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THE LIMITING BEHAVIOR OF A ONE-DIMENSIONAL RANDOM WALK IN A RANDOM MEDIUM

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1. Statement of the Problem and Formulation of Results

We consider the simplest random walk on the set of integer points of the straight line Z^1, under which a randomly moving point passes from $x \in Z^1$ to $x \pm 1$ with probabilities $p(x)$ and $q(x) = 1 - p(x)$, respectively. The position of the point after n transitions is denoted by $x(n)$. It is assumed that $x(0) = 0$.

A knowledge of all the $p(x)$ determines the probability distribution of the random variables $x(n)$, $n > 0$. All the probabilities relating to events connected with the behavior of $x(n)$, $n > 0$, will be denoted below by P.

It is said that a random walk occurs in a *random medium* if the probabilities $p(x)$ themselves are realizations of some random process. The simplest version arises when $p(x)$ forms a sequence of independent random variables; for example, $p(x) = \frac{1}{2} + \varepsilon\xi(x)$, where $\xi(x) = \pm 1$ is a random sign, and the signs $\xi(x)$ for different x are mutually independent, $0 < \varepsilon < \frac{1}{2}$.

In this paper we mainly discuss the situation when the $p(x)$ are independent. Certain generalizations are indicated at the end. The probabilities relating to events depending on the realization of $p(x)$ are denoted by $\mathbf{P}$.

Problems connected with the recurrence and non-recurrence properties of a random walk in a random medium were examined in [1]–[3]. In this paper we investigate the behaviour of $x(n)$ as $n \to \infty$. The basic assumption is that $p(x)$, $q(x) \geqq \text{const} > 0$ and

$$\mathbf{E}_{\mathbf{P}} \log \frac{q(x)}{p(x)} = 0.$$

It will be shown that, under these conditions, in contrast to the ordinary random walk, for large n the variable $x(n)$ takes on values of order $\log^2 n$. But if $x(n)$ is normalized, i.e., the variable $x(n)/\log^2 n$ is considered, then as $n \to \infty$ the probability distribution for $x(n)/\log^2 n$ becomes localized, i.e., concentrated in an arbitrarily small neighborhood of some point depending on the realization $p = \{p(x)\}$. The basic result can be formulated more precisely in the following manner.

Let $\alpha > 0$, $\delta > 0$ be given. For all sufficiently large n there exist a set C_n in the space of realizations p and a point $m^{(n)} = m^{(n)}(p)$ for each $p \in C_n$ such that

$\mathbf{P}(C_n) \geqq 1-\alpha$, *and for* $p \in C_n$

$$P\left(\left|\frac{x(n)}{\log^2 n} - m^{(n)}\right| < \delta\right) \to 1$$

as $n \to \infty$ *uniformly in* $p \in C_n$. *As* $n \to \infty$ *the probability distributions for* $m^{(n)}$ *converge weakly to some limit distribution.*

The function $w^{(n)}(t)$ is basic for the analysis; it is formed from the transition probabilities $p(x)$ in the following way:

$$w^{(n)}(t) = \begin{cases} \dfrac{1}{\log n} \sum\limits_{0 \leqq x \leqq k} \log \dfrac{q(x)}{p(x)} & \text{for } t = \dfrac{k}{\log^2 n}, \qquad k = 1, 2, \dots, \\ \dfrac{1}{\log n} \sum\limits_{k \leqq x \leqq 0} \log \dfrac{q(x)}{p(x)} & \text{for } t = \dfrac{k}{\log^2 n}, \qquad k = -1, -2, \cdots. \end{cases}$$

For the remaining t the function $w^{(n)}(t)$ is defined with the aid of linear interpolation, $w^{(n)}(0) = 0$. In what follows it turns out that $m^{(n)}(p)$ is one of the local minima of the function $w^{(n)}$.

The proof of the basic result is carried out in Sections 4 and 5. In Sections 2 and 3, we shall conduct auxiliary constructions, and concluding remarks are made in Section 6.

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2. Exit Probabilities and Their Estimates

Let $[a, b]$ be a segment with end-points belonging to Z^1. We shall denote by $k^+_{[a,b]}(x)$ the P-probability of paths starting at x which hit b before a; $k^-_{[a,b]}(x) = 1 - k^+_{[a,b]}(x)$ is the P-probability of paths leaving x and hitting a before b. Then

$$k^+_{[a,b]}(x) = p(x) k^+_{[a,b]}(x+1) + q(x) k^+_{[a,b]}(x-1),$$
$$k^+_{[a,b]}(a) = 0, \qquad k^+_{[a,b]}(b) = 1$$

and

$$k^-_{[a,b]}(x) = p(x) k^-_{[a,b]}(x+1) + q(x) k^-_{[a,b]}(x-1),$$
$$k^-_{[a,b]}(a) = 1, \qquad k^+_{[a,b]}(b) = 0.$$

The solutions of these equations are easily found. Let us cite the corresponding results in a form suitable for what follows.

Lemma 1. *The following equations hold*:

$$k^+_{[a,b]}(x) = \left(\sum_{y=a+1}^{x} \exp\{\log n [w^{(n)}(y \log^{-2} n) - w^{(n)}(a \log^{-2} n)]\}\right)$$
$$\cdot \left(\sum_{y=a+1}^{b} \exp\{\log n [w^{(n)}(y \log^{-2} n) - w^{(n)}(a \log^{-2} n)]\}\right)^{-1},$$

$$k^-_{[a,b]}(x)=\left(\sum_{y=x+1}^{b}\exp\{\log n[w^{(n)}(y\log^{-2}n)-w^{(n)}(b\log^{-2}n)]\}\right)$$
$$\cdot\left(\sum_{y=a+1}^{b}\exp\{\log n[w^{(n)}(y\log^{-2}n)-w^{(n)}(b\log^{-2}n)]\}\right)^{-1}.$$

We shall state two consequences of these relations. Let us introduce the notation $T_{r,s}=[r\log^{-2}n, s\log^{-2}n]$.

