An Introduction to Signal Processing

Machine Vision

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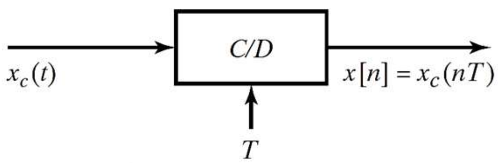
FT and FS

Time Duration		
Finite	Infinite	
Discrete FT (DFT) $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\omega_k n}$ $k = 0, 1, \ldots, N-1$	Discrete Time FT (DTFT) $X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$ $\omega \in [-\pi, +\pi)$	discr. time n
Fourier Series (FS) $X(k) = \frac{1}{P} \int_0^P x(t) e^{-j\omega_k t} dt$ $k = -\infty, \ldots, +\infty$	Fourier Transform (FT) $X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ $\omega \in (-\infty, +\infty)$	cont. time t
discrete freq. k	continuous freq. ω	

Sampling

$$x[n] = x_c(nT), \qquad -\infty < n < \infty.$$



$$f_s = 1/T. \qquad \Omega_s = 2\pi/T$$

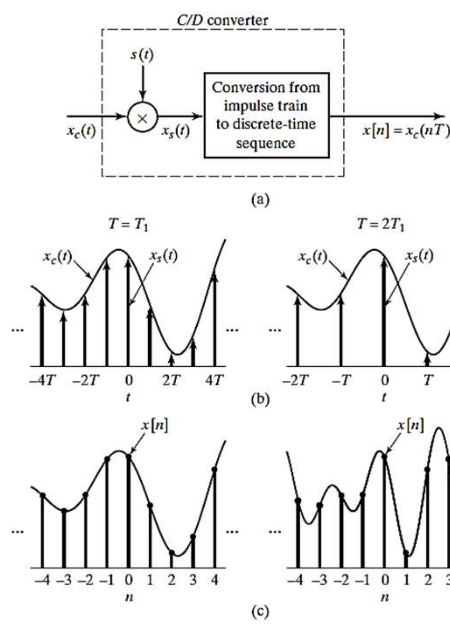
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) = x_c(t)s(t)$$

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT)$$

Sampling

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$



Frequency-Domain (Sampling)

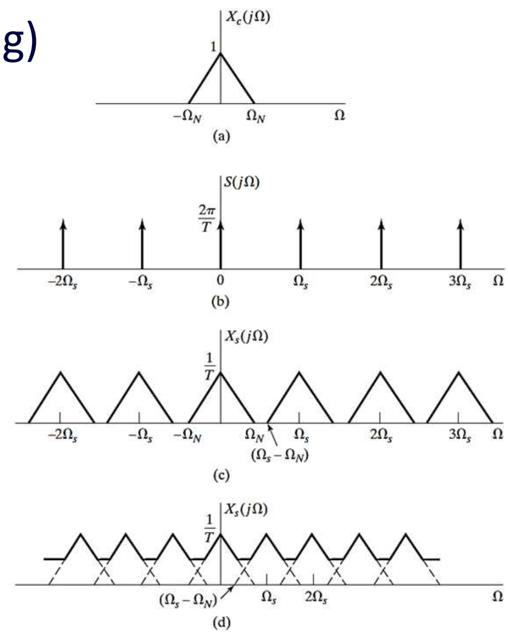
$$S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\,\Omega_s)$$

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)$$

$$X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega)$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\Omega - k\,\Omega_s))$$

$$\Omega_s - \Omega_N \geq \Omega_N, \quad \text{or} \quad \Omega_s \geq 2\Omega_N$$

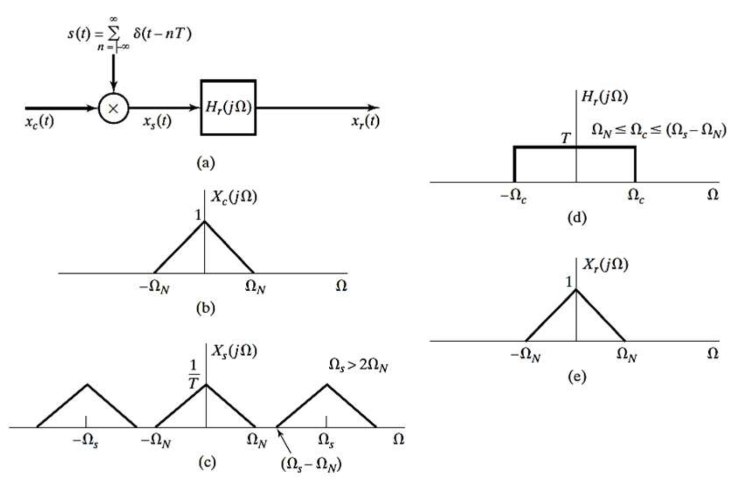


Frequency-Domain (Sampling)

$$X_r(j\Omega) = H_r(j\Omega)X_s(j\Omega)$$

$$\Omega_N \leq \Omega_c \leq (\Omega_s - \Omega_N)$$

$$X_r(j\Omega) = X_c(j\Omega)$$



Frequency-Domain (Sampling)

