2020 Winter Seminar

Linear Algebra

Chapter 2: Vector Spaces (2)

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Keyword: Graph, Incidence Matrix, $N(A^T)$, Linear Transformation

Chapter 1 Review

$$\left.\begin{array}{rcl} 2u + 5v + 5w &=& -5 \\ 4u - 6v &=& -2 \\ -2u + 7v + 2w &=& -9 \end{array}\right\} \text{System of linear equations}$$

$$\underset{n \times n}{\mathbf{A}} \; \mathbf{x} = \mathbf{b} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \mathbf{x}$$

# of eqns = # of unknowns

if a square matrix **A** is invertible (non-singular),
$\boldsymbol{Ax} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

else if (singular), infinitely many solution or no solution.

Chapter 2 Overview

U: Echelon form
R: Reduced row echelon form

$$\underset{m \times n}{\mathbf{A}} \mathbf{x} = \mathbf{b} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \mathbf{x}$$

Rectangular Matrix

$\mathbf{A} \rightarrow \mathbf{U} \rightarrow \mathbf{R}$

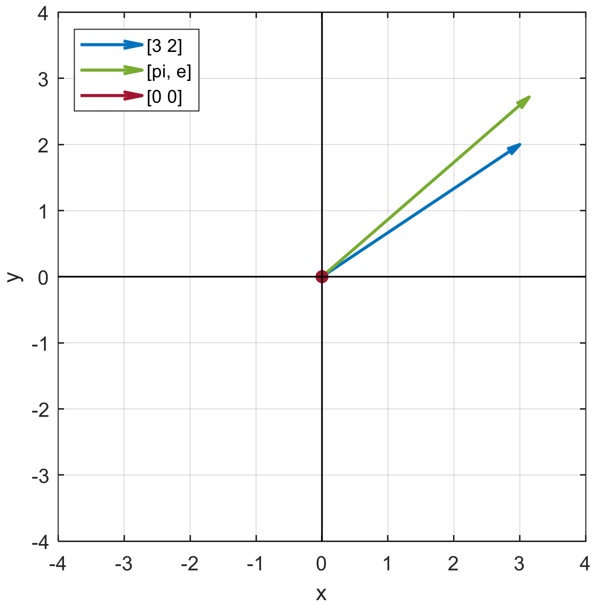
- unique solution
- infinitely many solution
- no solution

Contents

2.1 Vector Spaces and Subspaces

- Vector Spaces

The space R^n consists of all column vectors $\boldsymbol{v}$ with n real number components.



Ex)

R^2: All 2-dimentional real vectors ($x - y$ plane) $\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}$

R^3: All 3-dimentional real vectors with 3 components $\begin{bmatrix}3\\2\\0\end{bmatrix}$

$\vdots$

R^n: All n-dimentional real vectors with n components

2.1 Vector Spaces and Subspaces

- Vector Spaces

A vector space has to be closed under vector addition and scalar multiplication.

Thus, within all vector spaces, two operations are possible:
We can add any vectors in R^n, and we can multiply any vector $\boldsymbol{v}$ by any scalar c.

In other words, **Linear Combinations**.

2.1 Vector Spaces and Subspaces

- Vector Spaces

A set of vectors including **0** that satisfies requirements:

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in the subspace and c, d is any scalar, then

(i) $\boldsymbol{v} + \boldsymbol{w}$ is in the space

(ii) $c\boldsymbol{v}$ is in the space

(iii) all linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$ are in the space

2.1 Vector Spaces and Subspaces

- Vector Spaces

In the definition of a vector space, vector addition $\boldsymbol{x} + \boldsymbol{y}$ and scalar multiplication $c\boldsymbol{x}$ must obey the following eight rules:

(1) $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$

(2) $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$

(3) There is a unique "zero vector" such that $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$ for all $\boldsymbol{x}$

(4) For each $\boldsymbol{x}$ there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0}$

(5) 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$

(6) $(c_1 c_2)\boldsymbol{x} = c_1(c_2\boldsymbol{x})$

(7) $c(\boldsymbol{x} + \boldsymbol{y}) = c\boldsymbol{x} + c\boldsymbol{y}$

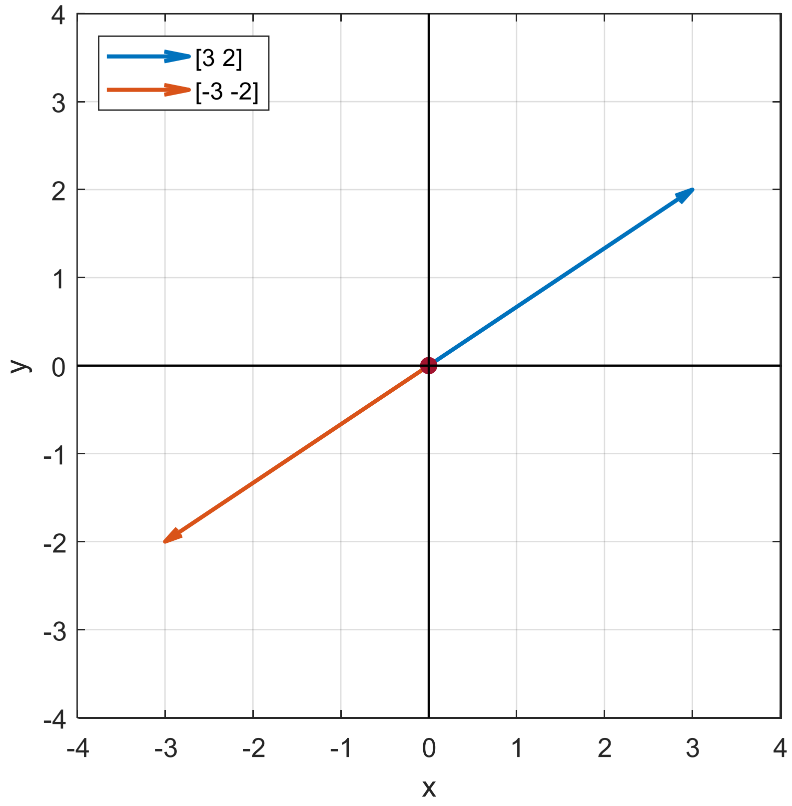
(8) $(c_1 + c_2)\boldsymbol{x} = c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

2.1 Vector Spaces and Subspaces

- Vector Spaces
  - Every vector space contains the zero vector

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = -\mathbf{v}_1 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

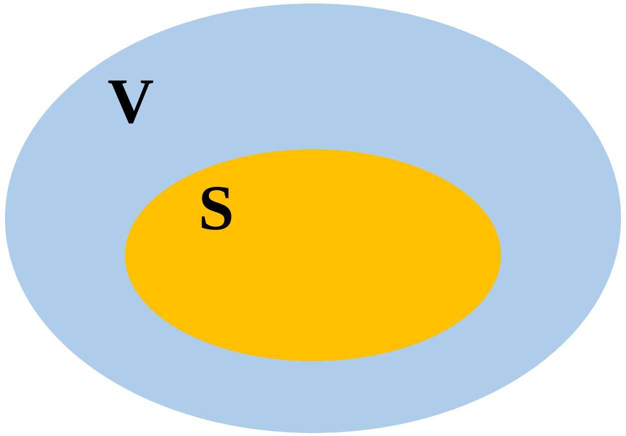
$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



2.1 Vector Spaces and Subspaces

- Subspaces $\mathbf{S} \subset \mathbf{V}$

  A subset of a vector space



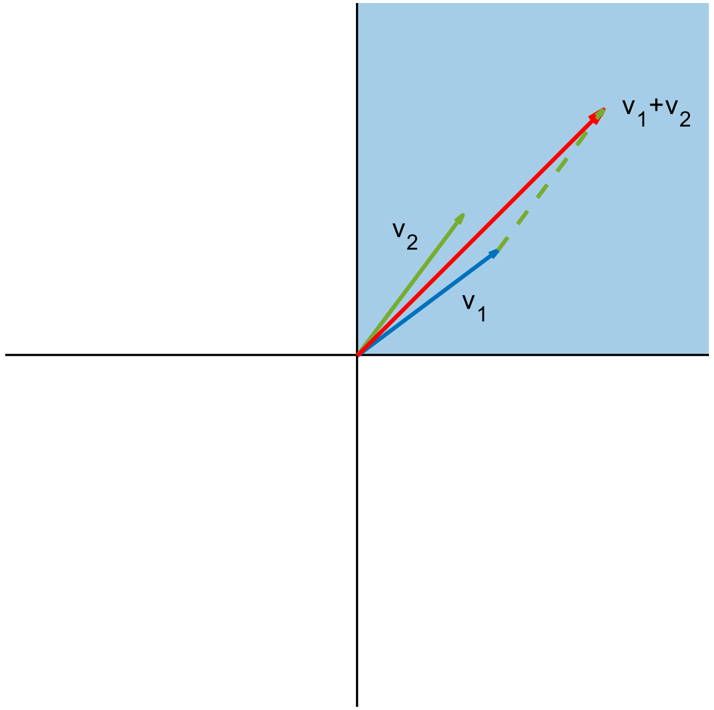
if $\boldsymbol{u}$, $\boldsymbol{v} \in S \subset V$, *any* scalar c, d

(i) $\boldsymbol{u} + \boldsymbol{v} \in S$
(ii) $c\boldsymbol{u} \in S$
(iii) $c\boldsymbol{u} + d\boldsymbol{v} \in S$