Corollary 1. *Let the variable x be such that*

$$w^{(n)}(x\log^{-2}n)=\max_{t\in T_{a,b}}w^{(n)}(t),\qquad w^{(n)}(a\log^{-2}n)=\min_{t\in T_{a,x}}w^{(n)}(t),$$
$$w^{(n)}(b\log^{-2}n)=\min_{t\in T_{x,b}}w^{(n)}(t).$$

Then $k^+_{[a,b]}(x)\geqq(b-a)^{-1}$, $k^-_{[a,b]}(x)\geqq(b-a)^{-1}$.

In fact,

$$\max_{y\in[a,b]}[w^{(n)}(y\log^{-2}n)-w^{(n)}(a\log^{-2}n)]=w^{(n)}(x\log^{-2}n)-w^{(n)}(a\log^{-2}n).$$

Therefore,

$$k^+_{[a,b]}(x)\geqq\exp\{\log n[w^{(n)}(x\log^{-2}n)-w^{(n)}(a\log^{-2}n)]\}$$
$$\cdot[(b-a)\max_{y\in[a,b]}\exp\{\log n[w^{(n)}(y\log^{-2}n)-w^{(n)}(a\log^{-2}n)]\}]^{-1}$$
$$=(b-a)^{-1}.$$

The estimate for $k^-_{[a,b]}(x)$ is obtained similarly.

Corollary 2. (a) *Let* $x=a+1$, $w^{(n)}(a\log^{-2}n)=\min_{t\in T_{a,b}}w^{(n)}(t)$, $w^{(n)}(b)=\max_{t\in T_{a,b}}w^{(n)}(t)$. *Then*

$$\text{const}\,(b-a)^{-1}\exp\{-\log n[w^{(n)}(b\log^{-2}n)-w^{(n)}(a\log^{-2}n)]\}$$
$$\leqq k^+_{[a,b]}(x)\leqq\text{const}\exp\{-\log n[w^{(n)}(b\log^{-2}n)-w^{(n)}(a\log^{-2}n)]\}.$$

(b) *Let* $x=b-1$, $w^{(n)}(b\log^{-2}n)=\min_{t\in T_{a,b}}w^{(n)}(t)$, $w^{(n)}(a\log^{-2}n)=\max_{t\in T_{a,b}}w^{(n)}(t)$. *Then*

$$\text{const}\,(b-a)^{-1}\exp\{-\log n[w^{(n)}(a\log^{-2}n)-w^{(n)}(b\log^{-2}n)]\}$$
$$\leqq k^-_{[a,b]}(x)\leqq\text{const}\exp\{-\log n[w^{(n)}(a\log^{-2}n)-w^{(n)}(b\log^{-2}n)]\}.$$

Let us prove, for example, assertion (a); assertion (b) is obtained similarly. We have

$$k^+_{[a,b]}(x)=q(a+1)(p(a+1))^{-1}$$
$$\cdot\left(\sum_{a+1}^{b}\exp\{\log n[w^{(n)}(y\log^{-2}n)-w^{(n)}(a\log^{-2}n)]\}\right)^{-1}.$$

According to the condition the numerator is contained between two constants, while

$$\max_{a\leqq y\leqq b}[w^{(n)}(y\log^{-2}n)-w^{(n)}(a\log^{-2}n)]=w^{(n)}(b\log^{-2}n)-w^{(n)}(a\log^{-2}n).$$

Therefore,

$$k^{+}_{[a,b]}(x)\leqq \text{const}\exp\{-\log n[w^{(n)}(b\log^{-2}n)-w^{(n)}(a\log^{-2}n)]\},$$

$$k^{+}_{[a,b]}(x)\geqq \frac{\text{const}}{b-a}\exp\{-\log n[w^{(n)}(b\log^{-2}n)-w^{(n)}(a\log^{-2}n)]\},$$

which it was required to show.

Now we shall consider the behavior for large n of the expectation $h_{[a,b]}(x)$ of the time a path, exiting from x, will reach the end-points of the segment $[a, b]$. It is clear that $h_{[a,b]}(a)=h_{[a,b]}(b)=0$ and, for $a<x<b$,

$$h_{[a,b]}(x)=p(x)h_{[a,b]}(x+1)+q(x)h_{[a,b]}(x-1)+1.$$

It is also not difficult to write out explicitly the solution of this chain of equations. That is,

$$\begin{aligned}h_{[a,b]}(x)=&\sum_{y=a+1}^{x}\left(\prod_{z=a+1}^{y-1}q(z)p^{-1}(z)\right)\\&\cdot\left(\sum_{a+1\leqq y_1<y_2\leqq b}(p(y_1))^{-1}\prod_{z=y_1+1}^{y_2-1}q(z)(p(z))^{-1}\right)\\&\cdot\left(\sum_{a+2\leqq y\leqq b}\prod_{z=a+1}^{y-1}q(z)(p(z))^{-1}\right)^{-1}\\&-\sum_{a+1\leqq y_1<y_2<x}(p(y_1))^{-1}\prod_{z=y_1+1}^{y_2-1}q(z)(p(z))^{-1}.\end{aligned}$$

With the aid of the function $w^{(n)}$ the latter expression can be rewritten in the following way:

$$\begin{aligned}h_{[a,b]}(x)=&\sum_{y=a+1}^{x}\exp\{\log n[w^{(n)}((y-1)\log^{-2}n)-w^{(n)}(a\log^{-2}n)]\}\\&\cdot\Bigg(\sum_{a+1\leqq y_1<y_2\leqq b}(p(y_1))^{-1}\\&\qquad\cdot\exp\{\log n[w^{(n)}((y_2-1)\log^{-2}n)-w^{(n)}(y_1\log^{-2}n)]\}\Bigg)\\&\cdot\left(\sum_{a+2\leqq y\leqq b}\exp\{\log n[w^{(n)}((y-1)\log^{-2}n)-w^{(n)}(a\log^{-2}n)]\}\right)^{-1}\\&-\sum_{a+1\leqq y_1<y_2<x}(p(y_1))^{-1}\\&\qquad\cdot\exp\{\log n[w^{(n)}((y_2-1)\log^{-2}n)-w^{(n)}(y_1\log^{-2}n)]\}.\end{aligned}$$

