MATH 15910 - Problem Set 1

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To show that if A, B, and C are sets and $A \subset B$ and $B \subset C$, then $A \subset C$, first suppose that $x \epsilon A$. $A \subset B$ means that if $x \epsilon A$, then $x \epsilon B$. Similarly, $B \subset C$ means that if $x \epsilon B$, then $x \epsilon C$. Thus, if $x \epsilon A$, then $x \epsilon C$, so $A \subset B$ and $B \subset C$ mean that $A \subset C$.

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2.1

To show $A \cup (B \cup C) = (A \cup B) \cup C$, we must show that $A \cup (B \cup C) \subset (A \cup B) \cup C$ and $(A \cup B) \cup C \subset A \cup (B \cup C)$. Starting with the former, assume $x \epsilon A \cup (B \cup C)$, which means that $x \epsilon A$ or $x \epsilon (B \cup C)$. This is equivalent to $x \epsilon A$ or ($x \epsilon B$ or $x \epsilon C$), which means $x \epsilon A$ or $x \epsilon B$ or $x \epsilon C$. This is the same as ($x \epsilon A$ or $x \epsilon B$) or $x \epsilon C$, which means $x \epsilon (A \cup B)$ or $x \epsilon C$, so $x \epsilon (A \cup B) \cup C$. Thus, we have shown that if $x \epsilon A \cup (B \cup C)$, then $x \epsilon (A \cup B) \cup C$, or that $A \cup (B \cup C) \subset (A \cup B) \cup C$.

For the latter, assume $x \epsilon (A \cup B) \cup C$, which means that $x \epsilon (A \cup B)$ or $x \epsilon C$. This is equivalent to ($x \epsilon A$ or $x \epsilon B$) or $x \epsilon C$, which means $x \epsilon A$ or $x \epsilon B$ or $x \epsilon C$. This is the same as $x \epsilon A$ or ($x \epsilon B$ or $x \epsilon C$), which means $x \epsilon A$ or $x \epsilon (B \cup C)$, so $x \epsilon A \cup (B \cup C)$. Thus, we have shown that if $x \epsilon (A \cup B) \cup C$, then $x \epsilon A \cup (B \cup C)$, or that $(A \cup B) \cup C \subset A \cup (B \cup C)$.

By showing that the these sets are subsets of each other, we have proven that $A \cup (B \cup C) = (A \cup B) \cup C$.

2.2

To show $A \cap (B \cap C) = (A \cap B) \cap C$, we must show that $A \cap (B \cap C) \subset (A \cap B) \cap C$ and $(A \cap B) \cap C \subset A \cap (B \cap C)$. Starting with the former, assume $x \epsilon A \cap (B \cap C)$, which means that $x \epsilon A$ and $x \epsilon (B \cap C)$. This is equivalent to $x \epsilon A$ and ($x \epsilon B$ and $x \epsilon C$), which means $x \epsilon A$ and $x \epsilon B$ and $x \epsilon C$. This is the same as ($x \epsilon A$ and $x \epsilon B$) and $x \epsilon C$, which means $x \epsilon (A \cap B)$ and $x \epsilon C$, so $x \epsilon (A \cap B) \cap C$. Thus, we have shown that if $x\epsilon A\cap(B\cap C)$, then $x\epsilon(A\cap B)\cap C$, or that $A\cap(B\cap C)\subset(A\cap B)\cap C$.

For the latter, assume $x\epsilon(A\cap B)\cap C$, which means that $x\epsilon(A\cap B)$ and $x\epsilon C$. This is equivalent to ($x\epsilon A$ and $x\epsilon B$) and $x\epsilon C$, which means $x\epsilon A$ and $x\epsilon B$ and $x\epsilon C$. This is the same as $x\epsilon A$ and ($x\epsilon B$ or $x\epsilon C$), which means $x\epsilon A$ and $x\epsilon(B\cap C)$, so $x\epsilon A\cap(B\cap C)$. Thus, we have shown that if $x\epsilon(A\cap B)\cap C$, then $x\epsilon A\cap(B\cap C)$, or that $(A\cap B)\cap C\subset A\cap(B\cap C)$.

By showing that the these sets are subsets of each other, we have proven that $A\cap(B\cap C)=(A\cap B)\cap C$.

