CMSC 27100 - Problem Set 6

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1

First, consider the case when $n = 0$. Then $(x_1 + ... + x_k)^0 = 1$. Also, $b_1, ..., b_k$ are all 0 since there is only one way to make $b_1 + ... + b_k = 0$. Given this, we know $\binom{0}{0!...0!} = 1$ and $\prod_{j=1}^{k} x_j^0 = 1$. Thus, the case for $n = 0$ holds true.

Now, consider $n > 0$. We can break this down into a few steps. We know that $(x_1 + ... + x_k)^n$ is equivalent to choosing one of $x_1, ..., x_k$ from each of the n groups of $(x_1 + ... + x_k)$ and multiplying them together for a single term.

- Each product will be in the form $x_1^{b_1}...x_k^{b_k}$, and the sum of b_1 to b_k must equal n since we are selecting one term from each of the n groups. Also, we add all the terms for the different $b_1 + ... + b_k = n$ combinations to get the summation.
- The $\prod_{j=1}^{k} x_j^{b_j}$ is the product of all x_j in a single term in the summation. This is essentially counting the number of ways we can select x_k from b_k distinct values.
- Then, for a single term, we have to distribute n over k exponents. For each combination of $b_1, ..., b_k$, if we took a different x_i term from each group (assuming we take a different term from each group, which is not necessarily the case or possible), there would be $n!$ different products following a specific arrangement of integers (exponents) that sum to n (for example, for $n = 4$ and $k = 4$, one such exponent combination would be $1, 1, 1, 1$, and we are finding all the ways to arrange these exponents). However, we may pick the same term from different groups, meaning we have to account for overcounting. So, instead of $n!$, there are $\binom{n}{b_1,...,b_k}$ ways to achieve this arrangement. This becomes the coefficient for $\prod_{j=1}^{k} x_j^{b_j}$ in the summation.

Thus, we have shown that the given identity holds.

2

We need to determine the number of permutations with a deck of $52 \cdot 3$ total cards, consisting of 3 cards of each type. First, assume all $52 \cdot 3$ cards are distinct. In this case, there are $(52 \cdot 3)!$ permutations. However, we need to account for overcounting since each of the 52 card types appear 3 times in the combined deck. We overcount each card type by $3!$, and repeat this process for all 52 card types. Thus, the answer is $\boxed{\dfrac{(52 \cdot 3)!}{(3!)^{52}}}$.

3

We must find the number of non-negative solutions to $x+y+z=10$. This is equivalent to selecting 10 elements from 3 piles of distinct elements. In this case, $n=3$ since we have 3 piles, each corresponding to one of x, y, z, and $r=10$ since we need to select 10 items total to acquire a sum of 10. We get $C(n+r-1, r) = C(3+10-1, 10) = C(12, 10) = C(12, 2)$.

We can also think of this problem using stars and bars. We have a total of 12 elements, consisting of 10 items (stars) and 2 dividers (bars). We have 2 bars because we divide 12 items into 3 groups, each corresponding to a variable. From 12 elements, we must choose 2 positions to be a divider position, and there are $C(12,2)$ ways to do so. Thus, the answer is $\boxed{C(12,2)}$.

4

We must find the number of positive solutions to $x+y+z=10$. This is equivalent to finding the number of non-negative solutions to $(a-1)+(b-1)+(c-1)=10$ or $a+b+c=7$. This is equivalent to selecting 7 elements from 3 piles of distinct elements. In this case, $n=3$ since we have 3 piles, each corresponding to one of a, b, c, and $r=7$ since we need to select 7 items total to acquire a sum of 7. We get $C(n+r-1, r) = C(3+7-1, 7) = C(9, 7) = C(9, 2)$.

We can also think of this problem using stars and bars. We have a total of 9 elements, consisting of 7 items (stars) and 2 dividers (bars). We have 2 bars because we divide 7 items into 3 groups, each corresponding to a variable. From 9 elements, we must choose 2 positions to be a divider position, and there are $C(9,2)$ ways to do so. Thus, the answer is $\boxed{C(9,2)}$.

The difference between problem 3 and 4 is that all solutions of problem 4 are also a solution of problem 3. Problem 3 requires non-negative integer solutions, which includes the positive integer solutions required by problem 4. Also, in problem 3, some of the terms can be 0, while in problem 4, each term must contribute positive value to the sum. Specifically, problem 3 has $C(12,2)=66$ solutions and problem 4 has $C(9,2)=36$ solutions, so there the difference between number of solutions is $66-36=30$. Again, we verified that problem 3 has more solutions as it includes the solutions of problem 4.