Graph, Network

Network Structure (Static)

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AI

- Characteristics of networks
  - Node degree distribution
  - Clusters
  - Diameter, sparse/dense, seachability
  - Attribute distributions
  - …
- How to generate networks with such properties?
  - Node generation
  - Edge generation

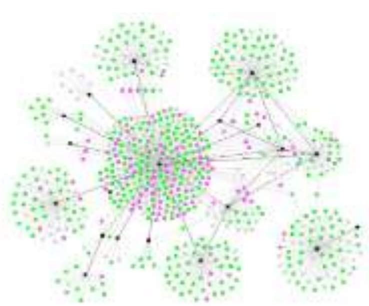
Dynamics of Networks

- How do nodes interact?
- Behaviors of nodes
  - Diffusion of information
  - Spread of virus
  - Vote
  - Reference (Citation)
  - …

Epidemiology

- Hippocrates
  - Epidemic & Endemic

Kermack, W.O. and McKendrick, A.G. (1927).
A contribution to the mathematical theory of epidemics I. *Proc.Roy.Soc. A* 115, 700-721.

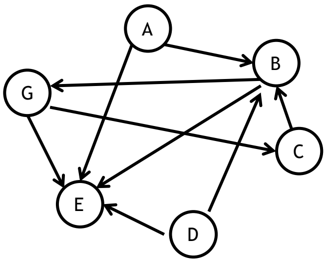


John Snow:
London Cholera, 1854

Representation

AI

- Text, table, …
  - Adam follows Bob and Eve
  - Bob follows Eve and Genie
  - Charlie follows Bob
  - Dave follows Bob and Eve
  - Genie follows Charlie and Eve
- Graphic



Matrix

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- Adjacency Matrix A: element A_{ij} is relation between node i and j

	A	B	C	D	F	G
A	0	1	0	0	1	0
B	0	0	0	0	1	1
C	0	1	0	0	0	0
D	0	1	0	0	1	0
E	0	0	0	0	0	0
G	0	0	1	0	0	1

Good for Analysis

Directed/Undirected Networks

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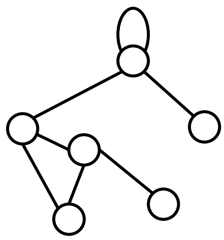
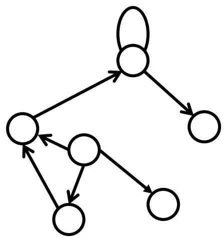
- Undirected networks
  - Links have no direction
  - Co-authoring of papers



- Directed networks
  - Directed links (arcs)
  - Following
  - Refer



Neighbors

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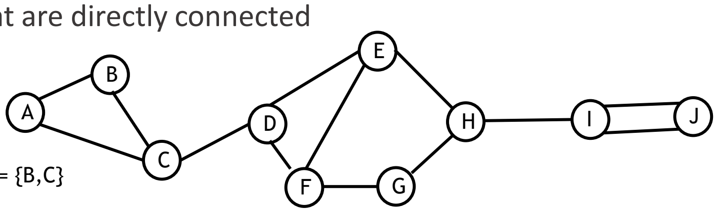
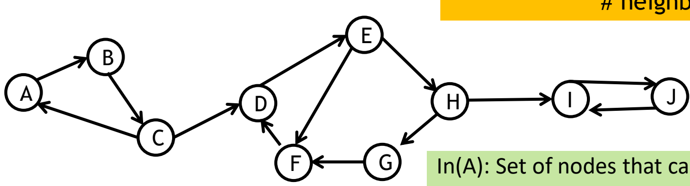
- Nodes that are directly connected



