CMSC 27100 - Problem Set 7

December 1, 2023

1

We flip 2 fair coins and define our sample space as $\{HH, HT, TH, TT\}$, all of which occur with an equal probability of $\frac{1}{4}$. Let A be the event that the first coin is H, B be the event that the second coin is H, and C be the event that one coin is H and one coin is T.

We can define the following events and calculate their probabilities:

- $\{A\} = \{HH, HT\}$, $Pr(A) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
- $\{B\} = \{HH, TH\}$, $Pr(B) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
- $\{C\} = \{HT, TH\}$, $Pr(C) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
- $\{A \cap B\} = \{HH\}$, $Pr(A \cap B) = \frac{1}{4}$
- $\{A \cap C\} = \{HT\}$, $Pr(A \cap C) = \frac{1}{4}$
- $\{B \cap C\} = \{TH\}$, $Pr(B \cap C) = \frac{1}{4}$
- $\{A \cap B \cap C\} = \{\}$, $Pr(A \cap B \cap C) = 0$

From here, we see the following:

- $Pr(A) \cdot Pr(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = Pr(A \cap B)$
- $Pr(A) \cdot Pr(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = Pr(A \cap C)$
- $Pr(B) \cdot Pr(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = Pr(B \cap C)$
- $Pr(A) \cdot Pr(B) \cdot Pr(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \neq 0 = Pr(A \cap B \cap C)$

Thus, we have shown that events A, B, and C are pairwise independent but not mutually independent.

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2.1

There is no restriction in how we distribute the gifts. So, for each of the 1000 gifts, we can give it to any of the 100 guests. Thus, the answer is $\boxed{100^{1000}}$.

2.2

We can consider ordering the 1000 gifts and using dividers to separate the gifts for 100 guests, where the first 10 gifts correspond to the first guest, and so on. There are 1000! different ways to order the gifts, but we need to account for overcounting. With partitions among 1000 gifts to separate 100 guests, we are overcounting by a factor of 10! per guest since a set of 10 gifts in different arrangements going to a single guest should count as one assignment. Then, for all guests combined, we are overcounting by $(10!)^{100}$. Thus, the answer is $\boxed{\frac{1000!}{(10!)^{100}}}$.

3

3.1

Since events A and B are independent, we know that $Pr(A \cap B) = Pr(A) \cdot Pr(B)$. Also, by definition, $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = Pr(A) + Pr(B) - Pr(A) \cdot Pr(B)$. Setting this equal to the definition of $Pr(A \cup B)$ given in the problem statement, we get:

- $2Pr(B) - Pr(A) = Pr(A) + Pr(B) - Pr(A) \cdot Pr(B)$
- $Pr(B) - 2Pr(A) = -Pr(A) \cdot Pr(B)$
- $Pr(B) + Pr(A) \cdot Pr(B) = 2Pr(A)$
- $Pr(B) \cdot (1 + Pr(A)) = 2Pr(A)$
- $Pr(B) = \frac{2Pr(A)}{1+Pr(A)}$
- $Pr(B) = \frac{2 \cdot \frac{1}{4}}{1+\frac{1}{4}}$
- $\boxed{Pr(B) = \frac{2}{5}}$

3.2

We can use the definition of conditional probability to find $Pr(A|B)$:

- $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$
- $Pr(A|B) = \frac{Pr(A) \cdot Pr(B)}{Pr(B)}$
- $Pr(A|B) = Pr(A)$
- $\boxed{Pr(A|B) = \frac{1}{4}}$

It makes sense that $P(A) = P(A|B)$ since A and B are independent, as the probability of these events do not depend on each other.

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4.1

From Markov's Inequality, we know that if X is a non-negative random variable, then $\forall a > 0$, $Pr[X \geq a] \leq \frac{E[X]}{a}$. We are given that $E[X] = 50$ and $a = 100$, so $Pr[X \geq 100] \leq \frac{50}{100}$. Thus, $\boxed{Pr[X \geq 100] \leq \frac{1}{2}}$.

4.2

From Chebyshev's Inequality, we know that if X is a random variable, then $\forall a > 0$, $Pr[|X - E[x]| \geq a] \leq \frac{Var[X]}{a^2}$. We are given that $E[X] = 50$ and $Var[X] = 25$, so $Pr[X \geq 100] \leq Pr[|X - 50| \geq 50] = Pr[X \geq 100] + Pr[X \leq 0] \leq \frac{25}{50^2}$. We know that $Pr[X \leq 0] = 0$. Thus, $\boxed{Pr[X \geq 100] \leq \frac{1}{100}}$.