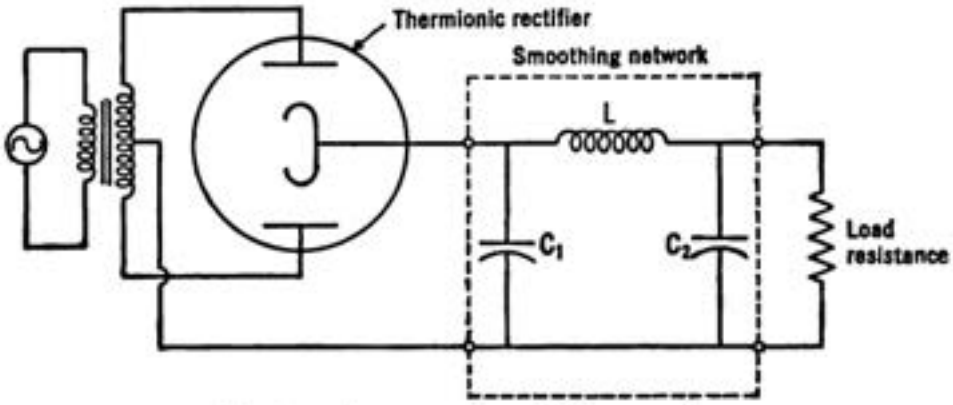

FIG. 2. A commonly used filter section.

The output voltage of the rectifying device, namely, that which appears across the input terminals of the filter section, will take the following general form:

$$v = V_{dc} + V_{m1} \sin (\omega_1 t + \alpha_1) + \text{higher harmonics}$$

where V_{dc} is the average value of the rectified wave and ω_1 is the angular velocity of the lowest-frequency component present in the voltage variation. A typical voltage input variation is shown in Oscillogram 1*a*.

If, for example, both halves of 60-cycle wave are rectified symmetrically, the lowest frequency component in the rectified voltage wave will be that of 120 cycles, in which case $\omega_1 = 754$ radians per second. In unsymmetrical rectification ω_1 is generally equal to the fundamental angular velocity of the alternating variation which is being rectified.

Under ideal conditions the filter section shown in Fig. 2 should pass

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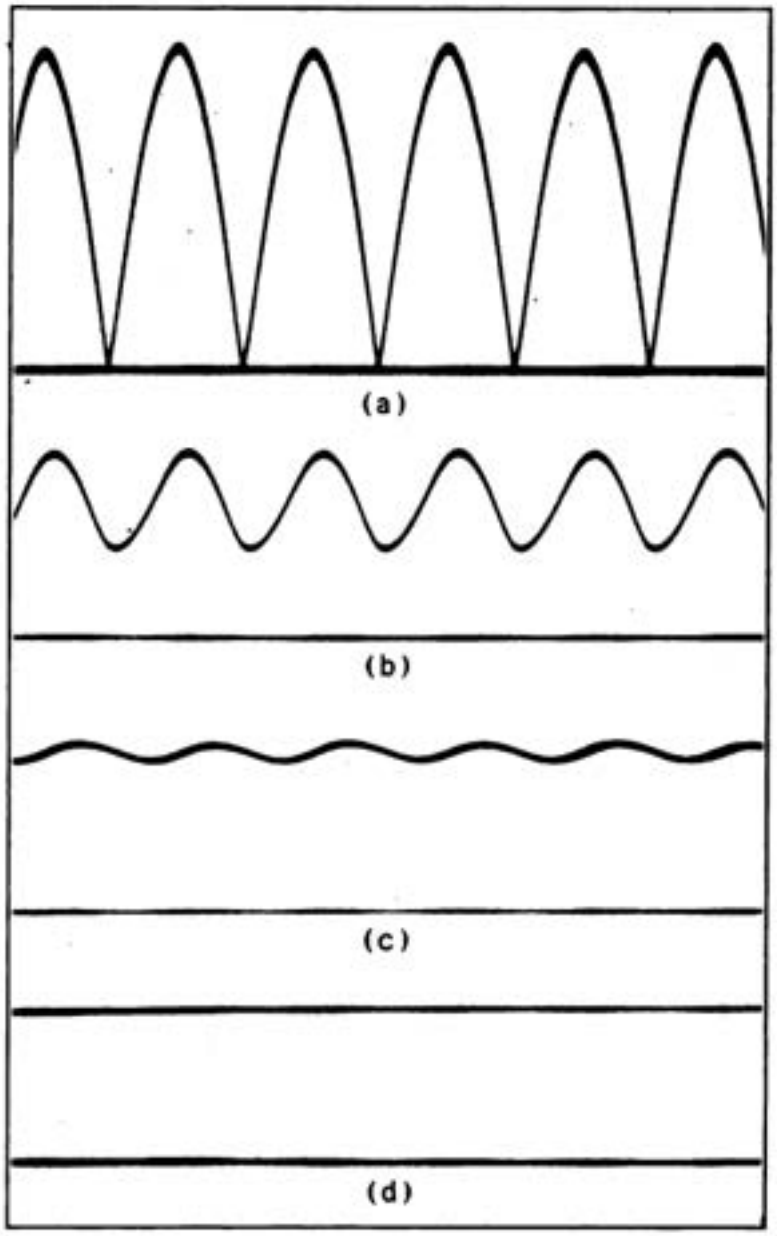
waves of zero frequency with no attenuation and absolutely stop waves which are of other than zero frequency. Obviously, these ideal conditions of operation can only be approached in practice, but the



OSCILLOGRAM 1.

(a) Rectified a-c wave, no filtering.
(b) Rectified a-c wave, choke filtering only.
(c) Rectified a-c wave, choke and input condenser filtering.
(d) Rectified a-c wave, complete π-section filtering. (See Fig. 2.)

difference between ideal operation and actual operation can be made exceedingly small by proper design. See Oscillogram 1*d*.

For full-wave, 60-cycle rectification satisfactory filtering can usually be obtained if C_1 and C_2 of Fig. 2 are about 4 or 5 μf each and L is

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about 30 or 40 henrys. The permissible voltage regulation will, to a large extent, determine the amount of resistance that can be present in the inductance coil in any particular instance. In any case R is very small as compared with $\omega_1 L$. The result is that, when the π-section is energized with a rectified voltage, it presents a relatively low impedance to zero-frequency current. The impedances offered to other than zero-frequency currents are relatively very high.

If, for example, $L = 30$ henrys and ω_1 is 754 radians per second, the series impedance of the filter section to the ω_1 component of current is approximately 22,600 ohms. The series impedances to the higher-frequency components are proportionately greater. The series impedance of the filter section to the d-c component of current may, in a particular case, be only 20 or 30 ohms. Therefore, the inductance coil acting by itself will tend to smooth out the rectified wave as shown in Oscillogram 1*b*.

The input condenser, C_1, is an important member of the filter section, although it is entirely possible to design a smoothing network which does not employ a condenser at the C_1 position shown in Fig. 2. It will be noted that C_1 is placed directly across the output terminals of the rectifying device. It provides a relatively low-impedance path for all a-c components. The anode-cathode impedance of the tube may be 10 or 20 times greater than $1/\omega_1 C_1$, in which case the voltage drop across C_1 is only a small fraction of the total drop due to the a-c components of the rectified voltage. This aids materially in the smoothing process but at the same time increases the actual plate current of the rectifying elements. Filter sections which employ a condenser directly across the terminals of the rectifying device are called condenser input sections.[1]

A complete analysis of the composite circuit shown in Fig. 2 is complicated by the presence of the transformer, tube, and load impedances and will not be undertaken at this time. Actually the smoothing network or ripple filter shown in Fig. 2 is a particular form of low-pass filter, the general theory of which is considered on pages 465–469 of the present chapter.

Image Impedances of Four-Terminal Networks. Most filter sections take the form of a four-terminal network, and as such they possess one pair of input terminals and one pair of output terminals. With this arrangement of terminals, a filter section can be inserted directly into a two-wire line.

General four-terminal network theory is rather elaborate and is not

[1] For details see "Electrical Engineers' Handbook: Electric Communication and Electronics," fourth edition, edited by Pender and McIlwain, pp. 7-106, 7-108, John Wiley & Sons, Inc., 1950.

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considered to be suitable first-course material. There are certain aspects of the subject, however, that are essential to a proper understanding of elementary filter theory. One of these is the concept of image impedances.

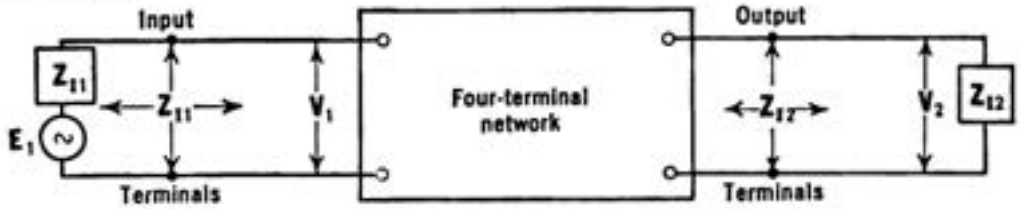

FIG. 3. Four-terminal network terminated on the image impedance basis.

The rectangle shown in Fig. 3 is assumed to be any form of four-terminal network, the internal circuit elements of which may or may not be accessible. It is also assumed that the individual circuit elements are linear. Circuit elements are linear if effects are proportional to causes, for example, if currents are proportional to applied voltages.

The image impedances of a four-terminal network are called $\mathbf{Z}_{I1}$ and $\mathbf{Z}_{I2}$ and are defined in the following manner. (Refer to Fig. 3.) If the impedance across the input terminals (looking into the network) is $\mathbf{Z}_{I1}$ when the output terminals are closed through $\mathbf{Z}_{I2}$, and if the impedance across the output terminals (looking into the network) is $\mathbf{Z}_{I2}$ when the input terminals are closed through $\mathbf{Z}_{I1}$, then $\mathbf{Z}_{I1}$ and $\mathbf{Z}_{I2}$ are called the image impedances of the network. If a four-terminal network is terminated in its image impedances, $\mathbf{Z}_{I1}$ and $\mathbf{Z}_{I2}$, the impedance looking either way from the input terminals is $\mathbf{Z}_{I1}$ and the impedance looking either direction from the output terminals is $\mathbf{Z}_{I2}$. The network is correctly matched when the input impedance is $\mathbf{Z}_{I1}$ and the output impedance is $\mathbf{Z}_{I2}$ and under these conditions the network is said to be terminated on the image basis.

A special case of image impedance termination is employed in elementary filter theory. The assumption is made that $\mathbf{Z}_{I1} = \mathbf{Z}_{I2}$, and this particular value of impedance is called the characteristic impedance of the filter section.

The image impedance at either end of a given network can be determined from the open-circuit and short-circuit impedances. By open-circuit impedance, $\mathbf{Z}_{o\text{-}c}$, is meant the impedance looking into one set of terminals when the other set of terminals is open-circuited. By short-circuit impedance, $\mathbf{Z}_{s\text{-}c}$, is meant the impedance looking into one set of terminals when the other set of terminals is short-circuited. It can be shown that image impedance at either end of a four-terminal network is the geometric mean of the open-circuit and short-circuit impedances.

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Thus in Fig. 3:

$$Z_{I1} = \sqrt{Z_{o\text{-}c1}Z_{s\text{-}c1}} \tag{1}$$

and

$$Z_{I2} = \sqrt{Z_{o\text{-}c2}Z_{s\text{-}c2}} \tag{2}$$

Generalized proofs of the above equations will not be given but it will be shown presently that the relations stated are correct when $Z_{I1} = Z_{I2}$, the condition which is of special importance in elementary filter theory.

Characteristic Impedances of T- and π-Sections. The basic units of elementary filter theory are the symmetrical T- and π-sections shown in Fig. 4. Although both of these sections are essentially three-terminal networks, they are usually inserted into a two-wire line in the same manner as a four-terminal network. Viewed as three-terminal networks, the T-section is a wye-connected set of impedances and the π-section is a delta-connected set of impedances. It should not be supposed that the Z_1 and Z_2 values given in Fig. 4*a* and Fig. 4*b* are, in

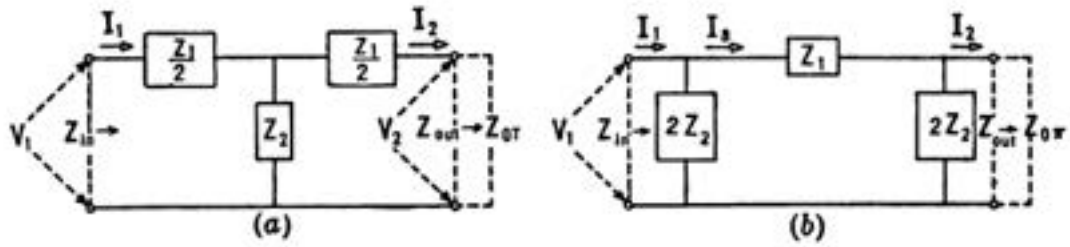

FIG. 4. Symmetrical T- and π-sections.

general, equivalent wye and delta values. The circuit elements are usually labeled as indicated in Fig. 4 in order to make the algebraic expressions for several of the filtering characteristics the same for both the T- and π-sections.

The series impedance of a symmetrical T-section is composed of two similar units, each of which is labeled $Z_1/2$ in Fig. 4*a*. The impedance labeled Z_2 in Fig. 4*a* is called the shunt impedance of the T-section. The shunt impedance of a symmetrical π-section is composed of two equal branches, each labeled $2Z_2$ in Fig. 4*b*, and these shunt branches are located on either side of the series impedance Z_1. If the series and shunt impedances are designated in accordance with Fig. 4, ladder structures formed by the cascade arrangement of successive sections are generally similar. (See Fig. 10 and Fig. 11.)

If the output terminals of the T-section shown in Fig. 4*a* are closed through an impedance Z_{oT}, the impedance across the input terminals

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(looking into the network) is:

$$Z_{in} = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2} + Z_{oT}\right)}{Z_2 + \frac{Z_1}{2} + Z_{oT}} \tag{3}$$

In order for Z_{in} to equal Z_{oT}, it follows that:

$$Z_{oT} = \frac{Z_1}{2} + \frac{\frac{Z_1 Z_2}{2} + Z_2 Z_{oT}}{\frac{Z_1}{2} + Z_2 + Z_{oT}} \tag{4}$$

The above equation may be solved for Z_{oT} and the result stated in terms of Z_1 and Z_2. Thus it can be shown that *the characteristic impedance of the* T-*section* is:

$$Z_{oT} = \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} = \sqrt{Z_1 Z_2\left(1 + \frac{Z_1}{4Z_2}\right)} \tag{5}$$

If the output terminals of the π-section shown in Fig. 4*b* are closed through an impedance $Z_{o\pi}$, the impedance across the input terminals (looking into the network) is:

$$Z_{in} = \frac{2Z_2\left(Z_1 + \frac{2Z_2 Z_{o\pi}}{2Z_2 + Z_{o\pi}}\right)}{2Z_2 + Z_1 + \frac{2Z_2 Z_{o\pi}}{2Z_2 + Z_{o\pi}}} \tag{6}$$

In order to determine the conditions under which Z_{in} is equal to $Z_{o\pi}$ it is simply necessary to set $Z_{in} = Z_{o\pi}$ in the above equation and solve the resulting equation for $Z_{o\pi}$. After Z_{in} has been set equal to $Z_{o\pi}$ and all fractions cleared, it will be found that:

$$Z_{o\pi}^2 (Z_1 + 4Z_2) = 4Z_1 Z_2^2$$

From which *the characteristic impedance of the* π-*section* is

$$Z_{o\pi} = \sqrt{\frac{4Z_1 Z_2^2}{Z_1 + 4Z_2}} = \sqrt{\frac{Z_1 Z_2}{\left(1 + \frac{Z_1}{4Z_2}\right)}} \tag{7}$$

Equations (5) and (7) are important relations in filter theory because they define the characteristic impedances Z_{oT} and $Z_{o\pi}$ in terms of the series and shunt elements out of which the T- and π-sections are com-

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posed. If a filter section is terminated in its characteristic impedance, the impedance across the input terminals (looking into the network) is the same as the receiving-end impedance. (The importance of designing filter sections to have particular characteristic impedances will become more evident after composite filter sections are studied.) It should be noted that a given filter section terminated at both ends in its characteristic impedance is terminated on the image basis and that in this particular case $\mathbf{Z}_{I1}$ and $\mathbf{Z}_{I2}$ are equal. (See Fig. 3.) Reference to equations (5) and (7) will show that:

$$\mathbf{Z}_{oT}\mathbf{Z}_{o\pi} = \mathbf{Z}_1\mathbf{Z}_2 \tag{8}$$

and

$$\mathbf{Z}_{o\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{oT}} \tag{9}$$

Equations (8) and (9) define a rather important relationship that exists between the characteristic impedances of T- and π-sections, the $\mathbf{Z}_1$'s and $\mathbf{Z}_2$'s of which are equal.

Filter theory is based upon $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_{oT}$, and $\mathbf{Z}_{o\pi}$ to such an extent that the physical significance of each of these four impedances should be clearly understood. The reader who is unfamiliar with filter theory nomenclature should at this stage review the definitions which have been given for $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_{oT}$, and $\mathbf{Z}_{o\pi}$. [See Fig. 4 and equations (5) and (7).]

Example 1. In Fig. 4*a*, let each $\mathbf{Z}_1/2$ take the form of an inductance coil, the inductance of which is 0.047 henry and the resistance of which is 1 ohm. The shunt arm, namely, $\mathbf{Z}_2$, is to take the form of a 300-μf condenser. (*Note:* This is an unconventional set of parameters for this type of filter section but since some of the experimental results which follow are based upon these particular values they will be used here to illustrate the calculation of $\mathbf{Z}_{oT}$.)

The method of calculating $\mathbf{Z}_{oT}$ at 50 cycles is as follows:

$$\frac{\mathbf{Z}_1}{2} = \frac{R_1}{2} + j\omega\frac{L_1}{2} = 1 + j14.77 = 14.8\,\underline{/86.1^\circ}\text{ ohms}$$

$$\mathbf{Z}_1 = 29.6\,\underline{/86.1^\circ}\text{ ohms} \quad \text{(Full series arm impedance.)}$$

$$\mathbf{Z}_2 = 0 - j\frac{1}{\omega C_2} = 0 - j10.61 = 10.61\,\underline{/-90^\circ}\text{ ohms}$$

$$\mathbf{Z}_{oT} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \frac{\mathbf{Z}_1^2}{4}}$$

$$= \sqrt{(29.6\,\underline{/86.1^\circ})(10.61\,\underline{/-90^\circ}) + \frac{(29.6\,\underline{/86.1^\circ})^2}{4}}$$

$$= 9.83\,\underline{/2.5^\circ} = 9.81 + j0.43\text{ ohms}$$

The physical significance of the above value of $\mathbf{Z}_{oT}$ is that, if an impedance of $9.83\,\underline{/2.5^\circ}$ ohms is placed across the output terminals of this symmetrical T-section, *the impedance looking into the input terminals is also* $9.83\,\underline{/2.5^\circ}$ *ohms.*

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Problem 1. Neglect the resistances of the two inductance coils that form the series impedance of the filter section in the illustrative example given above and find $\mathbf{Z}_{oT}$ at 50 cycles and at 100 cycles. (It may be of interest to know that this symmetrical T-section forms a low-pass filter that passes all frequencies up to 60 cycles and attenuates those above 60 cycles.)

Ans.: At 50 cycles, $\mathbf{Z}_{oT} = 9.76 \underline{/0^\circ}$ ohms.
At 100 cycles, $\mathbf{Z}_{oT} = 23.65 \underline{/90^\circ}$ ohms.

Problem 2. The series impedance, $\mathbf{Z}_1$, of a symmetrical π-section (like that shown in Fig. 4*b*) consists of a 0.02-henry inductance coil, the resistance of which is assumed to be negligibly small. Each of the shunt arms, namely, $2\mathbf{Z}_2$, is composed of a 2.0-μf condenser. (This symmetrical π-section forms a low-pass filter which passes all frequencies below 900 cycles without attenuation as will be shown later.)

Find the characteristic impedance of this section at 200 cycles and at 2000 cycles. Use equation (7) and recognize that

$$\mathbf{Z}_1 = 0.02\omega \underline{/90^\circ} \quad \text{and} \quad \mathbf{Z}_2 = \frac{10^6}{4\omega} \underline{/-90^\circ} \text{ ohms}$$

Ans.: At 200 cycles, $\mathbf{Z}_{o\pi} = 71.8 \underline{/0^\circ}$ ohms.
At 2000 cycles, $\mathbf{Z}_{o\pi} = 48 \underline{/-90^\circ}$ ohms.

Characteristic Impedance as a Function of Open-Circuit and Short-Circuit Impedances. Reference to Fig. 5*a* will show that the open-circuit impedance of a T-section (looking into the section) is:

$$\mathbf{Z}_{o\text{-}c} = \frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 \tag{10}$$

FIG. 5. $\mathbf{Z}_{oc}$ and $\mathbf{Z}_{sc}$ of a symmetrical T-section.

When the output terminals are short-circuited as shown in Fig. 5*b* the impedance of the T-section (looking into the section) is:

$$\mathbf{Z}_{s\text{-}c} = \frac{\mathbf{Z}_1}{2} + \frac{\frac{\mathbf{Z}_1}{2}\mathbf{Z}_2}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2} = \frac{\frac{\mathbf{Z}_1^2}{4} + \mathbf{Z}_1\mathbf{Z}_2}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2} \tag{11}$$

The geometric mean of $\mathbf{Z}_{o\text{-}c}$ and $\mathbf{Z}_{s\text{-}c}$ is:

$$\sqrt{\mathbf{Z}_{o\text{-}c}\mathbf{Z}_{s\text{-}c}} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \frac{\mathbf{Z}_1^2}{4}} \tag{12}$$

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It has already been shown that

$$Z_{oT} = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} \quad \text{[See equation (5).]}$$

Therefore,

$$Z_{oT} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} \tag{13}$$

The fact that Z_{oT} is equivalent to the geometric mean of $Z_{o\text{-}c}$ and $Z_{s\text{-}c}$ provides the basis for a simple experimental method of determining the characteristic impedance of a given section.

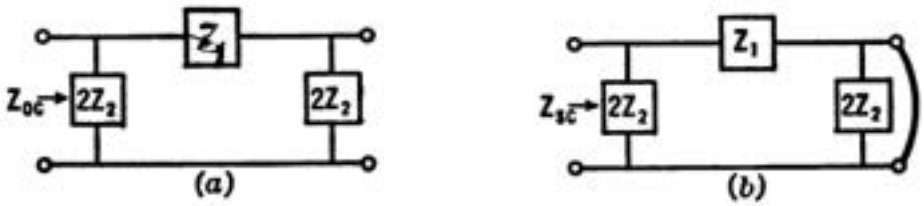

FIG. 6. Z_{oc} and Z_{sc} of a symmetrical π-section.

Reference to Fig. 6*a* will show that the open-circuit impedance of a symmetrical π-section (looking into the section) is:

$$Z_{o\text{-}c} = \frac{2Z_2(Z_1 + 2Z_2)}{Z_1 + 4Z_2} \tag{14}$$

If the output terminals of the π-section are short-circuited as shown in Fig. 6*b*, the input impedance is:

$$Z_{s\text{-}c} = \frac{2Z_2Z_1}{Z_1 + 2Z_2} \tag{15}$$

$$\sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} = \sqrt{\frac{4Z_1Z_2^2}{Z_1 + 4Z_2}} \tag{16}$$

Comparison of the above relation with equation (7) will show that:

$$Z_{o\pi} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} \tag{17}$$

Equations (13) and (17) indicate that the characteristic impedance of either the T- or π-section is equal to the geometric mean of their respective open- and short-circuit impedances. It should be evident that the symbols $Z_{o\text{-}c}$ and $Z_{s\text{-}c}$ in equations (13) and (17) refer to open- and short-circuit impedances of the particular section that is under investigation. In general $Z_{oT} \neq Z_{o\pi}$.

Problem 3. Referring to Fig. 7 find (*a*) $Z_{o\text{-}c}$, (*b*) $Z_{s\text{-}c}$, and (*c*) Z_{oT} at 200 cycles.
Ans.: (*a*) 186.2 $\angle -90°$, (*b*) 26.0 $\angle 90°$, and (*c*) 69.5 $\angle 0°$ ohms.

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Problem 4. Referring to Fig. 8, find (*a*) $\mathbf{Z}_{o-c}$, (*b*) $\mathbf{Z}_{s-c}$, and (*c*) $\mathbf{Z}_{o\pi}$ at 200 cycles.
Ans.: (*a*) 192.5 $\underline{/-90°}$, (*b*) 26.8 $\underline{/90°}$, and (*c*) 71.8 $\underline{/0°}$ ohms.

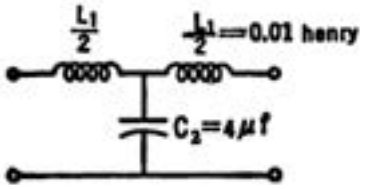

FIG. 7. A particular symmetrical T-section for use with Problem 3.

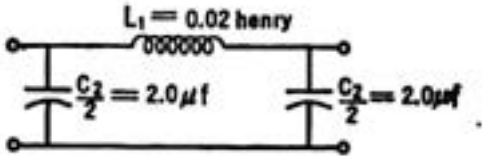

FIG. 8. A particular symmetrical π-section for use in connection with Problem 4.

Physical Operation of Symmetrical T- and π-Sections. The T- and π-sections shown in Fig. 4 possess some remarkable properties when their output terminals are connected to the characteristic impedances $\mathbf{Z}_{oT}$ and $\mathbf{Z}_{o\pi}$ respectively. Before considering the filtering properties of these sections, some of the basic relationships that follow directly from elementary circuit theory will be established.

The conditions imposed on equations (4) and (6), page 442, make $\mathbf{Z}_{in} = \mathbf{Z}_{out}$ for either type of section. Hence $\mathbf{I}_1 = \mathbf{V}_1/\mathbf{Z}_o$ and $\mathbf{I}_2 = \mathbf{V}_2/\mathbf{Z}_o$, where $\mathbf{Z}_o$ symbolizes the characteristic impedance of the particular type of section considered. It follows directly that

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{V}_1}{\mathbf{V}_2} \quad \text{and} \quad \frac{W_1}{W_2} = \frac{V_1 I_1 \cos\theta}{V_2 I_2 \cos\theta} = \frac{I_1^2}{I_2^2} \tag{18}$$

where the subscripts 1 refer to input quantities and the subscripts 2 refer to output quantities. Since the impedance looking into the input terminals is the same as the terminating impedance, the angle between $\mathbf{V}_1$ and $\mathbf{I}_1$ is equal to the angle between $\mathbf{V}_2$ and $\mathbf{I}_2$. This angle is symbolized as θ in equation (18) and is equal to $\tan^{-1}(X_o/R_o)$, where X_o and R_o are the reactive and resistive components of the characteristic impedance $\mathbf{Z}_o$. The basic relationships contained in equation (18) are illustrated photographically for a particular T-section in Oscillogram 2, page 447. These relationships will be used later in defining the attenuation of filter sections.

The next basic relationship to be established is that the ratio of input current to output current, namely, $\mathbf{I}_1/\mathbf{I}_2$, is completely defined by the series arm impedance ($\mathbf{Z}_1$) and the shunt arm impedance ($\mathbf{Z}_2$) out of which the symmetrical T- or π-section is composed. For the T-section shown in Fig. 4*a* it is plain from Kirchhoff's emf law that

$$\frac{\mathbf{Z}_1}{2}\mathbf{I}_1 + \frac{\mathbf{Z}_1}{2}\mathbf{I}_2 + \mathbf{Z}_{oT}\mathbf{I}_2 = \mathbf{V}_1 = \mathbf{Z}_{oT}\mathbf{I}_1 \tag{19}$$

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Whence

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{Z}_{oT} + \dfrac{\mathbf{Z}_1}{2}}{\mathbf{Z}_{oT} - \dfrac{\mathbf{Z}_1}{2}} \quad \text{(for T-sections)} \tag{20}$$

Referring to Fig. 4*b* for the π-section and remembering that $\mathbf{V}_1 = \mathbf{I}_1\mathbf{Z}_{o\pi}$ and that $\mathbf{V}_2 = \mathbf{I}_2\mathbf{Z}_{o\pi}$, the current $\mathbf{I}_{series}$ in the series arm is:

$$\mathbf{I}_{series} = \mathbf{I}_1 - \frac{\mathbf{I}_1\mathbf{Z}_{o\pi}}{2\mathbf{Z}_2} = \mathbf{I}_2 + \frac{\mathbf{I}_2\mathbf{Z}_{o\pi}}{2\mathbf{Z}_2} \tag{21}$$

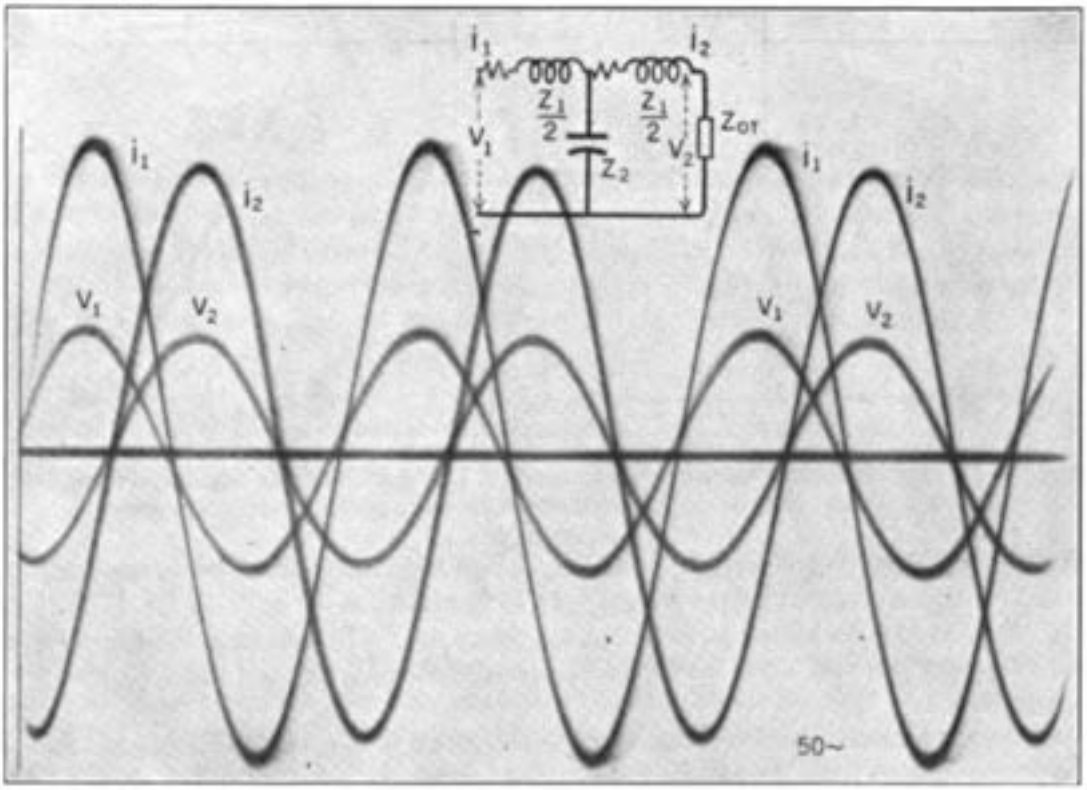
from which

$$\mathbf{I}_1\,\frac{(2\mathbf{Z}_2 - \mathbf{Z}_{o\pi})}{2\mathbf{Z}_2} = \mathbf{I}_2\,\frac{(2\mathbf{Z}_2 + \mathbf{Z}_{o\pi})}{2\mathbf{Z}_2} \tag{22}$$

and

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{2\mathbf{Z}_2 + \mathbf{Z}_{o\pi}}{2\mathbf{Z}_2 - \mathbf{Z}_{o\pi}} \tag{23}$$

Reference to equations (20) and (23) above and to equations (5) and (7), page 442, will show that the ratio $\mathbf{I}_1/\mathbf{I}_2$ is defined wholly in terms of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ for either T- or π-sections. It will be shown later that the



Oscillogram 2. Illustrating attenuation and phase shift in a symmetrical T-section. v_1 and i_1 are input voltage and current respectively. v_2 and i_2 are output voltage and current respectively.

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right members of equations (20) and (23) are identically equal when written wholly in terms of $\mathbf{Z}_1$ and $\mathbf{Z}_2$. For the present, equation (20) will be used to define the ratio $\mathbf{I}_1/\mathbf{I}_2$ in T-sections and equation (23) will be used to define this ratio in π-sections.

Example 2. Refer to the symmetrical T-section shown in Fig. 9*a*. Let it be required to evaluate the ratio $\mathbf{I}_1/\mathbf{I}_2$ at $f = 50$ cycles. Since this is the same T-section as described in example 1, page 443, the results of example 1 may be used here to define $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_{oT}$.

$$\frac{\mathbf{Z}_1}{2} = (1 + j14.77),\ \mathbf{Z}_2 = (0 - j10.61), \quad \text{and} \quad \mathbf{Z}_{oT} = (9.81 + j0.43) \text{ ohms}$$

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{Z}_{oT} + \dfrac{\mathbf{Z}_1}{2}}{\mathbf{Z}_{oT} - \dfrac{\mathbf{Z}_1}{2}} = \frac{(9.81 + j0.43) + (1 + j14.77)}{(9.81 + j0.43) - (1 + j14.77)}$$

$$= \frac{(10.81 + j15.20)}{(8.81 - j14.34)} = \frac{18.7\ \underline{/54.6^\circ}}{16.8\ \underline{/-58.4^\circ}} = 1.11\ \underline{/113^\circ}$$

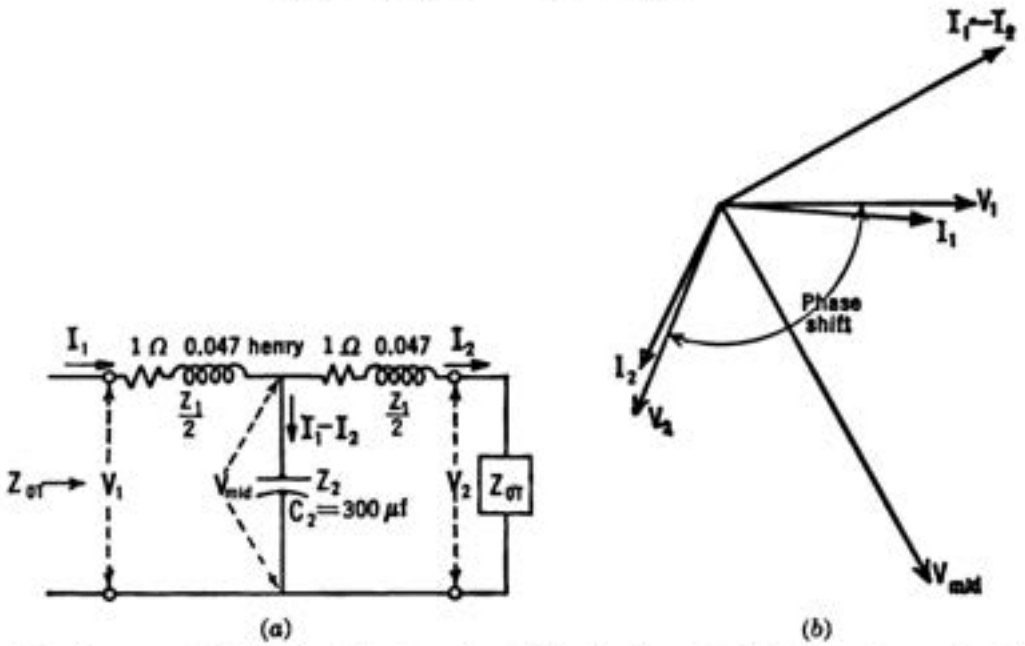

FIG. 9. A symmetrical T-section terminated in its characteristic impedance, together with a vector diagram of the currents and voltages in a particular case.

The physical significance of the above complex number is that the magnitude of $\mathbf{I}_1$ is 1.11 times as great as the magnitude of $\mathbf{I}_2$ and that $\mathbf{I}_1$ *leads* $\mathbf{I}_2$ by 113°. (See Fig. 9*b*.) It will be shown presently that the ratio I_1/I_2 defines the *attenuation* of the filter section and that the associated angle of $\mathbf{I}_1/\mathbf{I}_2$ defines the *phase shift* of the section.

A worthwhile exercise for the student at this stage is that of correlating the results given above with those determined by elementary circuit analysis. Let $\mathbf{V}_1$ of Fig. 9*a* $= 100\ \underline{/0^\circ}$ volts and solve for $\mathbf{I}_1$ and $\mathbf{I}_2$ by ordinary methods. The results are illustrated in Fig. 9*b* and in Oscillogram 2 which is a photographic record of v_1, i_1, v_2, and i_2 for the particular T-section shown in Fig. 9*a*.

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Example 3. Let it be required to find the ratio I_1/I_2 in Fig. 9a if the resistances of the inductance coils are neglected, assuming that the frequency of the supply voltage is 50 cycles.

$$\frac{Z_1}{2} = (0 + j14.77), \quad Z_2 = (0 - j10.61), \quad \frac{Z_1^2}{4} = (j14.77)^2$$

$$Z_{oT} = \sqrt{Z_1Z_2 + (Z_1/2)^2} = \sqrt{(j29.54)(-j10.61) + (j14.77)^2}$$
$$= \sqrt{313.4 - 218.2} = \sqrt{95.2} = 9.76\underline{/0^\circ} \text{ ohms}$$

Employing equation (20):

$$\frac{I_1}{I_2} = \frac{9.76 + j14.77}{9.76 - j14.77} = \frac{17.7\underline{/56.5^\circ}}{17.7\underline{/-56.5^\circ}} = 1\underline{/113^\circ}$$

Thus the output current I_2 is shown to be as great in magnitude as the input current I_1. This condition exists generally in symmetrical T- and π-sections when the resistances are negligibly small provided the characteristic impedance for the frequency considered is a pure ohmic resistance.

Example 4. Let it be required to find the characteristic impedance and the current ratio I_1/I_2 in Fig. 9a if the frequency of the supply is 100 cycles and if the resistances of the inductance coils are neglected. Under these conditions:

$$\frac{Z_1}{2} = (0 + j29.54), \quad Z_2 = (0 - j5.305), \quad \frac{Z_1^2}{4} = (j29.54)^2$$

$$Z_{oT} = \sqrt{(j59.08)(-j5.305) + (j29.54)^2}$$
$$= \sqrt{(313\underline{/0^\circ}) + (873\underline{/+180^\circ})}$$
$$= \sqrt{560\underline{/+180^\circ}} = 23.66\underline{/90^\circ} \text{ ohms}$$

The characteristic impedance of the filter section has changed from a pure resistance (of 9.76 ohms) to a pure inductive reactance of 23.66 ohms as a result of changing the frequency from 50 cycles to 100 cycles. *Note:* The values of L_1 and C_2 used in Fig. 9a make this section a low-pass filter section which starts to attenuate at 60 cycles, as will be shown later. See equation (55), page 466. At 100 cycles:

$$\frac{I_1}{I_2} = \frac{Z_{oT} + \frac{Z_1}{2}}{Z_{oT} - \frac{Z_1}{2}} = \frac{(23.66\underline{/90^\circ}) + (29.54\underline{/90^\circ})}{(23.66\underline{/90^\circ}) - (29.54\underline{/90^\circ})}$$

$$= \frac{53.2\underline{/90^\circ}}{5.88\underline{/-90^\circ}} = 9.04\underline{/+180^\circ}$$

It will be observed that, at 100 cycles, I_1 is 9.04 times as great as I_2 which indicates that marked attenuation is taking place. It will also be observed that the phase shift is 180°, a condition that always obtains in a resistanceless filter section which is operating in the attenuation band and which is terminated in its characteristic impedance.

The importance of the ratio I_1/I_2 has been emphasized in the foregoing

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examples because the physical operation of a filter section is concisely defined by this ratio.

Problem 5. Find the ratio I_1/I_2 of the symmetrical π-section shown in Fig. 8 page 446, at 200 cycles and at 2000 cycles. Neglect the resistance of the inductance coil and recognize that $Z_1 = (0 + j0.02\omega)$ is the full series arm and that $Z_2 = \left(0 - j\frac{10^6}{4\omega}\right)$ is the combined shunt arm since the total series inductance (L_1) is 0.02 henry and the combined shunt capacitance (C_2) is 4 μf. (See Fig. 4*b* and Fig. 8.) Note also that $2Z_2 = \left(0 - j\frac{10^6}{2\omega}\right)$ ohms.

Ans.: At 200 cycles $I_1/I_2 = 1\underline{/+20.5°}$.
At 2000 cycles $I_1/I_2 = 10.6\underline{/-180°}$.

Problem 6. Find the current ratio I_1/I_2 of the symmetrical T-section shown in Fig. 7, page 446, at 200 cycles and at 2000 cycles. Neglect the resistances of the inductance coils.

Ans.: At 200 cycles $I_1/I_2 = 1\underline{/20.5°}$.
At 2000 cycles $I_1/I_2 = 10.6\underline{/+180°}$.

Transmission Constant of a Filter Section. A transmission constant which applies to a generator feeding a resistance load has been defined in equation (80), page 91. It will be remembered that the reference used in that case was selected with a view toward including the effects of a possible mismatch between the resistance of the generator and the resistance of the load. Another transmission constant which applies to long lines was used in Chapter XII. In this case it was called the *propagation* constant, the term usually employed for the transmission constant of long lines.

Where a filter section or other four-terminal network is terminated on an image impedance basis as shown in Fig. 3, the impedance match between the generator and load is already effected and the definition of the transmission constant is somewhat different from that given in equation (80), page 91. Assuming that the filter section is terminated on an image impedance basis and that we wish to specify a measure of the attenuation and phase shift of the filter itself, we employ the following definition of the transmission constant:

$$\gamma = \alpha + j\beta = \log_\epsilon \frac{Z_T'}{Z_{I1}} = \log_\epsilon \frac{V_1/I_2}{V_1/I_1} = \log_\epsilon \frac{I_1}{I_2} \tag{24}$$

where Z_T' is the transfer impedance from the input terminals of the filter section to the output terminals, namely, V_1/I_2

Z_{I1} is the image impedance seen looking to the right of the input terminals, namely, V_1/I_1

α is called the attenuation of the filter section

β is called the phase-shift constant of the filter section.

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Actually the α and β defined in equation (24) apply to any four-terminal network which is terminated on an image impedance basis as shown in Fig. 3. As such they apply directly to a filter section which is terminated in its characteristic impedance, since characteristic impedance termination is but a special case of image impedance termination where $\mathbf{Z}_{I1} = \mathbf{Z}_{I2}$.

The attenuation, α, is a measure of the ratio of the power input to the power output of a filter section which is terminated in its characteristic impedance, since under these conditions the real part of equation (24) may be written as:

$$\alpha = \log_\epsilon \frac{\sqrt{I_1^2 R_0}}{\sqrt{I_2^2 R_0}} = \frac{1}{2} \log_\epsilon \frac{I_1^2 R_0}{I_2^2 R_0} = \frac{1}{2} \log_\epsilon \frac{W_1}{W_2} \tag{25}$$

where R_0 is the resistive component of $\mathbf{Z}_0$
W_1 is the power entering the input terminals
W_2 is the power leaving the output terminals.

From equation (24) it is plain that

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \epsilon^{(\alpha + j\beta)} = \epsilon^\alpha \epsilon^{j\beta} = K \underline{/\beta} \tag{26}$$

where $K = \epsilon^\alpha = I_1 / I_2$
β = angle of lead of $\mathbf{I}_1$ with respect to $\mathbf{I}_2$.

As applied to a series or cascade arrangement of filter sections like those shown in Fig. 10, page 453:

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{I}_2}{\mathbf{I}_3} = \frac{\mathbf{I}_3}{\mathbf{I}_4} \tag{27}$$

and the transmission constant (together with the attenuation and phase-shift) may be reckoned on a per section (or $\mathbf{I}_1 / \mathbf{I}_2$) basis or on a combined basis of $\mathbf{I}_1 / \mathbf{I}_4$, since both arrangements are presumably terminated on a characteristic impedance basis.

Units of Attenuation or Transmission Loss. Filter section attenuation is usually expressed in either nepers or decibels. (See pages 91–92.) These units of transmission loss are both defined on a logarithmic basis, since their greatest field of application is in the transmission of sound, the loudness of which is a logarithmic function of the sound energy.

The Neper. The general definition of attenuation expressed in nepers is:

$$(\text{Attenuation in nepers}) = \tfrac{1}{2} \log_\epsilon \frac{W_{(\text{general})}}{W_{(\text{reference})}} \tag{28}$$

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or

$$(\text{Attenuation in nepers}) = 1.151 \log_{10} \frac{W_{(\text{general})}}{W_{(\text{reference})}} \tag{29}$$

where $W_{(\text{general})}$ is any particular power level which might be under discussion

$W_{(\text{reference})}$ is the power level employed as reference from which $W_{(\text{general})}$ is to be measured.

Reference to equation (18) or to equation (25) will show that, for a filter section which is terminated in its characteristic impedance, the output power W_2 is employed as the reference power level and

$$(\text{Attenuation in nepers}) = \tfrac{1}{2} \log_\epsilon \frac{W_1}{W_2} = \tfrac{1}{2} \log_\epsilon \frac{I_1^2 R_0}{I_2^2 R_0} = \log_\epsilon \epsilon^\alpha = \alpha \tag{30}$$

If the filter section is not terminated in its characteristic impedance, equation (28) is employed and W_1 is used for $W_{(\text{general})}$ and W_2 is used for $W_{(\text{reference})}$.

The Decibel. The decibel is an arbitrarily defined unit of transmission loss (or gain) which has come into general use since about 1925.[2] The customary abbreviation is db. The general definition of attenuation expressed in decibels is

$$(\text{Attenuation in decibels}) = 10 \log_{10} \frac{W_{(\text{general})}}{W_{(\text{reference})}} \tag{31}$$

where $W_{(\text{general})}$ and $W_{(\text{reference})}$ have the same meanings as employed in connection with equation (28).

If the filter section is terminated on a characteristic impedance basis, reference to equation (18) or to equation (25) will show that

$$\begin{aligned}(\text{Attenuation in decibels}) &= 10 \log_{10} \left[\frac{I_1}{I_2}\right]^2 = 10 \log_{10} \epsilon^{2\alpha} \\ &= 20\alpha \log_{10} \epsilon = 8.686\alpha \end{aligned} \tag{32}$$

Comparison of equations (30) and (32) will show that the decibel is a transmission unit which is 1/8.686 times as large as the neper (or napier). In practice the decibel is used almost exclusively in the United States. Because of its rationality, the neper is widely used in theoretical derivations.

It should be noted that transmission loss (or attenuation) units define power ratios and under special conditions define current and voltage

[2] Originally the decibel was called the "transmission unit" (abbreviated TU). See "The Transmission Unit and Telephone Transmission Reference Systems," by W. H. Martin, *Bell System Technical Journal*, Vol. 3, p. 400.

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ratios. These units do not specify the actual loss (or gain) in either watts, amperes, or volts. If, for example, it is known that the ratio of power input to power output in a particular case is 3, the transmission loss or attenuation is:

$$\tfrac{1}{2}\log_e 3 = 0.55 \text{ neper} \quad \text{or} \quad 10\log_{10} 3 = 4.77 \text{ decibels}$$

If the current ratio is 3 and the input and output impedances are equal, the transmission loss is:

$$\tfrac{1}{2}\log_e 3^2 = 1.1 \text{ nepers} \quad \text{or} \quad 10\log_{10} 3^2 = 9.54 \text{ decibels}$$

The actual values of power or current are not specified in the statements given above, only logarithmic functions of the ratios.

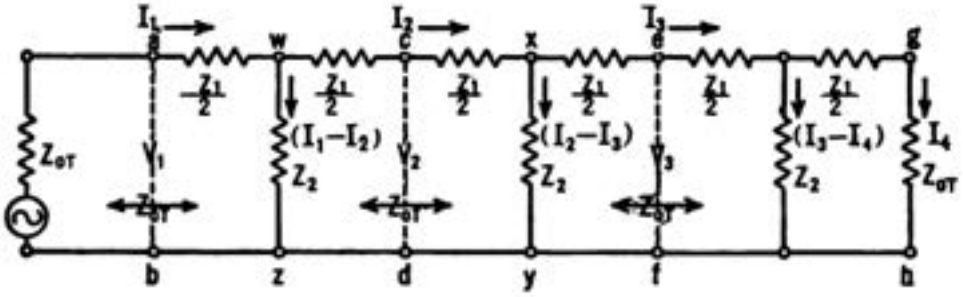

FIG. 10. Three symmetrical T-sections terminated on a characteristic impedance basis.

Example 5. If the vector current ratio per section of each of the three T-sections shown in Fig. 10 is $3\underline{/30^\circ}$ or $3\underline{/\pi/6 \text{ radian}}$:

$$\frac{I_1}{I_2} = \frac{I_2}{I_3} = \frac{I_3}{I_4} = e^{\gamma_1} = e^{\alpha_1}e^{j\beta_1} = 3\underline{/30^\circ}$$

from which

$$e^{\alpha_1} = 3 \quad \text{or} \quad \alpha_1 = \log_e 3 = 1.1 \text{ neper per section}$$
$$\beta_1 = 30^\circ \quad \text{or} \quad \pi/6 \text{ radian, phase shift of } I_2 \text{ behind } I_1$$

On a three-section basis:

$$\frac{I_1}{I_4} = e^{\gamma_3} = e^{\alpha_3}e^{j\beta_3} = 27\underline{/90^\circ}$$

From which the attenuation and phase shift of the three sections may be calculated as

$$e^{\alpha_3} = 27 \quad \text{or} \quad \alpha_3 = \log_e 27 = 3.3 \text{ nepers}$$
$$= 28.6 \text{ decibels}$$

$\beta_3 = 90^\circ$ or $\pi/2$ radians, phase shift of I_4 behind I_1.

Problem 7. The current ratio in a particular filter section is known to be $1.11\underline{/113^\circ}$ as in example 2, page 448. If the section is terminated in its characteristic impedance, find the attenuation in nepers and in decibels.

Ans.: 0.1043 neper, 0.905 decibel.

Problem 8. Calculate the attenuation in decibels and in nepers for the various power and current ratios indicated below. In the case of the current ratios, it is

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assumed that the filter sections to which they apply are terminated on a characteristic impedance basis. The few calculated values that appear in the table may be used as guide.

W_1/W_2	db	nepers	I_1/I_2	db	nepers
1	0	0	1	0	0
10	10	1.15	10	20	2.3
100			100		
1,000			1,000		
5,000			10,000		

With respect to a specified reference power level, any particular circuit power may be measured in plus or minus decibels, depending on whether the circuit power is greater or less than the reference power level. Several reference power levels have been used in sound engineering, namely, 6 milliwatts in telephone circuits, 12.5 milliwatts in public address systems, and a relatively new reference level which is designed to be generally applicable and which is specified as "1 milliwatt in 600 ohms." Thus, 6 milliwatts might be reckoned as $10 \log_{10} (6/1) = +7.78$ db with respect to a 1-milliwatt reference or as $10 \log_{10} (6/12.5) = -3.19$ db with respect to a 12.5-milliwatt reference.

General Considerations. Elementary filter theory concerns itself with uniform ladder structures which are composed of either conventional T- or π-sections. With the definitions which have been given to $\mathbf{Z}_1$ and $\mathbf{Z}_2$ in T- and π-sections, the ladder structures formed by cascade arrangements of these sections are equivalent except for their terminations.

Figure 10 illustrates a ladder structure composed of symmetrical T-sections which is midseries terminated. A ladder structure is said to be midseries terminated when it is terminated at the midpoint of a series arm such as wx. It will be noted that g is the midpoint of such a series arm. Under ideal conditions the structure is terminated at both sending and receiving ends in impedances which are equal to the characteristic impedance of a T-section, namely, $\mathbf{Z}_{oT}$. (Methods will be considered later whereby generating devices of one impedance can be properly matched to a load device of a different impedance.)

Figure 11 illustrates a ladder structure composed of symmetrical π-sections. Comparison of Fig. 10 and Fig. 11 will show the general circuit equivalence of T- and π-sections except for the terminations. The arrangement shown in Fig. 11 may be thought of as symmetrical T-sections which are terminated at planes such that the shunt arm $\mathbf{Z}_2$ is bisected longitudinally, leaving $2\mathbf{Z}_2$ directly across the input and output terminals. This form of termination is called midshunt ter-

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mination. It has a certain practical significance which will be discussed in a later article.

A *low-pass* filter is a network designed to pass currents of all frequencies below a critical or cut-off frequency and materially to reduce the amplitude of currents of all frequencies above this critical frequency. Under certain ideal conditions which will be considered, a low-pass filter will pass all frequencies from zero up to a predetermined number of cycles with theoretical zero attenuation. The frequency at which the theoretical attenuation takes on a finite value is called the cut-off frequency.

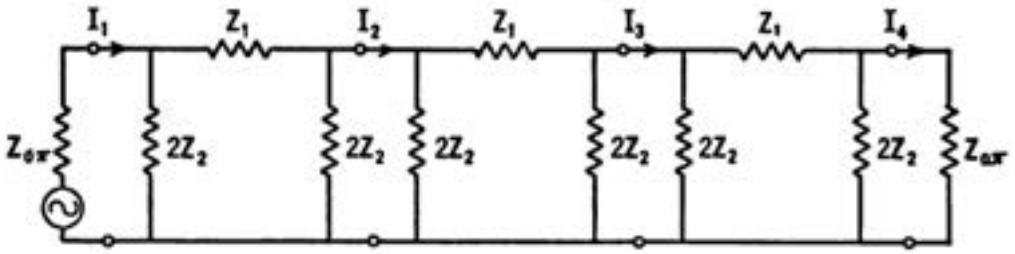

FIG. 11. Three symmetrical π-sections terminated on a characteristic impedance basis.

The general arrangements of circuit elements for elementary low-pass filter sections are illustrated in Fig. 14, page 466.

A *high-pass* filter is a network designed to pass currents of all frequencies above a critical or cut-off frequency and materially to reduce the amplitude of currents of all frequencies below this critical frequency. Under ideal conditions, a high-pass filter attenuates all frequencies from zero up to a predetermined number of cycles and transmits higher frequencies with theoretical zero attenuation. In a high-pass filter the lowest frequency at which theoretical zero attenuation obtains is called cut-off frequency. Elementary high-pass filter sections are shown in Fig. 16, page 469.

A Fundamental Filter Equation. An equation which defines the propagation constant of a filter section wholly in terms of an arbitrarily selected series arm ($\mathbf{Z}_1$) and an arbitrarily selected shunt arm ($\mathbf{Z}_2$) is necessary in the design of filter sections.

Reference to equations (20) and (23), page 447, and to equation (24), page 450, shows that

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \epsilon^{\gamma} = \frac{\mathbf{Z}_{oT} + \dfrac{\mathbf{Z}_1}{2}}{\mathbf{Z}_{oT} - \dfrac{\mathbf{Z}_1}{2}} \quad \text{(for T-sections)} \tag{33}$$

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \epsilon^{\gamma} = \frac{2\mathbf{Z}_2 + \mathbf{Z}_{o\pi}}{2\mathbf{Z}_2 - \mathbf{Z}_{o\pi}} \quad \text{(for } \pi\text{-sections)} \tag{34}$$

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After the value of Z_{oT} as given in equation (5), page 442, is substituted in equation (33), the following form may be obtained:

$$\frac{I_1}{I_2} = \epsilon^{\gamma} = \frac{\sqrt{1 + \frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2}}}{\sqrt{1 + \frac{Z_1}{4Z_2}} - \sqrt{\frac{Z_1}{4Z_2}}} \quad \text{(for T-sections)} \tag{35}$$

After substituting the value of $Z_{o\pi}$ as given in equation (7) into equation (34), the following form may be obtained:

$$\frac{I_1}{I_2} = \epsilon^{\gamma} = \frac{\sqrt{1 + \frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2}}}{\sqrt{1 + \frac{Z_1}{4Z_2}} - \sqrt{\frac{Z_1}{4Z_2}}} \quad \text{(for } \pi\text{-sections)} \tag{36}$$

Hence, for like values of Z_1 and Z_2;

$$\frac{I_1}{I_2} \text{ (for T-sections)} = \frac{I_1}{I_2} \text{ (for } \pi\text{-sections)} \tag{37}$$

Since $\epsilon^{\gamma} = \epsilon^{(\alpha+j\beta)}$, it follows that

$$\epsilon^{(\alpha+j\beta)} = \frac{\left(\sqrt{1 + \frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2}}\right)^2}{1} \tag{38}$$

Although equation (38) defines both α and β in terms of Z_1 and Z_2, a different form is usually employed in the actual evaluation process.[3] An algebraic rearrangement of the quantities involved in equation (38) will show that:

$$\alpha + j\beta = 2 \log_{\epsilon}\left(\sqrt{1 + \frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2}}\right) \tag{39}$$

The above relation is one form of fundamental filter equation, since the

[3] A fundamental filter equation which is sometimes called Campbell's equation (after G. A. Campbell who discovered the filtering properties of various lumped impedance networks) is:

$$\cosh \gamma = 1 + \frac{Z_1}{2Z_2} = (\cosh \alpha \cos \beta + j \sinh \alpha \sin \beta)$$

The above form need not be used here but, for the reader who is familiar with the manipulation of complex hyperbolic functions, Campbell's equation is much more elegant than is equation (39). See "Physical Theory of the Wave-Filter," by G. A. Campbell, *Bell System Technical Journal*, Vol. I, November, 1922.