N(A) = {B,C}

Node degree: # edges connected to a node
neighbor nodes ≅ |N(A)|

- Directed graph



In(A): Set of nodes that can reach to node A

Out(A): Set of nodes that can be reached from node A

Connectivity of Directed Networks

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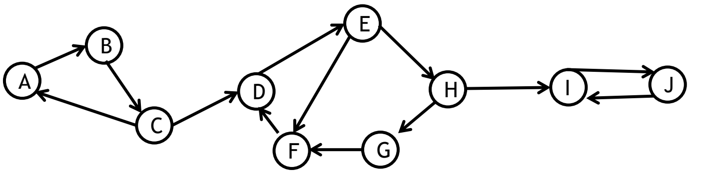
- Strongly connected
  - There are (directed) paths from one node to another node
- Weakly connected
  - Connected if edges are replaced by undirected edges



- **DAG** (Directed Acyclic Graph)
  - Directed graph without cycles
  - If there is a path from *u* to *v*, then there is no path from *v* to *u*

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SCC (Strongly Connected Component)

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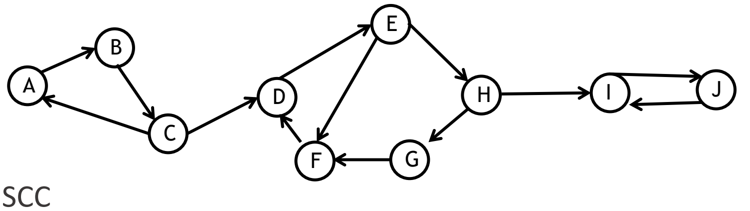
- A set *S* of nodes
  - Strongly connected
  - Largest set containing S with the strong connectivity property



- Find SCC
  - An SCC containing node A = In(A) ∩ Out(A)

Note: A node belongs to only one SCC

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The Structure of the Web

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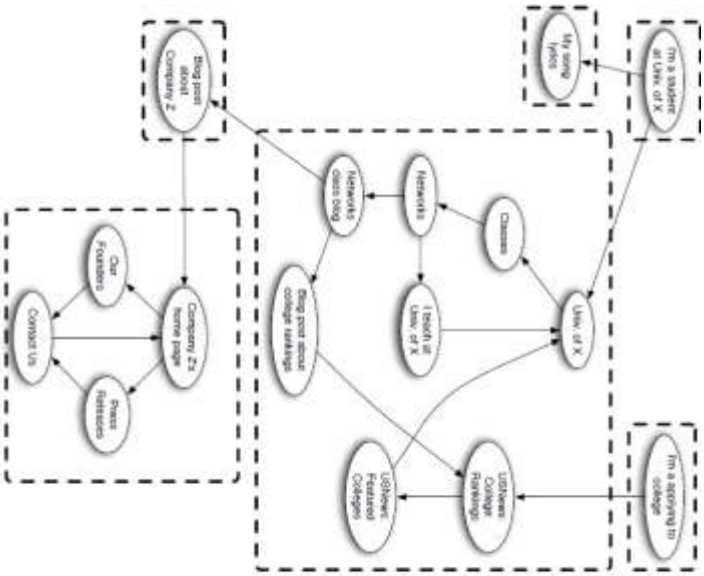

Weighted, Multigraph

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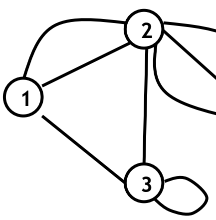
- Weighted graph
  - Links have weights

$$E = \sum \sum A_{ij}$$

$$\bar{k} = {}^{2E}/_{N}$$

- Multigraph
  - More than one links between a node pair
  - Weight = # of links

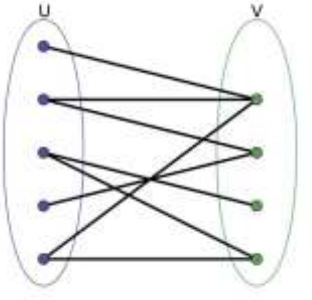


$$A_{ij} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$$

Bipartite Graph

- Two types of nodes
- Edge between nodes of different types
  - U, V are independent sets
- Examples
  - Actor-Movie
  - Author-paper
  - Person-Community
- Incident matrix

