Discrete Random Variables

Date

Name: Chong-kwon Kim

To Learn

- Concept of random variable
- Expectation
- Conditional expectation
- Several important discrete random variables (distribution)
  - Bernoulli
  - Binomial
  - Geometric
  - Poisson

Random Variable

- A **Random Variable** X is a real-valued function defined on sample space

  X: Ω → **R**

- Discrete random variable
  - Takes finite or countably infinite number of values
- Continuous random variable

- For a discrete rv X and value *a*
  - "X=*a*" is a set of the basic events in the sample space in which X is *a*
  - Set {s ∈ Ω | X(s) = a}
  - Pr(X = a) = $\sum_{s:X(s)=a} \Pr(s)$

Examples

- Flip a coin three times
- Ω = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

- Define X = Number of Heads in the three trials
  - X = 1, {HTT, THT, TTH} → Pr(X = 1)=3/8
  - X ≤ 1, {TTT, HTT, THT, TTH} → Pr(X ≤ 1)=1/2

X(HHH) = 3
X(HTH) = 2

- On the same sample space, we define X= # Heads - # Tails
  - X=-1, {HTT, TTH, THT} → Pr(X=-1)=3/8

X(HHH) = ?
X(HTH) = ?

Examples

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- Coin flips, X = Number of flips until the first heads

  H ➔ X = 1

  TH ➔ X = 2

  TTH ➔ X = 3

  …

  Pr(X=n) = ?

- Flip a coin N times, X = Number of heads in N trials

  HTTH ➔ X = 2

  Pr(X=k) = ?

- # babies born in a day, X = Number of babies born on June 15

  Pr(X=k) = ?

Independent Random Variable

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- Definition: Two random variables X and Y are independent iff

  Pr((X=a) ∩ (Y=b)) = Pr(X=a) Pr(Y=b) for all a and b

- Random variables X_1, X_2,…, X_k are independent iff

  for all subset I ⊆ [1,k] and any values x_i, i ∈ I

  $\Pr(\bigcap_{i\in I}(X_i = x_i)) = \prod_{i\in I} \Pr(X_i = x_i)$

Expectation

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- E[X]: Expectation of a rv X

  E[X] = $\sum_i x_i \cdot \Pr(X = x_i)$

  - Weighted average of values that the rv has
  - Weight: probability that the rv has the value

- Examples
  - Flip a coin three times
  - Ω = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
  - Define X = Number of Heads
    - E[X] = 0·Pr(X=0) + 1·Pr(X=1) + 2·Pr(X=2) + 3·Pr(X=3)
  - On the same sample space, we define X= # Heads - # Tails
    - E[X] = -3·Pr(X=-3) + … + 3·Pr(X=3)

Notations:
p(a) = Pr(X=a),
p_i= Pr(X=x_i)

Beating Casinos

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- One famous strategy to beat Casinos is **double betting**
  - Suppose you win \$Y with probability 2/5 and lose \$Y with 3/5 probability
  - Start from Y=1, every time you lose, double the bet
    1. Y=\$1
    2. Bet Y
    3. If Win, Stop
    4. If Loss, Y=2*Y and goto 2
  - Z: Result at the stop
  - E[Z]=(2/5)1 + (3/5)(2/5)(2-1) + (3/5)(3/5)(2/5)(4-2-1) + …

    = $\sum_{i=0}^{\infty} \left(\frac{3}{5}\right)^i \cdot \left(\frac{2}{5}\right) \cdot 1$

- E[Z] $\geq$ 0

Unbounded Expectation

Several (Most) random variables have bounded expectations
Some has unbounded expectations and/or variances
Ex: **Power Law** distribution

Daniel Bernoulli was a Dutch born Swiss mathematician, one of many in his family.

- St. Petersburg Paradox
  Daniel or Nicolas Bernoulli
  - A player flips a fair coin repeatedly until the first tails comes up
  - If the first tails comes up at the i-th flip, then the player receives $\$2^i$
  - How much will you pay to enter the game?
  - X: Your winnings
  - E[X] = $(1/2) \cdot 2^1 + (1/2)^2 \cdot 2^2 + (1/2)^3 \cdot 2^3 + \cdots$
    = $\sum 1 = \infty$

Expansion

- Let X = # Heads - # Tails in flipping a fair coin three times
  - Pr(X = -3) = 1/8, Pr (X = -1) = 3/8, Pr (X = 1) = 3/8, Pr(X = 3) = 1/8
  - Compute E[X^2]
- One solution **E[X^2] = $\sum_i x_i{}^2$Pr$(X = x_i)$**
  - E[X^2] = $(-3)^2$Pr(X=-3) + $(-1)^2$Pr(X=-1) +1^2Pr(X=1) + 3^2Pr(X=3)
    =
- Another solution
  - Let Y = X^2