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attenuation constant and the phase-shift constant are defined wholly in terms of the *full series arm impedance* ($\mathbf{Z}_1$) and the *full shunt arm impedance* ($\mathbf{Z}_2$). The analysis of any symmetrical T- or π-section composed of series and shunt arms of $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively, may be carried through with the aid of equation (39).

Since the right-hand member of equation (39) is, in general, a complex number, it is capable of defining both α and β of either T- or π-sections which are terminated on a characteristic impedance basis. In the manipulation of the factor $\mathbf{Z}_1/4\mathbf{Z}_2$ in equation (39), care should be exercised in determining the correct sign of the associated angle if the correct sign of β is desired.

Example 6. Let it be required to determine the attenuation and phase shift of a filter section whose full series arm is $565.6\underline{/60^\circ}$ ohms (at a particular frequency) and whose full shunt arm is $200\underline{/-90^\circ}$ ohms. *Note:* Characteristic impedance termination is implied in a case of this kind unless otherwise stated.

$$\mathbf{Z}_1 = 565.6\underline{/60^\circ} \quad \text{and} \quad \mathbf{Z}_2 = 200\underline{/-90^\circ} \text{ ohms}$$

$$\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}} = \sqrt{\frac{565.6\underline{/60^\circ}}{800\underline{/-90^\circ}}} = \sqrt{0.707\underline{/150^\circ}} = 0.841\underline{/75^\circ} = (0.2175 + j0.812)$$

$$\sqrt{1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}} = \sqrt{1\underline{/0^\circ} + 0.707\underline{/150^\circ}} = \sqrt{0.525\underline{/42.4^\circ}}$$

$$= 0.725\underline{/21.2^\circ} = (0.676 + j0.262)$$

$$\alpha + j\beta = 2\log_\epsilon[(0.676 + j0.262) + (0.2175 + j0.812)]$$

$$= 2\log_\epsilon(0.893 + j1.074)$$

$$= 2\log_\epsilon(1.396\underline{/50.25^\circ})$$

$$= (2\log_\epsilon 1.396) + j\frac{100.5}{57.3} = (0.668 + j1.76)$$

The attenuation of the filter section is 0.668 neper or 5.80 decibels. The vector input current is 1.76 radians or 100.5° ahead of the vector output current since $\alpha = 0.668$ neper and $\beta = 1.76$ radians.

In this example the resistance of the series arm is relatively high (565.6/2 ohms) and yet the attenuation is relatively low because the filter section is operating in its pass band.

Example 7. Let it be required to find the attenuation and phase shift of the π-section shown in Fig. 8, page 446, by means of equation (39). The resistances of the circuit elements are to be neglected and the frequency is assumed to be 200 cycles. At 200 cycles, $\omega = 1257$ radians per second and

$$\mathbf{Z}_1 = 0 + j\omega L_1 = 25.14\underline{/90^\circ} \text{ ohms}$$

$$2\mathbf{Z}_2 = 0 - j\frac{1}{\frac{\omega C_2}{2}} = 397.5\underline{/-90^\circ} \text{ ohms}$$

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$$4Z_2 = 795\underline{/-90^\circ} \text{ ohms}$$

$$\frac{Z_1}{4Z_2} = \frac{25.14\underline{/90^\circ}}{795\underline{/-90^\circ}} = 0.0316\underline{/+180^\circ}$$

$$\alpha + j\beta = 2\log_\epsilon [\sqrt{1\underline{/0^\circ} + 0.0316\underline{/+180^\circ}} + \sqrt{0.0316\underline{/+180^\circ}}]$$

$$= 2\log_\epsilon (1.0\underline{/10.25^\circ}) = (2\log_\epsilon 1.0) + \left(2j\frac{10.25}{57.3}\right)$$

$$= 0 + j0.358$$

Therefore $\alpha = 0$ and $\beta = 0.358$ radian or 20.5°. It will be noted that, as a result of neglecting the resistances of the circuit elements, the theoretical attenuation is zero.

Problem 9. A high-pass filter section is composed of two 7.96-μf condensers and a coil having an inductance of 0.0159 henry in the form of a T. The resistance of the inductance coil is assumed to be 4 ohms. (A condenser occupies each of the $Z_1/2$ positions in Fig. 4*a*, page 441, and the inductance coil occupies the Z_2 position in this T-section.) Find the attenuation and phase shift of this filter section at 200 cycles employing equation (39). At 200 cycles:

$$\omega = 1257 \text{ radians per second } \frac{Z_1}{2} = 100\underline{/-90^\circ} \quad Z_2 = 20.4\underline{/78.7^\circ} \text{ ohms}$$

Ans.: $\alpha = 17.8$ db; $\beta = -165^\circ$.

Problem 10. Evaluate α and β in equation (39) if $Z_1 = 200\underline{/90^\circ}$ ohms and $Z_2 = 50\underline{/-90^\circ}$ ohms. *Ans.:* $\alpha = 0$; $\beta = \pi$ radians.

Filter Section Analysis Assuming Zero Resistance. It is quite customary to neglect the resistive components of Z_1 and Z_2 in filter section analysis because the attenuation produced by these resistive components is incidental to the predominant filtering action that takes place. The discrepancy between theoretical results based on zero resistance and actual results will not be great if the resistances are relatively small compared with the reactances. Also the algebraic manipulations involved in filter design are greatly simplified by neglecting the resistive components of Z_1 and Z_2.

If the above resistances are neglected and if the filter sections are properly terminated, the pass bands are transmitted with zero attenuation while the stop bands experience certain varying degrees of attenuation. It will also be shown that the phase shift is 180° throughout the stop band under the conditions stated above. Before elaborating upon these customary generalizations, two examples based entirely upon equation (39) will be presented.

Example 8. Consider a symmetrical T-section in which $Z_1 = j\omega L_1$ and in which $Z_2 = -j\dfrac{1}{\omega C_2}$. Let it be required to predict the behavior of the filter section wholly

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in terms of the relationship stated in equation (39).

$$\frac{Z_1}{4Z_2} = \frac{-\omega^2 L_1 C_2}{4} = \frac{\omega^2 L_1 C_2}{4} \underline{/+180^\circ}$$

Since $Z_1/4Z_2$ possesses the general form given above, it will be convenient to reckon ω in $1/\sqrt{L_1C_2}$ units, thereby giving $Z_1/4Z_2$ definite numerical values for various different frequency units. The evaluation of the right-hand member of equation (39) for various frequencies is shown in tabular form in Table I.

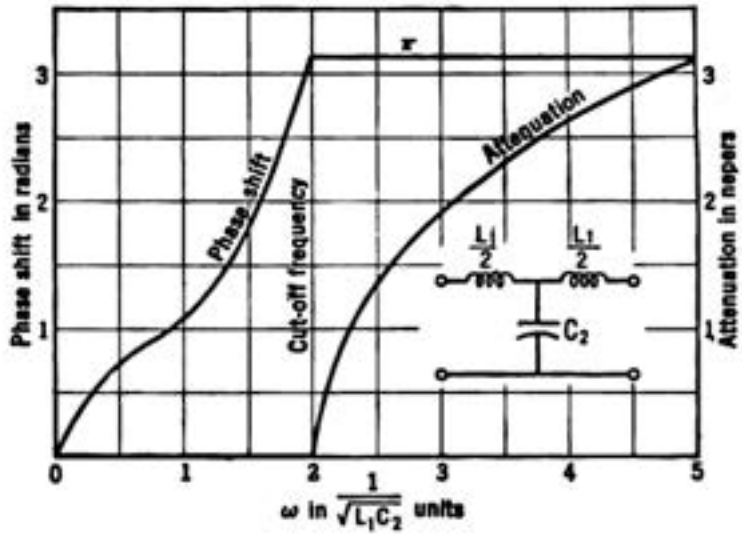

FIG. 12. Variations of phase shift and attenuation in a prototype low-pass filter section. (See Table I, page 460.)

The variations of attenuation and phase shift can readily be determined from an examination of columns (8) and (9) of the table. It will be observed that the filter section which is under discussion has theoretical zero attenuation between the limits of $\omega = 0$ and $\omega = 2/\sqrt{L_1C_2}$ radians per second. The section obviously operates as a low-pass filter. The arrangement of the series and shunt arms of this low-pass filter together with the general trends in the variations of attenuation and phase shift are shown in Fig. 12. The fact that the cut-off point occurs at $\omega = 2/\sqrt{L_1C_2}$ radians per second will be given more attention in a later article. The present example concerns itself primarily with the development of equation (39) in a particular case.

Example 9. Consider a symmetrical T-section in which $Z_1 = -j\dfrac{1}{\omega C_1}$ and $Z_2 = j\omega L_2$. Let it be required to predict the behavior of the filter section wholly in terms of equation (39). In the present case:

$$\frac{Z_1}{4Z_2} = -\frac{1}{4\omega^2 L_2 C_1} = \frac{1}{4\omega^2 L_2 C_1} \underline{/-180^\circ}$$

The same units of angular velocity as employed in example 8 are convenient units to employ in the present analysis. Also the evaluation of the right-hand member of equation (39) can be conveniently presented in tabular form. The calculations are indicated in Table II, and results are shown graphically in Fig. 13. T-sections

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TABLE I

$$\left[Z_1 = j\omega L_1 \quad \text{and} \quad Z_2 = -j\frac{1}{\omega C_2}\right] \qquad \frac{Z_1}{4Z_2} = \frac{\omega^2 L_1 C_2}{4}\underline{/+180^\circ}$$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
ω	$\frac{Z_1}{4Z_2}$	$\sqrt{\frac{Z_1}{4Z_2}}$	$\sqrt{\frac{Z_1}{4Z_2}+1}$	(3) + (4) Cartesian Form	(3) + (4) Polar Form	$2\log_e$ (6)	α nepers	β radians
0	0	0	1	$1 + j0$	$1\underline{/0^\circ}$	$0 + j0$	0	0
$1/\sqrt{L_1C_2}$	-0.25	$j0.50$	0.866	$0.866 + j0.5$	$1\underline{/30^\circ}$	$0 + j1.05$	0	1.05
$1.5/\sqrt{L_1C_2}$	-0.563	$j0.75$	0.662	$0.662 + j0.75$	$1\underline{/48.6^\circ}$	$0 + j1.70$	0	1.70
$2/\sqrt{L_1C_2}$	-1.0	$j1.00$	0.000	$0 + j1.0$	$1\underline{/90^\circ}$	$0 + j3.14$	0	3.14
$2.5/\sqrt{L_1C_2}$	-1.562	$j1.25$	$j0.75$	$0 + j2.0$	$2\underline{/90^\circ}$	$1.38 + j3.14$	1.38	3.14
$3/\sqrt{L_1C_2}$	-2.25	$j1.50$	$j1.13$	$0 + j2.63$	$2.63\underline{/90^\circ}$	$1.93 + j3.14$	1.93	3.14
$4/\sqrt{L_1C_2}$	-4.0	$j2.00$	$j1.73$	$0 + j3.73$	$3.73\underline{/90^\circ}$	$2.63 + j3.14$	2.63	3.14
$6/\sqrt{L_1C_2}$	-9.0	$j3.00$	$j2.83$	$0 + j5.83$	$5.83\underline{/90^\circ}$	$3.52 + j3.14$	3.52	3.14
$8/\sqrt{L_1C_2}$	-16.0	$j4.00$	$j3.87$	$0 + j7.87$	$7.87\underline{/90^\circ}$	$4.12 + j3.14$	4.12	3.14
∞	$-\infty$	$j\infty$	$j\infty$	$0 + j\infty$	$\infty\underline{/90^\circ}$	$\infty + j3.14$	∞	3.14

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TABLE II

$$\left[Z_1 = -j\frac{1}{\omega C_1} \quad \text{and} \quad Z_2 = j\omega L_2\right] \qquad \frac{Z_1}{4Z_2} = \frac{1}{4\omega^2 L_2 C_1}\angle -180^\circ$$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
ω	$\frac{Z_1}{4Z_2}$	$\sqrt{\frac{Z_1}{4Z_2}}$	$\sqrt{\frac{Z_1}{4Z_2}+1}$	(3) + (4) Cartesian Form	(3) + (4) Polar Form	$2\log_e$ (6)	α nepers	β radians
0	$-\infty$	$-j\infty$	$-j\infty$	$0 - j\infty$	$\infty\angle -90^\circ$	$\infty - j3.14$	∞	-3.14
$\frac{0.1}{\sqrt{C_1L_2}}$	-25	$-j5.0$	$-j4.9$	$0 - j9.9$	$9.9\angle -90^\circ$	$4.58 - j3.14$	4.58	-3.14
$\frac{0.25}{\sqrt{C_1L_2}}$	-4.0	$-j2.0$	$-j1.73$	$0 - j3.73$	$3.73\angle -90^\circ$	$2.63 - j3.14$	2.63	-3.14
$\frac{0.40}{\sqrt{C_1L_2}}$	-1.563	$-j1.25$	$-j0.75$	$0 - j2.00$	$2.0\angle -90^\circ$	$1.38 - j3.14$	1.38	-3.14
$\frac{0.50}{\sqrt{C_1L_2}}$	-1.00	$-j1.0$	0.00	$0 - j1.00$	$1.0\angle -90^\circ$	$0 - j3.14$	0	-3.14
$\frac{0.60}{\sqrt{C_1L_2}}$	-0.694	$-j0.833$	0.554	$0.554 - j0.833$	$1.0\angle -56.4^\circ$	$0 - j1.97$	0	-1.97
$\frac{1.0}{\sqrt{C_1L_2}}$	-0.250	$-j0.500$	0.866	$0.866 - j0.500$	$1.0\angle -30^\circ$	$0 - j1.05$	0	-1.05
$\frac{2.0}{\sqrt{C_1L_2}}$	-0.0625	$-j0.250$	0.968	$0.968 - j0.250$	$1.0\angle -14.5^\circ$	$0 - j0.506$	0	-0.506
$\frac{4.0}{\sqrt{C_1L_2}}$	-0.0156	$-j0.125$	0.992	$0.992 - j0.125$	$1.0\angle -7.2^\circ$	$0 - j0.246$	0	-0.246
∞	0	0	1.00	$1.00 - j0$	$1.0\angle 0^\circ$	$0 - j0$	0	0

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consisting of series-arm condensers and shunt-arm inductances are thus shown to operate effectively as high-pass filters.

The phase shift constant, β, in Table I represents a lag of section output voltage and current with respect to section input voltage and current. In Table II, β represents a lead of section output voltage and current with respect to section input voltage.

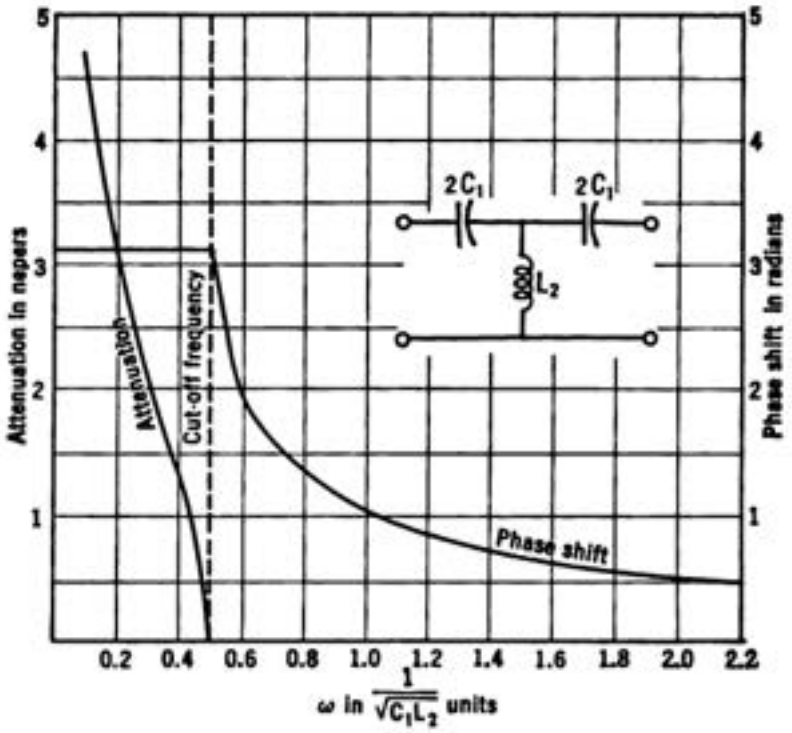

FIG. 13. Variations of phase shift and attenuation in a prototype high-pass filter section. (See Table II, page 461.)

Problem 11. Refer to Table I, page 460. Check all the values listed at $\omega = 1.5/\sqrt{L_1C_2}$ and at $\omega = 3/\sqrt{L_1C_2}$. Compare the results obtained for α and β with those plotted in Fig. 12, page 459.

Problem 12. Refer to Table II, page 461. Check all the values listed at $\omega = 0.25/\sqrt{C_1L_2}$ and at $\omega = 2.0/\sqrt{C_1L_2}$. Compare the results obtained for α and β with those plotted in Fig. 13, page 462.

The chief facts to be gained from the foregoing analyses are:

(1) α is equal to zero within the pass-band region.
(2) β is equal to $\pm\pi$ within the stop-band region.

A study of Tables I and II will show that the pass bands are limited to those regions where $\mathbf{Z}_1/4\mathbf{Z}_2$ possesses values between 0 and -1. These results might have been anticipated mathematically by investigating the possible values of α and β when $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are reactances of opposite types. Let

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = \mathbf{A}$$

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It is plain that $\mathbf{A} = A\underline{/\pm\pi}$ since

$$\frac{X_L\underline{/90^\circ}}{4X_C\underline{/-90^\circ}} \quad \text{or} \quad \frac{X_C\underline{/-90^\circ}}{4X_L\underline{/90^\circ}}$$

are complex numbers which have associated angles of $+\pi$ or $-\pi$ radians, respectively.

If

$$-1 \leqq A \leqq 0$$

$$\begin{aligned}\alpha + j\beta &= 2\log_e(\sqrt{1-A} + \sqrt{-A}) \\ &= 2\log_e(\sqrt{1-A} + j\sqrt{A}) \\ &= 2\left(\log_e\sqrt{1-A+A} + j\tan^{-1}\frac{\sqrt{A}}{\sqrt{1-A}}\right)\end{aligned}$$

Hence $\alpha = 0$ and $\beta = 2\tan^{-1}(\sqrt{A}/\sqrt{1-A})$ when $\mathbf{A} = \mathbf{Z}_1/4\mathbf{Z}_2$ lies between 0 and -1.

When $\mathbf{Z}_1/4\mathbf{Z}_2$ lies between -1 and $-\infty$ a similar analysis will show that for $\mathbf{Z}_1/4\mathbf{Z}_2 = A'\underline{/\pm\pi}$, A' being greater in magnitude than unity.

$$\begin{aligned}\alpha + j\beta &= 2\log_e(\sqrt{1-A'} + \sqrt{-A'}) \\ &= 2\log_e(j\sqrt{A'-1} + j\sqrt{A'}) \\ &= 2\log_e(\sqrt{A'-1} + \sqrt{A'}) + j(\pm\pi)\end{aligned}$$

Hence $\alpha = 2\log_e(\sqrt{A'-1} + \sqrt{A'})$ and $\beta = \pm\pi$ when $\mathbf{A}' = \mathbf{Z}_1/4\mathbf{Z}_2$ lies between -1 and $-\infty$.

The above analysis shows that the pass bands are limited to those regions where $\mathbf{Z}_1/4\mathbf{Z}_2$ takes on values between and including 0 and -1. Hence:

$$-1 \leqq \frac{\mathbf{Z}_1}{4\mathbf{Z}_2} \leqq 0 \tag{40}$$

defines the pass-band regions in terms of $\mathbf{Z}_1$ and $\mathbf{Z}_2$. The boundaries of a pass band in a particular case may be obtained by setting:

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0 \quad \text{and} \quad \frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1 \tag{41}$$

or by setting

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = 0 \quad \text{and} \quad \frac{\mathbf{Z}_1}{\mathbf{Z}_2} = -4 \tag{42}$$

Reference to equation (39) will show that $\alpha = 0$ when $\mathbf{Z}_1/4\mathbf{Z}_2 = 0$ and when $\mathbf{Z}_1/4\mathbf{Z}_2 = -1$.

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Example 10. Refer to the symmetrical π-section shown in Fig. 8, page 446. Let it be required to predict the pass-band boundaries in terms of the relationships stated in (42). The full series arm of Fig. 8 is $L_1 = 0.02$ henry and $\mathbf{Z}_1 = 0.02\omega\underline{/90^\circ}$ ohms. The full shunt arm is $C_2 = 4.0\ \mu f$ and $\mathbf{Z}_2 = (10^6/4\omega)\underline{/-90^\circ}$ ohms.

Setting $\mathbf{Z}_1/\mathbf{Z}_2 = 0$ yields

$$\frac{0.02\omega\underline{/90^\circ}}{\frac{10^6}{4\omega}\underline{/-90^\circ}} = 0 \quad \text{or} \quad \omega = 0 \quad \text{(one boundary)}$$

Setting $\mathbf{Z}_1/\mathbf{Z}_2 = -4$ yields

$$\frac{0.02\omega\underline{/90^\circ}}{\frac{10^6}{4\omega}\underline{/-90^\circ}} = \frac{-0.08\omega^2}{10^6} = -4$$

from which

$$\omega^2 = \sqrt{50 \times 10^6}$$

$$\omega = 7070 \text{ radians per second} \quad \text{(one boundary)}$$

The value of ω given above represents the cut-off angular velocity of this particular low-pass filter section and corresponds to a frequency of $7070/2\pi$ or 1125 cycles.

Cut-Off Frequencies of Elementary Low- and High-Pass Sections. The frequency limits of the pass band for an elementary low-pass filter without resistance may be obtained from equation (38). For a low-pass filter $\mathbf{Z}_1 = j\omega L_1$ and $\mathbf{Z}_2 = -j\dfrac{1}{\omega C_2}$. If these values are substituted in equation (38), the result, after a little algebraic simplification, is:

$$\epsilon^{\alpha+j\beta} = \epsilon^{\alpha}\epsilon^{j\beta} = 1 - \frac{2\omega^2 L_1 C_2}{4} + 2\sqrt{\frac{\omega^4 L_1^2 C_2^2}{16} - \frac{\omega^2 L_1 C_2}{4}} \tag{43}$$

For no attenuation $\alpha = 0$, and

$$\epsilon^{j\beta} = \cos\beta + j\sin\beta = 1 - \frac{2\omega^2 L_1 C_2}{4} + 2\sqrt{\frac{\omega^4 L_1^2 C_2^2}{16} - \frac{\omega^2 L_1 C_2}{4}} \tag{44}$$

Since the last term of equation (44) is the only one that may become imaginary, it follows that the real part must be $\cos\beta$. Therefore

$$\cos\beta = 1 - \frac{2\omega^2 L_1 C_2}{4} \tag{45}$$

Since $\cos\beta$ can vary from 1 to -1, the limits for ω may be obtained. Hence

$$\pm 1 = 1 - \frac{2\omega^2 L_1 C_2}{4} \tag{46}$$

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and

$$\omega = 0 \quad \text{or} \quad \frac{2}{\sqrt{L_1C_2}} \tag{47}$$

or

$$f_c = \frac{1}{\pi\sqrt{L_1C_2}} \qquad \text{(for low-pass)} \tag{48}$$

Equation (48) gives the upper or cut-off frequency for an elementary low-pass filter. In other words, any wave of frequency between zero and f_c is passed without attenuation provided that the filter section is terminated in the characteristic impedance for that frequency.

For a high-pass filter, $\mathbf{Z}_1 = -j\dfrac{1}{\omega C_1}$ and $\mathbf{Z}_2 = j\omega L_2$. If these values are substituted in equation (38), a similar method of analysis as used in obtaining equation (45) gives

$$\cos\beta = 1 - \frac{2}{4\omega^2C_1L_2} \tag{49}$$

Substituting the limits of ± 1 for $\cos\beta$, the upper limit of frequency is found to be ∞ while the lower limit or cut-off frequency is:

$$f_c = \frac{1}{4\pi\sqrt{C_1L_2}} \qquad \text{for high-pass} \tag{50}$$

Equation (50) gives the cut-off frequency for an elementary high-pass filter. This means that any frequency above the cut-off frequency f_c is passed with no attenuation if the filter section is terminated in the characteristic impedance for the particular frequency.

Constant-k Low-Pass Filter. Filter sections in which the series and shunt arms are inverse impedance functions possess a peculiar property. The product of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ is independent of frequency. Reference to either the T- or π-section of Fig. 14 will show that

$$\mathbf{Z}_{1k}\mathbf{Z}_{2k} = (j\omega L_{1k})\left(-j\frac{1}{\omega C_{2k}}\right) = \frac{L_{1k}}{C_{2k}} = R_k^2 \tag{51}$$

$\sqrt{L_{1k}/C_{2k}}$ is an important characteristic of the filter section, and inasmuch as

$$\sqrt{\frac{L_{1k}}{C_{2k}}} = R_k = \text{a constant}$$

filter sections of this type are called constant-k sections. There are many other types of filter sections, several of which are derived in one way or another from constant-k sections. For this reason the parame-

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ters of constant-k sections usually carry the subscript k in order to designate properly the type of filter section that is under discussion. The parameters of some of the more elaborate filter sections are specified directly in terms of L_k and C_k.

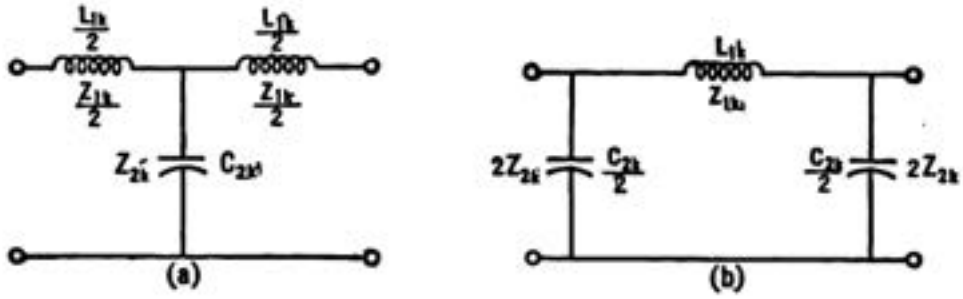

FIG. 14. Prototype or constant-k low-pass filter sections.

The general theory of the constant-k low-pass filter has already been presented. It remains only to develop the design equations for this type of filter.

$$\frac{\mathbf{Z}_{1k}}{\mathbf{Z}_{2k}} = \frac{j\omega L_{1k}}{-j\dfrac{1}{\omega C_{2k}}} = -\omega^2 L_{1k}C_{2k} \tag{52}$$

The boundaries of the pass band are determined by setting $\mathbf{Z}_{1k}/\mathbf{Z}_{2k}$ equal to -4 and equal to zero. [See equation (42), page 463.]

$$-\omega^2 L_{1k}C_{2k} = 0 \quad \text{yields} \quad \omega = 0 \tag{53}$$

$$-\omega^2 L_{1k}C_{2k} = -4 \quad \text{yields} \quad \omega_c = \frac{2}{\sqrt{L_{1k}C_{2k}}} \tag{54}$$

ω_c is the angular velocity at which cut-off takes place and as such forms the upper boundary of the pass band. The *cut-off frequency of a low-pass, constant-k-type* filter is:

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{\pi\sqrt{L_{1k}C_{2k}}} \tag{55}$$

It will be observed that f_c is governed wholly by the magnitude of the $L_{1k}C_{2k}$ product. The lower the cut-off frequency, the higher is the $L_{1k}C_{2k}$ product, and vice versa.

Another important consideration in either the theory or design of a filter section is the matter of correct terminating impedances. A single section can be properly matched to its sending and receiving ends if terminated on an image basis, as explained on page 440. If more than one filter section is to be employed between sending and receiving ends, it is desirable to design each section to have the same

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characteristic impedance. Under these conditions minimum reflection loss results when the various sections are arranged as shown in Fig. 10 or Fig. 11. A detailed analysis of these losses will not be given here since they are similar in nature to reflection losses on long lines. (See Chapter XII.)

For a constant-k, low-pass T-section:

$$Z_{oTk} = \sqrt{\frac{L_{1k}}{C_{2k}}\left(1 - \frac{\omega^2 L_{1k} C_{2k}}{4}\right)} \tag{56}$$

$$L_{1k}C_{2k} = \frac{4}{\omega_c^2} \quad \text{[See equation (54).]}$$

Therefore, for a constant-k, low-pass T-section:

$$\begin{aligned} Z_{oTk} &= \sqrt{\frac{L_{1k}}{C_{2k}}}\sqrt{1 - \frac{f^2}{f_c^2}} \\ &= R_k\sqrt{1 - \frac{f^2}{f_c^2}} \end{aligned} \tag{57}$$

For a constant-k, low-pass π-section:

$$Z_{o\pi k} = \frac{\sqrt{L_{1k}/C_{2k}}}{\sqrt{1 - \frac{f^2}{f_c^2}}} = \frac{R_k}{\sqrt{1 - \frac{f^2}{f_c^2}}} \tag{58}$$

The variations of Z_{oTk} and $Z_{o\pi k}$ from $f = 0$ to $f = f_c$ are illustrated in Fig. 15. The fact that the correct terminating impedance of a constant-k section varies over such wide limits is a very serious limitation in certain communication circuits. For a fixed receiving impedance it is plain that either the T- or π-section is correctly terminated at only one frequency. The opposite trends in Z_{oTk} and $Z_{o\pi k}$ are combined in one form of filter section to obtain a characteristic impedance which is reasonably constant over the frequency range of the pass band. (See m-derived filter sections, pages 481–485.)

The zero-frequency value of either Z_{oTk} or $Z_{o\pi k}$ is:

$$R_k = \sqrt{\frac{L_{1k}}{C_{2k}}} \quad \text{[See equations (57) and (58).]} \tag{59}$$

L_{1k} and C_{2k} can be related to one another through the value of R_k^2. [See equation (51).]

$$L_{1k} = R_k^2 C_{2k} \tag{60}$$

$$C_{2k} = \frac{L_{1k}}{R_k^2} \tag{61}$$

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The design values of L_{1k} and C_{2k} are usually specified in terms of cut-off frequency, f_c, and the zero-frequency value of the characteristic

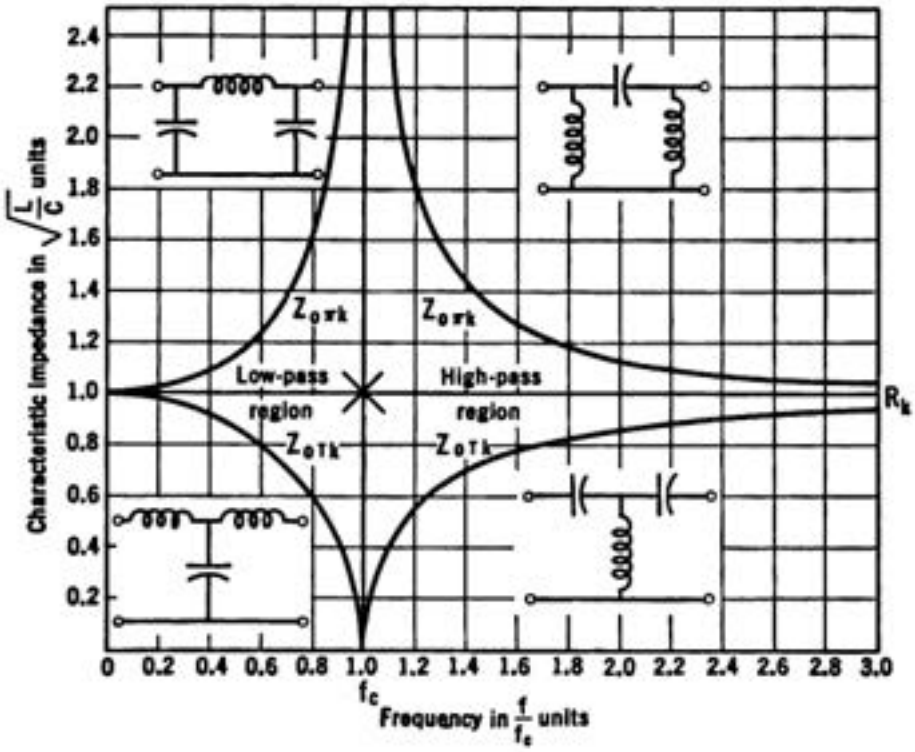

FIG. 15. Variations of the characteristic impedances of low-pass and high-pass constant-k filter sections.

impedance, R_k. It has been shown that:

$$f_c = \frac{1}{\pi\sqrt{L_{1k}C_{2k}}} \quad \text{[See equation (55).]}$$

Eliminating C_{2k} as given in equation (61) from the above equation yields:

$$f_c = \frac{1}{\pi\sqrt{L_{1k}^2/R_k^2}}$$

or

$$L_{1k} = \frac{R_k}{\pi f_c} \quad \text{(for low-pass filter)} \tag{62}$$

From equations (61) and (62) it is plain that:

$$C_{2k} = \frac{L_{1k}}{R_k^2} = \frac{1}{\pi R_k f_c} \quad \text{(for low-pass filter)} \tag{63}$$

Equations (62) and (63) specify the values of L and C to employ in a constant-k, low-pass filter section in terms of f_c and R_k.

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Problem 13. Design both T- and π-section, low-pass filters of the constant-k type which will have a zero-frequency characteristic impedance of 600 ohms and a cut-off frequency of 940 cycles. Draw the circuit arrangement in each case, indicating the particular values (in henrys or microfarads) of each circuit element. *Ans.:* The *full* series arm $L_{1k} = 0.203$ henry; and the *full* shunt arm $C_{2k} = 0.565\ \mu f$.

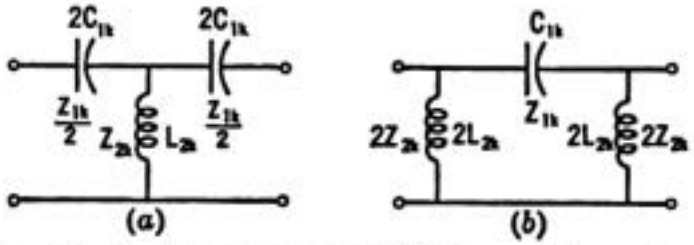

FIG. 16. Prototype or constant-k high-pass filter sections.

Constant-k High-Pass Filter. Prototype or constant-k, high-pass filter sections are illustrated in Fig. 16. In the present case:

$$\mathbf{Z}_{1k}\mathbf{Z}_{2k} = \left(-j\frac{1}{\omega C_{1k}}\right)(j\omega L_{2k}) = \frac{L_{2k}}{C_{1k}} = R_k^2 \tag{64}$$

and

$$\frac{\mathbf{Z}_{1k}}{\mathbf{Z}_{2k}} = \frac{-j\dfrac{1}{\omega C_{1k}}}{j\omega L_{2k}} = -\frac{1}{\omega^2 C_{1k}L_{2k}} \tag{65}$$

The boundaries of the pass band are again determined by setting $\mathbf{Z}_{1k}/\mathbf{Z}_{2k}$ equal to -4 and equal to zero. [See equation (42), page 463.]

$$-\frac{1}{\omega^2 C_{1k}L_{2k}} = 0 \quad \text{yields} \quad \omega = \infty \tag{66}$$

$$-\frac{1}{\omega^2 C_{1k}L_{2k}} = -4 \quad \text{yields} \quad \omega_c = \frac{1}{2\sqrt{C_{1k}L_{2k}}} \tag{67}$$

The cut-off frequency of a high-pass, constant-k filter is

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{4\pi\sqrt{C_{1k}L_{2k}}} \tag{68}$$

$\mathbf{Z}_{oT}$ and $\mathbf{Z}_{o\pi}$ may be expressed in terms of f_c, f, and $\sqrt{L_{2k}/C_{1k}}$ For a constant-k, high-pass T-section:

$$\mathbf{Z}_{oTk} = \sqrt{\frac{L_{2k}}{C_{1k}}} \times \sqrt{1 - \frac{f_c^2}{f^2}} \tag{69}$$

For a constant-k, high-pass π-section:

$$\mathbf{Z}_{o\pi k} = \sqrt{\frac{L_{2k}}{C_{1k}}} \div \sqrt{1 - \frac{f_c^2}{f^2}} \tag{70}$$

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General trends in Z_{oTk} and $Z_{o\pi k}$ in constant-k, high-pass filter sections are illustrated in Fig. 15. Both Z_{oTk} and $Z_{o\pi k}$ approach the common value $\sqrt{L_{2k}/C_{1k}}$ at $f = \infty$. Because it is a useful common base from which to work, $\sqrt{L_{2k}/C_{1k}}$ is given special designation, namely R_k. R_k is known as the infinite-frequency characteristic impedance. Since

$$\sqrt{\frac{L_{2k}}{C_{1k}}} = R_k \tag{71}$$

$$L_{2k} = R_k^2 C_{1k} \quad \text{and} \quad C_{1k} = \frac{L_{2k}}{R_k^2} \tag{72}$$

If the above values are substituted separately in equation (68), the following relationships are obtained:

$$C_{1k} = \frac{1}{4\pi R_k f_c} \quad \text{(for high-pass filter)} \tag{73}$$

$$L_{2k} = \frac{R_k}{4\pi f_c} \quad \text{(for high-pass filter)} \tag{74}$$

Equations (73) and (74) may be employed in the design of constant-k, high-pass filter sections which are to have a particular cut-off frequency and which are to have infinite-frequency characteristic impedances equal to R_k.

Problem 14. What are the cut-off frequency and infinite-frequency characteristic impedance of the high-pass filter section that can be constructed from two 1-μf condensers and one 15-millihenry inductance coil?

Ans.: $f_c = 919$ cycles; $R_k = 173$ ohms.

Tabulation and Review of Constant-k Filter Theory. The important features contained in equations (51) to (74) inclusive are summarized concisely in Table III, pages 472–473. The attenuation and phase shift in Table III are expressed in forms which derive directly from "Campbell's" equation. (See footnote 3 on page 456.) It has been shown in examples 8 and 9, pages 458–459, how the attenuation and phase shift may be calculated from equation (39), page 456, without the aid of hyperbolic functions. For the reader who is familiar with complex hyperbolic functions the following derivation and application of "Campbell's" equation may be of interest.

Derivation and Application of Campbell's Equation. The application of Kirchhoff's emf law to the *wxyz* loop of the filter sections shown in Fig. 10, page 453, yields

$$\mathbf{Z}_1\mathbf{I}_2 + \mathbf{Z}_2(\mathbf{I}_2 - \mathbf{I}_3) - \mathbf{Z}_2(\mathbf{I}_1 - \mathbf{I}_2) = 0 \tag{75}$$

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or

$$Z_1I_2 + 2Z_2I_2 - Z_2I_3 - Z_2I_1 = 0 \tag{76}$$

Dividing the above equation through by Z_2I_2 and transposing results in

$$\frac{I_1}{I_2} + \frac{I_3}{I_2} = 2 + \frac{Z_1}{Z_2} \tag{77}$$

Since

$$\frac{I_1}{I_2} = \epsilon^{\gamma} = \frac{I_2}{I_3}$$

it follows that

$$\frac{I_3}{I_2} = \epsilon^{-\gamma}$$

$$\epsilon^{\gamma} + \epsilon^{-\gamma} = 2 + \frac{Z_1}{Z_2} \tag{78}$$

$$\frac{\epsilon^{\gamma} + \epsilon^{-\gamma}}{2} = \cosh \gamma = 1 + \frac{Z_1}{2Z_2} \quad \text{(Campbell's equation)} \tag{79}$$

A more useful form for the purposes at hand may be derived as follows:

$$\cosh \gamma = \cosh (\alpha + j\beta) = \frac{\epsilon^{(\alpha+j\beta)} + \epsilon^{-(\alpha+j\beta)}}{2}$$

$$= \frac{\epsilon^{\alpha}\epsilon^{j\beta} + \epsilon^{-\alpha}\epsilon^{-j\beta}}{2} \tag{80}$$

Converting the $\epsilon^{j\beta}$ terms into their rectangular forms results in:

$$\cosh (\alpha + j\beta) = \frac{\epsilon^{\alpha} (\cos \beta + j \sin \beta) + \epsilon^{-\alpha} (\cos \beta - j \sin \beta)}{2}$$

$$= \frac{(\epsilon^{\alpha} + \epsilon^{-\alpha})}{2} \cos \beta + j \frac{(\epsilon^{\alpha} - \epsilon^{-\alpha})}{2} \sin \beta$$

From the analytical definitions of hyperbolic cosine and hyperbolic sine, it follows that

$$\cosh (\alpha + j\beta) = \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta$$

Therefore

$$\cosh \alpha \cos \beta + j \sinh \alpha \sin \beta = 1 + \frac{Z_1}{2Z_2} \tag{81}$$

The above form may be used directly to derive the attenuation and phase-shift expressions given in Table III, page 473.

In the stop band, $\beta = \pi$. Since $\cos \beta$ then becomes -1 in equation

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TABLE III

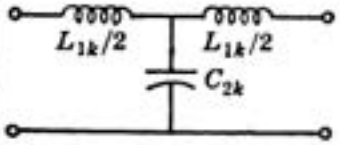
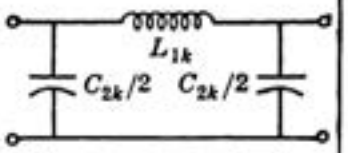
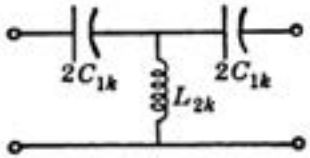
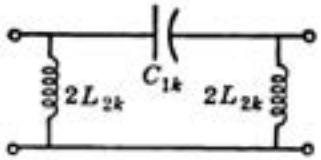
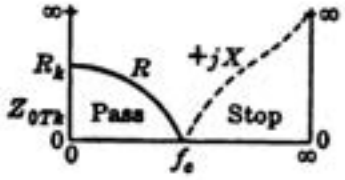
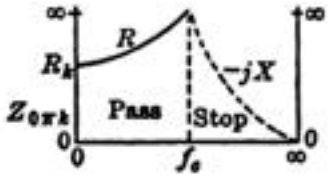
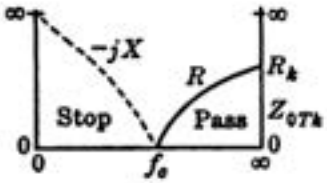
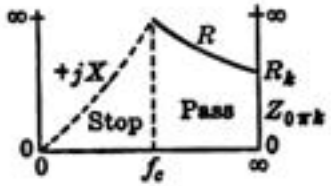
	Constant-k Low Pass		Constant-k High Pass	
	T-Section	π-Section	T-Section	π-Section
Circuit Configuration				
Cut-Off Frequency	$f_c = \dfrac{1}{\pi\sqrt{L_{1k}C_{2k}}}$		$f_c = \dfrac{1}{4\pi\sqrt{C_{1k}L_{2k}}}$	
Z_1Z_2 Base Characteristic Impedance	Zero-Frequency Characteristic Z $R_k = \sqrt{L_{1k}/C_{2k}}$		Infinite-Frequency Characteristic Z $R_k = \sqrt{L_{2k}/C_{1k}}$	
Z_0 With Respect To f	 $Z_{0Tk} = R_k \times \sqrt{1 - f^2/f_c^2}$	 $Z_{0\pi k} = R_k/\sqrt{1 - f^2/f_c^2}$	 $Z_{0Tk} = R_k \times \sqrt{1 - f_c^2/f^2}$	 $Z_{0\pi k} = R_k/\sqrt{1 - f_c^2/f^2}$

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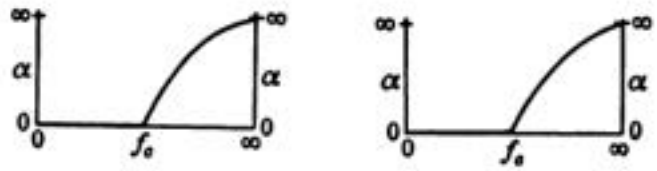
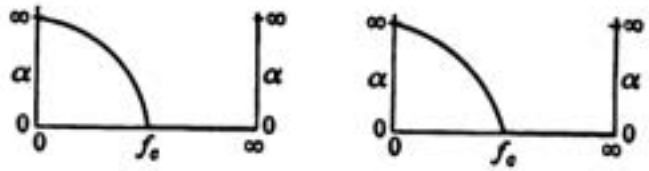
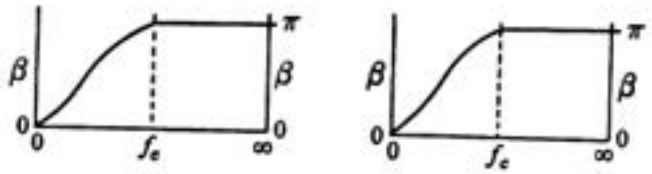
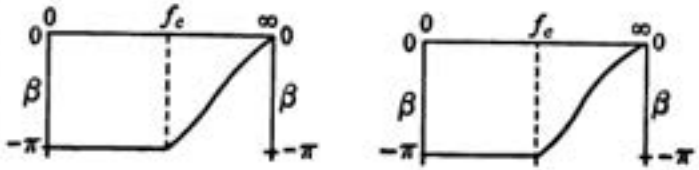
TABLE III (*Continued*)

α Attenuation (quantitative values given in Fig. 12, page 418.)	 In the stop-band: $\alpha = \cosh^{-1}\left(\frac{\omega^2 L_{1k} C_{2k}}{2} - 1\right)$	 $\alpha = \cosh^{-1}\left(\frac{1}{2\omega^2 C_{1k} L_{2k}} - 1\right)$
β Phase Shift (quantitative values given in Fig. 13, page 418.)	 In the pass-band: $\beta = \cos^{-1}\left(1 - \frac{\omega^2 L_{1k} C_{2k}}{2}\right)$	 $\beta = \cos^{-1}\left(1 - \frac{1}{2\omega^2 C_{1k} L_{2k}}\right)$
Values of Circuit Elements in Terms of R_k and f_c	$L_{1k} = \frac{R_k}{\pi f_c}$ $\quad C_{2k} = \frac{1}{\pi R_k f_c}$	$C_{1k} = \frac{1}{4\pi R_k f_c}$ $\quad L_{2k} = \frac{R_k}{4\pi f_c}$

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(81) and $\sin \beta = 0$:

$$-\cosh \alpha = 1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}$$

$$\alpha = \cosh^{-1}\left(-\frac{\mathbf{Z}_1}{2\mathbf{Z}_2} - 1\right) \quad \text{(in stop band)} \tag{82}$$

In the pass band, $\alpha = 0$. Since $\cosh 0 = 1$ and $\sinh 0 = 0$, equation (81) becomes:

$$\cos \beta = \left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)$$

$$\beta = \cos^{-1}\left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right) \quad \text{(in pass band)} \tag{83}$$

As applied to a constant-k low-pass filter section:

$$\mathbf{Z}_1 = \omega L_{1k}\underline{/90^\circ} \quad \mathbf{Z}_2 = \frac{1}{\omega C_{2k}}\underline{/-90^\circ} \quad \frac{\mathbf{Z}_1}{\mathbf{Z}_2} = -\omega^2 L_{1k}C_{2k}$$

Equation (82) then takes the form:

$$\alpha = \cosh^{-1}\left(\frac{\omega^2 L_{1k}C_{2k}}{2} - 1\right)$$

as shown in Table III. Equation (83) takes the form:

$$\beta = \cos^{-1}\left(1 - \frac{\omega^2 L_{1k}C_{2k}}{2}\right)$$

as shown in Table III. Corresponding expressions for α and β may be derived for the constant-k high-pass filter section. The results are shown in Table III.

Band-Pass and Band-Elimination Filters. Band-pass filters are networks which are designed to attenuate all frequencies except those in a specified band. A band-pass filter may be formed by placing a low-pass filter section (having a cut-off frequency of f_{cl}) in series with a high-pass filter section (having a cut-off frequency of f_{ch}). Then f_{cl} is made higher than f_{ch} by the specified band width, which is $f_{cl} - f_{ch}$. A study of the attenuation graphs shown in Table III will show how f_{cl} and f_{ch} should be adjusted to give a zero-attenuation band.

A band-pass filter may take the form of a single section as shown in Fig. 17. The section shown in Fig. 17 is called a constant-k band-pass filter when $L_2C_2 = L_1C_1$ because under these conditions:

$$\mathbf{Z}_1\mathbf{Z}_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = \text{a constant}$$

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An analysis of the band-pass filter will not be given here, although such an analysis may be carried through in a manner similar to those given for the low-pass and high-pass sections.

Band-elimination filters are networks which are designed to pass all frequencies except those in a specified band. A band-elimination filter may be formed by placing a low-pass section (having a cut-off frequency of f_{cl}) in parallel with a high-pass section (having a cut-off frequency of f_{ch}). Then f_{cl} is made lower than f_{ch} by the specified band width, which is $f_{ch} - f_{cl}$. All frequencies have a pass band (through one of the parallel sections) except where the two attenuation graphs overlap. (See attenuation graphs in Table III.)

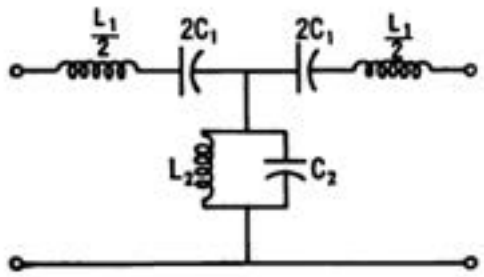

FIG. 17. Band-pass filter contained in a single section.

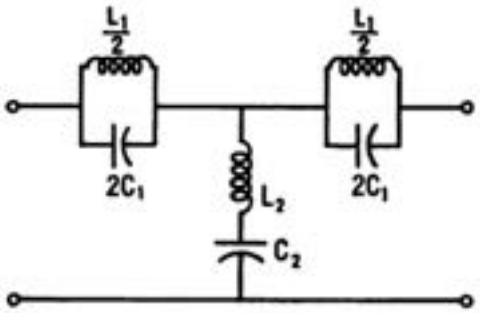

FIG. 18. Band-elimination filter contained in a single section.

A band-elimination filter may take the form of a single section as shown in Fig. 18. The section shown in Fig. 18 is called a constant-k band-elimination filter when $L_2C_2 = L_1C_1$ because under these conditions $\mathbf{Z}_1\mathbf{Z}_2$ is a constant. It will be observed that the arms of Fig. 18 are the reverse of those in Fig. 17.

Two Limitations of Constant-k Sections. The constant-k type of filter section has two rather serious shortcomings. First, its characteristic impedance is not sufficiently constant over the transmission band for certain classes of work. (See Fig. 15.) Second, the attenuation does not rise very abruptly at the boundary of the transmission band. (See Figs. 12 and 13.)

In order to overcome the inherent limitations of the constant-k type, Zobel[4] devised a filter section which he called the m-derived type. The m-derived half section may be employed to give practically uniform characteristic impedance over a large part of the pass band and at the same time increase the abruptness with which cut-off occurs. Full m-derived sections may be employed to give further increased attenuation near the cut-off point, and by proper adjustment of the parameter

[4] "Theory and Design of Uniform and Composite Electric Wave Filters," by O. J. Zobel, *Bell System Technical Journal*, January, 1923.

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m they can be made to meet any practical attenuation requirement in this region. When worked in conjunction with constant-k sections, the m-derived sections overcome both the aforementioned shortcomings of the constant-k sections. However, m-derived sections by them-

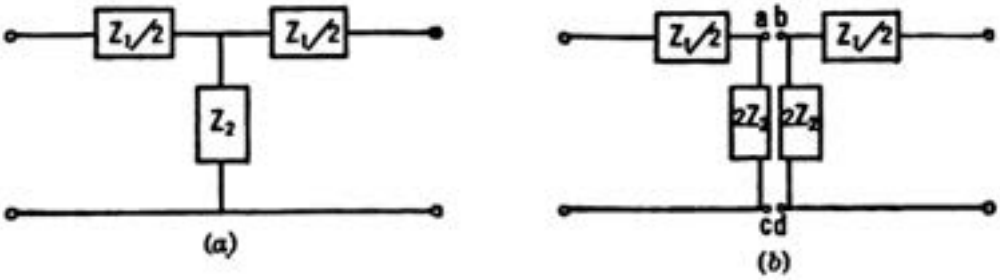

FIG. 19. Illustrating the circuit configuration of half sections formed by longitudinal bisection of shunt arm of a prototype T-section.

selves have certain limitations which will become apparent after the attenuation characteristics of these sections have been studied.

m-Derived Half Sections. If the full shunt arm of Fig. 19a is separated into two parallel paths of $2Z_2$ ohms each, the original T-section may

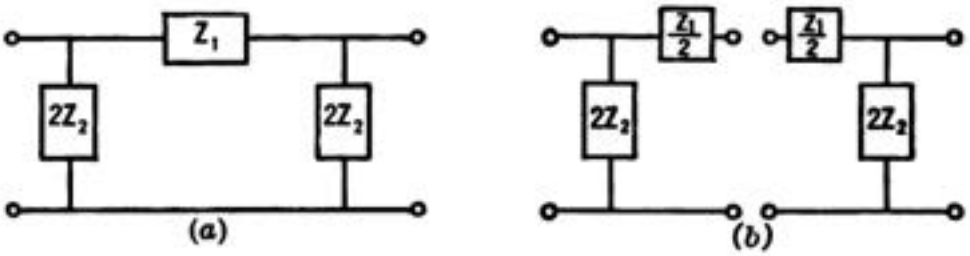

FIG. 20. Illustrating the circuit configuration of half sections formed by longitudinal bisection of the series arm of a prototype π-section.

be separated into two similar parts as shown in Fig. 19b. Each of these parts is known as a half section or as an L-type section. If the full series arm of the π-section shown in Fig. 20a is separated into two series elements of $Z_1/2$ ohms each, the original π-section can be separated into two half sections as shown in Fig. 20b. A comparison of Fig. 20b with Fig. 19b will show the equivalence of half sections formed by "halving" π-sections and those formed by "halving" T-sections.

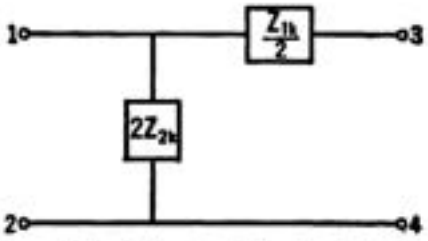

FIG. 21. Constant-k terminating half section.

The image impedances of the half section shown in Fig. 21 may be found from open-circuit and short-circuit conditions. Let the open-circuit and short-circuit impedances be known as Z_{o-c} and Z_{s-c}, respectively.

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The impedance looking into terminals 1 and 2 is:

$$Z_{12} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} = \sqrt{\frac{2Z_{1k}Z_{2k}^2}{\frac{Z_{1k}}{2} + 2Z_{2k}}}$$

from which

$$Z_{12} = \sqrt{\frac{Z_{1k}Z_{2k}}{1 + \frac{Z_{1k}}{4Z_{2k}}}} = Z_{o\pi k} \tag{84}$$

The impedance looking into terminals 3 and 4 is:

$$Z_{34} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} = \sqrt{\left(\frac{Z_{1k}}{2} + 2Z_{2k}\right)\frac{Z_{1k}}{2}}$$

or

$$Z_{34} = \sqrt{Z_{1k}Z_{2k}\left(1 + \frac{Z_{1k}}{4Z_{2k}}\right)} = Z_{oTk} \tag{85}$$

The half section shown in Fig. 21 has the impedance characteristics of a π-section between terminals 1 and 2 and the impedance characteristics of a T-section between terminals 3 and 4. It may, therefore, be used to match a π-section to a T-section. Also it may be used to match a filter section to a terminating impedance which differs from the characteristic impedance of the filter section or to change the impedance level at any point in a two-wire line. The proper values of $Z_{1k}/2$ and $2Z_{2k}$ to be employed in effecting any desired impedance transformation may be determined by solving equations (84) and (85) simultaneously for Z_{1k} and Z_{2k} in terms of Z_{12} and Z_{34}.

Some little difficulty is usually encountered in presenting m-derived filter theory to beginning students because certain anticipations have to be made at the outset of the investigation. Inasmuch as anticipations must be indulged in in any event, the actual circuit configuration of the m-derived half section will be accepted and its operating characteristics studied.

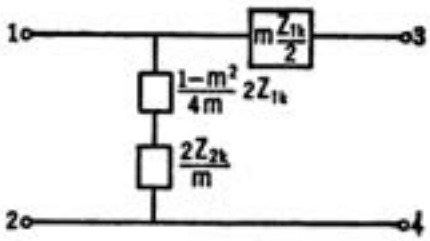

FIG. 22. m-derived terminating half section.

It will now be assumed that the half section shown in Fig. 21 takes the particular form shown in Fig. 22. A new parameter, m, has been arbitrarily introduced. It is simply a numeric which may, for the purposes at hand, range in value from zero to unity. The change in circuit configuration from Fig. 21 to Fig. 22

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may be interpreted as follows:

(a) $\frac{Z_{1k}}{2}$ of the constant-k half section is changed to some fractional part of $\frac{Z_{1k}}{2}$ in Fig. 22.

(b) $2Z_{2k}$ of Fig. 21 is changed to $\frac{2Z_{2k}}{m}$ in Fig. 22.

(c) In series with $\frac{2Z_{2k}}{m}$ in Fig. 22 is placed an impedance $\frac{1-m^2}{4m}2Z_{1k}$.

It may be shown that, if the change in (a) is made, the changes in (b) and (c) must be made if the two half sections shown in Figs. 21 and 22 are to have the same characteristic impedance looking into the 3–4 terminals.