2.3

To show $A\Delta(B\Delta C)=(A\Delta B)\Delta C$, we must show that $A\Delta(B\Delta C)\subset(A\Delta B)\Delta C$ and $A\Delta(B\Delta C)\subset(A\Delta B)\Delta C$. Each statement implies the next.

Statement 1. Before starting, we will demonstrate a set equality that justifies the equality shown (used later in proof). Let X and Y be sets. By distributing the $\cap$ in $(X^C\cup Y)\cap(X\cup Y^C)$, we get that it is equivalent to $(X^C\cap(X\cup Y^C))\cup(Y\cap(X\cup Y^C))$. Again, distributing the $\cap$, we get that is is equivalent to $(X^C\cap X)\cup(X^C\cap Y^C)\cup(Y\cap X)\cup(Y\cap Y^C)$. This simplifies to $(X\cap Y)\cup(X^C\cup Y^C)$.

Starting with the former, assume $x\epsilon A\Delta(B\Delta C)$.
Definition of Δ: $x\epsilon(A\cap(B\Delta C)^C)\cup(A^C\cap(B\Delta C))$
Definition of Δ: $x\epsilon(A\cap((B\cap C^C)\cup(B^C\cap C))^C)\cup(A^C\cap((B\cap C^C)\cup(B^C\cap C)))$
De Morgan's Law: $x\epsilon(A\cap((B\cap C^C)^C\cap(B^C\cap C)^C))\cup(A^C\cap((B\cap C^C)\cup(B^C\cap C)))$
De Morgan's Law: $x\epsilon(A\cap((B^C\cup C)\cap(B\cup C^C))\cup(A^C\cap((B\cap C^C)\cup(B^C\cap C)))$
Statement 1 (from above): $x\epsilon(A\cap((B\cap C)\cup(B^C\cap C^C)))\cup(A^C\cap((B\cap C^C)\cup(B^C\cap C)))$
Distribute $\cap$ over $\cup$: $x\epsilon((A\cap(B\cap C))\cup(A\cap(B^C\cap C^C)))\cup((A^C\cap(B\cap C^C))\cup(A^C\cap(B^C\cap C)))$
Simplify: $x\epsilon(A\cap B\cap C)\cup(A\cap B^C\cap C^C)\cup(A^C\cap B\cap C^C)\cup(A^C\cap B^C\cap C)$
Rearrange: $x\epsilon(((A\cap B^C)\cap C^C)\cup((A^C\cap B)\cap C^C))\cup(((A\cap B)\cap C)\cup((A^C\cap B^C)\cap C))$
Un-distribute $\cap$ over $\cup$: $x\epsilon(((A\cap B^C)\cup(A^C\cap B))\cap C^C)\cup(((A\cap B)\cup(A^C\cap B^C))\cap C)$
Statement 1 (from above): $x\epsilon(((A\cap B^C)\cup(A^C\cap B))\cap C^C)\cup(((A^C\cup B)\cap(A\cup B^C))\cap C)$
De Morgan's Law: $x\epsilon(((A\cap B^C)\cup(A^C\cap B))\cap C^C)\cup(((A\cap B^C)^C\cap(A^C\cap B)^C)\cap C)$
De Morgan's Law: $x\epsilon(((A\cap B^C)\cup(A^C\cap B))\cap C^C)\cup(((A\cap B^C)\cup(A^C\cap B))^C\cap C)$
Definition of Δ: $x\epsilon((A\Delta B)\cap C^C)\cup((A\Delta B)^C\cap C)$
Definition of Δ: $x\epsilon(A\Delta B)\Delta C$

Thus, $A\Delta(B\Delta C)\subset(A\Delta B)\Delta C$.

