CMSC 27100 - Problem Set 3

Sohini Banerjee

October 23, 2023

1

We can break the problem into two cases: n is odd and n is even.

An even integer can be written $2m$ where m is an integer. So, $n^2 + 2 = (2m)^2 + 2 = 4m^2 + 2$. $(4m^2 + 2) \pmod 4 = ((4m^2 \pmod 4)) + (2 \pmod 4)) \pmod 4 = (0 + 2) \pmod 4 = 2 \pmod 4 = 2 \neq 0$. Thus, $4 \nmid (n^2 + 2)$ when n is even.

An odd integer can be written as $2m+1$ where m is an integer. So, $n^2+2 = (2m+1)^2+2 = 4m^2+4m+3$. $(4m^2+4m+3) \pmod 4 = ((4m^2 \pmod 4))+(4m \pmod 4)) + (3 \pmod 4))) = (0 + 0 + 3) \pmod 4 = 3 \pmod 4 = 3 \neq 0$. Thus, $4 \nmid (n^2 + 2)$ when n is odd.

So, $\boxed{4 \nmid (n^2 + 2)}$ for any integer n.

2

We know that $a \equiv b \pmod p$ means $m \mid (a - b)$. From binomial expansion, we know that $a^n - b^n = (a - b) \cdot (a^{n-1} + a^{n-2}b + ... + ab^{n-2} + b^{n-1})$. This means that $m \mid (a^n - b^n)$ because $m \mid (a - b)$, so $a \equiv b \pmod p$ implies that $a^n \equiv b^n \pmod p$. We need to show that $7 \mid (2^n + 6 \cdot 9^n)$. Setting $a = 9$, $b = 2$, and $m = 7$, we get that $9 \equiv 2 \pmod 7$ implies $9^n \equiv 2^n \pmod 7$. So, $2^n + 6 \cdot 9^n \pmod 7 \equiv 2^n + 6 \cdot 2^n = 2^n(1 + 6) = 7 \cdot 2^n$. It is evident that $7 \mid 7 \cdot 2^n$, so $\boxed{7 \mid (2^n + 6 \cdot 9^n)}$.

3

Let $p = 2$, $q = 3$, and $x = 6$. 2 and 3 are coprime because $GCD(2, 3) = 1$ and $2 \mid 6$ and $3 \mid 6$, we know that $(2 \cdot 3) \mid 6$. Among n, $n + 1$, and $n + 2$, if at least one term is divisible by 2 and at least one is divisible by 3, we know that $n(n + 1)(n + 2)$ is divisible by 6.

We can break this into two cases for 2:

- Assume $n \equiv 0 \pmod 2$. Then $2 \mid n$.

- Assume $n \equiv 1 \pmod 2$. Then, $n+1 \equiv (1+1) \pmod 2 \equiv 2 \pmod 2 = 0 \pmod 2$. Therefore, $2 \mid n+1$.

We can break this into three cases for 3:

- Assume $n \equiv 0 \pmod 3$. Then $3 \mid n$.
- Assume $n \equiv 1 \pmod 3$. Then, $n+2 \equiv (1+2) \pmod 3 \equiv 3 \pmod 3 \equiv 0 \pmod 3$. Therefore, $3 \mid n+2$.
- Assume $n \equiv 2 \pmod 3$. Then $n+1 \equiv (2+1) \pmod 3 \equiv 3 \pmod 3 = 0 \pmod 3$. Therefore, $3 \mid n+1$.

Thus, among n, $n+1$, and $n+2$, at least one term is divisible by 2 and at least one is divisible by 3, so we know that $\boxed{6 \mid n(n+1)(n+2)}$.

4

Let $p = 3$, $q = 8$, and $x = 24$. 3 and 8 are coprime because $GCD(3,8) = 1$ and $3 \mid 24$ and $8 \mid 24$, we know that $(3 \cdot 8) \mid 24$. From above, we know that among n, $n+1$, $n+2$, and $n+3$, at least one term is divisible by 3. First, we show that the product of four integers is divisible by 4, and in all these cases, the product is also divisible by 8. If we show that the product is divisible by 3 and 8, then the product must also be divisible by 24.