Y: Another Random Variable, (# Heads - # Tails)^2
Y = 1 ➔ {TTH, THT, HTT, HHT, HTH, THH}
Y = 9 ➔ { TTT, HHH}

Pr(Y=1) = Pr(X=-1) + Pr(X=1)
Pr(Y=9) = Pr(X=-3) + Pr(X=3)

E[Y] = $\sum_i y_i$ Pr$(Y = y_i)$

  - E[X^2] = E[Y] = 1·Pr(Y=1) + 9·Pr(Y=9)
    =

Note that E[X] = 0 and E[X^2]≠E[X]^2

Expansion

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- Let Y=g(X), where g() is a real-valued function
- E[g(X)]=E[Y] = $\sum_j y_j \cdot (\Pr(Y = y_j))$

  = $\sum_j y_j \cdot (\sum_{i:g(x_i)=y_j} \Pr(x_i))$

  = $\sum_j \sum y_j \Pr(x_i)$

  = $\sum_j \sum g(x_i) \Pr(x_i)$

Reconsider rv X in the previous slide
Define g(X) = $X^2 + X$
Compute E[g(X)]

- For any constant E[c·X] = c · E[X]
- n-th **moment** of X:

  E[X^n] = $\sum_i {x_i}^n \, Pr(X = x_i)$

Linearity of Expectation

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- For any finite collection of discrete rv $X_1, X_2, \cdots, X_n$

  $E[\sum_{i=1}^{n} X_i]$ = $\sum_{i=i}^{n} E[X_i]$

- Proof
  - For two rv X and Y, prove that $E[X+Y] = E[X] + E[Y]$
  - $E[X+Y] = \sum_i \sum_j (i+j) \cdot \Pr((X=i) \cap (Y=j))$

    $=$

Jensen's Inequality

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- In general, E[X^2] ≠E[X]^2
- Claim: E[X^2] ≥ E[X]^2
- Proof
  - Consider Y= $(X-E[X])^2$
  - 0 ≤ E[Y] = E[$(X-E[X])^2$]
    = E[X^2 - 2XE[X] + E[X]^2]
    = E[X^2] - E[X]^2
- Definition: Convex
  - A function f is convex if, for any x1 and x2 and 0≤λ≤1,
    $f(\lambda\cdot x_1+(1-\lambda)\cdot x_2)\le\lambda\cdot f(x_1)+(1-\lambda)\cdot f(x_2)$

Convex function & Optimization
Optimization: Another important technique
Generally, we can ***easily*** find optimal points if functions are convex

Jensen's Inequality

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- Theorem: If f is convex, then E[f(X)] ≥ f(E[X])
- Proof
  - Let $\mu=E[X]$
  - By Taylor's theorem, there is c such that
    $$f(x)=f(\mu)+f'(\mu)\cdot(x-\mu)+\frac{f''(c)(x-\mu)^2}{2}$$
    $$\ge f(\mu)+f'(\mu)\cdot(x-\mu)$$
    Lemma: If f is convex, then $f''(x)$ ≥ 0
  - $E[f(X)]\ge E[f(\mu)+f'(\mu)\cdot(X-\mu)]$
    $=f(\mu)=f(E[X])$

Conditional Expectation

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- Definition: E[Y | Z=z] = $\sum_y y \cdot \Pr(Y = y \mid Z = z)$

  E[Y | E] = $\sum_y y \cdot \Pr(Y = y \mid E)$

- Example:
  - Roll two dice
  - X_1: Number on the first die
  - X_2: Number on the second die
  - X: X_1+X_2
  - E[X | X_1=2] = $\sum_{x_2=1}^{6}(2 + x_2)\Pr(X_2 = x_2 \mid X_1 = 2)$

    $= \sum_{x_2=1}^{6}(2 + x_2) \cdot \frac{1}{6}$
  - E[X_1 | X=5] =$\sum_{x_1} x_1 \Pr(X_1 = x_1 \mid X_1 + X_2 = 5)$

    $= \sum_{x_1=1}^{4} x_1 \Pr(X_1 = x_1 \mid X_1 + X_2 = 5)$

    $= \sum_{x_1=1}^{4} x_1 \frac{\Pr((X_1=x_1) \cap (X_1+X_2=5))}{\Pr(X_1+X_2=5)}$

Properties of Conditional Expectation

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- Lemma 2.5: For any random variables X and Y,