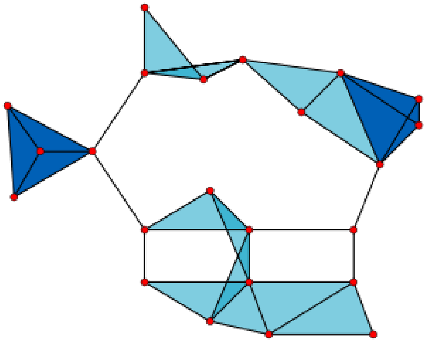
  Aij = 1 if member j belongs to group i

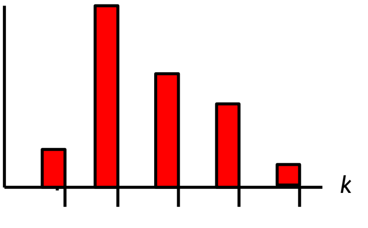


Network Property: Degree Dist.

- *p(k)* = Prob. that a node degree = *k*

$$N_k = \#\ of\ nodes\ with\ degree = k$$
$$p(k) = {}^{N_k}/_{N}$$





Power Law Distribution

Path

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- Definition:
  - Sequence of nodes that are directly connected
- Path length, h
  - Number of hops (≡ # of edges ≡ # of nodes -1)
- Directly connected node pair (u, v)
  - Path length =1
  - $A_{uv} = 1$
- u, v are connected by a path of length h
  - ${A^h}_{uv}$=1
- Shortest path
  - Path of smallest length

BFS (Breath First Search)

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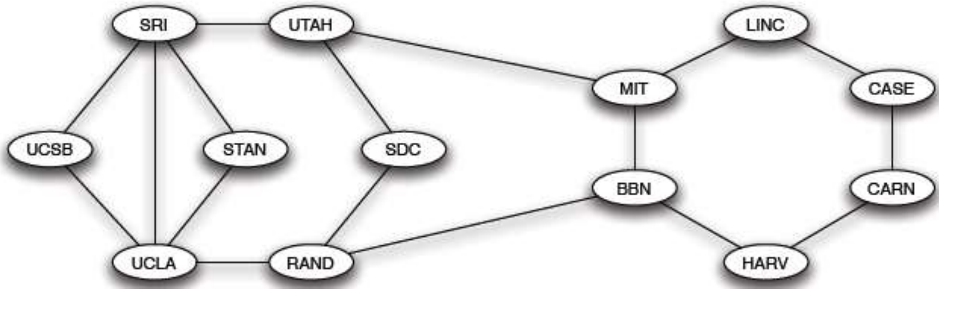
- Finding shortest paths in weighted graph
  - Dijkstra algorithms, Bellman-Ford, Floyd-Warshall, …
- In unweighted graph
  - BFS



BFS

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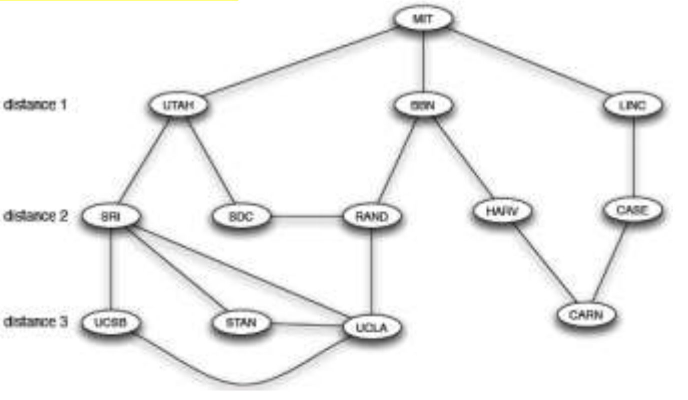
Shortest path tree from node u
Mark u, let h(u) = 0, enqueue(u)

While queue not empty
dequeue(v), retrieve its unmarked neighbor, w
h(w) = h(v)+1
mark w
enqueue(w)