We shall now deduce a series of corollaries. Let the segment $[a, b]$ and the point x be such that $x = a+1$,

$$w^{(n)}(a \log^{-2} n) = \min_{t \in T_{a,b}} w^{(n)}(t), \qquad w^{(n)}(b \log^{-2} n) = \max_{t \in T_{a,b}} w^{(n)}(t).$$

Then

$$\begin{aligned} h_{[a,b]}(x) = &\Big(\sum_{a+1 \leqq y_1 < y_2 \leqq b} (p(y_1))^{-1} \exp \{\log n [w^{(n)}((y_2 - 1) \log^{-2} n) \\ &- w^{(n)}(y_1 \log^{-2} n)]\} \Big) \\ &\cdot \Big(\sum_{a+2 \leqq y \leqq b} \exp \{\log n [w^{(n)}((y-1)\log^{-2} n) - w^{(n)}(a \log^{-2})]\} \Big)^{-1} \\ &\leqq \mathrm{const}\, (b-a)^2, \end{aligned}$$

since

$$\begin{aligned} \max_{a+1 \leqq y_1 \leqq y_2 \leqq b} &[w^{(n)}(y_2 \log^{-2} n) - w^{(n)}(y_1 \log^{-2} n)] \\ &= w^{(n)}(b \log^{-2} n) - w^{(n)}((a+1) \log^{-2} n). \end{aligned}$$

A similar inequality is also valid in the case when $x = b-1$,

$$w^{(n)}\Big(\frac{b}{\log^2 n}\Big) = \min_{a \leqq t \leqq b} w^{(n)}\Big(\frac{t}{\log^2 n}\Big), \qquad w^{(n)}\Big(\frac{a}{\log^2 n}\Big) = \max_{a \leqq t \leqq b} w^{(n)}\Big(\frac{t}{\log^2 n}\Big).$$

Let us now consider the situation when

$$w^{(n)}\Big(\frac{a}{\log^2 n}\Big) = \max_{a \leqq t \leqq b} w^{(n)}\Big(\frac{t}{\log^2 n}\Big), \qquad w^{(n)}\Big(\frac{b}{\log^2 n}\Big) = \min_{a \leqq t \leqq x} w^{(n)}\Big(\frac{t}{\log^2 n}\Big).$$

In this case,

$$h_{[a,b]}(x) \leqq \mathrm{const}\,(b-a)^2 \exp \Big\{ \log n \max_{a < y_1 \leqq y_2 < b} \Big[w^{(n)}\Big(\frac{y_2}{\log^2 n}\Big) - w^{(n)}\Big(\frac{y_1}{\log^2 n}\Big)\Big]\Big\}.$$

In what follows the difference $(b-a)$ takes on, as $n \to \infty$, values of order $\log n$, and the difference $w^{(n)}(y_2/\log^2 n) - w^{(n)}(y_1/\log^2 n)$ takes on values of order 1.

The variance of the time of reaching the boundary can be estimated similarly.

3. Local Maxima and Minima of the Function $w^{(n)}(t)$

In this section we shall introduce the maxima and minima of the curve $w^{(n)}(t)$ of interest to us. Our construction is carried out in a manner such that it also has meaning as $n \to \infty$, i.e., for typical realizations of a Wiener measure.

Let the curve $w^{(n)}(t)$, $-\infty < t < \infty$, (see Section 1) be given. We shall call a segment $[t_1, t_2]$ a *depression* if for $\bar{t}$ such that $w^{(n)}(\bar{t}) = \min_{t_1 \leqq t \leqq t_2} w^{(n)}(t)$,

$$w^{(n)}(t_1) = \max_{t_1 \leqq t \leqq \bar{t}} w^{(n)}(t), \qquad w^{(n)}(t_2) = \max_{\bar{t} \leqq t \leqq t_2} w^{(n)}(t).$$

We shall call $w^{(n)}(t_1)-w^{(n)}(\bar{t})(w^{(n)}(t_2)-w^{(n)}(\bar{t}))$ the left (right) depth of the depression. We shall call $\min(w^{(n)}(t_1)-w^{(n)}(\bar{t}),\ w^{(n)}(t_2)-w^{(n)}(\bar{t}))$ the depth of the depression $d([t_1,t_2])$. We shall call a set of points $\mathfrak{M}=\{M_0, m_1, M_1, m_2, \cdots, M_r, m_{r+1}, M_{r+1}\}$ such that each segment $[M_i, M_{i+1}]$, $0\leqq i\leqq r$, is a depression, $w^{(n)}(m_i)=\min_{t\in[M_i,M_{i+1}]} w^{(n)}(t)$, the *set* of maxima and minima of the function $w^{(n)}$. The set $\mathfrak{M}'\in\mathfrak{M}''$ if all points of $\mathfrak{M}'$ are contained among the points of $\mathfrak{M}''$.

We shall now define the operation of refining a depression. Let $\mathfrak{M}=\{M_0, m_1, M_1, \cdots, M_r, m_{r+1}, M_{r+1}\}$. We consider a segment $[m_i, M_i]$. We find t_1, t_2 such that $m_i\leqq t_1<t_2\leqq M_i$ and

$$w^{(n)}(t_1)-w^{(n)}(t_2)=\max_{m_i\leqq t'\leqq t''\leqq M_i}(w^{(n)}(t')-w^{(n)}(t'')).$$

It is not difficult to see that $[M_{i-1}, t_1]$ and $[t_1, M_i]$ will be depressions; therefore, adding the points t_1, t_2 to the set $\mathfrak{M}$ again reduces to a set of maxima and minima. We shall call the operation described a *right refinement operation*. In a similar manner let us take the segment $[M_i, m_{i+1}]$ and consider $t_1, t_2, M_i\leqq t_1\leqq t_2\leqq m_{i+1}$, such that $w^{(n)}(t_2)-w^{(n)}(t_1)=\max_{M_i\leqq t'\leqq t''\leqq m_{i+1}}(w^{(n)}(t'')-w^{(n)}(t'))$. It is directly verified that $[M_i, t_2]$ and $[t_2, M_{i+1}]$ are depressions, and adding the points t_1, t_2 to the set $\mathfrak{M}$ likewise leads us to a set of maxima and minima. We shall call this operation a *left refinement operation*.