The half section shown in Fig. 22 has some very desirable characteristics. Its characteristic impedance looking into terminals 3 and 4 is:

$$Z_{34m} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} = \sqrt{\left(m\frac{Z_{1k}}{2} + \frac{1-m^2}{2m}Z_{1k} + \frac{2Z_{2k}}{m}\right)m\frac{Z_{1k}}{2}}$$

$$= \sqrt{\frac{m^2Z_{1k}^2}{4} + \frac{mZ_{1k}^2}{4m} - \frac{m^2Z_{1k}^2}{4} + \frac{2mZ_{1k}Z_{2k}}{2m}}$$

$$= \sqrt{Z_{1k}Z_{2k} + \frac{Z_{1k}^2}{4}} = Z_{oTk} \tag{86}$$

The equation above shows that terminals 3 and 4 of the m-derived half section can be used to match the impedance of a constant-k T-section or any other equivalent impedance including the 3–4 terminal characteristic impedance of Fig. 21.

The characteristic impedance of the m-derived terminating half section looking into terminals 1 and 2 is:

$$Z_{12m} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}}$$

where

$$Z_{o\text{-}c} = \left[\frac{(1-m^2)}{2m}Z_{1k} + 2\frac{Z_{2k}}{m}\right]$$

$$Z_{s\text{-}c} = \frac{\left[\frac{(1-m^2)}{2m}Z_{1k} + 2\frac{Z_{2k}}{m}\right]\left[m\frac{Z_{1k}}{2}\right]}{\left[\frac{(1-m^2)}{2m}Z_{1k} + 2\frac{Z_{2k}}{m} + m\frac{Z_{1k}}{2}\right]}$$

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$$\mathbf{Z}_{12m} = \sqrt{\frac{\left(\frac{1-m^2}{2m}\mathbf{Z}_{1k} + 2\frac{\mathbf{Z}_{2k}}{m}\right)^2\left(\frac{m^2\mathbf{Z}_{1k}^2}{4}\right)}{\left(\frac{1-m^2}{2m}\mathbf{Z}_{1k} + 2\frac{\mathbf{Z}_{2k}}{m} + m\frac{\mathbf{Z}_{1k}}{2}\right)\left(\frac{m\mathbf{Z}_{1k}}{2}\right)}}$$

$$= \sqrt{\frac{\left(\frac{1-m^2}{4}\mathbf{Z}_{1k}^2 + \mathbf{Z}_{1k}\mathbf{Z}_{2k}\right)^2}{\frac{\mathbf{Z}_{1k}^2}{4} - \frac{m^2\mathbf{Z}_{1k}^2}{4} + \mathbf{Z}_{1k}\mathbf{Z}_{2k} + \frac{m^2\mathbf{Z}_{1k}^2}{4}}}$$

$$= \frac{\mathbf{Z}_{1k}\mathbf{Z}_{2k} + \frac{\mathbf{Z}_{1k}^2}{4}(1-m^2)}{\sqrt{\mathbf{Z}_{1k}\mathbf{Z}_{2k} + \frac{\mathbf{Z}_{1k}^2}{4}}}$$

$$= \frac{\mathbf{Z}_{1k}\mathbf{Z}_{2k}}{\mathbf{Z}_{oTk}}\left[1 + \frac{\mathbf{Z}_{1k}}{4\mathbf{Z}_{2k}}(1-m^2)\right] \tag{87}$$

or remembering (9):

$$\mathbf{Z}_{12m} = \mathbf{Z}_{ork}\left[1 + \frac{\mathbf{Z}_{1k}}{4\mathbf{Z}_{2k}}(1-m^2)\right] \tag{88}$$

In addition to being a function of $\mathbf{Z}_{1k}$ and $\mathbf{Z}_{2k}$, $\mathbf{Z}_{12m}$ is a function of m. With the proper choice of m, $\mathbf{Z}_{12m}$ can be made reasonably constant over about 90 per cent of the transmission band. The changes of $\mathbf{Z}_{ork}$ and the modifying factor $\left[1 + \frac{\mathbf{Z}_{1k}}{4\mathbf{Z}_{2k}}(1-m^2)\right]$ with respect to frequency combine in such a manner as to make $\mathbf{Z}_{12m}$ approximately constant over wide ranges of frequency.

Example 11. Consider the general trend of $\mathbf{Z}_{or}$ for the constant-k, low-pass section shown in Fig. 15. Instead of this rapidly rising curve, the change in the output characteristic impedance of a low-pass, m-derived half section at the 1–2 terminals is:

$$\mathbf{Z}_{12m} = \mathbf{Z}_{ork}\left[1 + \frac{j\omega L_{1k}}{-j4\frac{1}{\omega C_{2k}}}(1-m^2)\right]$$

or

$$\mathbf{Z}_{12m} = \mathbf{Z}_{ork}\left[1 - \frac{\omega^2 L_{1k}C_{2k}}{4}(1-m^2)\right]$$

Physically, m may be equal to any value between zero and unity. Mathematical experimentation shows that good results are obtained when $m = 0.60$. The calculated values of $\mathbf{Z}_{ork}$ and the modifying factor are shown in Table IV, and a graph of $\mathbf{Z}_{12m}$ for $m = 0.6$ is contained in Fig. 23. It will be remembered that f_c for a

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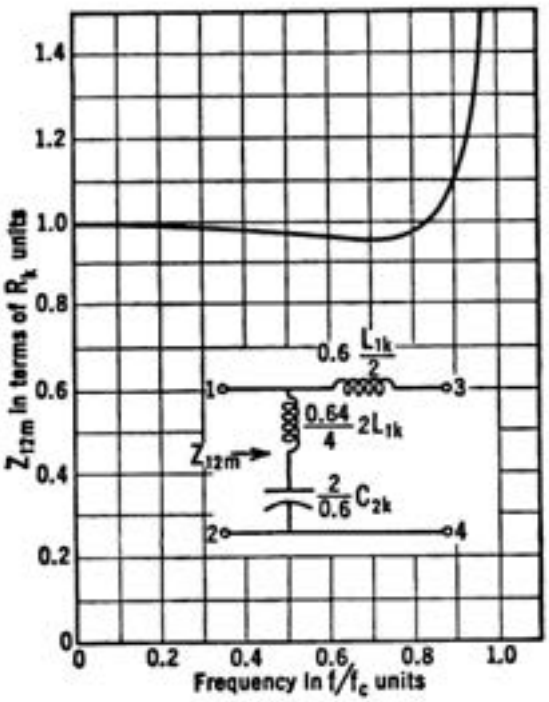

FIG. 23. Variation of Z_{12m} for $m = 0.6$.

TABLE IV

$$Z_{12m} = Z_{ork}\left[1 - \frac{f^2}{f_c^2}(1 - m^2)\right] \text{ for } m = 0.6$$

$$R_k = \sqrt{L_{1k}/C_{2k}}$$

$\frac{f}{f_c}$	$\left[1 - \frac{f^2}{f_c^2}(0.64)\right]$	Z_{ork}	Z_{12m}
0	1.000	R_k	R_k
0.10	0.994	1.005 R_k	0.999 R_k
0.20	0.974	1.02 R_k	0.993 R_k
0.40	0.898	1.09 R_k	0.979 R_k
0.60	0.770	1.25 R_k	0.963 R_k
0.80	0.590	1.67 R_k	0.963 R_k
0.90	0.482	2.30 R_k	1.108 R_k
0.95	0.424	3.16 R_k	1.34 R_k
1.00	0.360	∞	∞

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low-pass filter section is $1/\pi\sqrt{L_{1k}C_{2k}}$ and that $Z_{ork} = R_k/\sqrt{1-(f^2/f_c^2)}$. The expression for Z_{12m} in this particular case is, therefore, reducible to

$$Z_{12m} = \frac{R_k}{\sqrt{1-\dfrac{f^2}{f_c^2}}}\left[1-\frac{f^2}{f_c^2}(1-m^2)\right]$$

If it is necessary to work closer to the cut-off frequency than a value of $m = 0.6$ will permit, m may be made somewhat less than 0.60. However, these slightly lower values of m cause the Z_{12m} variation to be more irregular throughout the first 90 per cent of the transmission band. Numerical experimentation will show the effects caused by different values of m.

Problem 15. Plot, with respect to frequency, the variation of the characteristic output impedance of a low-pass, m-derived terminating half section (Z_{12m}) for $m = 0.55$. Reckon frequency in f/f_c units. (See Table IV and Fig. 23.)

Full m-Derived Sections. Full m-derived T-sections are shown in Fig. 24. As in the m-derived half section, the series and shunt arms are specified in terms of the constant-k impedances Z_{1k} and Z_{2k}. Any constant-k-type section may be altered to yield what is known as an m-derived section. Only the low-pass and high-pass, m-derived T-sections will be considered in detail. These are shown in Fig. 24*b* and 24*c*.

The variations of the characteristic impedance of full m-derived low-pass π-sections are generally similar to the curve shown in Fig. 23. A comparison of the characteristic impedance curves of different m-derived filter sections is shown in Fig. 25.

In establishing an m-derived T-section the parameters are so re-adjusted from the constant-k values that the m-derived section characteristic impedance is identical with the constant-k section characteristic impedance. This requires that

$$Z_{2m} = \left[\frac{1-m^2}{4m}Z_{1k} + \frac{Z_{2k}}{m}\right] \quad \text{if} \quad Z_{1m} = mZ_{1k}$$

as may be seen from the following algebraic steps:

$$Z_{oTm} = Z_{oTk} \quad \text{(imposed condition)} \tag{89}$$

Reference to equation (5) will show that, if $Z_{1m} = mZ_{1k}$:

$$\sqrt{(mZ_{1k})Z_{2m} + \frac{(mZ_{1k})^2}{4}} = \sqrt{Z_{1k}Z_{2k} + \frac{Z_{1k}^2}{4}} \tag{90}$$

Squaring both sides of the above equation and solving for Z_{2m}:

$$Z_{2m} = \frac{1-m^2}{4m}Z_{1k} + \frac{Z_{2k}}{m} \tag{91}$$

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One of the most important characteristics of a full m-derived section is its theoretical infinite attenuation near the point of cut-off.

Frequencies of Infinite Attenuation. Since $\mathbf{Z}_{1k}$ and $\mathbf{Z}_{2k}$ are different types of reactances, the shunt arm of Fig. 24*a* will, at some frequency, become resonant. If the shunt arm is in resonance, its impedance is theoretically equal to zero and the attenuation becomes infinitely large. The frequency at which these phenomena occur is know as f_∞,

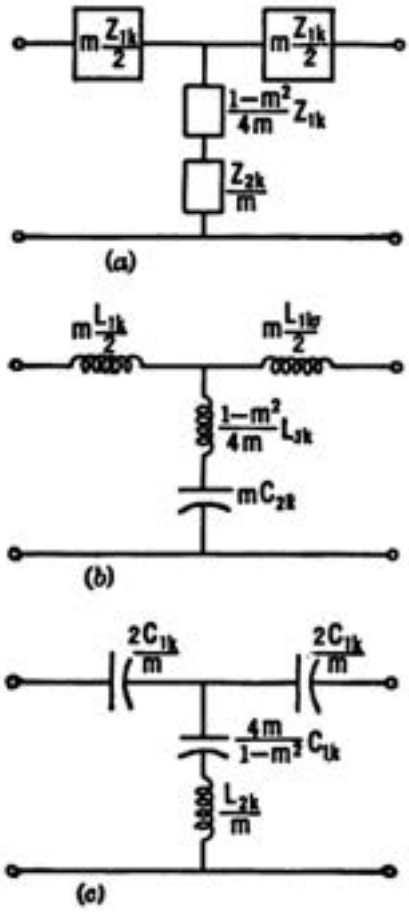

FIG. 24. m-Derived filter sections, with parameters specified in terms of constant-k filter-section parameters.

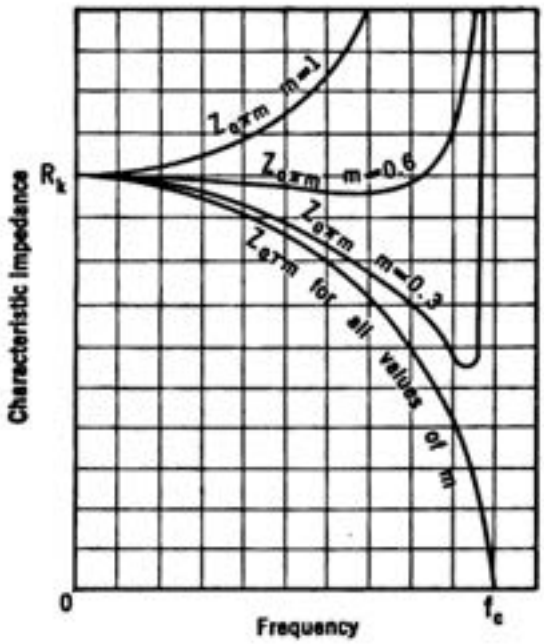

FIG. 25. Characteristic impedance curves for various low-pass m-derived filter sections.

and it may be calculated in any particular case by first setting the left-hand member of equation (91) equal to zero and then solving for f. In a low-pass, m-derived filter section:

$$f_\infty = \frac{1}{2\pi\sqrt{\frac{(1-m^2)}{4}L_{1k}C_{2k}}}$$

$$= \frac{1}{\pi\sqrt{L_{1k}C_{2k}}\sqrt{1-m^2}} \tag{92}$$

The cut-off frequency of the m-derived section is equal to the cut-off frequency of the constant-k section from which it is derived. (See

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Table V, page 486.) In the constant-k, low-pass section:

$$f_c = \frac{1}{\pi\sqrt{L_{1k}C_{2k}}} \quad \text{[See equation (55).]}$$

Therefore

$$f_\infty = \frac{f_c}{\sqrt{1 - m^2}} \tag{93}$$

from which

$$m = \sqrt{1 - \frac{f_c^2}{f_\infty^2}} \quad \text{(for low-pass section)} \tag{94}$$

In a similar manner it may be shown that for a high-pass, m-derived filter section:

$$f_\infty = f_c\sqrt{1 - m^2} \tag{95}$$

and

$$m = \sqrt{1 - \frac{f_\infty^2}{f_c^2}} \quad \text{(for high-pass section)} \tag{96}$$

Equations (94) and (96) illustrate the manner in which f_c and f_∞ determine the value of m that should be employed if theoretical infinite attenuation is to obtain at a specified f_∞. If, for example, a 1000-cycle cut-off frequency, low-pass filter is to have infinite attenuation at 1050 cycles, m is evaluated in accordance with equation (94). Thus:

$$m = \sqrt{1 - \frac{(1000)^2}{(1050)^2}} = 0.307 \quad \text{approximately}$$

The nearer f_∞ is to f_c, the lower will be the value of m. The reverse order of reasoning indicates that the lower the value of m, the sharper will be the cut-off. These facts are illustrated graphically in Fig. 26.

General Method of Analyzing m-Derived Filter Section Operation. Certain aspects of m-derived filter section operation may not be apparent from the cursory treatment that has been presented. The exact manner in which the phase shift and attenuation vary with respect to frequency can be obtained by subjecting the filter section to the "general" method of analysis. This method is summed up in equation (39), which, for convenience, is restated below.

$$\alpha + j\beta = 2 \log_\epsilon \left[\sqrt{\frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2} + 1}\right] \tag{39}$$

For the sake of illustration a low-pass, m-derived, T-section will be

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analyzed. From Fig. 24*b* it is evident that

$$\mathbf{Z}_{1m} = j\omega m L_{1k} = \mathbf{Z}_1$$

and

$$\mathbf{Z}_{2m} = j\left[\omega\frac{(1-m^2)}{4m}L_{1k} - \frac{1}{\omega m C_{2k}}\right] = \mathbf{Z}_2$$

Therefore, in the present case,

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = \frac{\omega m L_{1k}}{4\left[\omega\frac{(1-m^2)}{4m}L_{1k} - \frac{1}{\omega m C_{2k}}\right]}$$

$$= \frac{\omega^2 m^2 L_{1k}C_{2k}}{\omega^2(1-m^2)L_{1k}C_{2k} - 4} \tag{97}$$

The above expression is actually a complex number, the associated angle of which is 180° or 0°, depending upon whether $[\omega^2(1-m^2)L_{1k}C_{2k}]$ is less than or greater than 4. The foregoing statement follows directly

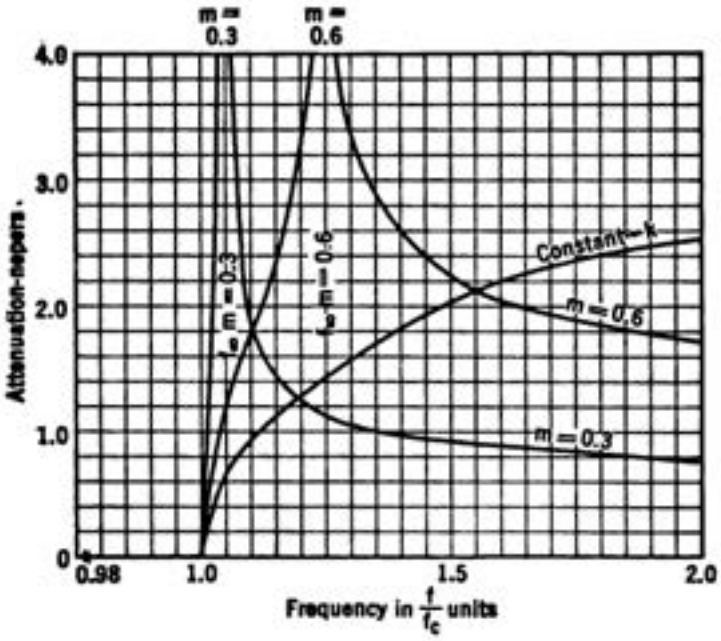

FIG. 26. Attenuation characteristics of two *m*-derived low-pass filter sections compared with those of a constant-*k* low-pass filter section.

from an inspection of $\mathbf{Z}_1/4\mathbf{Z}_2$ wherein all the factors are expressed in polar form. Let ω be arbitrarily reckoned in $1/\sqrt{L_{1k}C_{2k}}$ units. It should be observed that in this method of analysis the cut-off angular velocity or frequency is not necessarily anticipated by the choice of this convenient unit. Thus, for $\omega = 1/\sqrt{L_{1k}C_{2k}}$ radians per second, equation (39) reduces to

$$\alpha + j\beta = 2\log_e\left[\sqrt{\frac{m^2}{(1-m^2)-4}} - \sqrt{\frac{m^2}{(1-m^2)-4} + 1}\right]$$

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For a particular value of m it becomes a simple matter to evaluate α and β at any desired frequency. The calculations for $m = 0.6$ at various frequencies are shown in Table V. The variations of attenuation are represented graphically in Fig. 26 together with certain other attenuation curves. An inspection of column (9), Table V, will reveal the irregular manner in which the phase shift varies with frequency.

Problem 16. Graph the variation of attenuation with respect to frequency of a low-pass, m-derived T-section in which $m = 0.40$. The frequency may be indicated in terms of $1/\sqrt{L_{1k}C_{2k}}$ units of angular velocity. (See Table V, page 486.)

Comparison of Attenuation Characteristics. Constant-k and m-derived filter sections are sometimes worked in cascade because of the complementary nature of their respective attenuation characteristics. It has been shown that the attenuation of a constant-k, low-pass section is zero at cut-off frequency and that it increases gradually with increases of frequency above cut-off frequency. (See Fig. 12.) A similar situation holds for the constant-k, high-pass section except, of course, for the fact that the attenuation increases as the frequency decreases from the cut-off frequency. The attenuation characteristics of m-derived sections are radically different in character from those of constant-k sections. The differences are shown graphically in Fig. 26 for low-pass sections. Similar curves can be determined for high-pass sections.

It is plain from an inspection of Fig. 26 that a constant-k section can be combined with one or more m-derived sections to give high attenuation near cut-off as well as high attenuation in other regions of the stop band. In general, an m-derived section by itself will not give high attenuation in regions which are too widely removed from the point of theoretical infinite attenuation. (See Fig. 26.)

General Design Procedure. Filter sections are usually designed for a particular characteristic impedance and a particular cut-off frequency (or frequencies). Theoretically, at least, these conditions can be met accurately and straightforwardly. Usually certain attenuation requirements must also be met. These attenuation requirements are generally met by a method of successive approximations.

The *first step* in elementary filter design is the determination of the inductances and capacitances to be employed in a constant-k section. These values are found from the basic design equations.

The *second step* is the evaluation of the m-derived, terminating half-section inductances and capacitances. These values follow directly from the parameters of the constant-k section and the selected value of m. It is assumed here that the terminating half sections are required primarily for impedance-matching purposes, in which case the value of m will generally be 0.6.

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TABLE V

Evaluation of $\left[\alpha + j\beta = 2\log_e\left(\sqrt{\frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2}+1}\right)\right]$ where $\frac{Z_1}{4Z_2} = \frac{\omega^2 m^2 L_{1k}C_{2k}}{\omega^2(1-m^2)L_{1k}C_{2k} - 4}$

$m = 0.6$ and ω is reckoned in $\left[\frac{1}{\sqrt{L_{1k}C_{2k}}}\right]$ units which are called U. Under these conditions $\frac{Z_1}{4Z_2} = \frac{0.36U^2}{0.64U^2 - 4}$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
ω U units	$\frac{Z_1}{4Z_2}$	$\sqrt{\frac{Z_1}{4Z_2}}$	$\sqrt{\frac{Z_1}{4Z_2}+1}$	(3) + (4) Cartesian Form	(3) + (4) Polar Form	$2\log_e$ (6)	α nepers	β radians
0	0	0	1	$1 + j0$	$1\angle 0°$	$0 + j0$	0	0
1.0	$+0.107\angle 180°$	$j0.327$	0.945	$0.945 + j0.327$	$1\angle 19.2°$	$0 + j0.67$	0	0.67
1.5	$+0.316\angle 180°$	$j0.564$	0.826	$0.826 + j0.564$	$1\angle 34.3°$	$0 + j1.197$	0	1.20
2.0	$+1.00\angle 180°$	$j1.0$	0	$0 + j1.0$	$1\angle 90°$	$0 + j3.14$	0	3.14
2.1	$+1.35\angle 180°$	$j1.16$	$j0.59$	$0 + j1.75$	$1.75\angle 90°$	$1.12 + j3.14$	1.12	3.14
2.25	$+2.40\angle 180°$	$j1.55$	$j1.18$	$0 + j2.73$	$2.73\angle 90°$	$2.01 + j3.14$	2.01	3.14
2.50	∞	∞	∞	∞	∞	∞	∞	3.14 0.00
2.75	$+3.24\angle 0°$	1.80	2.06	$3.86 + j0$	$3.86\angle 0°$	$2.70 + j0$	2.70	0
3.0	$+1.84\angle 0°$	1.36	1.68	$3.04 + j0$	$3.04\angle 0°$	$2.22 + j0$	2.22	0
4.0	$+0.923\angle 0°$	0.961	1.39	$2.35 + j0$	$2.35\angle 0°$	$1.71 + j0$	1.71	0
6.0	$+0.683\angle 0°$	0.826	1.30	$2.13 + j0$	$2.13\angle 0°$	$1.51 + j0$	1.51	0

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If a sharp cut-off section is required, a full m-derived section, wherein m is about 0.2 or 0.3, can be employed. The evaluation of the inductances and capacitances to use in the full m-derived section constitutes the *third step* in the general design procedure.

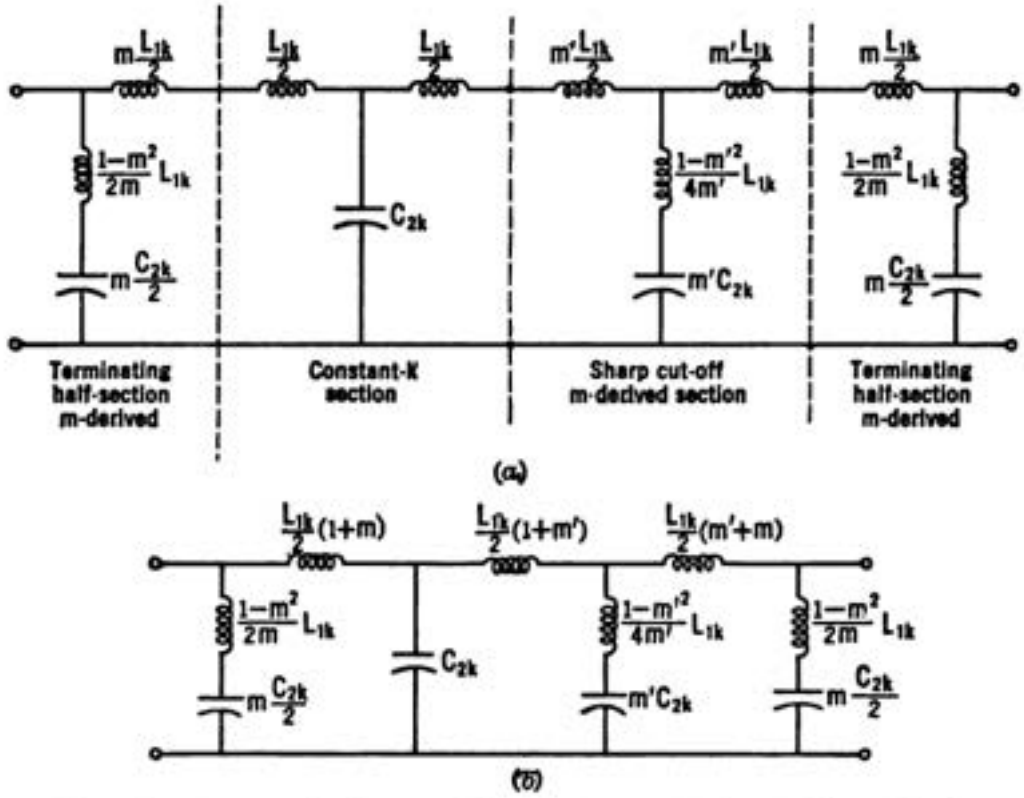

FIG. 27. A composite low-pass filter together with its equivalent circuit.

The *fourth step* is the predetermination of the attenuation characteristic of the composite filter and checking this against the actual attenuation requirements. Adjustments may then be made in the number or in the type of sections in order to meet the attenuation requirements in the most economical manner.

The method of combining a constant-k section, a full m-derived section, and m-derived terminating half sections to form a low-pass filter is illustrated in Fig. 27*a*. It will be noted that the assembly shown in Fig. 27*a* is reducible to that shown in Fig. 27*b*.

PROBLEMS

17. Consider a π-type filter section in which the full series arm, Z_1, consists of a 100-millihenry inductance coil the resistance of which is 50 ohms. Each of the two shunt arms consists of a 0.3-μf condenser the resistance of which is negligibly small.

(*a*) Find the open-circuit impedance, $Z_{o\text{-}c}$, and the short-circuit impedance, $Z_{s\text{-}c}$, of the section at 500 cycles.

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(*b*) Find the characteristic impedance at 500 cycles, at 1300 cycles, and at 2000 cycles.

18. Each of the series arms ($Z_1/2$) of a symmetrical T-section consists of a condenser the capacitance of which is 0.6 μf and the resistance of which is negligibly small. The shunt arm (Z_2) is a 200-millihenry inductance coil the resistance of which is 60 ohms.

(*a*) Find the characteristic impedance and the propagation constant of the section at 200 cycles.

(*b*) Find the characteristic impedance and the propagation constant of the section at 600 cycles.

19. The characteristic impedance of a filter section is to be measured. The measuring device is a 1-B Western Electric impedance bridge which indicates the R component of the impedance directly and the X component in terms of $+L$ or $-L$. Plus L readings indicate that $X = X_L = 2\pi fL$, and negative L readings indicate that $X = X_C = 2\pi f(-L)$. With the output terminals of the section open-circuited the bridge readings are: $R = 10$ ohms and $L = -190$ millihenrys at 400 cycles. With the output terminals of the section short-circuited the bridge readings are: $R = 20$ ohms and $L = +250$ millihenrys at 400 cycles. Find the characteristic impedance of the filter section at 400 cycles.

20. A resistanceless constant-k, low-pass, T-section has a cut-off frequency of 10,000 cycles and a zero-frequency characteristic impedance of 800 ohms. Evaluate the phase shift at 1000, 4000, 7000, and 10,000 cycles. Evaluate the attenuation at 11,000, 15,000, 20,000, and 25,000 cycles. Plot phase shift in degrees and attenuation in decibels against cycles per second.

21. Consider a symmetrical π-type section in which the inductance of the full series arm is 0.10 henry and the capacitance of each of the two condensers which go to form the π-section is 0.3 μf.

(*a*) Neglecting the resistive components of the circuit elements, find the propagation constant at 500 cycles, at 1300 cycles, and at 2000 cycles.

(*b*) What is the attenuation in decibels at each of the three frequencies referred to above?

22. (*a*) What is the decibel level of 0.00001 watt with respect to a 1-milliwatt reference power level?

(*b*) What is the decibel level of 6 watts with respect to a 1-milliwatt reference power level?

23. What is the cut-off frequency of a low-pass, constant-k, π-type filter section in which the inductance of the full series arm is 20 henrys and the capacitance of each condenser is 8.0 μf? What is the characteristic impedance of the section at 200 cycles?

24. A T-section filter has series arms $Z_1/2 = j100$ ohms and its shunt arm $Z_2 = -j1000$ ohms.

(*a*) Calculate the characteristic impedance.

(*b*) Calculate the attenuation in decibels and the phase shift in degrees.

(*c*) Are the reactances of the section for a frequency within the pass or stop band?

(*d*) Calculate the characteristic impedance of the section for 5 times the frequency for which the constants are given.

(*e*) Calculate the attenuation in decibels and phase shift for part (*d*).

25. A π-section filter has its series arm $Z_1 = -j100$ ohms and its shunt arms $2Z_2 = j500$ ohms.

(*a*) Calculate the characteristic impedance.

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(b) Calculate the attenuation in decibels and the phase shift in degrees.

(c) Are the reactances given for a frequency within the pass band or stop band?

(d) Repeat parts (a), (b), and (c) for a frequency of one-fifth of that for which the impedances are given.

26. Nine T-sections each having series arms of $Z_1/2 = j500$ ohms and shunt arms $Z_2 = -j200$ ohms are connected in series or cascade. If the input voltage is 100, find the output voltage of the ninth section and the output current, assuming characteristic termination.

27. Find the circuit element values of a high-pass, constant-k, T-type filter section which is to have a cut-off frequency of 5000 cycles and an infinite-frequency characteristic impedance of 600 ohms. Repeat for a π-type section. Draw circuit diagrams showing the configurations of the circuit elements and the values of each in millihenrys and microfarads.

28. A generator having an impedance of $800\underline{/0^\circ}$ ohms is to be connected to a load impedance of $100\underline{/0^\circ}$ through a half-section of the kind shown in Fig. 21, page 476. Find the value of $Z_{1k}/2$ (the series arm impedance) and of $2Z_{2k}$ (the shunt arm impedance) which will properly match the generator to the load. $Z_{1k}/2$ is arbitrarily taken as inductive.

29. Design a high-pass, m-derived, T-type filter section which will have a cut-off frequency of 5000 cycles, an infinite-frequency characteristic impedance of 600 ohms, and an infinite-attenuation frequency of 4500 cycles.

30. Design m-derived half sections which will properly match, at 800 cycles, a low-pass, constant-k, T-type section the full series arm of which is 0.30 henry and the full shunt arm of which is 0.03 μf. The value of m is to be taken as 0.60.

31. Consider an m-derived, low-pass, T-section in which Z_{1m} is mL_{1k} and Z_{2m} consists of $(1 - m^2/4m)L_{1k}$ in series with mC_{2k}. Let mL_{1k} be known as L_{1m}, $(1 - m^2/4m)L_{1k}$ be known as L_{2m}, and mC_{2k} be known as C_{2m}. Show that the cut-off frequency, namely, $1/\pi\sqrt{L_{1k}C_{2k}}$, can be written as $1/[\pi\sqrt{(L_{1m} + 4L_{2m})(C_{2m})}]$.

32. Refer to the composite low-pass filter shown in Fig. 27. The requirements to be met are: (1) zero-frequency characteristic impedance of 600 ohms, (2) cut-off frequency of 5000 cycles, (3) variation in characteristic impedance of not more than 30 ohms over the lower 80 per cent of the pass band, (4) attenuation of 40 decibels between the limits of 5242 and 10,000 cycles.

(a) Calculate the values of L_{1k} and C_{2k}.

(b) Design terminating half sections on the basis of $m = 0.60$.

(c) Design the full m-derived section to have theoretical infinite attenuation at 5242 cycles.

(d) Make a graph of the attenuation of the composite filter between the limits of 5242 and 10,000 cycles and compare the results with the attenuation requirements. Use the three attenuation graphs shown in Fig. 26, page 484, at $f/f_c = 1.05$, 1.10, 1.15, 1.20, 1.25, 1.30, 1.35, 1.40, 1.45, 1.50, 1.75, and 2 to obtain the composite attenuation graph.

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CHAPTER XIV

SYMMETRICAL COMPONENTS

Symmetrical components furnish a tool of great power for analytically determining the performance of certain types of unbalanced electrical circuits involving rotating electrical machines. It is particularly useful in analyzing the performance of polyphase electrical machinery when operated from systems of unbalanced voltages. Although it can be used to solve unbalanced static networks like those in Chapter IX, such application will in general be more cumbersome and laborious than the methods already considered. For unbalanced networks containing rotating machines, however, the method of symmetrical components provides the only practicable method of accounting for the unbalanced effects of these machines and is widely used in practice. It is also convenient for analyzing some types of polyphase transformer problems.

The method of "symmetrical components," in its most useful form, is founded upon Fortescue's[1] theorem regarding the resolution of unbalanced systems into symmetrical components. Although the present discussion will confine itself to three-phase systems, any unbalanced polyphase system of vectors can be resolved into balanced systems of vectors called "symmetrical components."

Fortescue's theorem, applied to a general three-phase system of vectors, is that any unbalanced three-phase system of vectors can be resolved into three balanced systems of vectors, namely:

1. A balanced system of three-phase vectors having the same phase sequence as the original unbalanced system of vectors. This balanced system is called the "positive-sequence system."

2. A balanced system of three-phase vectors having a phase sequence which is opposite to that of the original unbalanced system of vectors. This balanced system is called the "negative-sequence system."

3. A system of three single-phase vectors which are equal in magnitude and which have exactly the same time-phase position with respect to any given reference axis. This system of single-phase vectors is known as the zero-sequence or uniphase system.

A general proof of the resolution theorem will not be given because a little experience with the method will soon convince the reader that

[1] Fortescue, "Method of Symmetrical Co-ordinates Applied to the Solution of Polyphase Networks," *Transactions, A.I.E.E.*, Vol. 37, 1918.

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the theorem as stated is correct. In this respect Fortescue's theorem is similar to Fourier's theorem regarding complex waves. In Chapter VI it is shown that any complex wave may be resolved into definite harmonic components by the Fourier method. The ultimate proof of the theorem rests upon the fact that the components thus determined can be synthesized to form the original complex wave. In a similar manner it will be shown that any given unbalanced three-phase system of vectors may be resolved into the three balanced systems outlined above and that the composition of these balanced systems yields the original unbalanced system of vectors.

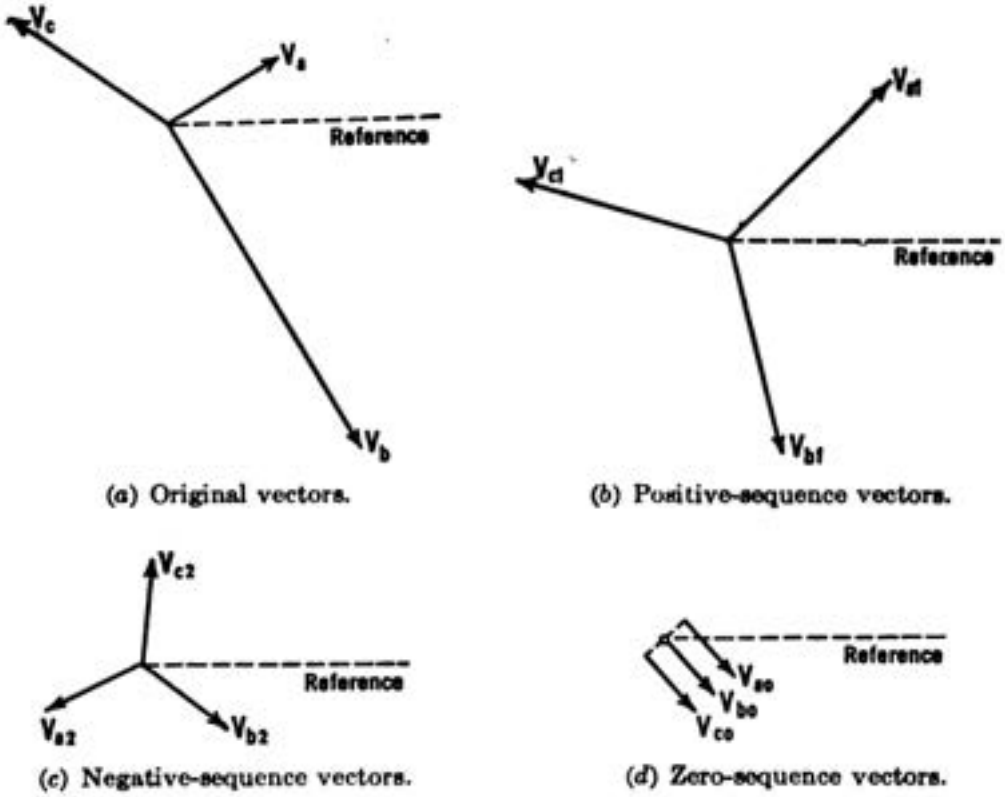

FIG. 1. Original set of three-phase vectors together with their symmetrical components.

The Original Unbalanced System of Vectors. Any number of vectors up to and including three may be considered as an unbalanced system of three-phase vectors. The vectors that form the unbalanced system may have any specified magnitude (including zero) and may possess any specified phase positions with respect to one another. In Fig. 1*a* is shown a set of three unbalanced vectors that will later be resolved into their symmetrical components. If the vectors that form the original unbalanced set come to us merely as three specified vectors, they can arbitrarily be assigned subscripts *a*, *b*, and *c* in the order shown in Fig. 1*a*. Thus the original vectors $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$ are arbitrarily assigned the *abc* phase sequence. (See Chapter IX, pages

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330–331.) Although the vectors shown in Fig. 1*a* are labeled as voltages, the proposed resolution applies equally well to a system of current vectors.

Owing to the fact that the symmetrical components will have to carry an additional subscript to designate the system to which they belong, single-subscript notation will be employed in connection with the original vectors wherever this can be done without loss of clarity. For complete specification, the positive circuit directions of the original three-phase voltages or currents must be indicated on a separate circuit diagram. The importance of complete specification will become apparent when numerical problems are considered.

The Positive-Phase Sequence System. As previously stated, the original unbalanced system of vectors is to be resolved into two balanced three-phase systems and one uniphase system. It will be shown presently that the balanced three-phase systems must be of opposite phase sequence. Therefore one balanced system has the same phase sequence as the original three-phase system and the other has a phase sequence opposite to that of the original system.

The balanced system of three-phase vectors that has the same phase sequence as the original system is called the positive-sequence system. If the original vectors are assigned the phase sequence of *abc*, then the phase sequence of the positive-sequence vectors is *abc* as shown in Fig. 1*b*. The positive-sequence vectors are completely determined when the magnitude and phase position of any one of them is known. A method of evaluating any one of the positive-sequence vectors in terms of the original vector values will be given presently. The positive-sequence vectors are designated as

$$\mathbf{V}_{a1}, \quad \mathbf{V}_{b1}, \quad \text{and} \quad \mathbf{V}_{c1}$$

The subscript 1 indicates that the vector thus labeled belongs to the positive-sequence system. The letters refer to the original vector of which the positive-sequence vector is a component part.

The vectors of any balanced three-phase system may be conveniently related to one another with the aid of the operator **a**. The general properties of this operator are considered in Chapter IV, page 77.

a1 is a unit vector 120° ahead of the reference axis. $\mathbf{a}^2 1$ is a unit vector 240° ahead of the reference axis. Thus:

$$\left.\begin{aligned} \mathbf{a}1 &= \epsilon^{j120^\circ} = -0.5 + j0.866 \\ \mathbf{a}^2 1 &= \epsilon^{j240^\circ} = -0.5 - j0.866 \end{aligned}\right\} \tag{1}$$

The operator **a** applied to any vector rotates that vector through 120° in the positive or counterclockwise direction. The operator $\mathbf{a}^2$ applied

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to any vector rotates that vector through 240° in the positive direction, which is, of course, equivalent to a rotation of 120° in the negative direction.

If, for example, $\mathbf{V}_{a1}$ has been determined, the positive-sequence system may be written simply as

$$\left.\begin{aligned} \mathbf{V}_{a1} &= \mathbf{V}_{a1} \\ \mathbf{V}_{b1} &= \mathbf{a}^2\mathbf{V}_{a1} = \mathbf{V}_{a1}\underline{/-120^\circ} \\ \mathbf{V}_{c1} &= \mathbf{a}\mathbf{V}_{a1} = \mathbf{V}_{a1}\underline{/-240^\circ} \end{aligned}\right\} \text{the positive-sequence system of vectors} \tag{2}$$

The Negative-Phase Sequence System. The balanced system of three-phase vectors which is opposite in phase sequence to that of the original vectors is called the negative-sequence system. If the original vectors have a phase sequence of *abc* the negative-sequence vectors have a phase sequence of *acb* as shown in Fig. 1*c*. Since the negative-sequence system is balanced, it is completely determined when one of the voltages is known. The negative-sequence vectors are designated as

$$\mathbf{V}_{a2}, \quad \mathbf{V}_{b2}, \quad \text{and} \quad \mathbf{V}_{c2}$$

Subscript 2 indicates that the vectors belong to the negative-sequence system. The *a*, *b*, and *c* subscripts indicate components of $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$ respectively. If $\mathbf{V}_{a2}$ is known, the negative-sequence system can be written in the following form:

$$\left.\begin{aligned} \mathbf{V}_{a2} &= \mathbf{V}_{a2} \\ \mathbf{V}_{b2} &= \mathbf{a}\mathbf{V}_{a2} = \mathbf{V}_{a2}\underline{/-240^\circ} \\ \mathbf{V}_{c2} &= \mathbf{a}^2\mathbf{V}_{a2} = \mathbf{V}_{a2}\underline{/-120^\circ} \end{aligned}\right\} \text{the negative-sequence system of vectors} \tag{3}$$

$\mathbf{V}_{a2}$, $\mathbf{V}_{b2}$, and $\mathbf{V}_{c2}$ are shown graphically in Fig. 1*c*.

The Zero-Phase Sequence System. The remaining system consists of three vectors, identical in magnitude and in time phase, as shown in Fig. 1*d*. These vectors form what is known as the uniphase or the zero-sequence system, and have special significance in certain physical problems. For the present it will be sufficient to think of the zero-sequence vectors as components of the original vectors $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$. The zero-sequence vectors are designated as

$$\mathbf{V}_{a0}, \quad \mathbf{V}_{b0}, \quad \text{and} \quad \mathbf{V}_{c0}$$

Since the above voltages are equal:

$$\left.\begin{aligned} \mathbf{V}_{a0} &= \mathbf{V}_{a0} \\ \mathbf{V}_{b0} &= \mathbf{V}_{a0} \\ \mathbf{V}_{c0} &= \mathbf{V}_{a0} \end{aligned}\right\} \text{the zero-sequence system of vectors} \tag{4}$$

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Graphical Composition of Sequence Vectors. It is evident that

$$(\mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0})$$
$$(\mathbf{V}_{b1} + \mathbf{V}_{b2} + \mathbf{V}_{b0})$$

and

$$(\mathbf{V}_{c1} + \mathbf{V}_{c2} + \mathbf{V}_{c0})$$

form a three-phase system of voltages which, in general, is unbalanced. The above-indicated compositions are carried out graphically in Fig. 2, employing the individual voltages contained in Figs. 1*b*, 1*c*, and 1*d*.

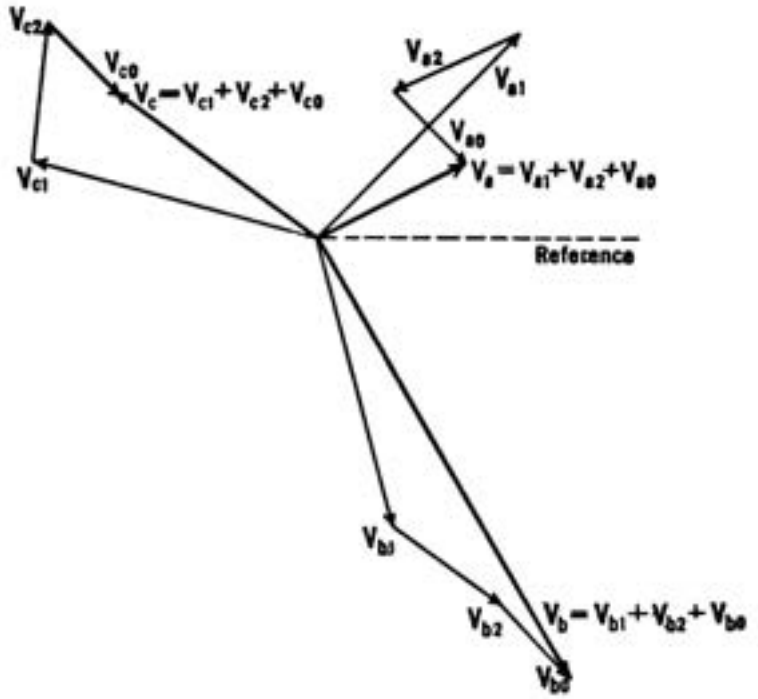

FIG. 2. Illustrating the manner in which the sequence components combine to form $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$.

The resultant system shown in Fig. 2 is identical with the unbalanced system shown in Fig. 1*a*. For the particular case considered it is plain that

$$\mathbf{V}_a = \mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0} \tag{5}$$

$$\mathbf{V}_b = \mathbf{V}_{b1} + \mathbf{V}_{b2} + \mathbf{V}_{b0} \tag{6}$$

$$\mathbf{V}_c = \mathbf{V}_{c1} + \mathbf{V}_{c2} + \mathbf{V}_{c0} \tag{7}$$

In terms of the operator **a**, the above relations may be stated as

$$\mathbf{V}_a = \mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0} \tag{8}$$

$$\mathbf{V}_b = \mathbf{a}^2\mathbf{V}_{a1} + \mathbf{a}\mathbf{V}_{a2} + \mathbf{V}_{a0} \tag{9}$$

$$\mathbf{V}_c = \mathbf{a}\mathbf{V}_{a1} + \mathbf{a}^2\mathbf{V}_{a2} + \mathbf{V}_{a0} \tag{10}$$

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An inspection of equations (8), (9), and (10) will show that the original system of vectors can be completely specified in terms of $\mathbf{V}_{a1}$, $\mathbf{V}_{a2}$, $\mathbf{V}_{a0}$, and the operator $\mathbf{a}$. The next step in the study of symmetrical components is the evaluation of $\mathbf{V}_{a1}$, $\mathbf{V}_{a2}$, and $\mathbf{V}_{a0}$ in terms of the original vectors $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$.

Evaluation of $\mathbf{V}_{a1}$. The resolution of an unbalanced system of vectors into its symmetrical components is essentially a geometric process, and different geometric methods have been devised whereby the resolution can be effected. However, none of the geometric methods thus far devised possesses the neat simplicity of the complex algebra method given below.

Before proceeding with the algebraic method it is well to understand that certain operations are performed solely for the purpose of obtaining the combination $(1 + \mathbf{a} + \mathbf{a}^2)$ which is equal to zero. Various simplifications may thus be made when quantities can be so collected as to possess the coefficient $(1 + \mathbf{a} + \mathbf{a}^2)$.

If equation (9) is multiplied by $\mathbf{a}$ the result is:

$$\mathbf{a}\mathbf{V}_b = \mathbf{a}^3\mathbf{V}_{a1} + \mathbf{a}^2\mathbf{V}_{a2} + \mathbf{a}\mathbf{V}_{a0}$$

or, since $\mathbf{a}^3 = 1$,

$$\mathbf{a}\mathbf{V}_b = \mathbf{V}_{a1} + \mathbf{a}^2\mathbf{V}_{a2} + \mathbf{a}\mathbf{V}_{a0} \tag{11}$$

If equation (10) is multiplied by $\mathbf{a}^2$, the result is:

$$\mathbf{a}^2\mathbf{V}_c = \mathbf{a}^3\mathbf{V}_{a1} + \mathbf{a}^4\mathbf{V}_{a2} + \mathbf{a}^2\mathbf{V}_{a0}$$

or, since $\mathbf{a}^4 = \mathbf{a}$,

$$\mathbf{a}^2\mathbf{V}_c = \mathbf{V}_{a1} + \mathbf{a}\mathbf{V}_{a2} + \mathbf{a}^2\mathbf{V}_{a0} \tag{12}$$

Adding equations (8), (11), and (12) yields

$$\mathbf{V}_a + \mathbf{a}\mathbf{V}_b + \mathbf{a}^2\mathbf{V}_c = 3\mathbf{V}_{a1} + (1 + \mathbf{a} + \mathbf{a}^2)(\mathbf{V}_{a2} + \mathbf{V}_{a0})$$

whence:

$$\mathbf{V}_{a1} = \tfrac{1}{3}(\mathbf{V}_a + \mathbf{a}\mathbf{V}_b + \mathbf{a}^2\mathbf{V}_c) = \tfrac{1}{3}(\mathbf{V}_a + \mathbf{V}_b\underline{/120^\circ} + \mathbf{V}_c\underline{/240^\circ}) \tag{13}$$

Geometrically speaking, the above equation means that $\mathbf{V}_{a1}$ is a vector one-third as large as the vector which results from the addition of the three vectors $\mathbf{V}_a$, $\mathbf{V}_b\underline{/120^\circ}$, and $\mathbf{V}_c\underline{/240^\circ}$.

Example 1. If the vectors shown in Fig. 1*a* are:

$$\mathbf{V}_a = 10\underline{/30^\circ}, \quad \mathbf{V}_b = 30\underline{/-60^\circ}, \quad \text{and} \quad \mathbf{V}_c = 15\underline{/145^\circ} \text{ units}$$

$$\begin{aligned}\mathbf{V}_{a1} &= \tfrac{1}{3}(10\underline{/30^\circ} + \mathbf{a}30\underline{/-60^\circ} + \mathbf{a}^2 15\underline{/145^\circ})\\ &= \tfrac{1}{3}(10\underline{/30^\circ} + 30\underline{/60^\circ} + 15\underline{/25^\circ})\\ &= 12.42 + j12.45 = 17.6\underline{/45^\circ.0} \text{ units}\end{aligned}$$

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Since $\mathbf{V}_{b1} = \mathbf{V}_{a1}\underline{/-120^\circ}$ and $\mathbf{V}_{c1} = \mathbf{V}_{a1}\underline{/+120^\circ}$, the positive-sequence system of vectors becomes:

$$\mathbf{V}_{a1} = 17.6\underline{/45^\circ},\ \mathbf{V}_{b1} = 17.6\underline{/-75^\circ},\ \text{and}\ \mathbf{V}_{c1} = 17.6\underline{/165^\circ}\ \text{units}$$

The above results are indicated graphically in Fig. 1*b*.

Evaluation of $\mathbf{V}_{a2}$. The negative-sequence component of $\mathbf{V}_{a2}$ can be evaluated in a manner almost identical with that given above for the evaluation of $\mathbf{V}_{a1}$. It is simply necessary to study equations (8), (9), and (10) with a view toward eliminating the $\mathbf{V}_{a1}$ and $\mathbf{V}_{a0}$ terms and at the same time retain the $\mathbf{V}_{a2}$ terms. The desired results can be obtained by multiplying equation (9) through by $\mathbf{a}^2$ and equation (10) through by $\mathbf{a}$. Equation (9) multiplied by $\mathbf{a}^2$ reduces to

$$\mathbf{a}^2\mathbf{V}_b = \mathbf{a}\mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{a}^2\mathbf{V}_{a0} \tag{14}$$

Equation (10) multiplied by $\mathbf{a}$ reduces to

$$\mathbf{a}\mathbf{V}_c = \mathbf{a}^2\mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{a}\mathbf{V}_{a0} \tag{15}$$

Adding equations (8), (14), and (15) yields

$$\mathbf{V}_a + \mathbf{a}^2\mathbf{V}_b + \mathbf{a}\mathbf{V}_c = 3\mathbf{V}_{a2} + (1 + \mathbf{a} + \mathbf{a}^2)(\mathbf{V}_{a1} + \mathbf{V}_{a0})$$

Since $(1 + \mathbf{a} + \mathbf{a}^2) = 0$,

$$\mathbf{V}_{a2} = \tfrac{1}{3}(\mathbf{V}_a + \mathbf{a}^2\mathbf{V}_b + \mathbf{a}\mathbf{V}_c) = \tfrac{1}{3}(\mathbf{V}_a + \mathbf{V}_b\underline{/240^\circ} + \mathbf{V}_c\underline{/120^\circ}) \tag{16}$$

$\mathbf{V}_{a2}$ is therefore a vector one-third the magnitude of [$\mathbf{V}_a$ + ($\mathbf{V}_b$ rotated through +240°) + ($\mathbf{V}_c$ rotated through +120°)].

Example 2. If $\mathbf{V}_a = 10\underline{/30^\circ}$, $\mathbf{V}_b = 30\underline{/-60^\circ}$, and $\mathbf{V}_c = 15\underline{/145^\circ}$ units:

$$\begin{aligned}\mathbf{V}_{a2} &= \tfrac{1}{3}(10\underline{/30^\circ} + \mathbf{a}^2 30\underline{/-60^\circ} + \mathbf{a}15\underline{/145^\circ})\\ &= \tfrac{1}{3}(10\underline{/30^\circ} + 30\underline{/180^\circ} + 15\underline{/265^\circ})\\ &= -7.55 - j3.32 = 8.25\underline{/-156.2^\circ}\ \text{units}\end{aligned}$$

$\mathbf{V}_{a2}$ for this particular case is shown in Fig. 1*c* together with $\mathbf{V}_{b2}$ and $\mathbf{V}_{c2}$. $\mathbf{V}_{b2} = \mathbf{V}_{a2}\underline{/120^\circ}$ and $\mathbf{V}_{c2} = \mathbf{V}_{a2}\underline{/-120^\circ}$.

Evaluation of $\mathbf{V}_{a0}$. The direct addition of equations (8), (9), and (10) will show that:

$$\mathbf{V}_a + \mathbf{V}_b + \mathbf{V}_c = \mathbf{V}_{a1}(1 + \mathbf{a}^2 + \mathbf{a}) + \mathbf{V}_{a2}(1 + \mathbf{a} + \mathbf{a}^2) + 3\mathbf{V}_{a0}$$

or

$$\mathbf{V}_{a0} = \tfrac{1}{3}(\mathbf{V}_a + \mathbf{V}_b + \mathbf{V}_c) \tag{17}$$

The zero-sequence component is simply a vector one-third as large as the vector obtained by adding $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$.

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Example 3. If $\mathbf{V}_a = 10\,\underline{/30^\circ}$, $\mathbf{V}_b = 30\,\underline{/-60^\circ}$, and $\mathbf{V}_c = 15\,\underline{/145^\circ}$ units:

$$\mathbf{V}_{a0} = \tfrac{1}{3}(10\,\underline{/30^\circ} + 30\,\underline{/-60^\circ} + 15\,\underline{/145^\circ})$$
$$= 3.79 - j4.13 = 5.60\,\underline{/-47.4^\circ} \text{ units}$$

The above value of $\mathbf{V}_{a0}$ together with corresponding values of $\mathbf{V}_{b0}$ and $\mathbf{V}_{c0}$ are shown in Fig. 1*d*.

Example 4. (*a*) The results obtained in the foregoing examples can be checked by comparing the complex expression for $(\mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0})$ with the complex expression of the original vector $\mathbf{V}_a$. The results of the foregoing examples are tabulated below.

$$\mathbf{V}_{a1} = 12.42 + j12.45 = 17.6\,\underline{/45^\circ} \text{ units}$$
$$\mathbf{V}_{a2} = -7.55 - j3.32 = 8.25\,\underline{/-156.2^\circ} \text{ units}$$
$$\mathbf{V}_{a0} = 3.79 - j4.13 = 5.60\,\underline{/-47.4^\circ} \text{ units}$$
$$(\mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0}) = 8.66 + j5.00 = 10\,\underline{/30^\circ} = \mathbf{V}_a$$

(*b*)
$$\mathbf{V}_{b1} = \mathbf{a}^2 17.6\,\underline{/45^\circ} = 17.6\,\underline{/-75^\circ} = 4.56 - j17.0 \text{ units}$$
$$\mathbf{V}_{b2} = \mathbf{a}8.25\,\underline{/-156.2^\circ} = 8.25\,\underline{/-36.2^\circ} = 6.66 - j4.87 \text{ units}$$
$$\mathbf{V}_{b0} = 5.60\,\underline{/-47.4^\circ} = 3.79 - j4.13 \text{ units}$$
$$(\mathbf{V}_{b1} + \mathbf{V}_{b2} + \mathbf{V}_{b0}) = 15.0 - j26.0 = 30.0\,\underline{/-60^\circ} = \mathbf{V}_b$$

(*c*)
$$\mathbf{V}_{c1} = \mathbf{a}17.6\,\underline{/45^\circ} = 17.6\,\underline{/165^\circ} = -17.0 + j4.56 \text{ units}$$
$$\mathbf{V}_{c2} = \mathbf{a}^2 8.25\,\underline{/-156.2^\circ} = 8.25\,\underline{/83.8^\circ} = 0.89 + j8.20 \text{ units}$$
$$\mathbf{V}_{c0} = 5.60\,\underline{/-47.4^\circ} = 3.79 - j4.13 \text{ units}$$
$$(\mathbf{V}_{c1} + \mathbf{V}_{c2} + \mathbf{V}_{c0}) = -12.32 + j8.63 = 15\,\underline{/145^\circ} = \mathbf{V}_c$$

Problem 1. Given the following three vector voltages: $\mathbf{V}_a = 150\,\underline{/0^\circ}$, $\mathbf{V}_b = 86.6\,\underline{/-90^\circ}$, and $\mathbf{V}_c = 86.6\,\underline{/90^\circ}$ volts.

