CMSC 27100 - Problem Set 2

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1

- injective but not surjective: $f : \mathbb{Z} \mapsto \mathbb{Z}$ such that $\boxed{f(x) = 2x}$
- surjective but not injective: $f : \mathbb{Z} \mapsto \mathbb{Z}$ such that $\boxed{f(x) = x^3 - x}$

2

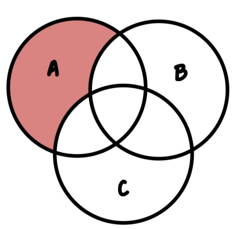
We must show that if f is a bijection, then f^{-1} exists, and that if f^{-1} exists, then f is a bijection.

- Starting with the former, assume f is a bijection. This means that for any $y \epsilon B$, there exists $x \epsilon A$ such that $f(x) = y$ (because f is surjective) and x is unique (because f is injective), allowing us to define $g : B \mapsto A$ such that $g(y) = x$. Given this, we know that for all $x \epsilon A$, $g(f(x)) = g(y) = x$ and for all $y \epsilon B$, $f(g(y)) = f(x) = y$. Thus, we have shown that if f is a bijection, then $g = f^{-1}$ exists.
- For the latter, assume f^{-1} exists. Defining $g : B \mapsto A$ as $g = f^{-1}$ we know that for all $x \epsilon A$, $g(f(x)) = x$ and for all $y \epsilon B$, $f(g(y)) = y$. Assume $x_1 \epsilon A$ and $x_2 \epsilon A$. If $f(x_1) = f(x_2)$ where $f(x_1) = y$ and $f(x_2) = y$, we know that $g(f(x_1)) = x_1$ and $g(f(x_2)) = x_2$, so $g(y) = x_1$ and $g(y) = x_2$. Since g is a well-defined function, each element $y \epsilon B$ can only be mapped to a single element $x \epsilon A$, so $x_1 = x_2$. Thus, f is injective. For all $y \epsilon B$, we know there exists $x \epsilon A$ such that $g(y) = x$, and since $g = f^{-1}$, then $f(g(y)) = f(x) = y$. Thus, f is surjective. Since f is both injective and surjective, it is bijective.

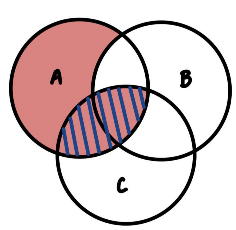
We have shown that if f is a bijection, then f^{-1} exists, and that if f^{-1} exists, then f is a bijection. This means that a function has an inverse if and only if it is a bijection.

3

$(A \setminus B) \setminus C = A^c \cap B^c \cap C^c$



$A \setminus (B \setminus C) = A \cap (B \cap C^c)^c$



$A \cap C = \emptyset$

The venn diagram shows that with no restrictions for A, B, and C, $(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$, unless $A \cap C = \emptyset$. We can formalize this proof as follows:

$(A \setminus B) \setminus C)$ is equivalent to $(A \cap B^C) \cap C^C$, which is the same as $A \cap B^C \cap C^C$.

$A \setminus (B \setminus C)$ is equivalent to $A \cap (B \cap C^C)^C$, which is the same as $A \cap (B^C \cup C)$. Distributing the $\cap$, we get $(A \cap B^C) \cup (A \cap C)$. We can rewrite this as $((A \cap B^C) \cap C) \cup ((A \cap B^C) \cap C^C) \cup ((A \cap C) \cap B) \cup ((A \cap C) \cap B^C)$. This is equivalent to $(A \cap B^C \cap C) \cup (A \cap B^C \cap C^C) \cup (A \cap B \cap C) \cup (A \cap B^C \cap C)$, which can be simplified to and rewritten as $(A \cap B^C \cap C^C) \cup (A \cap (B^C \cap C)) \cup ((A \cap (B \cap C))$. This is equivalent to $(A \cap B^C \cap C^C) \cup (A \cap ((B^C \cap C) \cup (B \cap C)))$. This simplifies to $(A \cap B^C \cap C^C) \cup (A \cap C)$.

For $A \cap B^C \cap C^C = (A \cap B^C \cap C^C) \cup (A \cap C)$, $A \cap C = \emptyset$. Thus, $\boxed{A \cap C = \emptyset}$ is a necessary and sufficient condition for $(A \setminus B) \setminus C = A \setminus (B \setminus C)$ to hold.

4

GCD(84, 63) = GCD(63, 84 **mod** 63) = GCD(63, 21) = GCD(21, 63 **mod** 21) = GCD(21, 0) = $\boxed{21}$

5

Let $x = 20$, $y = 5$, and $z = 8$. Then, $x \mid (y \cdot z)$ is true because $20 \mid (5 \cdot 8)$ or $20 \mid 40$. However, $x \mid y \vee x \mid z$ is false because $20 \mid 5 \vee 20 \mid 8$ is false. Therefore, the statement $x \mid (y \cdot z) \Rightarrow x \mid y \vee x \mid z$ is $\boxed{\text{false}}$.

6

We need to prove that for all integers a, b, if $3a - 5b = 31$, then GCD(a, b) $\neq 17$. We can prove this by proving the contrapositive. Assume GCD(a, b) = 17. By definition, $17 \mid a$ and $17 \mid b$, so there exists integers d_1 and d_2 such that $17 \cdot d_1 = a$ and $17 \cdot d_2 = b$. Substituting a and b into $3a - 5b = 31$, we get that $3(17 \cdot d_1) - 5(17 \cdot d_2) = 31$, which we can rewrite as $17(3d_1 - 5d_2) = 31$. We defined d_1 and d_2 as divisors, so $3d_1 - 5d_2$ must be an integer, which implies $17 \mid 31$, which is false, so $3a - 5b = 31$ is false. Thus, we have proven the contrapositive true, and so for all integers a, b, if $3a - 5b = 31$, then GCD(a, b) $\neq 17$ is $\boxed{\text{true}}$.