Let M^+ (M^-) be the smallest (largest) $t>0$ $(t<0)$ for which $w^{(n)}(t)=1$, and let m_0 be such that $w^{(n)}(m_0)=\min_{M^-\leqq t\leqq M^+} w^{(n)}(t)$. Then $[M^+, M^+]$ is a depression and $\{M^-, m_0, M^+\}$ is a set of maxima and minima.

Lemma 2. *Let $\alpha>0$, $\delta>0$ be given. Then there exist, for all sufficiently large n, a set C_n and numbers r and δ_1 such that $\mathbf{P}(C_n)\geqq 1-\alpha$, and for any choice of $p\in C_n$ from the set $\{M^-, m_0, M^+\}$ a set of maxima and minima $\mathfrak{M}=\{M^-=M_0, m_1, M_1, \cdots, M_r, m_r, M_{r+1}=M^+\}$ can be obtained using not more than r left and right refinement operations, such that*

$$\max_{0\leqq j\leqq r}(M_{j+1}-M_j)\leqq\delta,\qquad \min_{\substack{t',t''\in\mathfrak{M}\\ t'\neq t''}}|w^{(n)}(t')-w^{(n)}(t'')|\geqq\delta_1,$$

$$\min_{\substack{t\in\mathfrak{M}\\ t\neq M_0M_{r+1}}}|w^{(n)}(t)-1|\geqq\delta_1,\qquad \min_{t\in\mathfrak{M}}|t|\geqq\delta_1,\qquad M_{r+1}-M_0\leqq\delta_1^{-1}.$$

The proof of the lemma follows from the fact that the assertion of the lemma is valid for a Wiener measure, and a measure in the space of paths $w^{(n)}$ converges weakly to a Wiener measure.

Lemma 3. *Let $M_{i_1}\in\mathfrak{M}$, $M_{i_2}\in\mathfrak{M}$ and $[M_{i_1}, M_{i_2}]$ be a depression, $d([M_{i_1}, M_{i_2}])>1$ and $0\in[M_{i_1}, M_{i_2}]$. Then*

$$P\{x(m)\in[M_{i_1}\log^2 n, M_{i_2}\log^2 n]\text{ for all }0\leqq m\leqq n\}\to 1\quad as\ n\to\infty.$$

PROOF. Let m_j be a minimum of the depression under consideration, $m_j\in\mathfrak{M}$, and for definiteness $m_j<0$. Also let the point 0 be contained in the depression

$[M_{i_0}, M_{i_0+1}]$. We take a segment $[m_j \log^2 n, M_{i_2} \log^2 n]$. From Lemma 1 for $p \in C_n$

$$k^+_{[m_j\log^2 n, M_{i_2}\log^2 n]}(0) \leqq \exp\{-\log n[w^{(n)}(M_{i_2}) - w^{(n)}(m_j)]\}$$
$$\cdot |M_{i_2} - m_j| \log^2 n \exp\{\log n[w^{(n)}(M_{i_0}) - w^{(n)}(m_j)]\}$$
$$\leqq \delta_1^{-1} \log^2 n \exp\{-(\log n)\delta_1\} \to 0$$

as $n \to \infty$. Thus, as $n \to \infty$, a random point hits $m_j \log^2 n$ earlier than $M_{i_2} \log^2 n$ with probability tending to 1.

Let us introduce the P-probability $\pi_{[a,b]}(x)$ that a random point, leaving x, hits one of the end-points of $[a, b]$ without getting to x on the way. It is clear that

$$\pi_{[a,b]}(x) = p(x)k^+_{[x,b]}(x+1) + q(x)k^-_{[a,x]}(x-1).$$

We shall now estimate $\pi_{[M_{i_1}\log^2 n, M_{i_2}\log^2 n]}(m_j \log^2 n)$. By virtue of Lemma 1,

$$k^+_{[m_j\log^2 n, M_{i_2}\log^2 n]}(m_j \log^2 n + 1) \leqq \text{const} \exp\{-\log n[w^{(n)}(M_{i_2}) - w^{(n)}(m_j)]\},$$
$$k^-_{[M_{i_1}\log^2 n, m_j\log^2 n]}(m_j \log^2 n - 1) \leqq \text{const} \exp\{-\log n[w^{(n)}(M_{i_1}) - w^{(n)}(m_j)]\}.$$

Hence it follows that $\pi_{[M_{i_1}\log^2 n, M_{i_2}\log^2 n]}(m_j \log^2 n) \leqq \text{const}/n^{1+\delta_1}$. From the latter inequality it follows that the P-probability that the random point, leaving m_j, returns to m_j at least n times before it hits M_{i_1} or M_{i_2} is equal to

$$(1 - \pi_{[M_{i_1}\log^2 n, M_{i_2}\log^2 n]}(m_j \log^2 n))^n \geqq \left(1 - \frac{\text{const}}{n^{1+\delta_1}}\right)^n \to 1$$

as $n \to \infty$. The lemma is proved.

Let $[M_{i_1'}, M_{i_2'}]$, $[M_{i_1''}, M_{i_2''}]$ be two depressions satisfying the conditions of Lemma 3. We shall show that their intersection $D = [M_{i_1'}, M_{i_2'}] \cap [M_{i_1''}, M_{i_2''}]$ is again a depression satisfying the conditions of Lemma 3. We shall consider only the case when $D = [M_{i_1''}, M_{i_1'}]$, since the remaining cases are simpler.

