CMSC 27100 - Problem Set 4

Sohini Banerjee

October 30, 2023

1

Let $\phi(n) = 1$ if and only if for all real numbers $x \neq 1$ and a non-negative integer n, $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$. First, we show that the base case $n = 0$ holds. The LHS is $\sum_{i=0}^{0} x^i = x^0 = 1$. The RHS is $\frac{x^{0+1}-1}{x-1} = \frac{x-1}{x-1} = 1$. $1 = 1$ and thus, $\phi(0) = 1$ and the base case is true.

The inductive hypothesis is that we assume for all real numbers $x \neq 1$, $\phi(n) = 1$ for a non-negative integer n, which means $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$. We must show that this implies $\phi(n+1) = 1$.

- $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$ (by inductive hypothesis)
- $\sum_{i=0}^{n} x^i + x^{n+1} = \frac{x^{n+1}-1}{x-1} + x^{n+1}$ (adding x^{n+1} to both sides)
- $\sum_{i=0}^{n+1} x^i = \frac{x^{n+1}-1+(x-1)x^{n+1}}{x-1}$
- $\sum_{i=0}^{n+1} x^i = \frac{x^{n+1}-1+x^{n+2}-x^{n+1}}{x-1}$
- $\sum_{i=0}^{n+1} x^i = \frac{x^{n+2}-1}{x-1}$
- $\sum_{i=0}^{n+1} x^i = \frac{x^{(n+1)+1}-1}{x-1}$

This shows that $\phi(n) = 1$ implies $\phi(n+1) = 1$ for non-negative integers n. Thus, $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$ for all non-negative integers n and all real numbers $x \neq 1$.

2

Let $\phi(n) = 1$ if and only if for an integer $n \geq 2$, $n! \cdot 2^n < (2n)!$. First, we show that the base case $n = 2$ holds. The LHS is $2! \cdot 2^2 = 2 \cdot 4 = 8$. The RHS is $(2 \cdot 2)! = 4! = 24$. $8 < 24$, and thus, $\phi(2) = 1$ and the base case is true.

The inductive hypothesis is we assume $\phi(n) = 1$ for an integer $n \geq 2$, which means $n! \cdot 2^n < (2n)!$. We must show that this implies $\phi(n+1) = 1$.

- $n! \cdot 2^n < (2n)!$ (by inductive hypothesis)

- $n! \cdot 2^n < (2n)! \cdot (2n+1)$ (because $(2n)! \cdot (2n+1) > (2n)!$ since $n \geq 2$)
- $(n+1) \cdot n! \cdot 2^n \cdot 2 < (2n)! \cdot (2n+1) \cdot 2 \cdot (n+1)$ (multiplying both sides by $2 \cdot (n+1)$)
- $(n+1) \cdot n! \cdot 2^n \cdot 2 < (2n)! \cdot (2n+1) \cdot (2n+2)$
- $(n+1) \cdot n! \cdot 2^n \cdot 2 < (2n+2)!$
- $(n+1)! \cdot 2^{n+1} < (2 \cdot (n+1))!$

This shows that $\phi(n) = 1$ implies $\phi(n+1) = 1$ for integers $n \geq 2$. Thus, $n! \cdot 2^n < (2n)!$ for all integers $n \geq 2$.

3

Let $\phi(n) = 1$ if and only if an integer $n > 1$ can be written as a product of primes. $\phi(2) = 1$ because 2 is a prime number. Thus, we have shown that the base case holds, as 2 is the smallest integer $n > 1$. The inductive hypothesis is we assume $\phi(k) = 1$ for $2 \leq k < n$. If n is prime, we are done, since it can be written as product of primes, namely itself. So, assume n is not prime. Then, we can write $n = a \cdot b$ for $1 < a, b < n$. The inductive hypothesis demonstrates the following:

- Since $a < n$, $\phi(a) = 1$. This means there exists a set of prime numbers $c_1, ..., c_x$ such that $c_1 \cdot ... \cdot c_x = a$ and $c_i \leq a < n$.
- Since $b < n, \phi(b) = 1$. This means there exists a set of prime numbers $d_1, ..., d_y$ such that $d_1 \cdot ... \cdot d_y = b$ and $d_j \leq b < n$.

So, since $a = c_1 \cdot ... \cdot c_x$ and $b = d_1 \cdot ... \cdot d_y$ where c_i and d_j are prime numbers, we can rewrite $n = a \cdot b = c_1 \cdot ... \cdot c_x \cdot d_1 \cdot ... \cdot d_y$. This means n can be written as a product of primes. This shows that for $2 \leq k < n$, $\phi(k) = 1$ implies $\phi(n) = 1$. Thus, all integers $n > 1$ can be written as a product of primes.

With weak induction, we have to show that if $n-1$ has a prime factorization, then n has a prime factorization. This is a problem because for a non-prime number n, the two factors $1 < a, b < n$ are not necessarily equal to $n-1$, so we cannot conclude that $\phi(n-1)$ implies $\phi(n)$. Specifically, an integer having a prime factorization does not clearly imply that the subsequent integer also has a prime factorization, which is what weak induction requires. Instead, we need strong induction because knowing that all $2 \leq k < n$ allows us to conclude that the two factors a and b of n have the strong induction hypothesis applied to it, allowing us to conclude that n has a prime factorization since it is the product of a and b.

4

Wilson's theorem states that for all p where p is a prime number, $(p-1)! \cong_p -1$. We know that 71 is a prime number because $Div(71) = \{1, 71\}$. So, $(71-1)! \cong_{71} -1$, or $70! \cong_{71} -1$. This means $70 \cdot 69 \cdot 68 \cdot 67! \cong_{71} -1$. So, $70 \pmod{71} \cdot 69 \pmod{71} \cdot 68 \pmod{71} \cdot 67! \cong_{71} -1$.

- $70 \cong_{71} -1$
- $69 \cong_{71} -2$
- $68 \cong_{71} -3$

This means $(-1)(-2)(-3)(67!) \cong_{71} -1$, so $-6 \cdot 67! \cong_{71} -1$ or $6 \cdot 67! \cong_{71} 1$. The multiplicative inverse for $6 \pmod{71}$ is 12 because $12 \cdot 6 = 72$ and $72 \pmod{71} = 1$, so $12 \cdot 6 \cong_{71} 1$. Again, we know that $12 \cdot 6 \cdot 67! \cong_{71} 12 \cdot 1$, and since $12 \cdot 6 \cong_{71} = 1$, then $67! \cong_{71} 12$. Thus, the remainder of 67! when divided by 71 is $\boxed{12}$.