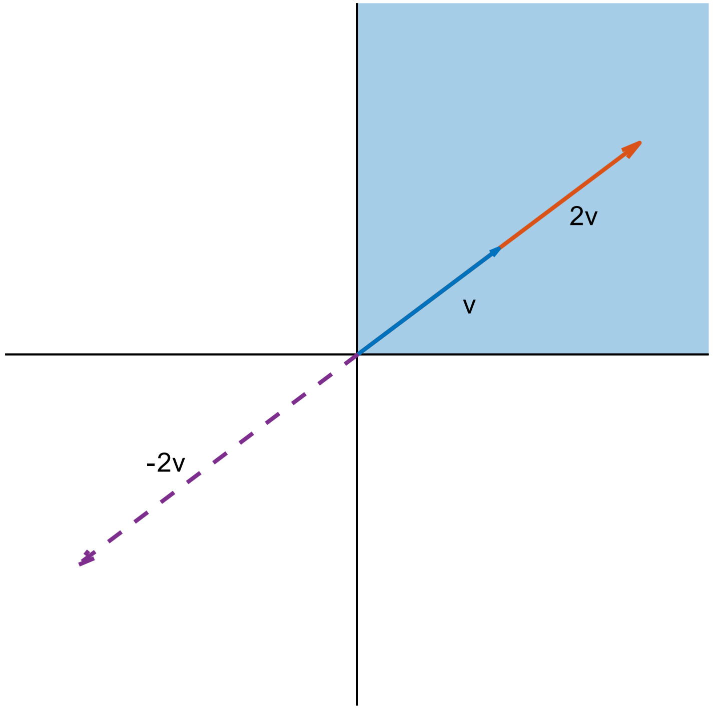
2.1 Vector Spaces and Subspaces

- Subspaces
  - case) not a vector space

closure under vector addition

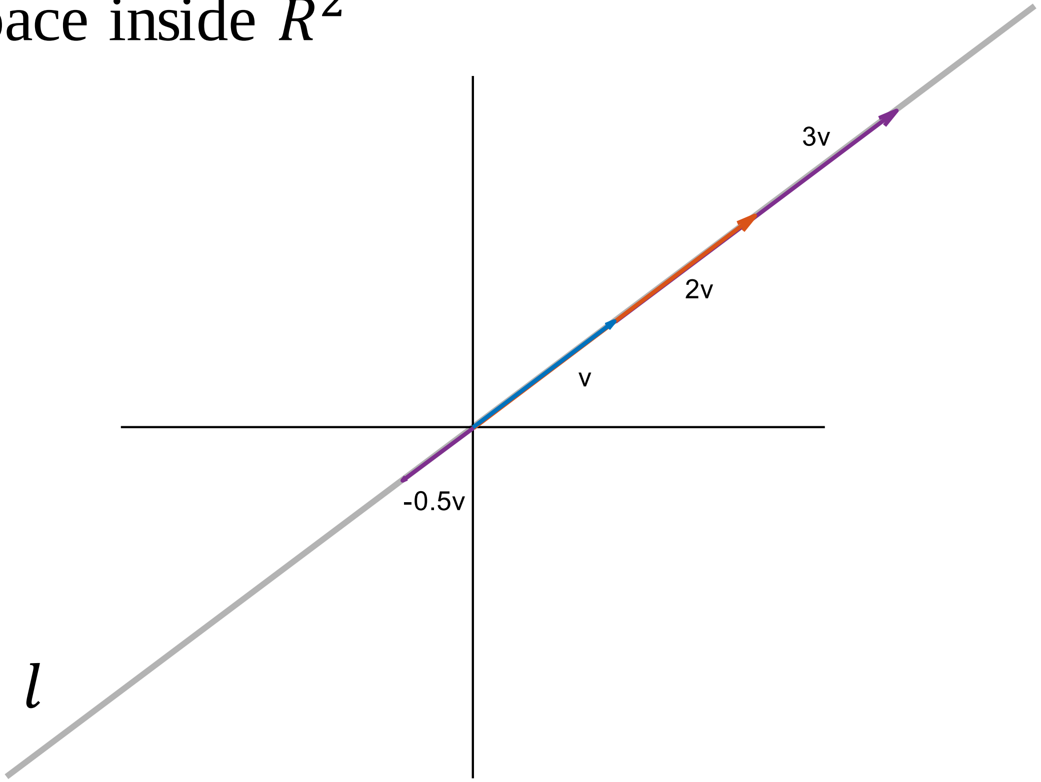


not closure under scalar multiplication



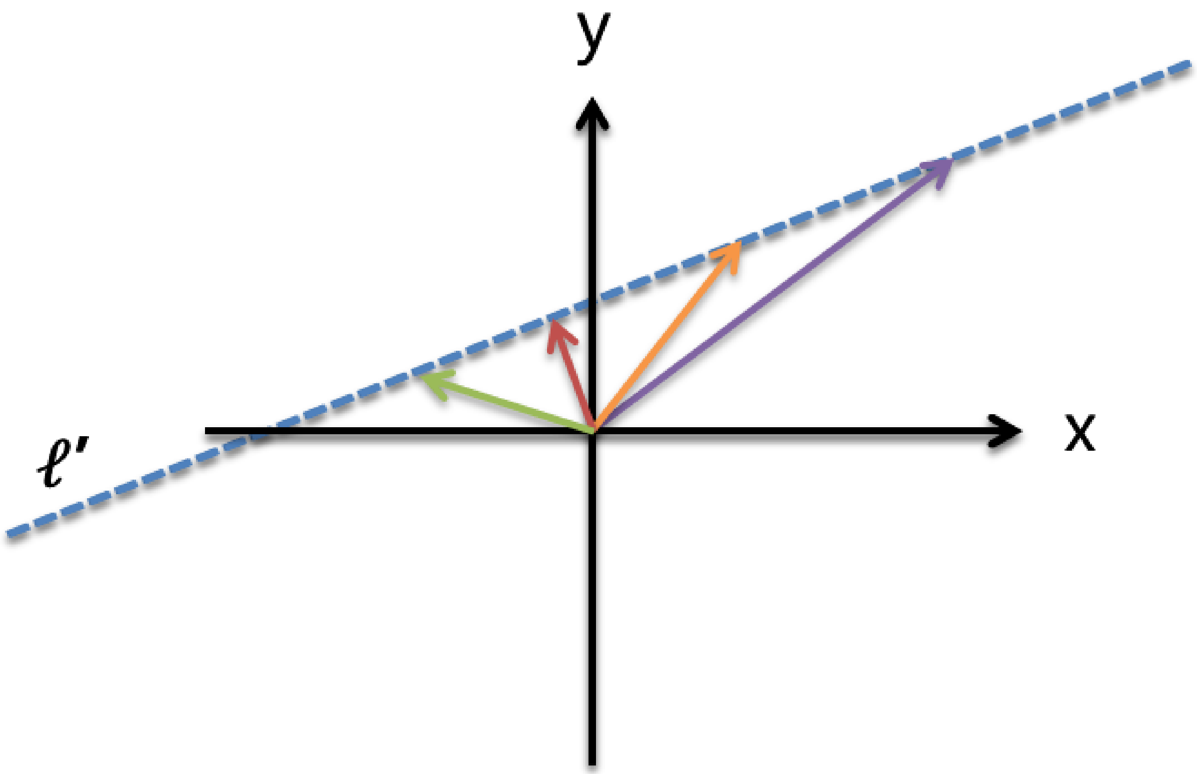
2.1 Vector Spaces and Subspaces

- Subspaces
  - case) a vector space inside R^2



2.1 Vector Spaces and Subspaces

- Subspaces
  - Every subspace contains the zero vector



2.1 Vector Spaces and Subspaces

- Subspaces

Subspace of R^2

① all of R^2
② any line through zero vector
③ the zero vector alone

Subspace of R^3

① all of R^3
② any plane through the origin **P**
③ any line through the origin **L**
④ the origin (zero vector) alone **Z**

2.1 Vector Spaces and Subspaces

$$A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \dots & A_n \\ | & | & & | \end{bmatrix}$$

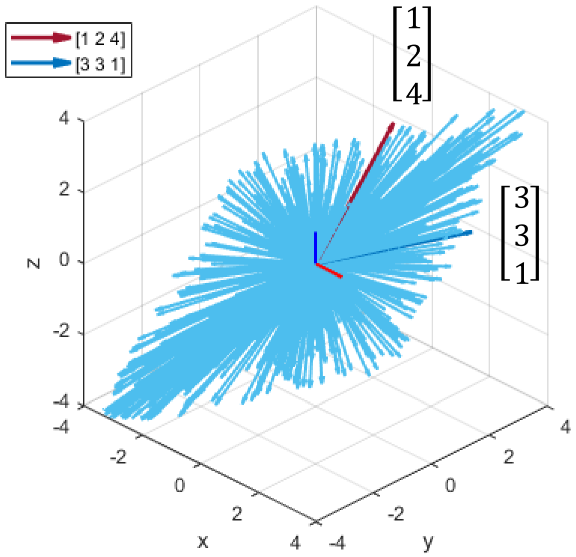
- The Column Space of A

All linear combinations of columns form a subspace of R^m $\Rightarrow$ column space C(**A**)

The column space of **A** is the plane through the origin in R^m containing $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$.

C(A) is somewhere between the zero space and the whole space R^m.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$



2.1 Vector Spaces and Subspaces

- The Column Space of A: Solving **Ax = b**

Does **Ax = b** have a solution for every **b**? No. Why? Independent / Dependent

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \Rightarrow \quad x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\mathbf{A} \quad \mathbf{x} = \mathbf{b} \qquad\qquad \boldsymbol{A_1} \quad \boldsymbol{A_2} \quad \boldsymbol{A_3}$$

C(A) = subspace of R^4
= all linear combinations of columns
$= x_1\, \boldsymbol{A_1} + x_2\, \boldsymbol{A_2} + x_3\, \boldsymbol{A_3}$
= **b** Can solve **Ax = b** when **b** is in C(**A**)

if **b** ∈ C(**A**), there are solutions.
else if **b** ∉ C(**A**), no solution.

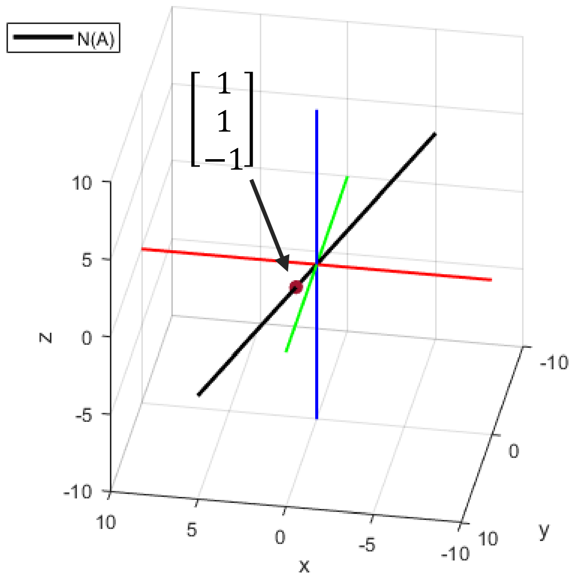
2.1 Vector Spaces and Subspaces

- The Nullspace of A: Solving $\mathbf{Ax} = \mathbf{0}$

All solutions $\mathbf{x}$ to $\mathbf{Ax} = \mathbf{0}$ form a subspace of R^n $\Rightarrow$ Nullspace N(**A**)

$$\underset{\mathbf{A}}{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}} \underset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\mathbf{0}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \ldots$$

$$\Rightarrow \quad \mathbf{x} = c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$



2.1 Vector Spaces and Subspaces

- The Nullspace of A

Requirement:

(i) If $\mathbf{Av} = \mathbf{0}$ and $\mathbf{Aw} = \mathbf{0}$ then $\mathbf{A(v + w) = 0}$

(ii) If $\mathbf{Ax} = \mathbf{0}$ then $\mathbf{A}(c\mathbf{x}) = \mathbf{0}$

2.1 Vector Spaces and Subspaces

- The Subspace of Solutions When $\mathbf{b} \neq \mathbf{0}$

Do the solutions $\mathbf{x}$ form a vector space? No.

$$\underset{\mathbf{A}}{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}} \underset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\mathbf{b}}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Why? The zero vector is not a solution.

A plane/line that doesn't go through the origin.