Let us denote by $m_{j'}$, $m_{j''}$ the minima corresponding to the initial depressions. We shall show that D must contain at least one of the points $m_{j'}$, $m_{j''}$. Let us assume that this is not so, and let $w^{(n)}(M_{i_2'}) > w^{(n)}(M_{i_2''})$. But $M_{i_1''} < M_{i_2'} < m_{j''}$, and we obtain a contradiction with the fact that $[M_{i_1''}, M_{i_2''}]$ is a depression. If $w^{(n)}(M_{i_1''}) > w^{(n)}(M_{i_2'})$, then we obtain a contradiction with the fact that $[M_{i_1'}, M_{i_2'}]$ is a depression.

Assume that $m_{j'} \in D$ and $M_{i_1'} < M_{i_1''} < m_{j'}$. If $w^{(n)}(M_{i_1''}) - w^{(n)}(m_{j'}) < 1$, then $w^{(n)}(m_{j'}) > w^{(n)}(m_{j''})$, since $w^{(n)}(M_{i_1''}) - w^{(n)}(m_{j''}) > 1$. Hence $M_{i_2'} > m_{j''}$, since otherwise $w^{(n)}(M_{i_1''}) < w^{(n)}(M_{i_2'})$, and we again obtain a contradiction with the fact that $[M_{i_1''}, M_{i_2''}]$ is a depression. Hence $M_{i_2'} > m_{j''}$ and both minima belong to D. Then $m_{j''}$ is a minimum of D and $w^{(n)}(M_{i_1'}) - w^{(n)}(m_{j''}) \geqq w^{(n)}(M_{i_1'}) - w^{(n)}(m_{j'}) > 1$. Thus we see that D is a depression satisfying the conditions of Lemma 3.

We can now derive the smallest depression $[M_-^{(0)} = M_{i_1}, M_+^{(0)} = M_{i_2}] \subset \mathfrak{M}$ satisfying the conditions of Lemma 3. We shall call this depression *basic*, and the set of maxima and minima $\mathfrak{M}_1 = \{M_{i_1}, m_{i_1+1}, M_{i_1+1}, \cdots, M_{i_2-1}, m_{i_2}, M_{i_2}\}$ the *basic set*. Let $m^{(0)} = m_{j_0} \in \mathfrak{M}_1$ be a minimum of the basic depression. We can now formulate the basic result of this paper more precisely in the form of the following theorem.

Theorem 1. *For all $p \in C_n$ (see Lemma 2) the relation*

$$P\left(\frac{x(n)}{\log^2 n} \in [M_{j_0-1}, M_{j_0}]\right) \to 1 \quad \text{as } n \to \infty$$

holds uniformly in $p \in C_n$.

4. Auxiliary Process

Let $\mathfrak{M}_1 = \{M_-^{(0)} = M_{i_1}, m_{i_1+1}, M_{i_1+1}, \cdots, m^{(0)} = m_j, M_j, \cdots, m_{i_2}, M_{i_2} = M_+^{(0)}\}$ be the basic set and $m^{(0)}$ the absolute minimum for $\mathfrak{M}_1$. Let us fix an increasing sequence of sets of maxima and minima $\mathfrak{M}^{(1)} \subset \mathfrak{M}^{(2)} \subset \cdots \subset \mathfrak{M}^{(r)} = \mathfrak{M}_1$, where $\mathfrak{M}^{(1)} = \{M_-^{(0)}, m^{(0)}, M_+^{(0)}\}$ and $\mathfrak{M}^{(i+1)}$ is obtained from $\mathfrak{M}^{(i)}$ with the aid of the (right or left) refinement operation.

Let n be fixed and let $x = \{x(i), i \geqq 0\}$ be the path of the random walk under consideration. During the walk the point $x(i)/\log^2 n$ passes through points of any set $\mathfrak{M}^{(l)}$, $1 \leqq l \leqq r$, at certain times. Let us write out these points in succession until the path of $x/\log^2 n$ has passed through $m^{(0)}$ n times: $\omega = (\omega_1, \cdots, \omega_t)$, $\omega_i \in \mathfrak{M}^{(l)}$, $\omega_t = m^{(0)}$, and the symbol $m^{(0)}$ is encountered n times in the *message* ω. It is clear that $t \geqq n$. We shall denote by $\Omega_{\mathfrak{M}^{(l)}}$ the space of such messages. The probability distribution on paths of x induces a probability distribution on messages $\omega \in \Omega_{\mathfrak{M}^{(l)}}$. The form of this distribution will be described shortly. Let $\mathfrak{M}^{(i)}$ be one of the sets and let $\mathfrak{M}^{(i+1)}$ be obtained with the aid of the refinement operation, for definiteness the right one. This means that for some $m_k, M_k \subset \mathfrak{M}^{(i)}$ the points M'', m'' are added in order to obtain $\mathfrak{M}^{(i+1)}$.

We take $\omega' \in \Omega_{\mathfrak{M}^{(i)}}$ and consider all possible pairs of the form (m_k, m_k), (m_k, M_k), (M_k, m_k), (M_k, M_k). Then the message $\omega'' \in \Omega_{\mathfrak{M}^{(i+1)}}$ is obtained from ω' by a refinement inside each pair of a certain number of symbols m'', M''. The refinement m'', M'' inside the pair m_k, m_k means that the path of x, leaving m_k, will be in the states m'', M'' up to the following return to m_k in accordance with the refined set. The refinement inside the other pairs has a similar meaning.