E[X] = $\sum_y \Pr(Y = y) \cdot E[X|Y = y]$

Important lemma
In many cases, E[X|Y=y] is easier to compute than E[X]

- Proof:
  - E[X] =$\sum_i x_i \cdot \Pr(X = x_i)$

    Theorem 1.6: Law of Total Probability

    = $\sum_i x_i \cdot \sum_y \Pr(X = x_i|Y = y) \cdot \Pr(Y = y)$

    = $\sum_y \underbrace{\sum_i x_i \cdot \Pr(X = x_i|Y = y)}_{\equiv E[X|Y=y]} \cdot \Pr(Y = y)$

Linearity of Conditional Expectation

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- **Linearity**: For any finite collection of rv X1, X2,..., Xn, and for any random variable Y,

  $E[\sum_i X_i \,|Y = y] = \sum_i E[X_i|Y = y]$

- Example
  - Roll two dice and let X_1, X_2 be the numbers on the first and second die, respectively
  - $E[X_1+X_2 \mid X_1=2] = E[X_1 \mid X_1=2] + E[X_2 \mid X_1=2]$

RV Conditional Expectation

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- Definition: Expression E[Y | Z] is a r.v. g(Z) that takes on the value E[Y | Z=z] when Z=z
- Example
  - $\mathrm{E}[X|X_1] = \sum_{x_2}(X_1 + x_2) \cdot \Pr(X = X_1 + x_2|X_1)$

    $= X_1 + \sum_{x_2} x_2 \cdot \Pr(X = X_1 + x_2|X_1)$

    $= X_1 + \frac{7}{2}$
  - Now $E[E[X|X_1]] = E[X_1+ 7/2] = E[X_1]+ 7/2$

Slide #15
Roll two dice
X_1: Number on the first die
X_2: Number on the second die
X: X_1+X_2

- Theorem: E[Y] = E[E[Y | Z]}
- Proof:

  g(Z)
  - $\mathrm{E}[Y|Z] = \sum_i y_i \cdot \Pr(Y = y_i|Z)$
  - $\mathrm{E}[\mathrm{E}[Y|Z]] = \sum_j(\sum_i y_i \cdot \Pr\big(Y = y_i\big|Z = z_j\big)\,) \cdot \Pr(Z = z_j)$

    $= \sum_j E[Y|Z = z_j] \cdot \Pr(Z = z_j)$

    $= E[Y]$

    Lemma 2.5

Bernoulli RV

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- Run an experiment
  - Success probability = p and Failure probability = (1-p)
- **Bernoulli (Indicator)** random variable Y is
  - $Y = \begin{cases} 1, & \text{if success} \\ 0, & \text{if failure} \end{cases}$
- Examples
  - A toss of an unfair coin with Pr(H)=p
  - $Y = \begin{cases} 1, & \text{if Heads} \\ 0, & \text{if Tails} \end{cases}$
- Expectation
  - E[Y] = p = Pr(Y=1)

Binomial R.V.

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- Repeat the same (Bernoulli) experiments n times
- Random Variable X= the number of successes in n experiments
- Definition: **Binomial** random variable X with parameter n and p, **B(n,p)**, is

  Pr(X=j) = $\binom{n}{j} \cdot p^j (1-p)^{n-j}$
- Example
  - Toss an unfair coin with Pr(H)=p n times
  - X = # Heads among n tosses
- Prove that $\sum_{i=0}^{n} \Pr(X = i) = 1$

E[X] of Binomial RV

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- $E[X] = \sum i \cdot \Pr(X = i)$

  $= \sum i \cdot \binom{n}{i} \cdot p^i (1-p)^{n-i}$

- Another method
  - $X = X_1 + X_2 + \cdots + X_n$ where X_i is the indicator function (Bernoulli rv) of i-th experiment

Geometric Distribution

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- Definition: A **Geometric** random variable X with parameter p is given by the following probability distribution for n=1, 2,…

  Pr(X=n) = $(1-p)^{n-1} \cdot p$
- Example
  - X = # coin flips until the first heads where Pr(H)=p
- First, note that $\sum_{n\geq 1}$ Pr(X=n) = 1
- **Memoryless property**: Given you tried k times w/o heads, how many more trials until the first success?
- Lemma: Pr(X=n+k | X>k) = Pr(X=n)
- Proof
  - Pr(X=n+k | X>k) = $\frac{\Pr(X=n+k \cap X>k)}{\Pr(X>k)}$