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega T n}$$

$$x[n] = x_c(nT)$$

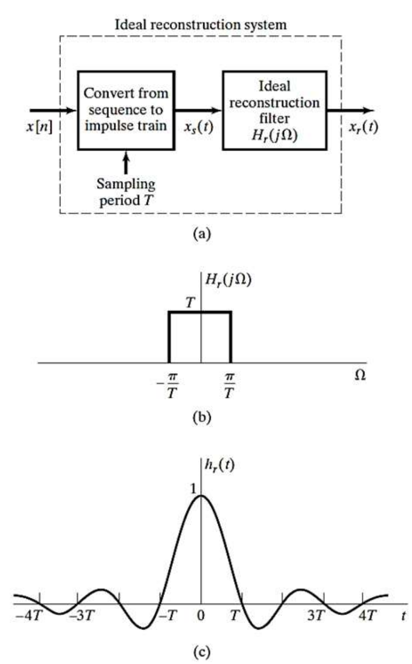
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T})$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\,\Omega_s))$$

$$\boxed{\begin{aligned} X(e^{j\Omega T}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\,\Omega_s)) \\ X(e^{j\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right] \end{aligned}}$$

Signal Reconstruction



$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

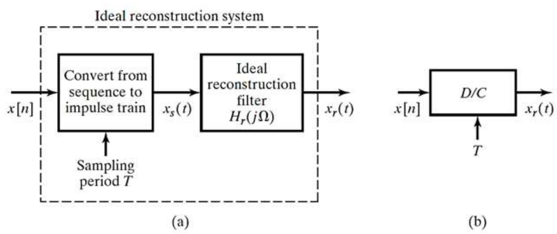
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

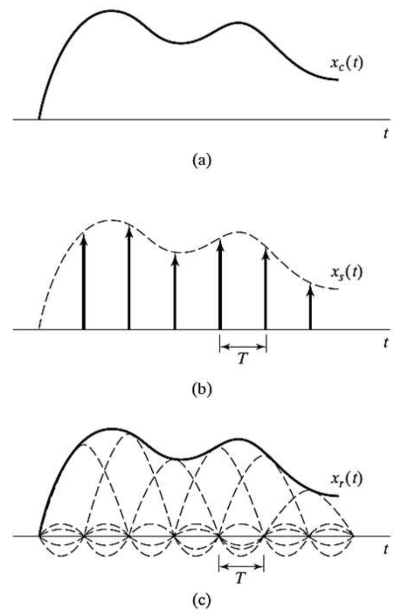
$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

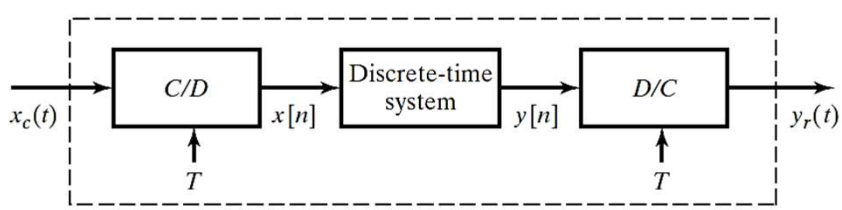
$$h_r(0) = 1$$

Signal Reconstruction





Discrete-Time Processing



Changing the Sampling Rate

$$x[n] = x_c(nT)$$

$$x_1[n] = x_c(nT_1) \qquad T_1 \neq T$$

Downsampling

$$x_d[n] = x[nM] = x_c(nMT) \qquad T_d = MT$$

$$\pi/T_d = \pi/(MT) \geq \Omega_N$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

$$X_d(e^{j\omega}) = \frac{1}{T_d}\sum_{r=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T_d} - \frac{2\pi r}{T_d}\right)\right]$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right]$$

Downsampling

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right] \qquad r = i + kM \qquad \begin{array}{l} 0 \le i \le M-1 \\ -\infty < k < \infty \end{array}$$

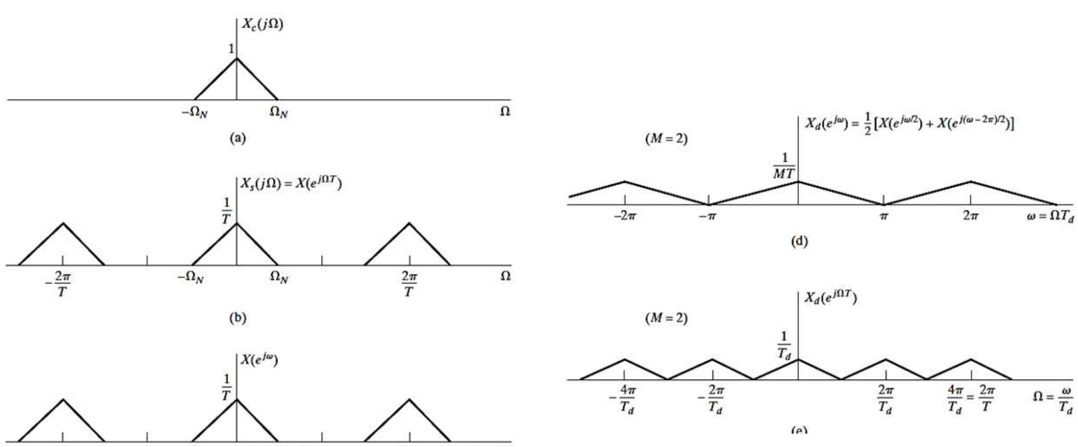
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1}\left\{\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)\right]\right\}$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