For the latter, assume $x\epsilon(A\Delta B)\Delta C$.
Definition of Δ: $x\epsilon((A\Delta B)\cap C^C)\cup((A\Delta B)^C\cap C)$
Definition of Δ: $x\epsilon(((A\cap B^C)\cup(A^C\cap B))\cap C^C)\cup(((A\cap B^C)\cup(A^C\cap B))^C\cap C)$
De Morgan's Law: $x\epsilon(((A\cap B^C)\cup(A^C\cap B))\cap C^C)\cup(((A\cap B^C)^C\cap(A^C\cap B)^C)\cap C)$
De Morgan's Law: $x\epsilon(((A\cap B^C)\cup(A^C\cap B))\cap C^C)\cup(((A^C\cup B)\cap(A\cup B^C))\cap C)$
Statement 1 (from above): $x\epsilon(((A\cap B^C)\cup(A^C\cap B))\cap C^C)\cup(((A\cap B)\cup(A^C\cap B^C))\cap C)$
Distribute $\cap$ over $\cup$: $x\epsilon(((A\cap B^C)\cap C^C)\cup((A^C\cap B)\cap C^C))\cup(((A\cap B)\cap C)\cup((A^C\cap B^C)\cap C))$
Simplify: $x\epsilon(A\cap B^C\cap C^C)\cup(A^C\cap B\cap C^C)\cup(A\cap B\cap C)\cup(A^C\cap B^C\cap C)$
Rearrange: $x\epsilon((A\cap(B\cap C))\cup(A\cap(B^C\cap C^C)))\cup((A^C\cap(B\cap C^C))\cup(A^C\cap(B^C\cap C)))$
Un-distribute $\cap$ over $\cup$: $x\epsilon(A\cap((B\cap C)\cup(B^C\cap C^C)))\cup(A^C\cap((B\cap C^C)\cup(B^C\cap C)))$
Statement 1 (from above): $x\epsilon(A\cap((B^C\cup C)\cap(B\cup C^C))\cup(A^C\cap((B\cap C^C)\cup(B^C\cap C)))$
De Morgan's Law: $x\epsilon(A\cap((B\cap C^C)^C\cap(B^C\cap C)^C))\cup(A^C\cap((B\cap C^C)\cup(B^C\cap C)))$
De Morgan's Law: $x\epsilon(A\cap((B\cap C^C)\cup(B^C\cap C))^C)\cup(A^C\cap((B\cap C^C)\cup(B^C\cap C)))$
Definition of Δ: $x\epsilon(A\cap(B\Delta C)^C)\cup(A^C\cap(B\Delta C))$
Definition of Δ: $x\epsilon A\Delta(B\Delta C)$.

Thus, $A\Delta(B\Delta C)\subset(A\Delta B)\Delta C$.

By showing that the these sets are subsets of each other, we have proven that $A\Delta(B\Delta C)=(A\Delta B)\Delta C$.

2.4

To show $A\cup\emptyset=A$, we must show that $A\cup\emptyset\subset A$ and $A\subset A\cup\emptyset$. Starting with the former, assume $x\epsilon A\cup\emptyset$, then $x\epsilon A$ or $x\epsilon\emptyset$. The empty set cannot contain any elements, so the element $x\notin\emptyset$. This means that if $x\epsilon A\cup\emptyset$, then $x\epsilon A$, so $A\cup\emptyset\subset A$.

For the latter, assume $x\epsilon A$. The set $A\cup\emptyset$ contains all elements in A and $\emptyset$. This means that if $x\epsilon A$, then $x\epsilon A\cup\emptyset$, so $A\subset A\cup\emptyset$.

By showing that the these sets are subsets of each other, we have proven that $A\cup\emptyset=A$.

2.5

To show that if $A\cap B=X$, then $A=X$ and $B=X$, assume $x\epsilon X$. Since $X=A\cap B$, $x\epsilon(A\cap B)$, meaning $x\epsilon A$ and $x\epsilon B$. So, if $x\epsilon X$, then $x\epsilon A$ and $x\epsilon B$. This means that $X\subset A$ and $X\subset B$. From the problem statement, we know that A and B are subsets of X, so $A\subset X$ and $B\subset X$. Thus, since $X\subset A$ and $A\subset X$, $A=X$. Similarly, since $X\subset B$ and $B\subset X$, $B=X$.