Lemma 4. *The conditional distribution on messages ω'' under the condition ω' is such that sets of m'', M'', refined inside separate pairs, are independent of one another.*

The proof follows directly from the definitions. We investigate the probability distribution for the refined set in more detail. We shall dwell on the case of the pair (m_k, m_k). The probability that not a single symbol is refined is equal to

$$q(m_k \log^2 n) k^+_{[M_{k-1}\log^2 n, m_k \log^2 n]}(m_k \log^2 n + 1)$$
$$+ p(m_k \log^2 n) k^-_{[m_k \log^2 n, M''\log^2 n]}(m_k \log^2 n + 1).$$

The probability that exactly one symbol M'' is refined is equal to

$$p(m_k \log^2 n) k^+_{[m_k \log^2 n, M''\log^2 n]}(m_k \log^2 n + 1) q(M''\log^2 n)$$
$$\cdot k^-_{[m_k \log^2 n, M''\log^2 n]}(M''\log^2 n - 1).$$

From Lemma 1 and Corollary 1 it follows that this probability is contained between

$$\text{const}\exp\{-\log n[w^{(n)}(M'')-w^{(n)}(m_k)]\}=\text{const}\, n^{-(w^{(n)}(M'')-w^{(n)}(m_k))}$$

and

$$\text{const}\,(\log n)^{-2} n^{-(w^{(n)}(M'')-w^{(n)}(m_k))}.$$

Further, any refined set consists of a series of symbols M'' followed by a series of symbols m'', then again by a series of M'', and so on. At the end one has invariably a series of symbols M''. The lengths of various series are mutually independent. An important remark concerns the length of the series of m''. We shall find a conditional probability π of passing from m'' to m'' without getting to M'' and M_k. Obviously we can write

$$\begin{aligned}1-\pi&=q(m''\log^2 n)k^-_{[M''\log^2 n,m''\log^2 n]}(m''\log^2 n-1)\\&\quad+p(m''\log^2 n)k^+_{[m''\log^2 n,M_k\log^2 n]}(m''\log^2 n+1),\end{aligned}$$

from which we obtain, with the aid of Lemma 1 and its corollaries;

$$\frac{\text{const}}{\log^2 n} n^{-(w^{(n)}(M'')-w^{(n)}(m''))}\leqq 1-\pi\leqq \text{const}\, n^{-(w^{(n)}(M'')-w^{(n)}(m''))}.$$

This inequality together with the preceding ones shows that the symbols M'', m'' are sparsely refined, with a probability approximately equal to $n^{-(w^{(n)}(M'')-w^{(n)}(m_k))}$; but if the refinement has occurred, then the length of the refined series of m'' is approximately equal to $n^{(w(M'')-w(m''))}$. Here the inequality $w^{(n)}(M'')-w^{(n)}(m_k)>w^{(n)}(M'')-w^{(n)}(m'')$ is of substantial importance, showing that on the whole a relatively small number of symbols m'', M'' is refined. In essence it is this circumstance which is basic for the validity of Theorem 1. We point out also that the length of each series of m'' or M'' has an exponential distribution and the lengths of various series are mutually independent. A similar analysis is carried out for the other pairs (m_k, M_k), (M_k, M_k), (M_k, m_k).

Let us introduce the set $\Xi^{(i+1)}$ of pairs (M'', m_k), (m_k, M''), (M'', M''), (M'', m''), (m'', m''), (m'', M''), (m'', M_k), (M_k, m''), and for $\omega''\in\Omega_{\mathfrak{M}^{(i+1)}}$ and $\xi\in\Xi^{(i+1)}$ denote by $\nu_\xi(r;\omega'')$ the number of places of the pair ξ appearing up to the r-th appearance of $m^{(0)}$. If $\omega'\in\Omega_{\mathfrak{M}^{(i)}}$ is a message from which ω'' was produced with the aid of a refinement of symbols, then for fixed ω' each $\nu_\xi(r;\omega'')$ is representable in the form of a sum of independent random variables. Therefore the conditional expectation and variance can be represented in the form of linear combinations of $\nu_\zeta(r;\omega')$, where $\zeta\in Z^{(i)}$ and $Z^{(i)}$ is the set of all possible pairs of symbols for $\mathfrak{M}^{(i)}$.

We fix the chain $\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(i)}=\omega'$, where the message $\omega^{(l+1)}$ has been obtained from $\omega^{(l)}$ by a refinement of the corresponding pairs of symbols, and $Z^{(l)}$ is the set of pairs of symbols for the set $\mathfrak{M}^{(l)}$ encountered in the messages $\omega\in\Omega_{\mathfrak{M}^{(l)}}$. Let us consider the conditional variance $\mathbf{D}(\nu_\xi(r;\omega'')|\omega')$. As was already stated, it is a linear combination with positive coefficients $\nu_\zeta(r;\omega^{(i)})$, $\zeta\in Z^{(i)}$.

From Chebyshev's inequality it follows that, with a probability tending to 1, as $n \to \infty$ and for $\frac{1}{2}n \leqq r \leqq n$, each $\nu_\zeta(r; \omega^{(i)})$ is equivalent (in the sense of the ratio tending to 1) to its conditional expectation under the condition $\omega^{(i-1)}$, which is a linear combination of $\nu_\xi(r; \omega^{(i-1)})$, $\zeta \in Z^{(i-1)}$. The coefficients of these linear combinations do not depend on the conditions. Moving farther on from $(i-1)$ to $(i-2)$, $(i-3)$ and so on, we see that with a probability tending to 1, as $n \to \infty$, the conditional variance $\mathbf{D}(\nu_\xi(r; \omega'')|\omega')$ is equivalent to some expression $D_\xi^{(n)}$ depending on n, but not depending on ω'.

The following assertion can be obtained by the usual methods of proof of the local limit theorem for sums of independent random variables. Let $\mathbf{E}(\nu_\xi(r; \omega''|\omega'))$ be the conditional expectation of $\nu_\xi(r; \omega'')$ under the condition ω', and A a constant number. Then for any set of integers z_ξ, $\xi \in \Xi^{(i+1)}$ such that $|z_\xi - \mathbf{E}(\nu_\xi(r; \omega'')|\omega')| \leqq A\sqrt{\mathbf{D}(\nu_\xi(r; \omega'')|\omega')}$

$$P(\nu_\xi(r; \omega'') = z_\xi|\omega') \sim \left(\prod_{\xi \in \Xi^{(i+1)}} D_\xi^{(n)}\right)^{-1/2} g\left(\frac{z-E}{\sqrt{D^{(n)}}}\right),$$

where z, E, $D^{(n)}$ are vectors with components z_ξ, $\mathbf{E}(\nu_\xi)$, $D_\xi^{(n)}$, respectively, and g is a non-singular eight-dimensional Gaussian distribution.