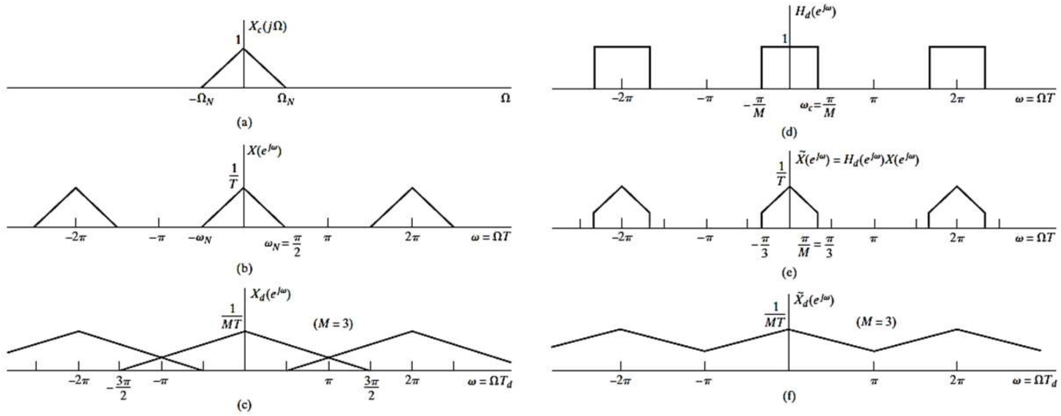
$$X(e^{j(\omega-2\pi i)/M}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega-2\pi i}{MT} - \frac{2\pi k}{T}\right)\right]$$

$$\boxed{X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega/M - 2\pi i/M)})}$$

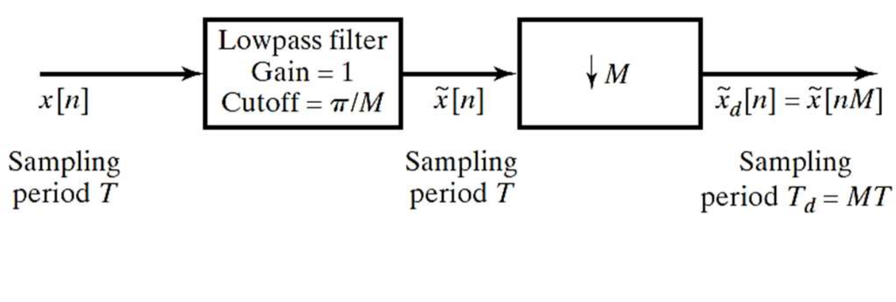
Downsampling



Downsampling



Downsampling



Upsampling

$$x[n] = x_c(nT)$$

$$x_i[n] = x_c(nT_i) \qquad\qquad T_i = T/L$$

$$x_i[n] = x[n/L] = x_c(nT/L) \qquad\qquad n = 0, \pm L, \pm 2L, \ldots.$$

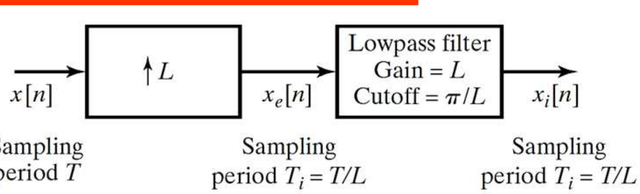
$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise,} \end{cases}$$

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL]$$

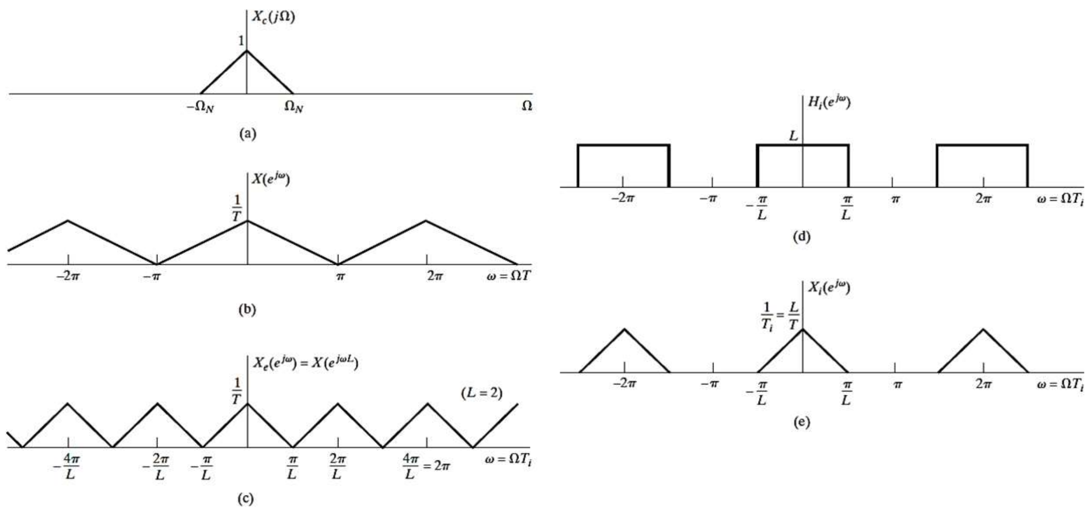
Upsampling

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\delta[n - kL] \right) e^{-j\omega n}$$

$$\boxed{= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega L k} = X(e^{j\omega L}).}$$