2.2 Solving $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax} = \mathbf{b}$

U: Echelon form
R: Reduced row echelon form

$$\underset{m \times n}{\mathbf{A}} \; \mathbf{x} = \mathbf{b}$$

Rectangular Matrix

$\mathbf{A} \rightarrow \mathbf{U} \rightarrow \mathbf{R}$

$\xrightarrow[\text{Elimination}]{\text{Gaussian}} \mathbf{x}$

- unique solution
- infinitely many solution
- no solution

Complete solution $\quad \mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$

$$\mathbf{A}\,\mathbf{x}_p = \mathbf{b} \quad \text{and} \quad \mathbf{A}\,\mathbf{x}_n = \mathbf{0} \quad \Longrightarrow \quad \mathbf{A}\,(\mathbf{x}_p + \mathbf{x}_n) = \mathbf{b}$$

$\mathbf{x}_p$: particular solution $\qquad \mathbf{x}_n$: special solution

2.2 Solving **Ax = 0** and **Ax = b**

Rank of A = # of pivots

- Computing The Nullspace ($\mathbf{Ax} = \mathbf{0}$)

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

Dependent

$\text{col}_2 = 2\,\text{col}_1$

$\text{row}_3 = \text{row}_1 + \text{row}_2$

$\text{row}_2 = \text{row}_2 - 2\,\text{row}_1$

$$= \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$\text{row}_3 = \text{row}_3 - 3\,\text{row}_1$

$$= \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$\text{row}_3 = \text{row}_3 - \text{row}_2$

$$= \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & \boxed{0} \end{bmatrix} = \mathbf{U} \text{ (echelon form)}$$

free columns

$$\mathbf{U} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

pivot variables: x_1, x_3

free variables: x_2, x_4

$$\mathbf{x} = x_2 \begin{bmatrix} \square \\ \square \\ \square \\ \square \end{bmatrix} + x_4 \begin{bmatrix} \square \\ \square \\ \square \\ \square \end{bmatrix}$$

2.2 Solving $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax} = \mathbf{b}$

Rank of A = # of pivots

- Computing The Nullspace ($\mathbf{Ax} = \mathbf{0}$)

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\ 2x_3 + 4x_4 &= 0 \end{aligned}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ 1 & 0 \end{array}$$

free columns

$$\mathbf{U} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

pivot variables: x_1, x_3

free variables: x_2, x_4

$$\mathbf{Ax} = \mathbf{Ux} \longleftarrow \mathbf{x} = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2.2 Solving **Ax = 0** and **Ax = b**

Rank of A = # of pivots

- Computing The Nullspace (**Ax = 0**)

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\ 2x_3 + 4x_4 &= 0 \end{aligned}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ 0 & 1 \end{array}$$

free columns

$$\mathbf{U} = \begin{array}{c} \begin{array}{cccc} & \downarrow & & \downarrow \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & & \uparrow & \end{array} \end{array}$$

pivot columns

pivot variables: x_1, x_3
free variables: x_2, x_4

$$\mathbf{Ax} = \mathbf{Ux} \longleftarrow \mathbf{x} = d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

2.2 Solving **Ax = 0** and **Ax = b**

Rank of A = # of pivots

- Computing The Nullspace ($\mathbf{Ax} = \mathbf{0}$)

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

free columns

$$\begin{array}{cc} & \begin{array}{cccc} & \downarrow & & \downarrow \end{array} \\ \mathbf{U} = & \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & \begin{array}{cccc} \uparrow & & \uparrow & \end{array} \end{array}$$

pivot columns

Nullspace of **A** & **U**

: All linear combination of special solution vectors

$$\mathbf{x} = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

A: $m \times n$

r = rank of A = # of pivot variables

$n - r$ = # of free variables = # of special solutions

2.2 Solving **Ax = 0** and **Ax = b**

- R = Reduced Row Echelon Form

  pivots = 1 and zeros above and below pivots

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{row}_1 = \text{row}_1 - \text{row}_2]{} \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{row}_2 = \text{row}_2 \,/\, \text{pivot } 2]{} \overset{\mathbf{R}}{\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\begin{array}{cc} & \begin{array}{cccc} & \text{free columns} & & \\ & \downarrow & & \downarrow \end{array} \\ \mathbf{R} = & \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & \begin{array}{cccc} \uparrow & & \uparrow & \\ & \text{pivot columns} & & \end{array} \end{array}$$

2.2 Solving **Ax = 0** and **Ax = b**

- R = Reduced Row Echelon Form

  pivots = 1 and zeros above and below pivots

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

free columns

$$\mathbf{F} = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{F} \\ 0 & 0 \end{bmatrix}$$

pivot columns

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If some rows of **A** are linearly dependent, the lower rows of **R** will be filled with zeros.

pivot variables: x_1, x_3

free variables: x_2, x_4

$$\begin{aligned} x_1 + 2x_2 \quad - 2x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -2x_2 + 2x_4 \\ x_3 &= \quad\quad -2x_4 \end{aligned}$$

$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

coefficient

$= -\mathbf{F}$

$= \mathbf{I}$

2.2 Solving $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax} = \mathbf{b}$

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $\mathbf{Ax} = \mathbf{Ux} = \mathbf{Rx} = \mathbf{0}$

$$\mathbf{x} = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Ax=0 \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Ux=0 \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$Rx=0 \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2.2 Solving $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax} = \mathbf{b}$

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Solving $\mathbf{Ax} = \mathbf{b}$, $\mathbf{Ux} = \mathbf{c}$, $\mathbf{Rx} = \mathbf{d}$

$\mathbf{Ax_p} = \mathbf{b}$ and $\mathbf{Ax_n} = \mathbf{0}$ $\rightarrow$ $\mathbf{A}(\mathbf{x_p} + \mathbf{x_n}) = \mathbf{b}$

$$(\,\mathbf{A} \mid \mathbf{b}\,) \rightarrow (\,\mathbf{U} \mid \mathbf{c}\,) \rightarrow (\,\mathbf{R} \mid \mathbf{d}\,)$$

Augmented Matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right) \rightarrow \underbrace{\left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right)}_{\text{}}$$

$(\,\mathbf{U} \mid \mathbf{c}\,)$

$\mathrm{row}_2 = \mathrm{row}_2 - 2\ \mathrm{row}_1$
$\mathrm{row}_3 = \mathrm{row}_3 - 3\ \mathrm{row}_1$

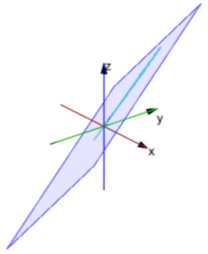
$\mathrm{row}_3 = \mathrm{row}_2 - \mathrm{row}_2$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 3b_1 - b_2 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 3b_1 - b_2 \\ 0 & 0 & 1 & 2 & \frac{1}{2}b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right)$$

$\mathrm{row}_1 = \mathrm{row}_1 - \mathrm{row}_2$

$\mathrm{row}_2 = \frac{1}{2}\ \mathrm{row}_2$

$(\,\mathbf{R} \mid \mathbf{d}\,)$



For solution to exist,
$b_3 - b_2 - b_1 = 0$ and $\mathbf{b} \in C(\mathbf{A})$

2.2 Solving $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax} = \mathbf{b}$

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Solving $\mathbf{Ax} = \mathbf{b}$, $\mathbf{Ux} = \mathbf{c}$, $\mathbf{Rx} = \mathbf{d}$

$$\mathbf{Ax_p} = \mathbf{b} \quad \text{and} \quad \mathbf{Ax_n} = \mathbf{0} \quad \rightarrow \quad \mathbf{A}(\mathbf{x_p} + \mathbf{x_n}) = \mathbf{b}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \qquad \underset{(\mathbf{R} \mid \mathbf{d})}{\left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 3b_1 - b_2 \\ 0 & 0 & 1 & 2 & \frac{1}{2}b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right)} = \left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & -2 \\ 0 & 0 & 1 & 2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

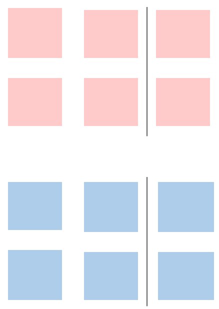
$$\begin{aligned} x_1 + 2x_2 \qquad - 2x_4 &= -2 \\ x_3 + 2x_4 &= 3/2 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} x_1 &= -2x_2 + 2x_4 - 2 \\ x_3 &= -2x_4 + 3/2 \end{aligned}$$

$$\mathbf{x} = \underbrace{x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{x_n} + \underbrace{\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}}_{x_p}$$

$$= x_n + x_p$$



$= -\mathbf{F} \mid \mathbf{d}$

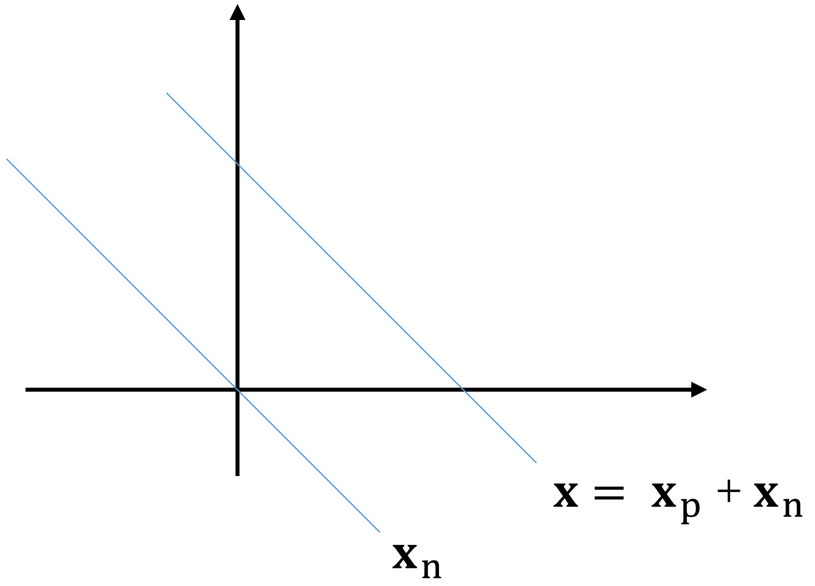
$= \mathbf{I} \mid \mathbf{0}$

2.2 Solving **Ax = 0** and **Ax = b**

- Complete solution $\mathbf{x} = \mathbf{x}_\mathrm{p} + \mathbf{x}_\mathrm{n}$

$\mathbf{x}_\mathrm{p}$: particular solution
$\mathbf{x}_\mathrm{n}$: special solution

$$\mathbf{A}\,\mathbf{x}_\mathrm{p} = \mathbf{b} \quad \text{and} \quad \mathbf{A}\,\mathbf{x}_\mathrm{n} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{A}\,(\mathbf{x}_\mathrm{p} + \mathbf{x}_\mathrm{n}) = \mathbf{b}$$