2.6

To show that if $A\Delta B = \emptyset$, then $A = B$, we use the definition of Δ to show that $A\Delta B = (A-B)\cup(B-A)$. By definition, this is equivalent to $(A\cap B^C)\cup(B\cap A^C)$. We are given that $A\Delta B = \emptyset$, meaning both $A \cap B^C = \emptyset$ and $B \cap A^C = \emptyset$. For $A\cap B^C = \emptyset$, no element can exist in A and not exist in B, so $A \subset B$. Similarly, for $B \cap A^C = \emptyset$, no element can exist in B and not exist in A, so $B \subset A$. If $A \subset B$ and $B \subset A$, then $A = B$. Thus, we have shown that if $A\Delta B = \emptyset$, then $A = B$.

2.7

To show that $(A \cap B)^C = A^C \cup B^C$, we must show that $(A \cap B)^C \subset A^C \cup B^C$ and $A^C \cup B^C \subset (A \cap B)^C$. Starting with the former, assume $x\epsilon(A \cap B)^C$, then $x\epsilon X$ and $x \notin (A \cap B)$, which is equivalent to $x\epsilon X$ and ($x \notin A$ or $x \notin B$). This means that ($x\epsilon X$ and $x \notin A$) or ($x\epsilon X$ and $x \notin B$), so $x\epsilon A^C$ or $x\epsilon B^C$, which is equivalent to $x\epsilon(A^C \cup B^C)$. Thus, we have shown that if $(A \cap B)^C$, then $A^C \cup B^C$, or that $(A \cap B)^C \subset A^C \cup B^C$.

For the latter, assume $x\epsilon(A^C \cup B^C)$, then $x\epsilon X$ and $x\epsilon(A^C \cup B^C)$, which is equivalent to $x\epsilon X$ and ($x\epsilon A^c$ or $x\epsilon B^C$). This means that $x \notin A$ or $x \notin B$, so $x \notin (A \cap B)$, which is equivalent to $x\epsilon(A \cap B)^C$. Thus, we have shown that if $x\epsilon(A^C \cup B^C)$, then $(A \cap B)^C$, or that $x\epsilon(A^C \cup B^C) \subset (A \cap B)^C$.

By showing that the these sets are subsets of each other, we have proven that $(A \cap B)^C = A^C \cup B^C$.

3

To show that $A \times B = B \times A$ iff $A = B$, we need to show if $A \times B = B \times A$, then $A = B$ and if $A = B$, then $A \times B = B \times A$.

Starting with the former, assume $a\epsilon A$ and $b\epsilon B$. Then, $(a, b)\epsilon A \times B$. The two Cartesian products are equal, so $(a, b)\epsilon B \times A$. This means that $a\epsilon B$ and $b\epsilon A$. Thus, if $a\epsilon A$, then $a\epsilon B$, which means $A \subset B$. Similarly, if $b\epsilon B$, then $b\epsilon A$, which means $B \subset A$. Because $A \subset B$ and $B \subset A$, that means if $A \times B = B \times A$, then $A = B$.

For the latter, we know $A = B$. This means $A \times B = A \times A$. Similarly, $B \times A = A \times A$. Because both $A \times B$ and $B \times A$ are equivalent to $A \times A$, $A \times B = B \times A$. Thus, if $A = B$, then $A \times B = B \times A$.

By showing that if $A \times B = B \times A$, then $A = B$ and if $A = B$, then $A \times B = B \times A$, we have proven that $A \times B = B \times A$ iff $A = B$.

4

To prove an equivalence relation, we must check 3 conditions.

- Reflexive: For any $x \epsilon X$, $(x, x) \epsilon R$ must be true. This is true because the arbitrary book x in the set X has the same author first name as itself. Thus, (x, x) must exist in the relation.

- Symmetric: For $x \epsilon X$ and $y \epsilon X$, where $(x, y) \epsilon R$, $(y, x) \epsilon R$ must be true. This is true because if the arbitrary book x in the set X and arbitrary book y in the set X have the same author first name, they would still have the same author first name if the order or the books were switched. Thus, (y, x) must exist in the relation.

- Transitive: For $x \epsilon X$, $y \epsilon X$, and $z \epsilon X$, where $(x, y) \epsilon R$ and $(y, z) \epsilon R$, $(x, z) \epsilon R$ must be true. This is true because if the arbitrary book x in the set X and arbitrary book y in the set X have the same author first name, and book y and arbitrary book z in the set X have the same author first name, then book x and z must also have the same author first name. Thus, (x, z) must exist in the relation.