Upsampling



Upsampling

$$h_i[n] = \frac{\sin(\pi n/L)}{\pi n/L}$$

$$h_i[0] = 1,$$
$$h_i[n] = 0, \qquad n = \pm L, \pm 2L, \ldots.$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n - kL)/L]}{\pi(n - kL)/L}$$

$$x_i[n] = x[n/L] = x_c(nT/L) = x_c(nT_i) \qquad n = 0, \pm L, \pm 2L, \ldots$$

DTFT and DFT (Signal)

DTFT
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

DFT
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)kn}$$

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k})$$

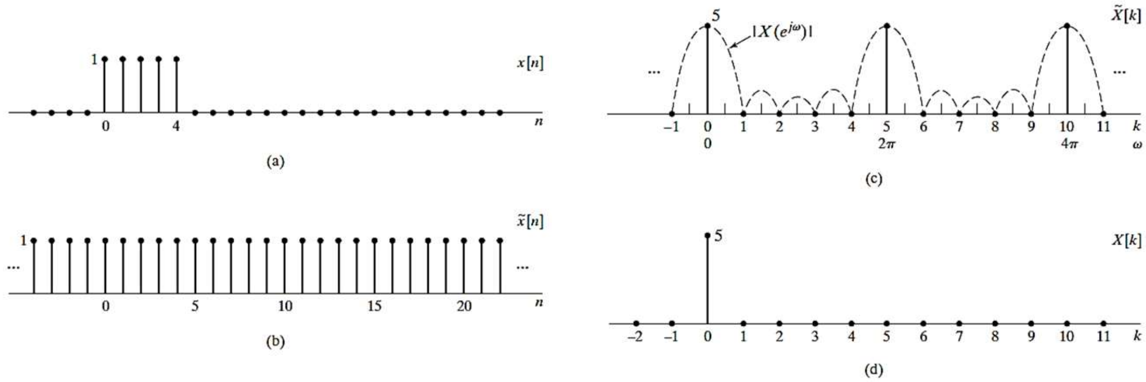
$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1, \\ 0, & \text{otherwise,} \end{cases}$$

DTFT and DFT (Image)

DTFT
$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m,n)e^{-j2\pi(mu+nv)}$$

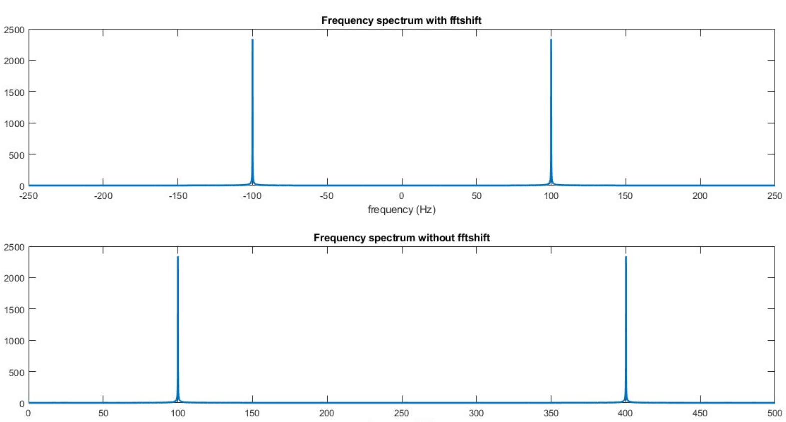
DFT
$$F(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)e^{-j2\pi(mk/M+nl/N)}$$

DFT



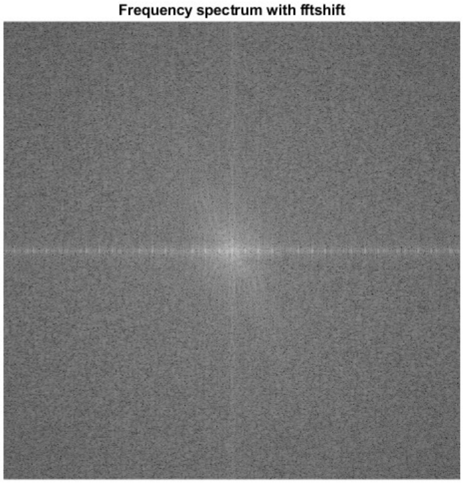
Shift Zero-Frequency (Signal)

```
fs = 500;
t = 0:1/fs:10;
x = sin(2*pi*t*100);
```

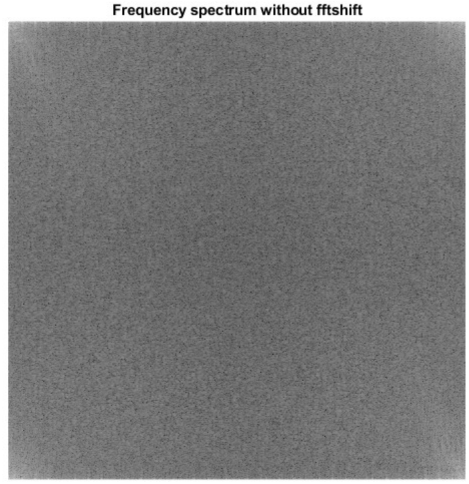


Shift Zero-Frequency (Image)

