Fourier Transform and Fourier Series

Machine Vision

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Periodic Signals (Continuous)

$$x(t) = x(t+T) \quad \text{for all } t$$

$$x(t) = \cos \omega_0 t \qquad T = 2\pi/\omega_0$$

$$x(t) = e^{j\omega_0 t} \qquad e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

Example

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t},$$

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

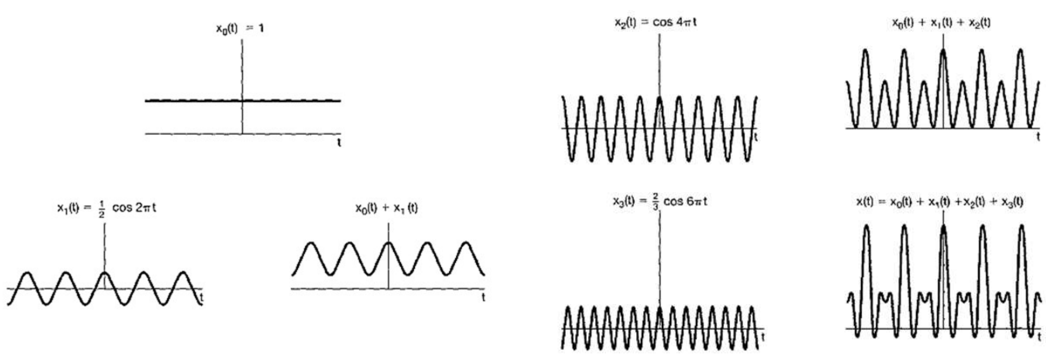
$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t}).$$

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

Example



Fourier Series (Continuous)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}\, dt$$

$$\int_0^T x(t)e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}\, dt\right]$$

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j\int_0^T \sin(k-n)\omega_0 t\, dt$$

Fourier Series (Continuous)

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j\int_0^T \sin(k-n)\omega_0 t\, dt$$

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\, dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\, dt$$

Example

$$x(t) = \sin \omega_0 t$$

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j}$$
$$a_k = 0, \qquad k \neq +1 \text{ or } -1$$

Dirichlet Conditions (FS)

1. Over any period, x(t) must be absolutely integrable; that is,

$$\int_T |x(t)| \, dt < \infty$$

$$|a_k| \leq \frac{1}{T} \int_T |x(t) e^{-jk\omega_0 t}| \, dt = \frac{1}{T} \int_T |x(t)| \, dt.$$

$$\int_T |x(t)| \, dt < \infty \longrightarrow |a_k| < \infty$$

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1$$

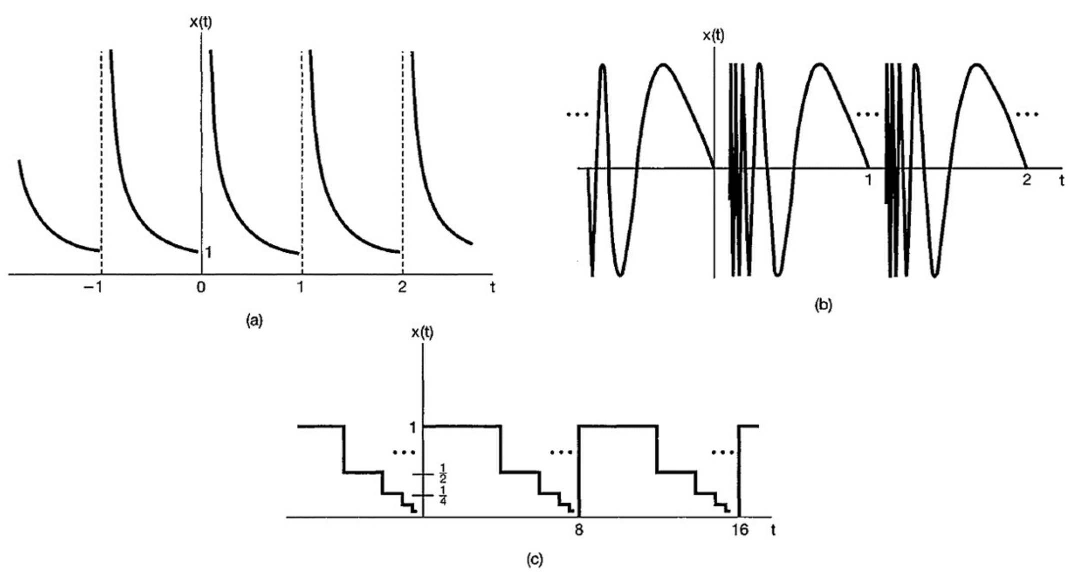
Dirichlet Conditions (FS)