2.3 Linear Independence, Basis, and Dimension

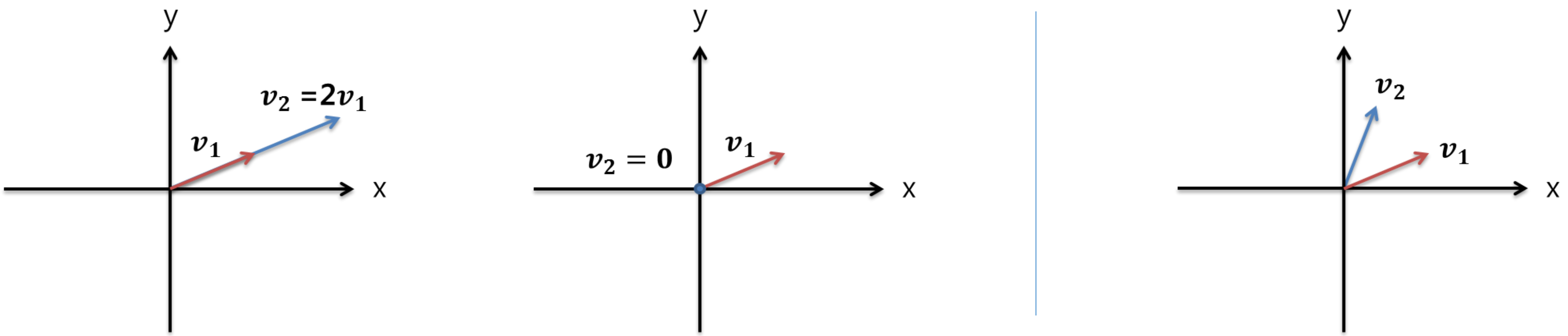
- Linear Independence

$c_1v_1 + c_2v_2 + \cdots + c_nv_n = \mathbf{0} \;\rightarrow\;$ Only $c_1 = c_2 = \cdots = c_n = 0$

If any other combination of the vectors gives zero, they are *dependent*.

The columns of **A** are *independent* exactly when N(**A**) = {zero vector} (except all $c_i = 0$)

dependent *independent*



2.3 Linear Independence, Basis, and Dimension

- Spanning

Set of vectors constructs a vector space by linear combinations

The set of vectors span the vector space

If given linearly independent vectors, $\Rightarrow$ Linear comb. is unique

linearly independent

(1) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

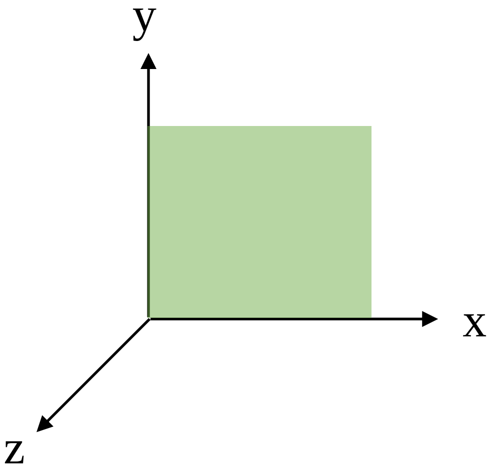
(2) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

linearly dependent

(3) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

	c_1	c_2	c_3
(i)	2	2	0
(ii)	0	0	2

$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$



2.3 Linear Independence, Basis, and Dimension

- Basis

minimum # of vectors to span a vector space

maximum # of linearly independent vectors

A basis for a vector space is a sequence of vectors $v_1, v_2, \ldots, v_\alpha$ with 2 properties:

1) The vectors are linearly independent
2) They span the space

$$\text{One basis for } R^3: \begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix}1\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\1\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{array} \qquad \text{Another basis} \quad \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\4\\8\end{bmatrix}$$

Basis vector is not unique for a vector space

2.3 Linear Independence, Basis, and Dimension

- Dimension of vector space

= # of linearly independent vectors

- Rank of A (r)

= # of independent column vectors

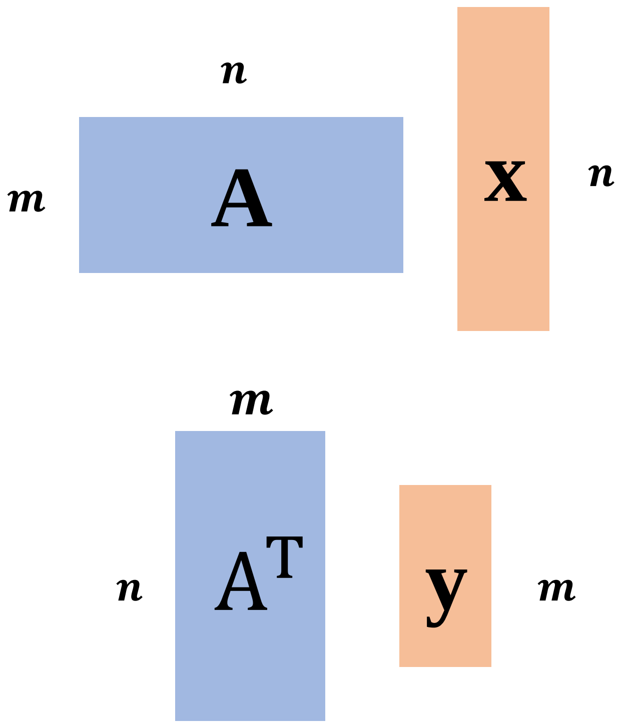
= # of independent row vectors

= # of pivots in Gaussian Elimination

2.4 The Four Fundamental Subspaces

- Column Space C(A): linear comb. of column vectors $\subset R^m$
- Null Space N(A): $\{x \mid Ax = 0\} \subset R^n$
- Row Space $C(A^T)$: linear comb. of row vectors $\subset R^n$
- Left Null Space $N(A^T)$: $\{y \mid A^T y = 0\} \subset R^m$

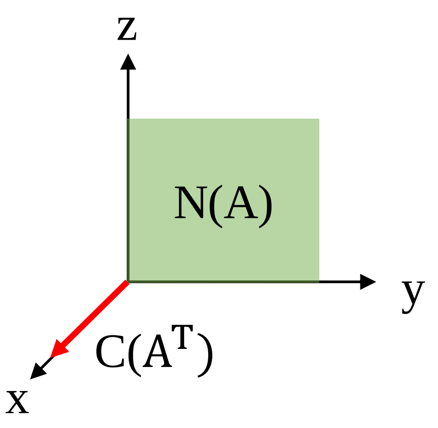
$$y^T A = 0^T$$



N(A) & $C(A^T)$ is subspace of R^n
$\rightarrow N(A) \perp C(A^T)$

C(A) & $N(A^T)$ is subspace of R^m
$\rightarrow C(A) \perp N(A^T)$

2.4 The Four Fundamental Subspaces



N(A) & C(A^T) is subspace of R^n
→ N(A) ⊥ C(A^T)

C(A) & N(A^T) is subspace of R^m
→ C(A) ⊥ N(A^T)

$$\boldsymbol{A = U = R =} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$ **일 때,**

$Dim(C(A)) = 1$ $\quad \boldsymbol{C(A)} = c\begin{bmatrix}1\\0\end{bmatrix}$, line in R^2 (xy평면의 x축)

$Dim\left(C(A^T)\right) = 1$ $\quad \boldsymbol{C(A^T) = c}\begin{bmatrix}1\\0\\0\end{bmatrix}$, line in R^3

$Dim(N(A)) = 2$ $\quad \boldsymbol{N(A) = c_1}\begin{bmatrix}0\\1\\0\end{bmatrix} + \boldsymbol{c_2}\begin{bmatrix}0\\0\\1\end{bmatrix}$, yz 평면

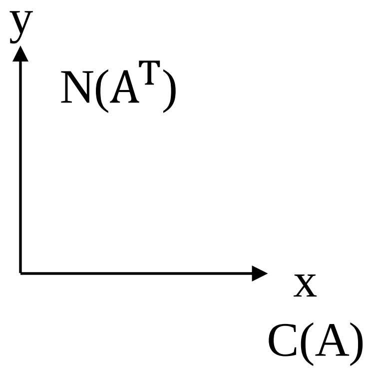
$Dim\left(N(A^T)\right) = 1$ $\quad \boldsymbol{N(A^T) = c}\begin{bmatrix}0\\1\end{bmatrix}$, line in R^2

$$\boldsymbol{Dim\left(C(A^T)\right) + Dim(N(A)) = n}$$

r m - r

$$\boldsymbol{Dim(C(A)) + Dim\left(N(A^T)\right) = m}$$

r n - r

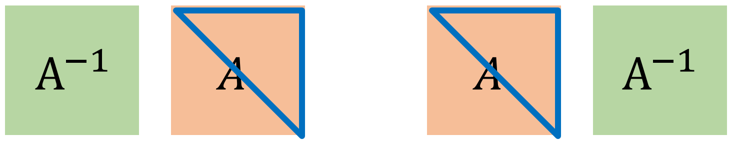


2.4 The Four Fundamental Subspaces

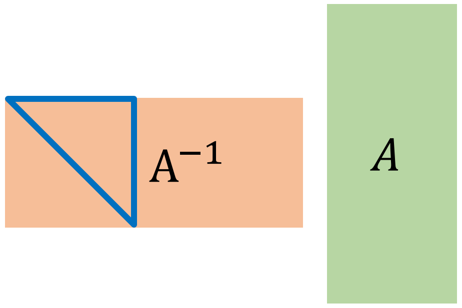
- Existence of Inverses

An inverse exists only when the rank is as large as possible.

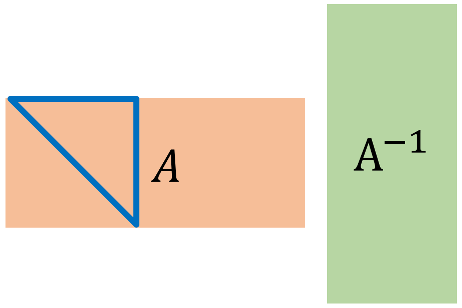
1) Two-sides inverse : $A^{-1}A = A\,A^{-1} = I$ $\quad$ r = m = n (Square case)
2) Left inverse : $A^{-1}A = I_{n \times n}$ $\quad$ r = n (m ≥ n)
3) Right inverse : $A\,A^{-1} = I_{m \times m}$ $\quad$ r = m (m ≤ n)



Two-sides inverse
(unique solution)



Left inverse
(unique solution)



Right inverse
(infinitely many solution)

2.4 The Four Fundamental Subspaces

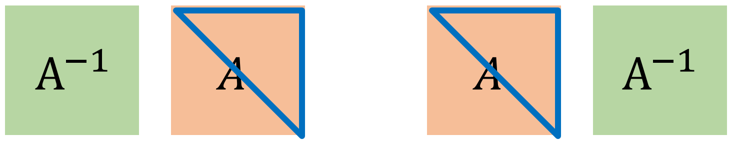
- Existence of Inverses

An inverse exists only when the rank is as large as possible.