Frequency spectrum with fftshift

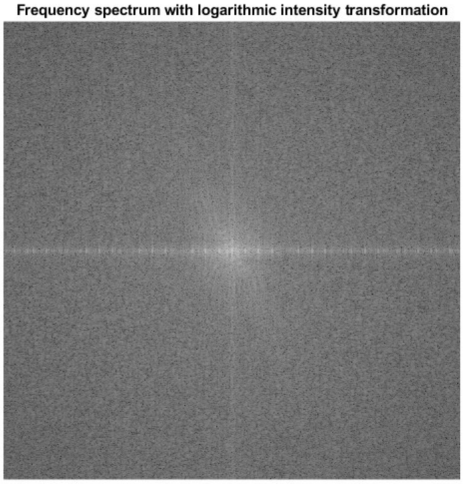


Frequency spectrum without fftshift

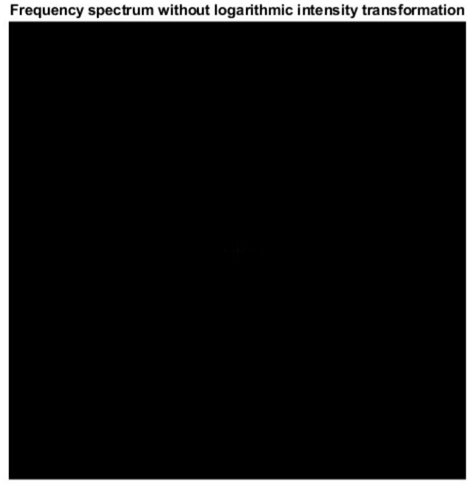


Intensity Transformation

Frequency spectrum with logarithmic intensity transformation

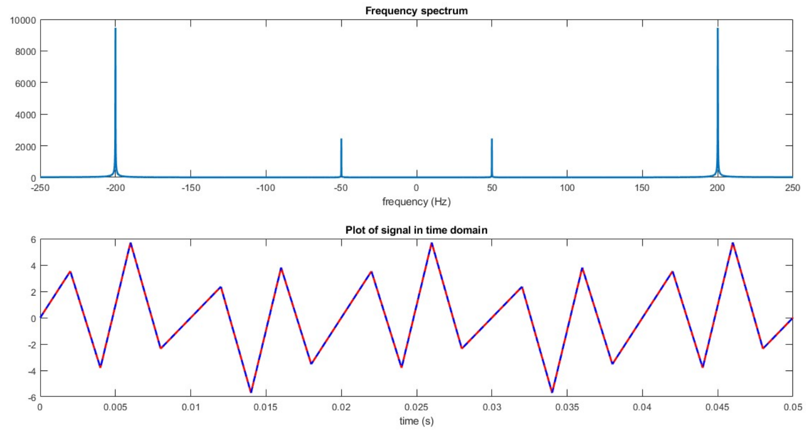


Frequency spectrum without logarithmic intensity transformation



Downsampling (Signal with M=1)

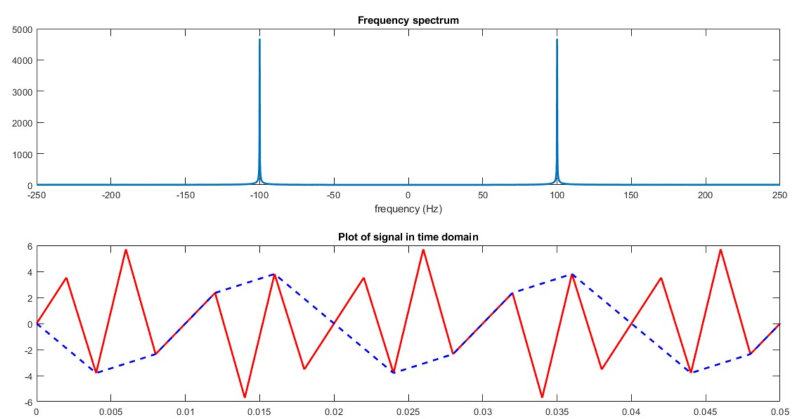
```
fs = 500;
t = 0:1/fs:10;
x = sin(2*pi*50*t)+5*sin(2*pi*200*t);
```



Downsampling (Signal with M=2)

```
fs = 500;
t = 0:1/fs:10;
x = sin(2*pi*50*t)+5*sin(2*pi*200*t);
```

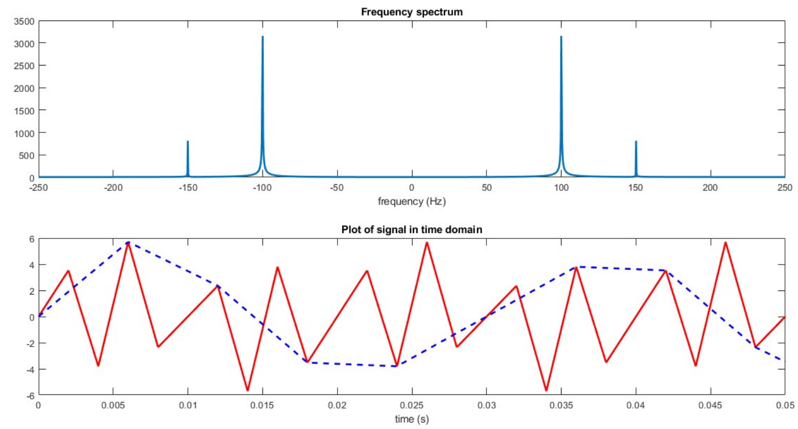
$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$