2. In any finite interval of time, x(t) is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1$$

3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Dirichlet Conditions (FS)



Properties of FS (Continuous)

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k$$

1. Linearity

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k$$

$$y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} b_k$$

$$z(t) = Ax(t) + By(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} c_k = Aa_k + Bb_k$$

2. Time Shifting

$$x(t - t_0) \stackrel{\mathfrak{F}S}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k$$

Properties of FS (Continuous)

3. Time Reversal

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k$$

$$x(-t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_{-k}$$

4. Time Scaling

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

5. Multiplication

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k$$

$$y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} b_k$$

$$x(t)y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Properties of FS (Continuous)

6. Conjugation and Conjugate Symmetry

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k$$

$$x^*(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a^*_{-k}$$

$$x(t) = x^*(t) \longrightarrow a_{-k} = a_k^* \longrightarrow |a_k| = |a_{-k}|$$

7. Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Summary of Properties of FS (Continuous)

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$, $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Example

Suppose we are given the following facts about a signal $x(t)$:

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k.
3. $a_k = 0$ for $|k| > 1$.
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2}a_{-k}$ is odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$.

$$x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1}e^{-j\pi t/2}$$

$$x(t) = a_0 + a_1 e^{j\pi t/2} + (a_1 e^{j\pi t/2})^* = a_0 + 2\Re e\{a_1 e^{j\pi t/2}\}$$

$$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = 1/2 \qquad |b_1|^2 + |b_{-1}|^2 = 1/2 \qquad b_1 = -b_{-1} \qquad |b_1| = 1/2$$

$$b_0 = 0 \qquad a_0 = 0 \qquad a_1 = e^{-j\pi/2}b_{-1} = -jb_{-1} = jb_1 \qquad x(t) = \cos(\pi t/2)$$

Periodic Signals (Discrete)

$$x[n] = x[n+N]$$

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \ k = 0, \pm 1, \pm 2, \ldots \qquad \omega_0 = 2\pi/N$$

$$\phi_k[n] = \phi_{k+rN}[n]$$

$$x[n] = \sum_k a_k\phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k\phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$k = 0, 1, \ldots, N-1$$

Fourier Series (Discrete)

$$x[0] = \sum_{k=\langle N\rangle} a_k,$$

$$x[1] = \sum_{k=\langle N\rangle} a_k e^{j2\pi k/N},$$

$$\vdots$$

$$x[N-1] = \sum_{k=\langle N\rangle} a_k e^{j2\pi k(N-1)/N}.$$

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n}$$

Fourier Series (Discrete)

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n}$$

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n} \qquad k = r$$

$$a_r = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n}$$

$$\boxed{\begin{aligned} x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \\ a_k &= \frac{1}{N} \sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n} \end{aligned}}$$

Summary of Properties of FS (Discrete)

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r] y[n-r]$	$N a_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)}) a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})} \right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Example

Suppose we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

$$a_0 = 1/3$$

$$(-1)^n = e^{-j\pi n} = e^{-j(2\pi/6)3n} \qquad a_3 = 1/6$$

$$P = \sum_{k=0}^{5} |a_k|^2 \qquad a_1 = a_2 = a_4 = a_5 = 0$$

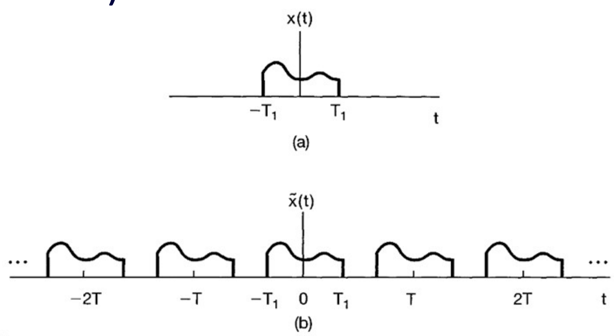
$$x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n$$

Fourier Transform (Continuous)

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$\tilde{x}(t) = x(t) \text{ for } |t| < T/2$$



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$a_k = \frac{1}{T} X(jk\omega_0)$$

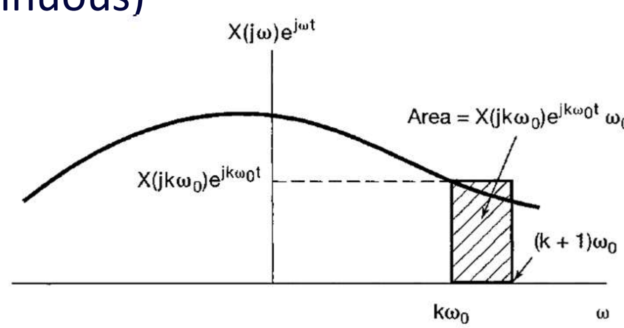
Fourier Transform (Continuous)

$$a_k = \frac{1}{T} X(jk\omega_0)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

$$2\pi/T = \omega_0$$



As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$.