(*a*) Find the symmetrical components of $\mathbf{V}_a$ and check the results by adding $\mathbf{V}_{a1}$, $\mathbf{V}_{a2}$, and $\mathbf{V}_{a0}$.

(*b*) Evaluate $\mathbf{V}_b$ and $\mathbf{V}_c$ in terms of the symmetrical components of $\mathbf{V}_a$ found in part (*a*).

(*c*) Draw a vector diagram illustrating all symmetrical components.

Ans.: (*a*) $\mathbf{V}_{a1} = 100\,\underline{/0^\circ}$, $\mathbf{V}_{a2} = 0$, $\mathbf{V}_{a0} = 50\,\underline{/0^\circ}$ volts.

Absence of Zero-Sequence Components. The zero-sequence components are non-existent in any system of voltages (or currents) if the vector sum of the original vectors is equal to zero. [See equation (17).] This fact may often be used advantageously in making numerical calculations because the original system of vectors is then directly reducible to two balanced three-phase systems of opposite phase sequence. An absence of zero-sequence components may have important physical significance in the analysis of practical problems. Some of the practical problems in which symmetrical-component analyses are used

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successfully will be referred to briefly in the following paragraphs and one of these problems will be treated in detail in the next chapter.

Three-Phase, Line-to-Line Voltages. The line-to-line voltages shown in Fig. 3 for either the wye or delta are:

$$\mathbf{V}_{ab} = (\mathbf{V}_{an} - \mathbf{V}_{bn}) \tag{18}$$

$$\mathbf{V}_{bc} = (\mathbf{V}_{bn} - \mathbf{V}_{cn}) \tag{19}$$

$$\mathbf{V}_{ca} = (\mathbf{V}_{cn} - \mathbf{V}_{an}) \tag{20}$$

For the delta the voltages to neutral are those of an equivalent wye. Regardless of the degree of unbalance in the line-to-line voltages

$$\mathbf{V}_{ab} + \mathbf{V}_{bc} + \mathbf{V}_{ca} = (\mathbf{V}_{an} - \mathbf{V}_{bn}) + (\mathbf{V}_{bn} - \mathbf{V}_{cn}) + (\mathbf{V}_{cn} - \mathbf{V}_{an}) = 0 \tag{21}$$

The zero-sequence components of the line-to-line voltages are non-existent because

$$\mathbf{V}_{ab0} = \mathbf{V}_{bc0} = \mathbf{V}_{ca0} = \tfrac{1}{3}(\mathbf{V}_{ab} + \mathbf{V}_{bc} + \mathbf{V}_{ca}) = 0 \tag{22}$$

Therefore three-phase, line-to-line voltages may be represented by a positive-sequence system and a negative-sequence system of voltages as represented by the voltage vector diagrams of Fig. 3. It should be realized that Fig. 3 shows a specific case. As has been previously stated, the relative magnitude of the positive: and negative-sequence voltages and the angle between $\mathbf{V}_{an1}$ and $\mathbf{V}_{an2}$ may take on an infinite number of different values in the most general case. The fact that unbalanced line-to-line voltages may be resolved into two balanced systems of opposite sequence is of considerable importance in the analyses of three-phase rotating machinery. When unbalanced voltages are applied to a three-phase induction motor, for example, the operation of the motor may be analyzed on the basis of balanced systems of voltages of opposite phase sequence.

The positive-sequence voltages and negative-sequence voltages shown in Fig. 3 are obtained in any particular case in terms of the vector values of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ as outlined in equations (13) and (16). In terms of the present notation

$$\mathbf{V}_{ab1} = \tfrac{1}{3}(\mathbf{V}_{ab} + \mathbf{V}_{bc}\underline{/120^\circ} + \mathbf{V}_{ca}\underline{/-120^\circ}) \tag{23}$$

$$\mathbf{V}_{ab2} = \tfrac{1}{3}(\mathbf{V}_{ab} + \mathbf{V}_{bc}\underline{/-120^\circ} + \mathbf{V}_{ca}\underline{/120^\circ}) \tag{24}$$

It will be observed from equation (23) that the positive-sequence component of the base vector ($\mathbf{V}_{ab}$ in this case) is obtained by advancing

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(through 120°) the vector which lags the base vector and retarding (through 120°) the vector which leads the base vector. Reversed operations are employed to secure the negative-sequence components as

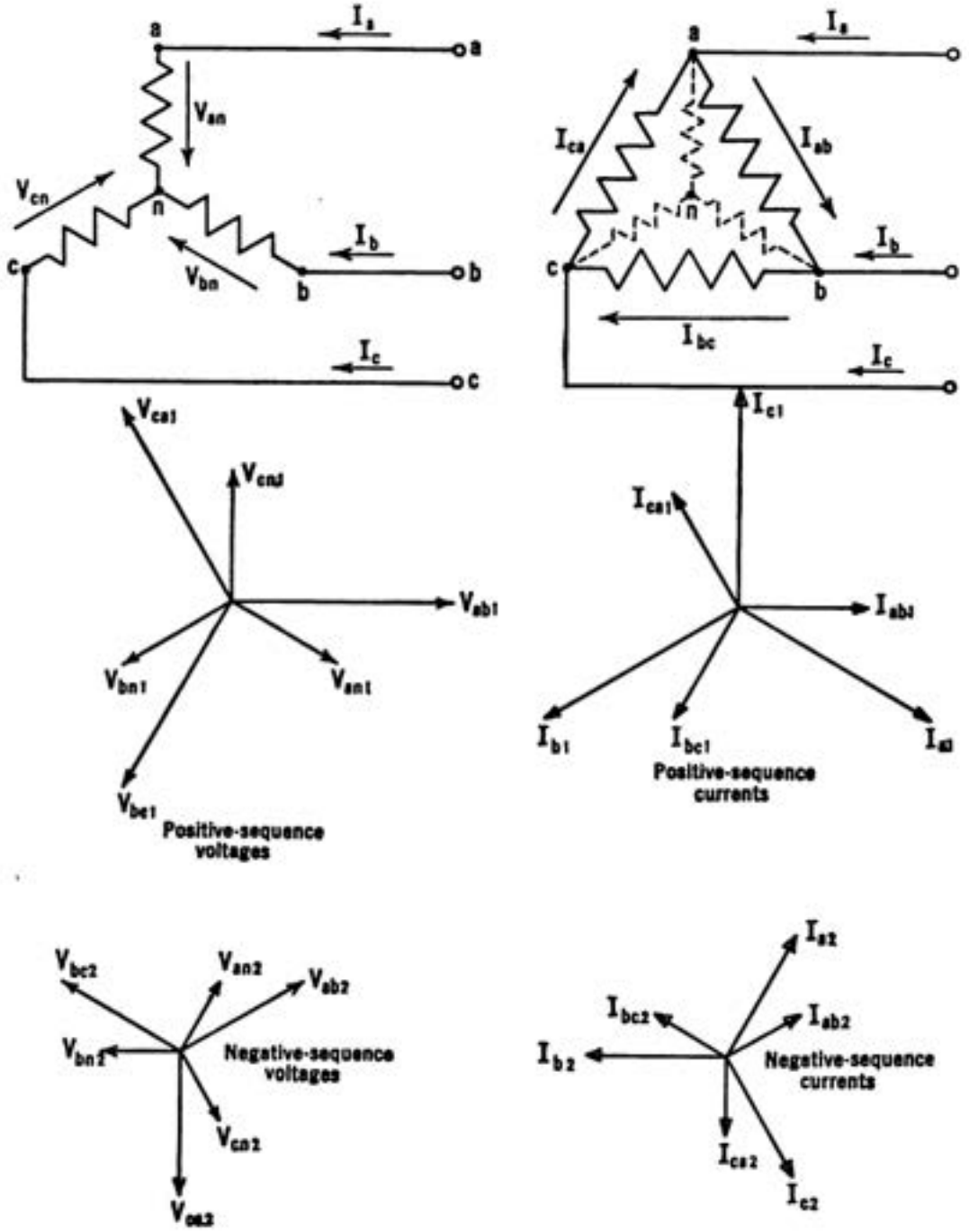

FIG. 3. Positive and negative systems of voltages and currents for a specific three-phase system.

shown in equation (24). If the general scheme is understood, neither changes in notation nor reversals of phase sequence (of the original vectors) can cause confusion.

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Problem 2. A three-phase system of line voltages, $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$, are unbalanced to the extent that $\mathbf{V}_{ab1} = 4000 \angle -60°$ and $\mathbf{V}_{ab2} = 2000 \angle 180°$ volts. ($\mathbf{V}_{ab0}$ is, of course, equal to zero.)

(*a*) Draw a common-origin vector diagram illustrating the positive-sequence voltages and the negative-sequence voltages of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$.

(*b*) Find the magnitudes of the three voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$.

Ans.: (*b*) $\mathbf{V}_{ab} = 3464$, $\mathbf{V}_{bc} = 3464$, $\mathbf{V}_{ca} = 6000$ volts.

Phase Voltages of Wye-Connected Loads. Reference to equation (21) will show that the phase voltages, $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$, may possess any vector values whatsoever and yet the vector sum of the line-to-line voltages is zero. In general, however,

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} \neq 0$$

The individual phase voltages will, therefore, generally possess zero-sequence components even though these components are absent in the line-to-line voltages. Under balanced conditions the phase voltages will, of course, possess no zero-sequence components.

Example 5. In Fig. 3, let

$$\mathbf{V}_{an} = 10 \angle 0° \quad \mathbf{V}_{bn} = 20 \angle -90° \quad \mathbf{V}_{cn} = 10 \angle 135° \text{ volts}$$

Under these conditions

$$\begin{aligned}
\mathbf{V}_{ab} &= (10 + j0) - (0 - j20) = 10 + j20 \\
\mathbf{V}_{bc} &= (0 - j20) - (-7.07 + j7.07) = 7.07 - j27.07 \\
\mathbf{V}_{ca} &= (-7.07 + j7.07) - (10 + j0) = -17.07 + j7.07 \\
\mathbf{V}_{ab0} &= \tfrac{1}{3}[(10 + j20) + (7.07 - j27.07) + (-17.07 + j7.07)] = 0 \\
\mathbf{V}_{an0} &= \tfrac{1}{3}[(10 + j0) + (0 - j20) + (-7.07 + j7.07)] \\
&= \tfrac{1}{3}(2.93 - j12.93) = 0.98 - j4.31 \text{ volts}
\end{aligned}$$

It will be noted that triple subscripts have been used in the above example in connection with the component voltages $\mathbf{V}_{ab0}$ and $\mathbf{V}_{an0}$. Where both line-to-line and phase voltages are involved in the same discussion, triple subscripts of this kind may be used advantageously. These subscripts tell whether line-to-line voltages or phase voltages are being considered, they specify the positive circuit direction of the voltages, and they designate the order of the system to which the component voltage belongs.

Delta-Wye Voltage Transformations. In symmetrical-component analyses it is very often particularly advantageous to consider delta-connected systems on an equivalent wye basis. If the delta-connected load shown in Fig. 3 is to be analyzed on an equivalent wye basis, the load impedances are first converted to their equivalent wye values in the conventional manner and then the line-to-line voltages are resolved into their symmetrical components as shown in equations (23) and (24).

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The remaining problem is that of finding the equivalent wye voltages in terms of the line-to-line voltages.

For a–b–c sequence:

$$\mathbf{V}_{bn1} = \mathbf{V}_{an1}\underline{/-120^\circ} \quad \text{and} \quad \mathbf{V}_{an1} - \mathbf{V}_{bn1} = \mathbf{V}_{ab1}$$

It follows that

$$\mathbf{V}_{an1} - \mathbf{V}_{an1}\underline{/-120^\circ} = \mathbf{V}_{ab1}$$

$$\mathbf{V}_{an1}[1 - (-0.5 - j0.866)] = \mathbf{V}_{ab1}$$

Hence:

$$\mathbf{V}_{an1} = \frac{\mathbf{V}_{ab1}}{\sqrt{3}\underline{/30^\circ}} = \frac{\mathbf{V}_{ab1}}{\sqrt{3}}\underline{/-30^\circ} \tag{25}$$

The complete positive-sequence system of voltages is shown in Fig. 3.

In a corresponding manner it may be shown that

$$\mathbf{V}_{an2} = \frac{\mathbf{V}_{ab2}}{\sqrt{3}\underline{/-30^\circ}} = \frac{\mathbf{V}_{ab2}}{\sqrt{3}}\underline{/30^\circ} \tag{26}$$

The complete negative-sequence system of voltages is shown in Fig. 3.

Equations (25) and (26) are useful in the analysis of either wye- or

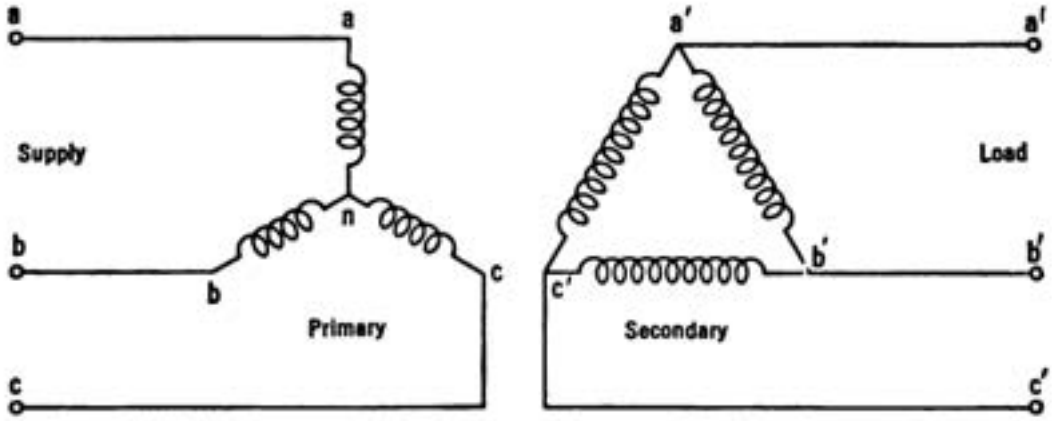

FIG. 4. Wye-delta transformer bank. The windings of transformer a are $a'b'$ and an, transformer b, $b'c'$ and bn, and transformer c, $c'a'$ and cn.

delta-connected loads where the line-to-line voltages are specified. They are also important in the analysis of delta-wye transformer banks like that shown in Fig. 4.

It should be noted in passing that $\mathbf{V}_{an0}$ may possess a finite value even though the zero-sequence components of the line-to-line voltages are of zero value. The fact that $\mathbf{V}_{an0}$ cannot be evaluated in terms of the line-to-line voltages presents no serious handicap as will be shown later, but it does preclude the possibility of immediately evaluating the voltage to neutral ($\mathbf{V}_{an} = \mathbf{V}_{an1} + \mathbf{V}_{an2} + \mathbf{V}_{an0}$).

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Problem 3. In the wye-delta transformer bank shown in Fig. 4, the operation of the three transformers, and the polarities of the windings are such that

$$\mathbf{V}_{a'b'} = n\mathbf{V}_{an} \qquad \text{(transformer } a\text{)}$$
$$\mathbf{V}_{b'c'} = n\mathbf{V}_{bn} \qquad \text{(transformer } b\text{)}$$
$$\mathbf{V}_{c'a'} = n\mathbf{V}_{cn} \qquad \text{(transformer } c\text{)}$$

where n is the voltage transformation ratio of the transformers. The primary line-to-line voltages are unbalanced in magnitude to the extent that $\mathbf{V}_{ab1} = 4000\,\underline{/-60^\circ}$ and $\mathbf{V}_{ab2} = 1000\,\underline{/-90^\circ}$ volts. ($\mathbf{V}_{ab1}$ and $\mathbf{V}_{ab2}$ are, of course, written with respect to a common reference axis.) The sequence of the primary line-to-line voltages is assumed to be ab–bc–ca, and $\mathbf{V}_{an0}$ is to be taken as zero.

(*a*) Find the magnitude and vector position of $\mathbf{V}_{ab}$ and of $\mathbf{V}_{bc}$.

(*b*) If the transformation ratio of the transformers is 10, find the magnitude and vector position of $\mathbf{V}_{a'b'}$ and of $\mathbf{V}_{b'c'}$.

Ans.: $\mathbf{V}_{ab} = 4890\,\underline{/-65.85^\circ}$, $\mathbf{V}_{bc} = 3173\,\underline{/170.94^\circ}$ volts.
$\mathbf{V}_{a'b'} = 28{,}230\,\underline{/-84.14^\circ}$, $\mathbf{V}_{b'c'} = 23{,}800\,\underline{/135.95^\circ}$ volts.

Problem 4. Find the relative vector positions of $\mathbf{V}_{ab}$ and $\mathbf{V}_{a'b'}$ of the wye-delta transformer bank of Fig. 4 if $\mathbf{V}_{ab2} = 0$ and $\mathbf{V}_{an0} = 0$. Find the relative vector positions of $\mathbf{V}_{bc}$ and $\mathbf{V}_{b'c'}$ under the same conditions. (The sequence of the supply voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ is assumed to be ab–bc–ca.)

Ans.: $\mathbf{V}_{a'b'}$ lags $\mathbf{V}_{ab}$ by 30°; $\mathbf{V}_{b'c'}$ lags $\mathbf{V}_{bc}$ by 30°.

The supply voltages are balanced and the positive-sequence voltage vector diagram of Fig. 3 applies directly since $\mathbf{V}_{a'b'} = n\mathbf{V}_{an}$, and $\mathbf{V}_{b'c'} = n\mathbf{V}_{bn}$.

Three-Phase, Three-Wire Line Currents and Associated Delta-Phase Currents. The line currents of a three-phase, three-wire system can contain no zero-sequence components regardless of whether the system is wye- or delta-connected. Reference to the wye-connected load given in Fig. 3 will show that at the junction n:

$$\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0$$

Therefore,

$$\mathbf{I}_{a0} = \tfrac{1}{3}(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = 0 \tag{27}$$

Reference to the delta-connected load given in Fig. 3 will show that

$$\mathbf{I}_a = \mathbf{I}_{ab} - \mathbf{I}_{ca} \tag{28}$$

$$\mathbf{I}_b = \mathbf{I}_{bc} - \mathbf{I}_{ab} \tag{29}$$

$$\mathbf{I}_c = \mathbf{I}_{ca} - \mathbf{I}_{bc} \tag{30}$$

Hence:

$$\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = (\mathbf{I}_{ab} - \mathbf{I}_{ca}) + (\mathbf{I}_{bc} - \mathbf{I}_{ab}) + (\mathbf{I}_{ca} - \mathbf{I}_{bc}) = 0 \tag{31}$$

Regardless of the degree of unbalance of the individual phase currents, $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, and $\mathbf{I}_{ca}$, the vector sum of the line currents, $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$, is equal to zero and therefore no zero-sequence components are present in the line currents.

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The individual delta-phase currents will, in general, possess zero-sequence components since $(\mathbf{I}_{ab} + \mathbf{I}_{bc} + \mathbf{I}_{ca})$ is, in general, not equal to zero. The zero-sequence components of the phase currents in a delta-connected system cannot be evaluated in terms of the line currents.

For a–b–c sequence of line currents:

$$\mathbf{I}_{ab1} - \mathbf{I}_{ca1} = \mathbf{I}_{a1} \quad \text{and} \quad \mathbf{I}_{ca1} = \mathbf{I}_{ab1}\underline{/120^\circ}$$

Employing the same type of derivation as that employed in the derivation of equation (25), it is easy to show that

$$\mathbf{I}_{ab1}[1 - (-0.5 + j0.866)] = \mathbf{I}_{a1}$$

$$\mathbf{I}_{ab1} = \frac{\mathbf{I}_{a1}}{\sqrt{3}}\underline{/30^\circ} \tag{32}$$

A complete positive-sequence system of currents is shown in Fig. 3. The vector diagram of the positive-sequence currents shows that $\mathbf{I}_{ab1}$ is $1/\sqrt{3}$ as large as $\mathbf{I}_{a1}$ and 30° in advance of $\mathbf{I}_{a1}$.

In a corresponding manner it may be shown that

$$\mathbf{I}_{ab2} = \frac{\mathbf{I}_{a2}}{\sqrt{3}}\underline{/-30^\circ} \tag{33}$$

In a wye-delta transformer bank like that shown in Fig. 4 where no zero-sequence components of current can exist in the wye primary windings, no zero-sequence currents will be present in the delta secondary windings since $N_pI_p = N_sI_s$. In this connection, N_p represents the primary turns and N_s the secondary turns of one transformer. (The magnetizing current is neglected in the statement $N_pI_p = N_sI_s$ or else I_p represents simply the load component of the primary current.) The fact that a transformer bank like that shown in Fig. 4 eliminates zero-sequence currents is of importance in power network short-circuit studies.

Problem 5. Find the line current, $\mathbf{I}_a$, in the delta-connected system shown in Fig. 3 if

$$\mathbf{I}_{ab1} = 10\underline{/0^\circ}, \quad \mathbf{I}_{ab2} = 5\underline{/60^\circ}, \quad \text{and} \quad \mathbf{I}_{ab0} = 7\underline{/19.5^\circ} \text{ amperes}$$

Ans.: $\mathbf{I}_a = 15\underline{/0^\circ}$ amperes.

Three-Phase Line Currents Associated with a Neutral Return. If a wye-wye system operates with grounded neutrals or with a connecting wire between neutrals, the vector sum of the line currents will not, in general, be equal to zero. In this case:

$$\mathbf{I}_{a0} = \mathbf{I}_{b0} = \mathbf{I}_{c0} = \tfrac{1}{3}(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) \tag{34}$$

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It will be noted that the ground or neutral return current, namely, $(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c)$, is three times as large as the individual zero-sequence components of the line currents. Each line wire carries a component of current which is equal in magnitude and in time phase with similar components in the other two lines. These zero-sequence components are sometimes called *uniphase* components and have important physical significance in connection with the inductive interference between three-phase power lines and paralleling telephone lines.

Where the line currents possess uniphase components, no manner of transposition of the power system line wires will prevent these components from establishing inductive interference in paralleling telephone lines, the reason being that the uniphase components in the three line wires establish similarly directed magnetic interference. In a case of this kind, transposition of the telephone wires themselves is required to balance out the undesirable emf's that are induced by the power system currents. Inductive interference studies usually refer to the uniphase or zero-sequence currents as *residuals* since they represent the component currents that remain after the positive- and negative-sequence components have been taken from the original unbalanced system of currents. The fact that the *residuals* can be separated from the two balanced systems of currents is an important feature in interference problems.

The zero-sequence components of the line currents of grounded or four-wire wye systems are also of importance in the evaluation of the short-circuit currents in power systems.

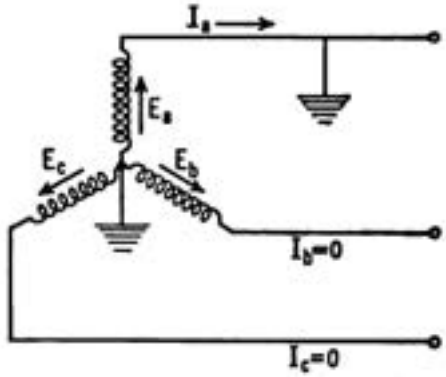

FIG. 5. A particular case of unbalanced three-phase line currents.

Example 6. A line-to-ground short circuit on a grounded wye-connected alternator is shown in Fig. 5. Let it be required to find the three-phase symmetrical components of the line currents $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$, where $\mathbf{I}_a = I\,\underline{/\alpha}$, $I_b = 0$, and $I_c = 0$. I is the magnitude of the short-circuit current, $\mathbf{I}_a$, and α is the angular displacement of this current from any arbitrary reference axis. The three line currents may be considered as an unbalanced three-phase system of currents even though two of the currents are equal to zero.

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The original system of currents is represented by:

$$\mathbf{I}_a = I\,\underline{/\alpha} \quad \mathbf{I}_b = 0 \quad \mathbf{I}_c = 0$$

The positive-sequence components of the above currents are:

$$\mathbf{I}_{a1} = \tfrac{1}{3}I\,\underline{/\alpha} \quad \mathbf{I}_{b1} = \tfrac{1}{3}I\,\underline{/\alpha - 120^\circ} \quad \mathbf{I}_{c1} = \tfrac{1}{3}I\,\underline{/\alpha + 120^\circ}$$

The negative-sequence components are:

$$\mathbf{I}_{a2} = \tfrac{1}{3}I\,\underline{/\alpha} \quad \mathbf{I}_{b2} = \tfrac{1}{3}I\,\underline{/\alpha + 120^\circ} \quad \mathbf{I}_{c2} = \tfrac{1}{3}I\,\underline{/\alpha - 120^\circ}$$

The zero-sequence components are:

$$\mathbf{I}_{a0} = \mathbf{I}_{b0} = \mathbf{I}_{c0} = \tfrac{1}{3}I\,\underline{/\alpha}$$

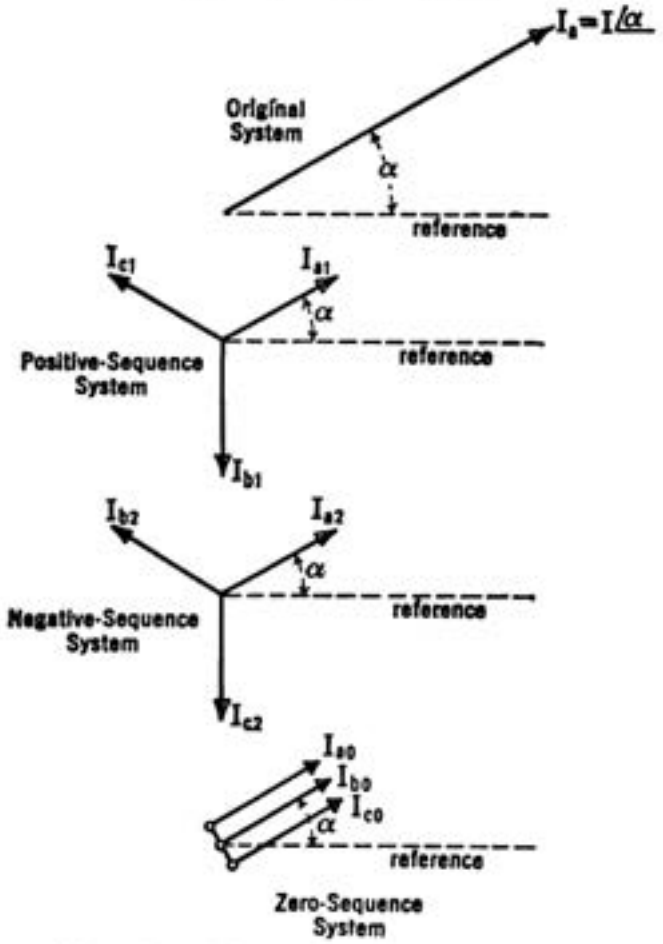

FIG. 6. The resolution of a single current $I\underline{/\alpha}$ into its three-phase symmetrical components.

Graphical representations of the above results are shown in Fig. 6. It will be observed that

$$\mathbf{I}_{a1} + \mathbf{I}_{a2} + \mathbf{I}_{a0} = \mathbf{I}_a = I\,\underline{/\alpha}$$
$$\mathbf{I}_{b1} + \mathbf{I}_{b2} + \mathbf{I}_{b0} = \mathbf{I}_b = 0$$
$$\mathbf{I}_{c1} + \mathbf{I}_{c2} + \mathbf{I}_{c0} = \mathbf{I}_c = 0$$

Symmetrical components of the kind given above are used in single line-to-ground, short-circuit current analyses and although this type of problem is not considered in the present chapter, a study of Fig. 6 at this stage will prove to be instructive.

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Problem 6. The three line currents in a four-wire wye system like that shown in Fig. 7, are:

$$\mathbf{I}_{a'a} = \mathbf{I}_a = 20\,\underline{/-60^\circ},\ \mathbf{I}_{b'b} = \mathbf{I}_b = 12\,\underline{/-100^\circ},\ \text{and}\ \ \mathbf{I}_{c'c} = \mathbf{I}_c = 10\,\underline{/75^\circ}\ \text{amperes}$$

Find the positive-, negative-, and zero-sequence components of the above line currents and check the results either graphically or by the vector addition of the symmetrical components.

Ans: $\mathbf{I}_{a1} = 9.45 - j6.76 = 11.62\,\underline{/-35.6^\circ}$ amperes
$\mathbf{I}_{b1} = -10.58 - j4.80 = 11.62\,\underline{/-155.6^\circ}$ amperes
$\mathbf{I}_{c1} = 1.136 + j11.58 = 11.62\,\underline{/84.4^\circ}$ amperes
$\mathbf{I}_{a2} = -2.95 - j4.07 = 5.03\,\underline{/-125.9^\circ}$ amperes
$\mathbf{I}_{b2} = 5.0 - j0.517 = 5.03\,\underline{/-5.9^\circ}$ amperes
$\mathbf{I}_{c2} = -2.05 - j4.59 = 5.03\,\underline{/114.1^\circ}$ amperes
$\mathbf{I}_{a0} = 3.503 - j6.49 = 7.375\,\underline{/-61.65^\circ}$ amperes

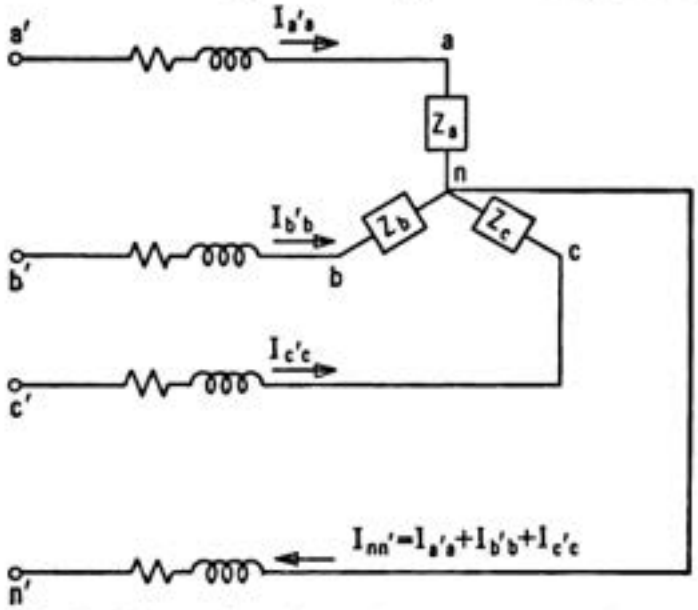

FIG. 7. Three-phase four-wire system for Problem 6.

Power from Symmetrical Components. For any unbalanced three-phase system the total power consumed is the sum of the powers absorbed in each phase. Thus

$$P = P_a + P_b + P_c = V_aI_a \cos \theta_{\mathbf{V}_a}^{\mathbf{I}_a} + V_bI_b \cos \theta_{\mathbf{V}_b}^{\mathbf{I}_b} + V_cI_c \cos \theta_{\mathbf{V}_c}^{\mathbf{I}_c}$$

If the voltage of a given phase, say $\mathbf{V}_a$, is resolved into several components, the power for that phase may be obtained by adding the products of each component of voltage by the current times the cosine of the angle between the particular voltage component and the current. Reference to Fig. 8 will make this evident. Here

$$P_a = I_a(V_a \cos \theta) = I_a(V_1 \cos \theta_1 + V_2 \cos \theta_2 + V_0 \cos \theta_0)$$
$$= I_aV_1 \cos \theta_{\mathbf{I}_a}^{\mathbf{V}_1} + I_aV_2 \cos \theta_{\mathbf{I}_a}^{\mathbf{V}_2} + I_aV_0 \cos \theta_{\mathbf{I}_a}^{\mathbf{V}_0}$$

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Similarly, if the current is divided into components, the power is the sum of the products of voltage by the current times the cosine of the phase angle between the respective components of current and the voltage. From these facts it should be apparent that if both voltage and current are resolved into components, the power will be the sum of the products of each component of voltage by each component of current times the cosine of the angle between the particular component of voltage and current appearing in each of the products.

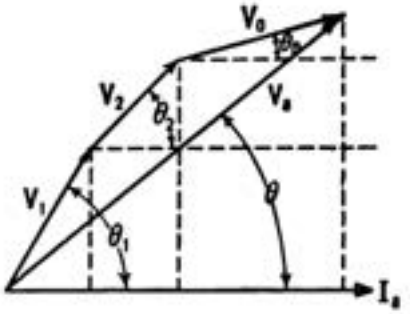

FIG. 8. In-phase component of $\mathbf{V}_a$ with respect to $\mathbf{I}_a$ is the sum of the in-phase components of each of the component voltages of $\mathbf{V}_a$.

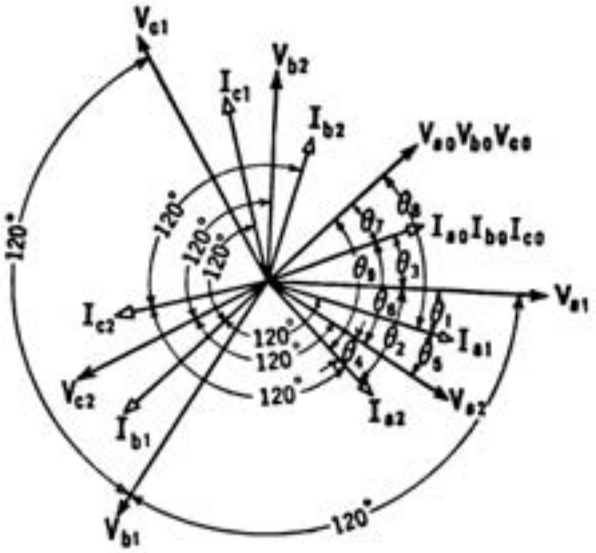

FIG. 9. Symmetrical components of voltages and currents of a general three-phase system.

Figure 9 shows the symmetrical components of currents and voltages for any three-phase system. The subscripts a, b, and c denote the phase while 0, 1, and 2 are the usual symbols denoting the sequence components. In terms of the components shown for phase a, the power is

$$\begin{aligned} P_a = {} & V_{a1}I_{a1}\cos\theta_1 + V_{a1}I_{a2}\cos\theta_2 + V_{a1}I_{a0}\cos\theta_3 + V_{a2}I_{a2}\cos\theta_4 \\ & + V_{a2}I_{a1}\cos\theta_5 + V_{a2}I_{a0}\cos\theta_6 + V_{a0}I_{a0}\cos\theta_7 + V_{a0}I_{a1}\cos\theta_8 \\ & + V_{a0}I_{a2}\cos\theta_9 \end{aligned} \tag{35}$$

For phase b

$$\begin{aligned} P_b = {} & V_{b1}I_{b1}\cos\theta_1 + V_{b1}I_{b2}\cos(120^\circ + \theta_2) + V_{b1}I_{b0}\cos(120^\circ + \theta_3) \\ & + V_{b2}I_{b2}\cos\theta_4 + V_{b2}I_{b1}\cos(120^\circ + \theta_5) + V_{b2}I_{b0}\cos(120^\circ - \theta_6) \\ & + V_{b0}I_{b0}\cos\theta_7 + V_{b0}I_{b1}\cos(120^\circ + \theta_8) \\ & + V_{b0}I_{b2}\cos(120^\circ - \theta_9) \end{aligned} \tag{36}$$

For phase c

$$\begin{aligned} P_c = {} & V_{c1}I_{c1}\cos\theta_1 + V_{c1}I_{c2}\cos(240^\circ + \theta_2) + V_{c1}I_{c0}\cos(240^\circ + \theta_3) \\ & + V_{c2}I_{c2}\cos\theta_4 + V_{c2}I_{c1}\cos(240^\circ + \theta_5) + V_{c2}I_{c0}\cos(240^\circ - \theta_6) \\ & + V_{c0}I_{c0}\cos\theta_7 + V_{c0}I_{c1}\cos(240^\circ + \theta_8) \\ & + V_{c0}I_{c2}\cos(240^\circ - \theta_9) \end{aligned} \tag{37}$$

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It should be remembered that only magnitudes of voltages and currents appear in equations (35), (36), and (37), and that $V_{a1} = V_{b1} = V_{c1}$, $V_{a2} = V_{b2} = V_{c2}$, $V_{a0} = V_{b0} = V_{c0}$, $I_{a1} = I_{b1} = I_{c1}$, $I_{a2} = I_{b2} = I_{c2}$, and $I_{a0} = I_{b0} = I_{c0}$. Under these conditions if equations (35), (36), and (37) are added, the terms containing θ_2 add to zero because they represent three equal quantities at 120-degree angles. Similarly, the terms containing θ_3, θ_5, θ_6, θ_8, and θ_9 add to zero. Dropping reference to particular phases, this leaves

$$P = P_a + P_b + P_c = 3V_1I_1 \cos\theta_1 + 3V_2I_2 \cos\theta_4 + 3V_0I_0 \cos\theta_7 \quad (38)$$

It will be noted that $\cos\theta_1 = \cos\theta_{V_1}^{I_1}$, $\cos\theta_4 = \cos\theta_{V_2}^{I_2}$, and $\cos\theta_7 = \cos\theta_{V_0}^{I_0}$. Hence

$$P = 3V_1I_1 \cos\theta_{V_1}^{I_1} + 3V_2I_2 \cos\theta_{V_2}^{I_2} + 3V_0I_0 \cos\theta_{V_0}^{I_0} \quad (39)$$

Equation (39) shows that the total power consumed by an unbalanced three-phase system is the sum of the powers represented by each of the symmetrical component systems. Hence, to obtain total power the algebraic sum of the total positive-, total negative-, and total zero-sequence powers may be calculated.

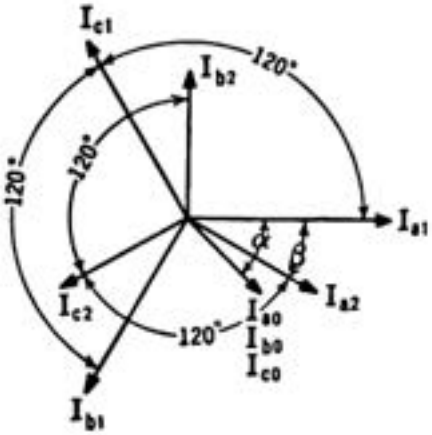

FIG. 10. Symmetrical components of currents in a general three-phase system.

Copper Losses in Terms of Symmetrical Components. The copper loss for any unbalanced three-phase system is:

$$P = I_a^2R_a + I_b^2R_b + I_c^2R_c \quad (40)$$

where phase currents and the corresponding phase resistances are used.

By referring to Fig. 10 and by remembering that

$$\mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} + \mathbf{I}_{a0}$$

it follows that

$$I_a^2 = (I_{a1} + I_{a2} \cos\beta + I_{a0} \cos\alpha)^2 + (I_{a2} \sin\beta + I_{a0} \sin\alpha)^2 \quad (41)$$

Similarly,

$$\begin{aligned} I_b^2 = {} & [I_{b1} \cos 240° + I_{b2} \cos (120° - \beta) + I_{b0} \cos\alpha]^2 \\ & + [I_{b1} \sin 240° + I_{b2} \sin (120° - \beta) + I_{b0} \sin\alpha]^2 \end{aligned} \quad (42)$$

$$\begin{aligned} I_c^2 = {} & [I_{c1} \cos 120° + I_{c2} \cos (240° - \beta) + I_{c0} \cos\alpha]^2 \\ & + [I_{c1} \sin 120° + I_{c2} \sin (240° - \beta) + I_{c0} \sin\alpha]^2 \end{aligned} \quad (43)$$

When R_a, R_b, and R_c are different the sequence components of current should be combined to obtain I_a, I_b, and I_c, and equation (40) used to

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calculate the copper loss. If, however, $R_a = R_b = R_c = R$, substitution of equations (41), (42), and (43) in equation (40), dropping reference to phase, and expanding and combining terms algebraically give

$$P = 3I_1^2R + 3I_2^2R + 3I_0^2R$$
$$= 3(I_1^2 + I_2^2 + I_0^2)R \tag{44}$$

Equation (44) shows that the total copper loss due to the resultant currents is the same as the sum of the copper losses due to the sequence components calculated separately.

If the resistances to the positive-, negative-, and zero-sequence currents are different, the copper loss may be determined from

$$P = 3I_1^2R_1 + 3I_2^2R_2 + 3I_0^2R_0 \tag{45}$$

where R_1, R_2, and R_0 are respectively the *resistance to* the positive-, negative-, and zero-sequence components of current. In using equation (45) it must be remembered that each of the sequence resistances must be the same for all three phases, since equality of phase resistances was assumed in obtaining equation (44), of which (45) is a modification.

Positive-, Negative-, and Zero-Sequence Impedance Components. For purposes of some analyses, three self-impedances may be separated or resolved into their symmetrical components exactly like three voltages or currents. If the voltages or currents which are to be associated with these component impedances are resolved in the order a–b–c, then the impedances should be resolved in the same order. [See equations (13), (16), and (17).] The term self-impedance implies that no mutual coupling exists between the individual impedances. In order to distinguish the components of self-impedance from the components of mutual impedance which are considered later, double subscripts of the kind given below will be used.

The symmetrical components of three self-impedances, $\mathbf{Z}_{aa}$, $\mathbf{Z}_{bb}$, and $\mathbf{Z}_{cc}$ are:

$$\mathbf{Z}_{aa1} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb}\,\underline{/120^\circ} + \mathbf{Z}_{cc}\,\underline{/-120^\circ}) \tag{46}$$

$$\mathbf{Z}_{aa2} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb}\,\underline{/-120^\circ} + \mathbf{Z}_{cc}\,\underline{/120^\circ}) \tag{47}$$

$$\mathbf{Z}_{aa0} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb} + \mathbf{Z}_{cc}) \tag{48}$$

As above defined $\mathbf{Z}_{aa1}$, $\mathbf{Z}_{aa2}$, and $\mathbf{Z}_{aa0}$ are called positive-sequence impedance, negative-sequence impedance, and zero-sequence impedance

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respectively. These component impedances have little physical significance but they are useful in a general mathematical formulation of symmetrical-component theory. It should be pointed out at this stage that the resistance (or in-phase) parts of the component impedances may possess negative signs even though the real parts of $\mathbf{Z}_{aa}$, $\mathbf{Z}_{bb}$, and $\mathbf{Z}_{cc}$ are all positive.

The above symmetrical components of an unbalanced *set of impedances* should not be confused with impedance to positive-, negative-, and zero-sequence currents which are defined as follows:

$$\text{Impedance to positive-sequence, } \mathbf{Z}_{a1} = \frac{\mathbf{V}_{a1}}{\mathbf{I}_{a1}}$$

$$\text{Impedance to negative-sequence, } \mathbf{Z}_{a2} = \frac{\mathbf{V}_{a2}}{\mathbf{I}_{a2}}$$

$$\text{Impedance to zero-sequence, } \mathbf{Z}_{a0} = \frac{\mathbf{V}_{a0}}{\mathbf{I}_{a0}}$$

These impedances to sequence component currents are usually applied to systems where the impedances of all phases are the same or balanced. In order to avoid confusion a *double-letter subscript* will be used on positive-, negative-, and zero-sequence *components of impedance*. For *impedance* to positive-, negative-, and zero-sequence currents a *single-letter* subscript will be used. In both cases the figure subscripts 1, 2, and 0 will denote positive, negative, and zero sequence, respectively.

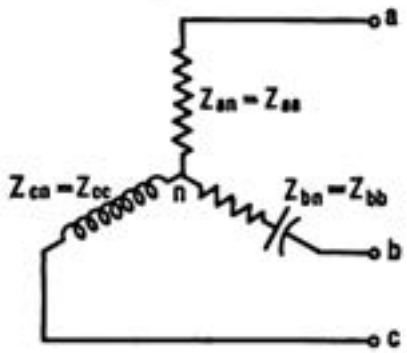

FIG. 11. See example 7.

Example 7. Let the wye-connected impedances of Fig. 11 be:

$$\mathbf{Z}_{aa} = (6 + j0) \quad \mathbf{Z}_{bb} = (5.2 - j3) \quad \mathbf{Z}_{cc} = (0 + j12) \text{ ohms}$$

Employing equations (46), (47), and (48), the component impedances are:

$$\begin{aligned}\mathbf{Z}_{aa1} &= \tfrac{1}{3}[(6 + j0) + (5.2 - j3)(-0.5 + j0.866) + (0 + j12)(-0.5 - j0.866)]\\ &= \tfrac{1}{3}[(6 + j0) + (0 + j6) + (10.4 - j6)]\\ &= \tfrac{1}{3}(16.4 + j0) = 5.47 + j0 \text{ ohms}\end{aligned}$$

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$$\begin{aligned}\mathbf{Z}_{aa2} &= \tfrac{1}{3}[(6 + j0) + (5.2 - j3)(-0.5 - j0.866) + (0 + j12)(-0.5 + j0.866)]\\ &= \tfrac{1}{3}[(6 + j0) + (-5.2 - j3) + (-10.4 - j6)]\\ &= \tfrac{1}{3}(-9.6 - j9) = -3.2 - j3 \text{ ohms}\end{aligned}$$

$$\begin{aligned}\mathbf{Z}_{aa0} &= \tfrac{1}{3}[(6 + j0) + (5.2 - j3) + (0 + j12)]\\ &= \tfrac{1}{3}(11.2 + j9) = 3.73 + j3 \text{ ohms}\end{aligned}$$

In accordance with previous considerations, it follows that

$$\mathbf{Z}_{bb1} = \mathbf{Z}_{aa1}\underline{/-120^\circ} \qquad \mathbf{Z}_{cc1} = \mathbf{Z}_{aa1}\underline{/120^\circ}$$

$$\mathbf{Z}_{bb2} = \mathbf{Z}_{aa2}\underline{/120^\circ} \qquad \mathbf{Z}_{cc2} = \mathbf{Z}_{aa2}\underline{/-120^\circ}$$

$$\mathbf{Z}_{bb0} = \mathbf{Z}_{aa0} \qquad \mathbf{Z}_{cc0} = \mathbf{Z}_{aa0}$$

The sum of the impedance components of one phase equals the actual impedance of that phase. For example,

$$\mathbf{Z}_{aa} = (5.47 + j0) + (-3.2 - j3) + (3.73 + j3) = 6 + j0 \text{ ohms}$$

Problem 7. Find $\mathbf{Z}_{bb1}$, $\mathbf{Z}_{bb2}$, and $\mathbf{Z}_{bb0}$ in the above example, employing the values of $\mathbf{Z}_{aa1}$, $\mathbf{Z}_{aa2}$, and $\mathbf{Z}_{aa0}$ which have been evaluated. Repeat for $\mathbf{Z}_{cc1}$, $\mathbf{Z}_{cc2}$, and $\mathbf{Z}_{cc0}$.

Ans.: $\mathbf{Z}_{bb} = \mathbf{Z}_{bb1} + \mathbf{Z}_{bb2} + \mathbf{Z}_{bb0}$
$= (-2.73 - j4.73) + (4.20 - j1.27) + (3.73 + j3.0)$
$= (5.2 - j3.0)$ ohms.

Problem 8. Given three wye-connected impedances:

$$\mathbf{Z}_{an} = (15 + j0) \quad \mathbf{Z}_{bn} = (6 - j3.464) \quad \mathbf{Z}_{cn} = (6 + j3.464) \text{ ohms}$$

(*a*) Find the symmetrical components of $\mathbf{Z}_{an}$ in accordance with the resolutions given in equations (46), (47), and (48).

(*b*) Find $\mathbf{Z}_{bn1}$, $\mathbf{Z}_{bn2}$, and $\mathbf{Z}_{bn0}$ in terms of the symmetrical components of $\mathbf{Z}_{an}$ and check $(\mathbf{Z}_{bn1} + \mathbf{Z}_{bn2} + \mathbf{Z}_{bn0})$ with the given value of $(6 - j3.464)$ ohms.

Ans.: (*a*) $\mathbf{Z}_{an1} = 5\underline{/0^\circ}$; $\mathbf{Z}_{an2} = 1\underline{/0^\circ}$; $\mathbf{Z}_{an0} = 9\underline{/0^\circ}$ ohms.

Sequence Rule as Applied to Component Voltages. If the voltage drop across one phase, say phase *a*, is written in terms of the symmetrical components of both current and impedance, nine component voltages appear. That is,

$$\begin{aligned}\mathbf{V}_a = \mathbf{I}_a\mathbf{Z}_a &= (\mathbf{I}_{a1} + \mathbf{I}_{a2} + \mathbf{I}_{a0})(\mathbf{Z}_{aa1} + \mathbf{Z}_{aa2} + \mathbf{Z}_{aa0})\\ &= \mathbf{I}_{a1}\mathbf{Z}_{aa1} + \mathbf{I}_{a1}\mathbf{Z}_{aa2} + \mathbf{I}_{a1}\mathbf{Z}_{aa0} + \mathbf{I}_{a2}\mathbf{Z}_{aa1} + \mathbf{I}_{a2}\mathbf{Z}_{aa2} + \mathbf{I}_{a2}\mathbf{Z}_{aa0}\\ &\quad + \mathbf{I}_{a0}\mathbf{Z}_{aa1} + \mathbf{I}_{a0}\mathbf{Z}_{aa2} + \mathbf{I}_{a0}\mathbf{Z}_{aa0} \qquad (49)\end{aligned}$$

These nine component voltages may be grouped in such a manner as to form the positive-, negative-, and zero-sequence components of $\mathbf{V}_a$, and this grouping may be made in accordance with an easily remembered rule.

The Sequence Rule

The order of the voltage system to which an IZ drop belongs is equal to the sum of the orders of the systems to which I and Z belong individually.

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In the application of the sequence rule, positive-sequence terms are of first order, negative-sequence terms are of second order, and zero-sequence terms are of zero or third order. In summing the orders both (1 + 0) and (2 + 2) are considered as belonging to the first order, since order 4 is considered as order 1, there being only three orders. In this connection, the zero in (1 + 0) may be reckoned either as zero or three. Also (1 + 2) is of order 3, or a zero-sequence term. As applied to the component voltages of equation (49), the sequence rule states:

$$\mathbf{V}_{a1} = \mathbf{I}_{a1}\mathbf{Z}_{aa0} + \mathbf{I}_{a2}\mathbf{Z}_{aa2} + \mathbf{I}_{a0}\mathbf{Z}_{aa1} \tag{50}$$

$$\mathbf{V}_{a2} = \mathbf{I}_{a1}\mathbf{Z}_{aa1} + \mathbf{I}_{a2}\mathbf{Z}_{aa0} + \mathbf{I}_{a0}\mathbf{Z}_{aa2} \tag{51}$$

$$\mathbf{V}_{a0} = \mathbf{I}_{a1}\mathbf{Z}_{aa2} + \mathbf{I}_{a2}\mathbf{Z}_{aa1} + \mathbf{I}_{a0}\mathbf{Z}_{aa0} \tag{52}$$

Obviously the real basis upon which the above equations are written is that, as written, they satisfy the definitions which were originally attached to $\mathbf{V}_{a1}$, $\mathbf{V}_{a2}$, and $\mathbf{V}_{a0}$. To satisfy these definitions, $\mathbf{V}_{a1}$ must be the positive-sequence component of the base vector $\mathbf{V}_a$, $\mathbf{V}_{a2}$ must be the negative-sequence component of the base vector $\mathbf{V}_a$, and $\mathbf{V}_{a0}$ must be the zero-sequence component. The proof that $\mathbf{V}_{a1}$, as written in equation (50), satisfies the definition of a positive-sequence voltage is outlined below.

Applying equation (50) to the b phase and making appropriate substitutions:

$$\begin{aligned}\mathbf{V}_{b1} &= \mathbf{I}_{b1}\mathbf{Z}_{bb0} + \mathbf{I}_{b2}\mathbf{Z}_{bb2} + \mathbf{I}_{b0}\mathbf{Z}_{bb1}\\ &= (\mathbf{I}_{a1}\underline{/-120^\circ})\mathbf{Z}_{aa0} + (\mathbf{I}_{a2}\underline{/120^\circ})\mathbf{Z}_{aa2}\underline{/120^\circ} + \mathbf{I}_{a0}\mathbf{Z}_{aa1}\underline{/-120^\circ}\\ &= \mathbf{I}_{a1}\mathbf{Z}_{aa0}\underline{/-120^\circ} + \mathbf{I}_{a2}\mathbf{Z}_{aa2}\underline{/-120^\circ} + \mathbf{I}_{a0}\mathbf{Z}_{aa1}\underline{/-120^\circ}\end{aligned} \tag{50a}$$

Comparison of equations (50a) and (50) will show that $\mathbf{V}_{b1}$ is equal in magnitude to $\mathbf{V}_{a1}$ and 120° behind $\mathbf{V}_{a1}$, as, of course, it should be if $\mathbf{V}_{a1}$, $\mathbf{V}_{b1}$, and $\mathbf{V}_{c1}$ are to form a positive-sequence system.

Applying equation (50) to the c phase and making appropriate substitutions:

$$\begin{aligned}\mathbf{V}_{c1} &= \mathbf{I}_{c1}\mathbf{Z}_{cc0} + \mathbf{I}_{c2}\mathbf{Z}_{cc2} + \mathbf{I}_{c0}\mathbf{Z}_{cc1}\\ &= (\mathbf{I}_{a1}\underline{/120^\circ})\mathbf{Z}_{aa0} + (\mathbf{I}_{a2}\underline{/-120^\circ})\mathbf{Z}_{aa2}\underline{/-120^\circ} + \mathbf{I}_{a0}\mathbf{Z}_{aa1}\underline{/120^\circ}\\ &= \mathbf{I}_{a1}\mathbf{Z}_{aa0}\underline{/120^\circ} + \mathbf{I}_{a2}\mathbf{Z}_{aa2}\underline{/120^\circ} + \mathbf{I}_{a0}\mathbf{Z}_{aa1}\underline{/120^\circ}\end{aligned} \tag{50b}$$

Comparison of equations (50b) and (50) will show that $\mathbf{V}_{c1}$ is equal in magnitude to $\mathbf{V}_{a1}$ and 120° ahead of $\mathbf{V}_{a1}$, which is the necessary requirement that $\mathbf{V}_{a1}$, $\mathbf{V}_{b1}$, and $\mathbf{V}_{c1}$ form a positive-sequence system of voltages.

In a manner similar to that outlined above, $\mathbf{V}_{a2}$ of equation (51) may be shown to be a member of a balanced negative-sequence system of voltages $\mathbf{V}_{a2}$, $\mathbf{V}_{b2}$, and $\mathbf{V}_{c2}$.

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In the following problem the reader is asked to analyze equation (52) with a view toward showing that the **IZ** components of that equation are correctly chosen to form a zero-sequence system of voltages.

Problem 9. Prove that $\mathbf{V}_{a0}$ (equal to $\mathbf{I}_{a1}\mathbf{Z}_{aa2} + \mathbf{I}_{a2}\mathbf{Z}_{aa1} + \mathbf{I}_{a0}\mathbf{Z}_{aa0}$) is equal in magnitude and in time phase, with

$$\mathbf{V}_{b0} = \mathbf{I}_{b1}\mathbf{Z}_{bb2} + \mathbf{I}_{b2}\mathbf{Z}_{bb1} + \mathbf{I}_{b0}\mathbf{Z}_{bb0}$$

and with

$$\mathbf{V}_{c0} = \mathbf{I}_{c1}\mathbf{Z}_{cc2} + \mathbf{I}_{c2}\mathbf{Z}_{cc1} + \mathbf{I}_{c0}\mathbf{Z}_{cc0}$$

Application of the Sequence Rule to Unbalanced Three-Wire Loads. The foregoing theory may be applied to any three-wire load which consists of individual or non-coupled phase impedances. Since the individual phases of three-phase rotating equipment are closely coupled magnetically, the present method of analysis does not apply directly to rotating equipment. (A method of accounting for the mutual impedances of rotating equipment is given in Chapter XV, and a general method of accounting for mutual impedance effects is given later in the present chapter.)

In applying equations (50), (51), and (52) to the *a* phase of a wye-connected load like that shown in Fig. 11, it is noted that, since $\mathbf{I}_{a0} = 0$,

$$\mathbf{V}_{an1} = \mathbf{I}_{a1}\mathbf{Z}_{an0} + \mathbf{I}_{a2}\mathbf{Z}_{an2} \tag{53}$$

$$\mathbf{V}_{an2} = \mathbf{I}_{a1}\mathbf{Z}_{an1} + \mathbf{I}_{a2}\mathbf{Z}_{an0} \tag{54}$$

$$\mathbf{V}_{an0} = \mathbf{I}_{a1}\mathbf{Z}_{an2} + \mathbf{I}_{a2}\mathbf{Z}_{an1} \tag{55}$$

If the line-to-line voltages, namely, $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$, are known, $\mathbf{V}_{an1}$ and $\mathbf{V}_{an2}$ may be evaluated directly from equations (25) and (26). (See page 501.) If $\mathbf{V}_{an1}$ and $\mathbf{V}_{an2}$ are known, $\mathbf{I}_{a1}$ and $\mathbf{I}_{a2}$ may be determined directly from equations (53) and (54), provided that $\mathbf{Z}_{an1}$, $\mathbf{Z}_{an2}$, and $\mathbf{Z}_{an0}$ are known.

Since $\mathbf{I}_{a0} = 0$,

$$\mathbf{I}_{an} = \mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} \tag{56}$$

$$\mathbf{I}_{bn} = \mathbf{I}_b = \mathbf{I}_{a1}\underline{/-120^\circ} + \mathbf{I}_{a2}\underline{/120^\circ} \tag{57}$$

$$\mathbf{I}_{cn} = \mathbf{I}_c = \mathbf{I}_{a1}\underline{/120^\circ} + \mathbf{I}_{a2}\underline{/-120^\circ} \tag{58}$$

Even though $\mathbf{I}_{a0} = 0$, $\mathbf{V}_{an0}$ will, in general, possess a finite value since by equation (55) $\mathbf{V}_{an0} = \mathbf{I}_{a1}\mathbf{Z}_{an2} + \mathbf{I}_{a2}\mathbf{Z}_{an1}$.