Network Diameter

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- Diameter Definition 1:
  - The length of the longest path in a graph
- Diameter Definition 2:
  - Average path length of connected pairs

$$\bar{h} = \frac{1}{N(N-1)} \sum \sum h_{ij}$$

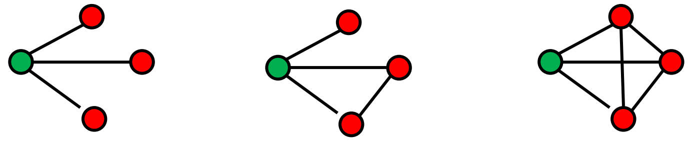
Complex
Use sampling to reduce complexity

Clustering Coefficient

AI

triangles

- Clustering coefficient
  - Proportion of directly connected neighbor pairs
  - $C_i = \frac{2\,e_i}{k_i(k_i-1)}$ where k_i is the degree of node i and e_i is the number of directed connected neighbor pairs



- $Ci = 0$ $Ci = 1/3$ $Ci = 1$

- Average clustering coefficient

$$C = \frac{1}{N}\sum C_i$$

Random Network & Basic Graph Theory

C. Kim

Erdös-Renyi Random Graph

- Erdös-Renyi - 1960

  http://en.wikipedia.org/wiki/Paul_Erd%C5%91s

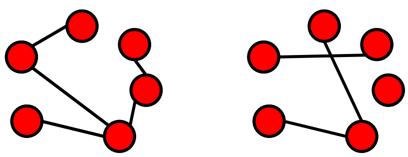
  http://en.wikipedia.org/wiki/File:Alfred_Renyi.jpg

- Two variants
  - $G_{n,p}$: Undirected graph on n nodes each edge (u,v) appears i.i.d. with probability p
  - $G_{n,m}$: Undirected graph with n nodes and m uniformly and randomly picked edges

Random Graph, $G_{n,p}$

AI

- Construction
  - Connect each n·(n-1)/2 node pairs with prob. p



Two examples of N=6, p=0.3

Random networks vs Real networks

Properties of Random Graph

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- Node degree distribution
- Clustering coefficient
- Size of giant components, prob. of isolated nodes
- Diameter
- …

Properties of $G_{n,p}$ – Clustering Coefficient

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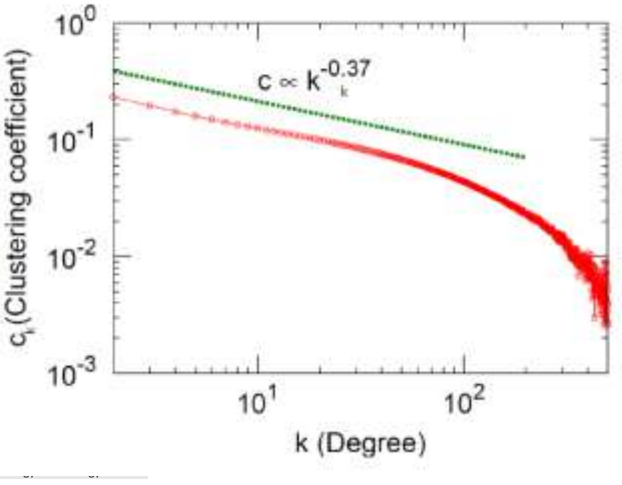
- Expected clustering coefficient ?
- Real networks