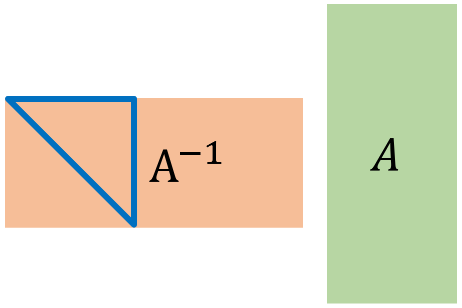
$$Ax = b$$

Left inverse : $A^{-1}Ax = A^{-1}b$ $\quad$ $x = A^{-1}b$ unique $\quad$ (# of eqns > # of unknowns)

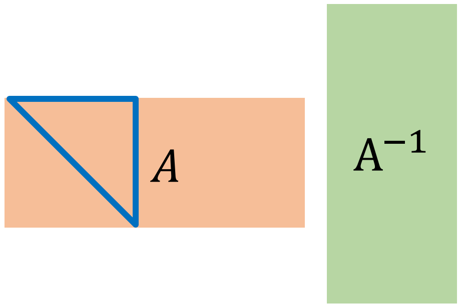
Right inverse : $A\,A^{-1}x = A^{-1}b$ $\quad$ $x =$ infinitely many solution



Two-sides inverse
(unique solution)



Left inverse
(unique solution)



Right inverse
(infinitely many solution)

Homework

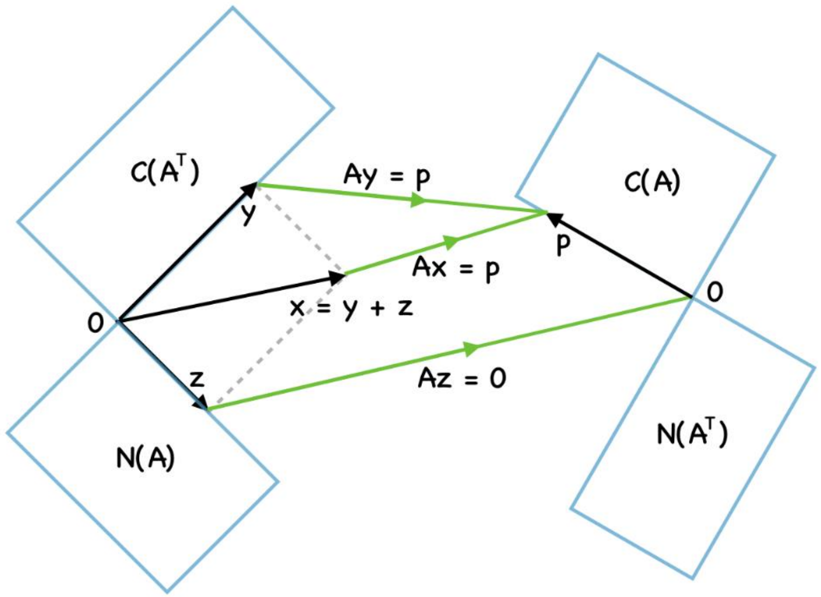
Summary

- A Geometric View of Gauss – Jordan Elimination

$$\mathbf{Ax = Ux = Rx = b}$$

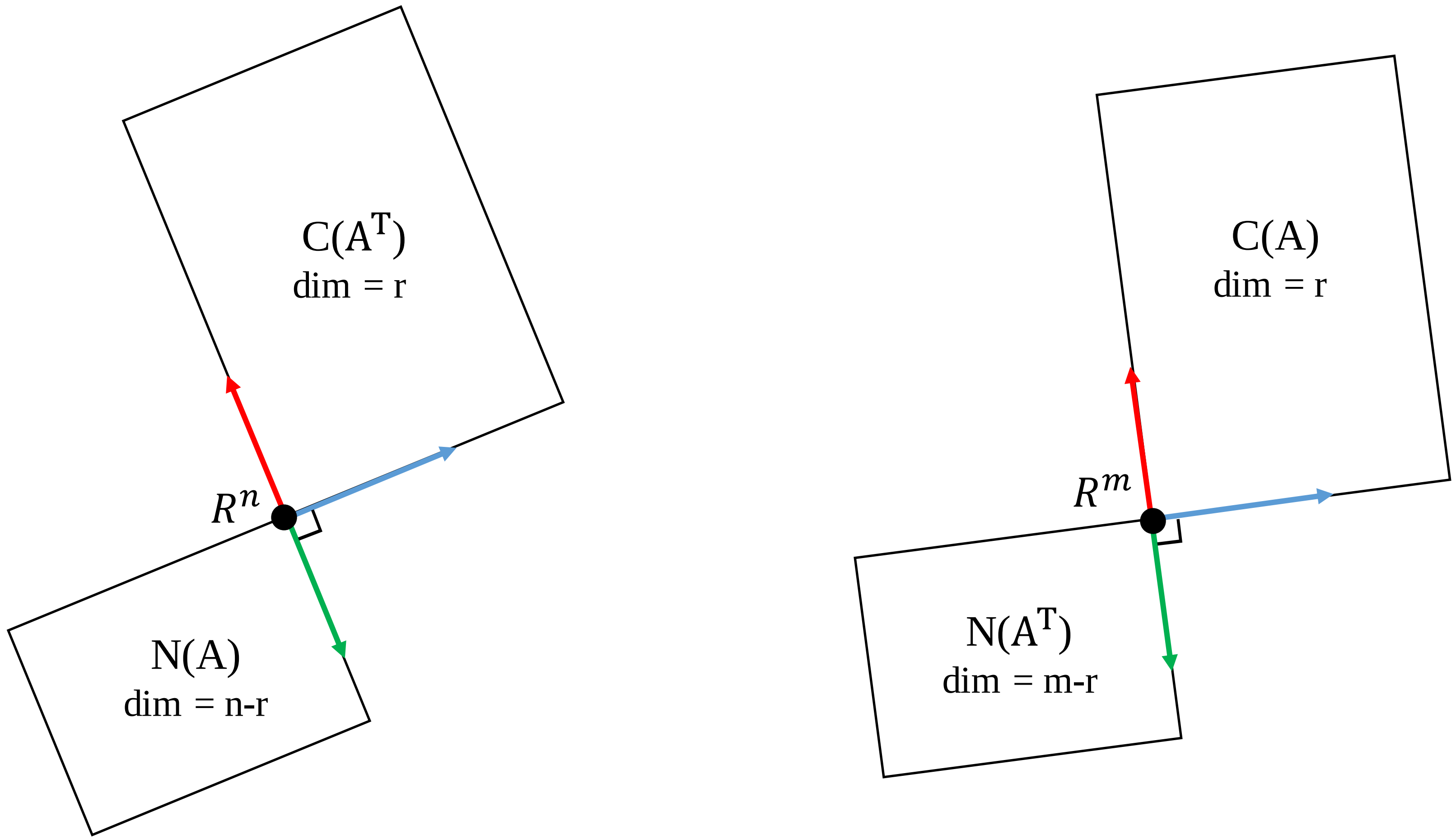
- 4 Fundamental Subspace 변환식 관계 설명

Column Space	$C(A) \subset R^m$
Null Space	$N(A) \subset R^n$
Row Space	$C(A^T) \subset R^n$
Left Null Space	$N(A^T) \subset R^m$



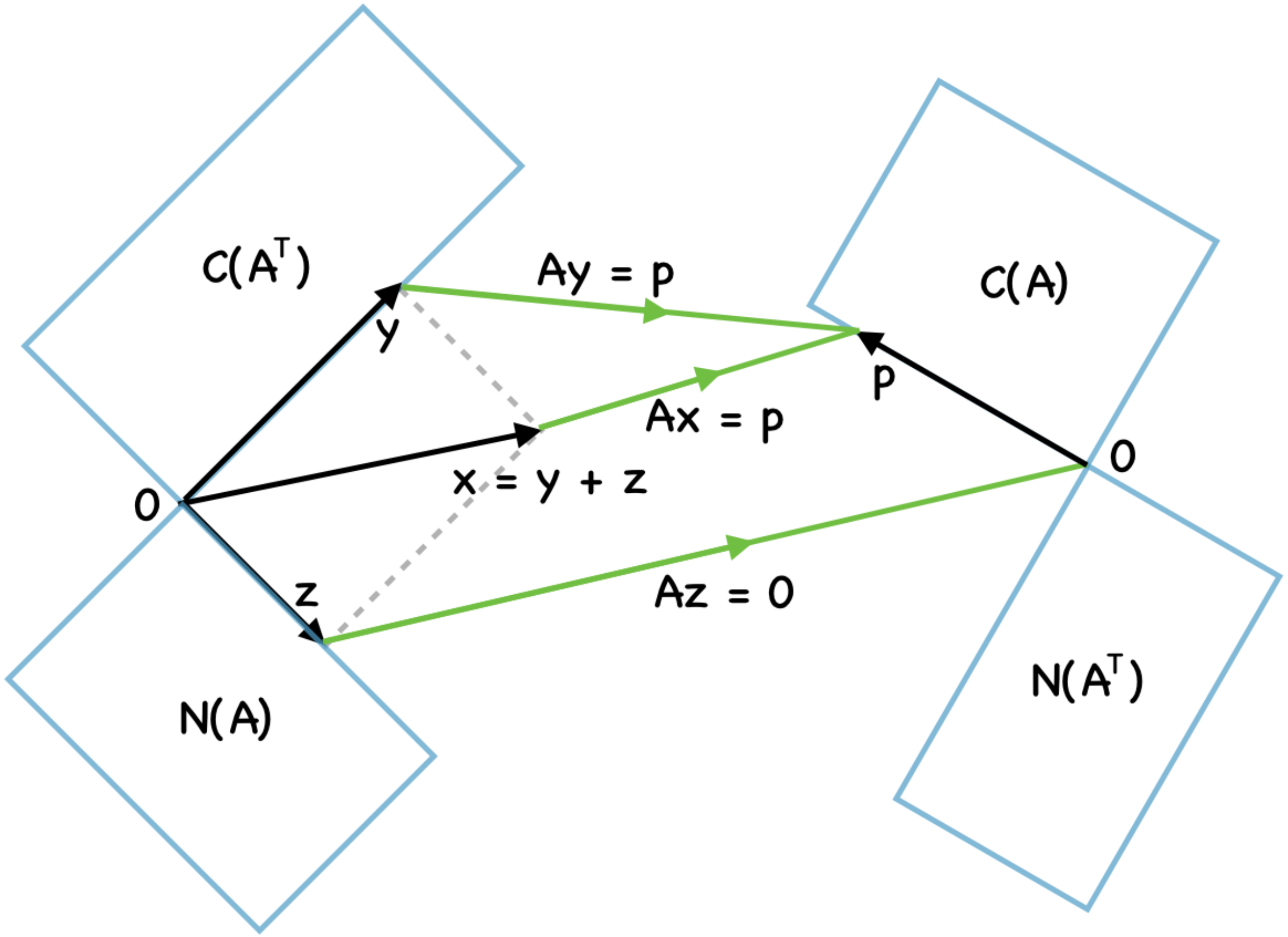
Homework

Column Space	$C(A) \subset R^m$
Null Space	$N(A) \subset R^n$
Row Space	$C(A^T) \subset R^n$
Left Null Space	$N(A^T) \subset R^m$