Example 8. Let the line-to-line voltages and the phase impedances of the wye-connected load shown in Fig. 11 be as follows:

$$V_{ab} = 200 \quad V_{bc} = 141.4 \quad V_{ca} = 141.4 \text{ volts}$$

$$\mathbf{Z}_{an} = (6 + j0) \quad \mathbf{Z}_{bn} = (5.2 - j3) \quad \mathbf{Z}_{cn} = (0 + j12) \text{ ohms}$$

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If the voltage sequence is *ab-bc-ca* and if $\mathbf{V}_{ab}$ is taken as reference,

$$\mathbf{V}_{ab} = 200\angle 0^\circ \quad \mathbf{V}_{bc} = 141.4\angle -135^\circ \quad \mathbf{V}_{ca} = 141.4\angle 135^\circ \text{ volts}$$

Resolution of the above line-to-line voltages into symmetrical components yields

$$\mathbf{V}_{ab1} = \tfrac{1}{3}[200\angle 0^\circ + 141.4\angle -15^\circ + 141.4\angle 15^\circ] = 157.8\angle 0^\circ \text{ volts}$$
$$\mathbf{V}_{ab2} = \tfrac{1}{3}[200\angle 0^\circ + 141.4\angle 105^\circ + 141.4\angle -105^\circ] = 42.3\angle 0^\circ \text{ volts}$$
$$\mathbf{V}_{ab0} = \tfrac{1}{3}[200\angle 0^\circ + 141.4\angle -135^\circ + 141.4\angle 135^\circ] = 0$$

From equations (25) and (26):

$$\mathbf{V}_{an1} = \frac{157.8\angle 0^\circ}{\sqrt{3}}\angle -30^\circ = 91\angle -30^\circ \text{ volts}$$
$$\mathbf{V}_{an2} = \frac{42.3\angle 0^\circ}{\sqrt{3}}\angle 30^\circ = 24.4\angle 30^\circ \text{ volts}$$

The symmetrical components of the phase impedances are:

$$\mathbf{Z}_{an1} = 5.47\angle 0^\circ, \quad \mathbf{Z}_{an2} = (-3.2 - j3) = 4.38\angle -136.8^\circ$$

and

$$\mathbf{Z}_{an0} = (3.73 + j3) = 4.78\angle 38.8^\circ \text{ ohms} \quad \text{(See example 7, page 510.)}$$

From equations (53) and (54):

$$\mathbf{I}_{a1} = \frac{\begin{vmatrix} \mathbf{V}_{an1} & \mathbf{Z}_{an2} \\ \mathbf{V}_{an2} & \mathbf{Z}_{an0} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{an0} & \mathbf{Z}_{an2} \\ \mathbf{Z}_{an1} & \mathbf{Z}_{an0} \end{vmatrix}} = \frac{\begin{vmatrix} 91\angle -30^\circ & 4.38\angle -136.8^\circ \\ 24.4\angle 30^\circ & 4.78\angle 38.8^\circ \end{vmatrix}}{\begin{vmatrix} 4.78\angle 38.8^\circ & 4.38\angle -136.8^\circ \\ 5.47\angle 0^\circ & 4.78\angle 38.8^\circ \end{vmatrix}} = \frac{491\angle 20.15^\circ}{44.8\angle 59.95^\circ}$$

$$\mathbf{I}_{a1} = 10.95\angle -39.8^\circ = 8.42 - j7.02 \text{ amperes}$$

$$\mathbf{I}_{a2} = \frac{\begin{vmatrix} \mathbf{Z}_{an0} & \mathbf{V}_{an1} \\ \mathbf{Z}_{an1} & \mathbf{V}_{an2} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{an0} & \mathbf{Z}_{an2} \\ \mathbf{Z}_{an1} & \mathbf{Z}_{an0} \end{vmatrix}} = \frac{\begin{vmatrix} 4.78\angle 38.8^\circ & 91\angle -30^\circ \\ 5.47\angle 0^\circ & 24.4\angle 30^\circ \end{vmatrix}}{44.8\angle 59.95^\circ} = \frac{528\angle 137.5^\circ}{44.8\angle 59.95^\circ}$$

$$\mathbf{I}_{a2} = 11.8\angle 77.45^\circ = 2.56 + j11.5 \text{ amperes}$$

$$\mathbf{I}_{an} = \mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} = (8.42 - j7.02) + (2.56 + j11.5)$$
$$= 10.98 + j4.48 \text{ amperes}$$

In polar form:

$$\mathbf{I}_a = 11.83\angle 22.2^\circ \text{ amperes}$$

After $\mathbf{I}_{a1}$ and $\mathbf{I}_{a2}$ have been evaluated, $\mathbf{I}_{b1}$, $\mathbf{I}_{b2}$, $\mathbf{I}_{c1}$, and $\mathbf{I}_{c2}$ follow directly, and hence $\mathbf{I}_{bn}$ and $\mathbf{I}_{cn}$ may be determined from the values of $\mathbf{I}_{a1}$ and $\mathbf{I}_{a2}$.

If the value of $\mathbf{V}_{an}$ is to be determined by the method of symmetrical components,

$$\mathbf{V}_{an} = \mathbf{V}_{an1} + \mathbf{V}_{an2} + \mathbf{V}_{an0}$$

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where, from equation (55), $\mathbf{V}_{an0} = \mathbf{I}_{a1}\mathbf{Z}_{an2} + \mathbf{I}_{a2}\mathbf{Z}_{an1}$. In this case:

$$\mathbf{V}_{an0} = (10.95\angle{-39.8^\circ})\,(4.38\angle{-136.8^\circ}) + (11.8\angle{77.45^\circ})\,(5.47\angle{0^\circ})$$
$$= -34 + j60.2 \text{ volts}$$
$$\mathbf{V}_{an} = (78.85 - j45.5) + (21.15 + j12.2) + (-34 + j60.2)$$
$$= 66 + j26.9 \text{ volts}$$

Problem 10. Study through the details of the above example and evaluate $\mathbf{I}_b$, $\mathbf{I}_c$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ by the method of symmetrical components. Check $\mathbf{V}_{an} - \mathbf{V}_{bn}$ against the given value of $\mathbf{V}_{ab} = 200\angle{0^\circ}$ volts, recognizing that slide-rule calculations were employed in the evaluations of $\mathbf{I}_{a1}$, $\mathbf{I}_{a2}$, and $\mathbf{V}_{an}$.

Ans.: $\mathbf{I}_b = -21.53 - j7.31 = 22.7\angle{-161.2^\circ}$ amperes.

Magnetic Coupling between Phases. If the three phases (including the line wires) possess mutual coupling of the kind shown in Fig. 12, the voltage drop in phase *an* due to its mutual coupling with phases *bn* and *cn* is:

$$\mathbf{V}_{am} = \mathbf{I}_b\mathbf{Z}_{ab} + \mathbf{I}_c\mathbf{Z}_{ac} \qquad (59)$$

where subscript *m* designates the fact that this voltage drop excludes the self-impedance voltage drop, namely, $\mathbf{I}_a\mathbf{Z}_{aa}$. If simple magnetic coupling is involved,

$$\mathbf{Z}_{ab} = jX_{ab} = \pm j\omega M_{ab} \qquad (60)$$
$$\mathbf{Z}_{ac} = jX_{ac} = \pm j\omega M_{ac} \qquad (61)$$

(See Chapter VII.)

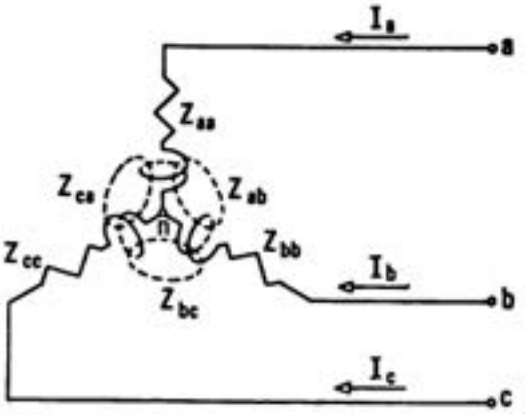

FIG. 12. Impedance in wye with mutual coupling between phases.

The signs of the mutual reactances are defined by the assigned directions of current flow and the modes of winding the mutually coupled coils.

The impedance drop in phase *an* due to the self-impedance of that phase will be called $\mathbf{V}_{aa}$, and the total voltage drop in phase *an* then becomes:

$$\mathbf{V}_{an} = \mathbf{V}_{aa} + \mathbf{V}_{am} = \mathbf{I}_a\mathbf{Z}_{aa} + \mathbf{I}_b\mathbf{Z}_{ab} + \mathbf{I}_c\mathbf{Z}_{ac} \qquad (62)$$

The problem of expressing the impedance drops of equation (62) in terms of symmetrical components will now be undertaken. Obviously $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ may be expressed in terms of the symmetrical components of any one of these currents and $\mathbf{Z}_{aa}$ may be resolved into symmetrical components if the other self-impedances $\mathbf{Z}_{bb}$ and $\mathbf{Z}_{cc}$ are known. In this connection:

$$\mathbf{Z}_{aa1} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb}\angle{120^\circ} + \mathbf{Z}_{cc}\angle{-120^\circ}), \quad \text{etc.}$$

if the other resolutions are effected in the *a-b-c* order.

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The self-impedance voltage drop in phase *an* may be written in terms of symmetrical components in accordance with the sequence rule.

$$\mathbf{V}_{aa} = \mathbf{V}_{aa1} + \mathbf{V}_{aa2} + \mathbf{V}_{aa0} \tag{63}$$

where

$$\mathbf{V}_{aa1} = \mathbf{I}_{a1}\mathbf{Z}_{aa0} + \mathbf{I}_{a2}\mathbf{Z}_{aa2} + \mathbf{I}_{a0}\mathbf{Z}_{aa1} \tag{64}$$

$$\mathbf{V}_{aa2} = \mathbf{I}_{a1}\mathbf{Z}_{aa1} + \mathbf{I}_{a2}\mathbf{Z}_{aa0} + \mathbf{I}_{a0}\mathbf{Z}_{aa2} \tag{65}$$

$$\mathbf{V}_{aa0} = \mathbf{I}_{a1}\mathbf{Z}_{aa2} + \mathbf{I}_{a2}\mathbf{Z}_{aa1} + \mathbf{I}_{a0}\mathbf{Z}_{aa0} \tag{66}$$

There remains the problem of resolving the mutual impedances $\mathbf{Z}_{ab} = \mathbf{Z}_{ba}$, $\mathbf{Z}_{bc} = \mathbf{Z}_{cb}$, and $\mathbf{Z}_{ca} = \mathbf{Z}_{ac}$ into symmetrical components that can be advantageously associated with $\mathbf{I}_{a1}$, $\mathbf{I}_{a2}$, and $\mathbf{I}_{a0}$ to account for the presence of $\mathbf{I}_b\mathbf{Z}_{ab}$ and $\mathbf{I}_c\mathbf{Z}_{ac}$ in equation (62). At this stage of the development it is rather difficult to say which of the three mutual impedances should be considered as the base mutual impedance. It turns out that the symmetrical components of $\mathbf{Z}_{bc}$ can best be associated with $\mathbf{I}_{a1}$, $\mathbf{I}_{a2}$, and $\mathbf{I}_{a0}$. See equation (72).

Resolving the mutual impedances into symmetrical components with $\mathbf{Z}_{bc}$ as base yields

$$\mathbf{Z}_{bc1} = \tfrac{1}{3}(\mathbf{Z}_{bc} + \mathbf{Z}_{ca}\underline{/120^\circ} + \mathbf{Z}_{ab}\underline{/-120^\circ}) \tag{67}$$

$$\mathbf{Z}_{bc2} = \tfrac{1}{3}(\mathbf{Z}_{bc} + \mathbf{Z}_{ca}\underline{/-120^\circ} + \mathbf{Z}_{ab}\underline{/120^\circ}) \tag{68}$$

$$\mathbf{Z}_{bc0} = \tfrac{1}{3}(\mathbf{Z}_{bc} + \mathbf{Z}_{ca} + \mathbf{Z}_{ab}) = \mathbf{Z}_{ca0} = \mathbf{Z}_{ab0} \tag{69}$$

$$\mathbf{Z}_{ab1} = \mathbf{Z}_{bc1}\underline{/120^\circ} \quad \mathbf{Z}_{ca1} = \mathbf{Z}_{bc1}\underline{/-120^\circ} \tag{70}$$

$$\mathbf{Z}_{ab2} = \mathbf{Z}_{bc2}\underline{/-120^\circ} \quad \mathbf{Z}_{ca2} = \mathbf{Z}_{bc2}\underline{/120^\circ} \tag{71}$$

In terms of symmetrical components,

$$\begin{aligned}\mathbf{V}_{am} &= \mathbf{I}_b\mathbf{Z}_{ab} + \mathbf{I}_c\mathbf{Z}_{ac}\\ &= (\mathbf{I}_{a1}\underline{/-120^\circ} + \mathbf{I}_{a2}\underline{/120^\circ} + \mathbf{I}_{a0})(\mathbf{Z}_{bc1}\underline{/120^\circ} + \mathbf{Z}_{bc2}\underline{/-120^\circ}\\ &\quad + \mathbf{Z}_{bc0}) + (\mathbf{I}_{a1}\underline{/120^\circ} + \mathbf{I}_{a2}\underline{/-120^\circ} + \mathbf{I}_{a0})(\mathbf{Z}_{bc1}\underline{/-120^\circ}\\ &\quad + \mathbf{Z}_{bc2}\underline{/120^\circ} + \mathbf{Z}_{bc0})\end{aligned} \tag{72}$$

Eighteen component voltages appear if the multiplications indicated in equation (72) are carried out. These components may be grouped into positive-, negative-, and zero-sequence terms in accordance with the sequence rule. For example, the component voltages of the first order are:

$$\begin{aligned}\mathbf{I}_{a1}\mathbf{Z}_{bc0}\underline{/-120^\circ} + \mathbf{I}_{a2}\mathbf{Z}_{bc2} + \mathbf{I}_{a0}\mathbf{Z}_{bc1}\underline{/120^\circ} + \mathbf{I}_{a1}\mathbf{Z}_{bc0}\underline{/120^\circ} + \mathbf{I}_{a2}\mathbf{Z}_{bc2}\\ + \mathbf{I}_{a0}\mathbf{Z}_{bc1}\underline{/-120^\circ} = \mathbf{V}_{am1}\end{aligned} \tag{73}$$

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If the like terms in the above equations are further grouped, the following form results:

$$\mathbf{V}_{am1} = -\mathbf{I}_{a1}\mathbf{Z}_{bc0} + 2\mathbf{I}_{a2}\mathbf{Z}_{bc2} - \mathbf{I}_{a0}\mathbf{Z}_{bc1} \tag{74}$$

The negative-sequence or second-order terms of equation (72) may be combined to form

$$\mathbf{V}_{am2} = 2\mathbf{I}_{a1}\mathbf{Z}_{bc1} - \mathbf{I}_{a2}\mathbf{Z}_{bc0} - \mathbf{I}_{a0}\mathbf{Z}_{bc2} \tag{75}$$

The zero-sequence terms of equation (72) may be combined to form

$$\mathbf{V}_{am0} = -\mathbf{I}_{a1}\mathbf{Z}_{bc2} - \mathbf{I}_{a2}\mathbf{Z}_{bc1} + 2\mathbf{I}_{a0}\mathbf{Z}_{bc0} \tag{76}$$

Equations (74), (75), and (76) contain all eighteen component voltages represented in equation (72), and these equations may be combined systematically with equations (64), (65), and (66) to yield the positive-, negative-, and zero-sequence components of the complete phase voltage, namely, $\mathbf{V}_{an} = \mathbf{V}_{aa} + \mathbf{V}_{am}$.

Adding equations (64) and (74), equations (65) and (75), and equations (66) and (76) results in:

$$\mathbf{V}_{an1} = \mathbf{I}_{a1}(\mathbf{Z}_{aa0} - \mathbf{Z}_{bc0}) + \mathbf{I}_{a2}(\mathbf{Z}_{aa2} + 2\mathbf{Z}_{bc2}) + \mathbf{I}_{a0}(\mathbf{Z}_{aa1} - \mathbf{Z}_{bc1}) \tag{77}$$

$$\mathbf{V}_{an2} = \mathbf{I}_{a1}(\mathbf{Z}_{aa1} + 2\mathbf{Z}_{bc1}) + \mathbf{I}_{a2}(\mathbf{Z}_{aa0} - \mathbf{Z}_{bc0}) + \mathbf{I}_{a0}(\mathbf{Z}_{aa2} - \mathbf{Z}_{bc2}) \tag{78}$$

$$\mathbf{V}_{an0} = \mathbf{I}_{a1}(\mathbf{Z}_{aa2} - \mathbf{Z}_{bc2}) + \mathbf{I}_{a2}(\mathbf{Z}_{aa1} - \mathbf{Z}_{bc1}) + \mathbf{I}_{a0}(\mathbf{Z}_{aa0} + 2\mathbf{Z}_{bc0}) \tag{79}$$

The above set of equations represents a powerful tool in the field of circuit analysis because with the aid of this set of equations any degree of unbalance and any degree of magnetic or capacitive coupling may be handled on a symmetrical-component basis. Equations (77), (78), and (79) are particularly useful in accounting for transmission line reactance voltage drops because these voltage drops result from mutual coupling between the line wires. These equations are also useful in accounting for the mutual impedance of the fourth wire of a four-wire, three-phase system.

Example 9. Let it be required to find the current $\mathbf{I}_a$ in Fig. 13 by the method of symmetrical components, if $\mathbf{V}_{ab} = \mathbf{V}_{bc} = \mathbf{V}_{ca} = 100$ volts and the sequence of these voltages is *ab-bc-ca*. From previous considerations, it is plain that

$$\mathbf{V}_{an1} = \frac{\mathbf{V}_{ab1}}{\sqrt{3}}\underline{/-30^\circ} \quad \text{and} \quad \mathbf{V}_{an2} = 0$$

If $\mathbf{V}_{ab}$ is chosen as the reference vector,

$$\mathbf{V}_{an1} = \frac{100}{\sqrt{3}}\underline{/-30^\circ} = (50 - j28.9) \text{ volts}$$

The self-impedances [$\mathbf{Z}_{aa} = (0 + j1)$, $\mathbf{Z}_{bb} = (2 + j0)$, and $\mathbf{Z}_{cc} = (0 + j3)$] may be

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resolved into symmetrical components in the usual manner.

$$Z_{aa1} = \tfrac{1}{3}(Z_{aa} + Z_{bb}\angle 120^\circ + Z_{cc}\angle -120^\circ) = (0.533 + j0.411) \text{ ohms}$$
$$Z_{aa2} = \tfrac{1}{3}(Z_{aa} + Z_{bb}\angle -120^\circ + Z_{cc}\angle 120^\circ) = (-1.199 - j0.744) \text{ ohms}$$
$$Z_{aa0} = \tfrac{1}{3}(Z_{aa} + Z_{bb} + Z_{cc}) = (0.667 + j1.33) \text{ ohms}$$

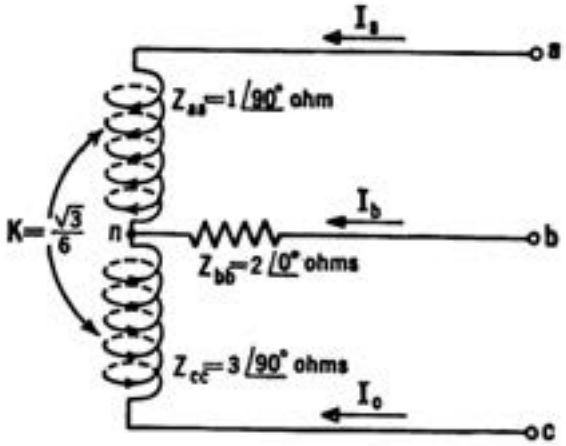

FIG. 13. See example 9.

As indicated on the circuit diagram (Fig. 13), the coefficient of coupling between the two inductance coils is $\sqrt{3}/6$. This coefficient is interpreted to mean that

$$\omega M_{ca} = \omega M_{ac} = \frac{\sqrt{3}}{6}\sqrt{\omega L_{aa} \times \omega L_{cc}} = \frac{\sqrt{3}}{6}\sqrt{1 \times 3} = 0.5 \text{ ohm}$$

If the modes of winding and the space positions of the coils are as represented in Fig. 13,

$$Z_{ca} = (0 - j\omega M_{ca}) = (0 - j0.5) \text{ ohm}$$

Z_{ab} and Z_{bc} are both zero because no coupling exists between phases a and b or between phases b and c under the specified conditions.

In accordance with equations (67), (68), and (69),

$$Z_{bc1} = \tfrac{1}{3}(0.5\angle -90^\circ + 120^\circ) = 0.144 + j0.083 \text{ ohm}$$
$$Z_{bc2} = \tfrac{1}{3}(0.5\angle -90^\circ - 120^\circ) = -0.144 + j0.083 \text{ ohm}$$
$$Z_{bc0} = \tfrac{1}{3}(0.5\angle -90^\circ) = 0 - j0.167 \text{ ohm}$$

Since I_{a0} is equal to zero, it follows from equations (77) and (78) that

$$V_{an1} = 50 - j28.9 = I_{a1}(0.667 + j1.50) + I_{a2}(-1.487 - j0.578)$$
$$V_{an2} = 0 \qquad = I_{a1}(0.821 + j0.578) + I_{a2}(0.667 + j1.50)$$

The above equations may be solved simultaneously for I_{a1} and I_{a2}

$$I_{a1} = \frac{\begin{vmatrix} (50 - j28.9) & (-1.487 - j0.578) \\ 0 & (0.667 + j1.50) \end{vmatrix}}{\begin{vmatrix} (0.667 + j1.50) & (-1.487 - j0.578) \\ (0.821 + j0.578) & (0.667 + j1.50) \end{vmatrix}} = \frac{76.6 + j55.7}{-0.918 + j3.33}$$
$$= (9.63 - j25.6) \text{ amperes}$$

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$$\mathbf{I}_{a2} = \frac{\begin{vmatrix} (0.667 + j1.50) & (50 - j28.9) \\ (0.821 + j0.578) & 0 \end{vmatrix}}{(-0.918 + j3.33)} = \frac{-57.7 - j5.1}{-0.918 + j3.33}$$

$$= (3.01 + j16.46) \text{ amperes}$$

$$\mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} = (9.63 - j25.6) + (3.01 + j16.46)$$

$$= (12.64 - j9.14) = 15.6\angle{-35.85^\circ} \text{ amperes}$$

PROBLEMS

11. The line-to-neutral voltages of a four-wire, three-phase system are represented by the following vector expressions: $\mathbf{V}_a = 200\angle{0^\circ}$, $\mathbf{V}_b = 150\angle{-150^\circ}$, and $\mathbf{V}_c = 100\angle{-75^\circ}$. Find the positive-, negative-, and zero-sequence components of the above voltages, and check the results obtained by graphical additions of the symmetrical components.

12. The three line currents of a four-wire wye load (like that shown in Fig. 7, page 506) directed to the common junction are

$$\mathbf{I}_{an} = 15 - j20 \text{ amperes}$$
$$\mathbf{I}_{bn} = -8 + j15 \text{ amperes}$$
$$\mathbf{I}_{cn} = 8 - j25 \text{ amperes}$$

Find $\mathbf{I}_{an1}$, $\mathbf{I}_{an2}$, and $\mathbf{I}_{an0}$.

13. The three wye-connected impedances through which the currents of Problem 12 flow are, respectively,

$$\mathbf{Z}_{an} = 20 - j20 \text{ ohms}$$
$$\mathbf{Z}_{bn} = 30 + j10 \text{ ohms}$$
$$\mathbf{Z}_{cn} = 10 - j20 \text{ ohms}$$

Find $\mathbf{Z}_{an1}$, $\mathbf{Z}_{an2}$, and $\mathbf{Z}_{an0}$.

14. Employing the symmetrical components $\mathbf{I}_{an1}$, $\mathbf{I}_{an2}$, $\mathbf{I}_{an0}$, $\mathbf{Z}_{an1}$, $\mathbf{Z}_{an2}$, and $\mathbf{Z}_{an0}$ determined in Problems 12 and 13, evaluate $\mathbf{V}_{an} = \mathbf{I}_{an}\mathbf{Z}_{an}$ in terms of symmetrical components and check the result against the known value of $\mathbf{I}_{an}\mathbf{Z}_{an}$.

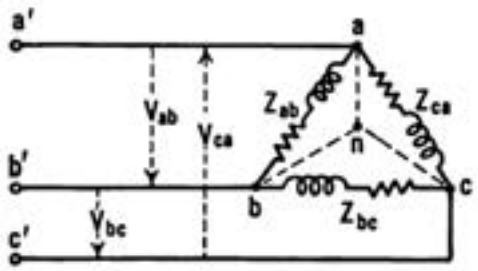

FIG. 14. See Problem 15.

15. Assume that the three-phase line voltages shown in Fig. 14 are

$$\mathbf{V}_{bc} = 200\angle{0} \quad \mathbf{V}_{ca} = 100\angle{120^\circ} \quad \mathbf{V}_{ab} = 173.2\angle{210^\circ}$$

(*a*) Find $\mathbf{V}_{bc1}$, $\mathbf{V}_{bc2}$, and $\mathbf{V}_{bc0}$.

(*b*) Find $\mathbf{V}_{nc1}$, $\mathbf{V}_{nc2}$, and $\mathbf{V}_{nc0}$. Employ phase sequence *bc*, *ab*, *ca*.

16. The three line-to-line voltages shown in Fig. 14 are:

$$\mathbf{V}_{ab} = 100 \quad \mathbf{V}_{bc} = 150 \quad \mathbf{V}_{ca} = 175 \text{ volts}$$

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Sequence *ab-bc-ca*.

(*a*) Find $\mathbf{V}_{ab1}$, $\mathbf{V}_{ab2}$, and $\mathbf{V}_{ab0}$.

(*b*) Find $\mathbf{V}_{an1}$ and $\mathbf{V}_{an2}$, the equivalent wye voltages of the delta load shown in Fig. 14.

(*c*) Explain how the line currents may be determined from $\mathbf{V}_{an1}$, $\mathbf{V}_{an2}$, and the delta load impedances.

17. The line-to-line voltages of a three-wire, three-phase system are $\mathbf{V}_{ab} = 200$ volts, $\mathbf{V}_{bc} = 141.4$ volts, and $\mathbf{V}_{ca} = 141.4$ volts. The sequence of the voltages is *ab-ca-bc*. A wye-connected set of static impedances ($\mathbf{Z}_{an} = 20\angle 0°$ ohms, $\mathbf{Z}_{bn} = 30\angle 60°$ ohms, and $\mathbf{Z}_{cn} = 20\angle 0°$ ohms) is connected to the three lines *a*, *b*, and *c* in the order indicated by the subscripts. Find the line currents $\mathbf{I}_{an}$, $\mathbf{I}_{bn}$, and $\mathbf{I}_{cn}$ by the method of symmetrical components.

18. Solve for $\mathbf{I}_a$ in Fig. 13 by the method of symmetrical components if $\mathbf{V}_{ab} = 200$, $\mathbf{V}_{bc} = 173.2$, and $\mathbf{V}_{ca} = 100$ volts. The sequence of the line-to-line voltages is *ab-bc-ca*.

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CHAPTER XV

POWER SYSTEM SHORT-CIRCUIT CALCULATIONS

Power systems are subject to three kinds of short circuits. First, all three lines of a three-phase system may become electrically connected. This is known as a three-phase short circuit. Second, only two lines may be electrically connected, which constitutes a line-to-line short circuit. Third, a single wire may be electrically connected to ground. This is called a line-to-ground short circuit. Although the electrical connections referred to may be of varying impedance, short-circuit calculations are based upon zero impedance at the point of short circuit. In other words, a perfect short circuit is assumed. Short circuits on systems are usually called faults.

A distribution system should be protected in such a way that a faulty or short-circuited section will be isolated from the rest of the system. This is accomplished through the use of relays which operate circuit breakers. To protect a system, relays are set to trip in a certain length of time after the fault occurs. By varying the amount of time required for a relay to operate, certain selective operation of circuit breakers may be obtained. After proper adjustments are made, this selective operation causes only the faulty section of the line to be isolated. In order to determine the proper time settings of these relays and in order to determine the sizes of circuit breakers necessary, the magnitudes of the short-circuit currents that these devices are to handle must be known. In general, different values of short-circuit current occur for the three-phase symmetrical, line-to-line, and line-to-ground short circuits. Usually the three-phase symmetrical short circuit yields the lowest value of short-circuit current (except when the system has practically no grounds). Hence relay settings are usually based upon three-phase symmetrical faults because it is desirable to protect a system for the minimum fault current. If the relay trips a circuit breaker for minimum fault current, it will obviously open the breaker for the highest fault current, but the converse is not true. Since a breaker must interrupt the largest short-circuit current that can possibly exist, the size of a circuit breaker is determined by the largest possible fault current. The greatest current usually occurs for either the line-to-line or line-to-ground fault. Obviously, the determination of short-circuit currents in power systems is required if the proper settings of relays and proper selection of circuit-breaker sizes are to be made.

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Bases for Short-Circuit Calculations. A distribution network consists of many lines which may be connected by transformers and which, in general, operate at different nominal voltages. To establish a simple network for purposes of calculation, the impedances of all lines and transformers are expressed in ohms referred to a common voltage base or in percentage referred to a common kilovolt-ampere base. The former generally appears simpler to the beginner, but the latter method is actually the better and is to be preferred. The two methods yield identical results.

Method Using Ohms on a Kilovolt Base. In general, various branches of an electrical distribution system operate at different potentials. In representing such a system by a system of impedances, it is desirable to employ a scheme which permits the combination of the different impedances so that the network can be represented by a single impedance between the source and the fault. This requires the determination of an impedance, Z_2, which may be used with an arbitrarily selected voltage, V_2, such that the same kva will be taken as when the actual impedance, Z_1, is used with the actual voltage V_1. Stated algebraically,

$$\left(\frac{V_2}{Z_2}\right) V_2 = \frac{V_1}{Z_1} V_1$$

or

$$Z_2 = Z_1 \left(\frac{V_2}{V_1}\right)^2 \tag{1}$$

Equation (1) shows that the original impedance must be multiplied by the square of the ratio of voltage to be used to the nominal operating voltage for the impedance. To illustrate, suppose that 1000 volts are impressed on an impedance of 100 ohms and that it is desired to find the current and kva taken.

$$I_1 = \frac{1000}{100} = 10 \text{ amperes}$$

$$va = 1000 \times 10 = 10{,}000$$

Now assume that it is desired to work the same problem when all values are referred to a 2000-volt base. Then

$$Z_2 = \left(\frac{2000}{1000}\right)^2 \times 100 = 400 \text{ ohms}$$

$$I_2 = \frac{2000}{400} = 5 \text{ amperes}$$

$$va = 2000 \times 5 = 10{,}000$$

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The foregoing example shows that there is no difference between calculating the volt-amperes for the actual voltage and impedance and for some other selected voltage and an equivalent impedance found by multiplying the original impedance by the square of the ratio of the selected voltage to the original. The current on the actual voltage base is then found by multiplying the result calculated on the selected voltage base by the ratio of the voltages. Thus the actual current at 1000 volts is:

$$I_1 = 5 \times \frac{2000}{1000} = 10 \text{ amperes}$$

This procedure is evident from the following relationship.

$$V_1 I_1 = V_2 I_2$$

or

$$I_1 = \frac{V_2}{V_1} I_2$$

Example 1. Calculate the short-circuit current for the system shown in Fig. 1. A 10 to 1 ratio wye-wye connected transformer bank is represented at A. A transformer has resistance and leakage reactance which may be referred to either side as

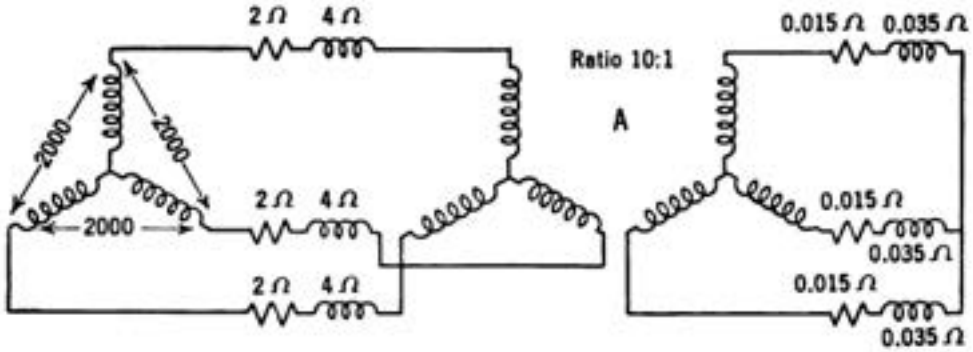

Fig. 1. Elementary three-phase system. See example 1.

was shown in Chapter VII. The transformer impedance in this case is $1 + j2$ ohms per phase when referred to the high-voltage side. The line impedance $2 + j4$ is assumed to include the phase impedance of the generator. Since Fig. 1 represents a

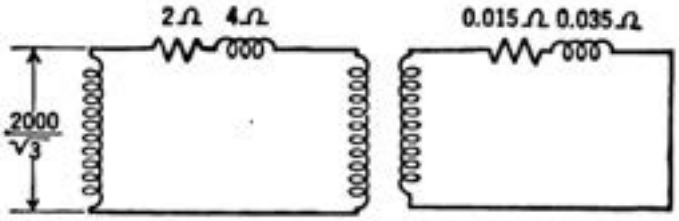

Fig. 2. Equivalent circuit per phase of Fig. 1.

balanced circuit, all calculations will be made per phase. The equivalent circuit for one phase to neutral is shown in Fig. 2, and the corresponding one-line diagram is

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shown in Fig. 3. A short line at the generator neutral is used to represent the neutral bus, and a cross at the end of the line denotes the point of short circuit. The per phase voltage is impressed between the neutral bus and the point X. The trans-

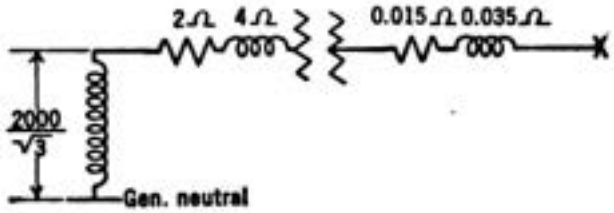

FIG. 3. One-line diagram of Fig. 2 and Fig. 1.

former impedance causes a drop in voltage from its primary to its secondary side and therefore acts like a series impedance. Transferring the impedance of the secondary line to its equivalent value on a 2000-volt base (the primary line-to-line voltage), or

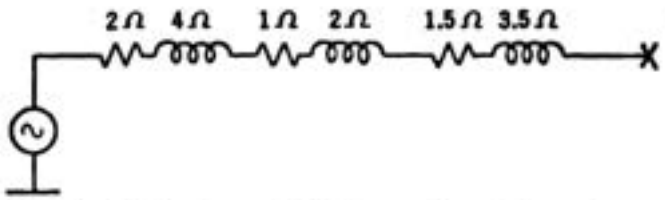

FIG. 4. Reduction of Fig. 3 to a series of impedances.

to a $2000/\sqrt{3}$ volts to neutral base which is the same, and inserting the transformer equivalent impedance, reduces the one-line diagram to the equivalent circuit shown in Fig. 4. Then

$$\mathbf{I} = \frac{2000/\sqrt{3}}{(2 + j4) + (1 + j2) + (1.5 + j3.5)}$$
$$= 47 - j99.2 \quad \text{or} \quad 109.8 \text{ amperes}$$

The actual current at the fault is found by referring the current to the voltage of the faulty line.

$$\text{Fault current} = 109.8 \times 10 = 1098 \text{ amperes}$$

Problem 1. A wye-connected generator rated at 2200 terminal volts has 0.2 ohm resistance and 2 ohms reactance per phase. The generator is connected by lines each having an impedance of $2.06 \angle 29.05°$ ohms to a wye-wye transformer bank. Each transformer has a total equivalent impedance referred to the high side of $100 \angle 60°$ ohms, and the transformer bank is connected to a load through lines each of which has a resistance of 50 ohms and an inductive reactance of 100 ohms. If the ratio of transformation is 6 and the low-voltage side is connected to the generator lines, calculate the actual fault current for a three-phase symmetrical short circuit at the load.

Ans.: 22.3 amperes.

Percentage Method. In general, short-circuit calculations are made through the use of percentage resistances and reactances. Percentage reactance is defined as the percentage of the rated voltage which is con-

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sumed in the reactance drop when rated current flows. Expressed algebraically,

$$\% \text{ reactance} = \frac{I_{\text{rated}} \times \text{ohms}}{V_{\text{rated}}} \times 100 \tag{2}$$

Percentage resistance is similarly defined. Percentage values are manipulated like ohmic values. When percentage values are employed, a common kva base is used instead of a common voltage base as employed in the ohmic method. The derivation of the method for determining the percentage reactance on different kva bases follows. Three-phase will be assumed since it is the most common.

Let p be the percentage reactance based on a particular 3-phase kva.

kv = the voltage between the three-phase lines in kilovolts.
X = the reactance in ohms.

Then
$$IX = \frac{X \text{ kva } 10^{-3}}{\sqrt{3} \text{ kv}} \text{ kilovolts}$$

$$p = \frac{100IX}{\text{kv}/\sqrt{3}} = \frac{\dfrac{100X \text{ kva } 10^{-3}}{\sqrt{3} \text{ kv}}}{\text{kv}/\sqrt{3}} = \frac{X \text{ kva}}{\text{kv}^2\, 10} \tag{3}$$

Equation (3) shows that percentage reactance varies directly with the kva when the rest of the factors remain constant. A similar relation holds true for percentage resistance. Although equation (3) was derived on the assumption of three-phase it is equally applicable to single-phase.

Example 2. By way of illustrating the use of percentage resistance and reactance, example 1, which was worked on the ohmic basis, will be reworked employing the percentage method. Ordinarily, much of the data on a system is expressed in percentage and no transformation from ohmic to percentage impedance is necessary. Since the parameters in the previous example are given in ohms, the transformation to percentage will be shown. Also, to illustrate changing to a common base, the percentage impedance of the lines on the generator side of the transformer and the transformer will be found on a 10,000-kva base, while that on the secondary side will be found on a 100-kva base.

For the lines on generator side of transformer:

$$\text{Base current } I = \frac{10{,}000{,}000}{\sqrt{3}\, 2000} = 2885 \text{ amperes}$$

$$\% \; IX \text{ drop due to base current} = 100 \times \frac{2885 \times 4}{2000/\sqrt{3}} = 1000, \text{ or } 1000\% \text{ reactance}$$

$$\% \; IR \text{ drop due to base current} = 100 \times \frac{2885 \times 2}{2000/\sqrt{3}} = 500, \text{ or } 500\% \text{ resistance}$$

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Transformer impedance on 10,000-kva base:

$$\% \, IR \text{ drop} = \frac{100 \times 2885 \times 1}{2000/\sqrt{3}} = 250$$

$$\% \, IX \text{ drop} = \frac{100 \times 2885 \times 2}{2000/\sqrt{3}} = 500$$

The line impedance on the secondary side of the transformer based on 100 kva is determined as follows:

$$\text{Nominal rated voltage on secondary } \frac{2000}{10} = 200 \text{ volts}$$

$$\text{Base current } I = \frac{100{,}000}{\sqrt{3}\,200} = 288.5 \text{ amperes}$$

$$\% \, IX \text{ drop} = \frac{100 \times 288.5 \times 0.035}{200/\sqrt{3}} = 8.75$$

$$\% \, IR \text{ drop} = \frac{100 \times 288.5 \times 0.015}{200/\sqrt{3}} = 3.75$$

The circuit of Fig. 1 with parameters expressed in percentage is shown in Fig. 5. It is common to receive data on distribution networks expressed like those in Fig. 5.

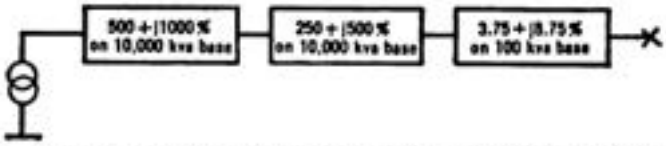

FIG. 5. One-line diagram of Fig. 1 with parameters expressed on a percentage basis.

Before simplifying, a common kva base is chosen to which all constants are referred. This base may be any arbitrarily selected. A 1000-kva base is chosen for this example because it yields convenient numerical quantities.

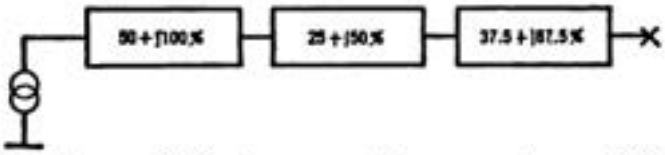

FIG. 6. Impedances of Fig. 5 expressed in per cent on a 1000-kva base.

It was shown that percentage reactance and resistance, and hence impedance, vary directly with the kva base. Employing this principle yields the circuit shown in Fig. 6. The combined impedance to the fault is

$$50 + j100 + 25 + j50 + 37.5 + j87.5 = 112.5 + j237.5\%$$

or

$$\sqrt{112.5^2 + 237.5^2} = 263\%$$

This result indicates that 263 per cent of the rated voltage is necessary to cause 1000 kva to be delivered by the generator Since only rated voltage, or 100 per cent voltage, is available, the total short-circuit kva must be $\frac{100}{263} \times 1000 = 380.5$ kva. If

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the fault current is desired at the actual voltage of the faulty line, namely, 200 volts, it is found as follows:

$$I_{\text{fault}} = \frac{380.5 \times 1000}{\sqrt{3} \times 200} = 1098 \text{ amperes}$$

Problem 2. Rework Problem 1, page 524, employing percentage values.

Per Unit Method. A study of the percentage method will show that problems could be worked by using percentage values expressed in hundredths, which would be equivalent to moving the decimal point two places to the left in the calculations shown in example 2. In other words, quantities could be expressed on a per unit basis instead of on a per hundred basis as in the percentage method. Thus instead of a reactance of 15 per cent a value of 0.15 would be used. A little experience with both schemes shows relatively little difference in the methods. Both methods are used according to personal preferences.

Accuracy of Short-Circuit Calculations. In general, extreme accuracy in the determination of short-circuit currents in distribution systems is not required. Because the resistance of most synchronous apparatus is low compared to the reactance, the final impedance to the fault in many cases is about the same as the reactance. For this reason, and because of the resulting simplification of the calculations, only reactances are generally used. An exception to these statements occurs when stability studies of systems are made. It then becomes necessary to consider phase angles, and then both resistance and reactance must be considered.

When several sources of current are in parallel, it is customary to assume that all the generated voltages are in phase and equal in magnitude at the time of short circuit. Load currents on the system are neglected. All synchronous apparatus like generators, synchronous motors, and rotary converters are considered as sources of short-circuit current. The kinetic energy of these rotating machines causes them to act like generators during the first few cycles of short circuit. In spite of all these approximations, tests have shown that calculations based upon these assumptions are usually within about 5 per cent of the correct values. From 5 to 10 per cent error in the values of short-circuit currents is usually tolerable in the determination of circuit-breaker sizes and relay settings.

Three-Phase Short Circuits. Three-phase short-circuit currents are determined by means of the same principles employed in the analysis of balanced three-phase systems. The method is best shown by an example.

Example 3. It is desired to find the short-circuit current for the system shown in Fig. 7. The data for the system are shown in Table I. A symmetrical three-phase

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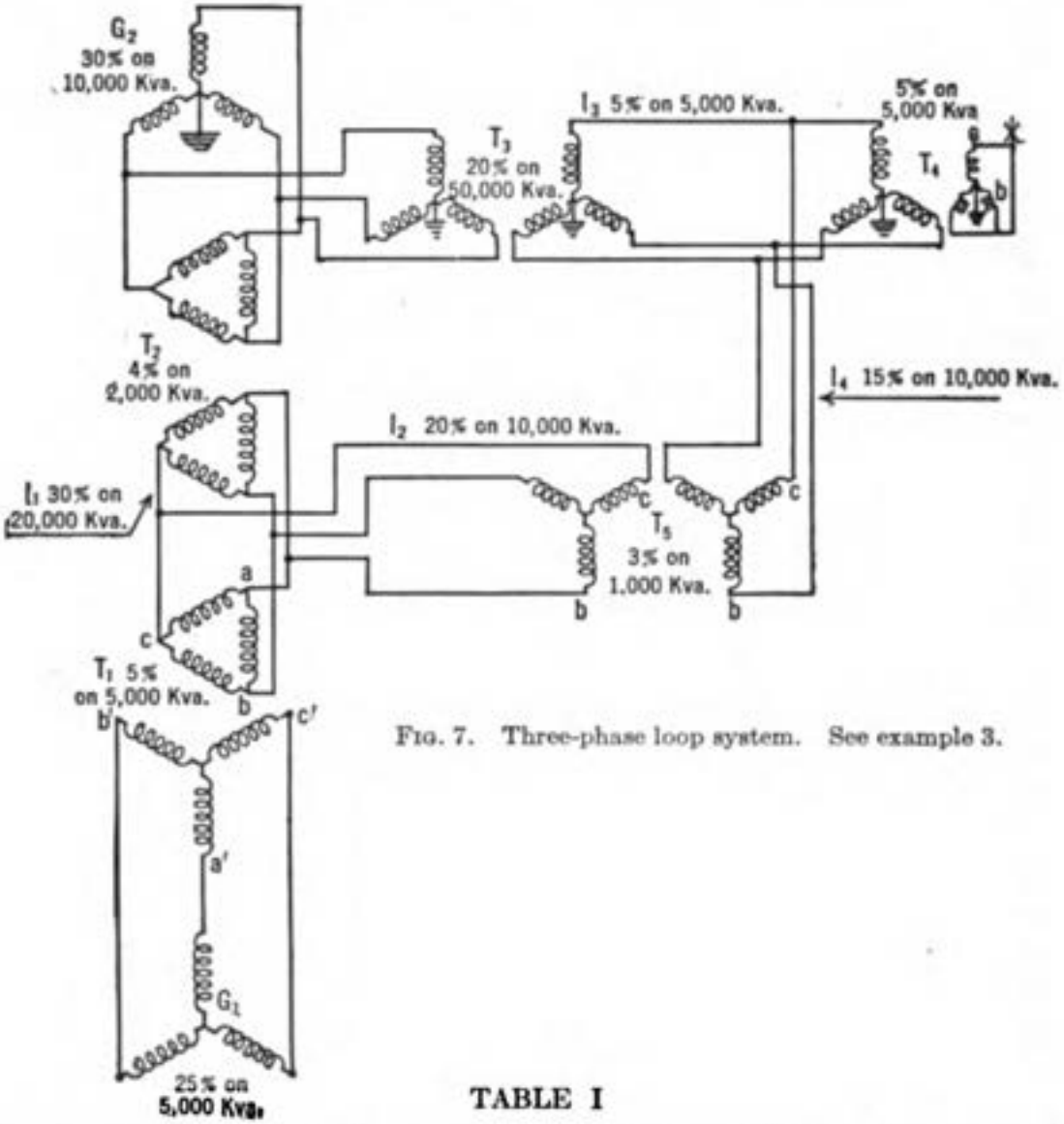

FIG. 7. Three-phase loop system. See example 3.

TABLE I

Apparatus	Rating kva	% Reactance	% Reactance based on kva
Generator 1	5,000	25	5,000
Generator 2	10,000	30	10,000
Transformer 1	4,000	5	5,000
Transformer 2	2,000	4	2,000
Transformer 3	5,000	20	50,000
Transformer 4	5,000	5	5,000
Transformer 5	1,000	3	1,000
Line 1		30	20,000
Line 2		20	10,000
Line 3		5	5,000
Line 4		15	10,000

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short circuit is assumed at the point denoted by the cross in the upper right-hand corner of the circuit diagram.

The following represents a satisfactory procedure.

1. A one-line diagram of the system as shown in Fig. 8 is drawn.

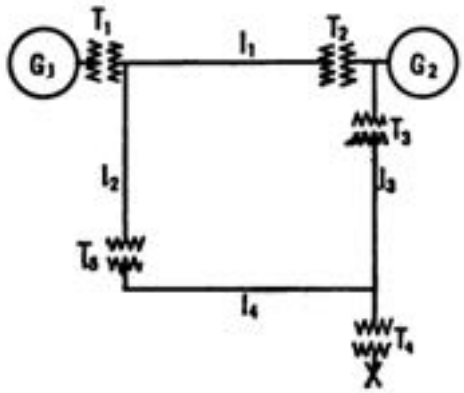

FIG. 8. One-line diagram of Fig. 7.

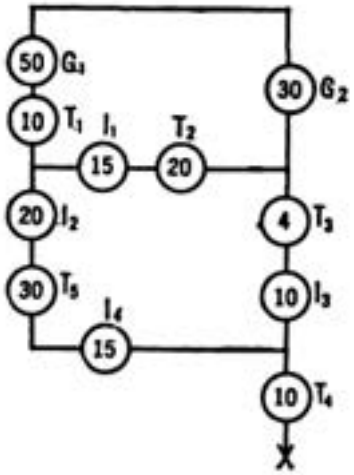

FIG. 9. One-line diagram of Fig. 8 where G_1 and G_2 are connected to a common neutral bus and all reactances are shown on a 10,000-kva base.

2. A common kva base upon which all reactances are based is chosen. Any convenient base may be used; here a 10,000-kva base is selected.

3. A one-line diagram is drawn in which all sources of current are connected to a so-called neutral bus. Circles represent reactances, and the value of the various reactances referred to the selected common kva base is placed in the circle as shown in Fig. 9.

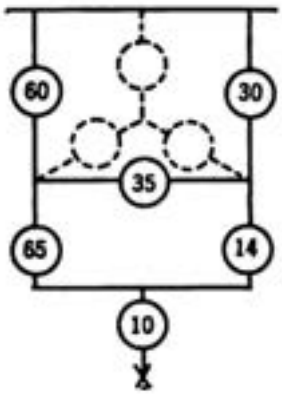

FIG. 10.

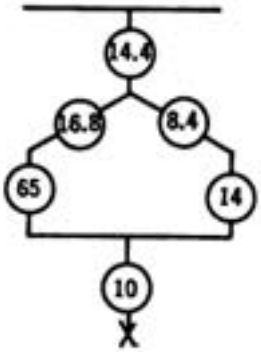

FIG. 11.

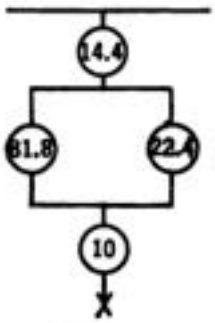

FIG. 12.

FIG. 13.

4. Reactances are combined according to laws of series or parallel circuits, and substitution of wyes for deltas or the reverse are made so as to obtain a single reactance between the neutral bus and the point of short circuit. These steps are illustrated in the successive Figs. 10, 11, 12, 13, and 14. The dotted lines and circles indicate the circuit arrangement to be employed in replacing an existing circuit arrangement. The resultant reactance to the fault based on

FIG. 14. Resultant percentage of reactance on a 10,000-kva base of Fig. 7 to the point of short circuit.

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10,000 kva is 41.95 per cent.

$$\text{Short-circuit kva} = \frac{100}{41.95} \times 10{,}000 = 23{,}800$$

If the nominal voltage of the line at the short circuit is 12,000 volts, the current at the fault is

$$\frac{23{,}800 \times 1000}{\sqrt{3}\ 12{,}000} = 1144 \text{ amperes}$$

The distribution of currents throughout the network may be determined by retracing the steps and using the percentage values just exactly as though they were ohmic quantities. For example, the currents in the divided circuit of Fig. 12 may be determined as follows. To indicate the branch under consideration, a subscript which is the same as the branch impedance is used.

$$V_{17.55} = 17.55 \times 1144 \text{ volts}^1$$

$$I_{81.8} = \frac{17.55 \times 1144}{81.8} = 246 \text{ amperes}$$

$$I_{22.4} = \frac{17.55 \times 1144}{22.4} = 898 \text{ amperes}$$

If the nominal voltage of any line differs from the 12,000-volt base used above, the actual current is determined by multiplying the current calculated on the 12,000-volt base by the ratio of 12,000 to the nominal voltage for the line in question.

Problem 3. Find the actual currents delivered by generators G_1 and G_2.
Ans.: $I_{G1} = 344$ amperes, $I_{G2} = 800$ amperes.

Line-to-Line Short Circuits. Line-to-line short-circuit currents may be determined in accordance with the principles set forth in Chapter X, or they may be calculated by the method of symmetrical components. The method of symmetrical components possesses the advantage of accounting in a measure for the change in the impedance of synchronous machines when the loading is changed from balanced three-phase to single-phase line-to-line loading. Furthermore the method of symmetrical components reduces the calculations to the solution of balanced three-phase systems. Certain modifications of the network parameters are necessary in employing the method of symmetrical components, and in addition the combination of the balanced systems solutions must be properly made to obtain the final result.

The method of symmetrical components for effecting a solution of the line-to-line short-circuit problem will be developed with reference to Fig. 15. The fundamental objective is to determine the positive- and negative-sequence components of current in terms of (the known quanti-

[1] This number of volts is only proportional to the actual voltage and is used merely as a convenient means to determine the distribution of currents.

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ties) the induced voltage and impedance. The following symbols are used:

E, generated voltage per phase
n', electrical neutral at the point of short circuit
V_1, positive-sequence voltage to neutral at the short circuit
V_2, negative-sequence voltage to neutral at the short circuit
V_0, zero-sequence voltage to neutral at the short circuit
Z_1, impedance to positive sequence
Z_2, impedance to negative sequence

According to Kirchhoff's voltage law, the positive-sequence voltage to neutral at the short circuit must be the positive-sequence generated voltage minus the positive-sequence drop. A similar relation obtains

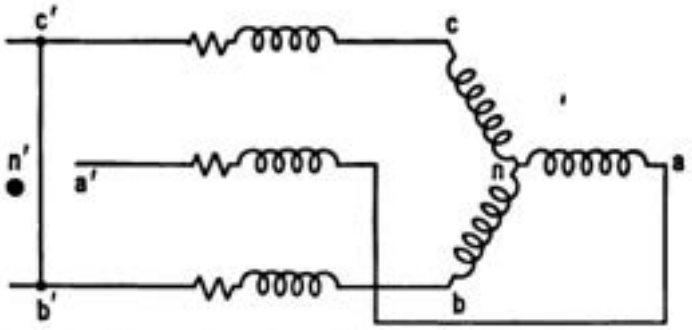

FIG. 15. Line-to-line short circuit on a three-phase system.

for the negative sequence. Since all generated voltages at the generator are assumed to be balanced, the positive-sequence generated voltage is E. The negative-sequence generated voltage is zero. Hence for any particular phase

$$\mathbf{V}_1 = \mathbf{E} - \mathbf{I}_1\mathbf{Z}_1 \tag{4}$$

$$\mathbf{V}_2 = 0 - \mathbf{I}_2\mathbf{Z}_2 \tag{5}$$

Since there is no ground return or fourth wire in Fig. 15, there can be no zero-sequence current in this system. At the short circuit

$$\mathbf{V}_{b'c'} = \mathbf{V}_{b'n'} + \mathbf{V}_{n'c'} = 0 \tag{6}$$

or

$$\mathbf{V}_{n'b'} = \mathbf{V}_{n'c'} \tag{7}$$

The three voltages to neutral at the short circuit in terms of their symmetrical components are (assuming *ab-bc-ca* sequence)

$$\mathbf{V}_{n'a'} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_0 \tag{8}$$

$$\mathbf{V}_{n'b'} = \mathbf{V}_1\underline{/-120^\circ} + \mathbf{V}_2\underline{/120^\circ} + \mathbf{V}_0 \tag{9}$$

$$\mathbf{V}_{n'c'} = \mathbf{V}_1\underline{/120^\circ} + \mathbf{V}_2\underline{/-120^\circ} + \mathbf{V}_0 \tag{10}$$

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Substituting equations (9) and (10) in equation (7),

$$\mathbf{V}_1\underline{/-120^\circ} + \mathbf{V}_2\underline{/120^\circ} + \mathbf{V}_0 = \mathbf{V}_1\underline{/120^\circ} + \mathbf{V}_2\underline{/-120^\circ} + \mathbf{V}_0$$

$$\mathbf{V}_1(\underline{/-120^\circ} - \underline{/120^\circ}) = \mathbf{V}_2(\underline{/-120^\circ} - \underline{/120^\circ})$$

or

$$\mathbf{V}_1 = \mathbf{V}_2 \tag{11}$$

Equation (11) shows that equations (4) and (5) are equal. Therefore

$$\mathbf{E} - \mathbf{I}_1\mathbf{Z}_1 = -\mathbf{I}_2\mathbf{Z}_2 \tag{12}$$

If $\mathbf{I}_2$ can be expressed in terms of $\mathbf{I}_1$, the sequence components of currents can be found. Since no zero-sequence current can exist in the circuit of Fig. 15, $\mathbf{I}_1$ and $\mathbf{I}_2$ are found as shown below.

$$\mathbf{I}_{na} = \mathbf{I}_1 + \mathbf{I}_2 = 0 \quad \text{(Line } na \text{ is open.)} \tag{13}$$

$$\mathbf{I}_{nb} = \mathbf{I}_1\underline{/-120^\circ} + \mathbf{I}_2\underline{/120^\circ} \tag{14}$$

$$\mathbf{I}_{nc} = \mathbf{I}_1\underline{/120^\circ} + \mathbf{I}_2\underline{/-120^\circ} \tag{15}$$

Because of the short circuit,

$$\mathbf{I}_{nb} = \mathbf{I}_{cn} = -\mathbf{I}_{nc} \tag{16}$$

Substituting equations (14) and (15) in equation (16),

$$\mathbf{I}_1\underline{/-120^\circ} + \mathbf{I}_2\underline{/120^\circ} = -\mathbf{I}_1\underline{/120^\circ} - \mathbf{I}_2\underline{/-120^\circ}$$

$$\mathbf{I}_1(\underline{/-120^\circ} + \underline{/120^\circ}) + \mathbf{I}_2(\underline{/-120^\circ} + \underline{/120^\circ}) = 0$$

$$\mathbf{I}_1 = -\mathbf{I}_2 \tag{17}$$

Substituting equation (17) in equation (12) yields

$$\mathbf{E} - \mathbf{I}_1\mathbf{Z}_1 - \mathbf{I}_1\mathbf{Z}_2 = 0$$

$$\mathbf{I}_1 = \frac{\mathbf{E}}{\mathbf{Z}_1 + \mathbf{Z}_2} \tag{18}$$

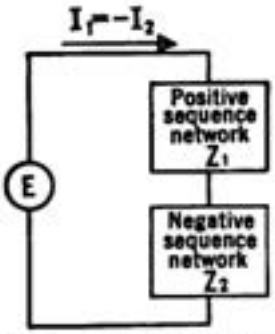

FIG. 16. Arrangement of sequence networks for determination of positive- and negative-sequence currents for a line-to-line short circuit.