    $= \frac{(1-p)^{n+k-1} \cdot p}{\sum_{i=k}(1-p)^{i} \cdot p}$

Geometric - Expectation

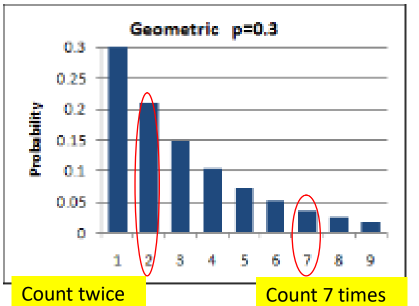
- Claim: E[X] = $\sum_{i=1}^{\infty} \Pr(X \geq i)$
- Proof:

$\Pr(X = 1)$+$\Pr(X = 2)$+$\Pr(X = 3)$+…

  – $\sum_{i=1}^{\infty} \Pr(X \geq i) = \Pr(X \geq 1) + \Pr(X \geq 2) + \Pr(X \geq 3) + \cdots$
  $= \sum_{i=1}^{\infty} \mathrm{i} \cdot \Pr(X = i)$
  $= E[X]$

- Note $\Pr(X \geq i) = \sum_{n=i}^{\infty} (1-p)^{n-1} \cdot p$
  $= (1-p)^{i-1}$

  ➔ E[X] = $\sum_{i=1}^{\infty} (1-p)^{i-1}$ $= {}^{1}/_{p}$



Count twice

Count 7 times

Geometric - Expectation

- Another Approach to Compute E[X]
  – Remember: E[X] = E[E[X|Y]]
  – Y: result of the first flip = {0, 1}

  – E[X]= E[X | Y=0] Pr(Y=0) + E[X | Y=1] Pr(Y=1)
  = E[X+1]$\cdot$ $(1-p)$+ 1$\cdot$ p

  ➔ E[X] =${}^{1}/_{p}$

Coupon Collector's Problem

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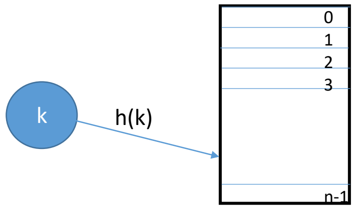
- Setting
  - There are N different types of coupon
  - Receive a coupon that is any one of N types
  - Any similar problems?
    ➔ Exactly same as "Hash Table"



Collect All Coupon Types

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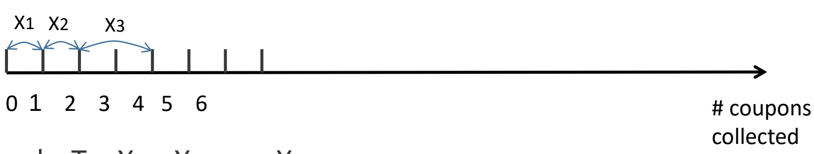
- Interested in random variable T: # coupons need to be collected until at least one from every type of coupon is collected
  - E[T]??
  - Xi: Given that (i-1) types of coupon are collected, how many more to collect to obtain the i-th type



- Clearly, T = $X_1 + X_2 + \ldots + X_N$
  - Xi: Geometric r. v. with p_i = (1 - $(i-1)$/N)= $(N-i+1)/N$
  - E[Xi] = 1 /p_i = N / $(N-i+1)$
  - E[T] = $\sum_i E[X_i]$

    = $\sum_i \frac{N}{N-i+1}$

    = N · $\sum_i {}^{1}/_{i}$

Harmonic number H(N) = ln N + Θ(1)

Collect All Coupon Types

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- Another Approach
  - Collect n coupons
  - Ai: Type i is not included in the n coupons
  - Pr(Ai) = $(\frac{N-1}{N})^n$

A_{j_1} and A_{j_2} indpendent?
Pr $(A_{j_1} \cap A_{j_2})$?= Pr (A_{j_1}) Pr (A_{j_2})
No!!
Pr $(A_{j_1} \cap A_{j_2})$ = $(\frac{N-2}{N})^n$
Pr$(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k})$ = $(\frac{N-k}{N})^n$

  - Pr(T>n) = Pr($\cup_{j=1}^{N} A_j$)
    = …
    = $\sum_i^{N-1}\binom{N}{i}(\frac{N-i}{N})^n(-1)^{i+1}$

Inclusion-Exclusion Rule

- … Continue

  Now, Pr(T=n) = Pr(T>n-1) - Pr(T>n)

Types in n Coupons

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- Another interesting random variable, Dn: # coupon types covered by n coupons
  - Pr(Dn=k)
  - Fix k types
  - Define A: each coupon is one of these k types, and