Homework

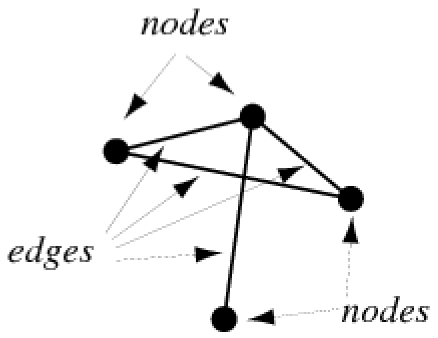
Column Space $C(A) \subset R^m$
Null Space $N(A) \subset R^n$
Row Space $C(A^T) \subset R^n$
Left Null Space $N(A^T) \subset R^m$

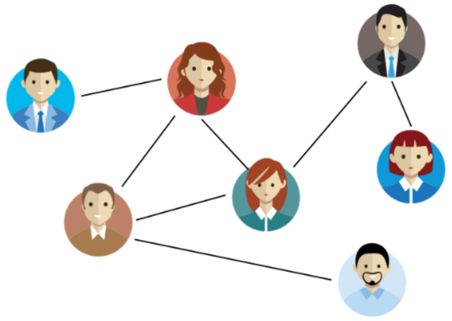


2.5 Graphs and Networks

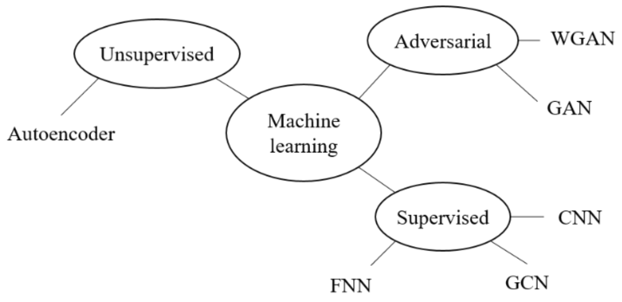
- Graph

A graph is mathematical structures used to model pairwise relations between objects. A graph consists of a set of vertices or *nodes*, and a set of *edges* that connect them.

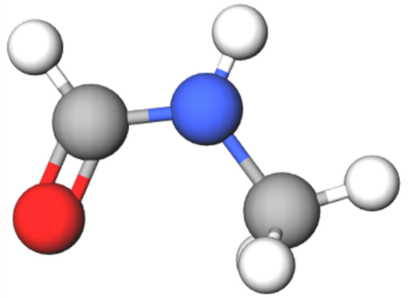




(a) 소셜 네트워크



(b) 관계형 데이터베이스

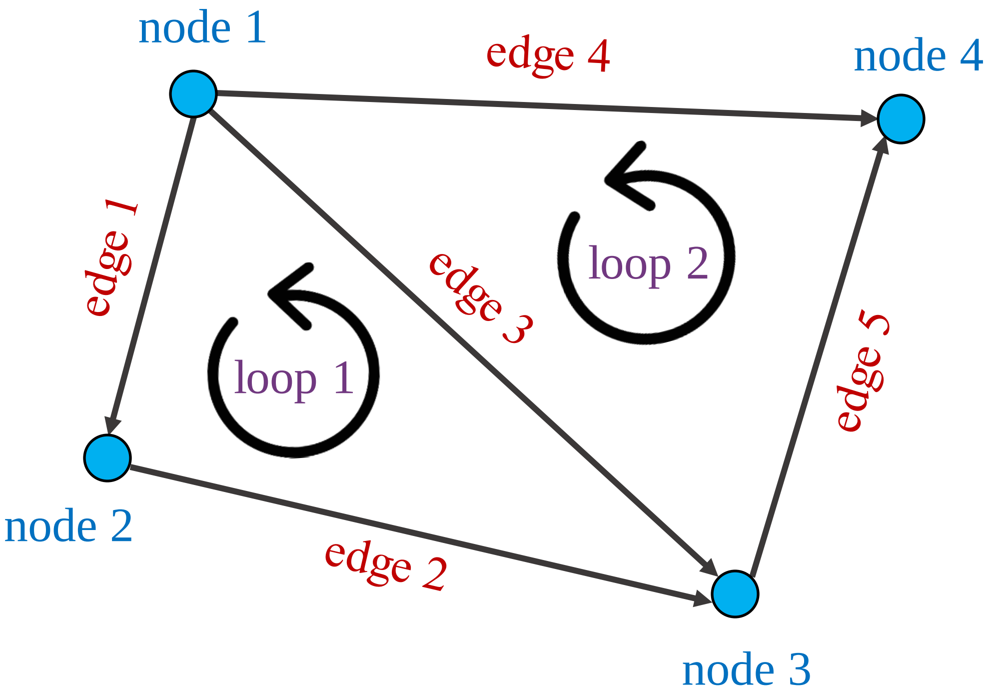


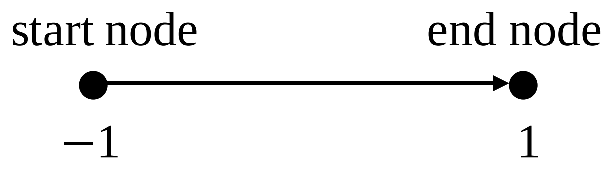
(c) 분자 구조

2.5 Graphs and Networks

- Graph = {nodes, edges} → Incidence Matrix

n = 4 m = 4





node 1	node 2	node 3	node 4		dependent
−1	1	0	0	edge 1	
0	−1	1	0	edge 2	loop 1
−1	0	1	0	edge 3	
−1	0	0	1	edge 4	loop 2
0	0	−1	1	edge 5	

2.5 Graphs and Networks

column vectors
$\longrightarrow$ independent
$N(A) = \{ \text{zero vector} \}$
$\longrightarrow$ dependent
$N(A) = \{ x \mid Ax = 0 \}$: set of solutions

- Null Space of A $\Rightarrow$ column vectors: Independent / Dependent

potentials at nodes

$$\mathbf{Ax} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

potentials differences between nodes

$$\mathbf{x} = c \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$\dim(N(A)) = n - \text{rank}(A) = 1$
$\text{rank}(A) = 3$

2.5 Graphs and Networks

- Null Space of A

$$\mathbf{Ax} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{Ax} = x_1 \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

dim(N(A)) = n – r = 1

r = dim(C(A)) = # of pivot = 3

$x_4 = 0$, ground

2.5 Graphs and Networks

- Null Space of A^T $\quad N(A^T)$: $\{y \mid A^T y = 0\} \Rightarrow$ Kirchhoff's current law

$$\mathbf{A^T y} = \begin{pmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\uparrow$
current on edges

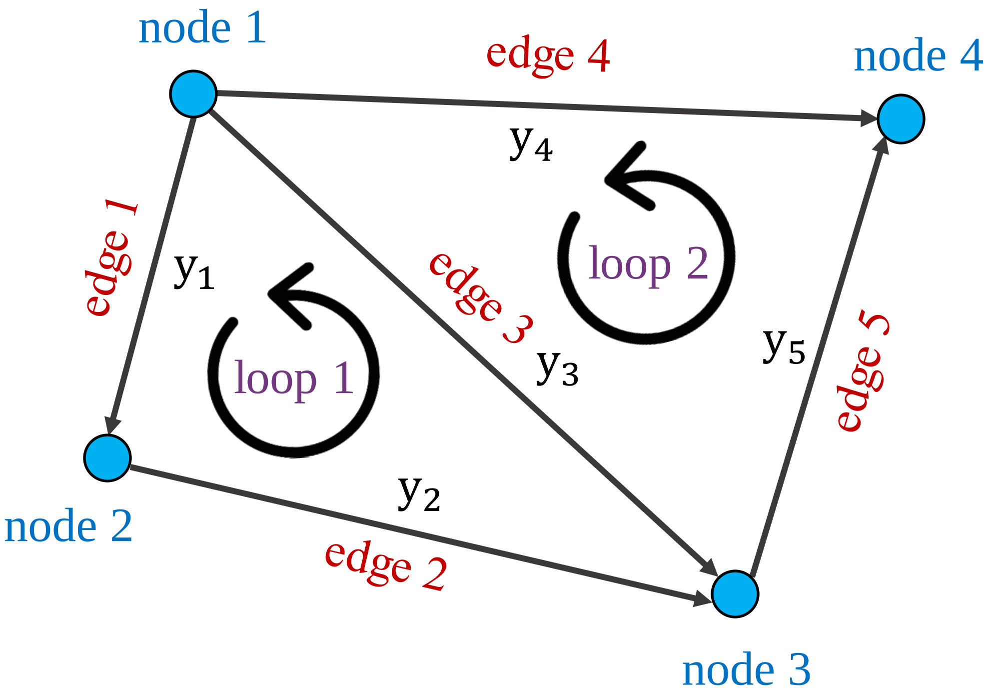
$$\underset{5 \times 4}{\mathbf{A}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$\dim(C(A)) = r = 3$
$\dim(N(A)) = n - r = 1$

$\dim(C(A^T)) = r = 3$ = # pivot columns
$\dim(N(A^T)) = m - r = 2$ = # free columns

2.5 Graphs and Networks

- Null Space of A^T

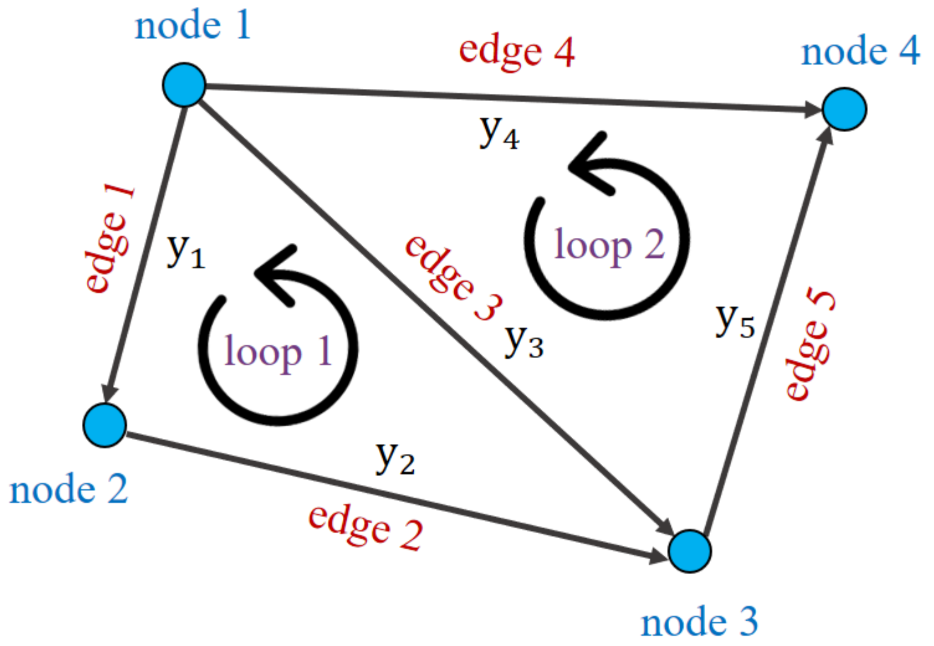


$$\mathbf{A}^{\mathbf{T}}\mathbf{y} = \begin{pmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{array}{l} \text{node 1} \\ \text{node 2} \\ \text{node 3} \\ \text{node 4} \end{array} \begin{pmatrix} -y_1 - y_3 - y_4 \\ y_1 - y_2 \\ y_2 + y_3 - y_5 \\ y_4 + y_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Kirchhoff's Current Law
All the currents at a node sum to zero