Avg. clustering of the MSN: C = 0.1140

Avg. clustering of corresponding G_{np}: $C = \bar{k}/n \approx 8 \cdot 10^{-8}$

Properties of $G_{n,p}$ – Node Degree

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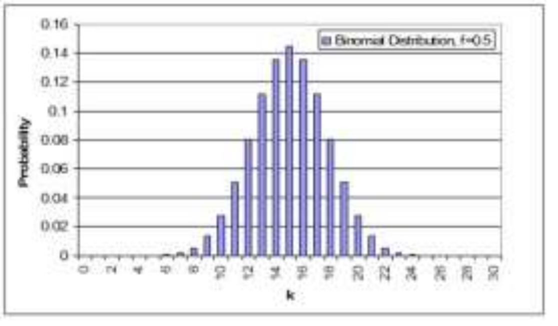
- Prob. that $G_{n,p}$ generates a graph with E edges

$$P(E) = \binom{E^{max}}{E} p^E (1-p)^{E^{max}-E}$$

➔ Binomial distribution

- Pr. that a node has k edges

$$P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$



Real Networks – Degree Distribution

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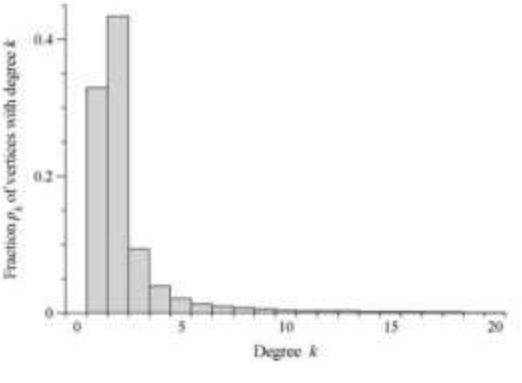

Figure 8.3: The degree distribution of the Internet. A histogram of the degree distribution of the vertices of the Internet graph at the level of autonomous systems.

Real Networks – Degree Distribution

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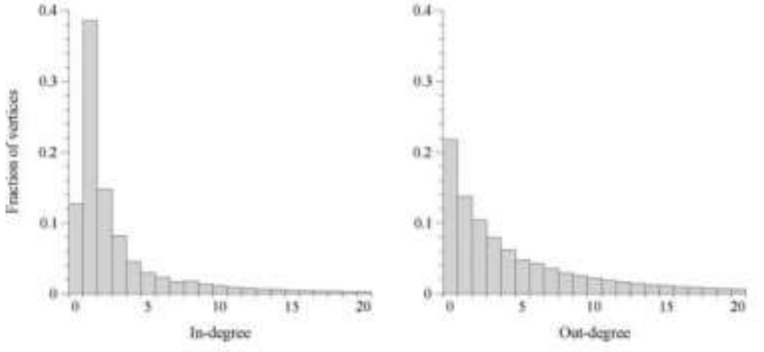

Figure 8.4: The degree distributions of the World Wide Web. Histograms of the distributions of in–and out-degrees of pages on the World Wide Web. Data are from the study by Broder *et al.* [56].

Properties of $G_{n,p}$ - Diameter

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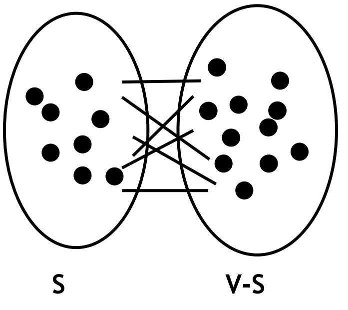
- First define expansion
- Graph G(V, E) has expansion *a*
  - If $\forall S \subseteq V$, # of edges leaving S ≥ a * min(|S|, |V-S|)

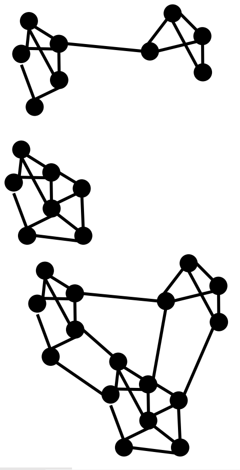
$$\rightarrow a = \min \frac{\#\, of\ edges\ leaving\ S}{\min(|S|,|V-S|)}$$