    B: each of these k types is represented

Fix the k types
Instead of collecting all types, collect the k types

  - Pr(A) = $(\frac{k}{N})^n$
  - Now consider Pr(B | A): Same as probability Pr(T $\leq$ n) with k replacing N
  - Pr(B | A) = 1-$\sum_i^{k-1}\binom{k}{i}(\frac{k-i}{k})^n(-1)^{i+1}$
  - Pr(Dn=k) = $\binom{N}{k}$Pr(A∩B) = $\binom{N}{k}$Pr(B | A) Pr(A)

QuickSort

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- Sorting problem
  - Given n comparable objects $x_1, x_2, \ldots, x_n$, arrange them in increasing order
    ➔ Let sorted result is $y_1, y_2, \ldots, y_n$

Note: Actual QuickSort implementations are slightly different
Refer to CLRS

QuickSort Algorithm

Given objects $x_1, x_2, \ldots, x_n$
1. Pick a pivot element x_t, $1 \le t \le n$
2. Partition on x_t
   $S1 = \{x_i: x_i \le x_t\}$
   $S2 = \{x_i: x_i > x_t\}$
   $S1 \le x_t < S2$
3. Sort S1 & S2, respectively
4. Combine
   $y_1, y_2, \ldots, y_p, x_t, y_{p+1}, y_{p+2}, \ldots, y_n$

Objects in S1 won't be compared to objects in S2

QuickSort - Example

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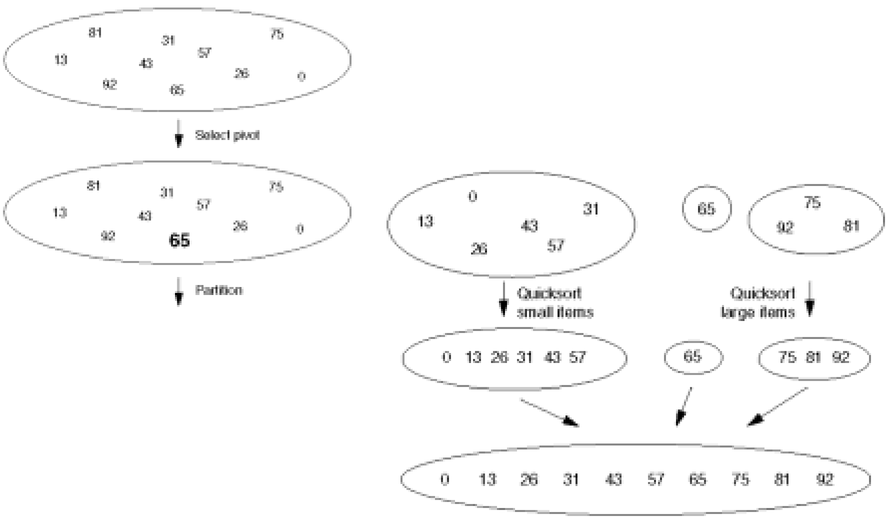

Complexity of QuickSort

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- Complexity of quick sort
  - T(N) = T(|S1|) + T(|S2|) + O(N)
  - Running time depends on the choice of the pivot
  - Worst case
    - T(N) = T(N-1) + O(N)
      = O(N^2)
  - Best case
    - T(N) = 2 T(N/2) + O(N)
      = O(N log N)
- Average case analysis (Probabilistic Analysis)
  - All N! permutations of the sorted order are equally likely
  - Always pick an element with a fixed index, say x_1, as a pivot
    - Pi = probability that x1 is the i-th element in the sorted order
      = 1/N
  - C_N = Average number of operations for sorting a table of size N
    = 1/N ∑(Ci-1 + Cn-i) + a N
    = 2/N ∑Ci + a N
    = O(Nlog N)

Refer to CLRS
We obtain a recurrence equation
Guess that $C_N \leq \alpha \cdot N \cdot logN$

Randomized QuickSort

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- Randomized Algorithm
  - Select pivot numbers uniformly at random among the candidates
- Theorem: For any input, the expected number of comparisons made by randomized QuickSort is *2N · ln N*+ Θ(N)
- Proof
  - Let $y_1, y_2, \dots, y_N$ be the sorted sequence
  - For i < j, define random variable Xij such that
  - Xij = $\begin{cases} 1, & if\ y_i\ and\ y_j\ are\ compared \\ 0, & ow \end{cases}$
  - Total number of comparisons X = $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}$
  - Pr(Xij) ??
  - E[X] = E[$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}$]
    = $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E[X_{ij}]$

What is the probability that y_1 and y_N are compared?
How about yi and y_{i+1}?

Randomized QuickSort

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– $\Pr(X_{ij} = 1) = \frac{2}{(j-i+1)}$ Why?

– $E[X] = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{2}{(j-i+1)}$

=