Equations 17 and 18 show that the arrangement illustrated in Fig. 16 may be used to calculate the positive- and negative-sequence currents at the fault for a line-to-line short circuit.

Impedances to Positive and Negative Sequence. Before equation (18) can be applied, the values of the impedances to positive and negative sequence must be known. The impedance to positive sequence is the impedance offered to a system whose voltages a, b, and c, respectively, lag the one preceding it by 120°. The impedance to negative sequence

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is the impedance offered to a system whose voltages a, b, and c, respectively, lead the one preceding it by 120°. It should be apparent, and it can be demonstrated by test, that the impedances of lines and transformers are no different for a polyphase system of voltages when two lines are interchanged (opposite sequence). Hence impedances to positive and negative sequence for all lines and static machinery like transformers are the same. For a synchronous generator it would seem that these impedances are different since one system causes a reaction from the armature that rotates in the same direction as the rotating field structure, whereas the other causes an armature reaction that rotates in a direction opposite to the field structure. The values of Z_1 and Z_2 may be obtained from a three-phase and a line-to-line short-circuit test. The relation between the line-to-line short-circuit current designated by I' and the three-phase short-circuit current represented by I''' is established for an alternator of voltage E_n to neutral as follows:

$$I''' = \frac{E_n}{Z_1} \tag{19}$$

For a line-to-line short circuit between terminals b and c at the generator, Fig. 15, a combination of equations (15), (17), and (18) gives the current:

$$\begin{aligned} \mathbf{I}' = \mathbf{I}_{nc} &= \frac{\mathbf{E}_n}{\mathbf{Z}_1 + \mathbf{Z}_2}\underline{/120^\circ} - \frac{\mathbf{E}_n}{\mathbf{Z}_1 + \mathbf{Z}_2}\underline{/-120^\circ} \\ &= \frac{\mathbf{E}_n}{\mathbf{Z}_1 + \mathbf{Z}_2}(\cos 120^\circ + j \sin 120^\circ - \cos 120^\circ + j \sin 120^\circ) \\ &= +j\sqrt{3}\,\frac{\mathbf{E}_n}{\mathbf{Z}_1 + \mathbf{Z}_2} \end{aligned} \tag{20}$$

Since $\mathbf{Z}_1 + \mathbf{Z}_2$ is practically $Z_1 + Z_2$ owing to the resistances in both cases being small compared to the reactance,[2] the magnitude of $\mathbf{I}'$ is:

$$I' = \frac{\sqrt{3}E_n}{Z_1 + Z_2} \tag{21}$$

Let $k = I'/I'''$. Then

$$k\frac{E_n}{Z_1} = \frac{\sqrt{3}E_n}{Z_1 + Z_2}$$

[2] In general the resistances of generators and transformers are sufficiently low in comparison with the corresponding reactances that it is customary to neglect resistances in making short-circuit calculations. For this reason reactances only are used in many of the subsequent computations even though the formulas are written in terms of impedances. If these facts are not kept in mind the rather loose use of the terms reactance and impedance may become confusing.

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or

$$Z_2 = \frac{\sqrt{3}Z_1}{k} - Z_1 \tag{22}$$

Equation (22) shows that Z_2 depends upon the ratio, k, of the line-to-line and three-phase short-circuit currents. When this ratio is known and the impedance to positive sequence is determined by the ordinary methods, Z_2 can be determined. One salient-pole machine with an amortisseur winding tested by one of the authors gave a value of 1.44 for k, while another non-salient pole machine without an amortisseur winding yielded a value of 1.46.[3]

TABLE II

IMPEDANCES AND REACTANCES TO DIFFERENT SEQUENCES OF SALIENT-POLE SYNCHRONOUS GENERATORS WITH DAMPER WINDINGS

Name of Reactance	Synchronous X_s	Positive-Sequence X_1	Negative-Sequence X_2	Zero-Sequence X_0
Per Cent Reactances	100	100	Approximate Range 25–50	Approximate Range 2–20
Name of Impedance	Synchronous Z_s	Positive-Sequence Z_1	Negative-Sequence Z_2	Zero-Sequence Z_0
Approximate Per Cent Impedances	100	100	Approximate Range 25–50	Approximate Range 3–20

Table II shows approximate ranges of impedances to positive-, negative-, and zero-sequence currents of one class of generators with reference to the synchronous impedance taken as 100.

Example 4. Each of the line reactances in Fig. 15 is 10 per cent based on 1000 kva, and the positive-sequence impedance of the alternator is 25 per cent based on 1000 kva. A value of 1.45 is assumed for k. The short-circuit currents in the three lines for a short circuit between lines b and c are to be determined. The nominal rated

[3] The values of reactances to negative sequence depend upon the size and the design of the machines and vary over rather wide limits for special cases. The reader is referred to Wagner and Evans, "Symmetrical Components," p. 99, McGraw-Hill Book Company, for extensive data on synchronous generator reactances to the different sequences.

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line voltage of the system is 2200 volts. For the generator

$$Z_2 = \frac{\sqrt{3}}{1.45} Z_1 - Z_1 = 1.2Z_1 - Z_1 = 0.2Z_1$$
$$= 0.2 \times 25 = 5\%$$

The positive- and negative-sequence circuits are shown in Figs. 17 and 18, respectively. The resultant impedances to positive and negative sequence are 35 per cent and 15 per cent, respectively. From equation (18) and Fig. 16,

$$\mathbf{I}_1 = \frac{1{,}000{,}000}{\sqrt{3} \times 2200} \times \frac{100}{(35 + 15)} = 525 \text{ amperes}$$
$$\mathbf{I}_2 = -\mathbf{I}_1 = -525\underline{/0^\circ} \text{ amperes}$$
$$\mathbf{I}_{na} = \mathbf{I}_1 + \mathbf{I}_2 = 525\underline{/0^\circ} - 525\underline{/0^\circ} = 0$$
$$\mathbf{I}_{nb} = \mathbf{I}_1\underline{/-120^\circ} + \mathbf{I}_2\underline{/120^\circ} = 525\underline{/-120^\circ} - 525\underline{/120^\circ} = -j910 \text{ amperes}$$
$$\mathbf{I}_{nc} = \mathbf{I}_1\underline{/120^\circ} + \mathbf{I}_2\underline{/-120^\circ} = 525\underline{/120^\circ} - 525\underline{/-120^\circ} = +j910 \text{ amperes}$$

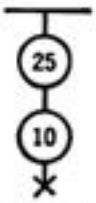

FIG. 17. Positive-sequence system of Fig. 15. See example 4.

Example 5. The short-circuit current for the system shown in Fig. 7 for a line-to-line short circuit is to be determined. The ratio k will be used as 1.45. Nominal line voltage at short circuit is 12,000 volts. The lines shorted are designated as b and c, and the fault is again assumed at the upper right-hand corner of the diagram.

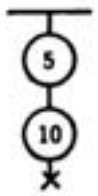

FIG. 18. Negative-sequence system of Fig. 15. See example 4.

Solution. A 10,000-kva base will be used. The positive-sequence network is the same as that shown in Fig. 9. The negative-sequence network shown in Fig. 19 is similar to the positive-sequence system except for the values of the generator reactances. For the generators

$$Z_2 = \frac{\sqrt{3}Z_1}{1.45} - Z_1 = 0.2Z_1$$

The resultant Z_1 (Fig. 14) is 41.95 per cent.

The resultant Z_2 (Fig. 23) is 26.17 per cent as obtained from the reductions indicated by Figs. 20, 21, 22, and 23.

$$\mathbf{I}_1 = -\mathbf{I}_2 = \frac{10{,}000{,}000}{\sqrt{3}\ 12{,}000} \times \frac{100}{41.95 + 26.17} + 706 \text{ amperes}$$

At the short circuit where currents in all three lines are considered in the same direction, that is, either to or from the short circuit,

$$\mathbf{I}_a = \mathbf{I}_1 + \mathbf{I}_2 = 706\underline{/0^\circ} - 706\underline{/0^\circ} = 0$$
$$\mathbf{I}_b = 706\underline{/-120^\circ} - 706\underline{/120^\circ} = -j1223 \text{ amperes}$$
$$\mathbf{I}_c = 706\underline{/120^\circ} - 706\underline{/-120^\circ} = +j1223 \text{ amperes}$$

To obtain the currents in the other lines, the positive- and negative-sequence currents should first be found by retracing the steps in each system as outlined for the three-

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phase short circuit. The current in the lines from the secondaries of transformer T_1 will be found in order to illustrate the procedure. The distribution of positive-

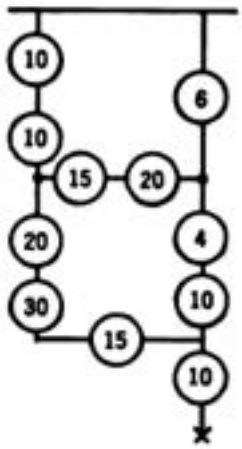

FIG. 19. Negative-sequence network of FIG. 7 for a line-to-line short circuit at the point indicated by the cross.

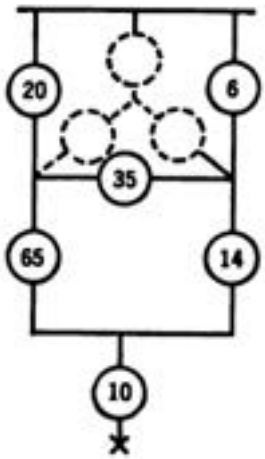

FIG. 20.

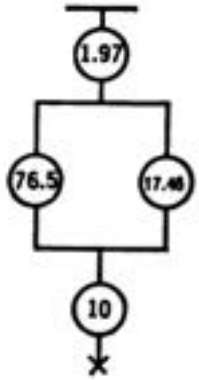

FIG. 21.

and negative-sequence components of current as shown in Figs. 24 and 25 are first found by retracing previous steps. If, when retracing the network from the short circuit, only transformers with both primary and secondary windings similarly connected are encountered, the actual current may be found by combining the sequence components as determined for Figs. 24 and 25. When a transformer like T_1 which is connected differently on primary and secondary is encountered, the symmetrical components in the lines on the primary side are no longer the same as those in the secondary lines. Failure to recognize this fact will introduce large errors in the short-circuit calculations. The short-circuit currents in the secondary lines from transformer T_1 are found from the sequence components shown in Figs. 24 and 25, as follows:

FIG. 22.

FIG. 23. Resultant percentage of reactance to negative sequence for a line-to-line short circuit at point indicated on Fig. 7.

$$\mathbf{I}_a = \mathbf{I}_1 + \mathbf{I}_2 = 212\underline{/0^\circ} - 144.7\underline{/0^\circ} = 67.3 \text{ amperes}$$

$$\mathbf{I}_b = 212\underline{/-120^\circ} - 144.7\underline{/120^\circ} = -33.65 - j308.8 \text{ amperes}$$

$$\mathbf{I}_c = 212\underline{/120^\circ} - 144.7\underline{/-120^\circ} = -33.65 + j308.8 \text{ amperes}$$

The currents in the lines on the primary of T_1, Fig. 7, are determined from the phase currents in the delta and are obviously equal to them if the ratio of each transformer is 1 to 1 and the magnetizing currents are neglected. If the impedances of all phases of a delta-connected bank of transformers like that shown in Fig. 26 are equal, and if the sum of the generated voltages of the three phases is zero, application of Kirchhoff's laws will yield the following equations:

$$\mathbf{I}_{aa'} + \mathbf{I}_{bb'} + \mathbf{I}_{cc'} = 0 \tag{23}$$

$$\mathbf{I}_{ba}\mathbf{Z}_{ba} + \mathbf{I}_{ac}\mathbf{Z}_{ac} + \mathbf{I}_{cb}\mathbf{Z}_{cb} = \mathbf{E}_{ba} + \mathbf{E}_{ac} + \mathbf{E}_{cb} = 0 \tag{24}$$

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Since $Z_{ba} = Z_{ac} = Z_{cb}$, equation (24) becomes

$$I_{ba} + I_{ac} + I_{cb} = 0 \tag{25}$$

Further application of Kirchhoff's current law gives

$$I_{aa'} = I_{ba} - I_{ac} \tag{26}$$

$$I_{bb'} = I_{cb} - I_{ba} \tag{27}$$

$$I_{cc'} = I_{ac} - I_{cb} \tag{28}$$

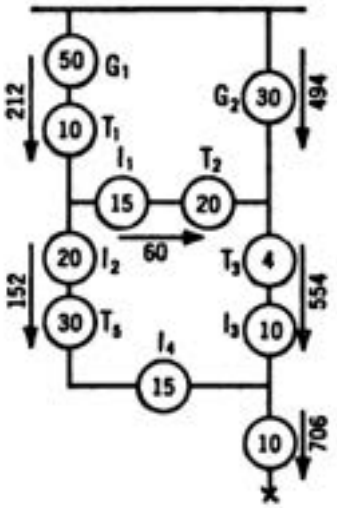

FIG. 24. Distribution of positive-sequence component currents for example 5.

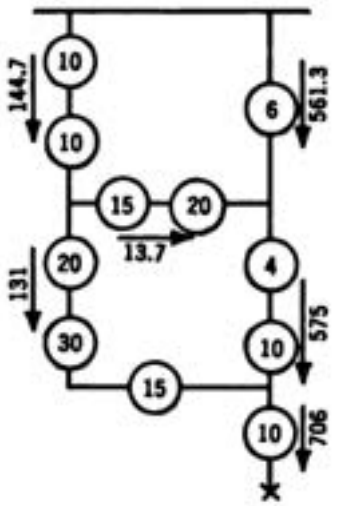

FIG. 25. Distribution of negative-sequence component currents for example 5.

Substituting I_{ac} from equation (25) in equation (26), then eliminating I_{cb} between this result and equation (27), and finally substituting the value of $I_{bb'}$ from equation (23), the following expression for I_{ba} results:

$$I_{ba} = \tfrac{2}{3}I_{aa'} + \tfrac{1}{3}I_{cc'} \tag{29}$$

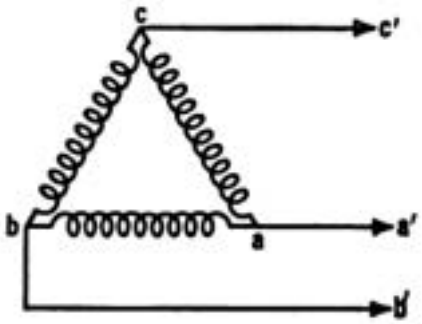

FIG. 26.

Similarly I_{ac} and I_{cb} are found to be, respectively,

$$I_{ac} = \tfrac{2}{3}I_{cc'} + \tfrac{1}{3}I_{bb'} \tag{30}$$

$$I_{cb} = \tfrac{2}{3}I_{bb'} + \tfrac{1}{3}I_{aa'} \tag{31}$$

The currents I_a, I_b, and I_c in the secondary lines of T_1 of Fig. 7 correspond to $I_{aa'}$,

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$\mathbf{I}_{bb'}$, and $\mathbf{I}_{cc'}$, respectively, in equations (29), (30), and (31).[4] Hence

$$\begin{aligned}
\mathbf{I}_{ba} &= \tfrac{2}{3}(67.3) + \tfrac{1}{3}(-33.65 + j308.8) \\
&= 33.7 + j102.9 = 108.2\underline{/71.8^\circ} \text{ amperes} \\
\mathbf{I}_{ac} &= \tfrac{2}{3}(-33.65 + j308.8) + \tfrac{1}{3}(-33.65 - j308.8) \\
&= -33.7 + j102.9 = 108.2\underline{/108.2^\circ} \text{ amperes} \\
\mathbf{I}_{cb} &= \tfrac{2}{3}(-33.65 - j308.8) + \tfrac{1}{3}(67.3) \\
&= -j205.8 = 205.8\underline{/-90^\circ} \text{ amperes}
\end{aligned}$$

On a 1 to 1 ratio, $\mathbf{I}_{ba}$, $\mathbf{I}_{ca}$, and $\mathbf{I}_{cb}$ above are the line currents from generator G_1, Fig. 7, or, in other words, the above currents are on a $\sqrt{3}$ 12,000-volt base. If the nominal voltage of the generator is 6600 volts, the currents in the three lines from the generator are

$$108.2 \times \frac{\sqrt{3}\,12{,}000}{6600} = 341 \text{ amperes}$$

$$108.2 \times \frac{\sqrt{3}\,12{,}000}{6600} = 341 \text{ amperes}$$

$$205.8 \times \frac{\sqrt{3}\,12{,}000}{6600} = 648 \text{ amperes}$$

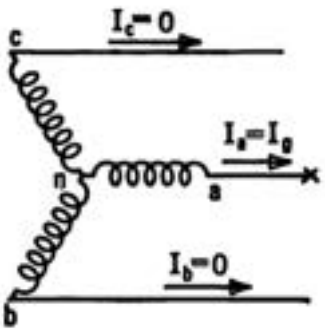

FIG. 27. Line-to-ground fault. Neutral n of the three-phase generator is assumed grounded.

Line-to-Ground Short Circuits. If a system has a number of wye-connected generators and transformers with grounded neutrals, there is a possibility of having a large short-circuit current for a line-to-ground fault. Such fault currents are most conveniently calculated with the aid of symmetrical components. An elementary circuit illustrating a line-to-ground fault is shown in Fig. 27. Application of equations (13), (16), and (17) of Chapter XIV gives the symmetrical components of the currents as

$$\mathbf{I}_0 = \tfrac{1}{3}(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = \frac{\mathbf{I}_a}{3} \tag{32}$$

$$\mathbf{I}_1 = \tfrac{1}{3}(\mathbf{I}_a + \mathbf{I}_b\underline{/120^\circ} + \mathbf{I}_c\underline{/-120^\circ}) = \frac{\mathbf{I}_a}{3} \tag{33}$$

$$\mathbf{I}_2 = \tfrac{1}{3}(\mathbf{I}_a + \mathbf{I}_b\underline{/-120^\circ} + \mathbf{I}_c\underline{/120^\circ}) = \frac{\mathbf{I}_a}{3} \tag{34}$$

Therefore

$$\mathbf{I}_0 = \mathbf{I}_1 = \mathbf{I}_2 = \frac{\mathbf{I}_a}{3} \tag{35}$$

[4] Equations other than (29), (30), and (31) for the currents in the transformer windings can be derived from the basic equations given.

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Let **E** be the induced voltage in phase a of the generator. According to Kirchhoff's emf law, the sum of all the drops must be equal to the sum of the emf's around a closed loop. Then

$$\mathbf{E} = \mathbf{I}_0\mathbf{Z}_0 + \mathbf{I}_1\mathbf{Z}_1 + \mathbf{I}_2\mathbf{Z}_2$$

Substituting equation (35) gives

$$\mathbf{E} = \mathbf{I}_0 (\mathbf{Z}_0 + \mathbf{Z}_1 + \mathbf{Z}_2) \tag{36}$$

Combining equations (35) and (36),

$$\mathbf{I}_0 = \frac{\mathbf{I}_a}{3} = \frac{\mathbf{E}}{\mathbf{Z}_0 + \mathbf{Z}_1 + \mathbf{Z}_2} \tag{37}$$

Equation (37) is the working equation for the line-to-ground fault.

Equations 35 and 37 show that the arrangement illustrated in Fig. 28 may be used to calculate the positive-, negative-, and zero-sequence currents at the fault for a line-to-ground short circuit.

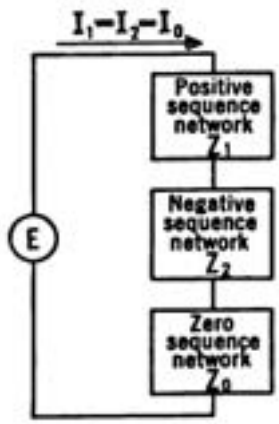

FIG. 28. Arrangement of sequence networks for calculating positive-, negative-, and zero-sequence currents for a line-to-ground short circuit.

The impedances to positive and negative sequence $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are exactly the same as those used for the line-to-line fault. The impedance to zero sequence, however, is different. Whereas the positive- and negative-sequence networks were alike in the number and arrangement of circuit elements, the zero-sequence network is radically different and usually much simpler.

Impedance to Zero Sequence for Generators. The determination of the impedance to zero sequence for generators is analogous to the determination of the impedance to negative sequence.

Let I_n represent the line-to-ground short-circuit current for a generator.

Let I''' represent the short-circuit current for a three-phase symmetrical short circuit.

Also let

$$\frac{I_n}{I'''} = k_n \tag{38}$$

From equation (37), if the ratio of X/R for all impedances is the same or if R is negligible compared to X, as is usual,

$$I_n = 3I_0 = \frac{3E}{Z_1 + Z_2 + Z_0} \tag{39}$$

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Also

$$I''' = \frac{E}{Z_1} \quad \text{and} \quad k_n I''' = I_n = k_n \frac{E}{Z_1}$$

Therefore

$$k_n \frac{E}{Z_1} = \frac{3E}{Z_1 + Z_2 + Z_0} \tag{40}$$

Solving equation (40) for Z_0 gives

$$Z_0 = Z_1 \left(\frac{3}{k_n} - 1\right) - Z_2 \tag{41}$$

The value of Z_0 thus depends upon the values of the impedances to positive and negative sequence and also upon the ratio of the line-to-ground and three-phase short-circuit currents. For example, k_n for the nonsalient-pole machine used in the previous example was shown by test to be about 2.4. For this machine

$$Z_0 = Z_1 \left(\frac{3}{2.4} - 1\right) - 0.2Z_1 = 0.05Z_1$$

The approximate range of impedance to zero sequence for one class of generators is shown in Table II on page 534. The values are given relative to the synchronous impedance taken as 100.

Impedance to Zero Sequence for Transformers. The impedance to zero sequence for transformers is either infinite or the ordinary leakage impedance,[5] depending upon the connection. Where the connection permits zero-sequence currents to flow, the impedance to zero sequence is the ordinary impedance of the transformer; otherwise it is infinite. Since the zero-sequence currents in the three lines of a three-phase system are all in phase, a fourth wire or ground connection on the neutral of transformers connected in wye is required to furnish a complete circuit for the return of the zero-sequence line currents. In addition, there must be another winding on the transformer to permit current to flow so that the resultant magnetomotive force acting upon the transformer core due to the zero-sequence current is zero (exciting current neglected). If these compensating currents are not permitted to exist, the inductive reactance of a single winding to the zero-sequence current is so high that the amount of this current which can flow is entirely negligible. The corresponding impedance may then be considered

[5] If the transformers have more than two windings which carry zero-sequence current, reactance due to certain mutual-inductance effects of the several windings should be included. For a discussion of the reactance of multiwinding transformers, see O. G. C. Dahl, "Electric Circuits," McGraw-Hill Book Company.

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infinite. A few examples as shown in Fig. 29 will illustrate these principles.

Transformer Bank A. No zero-sequence currents can flow since there is no return path. Therefore the impedance to zero sequence is infinite.

Transformer Bank B. Zero-sequence currents can flow. Winding p furnishes a path for the compensating currents of those in winding S. Hence the impedance to zero sequence is the ordinary leakage impedance.

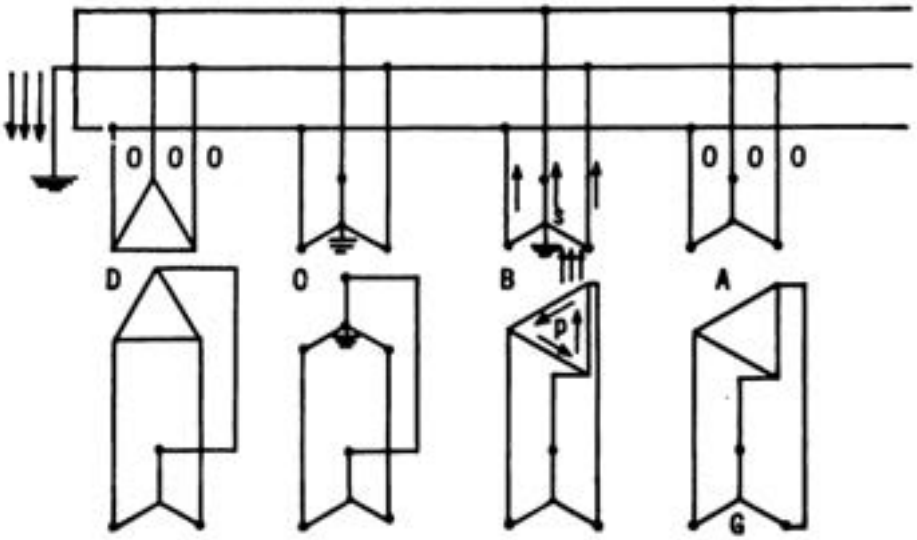

FIG. 29. Zero-sequence currents can flow in B but not in any of the other transformers.

Transformer Banks C and D. No zero-sequence currents can flow. The impedances to zero sequence are infinite. If the neutral of the wye-connected generator supplying transformer bank C were grounded, zero-sequence currents could flow in both primary and secondary of C. Under these conditions the impedance to zero sequence of transformer bank C would be the ordinary leakage impedance.

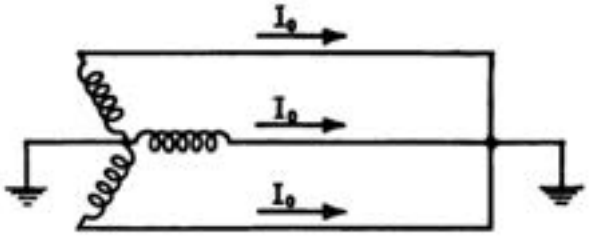

FIG. 30. Zero-sequence impedance of a transmission line is the impedance of the three conductors in parallel in series with a ground return.

Impedance to Zero Sequence of Transmission Lines. The impedance to zero sequence of a transmission line, Fig. 30, is the impedance of the three conductors in parallel with a ground return. The reactance depends upon the depth at which the return current appears to flow. A

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discussion sufficiently adequate to yield a working knowledge of the determination of reactance to zero sequence of transmission lines is somewhat involved and beyond the scope of this book. Those interested are referred to other works on the subject.[6] For purposes of illustration of the method of calculating line-to-ground fault currents in this book, certain values of reactance to zero sequence of a line are assumed.

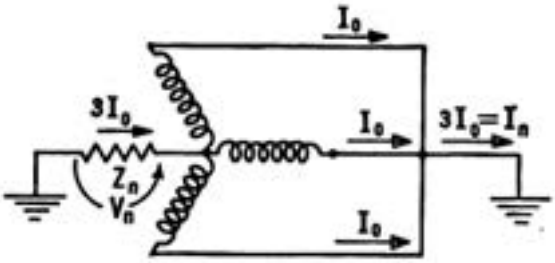

FIG. 31. Flow of zero-sequence currents through an impedance in the neutral.

If an impedance Z_n as shown in the neutral of the generator of Fig. 31 is encountered, it should be entered into the zero-sequence networks as $3\mathbf{Z}_n$. This may be shown as follows. The ordinary impedance $\mathbf{Z}_n$ is defined as the drop $\mathbf{V}_n$ across the impedance divided by the current through it. Hence

$$\mathbf{Z}_n = \frac{\mathbf{V}_n}{\mathbf{I}_n} \tag{42}$$

Since

$$\mathbf{I}_n = 3\mathbf{I}_0$$

$$\mathbf{Z}_n = \frac{\mathbf{V}_n}{3\mathbf{I}_0} \tag{43}$$

Since there are no positive- or negative-sequence currents in the neutral, $\mathbf{V}_n$ for this case is considered the zero-sequence voltage which is due to the zero-sequence current $\mathbf{I}_0$. Hence

$$\mathbf{Z}_0 = \frac{\mathbf{V}_0}{\mathbf{I}_0} = \frac{\mathbf{V}_n}{\mathbf{I}_0} \tag{44}$$

Substitution of $\mathbf{V}_n/\mathbf{I}_0$ from equation (43) in equation (44) gives

$$\mathbf{Z}_0 = 3\mathbf{Z}_n \tag{45}$$

Thus the impedance to zero sequence as defined in equation (45) is three times as large as the actual impedance in the conductor. Since the only zero-sequence current flowing in the zero-sequence network is $\mathbf{I}_0$, the

[6] See "Symmetrical Components" by Wagner and Evans and "Applications of the Method of Symmetrical Components" by Lyon, McGraw-Hill Book Company.

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value $\mathbf{Z}_0 = 3\mathbf{Z}_n$ should be entered into the zero-sequence network to yield the correct voltage drop.

Calculation of Line-to-Ground Fault Current. The system shown in Fig. 7, which was previously employed for three-phase and line-to-line short circuits, will be calculated for a line-to-ground fault on one of the secondary lines of transformer T_4. A determination of the reactance to zero-sequence of line l_3 is assumed to yield 20 per cent reactance on a 5000-kva base. The problem will be worked on a 10,000-kva base as before.

Solution. The positive- and negative-sequence networks are the same as those previously employed. They are shown in Figs. 24 and 25. The impedances to positive and negative sequence are the same for the line-to-ground fault solution and the general distribution of the positive- and negative-sequence currents is the same, but the actual magnitudes of the positive- and negative-sequence currents will be different because of the effect of the impedance to zero sequence in reducing the magnitude of the resultant positive- and negative-sequence currents. The resultant impedances to positive and negative sequence of 41.95 and 26.17 per cent, respectively, are still valid. An inspection of Fig. 7 shows that no zero-sequence current can exist in transformers T_5, T_2, T_1, or generator G_1. Therefore the zero-sequence network consists of G_2, T_3, and T_4 along with line l_3. The zero-sequence network is shown in Fig. 32. If $k_n = 2.4$ and $k_1 = 1.45$, substitution in equation (41) gives $Z_0 = 0.05Z_1$. For generator G_2

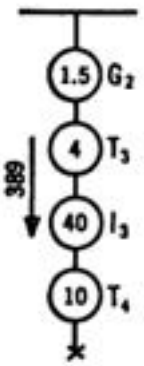

FIG. 32. Zero-sequence network for a line-to-ground fault on Fig. 7 at the point indicated by the cross.

$$Z_0 = 0.05 \times 30 = 1.5 \text{ per cent}$$

Resultant Z_0 for the zero-sequence network $= 1.5 + 4 + 40 + 10 = 55.5$ per cent.

$$\mathbf{I}_0 = \mathbf{I}_1 = \mathbf{I}_2 = \frac{\mathbf{I}_a}{3} = \frac{E}{Z_0 + Z_1 + Z_2}$$

In terms of percentage impedances,

$$\mathbf{I}_0 = \mathbf{I}_1 = \mathbf{I}_2 = \frac{\mathbf{I}_a}{3} = \frac{10{,}000{,}000}{\sqrt{3}\,12{,}000} \times \frac{100}{55.5 + 41.95 + 26.17}$$
$$= 389\underline{/0^\circ} \text{ amperes}$$

For a positive-sequence current of 389 amperes the distribution is shown in Fig. 33. These values are determined by multiplying the currents in Fig. 24 by 389/706. Similarly the negative-sequence current distribution is determined and shown in Fig. 34.

The currents on a 12,000-volt base are now found by combining the symmetrical components.

Fault current:

$$\mathbf{I}_a = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_0 = 3 \times 389\underline{/0^\circ} = 1167\underline{/0^\circ} \text{ amperes}$$
$$\mathbf{I}_b = 389\underline{/-120^\circ} + 389\underline{/120^\circ} + 389 = 0$$
$$\mathbf{I}_c = 389\underline{/120^\circ} + 389\underline{/-120^\circ} + 389 = 0$$

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Transformer T_3 and line l_3:

$\mathbf{I}_a = 305.2 + 316.8 + 389 = 1011$ $\qquad I_a = 1011$ amperes

$\mathbf{I}_b = 305.2\angle{-120^\circ} + 316.8\angle{120^\circ} + 389 = 78 + j10.5$ $\qquad I_b = 78.7$ amperes

$\mathbf{I}_c = 305.2\angle{120^\circ} + 316.8\angle{-120^\circ} + 389 = 78 - j10.5$ $\qquad I_c = 78.7$ amperes

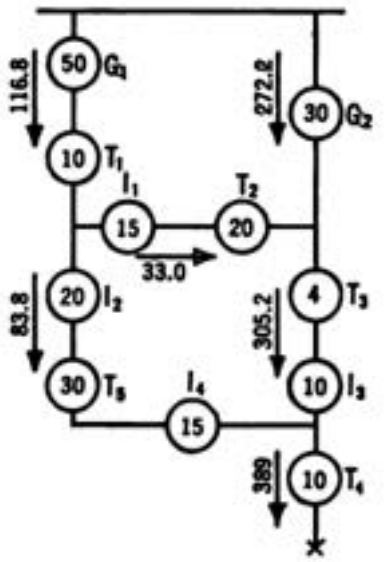

FIG. 33. Positive-sequence current distribution for line-to-ground fault on Fig. 7.

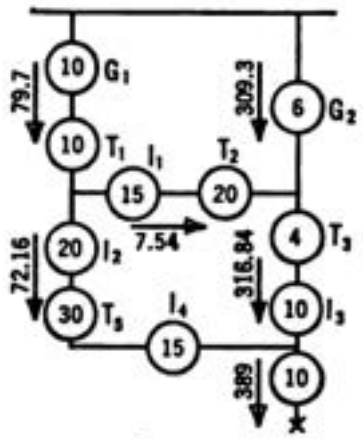

FIG. 34. Negative-sequence current distribution for line-to-ground fault on Fig. 7.

Line l_2, transformer T_5, and l_4:

$\mathbf{I}_a = 83.8 + 72.16 = 155.9$ $\qquad I_a = 155.9$ amperes

$\mathbf{I}_b = 83.8\angle{-120^\circ} + 72.16\angle{120^\circ} = -77.95 - j10$ $\qquad I_b = 78.6$ amperes

$\mathbf{I}_c = 83.8\angle{120^\circ} + 72.16\angle{-120^\circ} = -77.95 + j10$ $\qquad I_c = 78.6$ amperes

Line l_1 and transformer T_2:

$\mathbf{I}_a = 33.0 + 7.54 = 40.54$ $\qquad I_a = 40.5$ amperes

$\mathbf{I}_b = 33.0\angle{-120^\circ} + 7.54\angle{120^\circ} = -20.3 - j22.0$ $\qquad I_b = 30.0$ amperes

$\mathbf{I}_c = 33.0\angle{120^\circ} + 7.54\angle{-120^\circ} = -20.3 + j22.0$ $\qquad I_c = 30.0$ amperes

Secondary side of transformer T_1:

$\mathbf{I}_a = 116.8 + 79.7 = 196.5$ $\qquad I_a = 196.5$ amperes

$\mathbf{I}_b = 116.8\angle{-120^\circ} + 79.7\angle{120^\circ} = -98.2 - j32$ $\qquad I_b = 103.3$ amperes

$\mathbf{I}_c = 116.8\angle{120^\circ} + 79.7\angle{-120^\circ} = -98.2 + j32$ $\qquad I_c = 103.3$ amperes

Current in windings of transformer T_1 (see Fig. 7):

$\mathbf{I}_{ba} = \frac{2}{3}(196.5) + \frac{1}{3}(-98.25 + j32) = 98.3 + j10.7$ $\qquad I_{ba} = 98.9$ amperes

$\mathbf{I}_{ac} = \frac{2}{3}(-98.25 + j32) + \frac{1}{3}(-98.25 - j32)$
$= -98.3 + j10.7$ $\qquad I_{ac} = 98.9$ amperes

$\mathbf{I}_{cb} = \frac{2}{3}(-98.25 - j32) + \frac{1}{3}(196.5) = -j21.4$ $\qquad I_{cb} = 21.4$ amperes

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Since these are the delta transformer currents, on a 1 to 1 ratio they are also the currents in the phases of the wye primary, and therefore the currents in the lines from generator G_1 on a $\sqrt{3}$ 12,000 line voltage base.

Current in G_2:

$$\mathbf{I}_a = 272.2 + 309.3 + 389 = 970.5 \qquad I_a = 970.5 \text{ amperes}$$

$$\mathbf{I}_b = 272.2\angle{-120^\circ} + 309.3\angle{120^\circ} + 389 = 98.25 + j32 \qquad I_b = 103.3 \text{ amperes}$$

$$\mathbf{I}_c = 272.2\angle{120^\circ} + 309.3\angle{-120^\circ} + 389 = 98.25 - j32 \qquad I_c = 103.3 \text{ amperes}$$

PROBLEMS

4. Refer to Fig. 35. All circuit elements are assumed to have zero resistance. The reactances to positive sequence are the numbers preceded by j on the diagram. Generator A is a 3000-kva machine having a rated terminal voltage of 6600-volts. Generator B is a 6600-volt, 5000-kva machine.

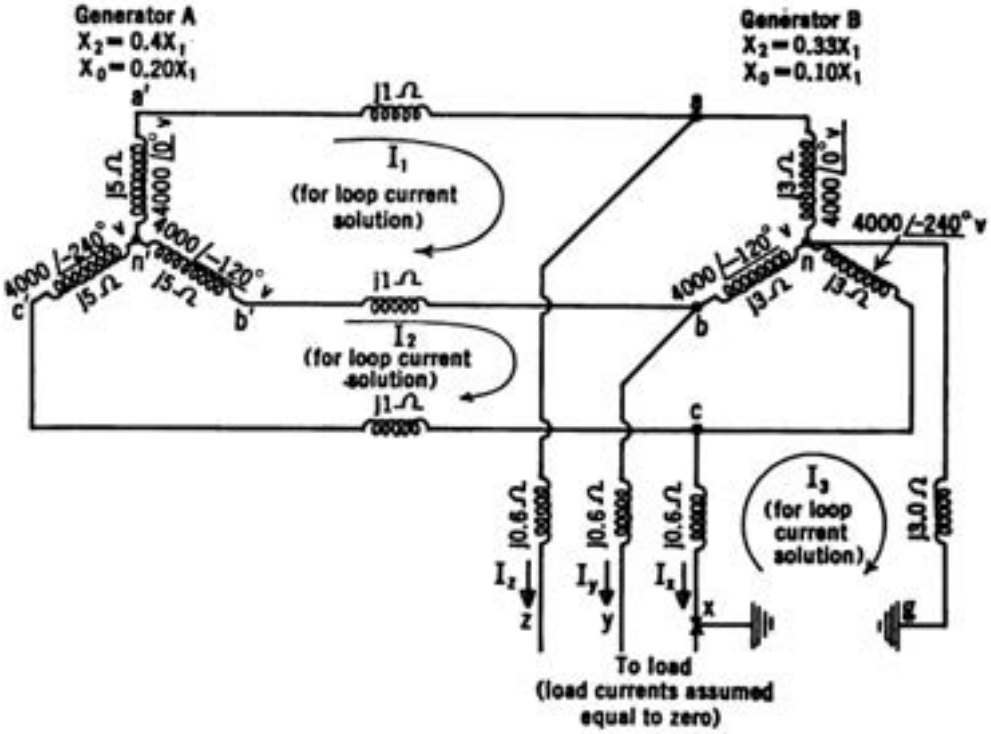

FIG. 35. See Problem 4.

(a) Solve for the currents in all branches by one of the methods considered in Chapter IX, assuming that the impedances shown on the diagram hold for any kind of unbalance.

(b) Solve for the currents in all branches by the method of symmetrical components, taking into account the difference in impedance to the positive, negative, and zero sequences. Impedances to positive sequence for the generators are those shown on the diagram.

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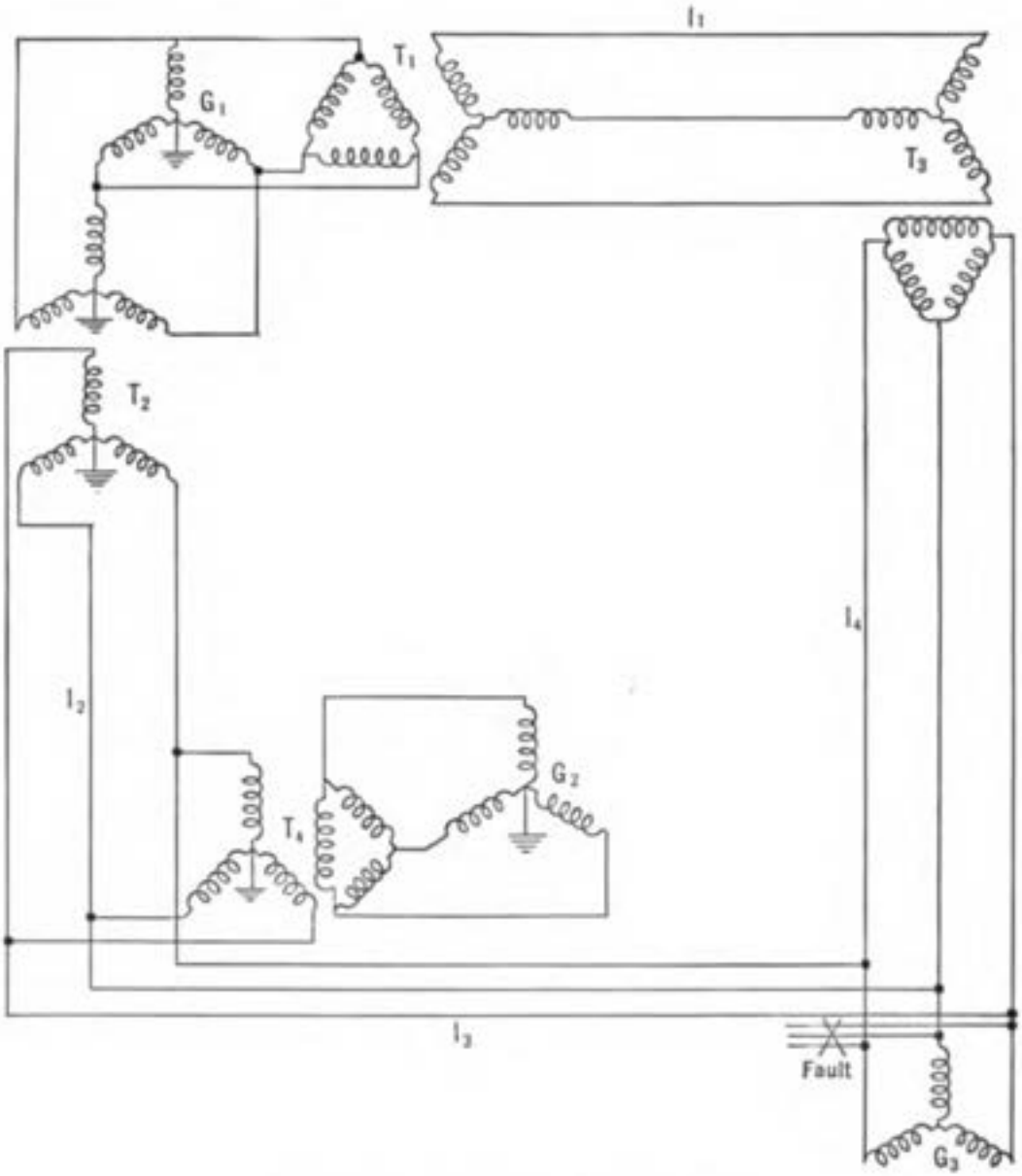

FIG. 36. See Problems 5, 6, and 7.

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5. The following data refer to Fig. 36.

Apparatus	Kva Rating	% Reactance	Kva Base for % Reactance
G_1	20,000	30	10,000
G_2	10,000	50	10,000
G_3	20,000	20	10,000
T_1	10,000	2	2,500
T_2	10,000	20	30,000
T_3	10,000	6	10,000
T_4	10,000	7	10,000
l_1		30	20,000
l_2		20	10,000
l_3		10	4,000
l_4		40	30,000

Generator	k	k_n
G_1	1.4	2.1
G_2	1.5	2.3
G_3	1.3	

All resistances are assumed negligible.

Calculate currents in all lines, transformers, and generators for a 3-phase symmetrical short circuit at the point marked fault. Express currents on a 33-kv base.

6. Calculate currents in all lines, transformers, and generators for a line-to-line fault at the point marked fault. Express currents on a 33-kv base.

7. Calculate currents in all lines, transformers, and generators for a line-to-ground fault at the point marked fault. Use 25 per cent based on 10,000 kva as the zero-sequence reactance of l_2 including lines and ground return. The zero-sequence reactance of l_3 including lines and ground return is 12 per cent based on 4,000 kva. Assume negligible resistance, and express currents on a 33-kv base.

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CHAPTER XVI

TRANSIENT CONDITIONS

The expressions which have thus far been derived for currents and voltages have carried with them certain tacit assumptions. All the alternating currents and voltages in any particular circuit have been assumed to be recurring, periodic functions of time; in other words, the circuit in question has been assumed to be in a steady-state condition.

Before a circuit (or machine) can arrive at a steady-state condition of operation which is different from some previous state, the circuit (or machine) passes through a transition period in which the currents and voltages are not recurring periodic functions of time. For example, immediately after the establishment of a circuit the currents and voltages have not, in general, settled into their steady-state conditions. The period required for the currents and voltages to adjust themselves to their steady-state modes of variation is called the transient period. During transient periods the mathematical expressions for the currents and voltages contain certain terms other than the steady-state terms. These additional terms are called transient terms, and they are usually of short duration, being damped out by certain damping factors which depend for their values upon the circuit parameters.

In general, any switching operation within the circuit itself or any voltage which is suddenly induced from an outside source will cause transient conditions to exist in the circuit. Although transient periods are generally of short duration, it is during these periods that some of the most serious and involved operating problems are encountered.

It should not be inferred that transient variations are always violent or that they always represent undesirable circuit conditions. Various devices actually operate by virtue of recurring transient phenomena. Notable among these devices are: (1) certain classes of sweep circuits, and (2) certain types of tube inverters. Sweep circuits are employed extensively to produce linear time axes in cathode-ray oscillographs and cathode-ray television tubes. Inverters are employed to convert direct to alternating current.

Examples of Elementary Transient Conditions. Example 1. In Fig. 1 it is assumed that the RL branch is suddenly energized with a constant potential difference by closing the switch S at $t = 0$. The

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general equation for voltage equilibrium in the resulting series circuit is:

$$L\frac{di}{dt} + Ri = E \tag{1}$$

If L, R, and E are constant the above equation may be solved explicitly for i in any one of several different ways. One of the most direct methods of solution in a simple case of this kind is to separate variables and integrate. Thus:

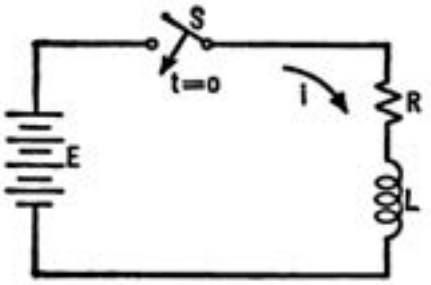

FIG. 1. A series RL branch which is suddenly energized by a constant potential difference E at $t = 0$.

$$L\frac{di}{(E - Ri)} = dt$$

or

$$L\int\frac{di}{(E - Ri)} = \int dt \tag{2}$$

Whence:

$$-\frac{L}{R}\log_\epsilon (E - Ri) = t + c$$

or

$$\log_\epsilon (E - Ri) = -\frac{Rt}{L} + c_1 \tag{3}$$

where ϵ is the base of the natural logarithms, namely, $2.71828 \cdots$, and c_1 is a constant of integration. From the definition of a logarithm it follows that:

$$E - Ri = \epsilon^{(-Rt/L)+c_1}$$

Therefore:

$$E - Ri = c_2\epsilon^{-Rt/L} \tag{4}$$

Solving the above equation for i yields:

$$i = \frac{E}{R} - c_3\epsilon^{-Rt/L} \tag{5}$$

The constant of integration c_3 must be evaluated in terms of the boundary conditions that surround the switching operation. Boundary conditions are usually specified in terms of the circuit currents and the condenser voltages that exist at the instant a given switching operation is performed. In general, the specification and incorporation of boundary conditions require an understanding of the natural characteristics of the circuit parameters involved. For example, if a circuit possesses inductance the current cannot change abruptly, that is, cannot become discontinuous with respect to time. Therefore the current in an inductive branch at the instant a given switching operation is

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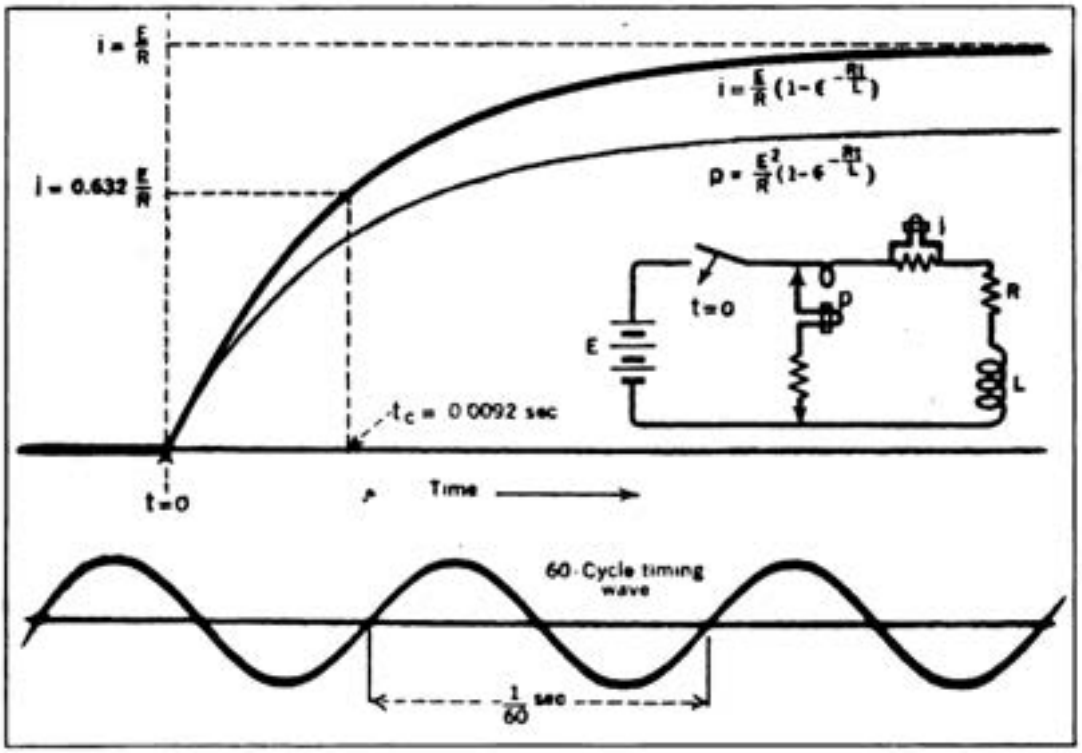
performed is equal to the current that exists in the branch just prior to switching operation. In the present case: $i = 0$ at $t = 0$, and this physical fact can be employed to determine the value of c_3 in equation (5). Imposing the boundary condition on equation (5) results in:

$$0 = \frac{E}{R} - c_3 \quad \text{or} \quad c_3 = \frac{E}{R} \tag{6}$$

The general expression for current becomes:

$$i = \underbrace{\frac{E}{R}}_{\text{steady-state term}} - \underbrace{\frac{E}{R}\epsilon^{-Rt/L}}_{\text{transient term}} \tag{7}$$

It will be noted that the complete expression for i consists of two terms: a steady-state term and a transient term. In general this distinct division of terms is present in complete current solutions. Under cer-



OSCILLOGRAM 1. Growth of current in an RL circuit which is suddenly energized with a constant potential difference, E. The instantaneous power delivered to the circuit is also shown.

tain conditions one or the other of the terms may be zero. The fact that the complete expression for current can be divided distinctly into a steady-state term and a transient term is of considerable importance. Under ordinary conditions the steady-state term can be evaluated in terms of elementary circuit concepts rather than by involved processes

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of integration. The transient term can usually be found in terms of simple exponential components if the circuit parameters are constant.

The time variations of the two terms of the current solution given in equation (7) can easily be visualized. The steady-state term, E/R, is independent of time; the transient term has a value of $(-E/R)$ at $t = 0$ and approaches zero exponentially as time increases. The two terms combine to form the current that actually flows in the RL circuit during the transient period. Oscillogram 1 illustrates the actual growth of current in an RL circuit when it is suddenly energized with a constant potential difference. It will be noted that the transition in current in this case is from zero to a steady d-c value equal to E/R.

In certain elementary types of circuits the length of time required for the current to make 63.2 per cent of its total transition is called the time constant of the circuit. The time constant of the RL circuit is L/R, as may be shown by direct substitution in equation (7). Thus if t is set equal to L/R in equation (7) it is simply a matter of algebra to show that:

$$[i]_{\text{at } t = L/R} = 0.632\frac{E}{R}$$

Example 2. The circuit shown in Fig. 2 is assumed to be carrying a steady current equal to E/R at $t = 0$. At $t = 0$, either the switch S is assumed to change from point a to point b in an infinitely short period of time or it is assumed that a dead short circuit occurs between the points a and b. In either event the RL branch is de-energized at $t = 0$ and left to subside through the short-circuit path. The basic voltage equation for the RL branch at and after $t = 0$ is:

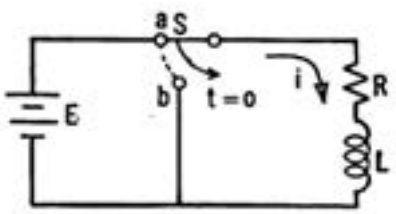

FIG. 2. An RL branch which is suddenly de-energized at $t = 0$.

$$L\frac{di}{dt} + Ri = 0 \tag{8}$$

From which:

$$i = \underbrace{0}_{\text{steady-state term}} + \underbrace{c_1 \epsilon^{-Rt/L}}_{\text{transient term}} \tag{9}$$

As previously mentioned, a current flowing through a circuit which has an appreciable amount of inductance cannot change its value instantaneously. Since $i = E/R$ just prior to $t = 0$, i is also equal to E/R at $t = 0$. Therefore:

$$\frac{E}{R} = c_1 \tag{10}$$

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and

$$i = \underset{\text{steady-state term}}{0} + \underset{\text{transient term}}{\frac{E}{R}\epsilon^{-Rt/L}} \tag{11}$$

It will be observed that the transition in current is from (E/R) at $t = 0$ to zero current at $t = \infty$ and that the rate of subsidence is governed by the ratio of R to L. The current actually comes to zero in a relatively short period of time because the driving voltage, $L\,di/dt$ or $N\,d\phi/dt$, becomes so small that it can no longer maintain a net movement of electrons in one direction. Thus when the energy of the collapsing magnetic field becomes so small that it cannot overcome the internal atomic forces that tend to prevent net drifts of electrons, the current actually becomes zero. The failure of theoretical equations to account for exceedingly minute effects of this kind is of no practical importance.

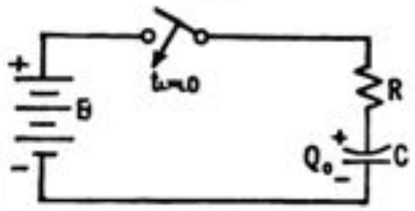

Fig. 3. A series RC circuit suddenly energized with a constant potential difference of E volts.

Example 3. If the condenser shown in Fig. 3 has a charge of Q_0 units of electrical charge at $t = 0$, the basic voltage equation at and after $t = 0$ is:

$$Ri + \frac{q}{C} = E \tag{12}$$

where

$$q = \int_0^t i\,dt + Q_0 \tag{13}$$

Differentiating equation (12) with respect to t and substituting i for dq/dt yields:

$$R\frac{di}{dt} + \frac{i}{C} = 0 \tag{14}$$

From which:

$$i = c_1\epsilon^{-t/RC} \tag{15}$$

The resultant voltage causing current to flow in the circuit at the instant of closing the switch is $(E - Q_0/C)$. Therefore the current instantly acquires a value $\dfrac{(E - Q_0/C)}{R}$ at $t = 0$ since the self-inductance is assumed to be negligibly small. In this connection it should be noted that the initial Q_0/C voltage of the condenser may possess either polarity with respect to the applied voltage E. For the case shown in Fig. 3

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the polarity of Q_0/C is opposite to that of the applied voltage E. Since

$$i = \frac{(E - Q_0/C)}{R} \quad \text{at} \quad t = 0$$

it follows that

$$c_1 = \frac{(E - Q_0/C)}{R} \tag{16}$$

and

$$i = \frac{(E - Q_0/C)}{R}\epsilon^{-t/RC} \tag{17}$$

Equation (17) is the mathematical expression for the direct charging current taken by a condenser when the self-inductance of the circuit is negligibly small.