Furthermore, $\omega_0 \to 0$ as $T \to \infty$

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \\ X(j\omega) &= \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \end{aligned}}$$

Dirichlet Conditions (FT)

1. x(t) be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |x(t)| dt < \infty$$

2. x(t) have a finite number of maxima and minima within any finite interval.

3. x(t) have a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Example

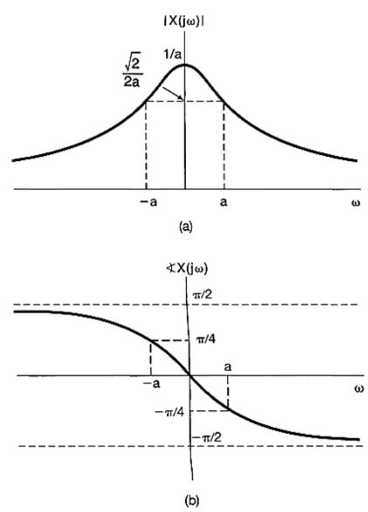
$$x(t) = e^{-at}u(t) \quad a > 0$$

$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt = -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \Big|_0^\infty$$

$$X(j\omega) = \frac{1}{a + j\omega}, \quad a > 0$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$\sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



Properties of FT (Continuous)

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega)$$

1. Linearity

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega)$$

$$y(t) \overset{\mathfrak{F}}{\longleftrightarrow} Y(j\omega)$$

$$ax(t) + by(t) \overset{\mathfrak{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$$

2. Time Shifting

$$x(t - t_0) \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega)$$

Properties of FT (Continuous)

3. Conjugation and Conjugate Symmetry

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega)$$

$$x^*(t) \overset{\mathfrak{F}}{\longleftrightarrow} X^*(-j\omega)$$

$$\left.\begin{array}{l} X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}] \\ X(j\omega) = \mathcal{R}e\{X(j\omega)\} + j\mathcal{I}m\{X(j\omega)\} \end{array}\right\} \begin{array}{l} \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \end{array}$$

$$x(t) = x_e(t) + x_o(t)$$

$$\mathfrak{F}\{x(t)\} = \mathfrak{F}\{x_e(t)\} + \mathfrak{F}\{x_o(t)\}$$

$$\mathcal{E}v\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} \mathcal{R}e\{X(j\omega)\}$$

$$\mathcal{O}d\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} j\mathcal{I}m\{X(j\omega)\}$$

Properties of FT (Continuous)

4. Differentiation and Integration

$$\frac{dx(t)}{dt} \stackrel{\mathfrak{F}}{\longleftrightarrow} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

5. Time and Frequency Scaling

$$x(at) \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

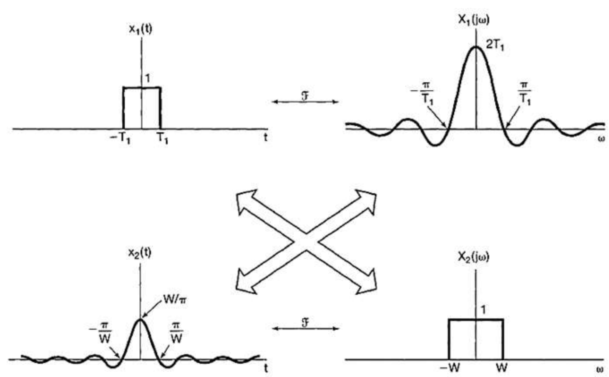
$$x(-t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(-j\omega)$$

Properties of FT (Continuous)

6. Duality

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega)$$

$$X(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} 2\pi x(-j\omega)$$



7. Parseval's Relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Properties of FT (Continuous)

8. Convolution

$$y(t) = h(t) * x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

9. Multiplication

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

Summary of Properties of FT (Continuous)