We can break this into four cases for 4:

- Assume $n \equiv 0 \pmod 4$. Then $4 \mid n$. This means $n = 4m$, so $n+2 = 4m+2$, and $n(n+2) = 4m(4m+2)$. We know that $4m(4m+2) = 16m^2 + 8m = 8(2m^2+1)$, so $8 \mid n(n+2)$, so 8 divides the product of four consecutive integers in this case.
- Assume $n \equiv 1 \pmod 4$. Then, $n+3 \equiv (1+3) \pmod 4 \equiv 4 \pmod 4 \equiv 0 \pmod 4$. Therefore, $4 \mid n+3$. This means $n+3 = 4m$, so $n+1 = 4m-2$, and $(n+1)(n+3) = (4m-2) \cdot 4m$. We know that $(4m-2) \cdot 4m = 16m^2 - 8m = 8(2m^2-1)$, so $8 \mid (n+1)(n+3)$, so 8 divides the product of four consecutive integers in this case.
- Assume $n \equiv 2 \pmod 4$. Then, $n+2 \equiv (2+2) \pmod 4 \equiv 4 \pmod 4 \equiv 0 \pmod 4$. Therefore, $4 \mid n+2$. This means $n+2 = 4m$, so $n = 4m-2$, and $n(n+2) = (4m-2) \cdot 4m$. We know that $(4m-2) \cdot 4m = 16m^2 - 8m = 8(2m^2-1)$, so $8 \mid n(n+2)$, so 8 divides the product of four consecutive integers in this case.
- Assume $n \equiv 3 \pmod 4$. Then, $n+1 \equiv (3+1) \pmod 4 \equiv 4 \pmod 4 \equiv 0 \pmod 4$. Therefore, $4 \mid n+1$. This means $n+1 = 4m$, so $n+3 = 4m+2$, and $(n+1)(n+3) = 4m(4m+2)$. We know that $4m(4m+2) = 16m^2+8m = 8(2m^2+1)$, so $8 \mid (n+1)(n+3)$, so 8 divides the product of four consecutive integers in this case.

Thus, $n(n+1)(n+2)(n+3)$ is always divisible by 3 and 8, so we know that $\boxed{24 \mid n(n+1)(n+2)(n+3)}$.

5

We need to solve the system: $x \cong 1 \pmod 2$, $x \cong 2 \pmod 3$, and $x \cong 3 \pmod 5$.

We know the following:

- $c_1 = 1$, $c_2 = 2$, $c_3 = 3$
- $m_1 = 2$, $m_2 = 3$, $m_3 = 5$

Therefore, $n = m_1 \cdot m_2 \cdot m_3 = 2 \cdot 3 \cdot 5 = 30$.

We can calculate each $n_i = \frac{n}{m_i}$:

- $n_1 = \frac{n}{m_1} = \frac{30}{2} = 15$
- $n_2 = \frac{n}{m_2} = \frac{30}{3} = 10$
- $n_3 = \frac{n}{m_3} = \frac{30}{5} = 6$

For each n_i, we need an a_i such that $n_i \cdot a_i \cong 1 \pmod n$:

- $n_1 \cdot a_1 \cong 1 \pmod 2$, so $15 \cdot a_1 = 1 \pmod 2$. Thus, $a_1 = 1$ since $15 \cdot 1 \cong 1 \pmod 2$.
- $n_2 \cdot a_2 \cong 1 \pmod 3$, so $10 \cdot a_2 = 1 \pmod 3$. Thus, $a_2 = 1$ since $10 \cdot 1 \cong 1 \pmod 3$.
- $n_3 \cdot a_3 \cong 1 \pmod 5$, so $6 \cdot a_3 = 1 \pmod 5$. Thus, $a_3 = 1$ since $6 \cdot 1 \cong 1 \pmod 5$.

Define $x_i = a_i \cdot n_i$ for all i:

- $x_1 = 1 \cdot 15 = 15$
- $x_2 = 1 \cdot 10 = 10$
- $x_3 = 1 \cdot 6 = 6$

Thus, $x = c_1 \cdot x_1 + c_2 \cdot x_2 + c_3 \cdot x_3 \pmod n = (1 \cdot 15) + (2 \cdot 10) + (3 \cdot 6) \pmod{30} = 15 + 20 + 18 \pmod{30} = 53 \pmod{30} = 23$. So, $\boxed{x = 23}$ satisfies the system of equations.