CMSC 27100 - Problem Set 5

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1

Let $\phi(n) = 1$ if and only if $n \geq 1$ can be written as a sum of distinct positive integers, each of which is a power of 2. The base case is $n = 1$, which holds because $n = 2^0$. For the strong induction hypothesis, we assume that $\phi(k) = 1$ for $1 \leq k < n$.

- n odd: We know that $\phi(n-1) = 1$ because $n - 1 < n$. We also know that $n - 1$ must be even, so 2^0 cannot be present in the sum of distinct integers for $n - 1$. This is because 2^0 is the only odd power of 2, so it must be paired with another 2^0 if the sum is even. However, this is not possible because we need distinct integers. So, the sum for $n - 1$ cannot contain 2^0. As a result, we can add 2^0 to the sum for $n - 1$ and get n, where all the integers in the sum are distinct. This means $\phi(n) = 1$ when n is odd.

- n even: We know that $\phi(\frac{n}{2}) = 1$ because $\frac{n}{2} < n$. We can enumerate the summation $\frac{n}{2} = 2^{x_1} + 2^{x_2} + ... + 2^{x_p}$, where $x_1, ..., x_p$ are distinct. This is equivalent to noting that $n = 2(2^{x_1} + 2^{x_2} + ... + 2^{x_p})$ or that $n = 2^{x_1+1} + 2^{x_2+1} + ... + 2^{x_p+1}$. Since $x_1, ..., x_p$ are distinct, we know that $x_1 + 1, ..., x_p + 1$ must also be distinct. This means $\phi(n) = 1$ when n is even.

By showing that $\phi(n) = 1$ when n is both odd and even, we can conclude that $\phi(n) = 1$ for all $n \geq 1$.

2

2.1

If there are no restrictions on the players we can choose, this is a standard combinations problem. We are selecting $r = 7$ players from $n = 11$ players. The total number of ways we can do so is $C(11, 7) = \frac{11!}{7!4!} = \boxed{330}$.

2.2

If a fixed player needs to be on the team, we are choosing them no matter what. This means they occupy a single position on the team, leaving 6 positions for us to fill. So, we reduce this problem to selecting $r = 6$ players from the $n = 10$ unfixed players. The total number of ways we can do so is $C(10, 6) = \frac{10!}{6!4!} = \boxed{210}$.

2.3

If a fixed player is never chosen, we still have 7 remaining positions to fill. However, instead of 11 players to choose from, we only have 10. So, we reduce this problem to selecting $r = 7$ players from $n = 10$ remaining players. The total number of ways we can do so is $C(10, 7) = \frac{10!}{7!3!} = \boxed{120}$.

3

To create 4 groups, we can select them one by one. From 20 people, we have $C(20, 4)$ ways to select the first group. Then, we have $C(16, 4)$ ways to select the second group because 4 players from the initial 20 have already been assigned to the first group. Following this pattern, there are $C(12, 4)$ ways to choose the third group and $C(8, 4)$ ways to choose the fourth group. Since all the groups are distinct, we do not have to consider an overcounting problem. Thus, the total number of ways to select 4 groups of 4 from 20 people where groups are distinct is $\boxed{C(20, 4) \cdot C(16, 4) \cdot C(12, 4) \cdot C(8, 4)}$.

4

We can first calculate the number of permutations if all the letters were distinct. There are 8! permutations with distinct letters because we have 8 options for the first letter, 7 for the second letter, and so on. However, because we have duplicate letters, we need to account for overcounting. All letters appear once except i appears 3 times and l appears 2 times. This means we are overcounting by 3! for the i and 2! for the l. So, the total number of permutations for Illinois (where letters are case insensitive) is $\frac{8!}{3!2!} = \boxed{3360}$.

5

If none of the points were colinear, then this would be a simple combinations problem of choosing $r = 3$ points from $n = 9$ total points to form a triangle. The number of ways to do so would be $C(9, 3) = \frac{9!}{3!6!} = 84$. However, not all these combinations are valid. In particular, we cannot form a triangle if all 3 points selected are the colinear ones. There is only one such combination where this occurs, so the total number of ways to create a triangle from 9 points where 3 are colinear is $C(9, 3) - 1 = 84 - 1 = \boxed{83}$.