2.5 Graphs and Networks



- Null Space of A^T

$$\mathbf{A}^T\mathbf{y} = \begin{pmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} -y_1 - y_3 - y_4 \\ y_1 - y_2 \\ y_2 + y_3 - y_5 \\ y_4 + y_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

How about big loop? $y_1 \to y_2 \to y_5 \to y_4$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

dependent
$\Rightarrow$ not basis

Basis for $N(A^T)$ $\underset{\text{loop 1}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\text{loop 2}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}}$

loop 1: y_1, y_2, y_3
node 1: $-y_1 - y_3 = 0$

loop 2: y_3, y_4, y_5
node 1: $-y_3 - y_4 = 0$

$\dim(C(A)) = r = 3$
$\dim(N(A)) = n - r = 1$

$\dim(C(A^T)) = r = 3$ $= \#$ of pivot vars
$\dim(N(A^T)) = m - r = 2$ $= \#$ of free vars $= \#$ of special solutions
$= \#$ loop

2.5 Graphs and Networks

- Column Space of A^T = Row Space

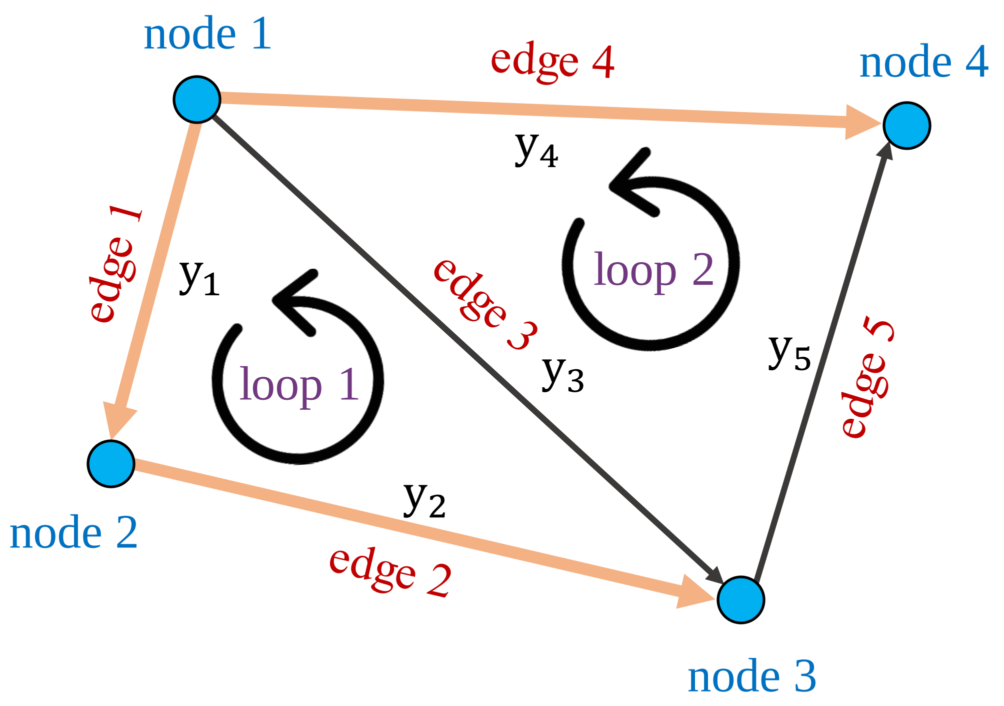
r = # pivot columns = 3

$$\mathbf{A^T y} = \begin{pmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}$$

$\uparrow \quad \uparrow \quad \quad \uparrow$

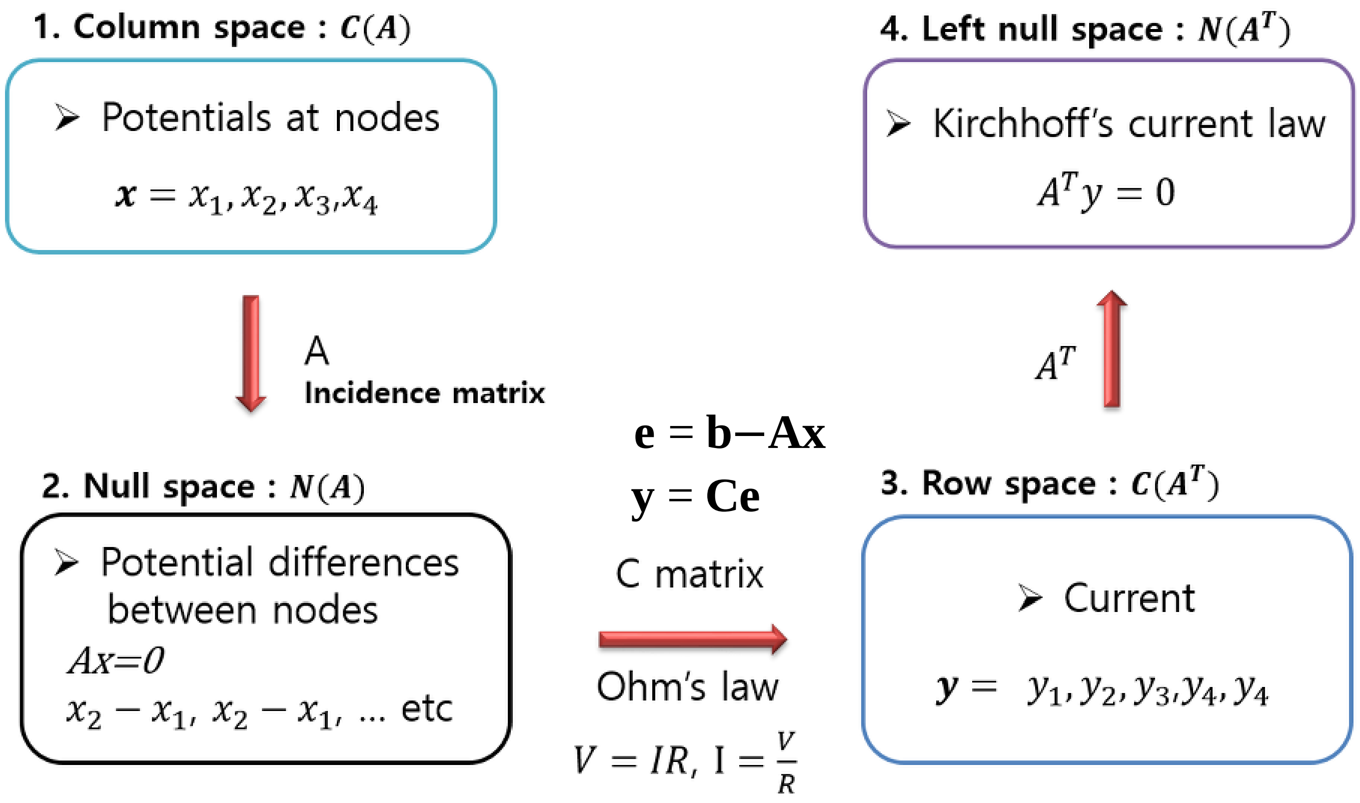
pivot columns

Tree = A graph without loops



2.5 Graphs and Networks

- Relationship between graph and the four fundamental subspaces



2.5 Graphs and Networks

- Euler's Formula

$$\underbrace{(\text{\# of nodes})}_{n} - \underbrace{(\text{\# of edges})}_{m} + (\text{\# of loops}) = 1$$

Null Space:	dim = n – r, contains x
Column Space:	dim = r = # of independent columns
Row Space:	dim = r = # of independent rows from any spanning tree
Left Null Space:	dim = m – r, contains y's from the loops

2.6 Linear Transformations

- Review

$$\mathbf{Ax = b}$$

Row picture:	system equation
	x = solution
Column picture:	linear combination of column vectors
	x = scalar coefficients

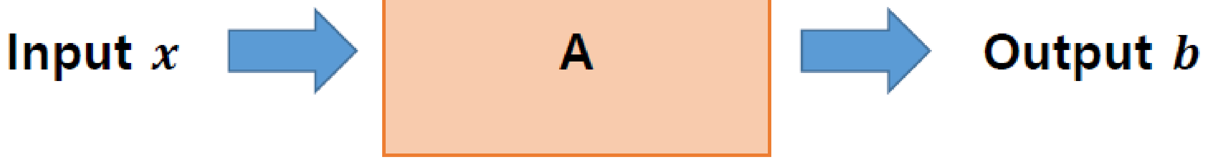
2.6 Linear Transformations

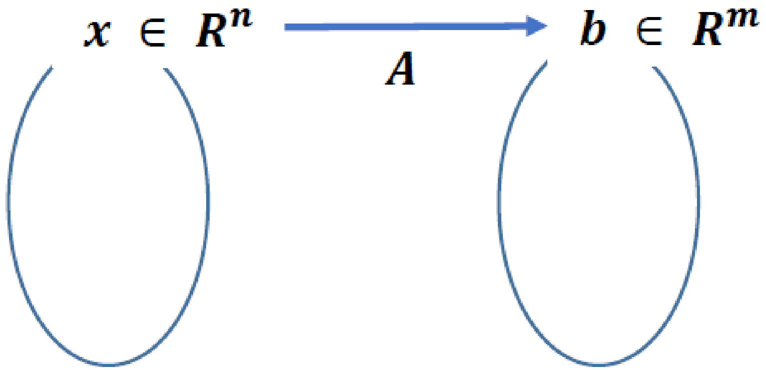
- Transformation T : $R^n \rightarrow R^m$

  mapping, function

$$\overset{(m\times n)}{\mathbf{A}}\,\overset{(n\times 1)}{\mathbf{x}} = \overset{(m\times 1)}{\mathbf{b}}$$

$$R^n \qquad\qquad R^m$$





2.6 Linear Transformations

- A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in the domain of T

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all **u** in the domain of T

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

$T(\mathbf{0}) = T(0\mathbf{u}) = 0\,T(u) = \mathbf{0}$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

- Superposition principle $T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_nT(\mathbf{v}_n)$