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2 d\omega$		

Example

$$h(t) = \delta(t - t_0).$$

$$H(j\omega) = e^{-j\omega t_0}$$

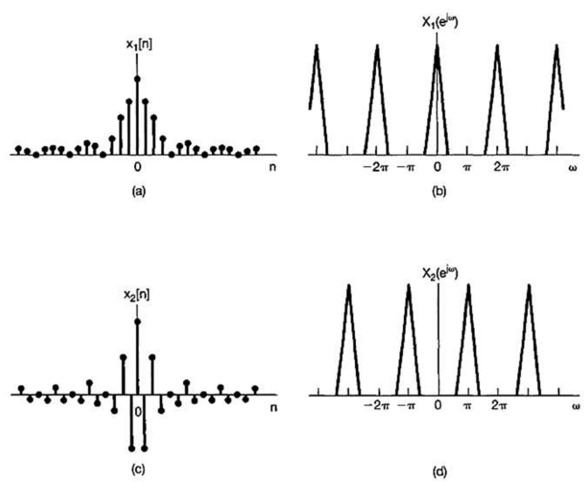
$$Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0}X(j\omega).$$

$$y(t) = x(t - t_0)$$

Fourier Transform (Discrete)

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega. \\ X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \end{aligned}}$$

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega}).$$

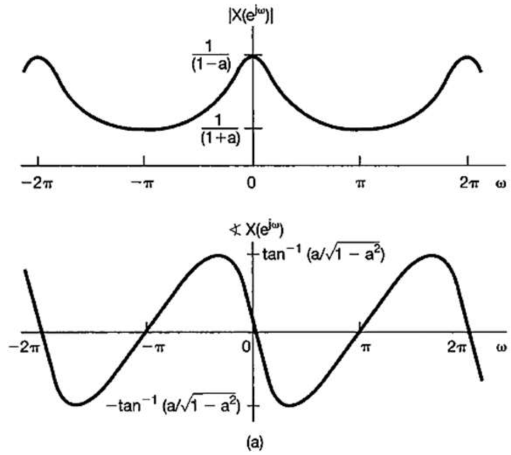


Example

$$x[n] = a^n u[n], \qquad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$



Properties of FT (Discrete)

1. Linearity

$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

$$x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega})$$

$$ax_1[n] + bx_2[n] \overset{\mathcal{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

2. Time Shifting and Frequency Shifting

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$x[n - n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega - \omega_0)})$$

Properties of FT (Discrete)

3. Periodicity

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

4. Conjugation and Conjugate Symmetry

$$x[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$x^*[n] \overset{\mathfrak{F}}{\longleftrightarrow} X^*(e^{-j\omega})$$

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n]\text{real}]$$

Properties of FT (Discrete)

5. Differencing and Accumulation

$$x[n] - x[n-1] \overset{\mathfrak{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega})$$

$$\sum_{m=-\infty}^{n} x[m] \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

6. Time Reversal

$$x[-n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{-j\omega})$$

Properties of FT (Discrete)

7. Time Expansion

$$x(at) \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$x_{(k)}[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{jk\omega})$$

8. Differentiation in Frequency

$$nx[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Properties of FT (Discrete)

9. Parseval's Relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

10. Convolution

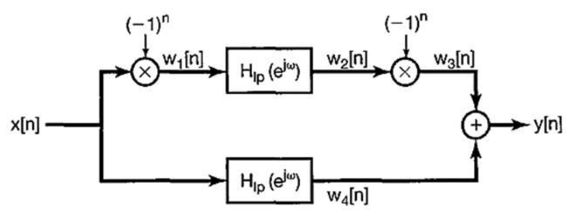
$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

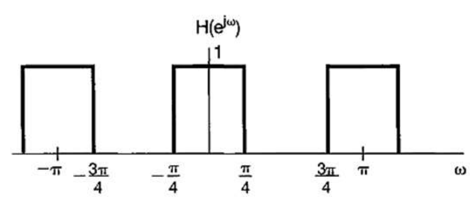
Summary of Properties of FT (Discrete)

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

Example



(a)



(b)

$$(-1)^n = e^{j\pi n}$$

$$w_1[n] = e^{j\pi n}x[n]$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

$$\begin{aligned} W_3(e^{j\omega}) &= W_2(e^{j(\omega-\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}) \end{aligned}$$

$$W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega})$$

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega})$$

$$\begin{aligned} Y(e^{j\omega}) &= W_3(e^{j\omega}) + W_4(e^{j\omega}) \\ &= [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]X(e^{j\omega}) \end{aligned}$$

$$H(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]$$

Reference

- *Alan V. Oppenheim, Alan S. Willsky, and S. Hamid Nawab. 1996. Signals & systems (2nd ed.). Prentice-Hall, Inc., USA. (Chapter 3-5)*