Downsampling (Signal with M=3)

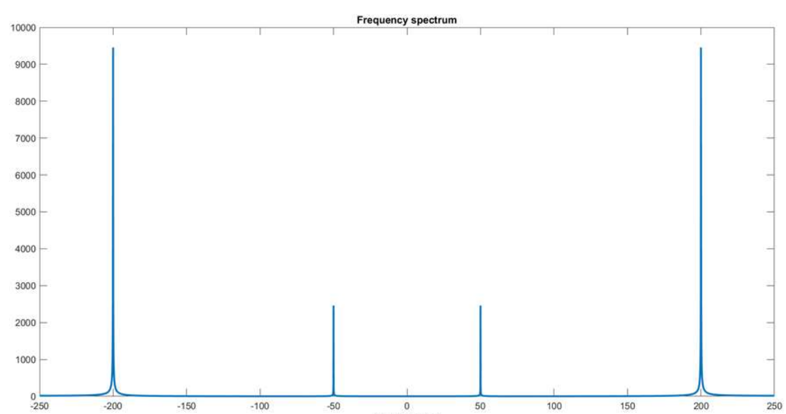
```
fs = 500;
t = 0:1/fs:10;
x = sin(2*pi*50*t)+5*sin(2*pi*200*t);
```

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\omega/M - 2\pi i/M)}\right)$$



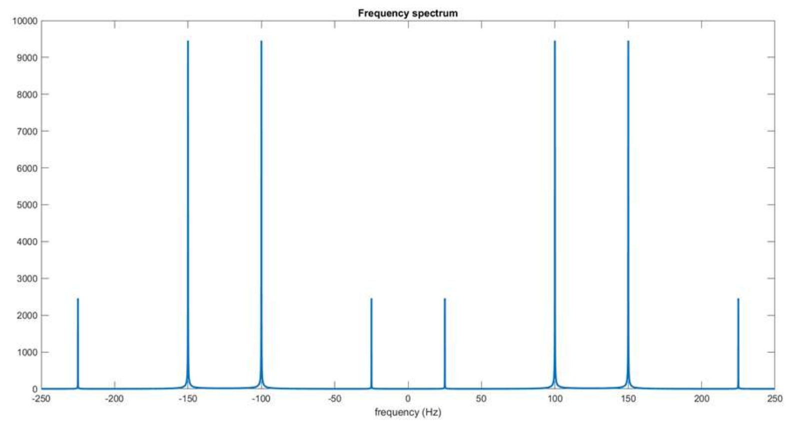
Upsampling (Signal with L=1)

```
fs = 500;
t = 0:1/fs:10;
x = sin(2*pi*50*t)+5*sin(2*pi*200*t);
```



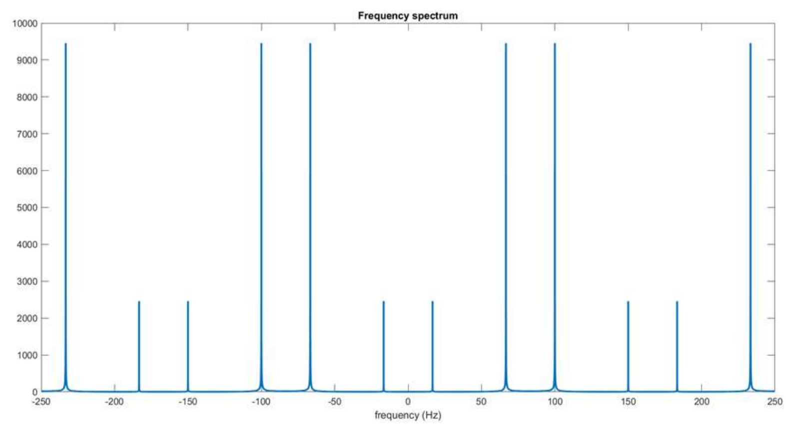
Upsampling (Signal with L=2)

```
fs = 500;
t = 0:1/fs:10;
x = sin(2*pi*50*t)+5*sin(2*pi*200*t);
```



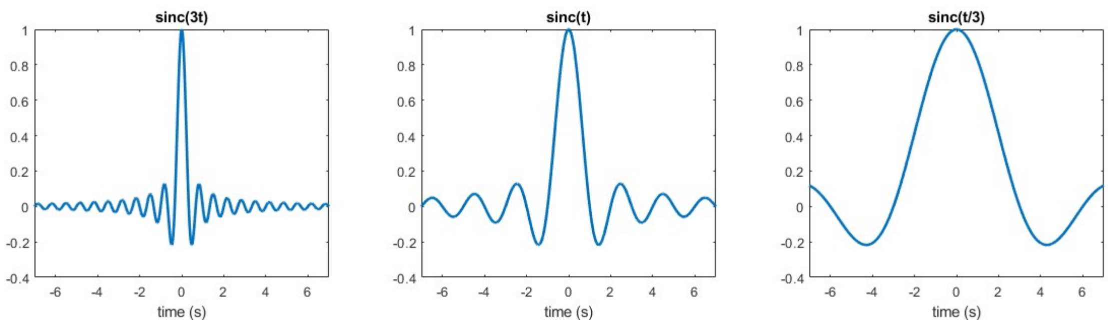
Upsampling (Signal with L=3)

```
fs = 500;
t = 0:1/fs:10;
x = sin(2*pi*50*t)+5*sin(2*pi*200*t);
```