Let us consider the pair $\zeta = (t', t'') \in Z^{(i)}$. It can be one of six types: 1) $t' = t''$ and $t' = m_j$; 2) $t' = m_j$ and $t'' = M_j$; 3) $t' = m_j$, $t'' = M_{j-1}$; 4) $t' = t'' = M_j$; 5) $t' = M_j$, $t'' = m_j$; 6) $t' = M_j$, $t'' = m_{j+1}$. The following assertion is easily obtained by induction on m: for any $\varepsilon > 0$ as $n \to \infty$,

in the 1st case

$$P(rn^{-(w^{(n)}(m_j) - w^{(n)}(m^{(0)})) - \varepsilon} \leqq \nu_\zeta(r; \omega) \leqq rn^{-(w^{(n)}(m_j) - w^{(n)}(m^{(0)})) + \varepsilon}) \to 1,$$

in the 2nd case

$$P(rn^{-(w^{(n)}(M_{j-1}) - w^{(n)}(m^{(0)})) - \varepsilon} \leqq \nu_\zeta(r; w) \leqq rn^{-(w^{(n)}(m_j) - w^{(n)}(m^{(0)})) + \varepsilon}) \to 1,$$

in the 3rd case

$$P(rn^{-(w^{(n)}(M_{j-1}) - w^{(n)}(m^{(0)})) - \varepsilon} \leqq \nu_\zeta(r; \omega) \leqq rn^{-(w^{(n)}(M_{j-1}) - w^{(n)}(m^{(0)})) + \varepsilon}) \to 1,$$

in the 4th case

$$P(rn^{-(w^{(n)}(M_j) - w^{(n)}(m^{(0)})) - \varepsilon} \leqq \nu_\zeta(r; \omega) \leqq rn^{-(w^{(n)}(M_j) - w^{(n)}(m^{(0)})) + \varepsilon}) \to 1,$$

in the 5th case

$$P(rn^{-(w^{(n)}(M_j) - w^{(n)}(m^{(0)})) - \varepsilon} \leqq \nu_\zeta(r; \omega) \leqq rn^{-(w^{(n)}(M_j) - w^{(n)}(m^{(0)})) + \varepsilon}) \to 1,$$

in the 6th case

$$P(rn^{-(w^{(n)}(M_j) - w^{(n)}(m^{(0)})) - \varepsilon} \leqq \nu_\xi(r; \omega) \leqq rn^{-(w^{(n)}(M_j) - w^{(n)}(m^{(0)})) + \varepsilon}) \to 1.$$

Since $\frac{1}{2}n \leqq r \leqq n$, while $w^{(n)}(t) - w^{(n)}(m^{(0)}) > \delta_1$ for $t \in \mathfrak{M}^{(i)} \setminus m^{(0)}$, these relations mean that the probable number of appearances of any symbol different from $m^{(0)}$ is small compared to n. We shall now derive the basic result of this section.

Let $\omega \in \Omega_{\mathfrak{M}_1}$ and ω_i be the i-th coordinate of the message ω.

Theorem 2. *Let $p \in C_n$. Then for any r with $\frac{1}{2}n \leqq r \leqq n$*

$$P(\omega_r = m^{(0)}) \to 1 \quad \text{as } n \to \infty$$

uniformly in $p \in C_n$.

PROOF. For any r_1 we denote by $\nu(r_1)$ the total number of inserted symbols up to the r_1-th appearance of the symbol $m^{(0)}$. Then $\nu(r_1)$ is also a sum of independent random variables satisfying the conditions of the local central limit theorem. If $\mathbf{E}(r_1)$, $\mathbf{D}(r_1)$ are the expectation and variance, then there is a $\gamma > 0$ such that $r_1 n^{1-\gamma-\varepsilon} \leqq \mathbf{E}(r_1) \leqq r_1 n^{1-\gamma+\varepsilon}$ for any $\varepsilon > 0$ and $n \to \infty$, and

$$\text{const}\, r_1^{-1} \mathbf{D}(r_1) \leqq \left(\frac{1}{r_1} \mathbf{E} r_1)\right)^2 \leqq \text{const}\, r_1^{-1} \mathbf{D}(r_1).$$

We can now write

$$P(\omega_r = m^{(0)}) = \sum_{r_1} P(\nu(r_1) = r - r_1) \sim \sum_{r_1} \frac{1}{\sqrt{2\pi \mathbf{D}(r_1)}} \exp\left\{-\frac{(r - r_1 - \mathbf{E}(r_1))^2}{2\mathbf{D}(r_1)}\right\}.$$

In the latter sum one can restrict oneself to those r_1 for which the local limit theorem is valid. For such r_1 we obviously have $\mathbf{D}(r_1) \sim \mathbf{D}(r)$ and

$$\exp\left\{-\frac{(r - r_1 - \mathbf{E}(r_1))^2}{2\mathbf{D}(r_1)}\right\} \sim \exp\left\{-\frac{(r - r_1 - \mathbf{E}(r))^2}{2\mathbf{D}(r)}\right\},$$

since

$$\begin{aligned} \frac{|r - r_1|\, |\mathbf{E}(r_1) - \mathbf{E}(r)|}{\mathbf{D}(r)} &\leqq \text{const} \frac{(r - r_1)^2 |\mathbf{E}(r_1) - \mathbf{E}(r)|}{\mathbf{D}(r) |r - r_1|} \\ &\leqq A \text{ const} \frac{|\mathbf{E}(r) - \mathbf{E}(r_1)|}{|r - r_1|} \to 0 \end{aligned}$$

as $n \to \infty$. Therefore the latter sum is equivalent to

$$\sum_{r_1} \frac{1}{\sqrt{2\pi \mathbf{D}(r)}} \exp\left\{-\frac{(r - r_1 - \mathbf{E}(r))^2}{2\mathbf{D}(r)}\right\} \to 1$$

as $r \to \infty$. Thus Theorem 2 is proved.

REMARK. The proof of Theorem 2 recalls the equivalence proofs of various canonical ensembles in statistical mechanics (for example, see [4]). A similar remark relates to the arguments of the following section.