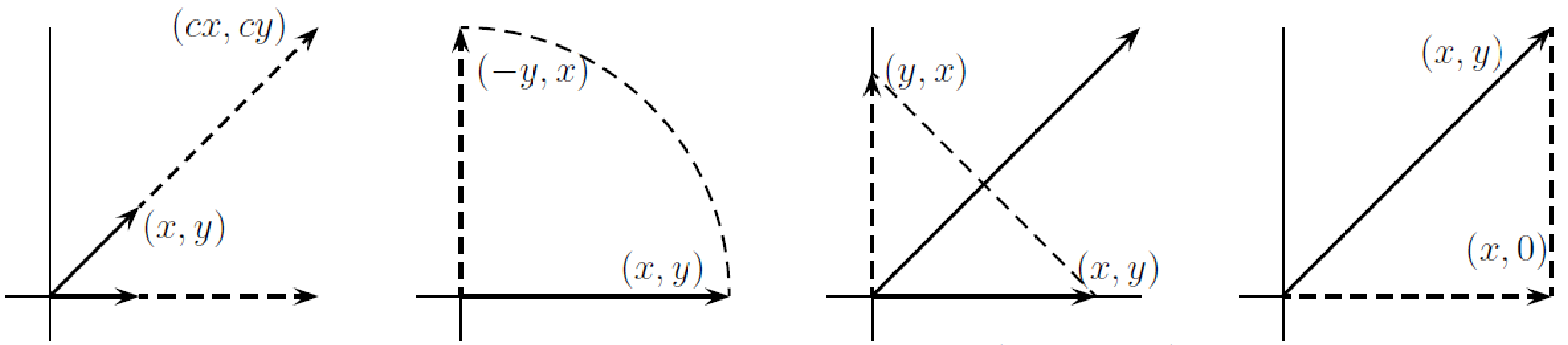
2.6 Linear Transformations

- Matrix Transofrmation

T(**x**) = **Ax**

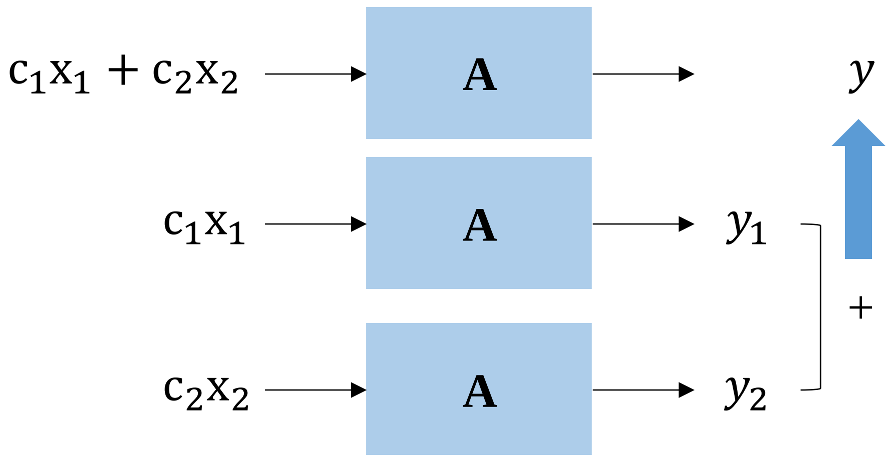
A(c**u** + d**v**) = c**Au** + d**Av** **Linear Transformation**

$$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



stretching / contracting — 90° rotation — reflection (45° mirror) — projection on axis

2.6 Linear Transformations



- The Matrix of A Linear Transformation

If given Ax for every basis vetors x,

$$\mathbf{A}\mathbf{x}_1 = \boldsymbol{a}_1, \qquad \mathbf{A}\mathbf{x}_2 = \boldsymbol{a}_2, \qquad \ldots, \qquad \mathbf{A}\mathbf{x}_n = \boldsymbol{a}_n$$

then, we can find any transform results in the vector space without **A**.

Linearity If $x = c_1 x_1 + \cdots + c_n x_n$ then $Ax = c_1(Ax_1) + \cdots + c_n(Ax_n).$

$$= c_1\,\boldsymbol{a}_1 + \cdots + c_n\boldsymbol{a}_n$$

2.6 Linear Transformations

- The Matrix of A Linear Transformation

Let T: $R^n \rightarrow R^m$ be a linear transformation. Then there exists a unique matrix **A** such that

$$\mathrm{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } R^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $\mathrm{T}(\boldsymbol{e}_j)$, where $\boldsymbol{e}_j$ is the jth column of the identity matrix in R^n:

$$\mathbf{A} = \begin{bmatrix} | & | & | & | \\ \mathrm{T}(\boldsymbol{e}_1) & \mathrm{T}(\boldsymbol{e}_2) & \cdots & \mathrm{T}(\boldsymbol{e}_n) \\ | & | & | & | \end{bmatrix}$$

2.6 Linear Transformations

- Standard Matrix of A Linear Transformation

If we know the **Ax** for the standard basis vectors,

$$\mathbf{x} = \mathbf{I}_\mathrm{n}\mathbf{x} = [\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \cdots \;\; \mathbf{e}_n]\,\mathbf{x} = \mathbf{x}_1\boldsymbol{e}_1 + \mathbf{x}_2\boldsymbol{e}_2 + \cdots + \mathbf{x}_n\boldsymbol{e}_n$$

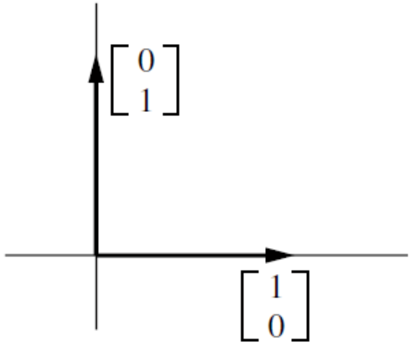
$$\mathbf{e} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\begin{aligned}
\mathrm{T}(\mathbf{x}) &= \mathrm{T}(\mathbf{x}_1\boldsymbol{e}_1 + \mathbf{x}_2\boldsymbol{e}_2 + \cdots + \mathbf{x}_n\boldsymbol{e}_n) \\
&= \mathbf{x}_1\mathrm{T}(\boldsymbol{e}_1) + \mathbf{x}_2\mathrm{T}(\boldsymbol{e}_2) + \cdots \mathbf{x}_n\mathrm{T}(\boldsymbol{e}_n) \\
&= \begin{bmatrix} | & | & & | \\ \mathrm{T}(\boldsymbol{e}_1) & \mathrm{T}(\boldsymbol{e}_2) & \cdots & \mathrm{T}(\boldsymbol{e}_n) \\ | & | & & | \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} \\
&= \mathbf{Ax}
\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} | & | & | & | \\ \mathrm{T}(\boldsymbol{e}_1) & \mathrm{T}(\boldsymbol{e}_2) & \cdots & \mathrm{T}(\boldsymbol{e}_n) \\ | & | & | & | \end{bmatrix}$$ Standard Matrix

2.6 Linear Transformations

- The Matrix of A Linear Transformation



Ex) The columns of $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose T is a linear transformation from R^2 into R^3 such that

$$\mathrm{T}(\boldsymbol{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \text{ and } \mathrm{T}(\boldsymbol{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

with no additional information, find a matrix **A**.

$\mathrm{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{x}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{x}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{x}_1 \boldsymbol{e}_1 + \mathbf{x}_2 \boldsymbol{e}_2$$

$$\mathrm{T}(\mathbf{x}) = \mathrm{T}(\mathbf{x}_1 \boldsymbol{e}_1) + \mathrm{T}(\mathbf{x}_2 \boldsymbol{e}_2) = \mathbf{x}_1 \mathrm{T}(\boldsymbol{e}_1) + \mathbf{x}_2 \mathrm{T}(\boldsymbol{e}_2) = \mathbf{x}_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + \mathbf{x}_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \overset{\mathbf{A}}{\begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

2.6 Linear Transformations

- The Matrix of A Linear Transformation

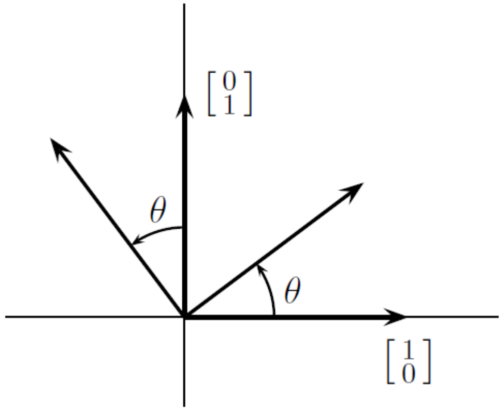
Ex) Find the standard matrix **A** for the dilation transformation T(**x**) = 3**x** for **x** in R^n

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathrm{T}(\boldsymbol{e}_1) = 3\boldsymbol{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \text{ and } \mathrm{T}(\boldsymbol{e}_2) = \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

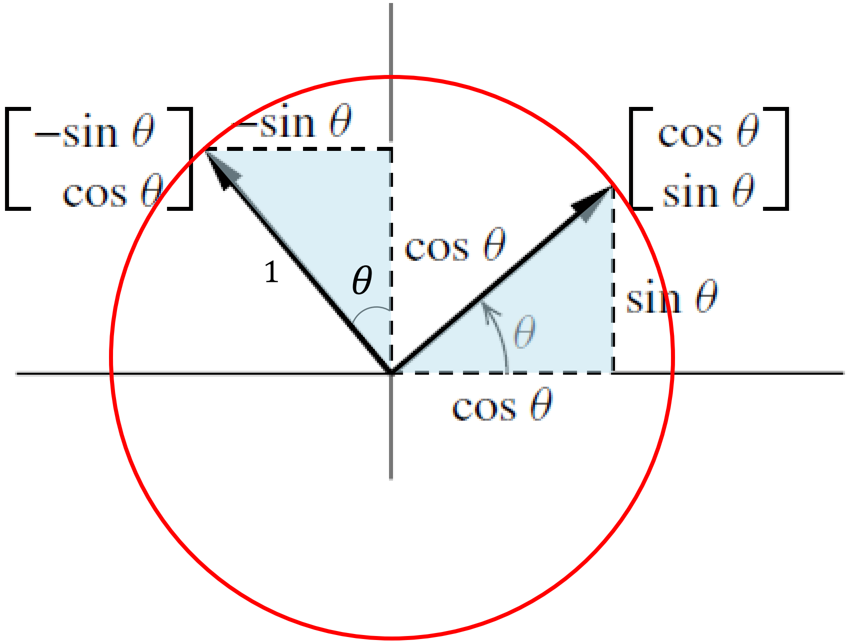
$$\mathbf{A} = \begin{bmatrix} | & | \\ \mathrm{T}(\boldsymbol{e}_1) & \mathrm{T}(\boldsymbol{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

2.6 Linear Transformations



- The Matrix of A Linear Transformation

Ex) Let T: $R^2 \rightarrow R^2$ be the transformation that rotates each point in R^2 about the origin through an angle θ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. Find the standard matrix A of this transformation.



$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{T}(\boldsymbol{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\mathrm{T}(\boldsymbol{e}_1) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\mathbf{A} = [\mathrm{T}(\boldsymbol{e}_1) \quad \mathrm{T}(\boldsymbol{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$