The variation of charge can be found by solving equation (12) for q and then substituting for i its value from equation (17). Thus

$$\begin{aligned} q &= CE - CRi \\ &= CE - (CE - Q_0)\epsilon^{-t/RC} \end{aligned} \tag{18}$$

If the initial charge $Q_0 = 0$, the variations of current and charge as given by equations (17) and (18) are shown in Fig. 4.

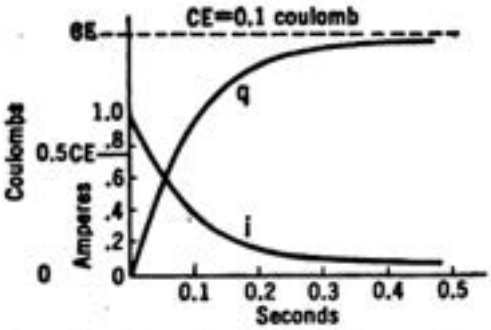

Fig. 4. Charging a condenser $C = 100\mu f$ through a resistance $R = 1000$ ohms from a d-c source of 1000 volts.

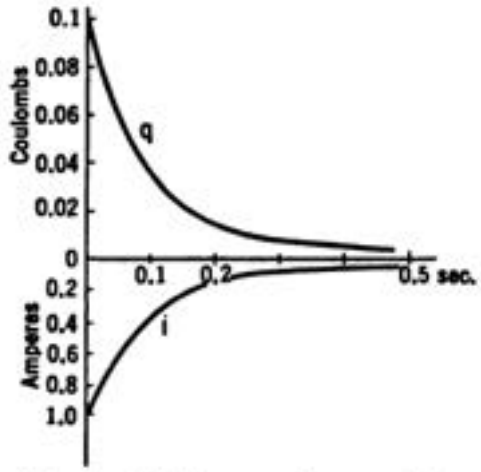

Fig. 5. Discharge of a condenser $C = 100\mu f$ through a resistance $R = 1000$ ohms. Initial charge at a potential of 1000 volts.

If a condenser of C units capacitance replaces the inductance L of Fig. 2, it is a simple matter to show that:

$$i = -\frac{E}{R}\epsilon^{-t/RC} \tag{19}$$

and

$$q = CE\epsilon^{-t/RC} \tag{20}$$

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Equation (19) is the expression for the discharge current in an RC circuit which contains a condenser initially charged to a potential difference of E volts. Equation (20) is the expression for the decay of charge under the same conditions. The variations of current and charge as given by equations (19) and (20) are shown in Fig. 5. Condenser charge and discharge currents are similar except for sign and are simple exponential variations. The steady-state current in either of the two cases is obviously equal to zero.

The time constants of the above RC circuits are both equal to RC since it is at this value of time that the current has made 63.2 per cent of its total change.

Sawtooth Wave Form Produced by Simple Transient Effects. Various forms of circuits have been devised to produce sawtooth wave forms or approximations thereto. One of the most elementary is

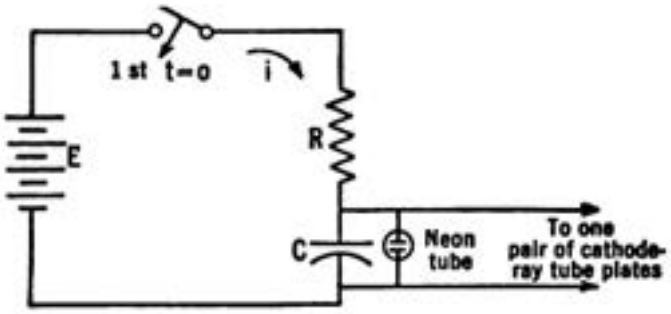

FIG. 6. An elementary form of sweep circuit the operation of which depends upon recurring transient phenomena.

shown in Fig. 6.[1] The operation of the device depends upon the natural behavior of the circuit elements, the details of which are listed below.

1. A transient voltage appears across the condenser due to the transient inrush of current to the main RC series circuit. Until a certain critical voltage is established across the condenser, the neon discharge tube remains un-ionized and acts practically as an open circuit.

2. When the condenser voltage has built up to a certain critical value, say E_1, the neon tube ionizes and suddenly places a low-resistance path across the condenser. The ionized tube thus provides a means of discharging the condenser because the time constant of the discharge path is relatively very small as compared with the time constant of the main RC series circuit. The voltage across the condenser drops from the value E_1 to some lower value, say E_2, in a very small fraction of the time required for the establishment of E_1.

[1] In practice the neon tube of Fig. 6 would probably be replaced by a gas triode which has an extremely low de-ionization time, for example, a type 885 tube. In this case the anode-cathode path of the triode replaces the neon tube of Fig. 6 and the grid of the triode can be used to control the starting of the discharge current.

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3. After the condenser has been discharged to the voltage E_2, the neon tube ceases to be a conducting path (becomes de-ionized) and permits the applied potential difference to recharge the condenser. The cycle of transient phenomena thus repeats itself, and the voltage e_c takes on an approximate sawtooth wave form.

During the charging period the condenser voltage is:

$$e_{c1} = \frac{q}{C} = \frac{\int_0^t i\,dt + Q_0}{C} \tag{21}$$

or

$$e_{c1} = \frac{\int_0^t i\,dt}{C} + E_2 \tag{22}$$

E_2 is the voltage left on the condenser from the previous cycle due to the discharge tube de-ionizing before zero condenser voltage is reached.

From equation (17) it is evident that

$$i = \frac{E - E_2}{R}\epsilon^{-t/RC} \tag{23}$$

Therefore,

$$e_{c1} = \frac{E - E_2}{RC}\int_0^t \epsilon^{-t/RC}\,dt + E_2 \tag{24}$$

or

$$e_{c1} = E - (E - E_2)\epsilon^{-t/RC} \tag{25}$$

The rising condenser voltage is in this case exponential in character rather than linear. However, when the actual change in voltage, $(E_1 - E_2)$, is small as compared with $(E - E_2)$ fairly satisfactory results can be obtained.

The condenser voltage continues to build up in accordance with equation (25) until the voltage E_1 is attained, at which time the neon tube discharges the condenser in the manner previously described. Obviously E must be greater than E_1.

A mathematical analysis of the conditions during the discharge period is complicated by the variability of the resistance of the discharge path. The exact behavior of the circuit during the discharge period is usually unimportant because the discharge period is of relatively short duration and does not represent the " working " part of the cycle. It should be recognized that the series resistance, R, is generally of the order of 10,000 times the value of the tube resistance when the tube is ionized. Therefore during the discharge period the tube cannot receive any appreciable percentage of the applied voltage. It is plain that the device would cease to function as a sawtooth-wave-form genera-

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tor, if, during an ionized period, the tube received a voltage sufficient to sustain ionization.

The general nature of the approximate sawtooth wave form produced is shown in Fig. 7. An obvious place for improvement is in the rising or building-up portion of the curve. The rising part of the curve can be made practically linear by replacing the constant resistance, R,

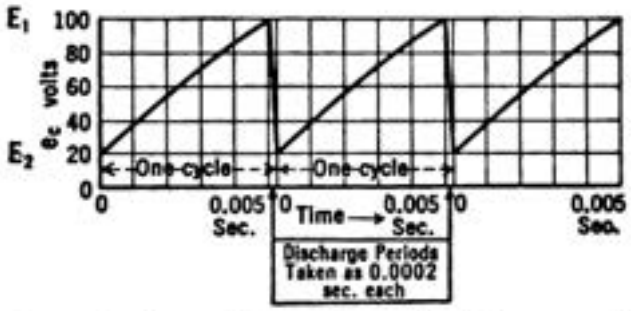

FIG. 7. Approximate sawtooth wave form as determined from equation (25) for the particular case of $E = 220$ volts, $E_1 = 100$ volts, $E_2 = 20$ volts, $R = 100{,}000$ ohms, and $C = 0.1\ \mu f$. The overall time of one cycle under these conditions is approximately 0.0052 second.

with a resistance that varies inversely as the amount of current passing through it. Many of the modern vacuum tubes, particularly the pentodes, possess this resistance characteristic from plate to cathode, provided they are worked between certain limits as regards plate-to-cathode voltage.

If the transient current inrush is maintained constant at I amperes by means of a variable resistance, then

$$e_{c1} = K \int_0^t I\,dt + E_2$$

$$= KIt + E_2$$

Under the conditions stated above, the rising part of the voltage curve shown in Fig. 7 would become linear with respect to time.

In addition to the use of a pentode type tube for maintaining constant charging or discharging current, some sweep circuits employ a grid-controlled mercury-vapor discharge tube as a starting and stopping valve. Various other combinations of electron tubes are also employed to produce sawtooth wave forms.

Oscillogram 2 is a photographic record of the wave form produced by a modern sweep circuit which employs a series of transient conditions to effect the desired result. In obtaining the photographic record the P_2 plates of the cathode-ray tube (see Fig. 3, page 365) are energized with one sweep-circuit potential difference and the P_1 plates are energized with the potential difference developed by an identical

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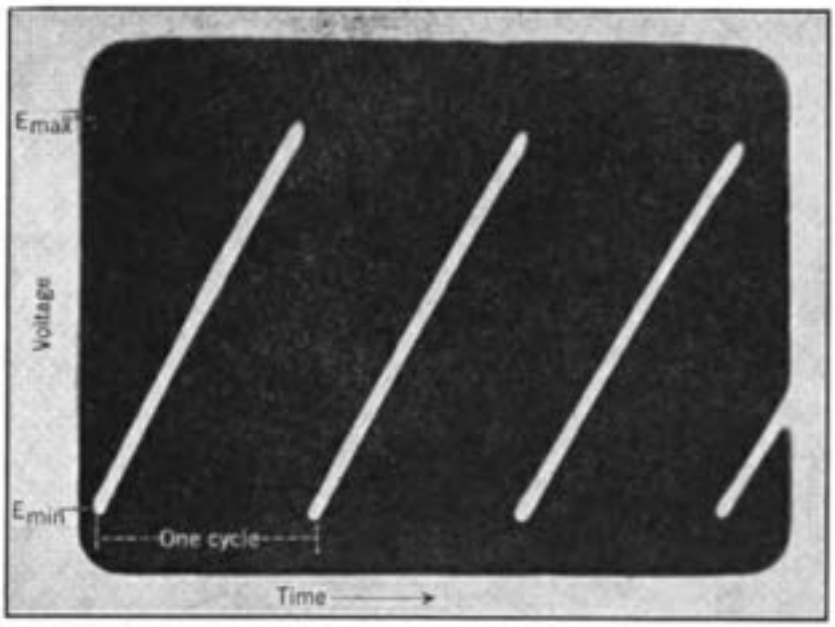
sweep circuit. The linearity of the sweep-circuit voltage is clearly shown.



OSCILLOGRAM 2. Illustrating the linearity of the potential difference developed by a modern sweep circuit. In this particular case the return time, that is, the time required for the voltage to return from E_{max} to E_{min}, is so short that the trace is not discernible on the photographic record.

The *RL* Circuit Energized with an Alternating Potential Difference. If an alternating potential difference replaces the battery shown in Fig. 1, the expression for dynamic equilibrium is:

$$L\frac{di}{dt} + Ri = E_m \sin(\omega t + \lambda) \tag{26}$$

or

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E_m}{L}\sin(\omega t + \lambda) \tag{27}$$

The symbol λ represents the phase of the voltage wave at which the switch of Fig. 1 is closed. Reference to Fig. 8 will show more clearly the exact meaning of λ. It is the angular displacement expressed in degrees or radians between the point $e = 0$ and the point $t = 0$ measured positively from the point where $e = 0$ and de/dt is positive.

The factor λ provides a convenient means of examining a-c transient conditions. In general, the magnitude of an a-c transient depends upon the time of the cycle at which the switching operation is performed. Most switching operations are performed with no regard for, or rather no knowledge of, the point on the voltage wave at which the transient period begins. Under these circumstances the investigator analyzes

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the effect of starting the transient disturbance at different points along the voltage wave. This is done by assigning different values to λ. In the case of surges or inrushes most attention is paid to those values of λ that produce the greatest currents or voltages.

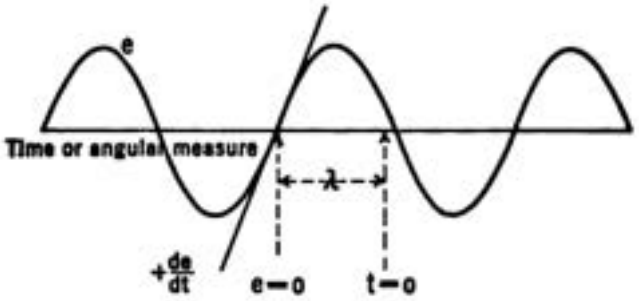

FIG. 8. Illustrating the physical significance of the symbol λ.

Equation (27) is representative of a general class of differential equations. The derivative of the dependent variable, (i), with respect to the independent variable, (t), added to the dependent variable, times some coefficient, equals some function of time. This form of equation defines the basic relationships involved in many physical problems, being particularly prevalent among the problems of electric circuit theory. The equation admits of relatively simple solution if all coefficients are constant and the right member is an exponential or sinusoidal function of time.

Let equation (27) be written as

$$\frac{di}{dt} + ai = h \sin (\omega t + \lambda) \tag{28}$$

where $a = R/L$ and $h = E_m/L$.

The solution of equation (28) takes the following form:

$$i = h\epsilon^{-at} \int \epsilon^{at} \sin (\omega t + \lambda)\, dt + c_1\epsilon^{-at} \tag{29}$$

The proof of the solution stated above rests in its ability to satisfy the original equation, namely, equation (28). In terms of the above solution:

$$\frac{di}{dt} = h\epsilon^{-at}\epsilon^{at} \sin (\omega t + \lambda) - ah\epsilon^{-at} \int \epsilon^{at} \sin (\omega t + \lambda)\, dt - ac_1\epsilon^{-at} \tag{30}$$

and

$$ai = ah\epsilon^{-at} \int \epsilon^{at} \sin (\omega t + \lambda)\, dt + ac_1\epsilon^{-at} \tag{31}$$

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Adding equations (30) and (31) will show that equation (29) is a general solution of equation (28). The solution stated in equation (29) is limited to those cases where a and h are constant. For the particular problem at hand this means that R, L, and E_m must be constant before equation (29) can be employed as a solution of (28).

The solution for current in an RL circuit with sinusoidal voltage applied is:

$$i = \underbrace{\frac{E_m}{L}\epsilon^{-Rt/L}\int \epsilon^{Rt/L}\sin(\omega t + \lambda)\,dt}_{\text{steady-state term}} + \underbrace{c_1\epsilon^{-Rt/L}}_{\text{transient term}} \tag{32}$$

The relative complexities of the two terms in the above solution should be noted. Mathematically, the steady-state term is known as the "particular integral," and the transient term as the "complementary function." The integration involved in the evaluation of the steady-state term can be carried out by the method of successive parts, but the algebraic simplification of the results is a tedious process.

With sinusoidal applied voltages, familiar algebraic methods may be employed to find the steady-state terms of general current solutions. Many of the disagreeable details connected with the evaluation of complete current solutions are thus avoided. For example, several lengthy mathematical relations are involved in the integration method of finding the steady-state term of equation (32) which is simply:

$$i_s = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) \tag{33}$$

where

$$Z = \sqrt{R^2 + \omega^2 L^2} \quad \text{and} \quad \theta = \tan^{-1}\omega L/R$$

Actually equation (33) can be thought of as following from two physical facts. The maximum value of the steady-state current is E_m/Z where $Z = \sqrt{R^2 + \omega^2 L^2}$, and the steady-state current wave lags the applied voltage wave by the angle whose tangent is $\omega L/R$. The complete expression for current becomes:

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + c_1\epsilon^{-Rt/L} \tag{34}$$

The constant of integration c_1 must be found from the initial conditions — those existing at the time of closing the switch. If the circuit current is zero just prior to closing the switch, then,

$$i = 0 \quad \text{at} \quad t = 0 \quad \text{(See page 549.)}$$

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Imposing the above condition on equation (34) yields

$$0 = \frac{E_m}{Z} \sin (\lambda - \theta) + c_1$$

From which:

$$c_1 = \frac{-E_m}{Z} \sin (\lambda - \theta) \tag{35}$$

and

$$i = \underbrace{\frac{E_m}{Z} \sin (\omega t + \lambda - \theta)}_{\text{steady-state term}} - \underbrace{\frac{E_m}{Z} \sin (\lambda - \theta) \epsilon^{-Rt/L}}_{\text{transient term}} \tag{36}$$

It will be noted from the above equation that the transient term is equal to zero when $(\lambda - \theta) = 0, \pi, 2\pi$, etc. If the RL branch is highly inductive the ratio of ωL to R is large, thereby causing θ to approach $\pi/2$ as an upper limit. In cases of this kind the transient term is zero when λ is approximately equal to $\pi/2$, $3\pi/2$, $5\pi/2$, etc. Physically this means that zero transient effects take place in highly inductive circuits when the circuit is energized at points of approximately maximum voltage on the voltage wave.

The transient term of equation (36) is maximum (for given values of R, L, ω, and E_m) when $(\lambda - \theta) = \pi/2$, $3\pi/2$, $5\pi/2$, etc. When θ is approximately equal to $\pi/2$ it is plain that the transient term is a maximum when λ is approximately equal to 0, π, 2π, etc. Therefore in a highly inductive circuit the transient term is maximum when the switch is closed at points of approximately zero voltage on the voltage wave. A detailed study of equation (36) will show that conditions which make for the maximum possible transient terms do not necessarily make for the maximum possible values of i. In highly inductive circuits the difference between the two sets of conditions is not large and maximum transient disturbance is usually assumed to be the result of those conditions that make $\sin (\lambda - \theta) = 1$ or $\sin (\lambda - \theta) = -1$.

The steady-state term and the transient term, together with the resultant current, are illustrated in Fig. 9 for the case of $\theta = 85°$ and for $(\lambda - \theta) = 3\pi/2$. Under these conditions:

$$\lambda = 270° + 85° = 355° = -5°$$

It will be noted that the switch is closed when the steady-state term is at a maximum (negative) value and that the transient term is at its maximum (positive) value. The transient term and the steady-state term combine at $t = 0$ to make the resultant current equal to zero, which of course must be the case in an inductive circuit which is at rest just prior to the application of a potential difference.

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Under the condition of constant R and L, the maximum value of the resultant current i is less than $2I_m$, where $I_m = E_m/Z$, the maximum value of the steady-state term. This fact may be easily substantiated

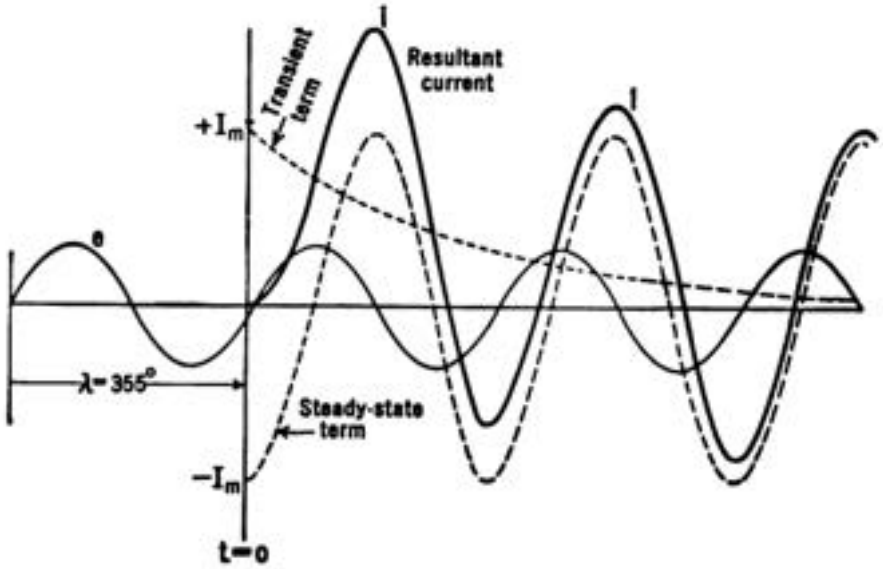

FIG. 9. Illustrating the manner in which the steady-state term and the transient term of equation (36) combine to form the resultant current. For the case shown, $\theta = 85°$ and $\sin(\lambda - \theta) = -1$.

from the graphs shown in Fig. 9. The effective value of the current during the early transient period is somewhat less than

$$\sqrt{I_{dc}^2 + I^2} = \sqrt{2I^2 + I^2} = \sqrt{3}I \quad \text{[See equation (28), page 190.]}$$

where $I_{dc} = I_m = \sqrt{2}I$ and I is the effective value of the steady-state term.

The transient term in an RL circuit is often referred to as the d-c component since it is unidirectional. This subsiding unidirectional component of current is of theoretical interest because it is partly responsible for the radical changes that take place in synchronous generator impedances during transient periods.

Oscillogram 3 illustrates the resultant current in a highly inductive circuit when $\lambda = 0$ and $\lambda = \pi/2$. The two current records are placed on the same oscillogram by means of superimposed exposures. In taking oscillograms of this kind it is necessary to employ some device for closing the circuit at the desired point on the voltage wave.

Problem 1. Plot the steady-state term and the transient term of equation (36) for two cycles of the steady-state variation under the following conditions:

(*a*) The applied voltage is a 60-cycle sinusoidal variation, the maximum value of which is 311 volts.

(*b*) $R = \omega L = 4$ ohms.

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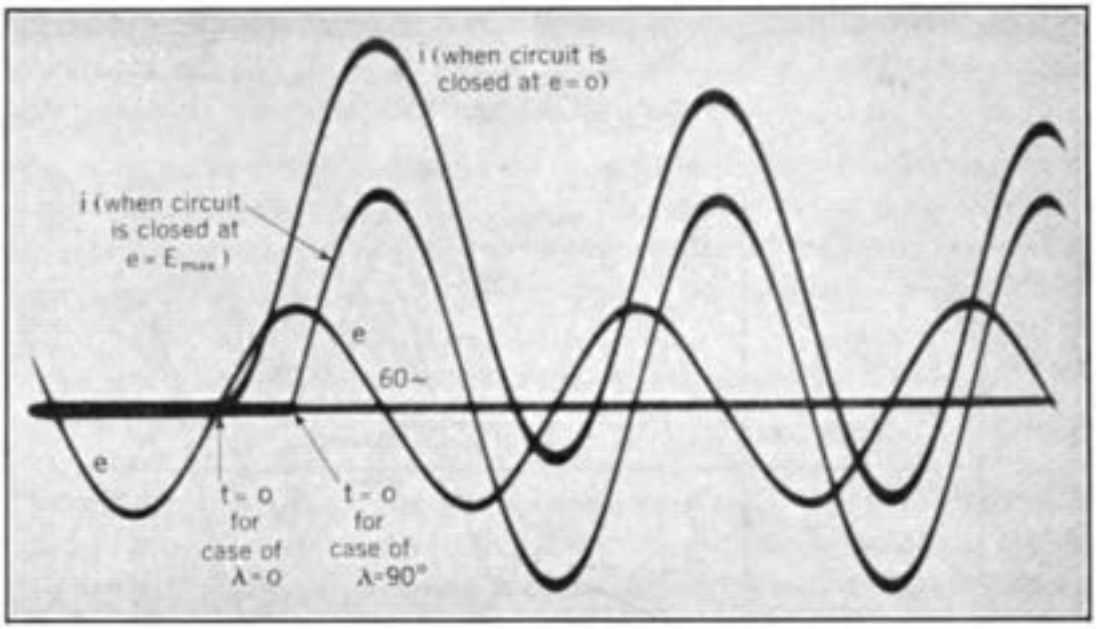

OSCILLOGRAM 3. Illustrating the current variations in an RL circuit which is suddenly energized with a p.d. of sinusoidal wave form. R and L, in this particular case, are sensibly constant. Two cases, namely, $\lambda = 0$ and $\lambda = 90°$, are shown.

(c) The switch is closed at such a time as to make the transient term acquire a negative maximum value.

Graph the resultant current i on the same plot.

Ans.: $i = 55 \sin (377t + 90°) - 55\epsilon^{-377t}$ amperes.

Problem 2. Analyze equation (36) for the case in which L is negligibly small.

Ans.: $i = \dfrac{E_m}{R} \sin (\omega t + \lambda)$.

The *RC* Circuit Energized with an Alternating Potential Difference. If an alternating potential difference replaces the battery shown in Fig. 3, the expression for dynamic equilibrium is:

$$Ri + \frac{q}{C} = E_m \sin (\omega t + \lambda) \tag{37}$$

Since $i = \dfrac{dq}{dt}$

$$R\frac{dq}{dt} + \frac{q}{C} = E_m \sin (\omega t + \lambda) \tag{38}$$

or

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E_m}{R} \sin (\omega t + \lambda) \tag{39}$$

Since equation (39) is a linear differential equation of the first order and first degree, the integrating factor[2] which makes the left-hand side

[2] Consult any standard book on differential equations.

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an exact derivative is:

$$\epsilon^{\int \frac{dt}{RC}} = \epsilon^{t/RC} \tag{40}$$

Multiplying equation (39) by $\epsilon^{t/RC}$ gives

$$\epsilon^{t/RC}\frac{dq}{dt} + \epsilon^{t/RC}\frac{q}{CR} = \epsilon^{t/RC}\frac{E_m}{R}\sin(\omega t + \lambda) \tag{41}$$

or

$$\epsilon^{t/RC}\,dq + \epsilon^{t/RC}\frac{q}{CR}\,dt = \epsilon^{t/RC}\frac{E_m}{R}\sin(\omega t + \lambda)\,dt \tag{42}$$

Integrating gives

$$q\epsilon^{t/RC} = \int \epsilon^{t/RC}\frac{E_m}{R}\sin(\omega t + \lambda)\,dt + K$$

or

$$q\epsilon^{t/RC} = \frac{E_m}{R}\left\{\frac{\epsilon^{t/RC}\left[\frac{1}{RC}\sin(\omega t + \lambda) - \omega\cos(\omega t + \lambda)\right]}{\frac{1}{R^2C^2} + \omega^2}\right\} + K \tag{43}$$

Dividing equation (43) through by $\epsilon^{t/RC}$, expressing the difference of the sine and cosine terms as a single cosine function, and making a few algebraic transformations give

$$q = -\frac{E_m}{\omega\sqrt{R^2 + \frac{1}{\omega^2C^2}}}\cos(\omega t + \lambda + \theta) + K\epsilon^{-t/RC} \tag{44}$$

where $\theta = \tan^{-1}\frac{1}{\omega CR} = \tan^{-1}\frac{X_c}{R}$.

Imposing the initial condition, namely, $q = Q_0$ when $t = 0$, and solving for K give

$$K = Q_0 + \frac{E_m}{\omega\sqrt{R^2 + \frac{1}{\omega^2C^2}}}\cos(\lambda + \theta) \tag{45}$$

Substituting (45) in (44) and replacing $\frac{1}{\omega C}$ by X_c

$$q = -\frac{E_m}{\omega\sqrt{R^2 + X_c^2}}\cos(\omega t + \lambda + \theta) + \left[Q_0 + \frac{E_m}{\omega\sqrt{R^2 + X_c^2}}\cos(\lambda + \theta)\right]\epsilon^{-t/RC} \tag{46}$$

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Equation (46) is the general equation for the charge on the condenser. If the initial charge is zero,

$$q = -\frac{E_m}{\omega\sqrt{R^2 + X_c^2}}\cos(\omega t + \lambda + \theta) + \frac{E_m e^{-t/RC}}{\omega\sqrt{R^2 + X_c^2}}\cos(\lambda + \theta) \tag{47}$$

The first term of the right-hand member of equation (47) is the steady-state term whereas the last term is the transient. It should be noted that at the time $t = 0$, the transient is always exactly equal and opposite

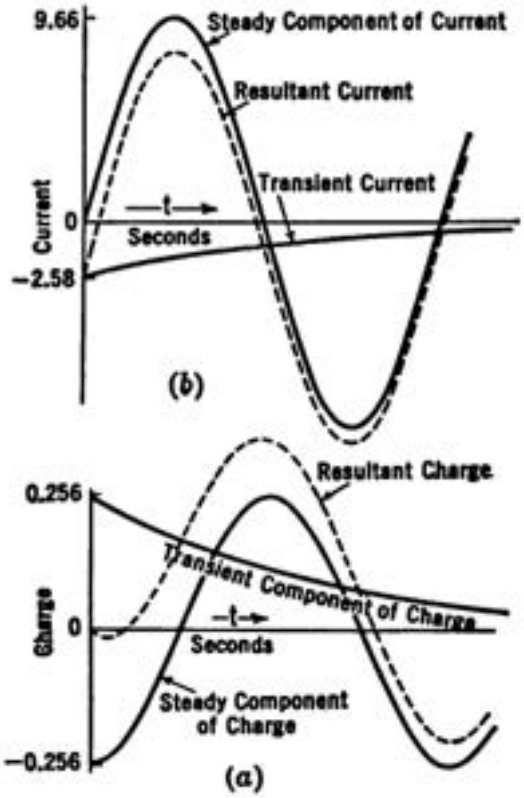

FIG. 10. Circuit containing $R = 100$ ohms, $C = 100$ μf when $e = 1000 \sin(377t - 14.95°)$ volts is impressed. Initial charge on condenser = 0.

to the steady-state component. These results are shown in Fig. 10a. This is the same relation that exists between the steady-state term and transient of *current* in the RL circuit.

The current in the RC circuit is obtained by differentiation of equation (47). Thus

$$i = \frac{dq}{dt} = \frac{E_m}{\sqrt{R^2 + X_c^2}}\sin(\omega t + \lambda + \theta) - \frac{E_m e^{-t/RC}}{RC\omega\sqrt{R^2 + X_c^2}}\cos(\lambda + \theta) \tag{48}$$

A study of equation (48) and the corresponding graph, Fig. 10b, reveals that there is no fixed relation between the transient and the

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steady-state component of current at the time $t = 0$. The relative magnitudes are dependent upon the ratio of $\frac{X_c}{R} = \frac{1}{RC\omega}$ and the time angle λ at which the switch is closed.

The *RLC* Series Circuit with a Constant Direct Voltage Suddenly Applied. Since the emf applied to the circuit must equal the sum of all the drops at every instant, the condition for dynamic equilibrium is:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E \tag{49}$$

Differentiating equation (49),

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0 \tag{50}$$

Employing the usual method of solving a second-order, first-degree linear differential equation,[3] the auxiliary equation is:

$$L\alpha^2 + R\alpha + \frac{1}{C} = 0$$

Hence,

$$\alpha = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Let

$$a = \frac{R}{2L} \quad \text{and} \quad b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

The complementary function is then

$$i = k_1\epsilon^{(-a+b)t} + k_2\epsilon^{(-a-b)t}$$

The complete solution is the sum of the complementary function and the particular integral, the latter being the steady-state current. Since this case involves a constant direct voltage on a condenser, the steady-state current is 0.

Hence the complete solution is:

$$i = k_1\epsilon^{(-a+b)t} + k_2\epsilon^{(-a-b)t} + 0 \tag{51}$$

The constants k_1 and k_2 must be evaluated by imposing certain known conditions. In this case when $t = 0$, $i = 0$, and $q = Q_0$, the latter being the initial charge on the condenser before closing the switch.

[3] See any standard book on differential equations.

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For $i = 0$ and $t = 0$ in equation (51)

$$0 = k_1 + k_2 \quad \text{or} \quad k_2 = -k_1 \tag{52}$$

From equation (49)

$$\frac{1}{C}\int i\,dt = \frac{q}{C} = E - L\frac{di}{dt} - Ri$$

and

$$q = CE - CL\frac{di}{dt} - CRi \tag{53}$$

Substituting (51) in (53) gives

$$q = CE - CL[k_1(-a + b)\epsilon^{(-a+b)t} + k_2(-a - b)\epsilon^{(-a-b)t}] \\ -CR\,k_1\epsilon^{(-a+b)t} - CR\,k_2\epsilon^{(-a-b)t} \tag{54}$$

Imposing[4] the condition that $q = Q_0$ when $t = 0$ on equation (54), substituting equation (52), and solving for k_1 give

$$k_1 = \frac{CE - Q_0}{2CLb} \tag{55}$$

From equation (52)

$$k_2 = -k_1 = -\frac{CE - Q_0}{2CLb} \tag{56}$$

The final equation for current is now obtained by substituting equations (55) and (56) in equation (51) and replacing b by its equal. Hence,

$$i = \frac{CE - Q_0}{\sqrt{R^2C^2 - 4LC}}[\epsilon^{(-a+b)t} - \epsilon^{(-a-b)t}] \tag{57}$$

If the values of k_1, k_2, a and b are substituted in equation (54), the expression for charge becomes

$$q = CE - (CE - Q_0)\left[\frac{RC + \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\epsilon^{(-a+b)t} \right. \\ \left. - \frac{RC - \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\epsilon^{(-a-b)t}\right] \tag{58}$$

Since $b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ [in equations (57) and (58)] may be real, imaginary or zero, there are three cases to be considered.

[4] It is important that initial conditions be imposed on the original equation rather than on one of the differentiated forms. Note that equation (54) comes from equation (49) without any differentiation of the original voltage equation (49).

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Case I. When $\frac{R^2}{4L^2} > \frac{1}{LC}$, the exponents of ϵ in equations (57) and (58) are real. When $t = 0$, the current is zero, and the quantity of electricity on the condenser is the initial charge before the switch was closed. Since $a = \frac{R}{2L}$ while $b = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$, $-a + b$ will be negative as long as $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$. Hence as t becomes infinite, the exponential terms become zero. The current therefore becomes zero and the charge on the condenser becomes CE. A graphical representation of the variation of current and charge is shown in Fig. 11. Both the current and charge are unidirectional and the phenomena are non-oscillatory.

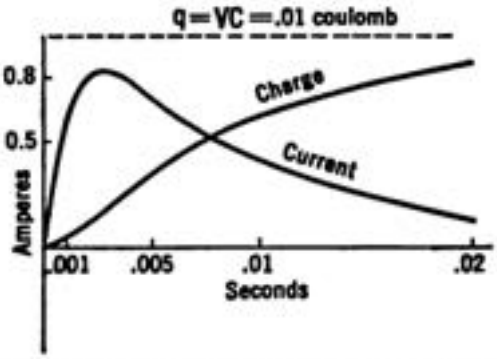

FIG. 11. Circuit containing $R = 100$ ohms, $C = 100\ \mu f$, $L = 0.1$ henry when a d-c voltage $V = 100$ volts is impressed. Initial charge = 0.

Case II. When $\frac{R^2}{4L^2} < \frac{1}{LC}$, b becomes imaginary. To evaluate the expression for b it may be written as

$$\sqrt{(-1)\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} = \sqrt{-1}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = j\beta$$

where
$$\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Equation (57) then becomes:

$$\begin{aligned} i &= \frac{CE - Q_0}{\sqrt{R^2C^2 - 4LC}}[\epsilon^{(-a+j\beta)t} - \epsilon^{(-a-j\beta)t}] \\ &= \frac{CE - Q_0}{\sqrt{R^2C^2 - 4LC}}[\epsilon^{-at}\epsilon^{j\beta t} - \epsilon^{-at}\epsilon^{-j\beta t}] \\ &= \frac{(CE - Q_0)\epsilon^{-at}}{\sqrt{R^2C^2 - 4LC}}[\cos\beta t + j\sin\beta t - \cos\beta t + j\sin\beta t] \\ &= \frac{(CE - Q_0)\epsilon^{-at}}{\sqrt{R^2C^2 - 4LC}}[2j\sin\beta t] \qquad (59) \end{aligned}$$

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For $\frac{R^2}{4L^2} < \frac{1}{LC}$, $R^2C^2 < 4LC$ and the denominator of equation (59) may be written as $j\sqrt{4LC - R^2C^2}$. Substituting $j\sqrt{4LC - R^2C^2}$ for $\sqrt{R^2C^2 - 4LC}$ in equation (59) gives the final expression for current in terms of all real quantities, as

$$i = \frac{2(CE - Q_0)\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin \beta t \tag{60}$$

Through a similar series of substitutions in and algebraic transformations of equation (58), the charge is found to be

$$q = CE - \frac{2(CE - Q_0)\sqrt{LC}\,\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin (\beta t + \theta) \tag{61}$$

where

$$\theta = \tan^{-1} \frac{\sqrt{4LC - R^2C^2}}{RC}$$

If the initial charge on the condenser is zero, the expressions for current and charge respectively are:

$$i = \frac{2CE\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin \beta t \tag{62}$$

$$q = CE - \frac{2CE\sqrt{LC}\,\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin (\beta t + \theta) \tag{63}$$

A graphical representation of equations (62) and (63) is shown in Fig. 12. Oscillogram 4 also shows the variation of current with time in another RLC circuit. It should be noted that the current is proportional to the slope dq/dt of the curve of charge variation at every instant. An examination of equation (62) shows that after an infinite time the current becomes zero which is the steady state. Also equation (63) reveals that the charge becomes CE after an infinite time has elapsed. For all practical purposes, however, these final or steady states are sensibly reached after a few seconds; in some cases in a few microseconds. (See page 552 for explanation.) From the time of closing the switch to the time of reaching the final state the current and quantity oscillate about their final values. Case II is therefore called the oscilla-

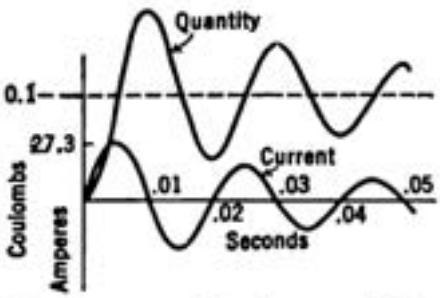

FIG. 12. Circuit containing $R = 5$ ohms, $C = 100$ μf, $L = 0.1$ henry, when a d-c voltage $V = 1000$ volts is impressed. Initial condenser charge = 0.

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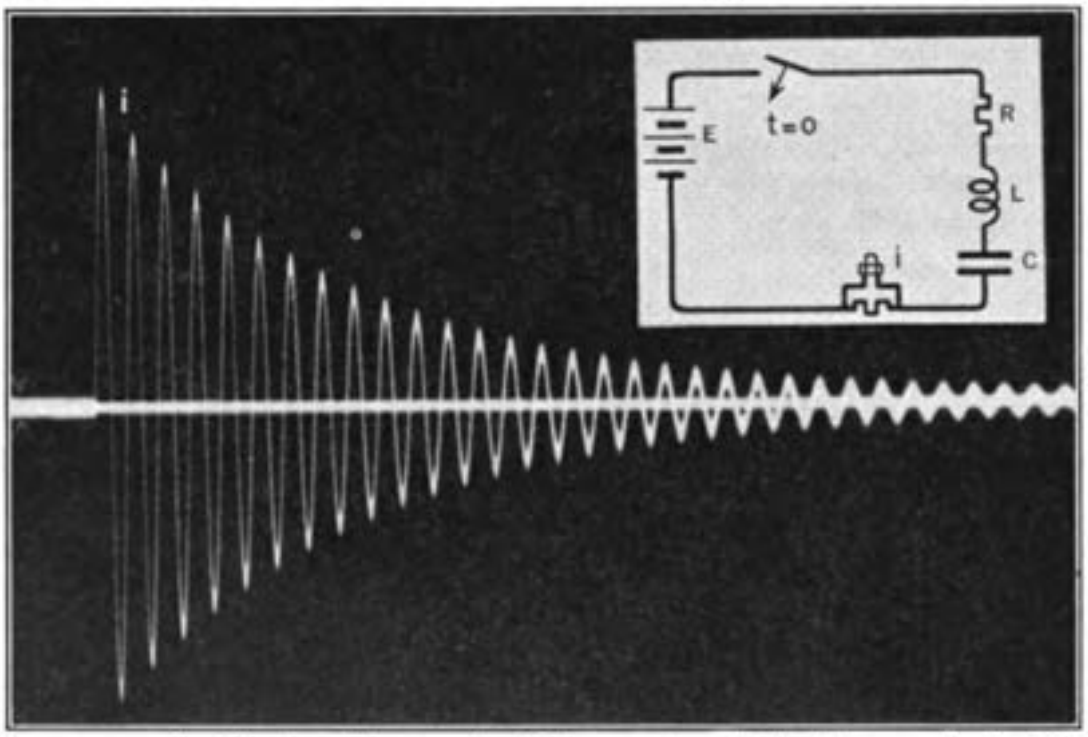
tory case. It is sometimes called the trigonometric case. Physically the current starts to flow and charges the condenser. Because of the low resistance compared with the inductance, the current continues to flow into the condenser when the magnetic field of the inductance collapses. The condenser charge thus overruns its final value and the potential drop across the condenser becomes higher than the impressed



OSCILLOGRAM 4. Photographic record of the current variation in a particular *RLC* series circuit which is suddenly energized with a constant potential difference.

voltage. The condenser then begins to discharge. These oscillations continue until the excess energy is dissipated in the resistance. The phenomenon is analogous to the case of a weight suspended from a spring with a low value of mechanical damping.

The frequency of the oscillation f_0 is obtained from equation (62) or (63). For a complete cycle βt must be 2π radians and since the time for a complete cycle is defined as the period T, we may write

$$\beta T = 2\pi$$

or

$$T = \frac{1}{f_0} = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \tag{64}$$

Hence

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{65}$$

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A comparison of the above equation with equation (10) on page 100 shows that the oscillatory frequency of the series *RLC* circuit when the resistance is zero is the same as the resonant frequency. Practically, they become the same when $R^2/4L^2$ is negligibly small compared with $1/LC$.

Case III. When $R^2/4L^2 = 1/LC$, $b = 0$ and the exponents of ϵ in equations (57) and (58) are real and negative as in case I. Hence the variations of current and charge are similar to those in case I. Case III is called the critical or limiting case and like case I is non-oscillatory.

Decay of Current and Charge in an *RLC* Circuit. The basic equation for this condition is:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0 \tag{66}$$

Equation (66) is obviously a special case of equation (49) where $E = 0$. Since equation (49) was solved in detail, the results of equation (66) will be found as special cases of equations (57), (58), (60), and (61) by making $E = 0$. It is plain that there will be three cases for the condition of zero voltage on (or short circuit of) the *RLC* circuit. These, as before, are the non-oscillatory case I where $R^2/4L^2 > 1/LC$, the oscillatory case 2 where $R^2/4L^2 < 1/LC$, and the critical case III, also non-oscillatory, where $R^2/4L^2 = 1/LC$.

Non-Oscillatory Case. The equations for current and charge for the non-oscillatory case are obtained from equations (57) and (58) respectively by setting $E = 0$. Thus

$$i = \frac{-Q_0}{\sqrt{R^2C^2 - 4LC}}[\epsilon^{(-a+b)t} - \epsilon^{(-a-b)t}] \tag{67}$$

and

$$q = Q_0\left[\frac{RC + \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\,\epsilon^{(-a+b)t} - \frac{RC - \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\,\epsilon^{(-a-b)t}\right] \tag{68}$$

A graphical representation of equations (67) and (68) is shown in Fig. 13. If desired, Q_0 can be replaced in the above equations by CV where V is the voltage drop across the condenser for the charge Q_0.

Oscillatory Case. If E is made equal to zero in equations (60) and (61), the equations for the decay of current and charge respectively are

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obtained as follows:

$$i = \frac{-2Q_0\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin \beta t \tag{69}$$

$$q = \frac{2Q_0\sqrt{LC}\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin (\beta t + \theta) \tag{70}$$

The variation of i and q as given by these equations is shown in Fig. 14. A comparison of equations (69) and (70) with equations (62) and (63) will show that the frequencies of oscillation for all of them are identical and are therefore given by equation (65).

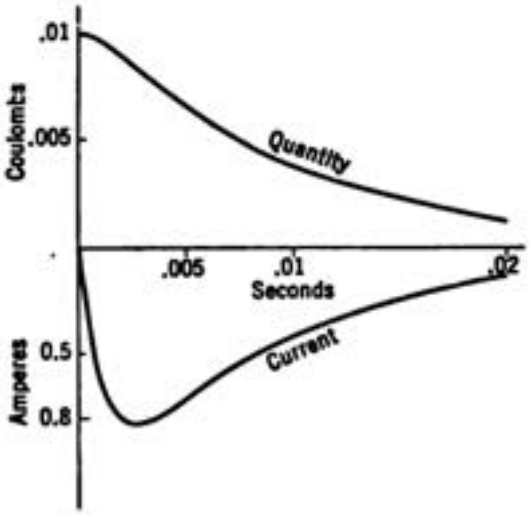

FIG. 13. Decay of current and charge or quantity in a circuit containing $R = 100$ ohms, $C = 100$ μf, and $L = 0.1$ henry when the initial charge on the condenser is 0.01 coulomb at a potential of 100 volts.

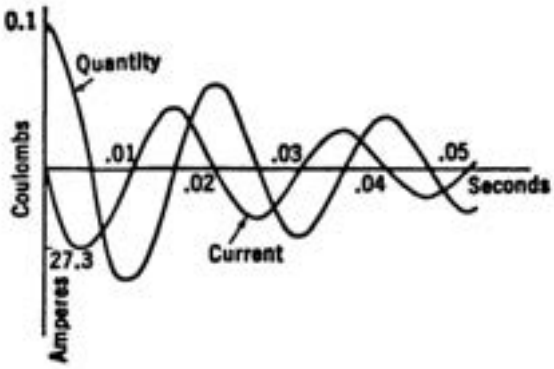

FIG. 14. Decay of current and charge or quantity in a circuit containing $R = 5$ ohms, $C = 100$ μf, and $L = 0.1$ henry when the initial charge on the condenser is 0.1 coulomb at a potential of 1000 volts.

Critical Case. Qualitatively this case is no different from the non-oscillatory case previously discussed. If b in equations (67) and (68) is made zero, the equations for the critical case result. Obviously Fig. 13 represents the general type of variation of current and charge for this condition.

The *RLC* Series Circuit with Alternating Voltage Suddenly Applied. The basic voltage equation of the *RLC* circuit shown in Fig. 15 is:

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E_m \sin (\omega t + \lambda) \tag{71}$$

The above equation can be put in terms of one dependent variable by differentiating the entire equation with respect to the independent

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variable, t. Differentiating as indicated above,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\frac{dq}{dt} = E_m\omega \cos(\omega t + \lambda) \tag{72}$$

Dividing through by L and substituting i for dq/dt results in

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{E_m\omega}{L}\cos(\omega t + \lambda) \tag{73}$$

Equation 73 is a linear differential equation of the second order, first degree, the solution of which consists of the sum of a complementary

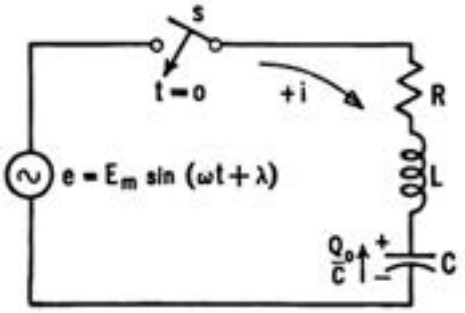

FIG. 15. An RLC series circuit energized with an alternating voltage at $t = 0$.

function or transient term and the particular integral or steady-state term. The former is obtained as indicated previously. The auxiliary equation is:

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0 \tag{74}$$

and

$$\alpha = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{75}$$

Let

$$a = \frac{R}{2L} \quad \text{and} \quad b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{76}$$

By definition

$$\alpha_1 = -a + b \quad \text{and} \quad \alpha_2 = -a - b \tag{77}$$

The transient term of the complete solution is:

$$i_t = c_1\epsilon^{(-a+b)t} + c_2\epsilon^{(-a-b)t} \tag{78}$$

The steady-state term of the complete solution is:

$$i_s = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) \tag{79}$$

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where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

The complete expression for current becomes:

$$i = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) + c_1 \epsilon^{(-a+b)t} + c_2 \epsilon^{(-a-b)t} \tag{80}$$

The two physical facts from which c_1 and c_2 can be evaluated are the state of current and the state of charge that exist in the circuit at the instant of closing the switch. Let it be assumed that

$$\left.\begin{aligned} i &= 0 \\ q &= Q_0 \end{aligned}\right\} \quad \text{at} \quad t = 0 \tag{81}$$

If the original voltage equation has been differentiated it is important that the initial conditions be imposed upon the original voltage equation rather than upon one of the differentiated forms. In the present case the initial conditions can be imposed upon equations (71) and (80).

Imposing the initial conditions upon equation (71) yields:

$$L\left[\frac{di}{dt}\right]_{t=0} + \frac{Q_0}{C} = E_m \sin \lambda$$

or

$$L\left[\frac{E_m \omega}{Z} \cos (\lambda - \theta) + c_1\alpha_1 + c_2\alpha_2\right] + \frac{Q_0}{C} = E_m \sin \lambda$$

From which

$$c_1\alpha_1 + c_2\alpha_2 = \frac{E_m}{L} \sin \lambda - \frac{Q_0}{LC} - \frac{E_m \omega}{Z} \cos (\lambda - \theta) \tag{82}$$

Imposing the initial conditions on equation (80) results in

$$0 = \frac{E_m}{Z} \sin (\lambda - \theta) + c_1 + c_2$$

or

$$c_1 + c_2 = -\frac{E_m}{Z} \sin (\lambda - \theta) \tag{83}$$

Equations (82) and (83) may be solved simultaneously for c_1 and c_2. From equation (83):

$$c_2 = -\frac{E_m}{Z} \sin (\lambda - \theta) - c_1 \tag{84}$$

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Substituting the above value of c_2 into equation (82) yields

$$c_1\alpha_1 - \left[\frac{E_m}{Z}\sin(\lambda-\theta)\right]\alpha_2 - c_1\alpha_2 = \frac{E_m}{L}\sin\lambda - \frac{Q_0}{LC} - \frac{E_m\omega}{Z}\cos(\lambda-\theta)$$

Whence

$$c_1(\alpha_1-\alpha_2) = \frac{1}{L}\left[E_m\sin\lambda - \frac{Q_0}{C} - \frac{E_m\omega L}{Z}\cos(\lambda-\theta)\right] + \alpha_2\frac{E_m}{Z}\sin(\lambda-\theta) \quad (85)$$

It will be remembered that

$$\alpha_1 = (-a+b) \quad \text{and} \quad \alpha_2 = (-a-b)$$

Therefore

$$\alpha_1 - \alpha_2 = 2b$$

and

$$\frac{\alpha_2}{\alpha_1-\alpha_2} = -\frac{a}{2b} - \frac{b}{2b} = -\frac{R}{4Lb} - \frac{1}{2}$$

Dividing equation (85) through by $(\alpha_1 - \alpha_2)$ and making substitutions for $(\alpha_1 - \alpha_2)$ and α_2,

$$c_1 = \frac{1}{2bL}\left[E_m\sin\lambda - \frac{Q_0}{C} - \frac{E_m\omega L}{Z}\cos(\lambda-\theta)\right] - \frac{R}{4bL}\frac{E_m}{Z}\sin(\lambda-\theta) - \frac{E_m}{2Z}\sin(\lambda-\theta)$$

Collecting the b terms in the above equation,

$$c_1 = \frac{1}{2bL}\left[E_m\sin\lambda - \frac{Q_0}{C} - \frac{E_m\omega L}{Z}\cos(\lambda-\theta) - \frac{E_mR}{2Z}\sin(\lambda-\theta)\right] - \frac{E_m}{2Z}\sin(\lambda-\theta) \quad (86)$$

From equation (83) it is evident that

$$c_2 = -c_1 - \frac{E_m}{Z}\sin(\lambda-\theta) \quad (87)$$

Therefore,

$$c_2 = -\frac{1}{2bL}\left[E_m\sin\lambda - \frac{Q_0}{C} - \frac{E_m\omega L}{Z}\cos(\lambda-\theta) - \frac{E_mR}{2Z}\sin(\lambda-\theta)\right] - \frac{E_m}{2Z}\sin(\lambda-\theta) \quad (88)$$

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For the sake of simplicity in writing, the following abbreviation will be adopted:

$$\left[E_m \sin \lambda - \frac{Q_0}{C} - \frac{E_m \omega L}{Z} \cos (\lambda - \theta) - \frac{E_m R}{2Z} \sin (\lambda - \theta)\right] = E_d \quad (89)$$

It will be observed that E_d is a voltage which is governed in magnitude by E_m, λ, Q_0, and the circuit parameters. The complete expression for current can now be written in terms of the applied voltage, the initial condenser charge, and the circuit parameters.

$$i = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) + \frac{E_d \epsilon^{-at}}{bL}\left[\frac{\epsilon^{bt} - \epsilon^{-bt}}{2}\right] - \frac{E_m}{Z} \sin (\lambda - \theta)\, \epsilon^{-at}\left[\frac{\epsilon^{bt} + \epsilon^{-bt}}{2}\right] \quad (90)$$

The transient component of the current consists of two terms, each of which is damped out with the damping factor ϵ^{-at} or $\epsilon^{-Rt/2L}$. The transient terms may be given different mathematical forms depending upon the nature of the symbol b. Since b is equal to $\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ it is evident that b may be either real or imaginary. A singular condition exists when b is equal to zero.

Case I. If $R^2/4L^2$ is greater than $1/LC$, b is a real number and the complete expression for current in the RLC series circuit may be written as

$$i = \underbrace{\frac{E_m}{Z} \sin (\omega t + \lambda - \theta)}_{\text{steady-state term}} + \underbrace{\frac{E_d}{bL} \epsilon^{-at} \sinh bt - \frac{E_m}{Z} \sin (\lambda - \theta) \epsilon^{-at} \cosh bt}_{\text{transient terms}} \quad (91)$$

The above expression follows directly from equation (90) since, if b is real,

$$\frac{\epsilon^{bt} - \epsilon^{-bt}}{2} = \sinh bt \quad \text{and} \quad \frac{\epsilon^{bt} + \epsilon^{-bt}}{2} = \cosh bt$$

Both transient terms are damped out by $\epsilon^{-Rt/2L}$. The damping constant $R/2L$ is relatively large when $\frac{R^2}{4L^2} > \frac{1}{LC}$ because of the relatively large value of $R/2L$. In general, the transient terms in this case are not predominantly large as compared with the steady-state term.

Case II. If $R^2/4L^2$ is less than $1/LC$, b takes the form of an imag-

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inary number and a change in notation becomes desirable. Let

$$b = j\beta \quad \text{where} \quad \beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

If $\frac{R^2}{4L^2} < \frac{1}{LC}$, β is a real number and b in equation (90) can be replaced by its equivalent, $j\beta$. In this connection j has its customary significance, namely, $\sqrt{-1}$.

$$i = \underbrace{\frac{E_m}{Z}\sin(\omega t + \lambda - \theta)}_{\text{steady-state term}} + \underbrace{\frac{E_d}{\beta L}\epsilon^{-at}\sin\beta t - \frac{E_m}{Z}\sin(\lambda - \theta)\epsilon^{-at}\cos\beta t}_{\text{transient terms}} \quad (92)$$

The above equation comes directly from equation (90) if it is recognized that the analytical expressions for $\sin\beta t$ and $\cos\beta t$ are:

$$\frac{\epsilon^{j\beta t} - \epsilon^{-j\beta t}}{2j} = \sin\beta t \quad \text{and} \quad \frac{\epsilon^{j\beta t} + \epsilon^{-j\beta t}}{2} = \cos\beta t$$

The two transient terms of equation (92) are exponentially damped sine and cosine terms of like frequency. Since the damping factors are identical, the sine and cosine terms can be combined by the method outlined on page 179. If the two transient terms are combined, equation (92) takes the following form:

$$i = \underbrace{\frac{E_m}{Z}\sin(\omega t + \lambda - \theta)}_{\text{steady-state term}} + \underbrace{I_t\,\epsilon^{-at}\sin(\beta t - \sigma)}_{\text{transient term}} \quad (93)$$

where

$$I_t = \sqrt{\left[\frac{E_d}{\beta L}\right]^2 + \left[\frac{E_m}{Z}\sin(\lambda - \theta)\right]^2}$$

and

$$\sigma = \tan^{-1}\frac{E_m\beta L\sin(\lambda - \theta)}{E_d Z}$$

In the present case the complete expression for current consists of two sinusoidal terms. The frequency of the steady-state term, $\omega/2\pi$, is determined solely by the frequency of the applied voltage; that of the transient term, $\beta/2\pi$, is governed entirely by the circuit parameters, R, L, and C. The frequency of the transient term may be less than, equal to, or greater than that of the applied voltage. In any event the transient oscillation disappears as soon as the damping factor, $\epsilon^{-Rt/2L}$, causes the transient term to become sensibly equal to zero.