5. Proof of Theorem 1

In this section we shall establish that, for any $\delta > 0$ and $p \in C_n$,

$$P\left(\left|\frac{x(n)}{\log^2 n} - m^{(0)}\right| \leqq \delta\right) \to 1 \quad \text{as } n \to \infty$$

uniformly in $p \in C_n$.

Let us consider the path of the random walk $x=\{x(t), t\geqq 0\}$ and the normalized path

$$x'=\frac{1}{\log^2 n}x=\left\{\frac{1}{\log^2 n}x(t), t\geqq 0\right\}.$$

To it corresponds the message $\omega\in\Omega_{\mathfrak{M}_1}$, $\omega=(\omega_1,\cdots,\omega_s)$. We denote by τ_i, $1\leqq i\leqq s$, the time from the i-th to the $(i+1)$st hitting of one of the points of $\mathfrak{M}_1$, and by τ_0 the time until first hitting of the path x' in $\mathfrak{M}_1$, having left from 0. Obviously, for a fixed ω the τ_i form a sequence of independent random variables. We shall be interested in a value of r such that $\tau_0+\tau_1+\cdots+\tau_r<n$, $\tau_0+\tau_1+\cdots+\tau_r+\tau_{r+1}>n$, and the probability corresponding to this event:

$$P=\sum_r\sum_{l_1<n}\sum_{l_2\geqq n-l}P(\tau_0+\cdots+\tau_r=l_1, \tau_{r+1}=l_2, \omega_r=m^{(0)}).$$

The assertion we need is equivalent to the fact that $P\to 1$ as $n\to\infty$.

For an assigned ω the probability distribution of τ_i, $i>0$, depends only on ω_i and ω_{i+1}. Let Z be the set of admissible pairs of symbols of $\mathfrak{M}_1$, and for $\zeta\in Z$, let $\nu_\zeta(r)$ be the number of appearances of the pair ζ up to the i-th appearance of the symbol $m^{(0)}$. Then

$$E_r=\mathbf{E}(\tau_1+\cdots+\tau_r|\omega)=\sum_{\zeta\in Z}h_\zeta\nu_\zeta(r),$$

$$D_r=\mathbf{D}(\tau_1+\cdots+\tau_r|\omega)=\sum_{\zeta\in Z}d_\zeta\nu_\zeta(r).$$

Here h_ζ and d_ζ are the expectation and variance of the transition time between states of the pair ζ. If $\zeta^{(0)}=(m^{(0)},m^{(0)})$, then $E_r\sim h_{\zeta^{(0)}}r$, $D_r\sim d_{\zeta^{(0)}}r$. Moreover, $|E_r-h_{\zeta^{(0)}}r|\leqq r^{1-\gamma}$ for some $\gamma>0$.

On the basis of the local limit theorem, the applicability of which is established here by the usual methods (for example, see [5]), we can now write, for typical ω,

$$P(\tau_0+\tau_1+\cdots+\tau_r=l|\omega, \omega_r=m^{(0)})\sim(2\pi n d_{\zeta^{(0)}})^{-1/2}\exp\left\{-\frac{(l-E_r)^2}{2D_r}\right\}$$

as $n\to\infty$, for $l=E_r+z\sqrt{D_r}$, $|z|\leqq A$, where A is an arbitrary fixed number. We have used the fact that the basic contribution to the variance of the sum for typical ω and $r\sim n$ is a term corresponding to the pair $\zeta^{(0)}$, since $\nu_{\zeta^{(0)}}(r)\sim r$. Further, we have,

$$P\sim\sum_r\sum_{l_1<n}(2\pi n d_{\xi^{(0)}})^{-1/2}\exp\left\{-\frac{(l_1-E_r)^2}{2nd}\right\}\sum_{l_1\geqq n-l}p_{l_1}, \tag{1}$$

where p_l is the probability distribution for the random variable τ_r corresponding to the pair $\xi^{(0)}$. Here $\sum_k\sum_{l_1\geqq k}p_{l_1}=h_{\zeta^{(0)}}$. We interchange the order of summation over l_1 and r. Let us use the fact that

$$\sum_r(2\pi nd)^{-1/2}\exp\left\{-\frac{(l_1-E_r)^2}{2nd}\right\}\sim\frac{1}{h_{\xi^{(0)}}}.$$

Therefore we obtain

$$P \sim \sum_{l_1<n} \sum_r (2\pi n d_{\xi^{(0)}})^{-1/2} \exp\left\{-\frac{(l_1-E_r)^2}{2nd}\right\} \sum_{l_2 \geqq n-l_1} p_{l_2}$$

$$\sim \frac{1}{h_{\xi^{(0)}}} \sum_{n-l_1 \geqq 1} \sum_{l_2 \geqq n-l_1} p_{l_2} = 1.$$

Our assertion is proved.

6. Concluding Remarks

1. The existence of a limit distribution for the point $m^{(0)}$ is easily derived from our constructions. The assertion of the recurrence of the random walk under consideration follows.

2. The basic result of this paper concerns the limiting behavior of $x(n)$ in probability. It would be interesting to clarify the nature of the behavior of $x(n)$ with probability 1. Apparently the random process $x(n)$ has a step-wise character. For a given n the path passes a great part of the time in the deepest minimum corresponding to this n. Thereupon, with an increase in n this minimum becomes insufficiently deep, and a deeper minimum appears into which the particle jumps, and so on.

3. The methods of this paper have been adapted for the analysis of random excitation of the simplest random walk. It would be interesting to extend them in order to include the more general case.

4. The condition of independence of probabilities $p(x)$ can be weakened appreciably. It suffices to require that the measure in the space of paths $w^{(n)}(t)$ converge weakly to a Wiener measure.

5. The problem considered above for random walks also arises naturally for diffusion processes. We are then led to the problem of the limiting behavior of diffusion processes for which the drift coefficient is a random time function. The case considered corresponds to the situation when the diffusion coefficient is constant or varies within restricted limits, while the drift coefficient is white noise, i.e., an arbitrary (of course, generalized) Wiener process. Our methods permit one to obtain results similar to those presented above for this case also.

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