Expansion

AI

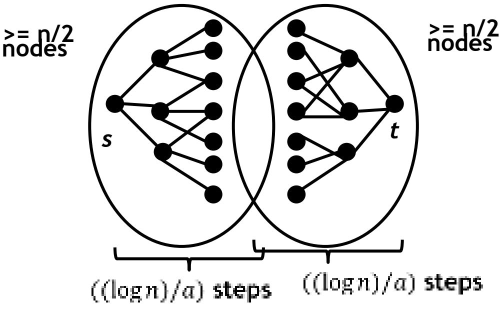
- Measure of Robustness
  - To partition *m* nodes, should cut at least *a***m* edges



Expansion & Diameter - 1

AI

- Theorem: The diameter of a graph on *n* nodes with expansion a is $O((\log n)/a)$
- Proof
  - To show the length of a shortest path between any node pair (s, t) is $O((\log n)/a)$



Expansion & Diameter - 2

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- Let S_j be a set of all nodes found within j steps of BFS from s

  → $|S_{j+1}| \geq |S_j| + \frac{a * |S_j|}{k}$

  $= |S_j|\left(1 + \frac{a}{k}\right)$

  $= (1 + \frac{a}{k})^{j+1}$

How many BFS steps are required to cover > n/2 nodes?

$(1 + \frac{a}{k})^j$ **>= n/2**

→ **Let** $j = \frac{k\, log_2 n}{a}$ **, then** $(1 + \frac{a}{k})^{\frac{k\, log_2 n}{a}}$ **>= n**

Real Network - Diameter

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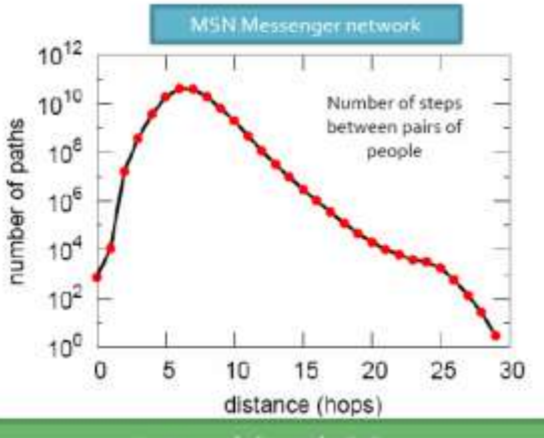


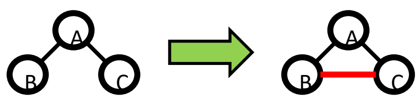
Avg. path length 6.6
90% of the people can be reached in < 8 hops