Sinc Function

$$\mathrm{sinc}(m) = \frac{\sin(\pi m)}{(\pi m)} \qquad \mathrm{sinc}(0) = 1$$

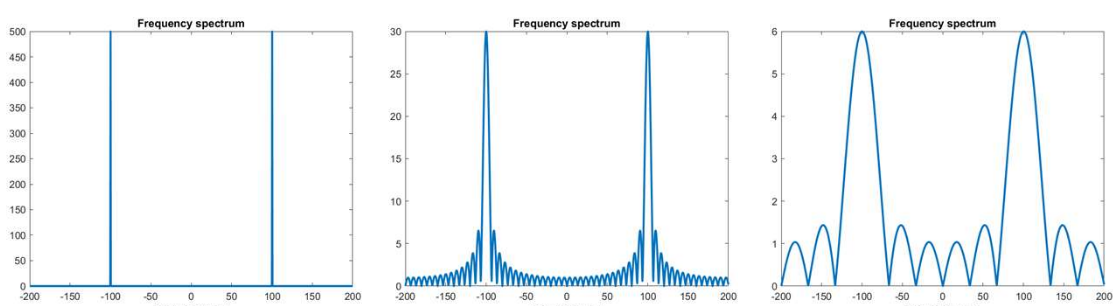


Signal Length

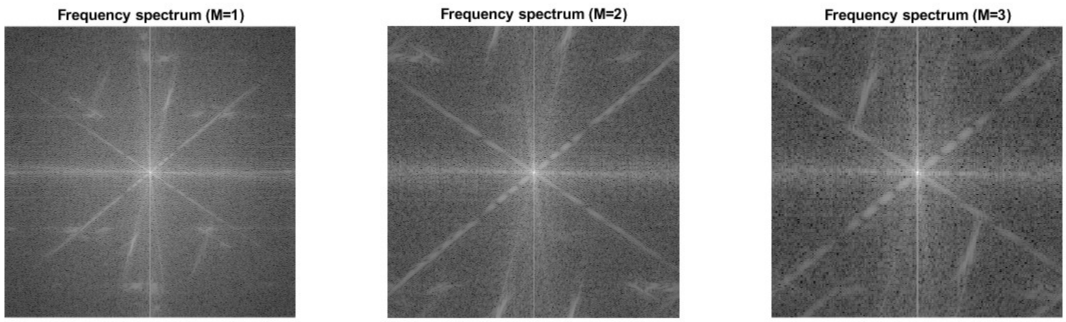
```
fs = 400;
t = 0:1/fs:5;
x = sin(2*pi*t*100);
```

```
fs = 400;
t = 0:1/fs:.15;
x = sin(2*pi*t*100);
```

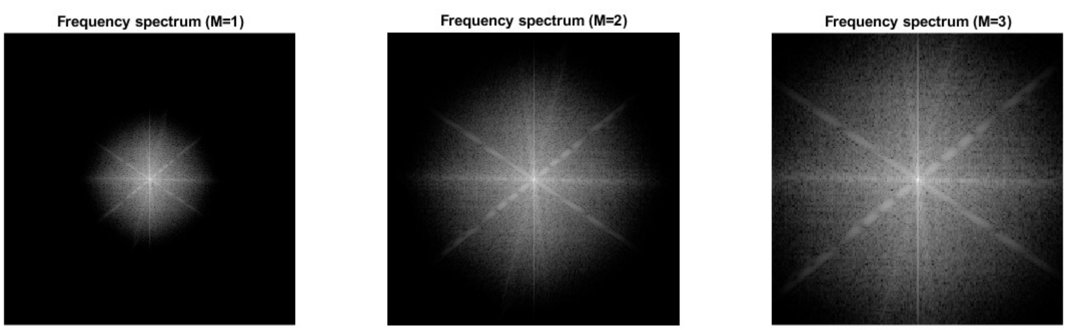
```
fs = 400;
t = 0:1/fs:.03;
x = sin(2*pi*t*100);
```



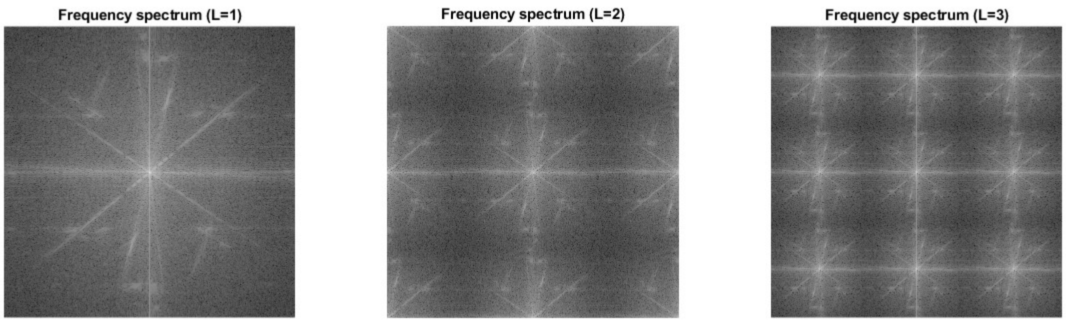
Downsampling (Image)



