Probability

Simpson's Paradox

KENTECH Korea Institute of Energy Technology AI

- Bob's GPAs in both Spring semester and Fall semester are better than Alice's GPAs in the same semesters. However, Alice's overall GPA is higher than Bob's.

	'22 Spring		'22 Fall		'22	
	GPA	Credits	GPA	Credits	GPA	Credit
Alice	3.5	5	4.0	20		25
Bob	3.7	20	4.1	5		25

- Do not believe anything blindly. Doubt everything.

Descartes Said "Cogito, ergo sum"
Originally, "Dubito, ergo cogito, ergo sum"

Better Doctor

AI

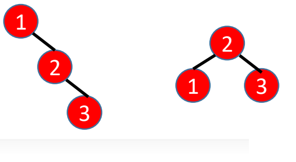
- An advertisement says that “The winning probability of our law firm is 90%”
- “Our hospital have successfully performed medical operations with 99%”
- “Every graduate of this department gets a job”

	Doctor A		Doctor B	
	Cases	Success	Cases	Success
Sucher	90	90	10	10
Cancer Operation	10	2	90	80
Total	100	92	100	90

Computing & Probability

AI

- Computer Science broadly uses the knowledge of probability & statistics in developing algorithms
  - Machine learning
  - Big data analyses
  - Networks, systems, ..
- Randomized algorithms
  - Use randomness in performing their procedures
  - Example: Select pivot elements randomly (Quick sort)
- Probabilistic analysis of algorithms
  - The performance of many algorithms depends on input
  - Average (or worst case) performance considering input probability
  - Example: BST (Binary Search Tree)



Probability

KENTECH Korea Institute of Energy Technology AI

Cardano (1501~1576) was an Italian polymath, gambler
He invented idea of probability (odds), independence, and binomial coefficients (that Pascal refined later)

- Cardano, a gambler & mathematician, first introduced the notion of probability
  - Roll two dice. How many times should you try until (6, 6) occurs?
- Probability
  - Trial: Roll two dice
  - Sample space: {(1,1), … (6, 6)}
  - Event: E1={6, 6}
  - Probability of E1 = 1/36

Axioms of Probability

KENTECH Korea Institute of Energy Technology AI

Kolmororov(1903~1987) was a Russian Mathematician
One of most important researchers in probability
At age 5, he discovered that sum of non-negative odd numbers is equal to square of a number (1+3+5+7 = 4^2)

- **Axioms of Probability**

  A1: $0 \le Pr(E) \le 1$

  A2: $Pr(\Omega) = 1$

  A3: If E_1 and E_2 are mutually exclusive ($E_1 \cap E_2 = \emptyset$),

  then $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2)$

  →For any sequence of pairwise mutually disjoint events $E_1, E_2, \cdots, E_n$

  $Pr(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} Pr(E_i)$

Lemmas

- From the axioms, we can easily derive following Lemmas
- Lemma 1.0
  - If E ⊆ F then Pr(E) ≤ Pr(F)
  - $\Pr(\bar{E}) = 1 - \Pr(E)$
- Lemma1.1
  - For any events E_1 & E_2

  $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$
- Lemma 1.2:
  - For any sequence of events E_i **Union Bound**
  - $Pr(\bigcup_{i\geq 1} E_i) \leq \sum_{i\geq 1} Pr(E_i)$
- Lemma 1.3: ***Inclusion-exclusion principle***
  - $Pr(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} Pr(E_i) - \sum_{i<j} Pr(E_i \cap E_j) + \cdots$
  $(-1)^{l+1} \sum_{i_1<i_2<\cdots<i_l} Pr(\bigcap_{r=1}^{l} E_{i_r}) + \cdots$

Birthday Problem

- What is the probability that none of n people share the same birthday?

365 days

- sample $|\Omega|$ = ? $\overbrace{365 \times 365 \times \cdots \times 365}^{n} = (365)^n$
- event $|E|$ = ? $365 \times (365-1) \times (365-2) \times \cdots \times (365-n+1)$
- Pr(no matching birthdays)

  = $\underbrace{(365)(364)\ldots(365-n+1)}_{|E|} / \underbrace{(365)^n}_{|\Omega|}$
- Cases
  - n= 23: Pr(no matching birthdays) < ½ (least such n)
  - n= 75: Pr(no matching birthdays) < 1/3,000
  - n= 100: Pr(no matching birthdays) < 1/3,000,000

Verifying Polynomial Identities

- Problem
  - Verify if F(x) ≡ G(x)
  - Where F(x) is given in a product of monomials form and
    G(x) is given in a canonical form
- Example
  - F(x) = (x+1)(x-2)(x+3)(x-4)(x+5)(x-6)
  - G(x) = $x^6 - 7x^3 + 25$
- Deterministic method
  - Convert F(x) to a canonical form and check if all coefficients are the same
- Complexity of the deterministic method
  - If F(x) = $\prod_{i=1}^{d}$ (x–a_i) then it takes $O(d^2)$ where *d* is the degree of the polynomial

Randomized Algorithm - Background

- If F(x) = G(x)
  - ➔ For all integers *r*, F(*r*) = G(*r*)
- Suppose F(x) ≠ G(x)
  - Compute F(r) and G(r) for a randomly selected integer r
  - Case 1: F(*r*) ≠ G(*r*) ➔ F(x) ≠ G(x)
  - Case 2: F(*r*) = G(*r*) ➔ F(x) = G(x)

Wrong Decision!!

- What is the probability of making a wrong decision?
- Consider F(x) – G(x)
  - There are at most *d* roots that yield F(x) - G(x) = 0

Randomized Algorithm

KENTECH
Korea Institute of Energy Technology
AI

Simple randomized algorithm

- Select a number *r*, uniformly at random from Ω = {1, 2, …, 100d}
- If F(r) = G(r), then conclude that F(x) = G(x)

Analysis of the simple randomized algorithm

- Probability of making wrong decision (given F(x) ≠ G(x))
  - Pr(Wrong Decision) = Pr(r is one of roots) ≤ $\frac{d}{100d} = \frac{1}{100}$

How do you improve the simple algorithm?

- Increase the sample space to {1, 2, …, 1000d}
- Any other methods?