Homophily in Networks

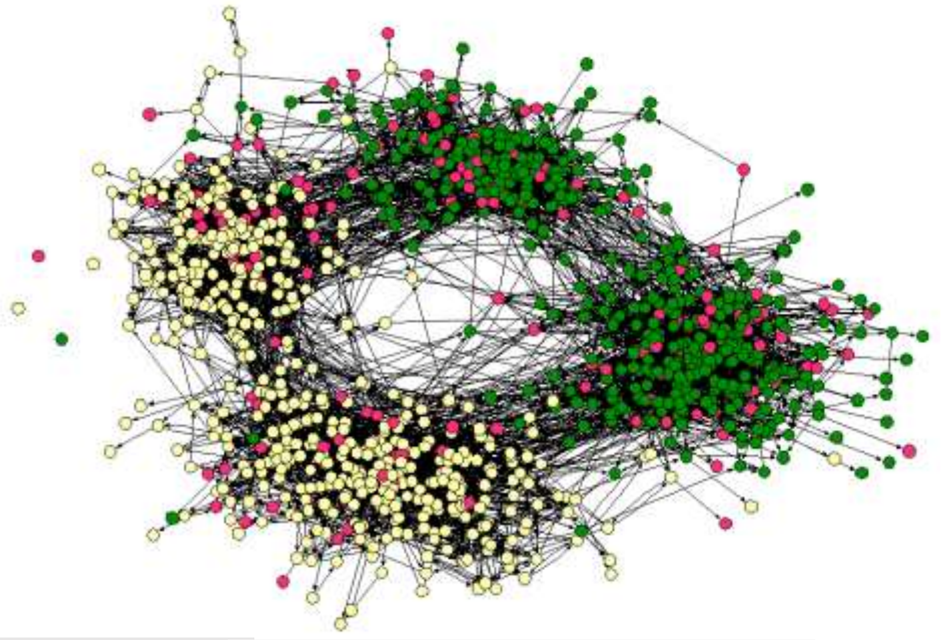
C. Kim

Homophily

- We observed triadic closures
  - People with common friends become friend
- Why?
  - Intrinsic to the network
    - Introduction by a common friend
  - Surrounding contexts (Property outside of network)
    - Same school, same region, …
    - A-B, A-C are similar → B-C are similar too



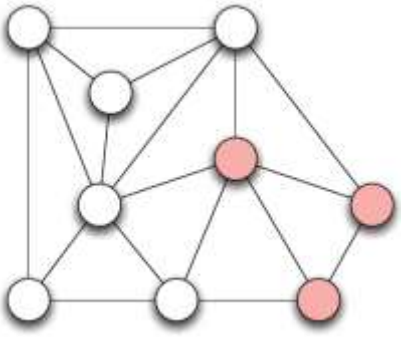
- Surrounding Context affects network formation
- Homophily:
  - Links are formed between nodes of similarity and compatibility
  - Plato "Similarity begets friendship"



Homophily Test

- Test if nodes of the same context form links
- Let populations of attribute A and B are p and q, resp.
  - If no homophily, then
    A-A link : B-B link : A-B link = $p^2 : q^2 : 2pq$
- Test
  - The fraction of A-B links < 2pq,
    then homophily exist
  - The fraction of A-B links > 2pq,
    reverse homophily

P=2/3, q=1/3
➔ 2pq= 4/9
|E|=18
Heterogeneous edge= 5



Selection & Social Influence

- Note: Links connect people with similar characteristics
- Why?

1. Selection: Select friends with similar characteristics
  - Individual characteristics drive the formation of links

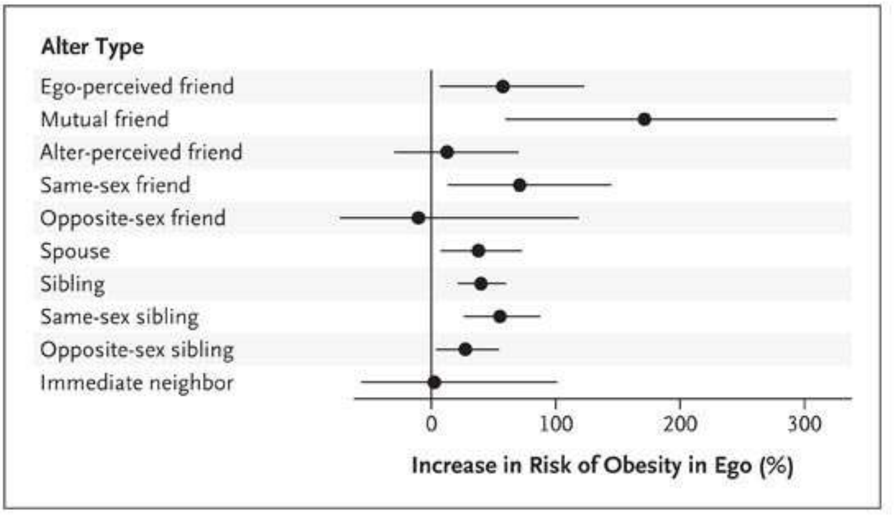
2. Social influence: Modify behaviors adapting to friends' behavior
  - Links shape characteristics

- What factor is more dominant?
- Longitudinal study
  - Social connections and behaviors within a group are traced over time

Obesity & Social Network

- Christakis & Fowler, "The spread of obesity in a large social network over 32 years", NEJM, 2007