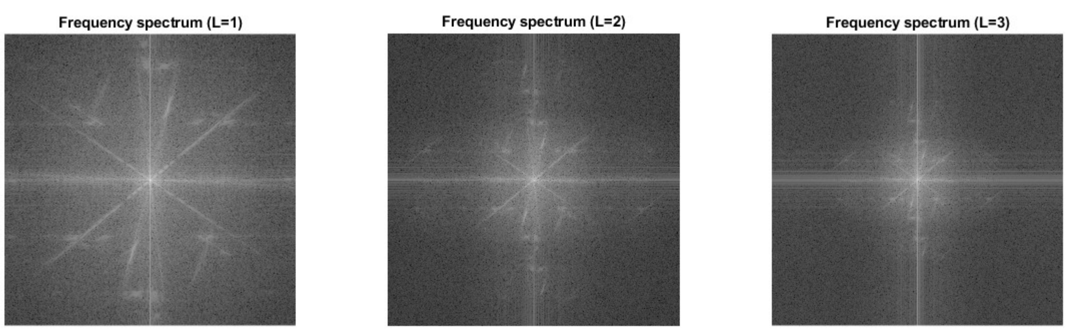
Downsampling (Image with LP Filter)



Upsampling (Image)



Upsampling (Image with interpolation)



Example

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(-1,1)	(0,1)	(1,1)
(-1,0)	(0,0)	(1,0)
(-1,-1)	(0,-1)	(1,-1)

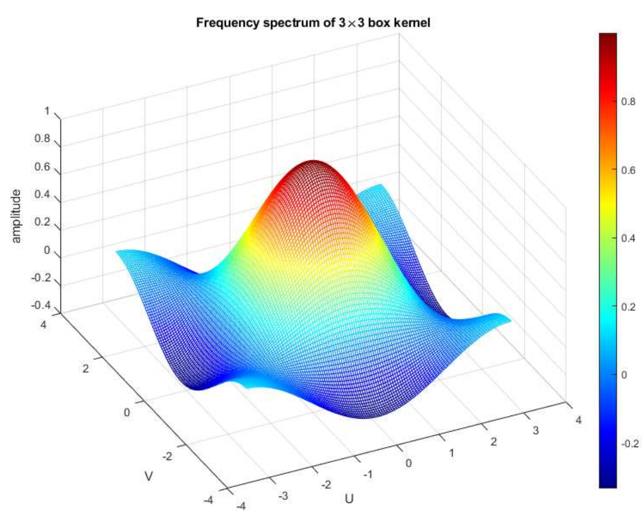
$$F(u,v)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n)e^{-j2\pi(mu+nv)}$$

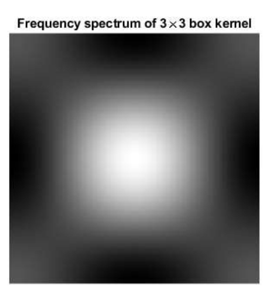
$$F(U,V)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n)e^{-j(Um+Vn)}$$

$$=\frac{1}{9}[e^{-j(-U+V)}+e^{-j(V)}+e^{-j(U+V)}+e^{-j(-U)}+e^{-j(0)}+e^{-j(U)}+e^{-j(-U-V)}+e^{-j(-V)}+e^{-j(U-V)}]$$

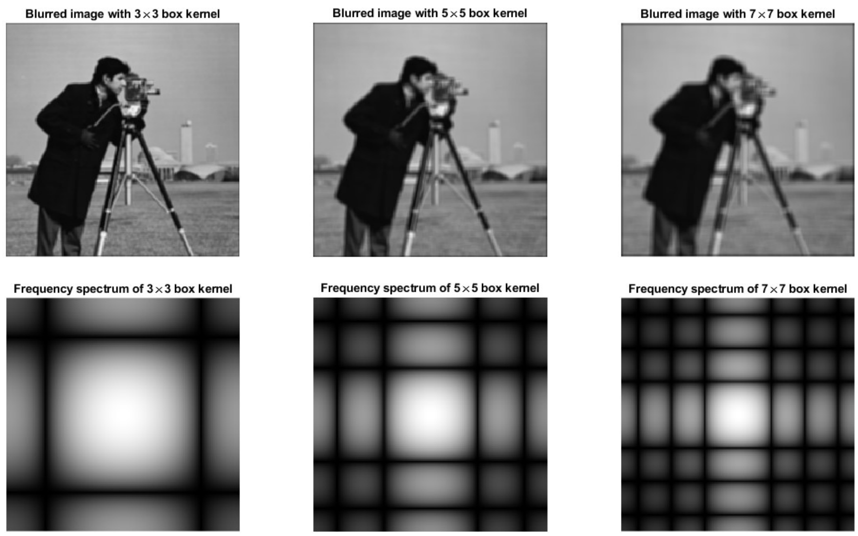
$$=\frac{1}{9}(2\cos(U-V)+2\cos(U+V)+2\cos(U)+2\cos(v)+1)$$

Example





Kernel Size



Reference

- *Alan V. Oppenheim and Ronald W. Schafer. 2009. Discrete-Time Signal Processing (3rd. ed.). Prentice Hall Press, USA. (Chapter 4 & 8)*

- *Rafael C. Gonzalez and Richard E. Woods. 2006. Digital Image Processing (3rd Edition). Prentice-Hall, Inc., USA. (Chapter 4)*