Oscillograms 5 and 6 illustrate the current variations in a particular RLC series circuit during transient periods. For the conditions shown,

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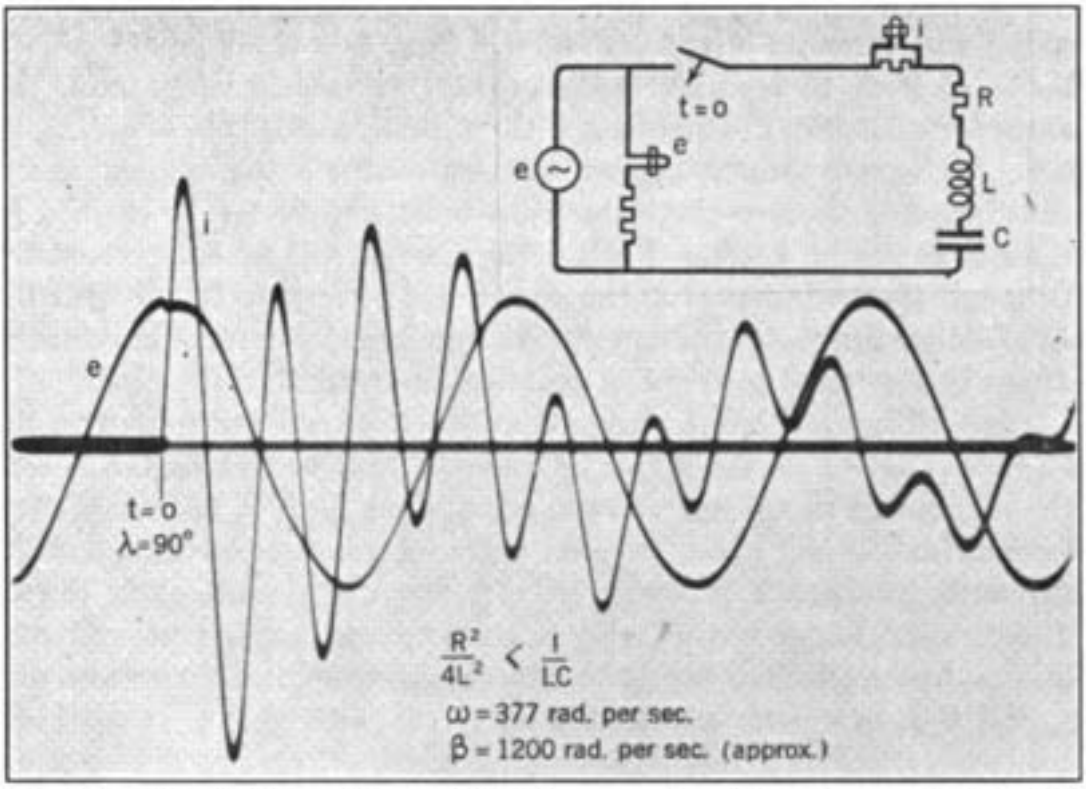
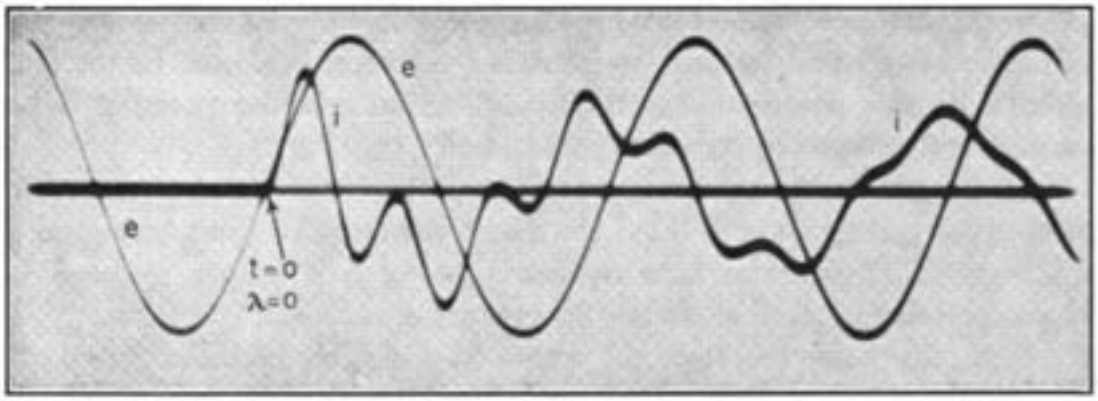
$\frac{R^2}{4L^2} < \frac{1}{LC}$ and $\beta > \omega$. The exponentially damped transient component can easily be discerned as the higher frequency variation which is superimposed on the 60-cycle steady-state variation. Also the effect



OSCILLOGRAM 5. Photographic record of the current variation in a particular *RLC* series circuit which is suddenly energized with an alternating potential difference *R*, *L*, and *C* are sensibly constant.



OSCILLOGRAM 6. Circuit arrangement and circuit parameters similar in every respect to those shown in connection with Oscillogram 5 except for the point on the voltage wave at which the circuit is energized. In the present case $\lambda = 0°$.

of closing the circuit at different points on the voltage wave can be observed by comparing Oscillograms 5 and 6. The transient term is shown to be several times as large in Oscillogram 5 as it is in Oscillogram 6.

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The Iron-Clad *RL* Circuit Energized by an Alternating Potential Difference. The mathematical analysis given in the article on page 557 for the case of constant R and L cannot, in general, be applied to an iron-clad circuit because of the wide variations of L that occur. For the iron-clad circuit, L in equation (26) is a function of i which in turn is an intricate function of time. The fact that L is variable makes both the coefficients of equation (27) or (28) variable. In general, the solution of differential equations with variable coefficients is a difficult task. It is plain that no general solution can be obtained because the variation of L in any particular case must necessarily be defined in terms of particular constants rather than in terms of arbitrary constant. Although the variation of L can sometimes be approximated with the aid of simple functions, the actual variation in many cases of importance cannot be expressed in terms of practical mathematical functions.

It is well known that L, being equal to $N\,d\phi/di$, depends upon the ϕ–i characteristic of the magnetic material that surrounds the L coil. The inductance that is operative in establishing an $L\,di/dt$ voltage drop depends for its value upon the exact degree of magnetic saturation of the surrounding magnetic material. Under any a-c condition the degree of saturation varies considerably with time and under transient conditions these variations are very often exaggerated. Reference to any typical B–H or ϕ–i curve will show that

$$L = N\frac{d\phi}{di}$$

is much greater over the straight portion of the curve than it is after the upper bend is reached. This fact plays an important role in determining the current inrush to iron-clad circuits, because, in general, circuits of this character are highly inductive and the variable L becomes an extremely influential parameter.

Circuit problems involving variable parameters can be solved by step-by-step methods provided the exact variation of the parameters is known. In the present case the variation of L is known if the $N\phi/i$ characteristic of the surrounding magnetic material is known. The data usually take the form of either the ϕ–i characteristic and the number of turns or the B–H characteristic, the dimensions of the magnetic circuit, and the number of turns. In any event it is somewhat more direct to substitute for $L\,di/dt$ [in equation (26)] its equivalent $N\,d\phi/dt$ value. The basic equation then becomes:

$$N\frac{d\phi}{dt} + Ri = E_m \sin(\omega t + \lambda) \tag{94}$$

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where ϕ is expressed in webers if the other quantities are expressed in practical units.

In many iron-clad circuits the maximum magnitude of the Ri term is of the order of 1 per cent of the maximum magnitude of the applied voltage. Under these conditions the $N\,d\phi/dt$ component of equation (94) is very nearly equal to the applied voltage and in approximate steady-state solutions the Ri drop can be neglected. The Ri drop cannot be entirely neglected in the transient solution of the problem because it is instrumental in helping to govern the maximum value of the initial current inrush. The resistance is also an important factor in governing the length of time required for the iron-clad circuit to adjust itself to steady-state operating conditions.

If the Ri drop is neglected and if it is assumed that $\lambda = 0$, equation (94) reduces to

$$N\frac{d\phi}{dt} = E_m \sin \omega t \tag{95}$$

from which

$$\phi = \frac{E_m}{N}\int \sin \omega t\, dt = -\frac{E_m}{\omega N}\cos \omega t + c_1 \tag{96}$$

The constant of integration c_1 may be evaluated in terms of the residual magnetism. ϕ may be at either positive or negative residual values at $t = 0$, and in general the exact state of residual magnetism is unknown. A compromise may be made by assuming that $\phi = 0$ at $t = 0$ unless the maximum possible current inrush is to be determined. In this case a maximum value of positive residual magnetism is assumed if the applied voltage is taken as $E_m \sin \omega t$. The manner in which residual magnetism helps to determine the initial current inrush will soon be apparent.

Assuming that $\phi = 0$ at $t = 0$, c_1 of equation (96) becomes:

$$c_1 = \frac{E_m}{\omega N} \tag{97}$$

Under these conditions:

$$\phi = \frac{E_m}{\omega N}(1 - \cos \omega t) \tag{98}$$

or

$$\phi = \phi_m (1 - \cos \omega t) \tag{99}$$

where $\phi_m = E_m/\omega N$, the approximate maximum value of the magnetic flux under steady-state operating conditions. Since ($\cos \omega t$) varies between $+1$ and -1, it is plain that the flux variation as defined by

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equation (99) varies from zero at $t = 0$ to $2\phi_m$ at $t = T/2$. In order to produce a flux equal to $2\phi_m$, the iron-clad inductance coil must draw a particular value of magnetizing current as defined by the ϕ–i characteristic of the magnetic circuit. For example, in the circuit shown in Fig. 16

$$\phi_m = \frac{155.5}{377 \times 80} = 0.00516 \text{ weber}$$

or

$$\phi_m = 0.00516 \times 10^5 = 516 \text{ kilolines}$$

Reference to the magnetization curve will show that the current required to establish ϕ_m is approximately 1.2 amperes, whereas current required to establish $2\phi_m$ is approximately 84 amperes. This great change in current is due primarily to the flattening out of the magnetization curve.

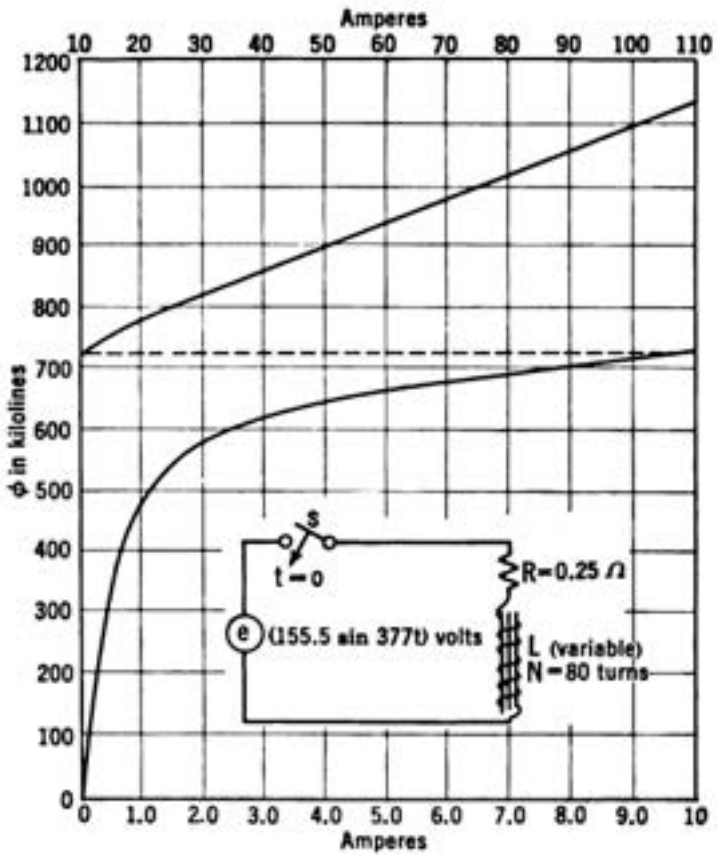

FIG. 16. Magnetization curve of a particular iron-clad *RL* circuit.

If the magnetic core referred to above had possessed a residual magnetism of, say, $+0.5\phi_m$, it is evident that a much larger current than the 84 amperes would finally be required to produce the $2\phi_m$ change in flux during the first half cycle. Actually the initial current inrush to an iron-clad circuit is somewhat less than that required to produce a $2\phi_m$ flux change.

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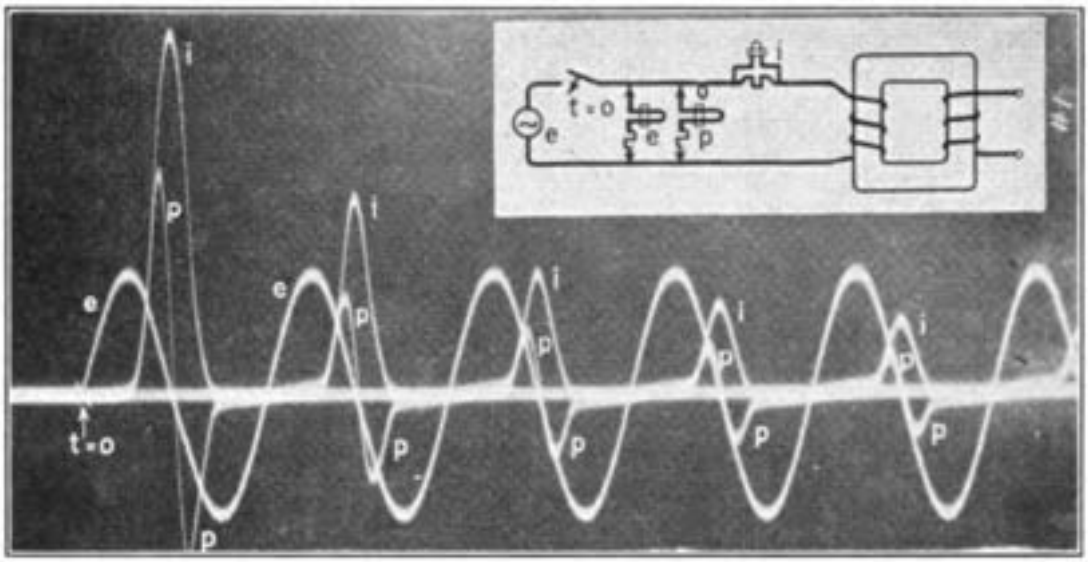
It will be remembered that equation (99) carries with it the assumption that the Ri drop is negligibly small. This assumption may be perfectly justified if the flux is worked between its normal steady-state values of $+\phi_m$ and $-\phi_m$. But in attempting to produce a $2\phi_m$ change in flux starting with zero flux, the circuit draws such a large current that the Ri drop becomes significantly large and must be taken into consideration. Under the above conditions the Ri drop consumes an appreciable portion of the applied voltage during the second quarter cycle after the switch is closed, thereby reducing the magnitude of the $N\,d\phi/dt$ component in this region. As a result, ϕ reaches a maximum value of something less than $2\phi_m$ shortly before $t = T/2$, and it is at this point that the maximum instantaneous current occurs.



OSCILLOGRAM 7. Iron-core transformer current and power inrushes when the primary is energized at the $e = 0$ point on the voltage wave.

e = 60-cycle applied emf	E (eff.) = 117 volts
i = instantaneous current	Peak i = 174 amperes
p = instantaneous power	Peak p = 10.5 kw

Steady-state conditions: $P_{av} = 30$ watts, $I_{eff} = 0.825$ ampere.
Transformer rating: 115 volts, 3 kva, 26.1 amperes, 60 cycles.

The ordinary iron-core transformer with open secondary operates as a simple iron-core *RL* circuit. Oscillogram 7 illustrates the nature of the starting current taken by the primary winding of an iron-core transformer when the secondary is open-circuited. In this particular case the initial peak current is considerably more than 100 times the steady-state maximum value of primary current when the secondary is open-circuited. However, the initial current inrush reaches a peak value which is only about 4.5 times the value of the maximum full-load current of the transformer. For the case shown in Oscillogram 7 the actual transient period is of approximately 0.5-second duration. Only the early part of the transient period is shown in the oscillogram.

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The Method of Finite Differences. Although it involves step-by-step calculations, the "method of finite differences" is very often employed in circuit analysis when variable parameters are encountered. The step-by-step calculations are based upon the assumption that the parameters remain sensibly constant over small finite intervals of time. Usually the basic voltage equation is rewritten so that all differentials take the form of finite increments. The circuit voltage and current are then assumed to remain constant over an arbitrarily assigned increment of time, Δt. As a first approximation the applied voltage and current are assumed to be constant at their "start-of-period" values. If, then, after assigning a particular value to Δt, only a single unknown incremental quantity remains in the equation, it can be solved for by methods of elementary algebra. The process can best be illustrated by means of an example.

The predetermination of the initial current inrush to an iron-clad circuit will serve to illustrate the details of the method of finite differences. If finite differences of ϕ and t are employed, equation (94) takes the following form:

$$N \frac{\Delta\phi}{\Delta t} + Ri = E_m \sin\left(\sum \Delta\rho + \lambda\right) \tag{100}$$

where $\sum \Delta\rho = \sum \omega\, \Delta t$, the angular displacement along the voltage wave of the point under investigation from the point of $t = 0$.

Judgment must be exercised in the choice of Δt in any particular case. The selection of the size of Δt in a-c circuits is governed largely by the magnitude of ω. If points every 10° along the voltage wave are desired, then each Δt is taken as $\frac{1}{18}$ of π/ω second. The choice of smaller increments will, of course, make for more accurate solutions. Δt should never be chosen so large that significant changes in the parameters take place within the time interval represented by Δt.

At the beginning of a period i and $E_m \sin\left(\sum \Delta\rho + \lambda\right)$ have particular values. Letting $E_m \sin\left(\sum \Delta\rho + \lambda\right)$ be written as e and solving equation (100) for $\Delta\phi$ results in

$$\Delta\phi = \frac{(e - Ri)\,\Delta t}{N} \tag{101}$$

If practical units of e, R, i, and t are employed in the above equation, $\Delta\phi$ is given in webers.

Various refinements can be employed to improve the accuracy of the method of finite differences as outlined above. Very often, however, the improved accuracy is not warranted because of the uncertainties that surround the initial conditions and other experimental data.

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Numerical Example. (1) The emf applied to the iron-clad RL circuit shown in Fig. 16 is:

$$e = \sqrt{2} \times 110 \sin 377t \text{ volts}$$

This signifies that a 60-cycle voltage, the effective value of which is 110 volts, is applied to the circuit at the point of zero voltage where de/dt is positive. A simpler way of expressing the same thing is to say that a 110-volt 60-cycle voltage is applied at $\lambda = 0$.

(2) $N = 80$ turns and $R = 0.25$ ohm as indicated in the circuit diagram of Fig. 16.

(3) The residual magnetism is zero, and the flux varies in accordance with the ϕ–i curve given in Fig. 16 for the first half cycle of the applied emf.

(4) Only the first maximum instantaneous value of current is to be determined. Therefore the hysteresis effects which occur after the first half cycle and which complicate the determination of succeeding maxima can be neglected. Let the numerical coefficients enumerated above be inserted into equation (100).

$$80 \frac{\Delta\phi}{\Delta t} + 0.25i = 155.5 \sin \sum \Delta\rho$$

or

$$\Delta\phi = \frac{(155.5 \sin \sum \Delta\rho - 0.25i)}{80} \Delta t \text{ webers}$$

It will be somewhat more convenient in the present example if $\Delta\phi$ is reckoned in kilolines.

$$\Delta\phi = \frac{(e - 0.25i)\,\Delta t}{80} \times 10^5 \text{ kilolines}$$

where $e = 155.5 \sin \sum \Delta\rho$.

Each time increment will be taken as 0.0005 second, a value which corresponds to an angular displacement along a 60-cycle wave of 10.8°.

The initial conditions are such as to make both e and i zero at $t = 0$. Assuming that both e and i maintain zero value throughout the first time interval, the change in flux during this period, $\Delta\phi_1$, is equal to zero.

At the beginning of the second period, $\sum \Delta t = 0.0005$ second and $e = 155.5 \sin 10.8°$ volts. For each interval i is assumed to have its "start-of-period" value, which in this case is zero.

$$\Delta\phi_2 = \frac{(29.1 - 0)\,0.0005}{80} \times 10^5$$
$$= 18.2 \text{ kilolines}$$

At the close of the second or the beginning of the third period the current is assumed to have acquired the value required for the establishment of $\Delta\phi_2$. Reference to the magnetization curve will show that the establishment of 18.2 kilolines requires approximately 0.03 ampere.

At the beginning of the third period, $\sum \Delta t = 0.001$ second and $e = 155.5 \sin 21.6°$ volts.

$$\Delta\phi_3 = \frac{(57.2 - 0.25 \times 0.03)\,0.0005}{80} \times 10^5$$
$$= 35.7 \text{ kilolines}$$

The current required to establish $\sum \Delta\phi$ [(18.2 + 35.7) kilolines], is approximately 0.09 ampere. Other $\Delta\phi$'s can be added by the step-by-step method outlined above. The results of a series of such calculations are shown in Table I.

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It will be noted that the current reaches a maximum value of approximately 72 amperes at $t = 0.008$ second. This corresponds to a point approximately 173° out along the voltage wave from the point at which the switch is closed, namely, the $e = 0$ point.

The general trend of the current variation is similar to that shown in Oscillogram 7. It will be observed that the current values are relatively very small during the first quarter cycle after the switch is closed. It is during this period that the Ri drop is negligibly small.

The change of flux that occurs during the period of negligible Ri drop can be calculated straightforwardly, and it may be of interest to compare the step-by-step results with a result which is very nearly accurate from a theoretical point of view. From equation (98)

$$\phi = \frac{155.5}{377 \times 80}(1 - \cos 377t) \text{ webers}$$

If t is taken as 0.0045 second, $(377t)$ is equal to approximately 1.7 radians or 97.2°. At $t = 0.0045$ second:

$$\phi = \frac{155.5}{377 \times 80}(1 - \cos 97.2°) \times 10^5 \text{ kilolines}$$

or

$$\phi = 579 \text{ kilolines} \quad \text{at} \quad t = 0.0045 \text{ second}$$

The value ϕ at $t = 0.0045$ second as determined by the step-by-step method is 626 kilolines. (See Table I.)

TABLE I

Period	$\Sigma\,\Delta t$ seconds	$\Sigma\,\Delta\rho$ degrees	$E_m \sin \Sigma\,\Delta\rho$ volts	Ri volts	$\Delta\phi$ kilolines	$\Sigma\,\Delta\phi$ kilolines	i amperes
1	0	0	0	0	0	0	0
2	0.0005	10.8	29.1	0	18.2	18.2	0.03
3	0.0010	21.6	55.7	Negligible	35.7	53.9	0.09
4	0.0015	32.4	83.3	Negligible	52.1	106.0	0.18
5	0.0020	43.2	106.0	Negligible	66.0	172.0	0.29
6	0.0025	54.0	126.0	Negligible	79.0	251.0	0.43
7	0.0030	64.8	141.0	Negligible	88.0	339.0	0.58
8	0.0035	75.6	151.0	Negligible	94.0	433.0	0.75
9	0.0040	86.4	155.0	Negligible	97.0	530.0	1.4
10	0.0045	97.2	154.0	Negligible	96.0	626.0	3.1
11	0.0050	108.0	148.0	0.78	92.0	718.0	9.0
12	0.0055	118.8	136.0	2.25	84.0	802.0	25.0
13	0.0060	129.6	120.0	6.25	71.0	873.0	44.5
14	0.0065	140.4	99.2	11.1	55.0	928.0	58.0
15	0.0070	151.2	74.9	14.5	38.0	966.0	66.5
16	0.0075	162.0	48.1	16.6	20.0	986.0	72.0
17	0.0080	172.8	19.5	18.0	1.0	987.0	72.0
18	0.0085	183.6	−9.8	18.0	−17.0	970.0	67.0
19	0.0090	194.4	−38.7	16.7	−35.0	935.0	59.0
20	0.0095	205.2	−66.2	14.7	−51.0	884.0	47.0

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PROBLEMS

3. (*a*) Find the current in a coil containing $L = 1$ henry and $R = 0.4$ ohm one second after applying a d-c voltage of 10 volts.

(*b*) What will the current be after 2.5 seconds?

(*c*) What is the value of the voltage accelerating the current after 1 second? after 2.5 seconds?

4. A coil has 0.1 henry and 1 ohm resistance and carries 10 amperes. If its terminals are suddenly short-circuited, what will be the value of current 0.1 second later? How long will it take the current to fall to 0.1 ampere?

5. Find the number of ohms resistance which may be placed in series with an inductance of 0.1 henry so as to permit the current in the circuit to reach 63.2 per cent of its final value in 2 seconds after the voltage is applied.

6. Ten volts direct current are applied to a 0.1-ohm resistance in series with a 1-henry inductance.

(*a*) Calculate the energy stored in the inductance 10 seconds after the voltage is applied. State units.

(*b*) Derive the expression for the energy dissipated in the resistance in the time t after the voltage is applied.

7. A 50-μf condenser with no initial charge is in series with a 1-megohm resistor. How long will it take to attain 63.2 per cent of its final charge?

8. A 50-μf condenser has stored 0.1 coulomb.

(*a*) If it is discharged through a 1000-ohm resistor how long will it take until it has 0.001 coulomb remaining?

(*b*) What will be the initial value of current?

(*c*) What will be the value of current when 0.001 coulomb remains on the condenser?

9. A 100-μf condenser has a charge of 0.1 coulomb. If it is discharged through a 10,000-ohm resistance, what will be the amount of energy in joules remaining in the condenser 1 second after the discharge is started?

10. A d-c voltage was applied to a resistance of 10,000 ohms in series with a 100-μf condenser. After 1 second there were 19.98 joules stored in the condenser which had no initial charge. How many volts were applied to the circuit?

11. A 1-megohm resistance is in series with a 1-μf condenser. A d-c voltage of 100 volts is suddenly applied to the circuit.

(*a*) Calculate the energy stored in the condenser 1 second after the voltage is applied.

(*b*) Derive the expression for the energy dissipated in the resistance during the first second after the voltage is applied.

(*c*) How much energy will be dissipated in the resistance in charging the condenser to full charge?

12. What fraction of total charge will the condenser in Problem 11 have after 2 seconds?

13. A voltage $e = 100 \sin [377t + (\pi/4)]$ is impressed on a 1-henry inductance coil containing 1 ohm resistance. What are the values of the steady and the transient components of current at $t = 0$?

14. A voltage $e = 100 \sin (377t + 30°)$ is impressed on a 100-μf condenser having no initial charge and containing 1 ohm resistance.

(*a*) What are the values of the steady and transient components of charge at $t = 0$?

(*b*) What are the corresponding values of current?

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15. A circuit contains $R = 100$ ohms, $C = 200$ μf, and $L = 0.1$ henry in series. If a d-c voltage of 50 volts is impressed, calculate the current and charge after 0.01 second, assuming no initial charge on the condenser.

16. A circuit contains $R = 5$ ohms, $L = 0.1$ henry, and $C = 200$ μf in series.

(*a*) Calculate the current and charge 0.01 second after 1000 volts are impressed if there was no initial charge on the condenser.

(*b*) Is the circuit oscillatory?

(*c*) If so, what is its frequency?

17. The condenser in the circuit of Problem 16 is charged to a potential of 1000 volts. If the circuit is connected upon itself, what will be the value of current and charge after 0.0125 second has elapsed?

18. Given an *RLC* series circuit which is suddenly energized with an alternating potential difference which is equal to

$$e = 141 \sin (377t - 45°) \text{ volts}$$

$R = 1.0$ ohm $L = 0.041$ henry $C = 18.7$ μf $Q_0 = 0$

(*a*) Write equation (93) for this particular case, employing numerical coefficients. The result is to be in the form:

$$i = k_1 \sin (k_2t + k_3) + k_4e^{k_5t} \sin (k_6t - k_7) \text{ amperes}$$

where all k's are expressed numerically.

(*b*) Make sketches of the steady-state term, the transient term, and the resultant current for the first three or four cycles of steady-state phenomena on the same plot. Show also the e variation.

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INDEX

Accuracy of short-circuit calculations, 527
Addition, of admittances, 113
  of complex waves, 199
  of currents, 83
  of impedances, 97, 98, 113
  of vectors, 68
  of voltages, 83
  of volt-amperes, 342, 343
Adjustment, full-load for watthour meters, 381
  light-load for watthour meters, 380
  for power-factor error of watthour meter, 378
Admittance, addition of, 113
  definition, 113
  surge, 419
Air-core transformer, 229, 252, 255
  vector diagram, 233, 253, 254
Akerman check, 389
Algebra of vectors, 63
Alternating current, amount used in United States, 1
  ampere value, 42
  average value, 43, 45
  beginning of systems, 1
  components of, 55
  definition, 3, 4
  effective value, 42
  equation of, 6, 7
  four-phase systems, 265
  generation of, 1, 2, 262
  lagging, 8, 9
  sine-wave representation, 6
  three-phase, 268, 271
  two-phase systems, 265
Alternating volt, definition, 43
Alternating voltage, *see also* Alternating current
  components of, 55
  definition, 3
Ammeter, 367, 369, 372
Amplitude factor, 46
Analysis of non-sinusoidal waves, 163
  analytical method, 166
  analyzing tables, 175–178
  graphical method, 171, 180
  rectangular wave, 169, 170
  triangular wave, 169, 170, 171
Analytical method for determining whether two waves are of the same form, 186–188
Angular frequency, 6
Angular velocity, 6
Attenuation, *see also* Filters, high- and low-pass
  band, 436
  constant or factor of a filter section, 450
  constant or factor of a long transmission line, 418
  decibels of, 92, 452
  frequency of infinite, 482
  general, 91
  method of expressing for filter, 450–454
  nepers of, 91, 451, 452
Auto-transformer, air-core, 255
Average value of a wave, 43, 45

Balanced delta load, 276
Balanced delta system of currents, 271, 272
Balanced systems, power in, 280, 285
Balanced wye load, 274
Balanced wye system of currents, 270
Band, side, 204
Band-eliminator filter, 474, 475
Band-pass filter, definition, 474
Bands, pass or transmission, 436
  stop or attenuation, 436
Bases, for expressing impedances, 522
Boundary conditions, 549
Bridge, elementary theory of methods, 403
  opposite-angle, 404, 405
  Owen's, 405
  similar-angle, 404

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Calibration of watthour meter, 277
Capacitance, distributed in coils, 403
effect of in a circuit, 21
energy stored in, 23
impedance due to, 21
tables of for transmission lines, 431
transmission line, 426
vector diagram, branch of, 50
Capacitive coupling, 214
Capacitive reactance, 21
Capacitive susceptance, 114–118
Carrier frequency, 202, 204
Cartesian form, of impedance, 81
of operator, 64, 65
Cathode-ray oscillograph, 365
Characteristic impedance, 440
determination from open- and short-circuit impedances, 444
low-pass, constant-k, π-section, 467
T-section, 467
π-section, 441
T-section, 441
variation of for constant-k sections, 468
Check, Akerman, 389
Circle diagram of series circuit, 110
Circuit, capacitive only, 21
coupled capacitively, 214
coupled conductively, 211
coupled mutual inductively, 216
definition, 211
direction, 1, 52, 84, 317
inductive only, 16, 50
parallel, 113
polyphase, 262
resistive, inductive, and capacitive, 31
resistive and capacitive, 33
resistive and inductive, 24
resistive only, 14, 50
series, 97
series-parallel, 128
sweep, 366, 556
Coefficient, of coupling, 215, 221
of mutual inductance, 217
of self-inductance, 217–218
Coil, iron-clad, 398
iron-core magnetizing current of, 162
Complex notation, 63
Complex waves, 161
Complex waves, addition of, 199
components of, 163
definition, 163
subtraction of, 199
Conductance, 114
Conductive coupling, types of, 211–214
Conductively coupled circuit, 211
Conjugates of voltage and current, 88, 89
Connection, checking of watthour meter, 388
Connection checks, for watthour meters, Akerman, 389
open-line, 388
open-potential coil, 389
for wattmeters, 288, 339
Connection of watthour meters, 344, 345, 385, 386
Conditions, boundary, 549
Constant-k filter, *see* Filters, constant-k type
Copper, comparison of requirements for transmission, 294
Coupled circuits, 211
design, 247
equations of voltage for, 220
impedance of, 237
loose, 221
maximum current in, 244
partial resonance of, 241
resonance of, 239–252
Coupling, capacitive, 214
coefficient of, 215, 221
conductive, 211, 214
critical, 244
inductive, 216
loose, 221
mutual inductance, 216
range of, 221
resistance, 214
Crest factor, 46
Critical coupling, 244
Current, alternating, definition, 3, 4
average value, 43, 45
components of, 55
effective value, 42, 45
fault, 521
instantaneous, 7, 35, 48, 199
in parallel branches, 113
resonance, 117
RMS value, 42, 45

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Current, in series branches, 97
for non-sinusoidal voltages, 194–200
in series-parallel branches, 128
for series resonance, 100
source, 140
virtual value of, 42
Current loci, for parallel branches, 119, 120
for series branches, 110, 111
Currents, addition of, 83
subtraction of, 83
Cut-off frequency, 455, 464, 466, 482
Cycle, 4

Decibel, definition, 452
Degrees of symmetry of waves, 184
Delta connection, balanced load, 276
current for, 271, 272
equivalent for wye, 145, 146
harmonics in, 301
symmetrical components of phase currents, 502, 503
unbalanced, current relations, 350, 351
solution of, 308, 327
voltage, 271
Delta-wye conversion, 145, 327
solution of unbalanced three-phase load by, 327
Delta and wye loads combined, solution of, 311
Deviation factor, definition of, 194
determination of, 194
use, 194
Directions, circuit, 1, 52, 84, 317
Direct wave of a transmission line, 418
Displaced a-c wave, effective value of, 561
Distorted waves, 161
Distributed capacitance of coils, 401, 402
Distributed parameters, 414
Division of vectors, 74
Dot-marked terminal, 224
Double-subscript notation, 264
Drain, high-frequency line, 436
low-frequency line, 437

Effective impedance, 393
Effective inductance, 400
Effective reactance, 397
Effective resistance, 395
Effective value, 42, 45
current, 42, 43, 45
displaced a-c wave, 561
measurement of, 368, 369
non-sinusoidal wave, 188
sinusoid, 45
voltage, 43, 45
Electric wave filters, 436
Electrodynamometer, ammeter, 367
voltmeter, 368
wattmeter, 373
Electromagnetic oscillograph, 364
Energy, stored in capacitance, 23
stored in inductance, 20
Energy measurements, general, 382
accuracy of, 384
Equivalent current source, 140
Equivalent delta for wye, 145
Equivalent equilateral spacing for transmission lines, 429
Equivalent plate circuit of vacuum tube, 144
Equivalent sine waves, 193
phase difference of, 193
Equivalent wye, for delta conversion, 145, 327
formula for, 146
Euler's equation, 66
Even harmonics, effect of, 184–186
Evolution of a vector, 77
Experimental determination, of currents of a delta load, 350
of vector relations, from oscillograms, 353
of three-phase line voltages, 347
of voltage of a wye load, 349
Exponential operator, 66

Factor, amplitude, 46
crest, 46
deviation, 194
form, 45
peak, 46
power, definition, 56; *see also* Power factor
reactive, 56, 281, 282
transfer, 238
Fault, defined, 521
Fault currents, solving for, 527, 530, 538
Faults, kinds of, 521

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Filters, attenuation constant, 450–454
band-eliminator, 474–475
definition, 475
band-pass, definition, 474
comparison of characteristics of constant-k and m-derived types, 485
constant-k type, definition, 465
high-pass, cut-off frequency, 469
design equations, 469–470
summary of characteristics, 472, 473
limitations, 475
low-pass, characteristic impedance, 467
cut-off frequency, 466
design equations, 467, 468
summary of characteristics, 472, 473
zero-frequency value of impedance, 467
definition, 436
electric wave, 436
full m-derived, 481
attenuation and phase shift, 484
effect of cut-off and infinite attenuation frequencies on value of m, 439
frequencies of infinite attenuation, 482
high-pass, 483
low-pass, 482, 483
fundamental equation of, 455–457
general design procedure, 485
half section, 476–480
high-pass, attenuation, 462
definition, 455
low-pass, attenuation, 459
definition, 455
m-derived half sections, 476
characteristic impedance, 478
general equations of, 478, 479
open-circuit impedance, 440, 444
phase constant β, 450, 451
phase shift, 456, 459, 462
π-section, 441
propagation constant, 450
short-circuit impedance, 440, 444
T-section, 441
without resistance, 458
Finite differences, method of solving equations by, 582
Flux, in iron core transformer, 579–584
leakage, 217, 252
mutual, 217, 252
Form factor, 45
Fourier analysis, general, 163
analytical method, 166
analyzing tables, 175–178
graphical method, 171, 180
rectangular wave, 169, 171
triangular wave, 166, 170
Fourier series, 163
coefficients of, 165
Four-phase systems, 265
Four-terminal network, 439
Four-wire, three-phase system, 268
Frequency, angular, 6
carrier, 202, 203
common for power use, 5
components of modulated waves, 202, 203
cut-off, 455, 464, 466, 469, 482
infinite attenuation for filters, 482
measurement, 389
meter, 389, 390
oscillatory, 569
range of pass band for RLC circuit, 107
resonant, for parallel branches, 121
for series circuit, 100
Full-load adjustment of watthour meter, 381

Generation of alternating current, 1, 2, 262
Geometric mean spacing of transmission conductors, 429
Graphical composition of symmetrical components, 494
Graphical method, for RMS value determination, 43
for wave analysis, 171, 180

Half-power points, 107
Half-wave resonance, 419
Half-wave symmetry, 184, 185
Harmonics, caused by variation of circuit parameters, 200
components of a complex wave, 163
in a delta system, 301
effect of even on a wave, 184–186

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Harmonics, in a wye system, 297
High-pass filter, *see* Filters

Image basis of termination of filter section, 443
Image impedance, 439
Impedance, base, 522
  capacitive, 21
  cartesian form of, 81
  characteristic, for filters, 440, *see also* Characteristic impedance
    for T- and π-sections, 441
  coupled circuit, 237
  determination of, 393
  diagram for series *RLC* branch, variable capacitance, 104
    variable frequency, 105
    variable inductance, 102
  effective, 393
  function, defined, 14
  functions, summary of, 34
  image, 439
  inductive, 18
  matching, 133
  mutual, 211, 219
  negative-sequence, 509–511, 532, 534
  negative-sequence component of, 509
  open circuit of a filter, 440, 444
  percentage of, 524
  polar form of, 79
  positive-sequence, 510, 532, 534
  positive-sequence component of, 509
  rectangular form of, *see* Impedance, cartesian form of
  referred to a common base, 522
  reflected, 237, 238
  resistive, 15
  resistive, inductive, and capacitive, 32
  resistive and capacitive, 33
  resistive and inductive, 25
  short-circuit, of filter, 440
  surge, 419
  transfer, 138
  transfer factor, 238
  transferred, 236
  zero-sequence, 534, 539
    of generators, 534, 539
    of transformers, 540
Impedance, zero-sequence, of transmission lines, 541
  zero-sequence component of, 509
Impedances, addition of, 97, 98, 113
  parallel, 113
  series, 97
  series-parallel, 128
Inductance, coefficient of mutual, 217
  coefficient of self-, 217
  effective, 400
  effect of, 16
  energy stored in, 20
  leakage, 254
  mutual, 217
    definition, 218
  transmission line, formula, 426
    general, 424
  transmission line table, 430
  true, 400
  vector diagram of branch, 50
Inductance coil, effect of distributed capacitance, 401
  self-resonance of, 402, 403
Induction, *see also* Inductance
  ammeters, 369
  voltmeters, 369
  watthour meter, 377, 383
  wattmeter, 374
Inductive coupling, 216
Inductive reactance, 18
Inductive susceptance, 114–117
Infinite-frequency characteristic impedance, 470
Instantaneous currents, combination of, 35, 48
Instantaneous power, 15, 18, 22, 28, 284
Instantaneous value of current and voltage, 6, 7
Instrument transformers, use, 372
Involution of a vector, 77
Iron-clad coil, 398
Iron-clad *RL* circuit, transients in, 578
Iron-core coil, 398
  current in, 162

j, operator, 63

Kilovolt-amperes, 55
Kirchhoff's law, application to alternating current, 312

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Ladder structures, 454
Lagging adjustment of watthour meter, 379
Lagging current, 9
Leakage flux, 217, 252
  inductance, 254
Light-load adjustment of watthour meter, 380
Line calculations, approximate methods, 411–414
  exact method, 414
Line constants, formulas for, 426–429
  tables of, 430, 431
Line drain, high-frequency, 436
  low-frequency, 437
Line-to-ground short circuit, 538
Line-to-line short circuit, 530
Logarithm of a vector, 78
Loop current solution, 212, 354; *see also* Mesh, current solution
Loose coupling, 221
Lower side-band, 204
Low-pass constant-k filter, *see* Filters, constant-k type
Low-pass filter, *see* Filters

Magnetic coupling or mutual inductive coupling, 216
Magnetic coupling between phases, 515
Magnetizing current of iron core coil, 162
Matching of impedances, 133
Maximum power transfer, 133
m-derived filter section, full, 481
m-derived half section, 476
Measurement, of frequency, 389
  of mutual inductance in terms of self-inductances, 401
  of power in balanced systems, 285
  of power factor, 376
  of reactive volt-amperes, 281, 290, 340, 344
  of three-phase power by three wattmeters, 334
  of three-phase power by two wattmeters, 285, 286, 334–336
Measurements, bridge, difficulty in making, 401
  elementary theory, 403
  opposite-angle bridge, 405
  Owen's bridge, 405
Measurements, similar-angle bridge, 404
Mesh, connection, general, 271, 272
  current solution, 212, 354
  four-phase, 267
  n-phase, 273
  three-phase, *see* Delta connection
Meters, ammeters, 367, 369, 372
  electrodynamometer, 367, 368
  frequency, 389
  induction ammeters and voltmeters, 369
  iron-vane, 369
  power-factor, 376
  reactive volt-ampere, 375, 376
  rectifier type, 372
  thermal, 372
  thermocouple, 372
  varmeter, *see* Reactive volt-amperes, meter
Mho, 114
Modulated waves, composition of, 202, 203
Modulation, definition of percentage, 203
Multiplication of vectors, 72
Mutual flux, 217, 252
Mutual inductance or induction, 217
  between parallel branches, 227
  between series branches, 224
  coefficient of, 217
  definition, 217
  impedance, 211, 219
  reactance due to, 219
  sign of, 222
  in terms of self-inductances, 401
Mutual induction voltage, 219
  direction of, 222
Mutual inductive coupling, 216, 217

Napier, definition, 452
Negative-sequence components, 491, 493
  evaluation, 496
  impedance component, 509
Negative-sequence impedances of rotating machines, 532–535
Neper, definition, 91, 451, 452
Network, image impedance, 439
  smoothing, 437, 439
  theorems, 135, 137, 138
Networks, 135
  π, 150

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Networks, T, 150
theorems, 135, 137, 138
Nodal method of circuit solution, 139
Non-linear parameters, step-by-step solutions, 582–584
Non-sinusoidal waves, 161
circuit analysis for, 163, 194, 196
effective value of, 188
expressing, 163
parallel circuit analysis for, 196
power calculations for, 190
power factor for, 192
series circuit analysis for, 194
shifting the phase of, 187
volt-amperes of, 192
Notation, double-subscript, 264
n-phase mesh, 273
n-phase star, 273

Ohmic method of short-circuit calculations, 522
Open-circuit impedance of a filter, 440, 444
Open-line check, 288, 388
Open-potential coil check, 389
Operator, cartesian form, 64, 65
exponential, 66
j, 63
polar form, 67
rectangular form, 65
Opposite-angle bridge, 404, 405
Oscillatory term of RLC circuit, 567, 568, 571, 576
frequency of, 569
Oscillograph, cathode-ray, 365
electromagnetic, 364
frequency range of, 365, 366
Owen's bridge, 405

Parallel, impedances in, 113
Parallel branch analysis, 113
for non-sinusoidal waves, 196
Parallel branch current loci, 118–121
Parallel branches, resonance in, 117
Parallel branch resonance for all frequencies, 124
Parallel resonance, 117–126
by varying C, 120
by varying f, 121
by varying L, 118
Parallel resonance, by varying R_c, 121
by varying R_L, 121
Parameters, determination of, 393
distributed, 414
effect of variation on wave form, 200
non-linear, step by step solution, 582
transmission line, 424
Partial resonance of coupled circuits, 241
Pass band, boundary of, for filters, 459–466, 474, 482
definition, for filters, 436
for series RLC circuit, 106
width of, for series RLC circuit, 107
Peak factor, 46
Percentage method of expressing parameters, 524
Percentage of modulation, 203
Percentage reactance, 525
Percentage resistance, 525
Period, definition, 4
Per unit method of expressing parameters, 527
Phase, 8
angle, 8
constant, of a filter, 450
of a long line, 419, 422
of current and voltage, 14, 18, 21, 25, 33
difference, 8
for equivalent sine waves, 193
magnetic coupling between phases, 515
n-phase mesh, 273
n-phase star, 273
sequence, 263, 264, 276
effects, 330
impedances, 509, 510, 532, 539
methods of checking, 331
shift, general, 92
of high-pass filter section, 462
of low-pass filter section, 459
shifting of a non-sinusoidal wave, 187
π-network, 150
π-section filter, 441
attenuation of, 456
characteristic impedance, 441, 444
phase shift, 456
Polar form, of impedance, 79
of operator, 67
Polarity marks, 224

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Polyphase, *see also* Mesh, Star connection, Delta connection, Wye, *n*-phase mesh
generation of voltages, 262
Polyphase power, *see also* Power
balanced systems, 280
comparison with single-phase power, 284
unbalanced systems, 334–339
Polyphase unbalanced, general, 308, 312
Positive directions of voltages and currents, 52, 84, 222
Positive-phase sequence system, 490, 492
evaluation of, 495, 532
impedance of rotating machines, 533, 534
impedance of static elements, 509
Power, half-power points, 107
instantaneous, 15, 18, 22, 28, 284
maximum from circle diagram, 112
maximum transfer in a series circuit, 133
measurement, of balanced polyphase, 285
in an *n*-wire system, 291, 338
by three-wattmeter method, 334
by two-wattmeter method 286, 334–337
in a purely capacitive branch, 22
in a purely inductive branch, 18
in a purely resistive branch, 15
reactive, in general, 53
real, in general, 53
in a resistive and inductive branch, 28
in an *RLC* branch, 33, 34
system short-circuit calculations, 521
three-phase, 280, 334–337
of a vector, 77
Power calculation, of balanced polyphase, 280
of balanced three-phase, 280
from complex forms, 86
from conjugate of V and I, 88
for non-sinusoidal waves, 190
of single and three-phase, 284
Power factor, balanced three-phase, 282
definition, 56, 280, 282, 341–343
error correction in a watthour meter, 379
error of watthour meter, 378
Power factor, measurement of, 376
meter, 376
for non-sinusoidal waves, 192
of series branch, 98
in unbalanced polyphase systems, 341
A.I.E.E. definition, 342
Power formula, for balanced three-phase, 280
for single-phase, 53
Propagation, velocity of, 420
of a wave, 418
Propagation constant, of a filter, 450
of a long line, 418
Proximity effect, 396

Q, of a coil, 108
of a parallel circuit, 125
of a series circuit, 108
Quarter-phase, *see* Four-phase systems
Quarter-wave resonance of a long line, 419
Quarter-wave symmetry, 185

Range of coupling, 221
RC branch, transients, 552
Reactance, capacitive, 21
effective, 397
inductive, 18
mutual, 219
percentage, 525
per unit, 524, 527
Reactive factor, 56, 281, 282
Reactive power, *see* Reactive volt-amperes
Reactive volt-amperes, 29, 53
balanced three-phase, 290
calculated from complex expressions, 87
determination from conjugates of V and I, 88
formula, for balanced three-phase, 290
for single-phase, 54
measurement, of balanced three-phase, 290
of single-phase, 374
in a 3-wire, 3-phase circuit, 376
in unbalanced 4-wire, 3-phase, 340
meter, 375, 376
sign of, 34, 54
Reciprocity theorem, 137

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Rectangular form, of impedance, 81
of operator, 64, 65
Rectangular wave, 169, 171
Rectifier, constants for a smoothing network of, 438, 439
smoothing network for, 437, 439
type meter, 372
Reflected impedance, 237, 238
Reflected wave, 418, 419
Reflections, terminal of a line, 420
Resistance, effective, 395
effect of, 14
impedance due to, 14
percentage of, defined, 525
in series, with capacitance, 33
with inductance, 24
with inductance and capacitance, 31
vector diagram of branch, 50
Resistance coupling, 214
Resonance, in coupled circuits, 239–252
of a long transmission line, half-wave, 419
quarter-wave, 419
three-quarter-wave, 419
in parallel circuits, 117; *see also* Parallel resonance
self-, of inductance coil, 403
series circuits, 99; *see also* Series resonance
RL branch transients and steady current in, for alternating current, 557
for direct current, 550
RLC branch, selective properties, *see* Selective *RLC* circuit
vector diagram, 51
RLC series branch, equation of current, 575
oscillatory term of, 576
transients in, 565, 570, 571
for $R^2/4L^2 > 1/LC$, 567, 570, 575
for $R^2/4L^2 < 1/LC$, 567, 570, 575
RL iron-clad circuit, transients in, 578
RMS value, 42, 45
of a displaced sinusoidal wave, 561
graphical determination of, 43
measurement of, 368, 369
of a sinusoidal wave, 45
Root of a vector, 77

Sawtooth wave form, 554
Selective properties of circuit elements, 436
high-frequency line drain, 436
low-frequency line drain, 437
Selective *RLC* circuit, 106
pass band, definition, 106
width of, 107
Selectivity of *RLC* series branch, 107, 108
Self-inductance, 217
voltage of, 220
Sequence, phase, 263, 264, 276
checking, 331
effects of, 330
negative-phase sequence system, 491, 493, 496
impedance, 509, 532–535
positive-phase sequence system, 490, 492, 495
impedance, 509, 533, 534
rule, 511, 513
zero-phase sequence system, 491, 493, 496
impedance, 509, 534, 539, 540, 541
Series, impedances in, 97
resonance, 99
Series branch, 97
circle diagram, 110
efficiency of for transmission, 112
Series circuit, 97
analysis for non-sinusoidal waves, 194
circle diagram of, 110
current loci, 110, 111
maximum power transfer of, 133
Series-parallel branches, 128
Series-parallel tuning, 130
Series resonance, 99
frequency for, 100
by varying capacitance, 103
by varying frequency, 104
by varying inductance, 101
voltage drops, 100
Shifting the reference point of a non-sinusoidal wave, 187
Short circuit, accuracy of calculations, 527
calculation, by ohmic method, 522
by percentage method, 524
in power systems, 521
impedance of a filter, 444
definition, 440, 444
line-to-ground, 538
calculation of, 538, 543

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Short circuit, line-to-line, 530
calculation of, 535
Side bands, 204
Sign of readings of wattmeters, 288, 339
Similar-angle bridge, 404
Sine wave, representation of alternating current, 6
Sine waves, equivalent, 193
Sinusoidal wave, average value of, 45
crest factor of, 46
effective value, 42, 45
form factor, 45
vector representation of, 47
Skin effect, 395
Smoothing network, 437, 439
constants for, 438
Star connection, general, 273
four-phase, 266
n-phase, 273
three-phase, 268, 269
Steady-state term, 550, 552, 559, 560, 564, 575
Step-by-step method solution for non-linear parameters, 582
Stop band, 436
Subtraction, of currents, 83
of vectors, 70
Superposition theorem, 135
Surge admittance, 419
Surge impedance, 419
Susceptance, 114–118
Sweep circuit, 366, 556
Symbolic notation, *see* Vectors, algebra of
Symmetrical components, 490
copper losses from, 508
delta-wye voltage transformations, 500
graphical composition of, 494
line-to-ground current representation, 504, 505
negative-sequence system, 491, 493
evaluation, 496, 532
impedance, for rotating machines, 509–511
for static elements, 509
of phase currents in delta-connected loads, 502
positive-sequence system, 490, 492
evaluation, 495, 532
impedance, for rotating machines, 533, 534
for static elements, 509
Symmetrical components, power from, 506–508
solution for fault currents, 521–545
unbalanced current representation, 502–506
unbalanced line voltage representation, 498, 502
use of, 490
zero-sequence system, 491, 493, 496
evaluation, 495–497, 532, 538
impedance, for rotating machines, 510, 532, 534, 539, 542
for static elements, 509
Symmetry, of complex waves, 184
degrees of, 184
half-wave, 184
non-sine waves, 184
positive and negative loops, 184
quarter-wave, 185
of wave about its 90° ordinate, 185
Systems, *see also* Alternating current
four-phase, 265
three-phase, 268–273
two-phase, 265

Tables, of capacitance for transmission lines, 431
of inductance for transmission lines, 430
Terminal reflections, 420
Termination of filter section on image basis, 443
Thermal meters, 372
Thermocouple meters, 372
Thévenin's theorem, 138
Three-origin vector diagram, 278
Three-phase, four-wire system, 268
Three-phase, three-wire system, 271
Three-phase power measurement, 334–337
Three-quarter-wave resonance of a long line, 419
Three-wire, two-phase system, 268
Time constant, of *RC* branch, 554
of *RL* branch, 551
T-network, 150
Transfer factor, 238
Transfer impedance, 90, 138
Transferred impedance, 236
Transformer, air-core, 229–236, 252–255
vector diagram of, 253, 254
air-core auto-, 255

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Transformer, instrument, 372
Transient, application of, 548
  defined, 548
  sawtooth wave, 554–557
Transients, of *RC* branch, 552
  of *RL* branch, 548, 577
  of *RLC* series circuit, 565–577
    oscillatory frequency of, 569, 571, 576
    $R^2/4L^2 > 1/LC$, 567, 570, 571
    $R^2/4L^2 < 1/LC$, 567, 570, 571
  of *RL* iron-clad circuit, 578
Transmission, 89
Transmission band, 436
Transmission line, general, 410
    approximate solution, by Steinmetz method, 413
    exact solution, 414
    geometric mean spacing of conductors, 429
    by π-line, 412
    representation of, 410
    by T-line, 411
Transmission line parameters, 424
  capacitance, formula, 429
    table, 431
  derivation of formulas, 424, 426
  inductance, formula, 426
    table, 430
Transmission of power, comparison of copper requirements, 294
Triangular waves, analysis of, 169–171
  equation of, 167, 170, 171
True inductance, 400
T-section, filter, 441
    attenuation of, 456, 459, 462
    characteristic impedance of, 440–442, 444
    phase shift, 450, 456, 459, 462
    series arm, 457
    shunt arm, 457
Tuning, series-parallel, 130
Two-phase systems, 265
Two-wattmeter method of measuring 3-phase power, 286, 334–337

Unbalanced currents, representation by symmetrical components, 502–506
Unbalanced polyphase circuits, general, 308, 312
    delta, solution of, 308, 327
    power factor of, 341–343
Unbalanced polyphase circuits, general, solution by Kirchhoff's laws, 312
    wye-wye with neutral solution, 326
    wye-wye without neutral solution, 319
Unbalanced voltage, representation by symmetrical components, 498–502
Upper side band, 204

Var, 29, 54
Varmeter, 30, 375; *see also* Reactive volt-amperes, meter
Vector diagrams, general, 49–52
    C-branch, 50
    L-branch, 50
    R-branch, 50
    RC-branch, 51
    RL-branch, 51
    RLC-branch, 51
    three-origin, 278
Vector relations, currents of an unbalanced delta, 350, 351
  line-to-line voltages from experiment, 347
  from oscillographic records, 353
  voltages of an unbalanced wye, 349
Vectors, addition of, 68
  algebra of, 63
  cartesian form of notation for, 64
  division of, 74
  evolution of, 77
  involution of, 77
  logarithm of, 78
  multiplication of, 72
  raising to powers, 77
  rectangular form of notation for, 64
  representing sine waves, 47
  root of, 77
  subtraction of, 70
Vector volt-amperes, 342, 343
Velocity, angular, 6
  of wave propagation, 420
Virtual value of current or voltage, 42
Volt, alternating defined, 43
Voltage, alternating, components of, 55
    definition, 1, 2
  drop, 52
  drops, across series-parallel branches, 128
    at series resonance, 100
  equations of coupled circuits, 220
  instantaneous, 7

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Voltage, mutual, direction of, 222
mutual induction, 217
resonance, 99
rise, 52
self-induced, 220
series branch for non-sinusoidal waves, 194
Voltages, addition of, 58, 59, 83
delta, 271
generation of polyphase, 262
vector addition of, 83
vector combination of, 58, 59, 83
vector subtraction of, 58, 59, 83
wye, 269
Volt-amperes, 55
addition of, 342, 343
balanced three-phase, 262
non-sinusoidal waves, for, 192
reactive, 29, 53, 87, 88, 290, 340, 374
in three-phase, 290, 342–346
vector, 342, 343
Voltmeters, electrodynamometer, 368
induction, 369
iron-vane, 369
rectifier, 372
thermal, 372
thermocouple, 372

Watthour meter, accuracy of, 384
calibration of, 381
checking connections of, 388
connections of three-phase, 385
with instrument transformers, 386
full-load adjustment, 381
induction, 377, 383
lagging adjustment of, 379
light-load adjustment, 380
power-factor error correction, 379
Wattmeters, checking signs of, 288, 339
connection in a circuit, 373
electrodynamometer, 373
induction, 374
loss in potential coils of, 374
Watt-ratio curve, 288
Wave, sawtooth, 554
Wave analysis, 163
analytical method, 166
graphical method, 171, 180
Wave form, 5, 184, 186, 189
of current in an iron-core coil, 162
Wave-length constant, of a filter, *see* Phase constant, of a filter
Wave-length constant, of a long line, 419, 422
Wave propagation, velocity of, 420
Waves, addition of complex, 199
attenuation of, 418
complex or non-sinusoidal, 161, 163
composite, 161, 163
direct, 418, 419
modulated, 201
propagation of, 418, 420
rectangular, 169, 171
reflected, 418, 419
of same shape or form, analytical method of determining, 186
sawtooth, 554
subtraction of, 199
triangular, 169, 171
Wave shape, 5, 6, 184–189
effect of shifting an harmonic on, 186, 188, 189
Wave trap, 123
Weber-turns, 219
Wye, general, 269, 273
and delta loads combined, solution of, 311
balanced load, 274
connection, voltages and currents in, 269, 270
magnetic coupling between phases of, 515
unbalanced voltages, vector relations, 349
Wye-delta, conversion, 149, 309, 310
unbalanced, solution of, 309, 312, 319–326
Wye system, currents in, 270
harmonics in, 297
Wye-wye system, unbalanced, with neutral, 326
without neutral, 320–325

Y, *see* Wye

Zero-frequency characteristic impedance, 467
Zero-sequence component of impedance, 509
Zero-sequence impedance, of rotating machines, 534, 539
of transformers, 540
of transmission lines, 541
Zero-sequence system, 491, 493, 496
evaluation of, 496, 497

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