Obesity & Social Network - 2



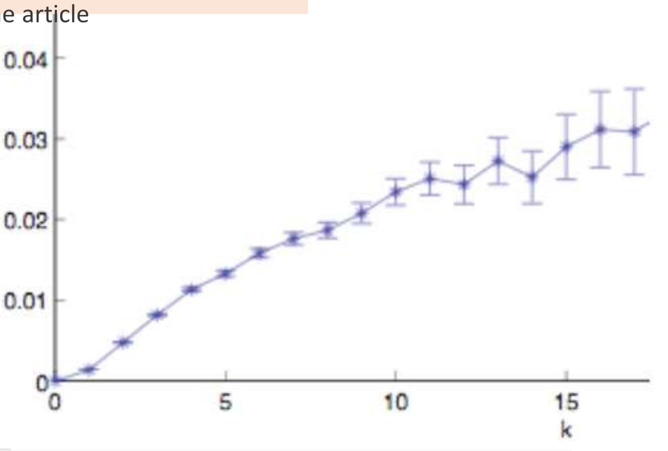
Context, Characteristics

- Immutable
  - Race, Ethnicity
- Mutable
  - Behavior, activity, interest, belief
  - Modify one's characteristics to align with neighbors

Network Evolution –Membership Closure

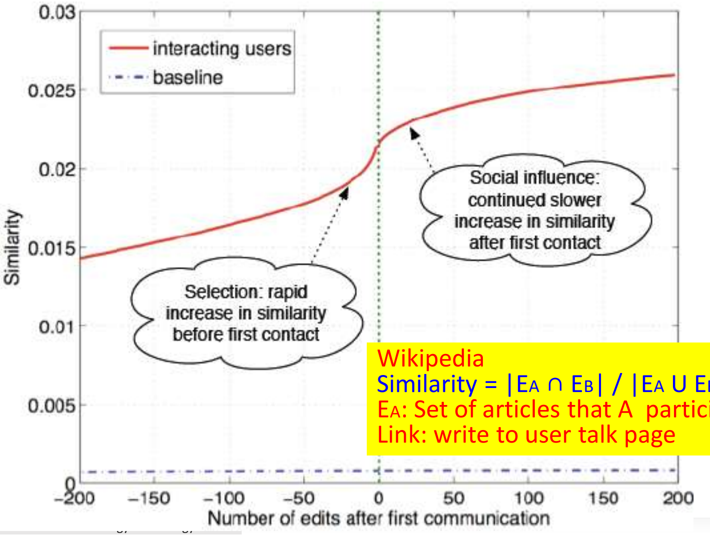
AI

Leskovec
Wikipedia
Friendship: User talk page
Foci: Edit the same article



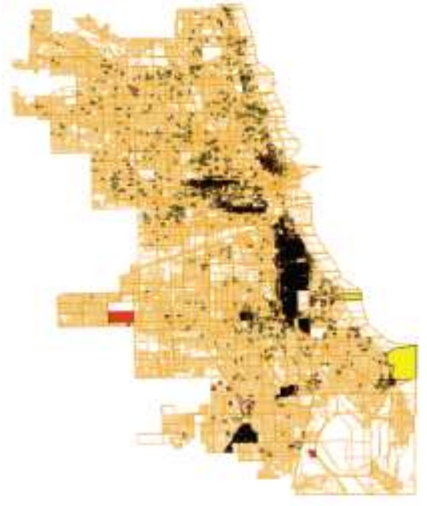
Selection OR Social Influence

AI



Wikipedia
Similarity = |EA ∩ EB| / |EA U EB|
EA: Set of articles that A participates in editing
Link: write to user talk page

Spatial Segregation


(a) Chicago, 1940


(b) Chicago, 1960

Spatial Segregation

- **Segregation** is a result of **homophily**
- Segregation
  - Global pattern
- Homophily
  - Local decision (individual desire)
- Shelling explained the relation using a simple model

SN SCONE lab.