MATH 15910 - Problem Set 7

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1

A set is countable if and only if its elements can be listed one by one.

- The set $\{x\}$ contains a single object, so it is countable because we can list its elements as $\{x\}$. It is important to note that we treat x as a single object, so whether x is a set (finite or infinite) or another element altogether does not matter.
- Since A is countable, we can list out its elements.
  - If A is finite, we can list the elements of A as $\{a_1, a_2, ..., a_n\}$, so $A \cup \{x\}$ can be listed as $\{x, a_1, a_2, ..., a_n\}$, where there are n elements in A.
  - If A is infinite, we can list the elements of A as $\{a_1, a_2, ...\}$, so $A \cup \{x\}$ can be listed as $\{x, a_1, a_2, ...\}$.

In both cases, $A \cup \{x\}$ can have its elements listed out, so $A \cup \{x\}$ is countable. We can also get this result by knowing that the union of countable sets is countable. We know $\{x\}$ is countable (single object x) and A is countable (given in the problem).

2

First, we construct a bijection from $\mathbb{Q}$ to $\mathbb{Z}$:

- From class, we know that there exists a bijection from $\mathbb{Q}_+$ to $\mathbb{Z}_+$.
- By symmetry, we know that there exists a bijection from $\mathbb{Q}_-$ to $\mathbb{Z}_-$ since each element of $\mathbb{Q}_+$ can be multiplied by -1 and the cardinality will not be changed. The same applies for $\mathbb{Z}_-$. We can map the negative of every element of $\mathbb{Q}_+$ to the negative of what it is mapped to in $\mathbb{Z}_+$ to get this bijection.
- The only element of $\mathbb{Q}$ not included in either $\mathbb{Q}_+$ or $\mathbb{Q}_-$ is 0, which we can map to 0 in $\mathbb{Z}$, which is not in $\mathbb{Z}_+$ or $\mathbb{Z}_-$.

Combining these, we get that there exists a bijection from $\mathbb{Q}$ to $\mathbb{Z}$. We know that $\mathbb{Q}_-$, $\{0\}$, and $\mathbb{Q}_+$ are disjoint. Similarly, $\mathbb{Z}_-$, $\{0\}$, and $\mathbb{Z}_+$ are disjoint. This means we can construct a bijection from $\mathbb{Q}$ to $\mathbb{Z}$ by combining the bijection from $\mathbb{Q}_+$ to $\mathbb{Z}_+$, $\mathbb{Q}_-$ to $\mathbb{Z}_-$, and $0 \epsilon \mathbb{Q}$ to $0 \epsilon \mathbb{Z}$.

Now, we use the fact that exists a bijection from $\mathbb{Z}$ to $\mathbb{Z}_+$:

- From class, we know that there exists a bijection from $\mathbb{Z}$ to $\mathbb{Z}_+$.
- With a bijection from $\mathbb{Q}$ to $\mathbb{Z}$ and $\mathbb{Z}$ to $\mathbb{Z}_+$, transitivity tells us that there exists a bijection from $\mathbb{Q}$ to $\mathbb{Z}_+$. If $f : \mathbb{Q} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}_+$, we know that $g \circ f : \mathbb{Q} \to \mathbb{Z}_+$.

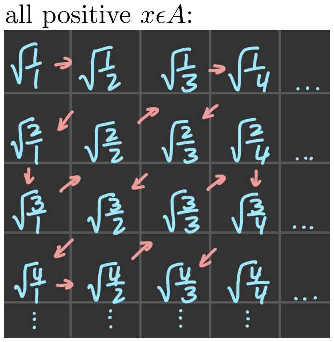
– From previous assignments/class, we know that if f is a bijective function and g is a bijective function, then $g \circ f$ is also a bijective function.

Since there exists a bijection from $\mathbb{Q}$ to $\mathbb{Z}_+$, $\#\mathbb{Q} = \#\mathbb{Z}_+$, and so, $\mathbb{Q}$ must be countable by definition.

3

3.1

Let $A = \{x\epsilon\mathbb{R} \mid x^2\epsilon\mathbb{Q}\}$. We know that for all $x\epsilon A$, $x^2\epsilon\mathbb{Q}$, so x^2 where x is positive can be enumerated using the table from class that traverses all positive rational numbers $\mathbb{Q}_+$. We take the square root of the table to get the traversal for x where x is positive. The following demonstrates the bijection for all positive $x\epsilon A$:



Using symmetry, we can construct a bijection for all negative $x\epsilon A$ to $\mathbb{Z}_-$. So, we have a bijection from positive $x\epsilon A$ to $\mathbb{Z}_+$ (diagram) and a bijection from negative $x\epsilon A$ to $\mathbb{Z}_-$. We also do this mapping from $0\epsilon\mathbb{Q}$ to $0\epsilon\mathbb{Z}$. Using what we showed in problem 2, we combine these bijections to get a bijection from A to $\mathbb{Z}$, and the bijection from $\mathbb{Z}$ to $\mathbb{Z}_+$ means there exists a bijection from A to $\mathbb{Z}_+$. By definition, if there exists a bijection between a set A and $\mathbb{Z}_+$, then A is countable.

3.2

We must show that the set of all finite subsets of $\mathbb{Z}$ is countable. Let $A = \{\text{finite subsets of } \mathbb{Z}\}$. Furthermore, we can let $A(n) = \{S \subset \mathbb{Z} \mid s\epsilon S \implies |s| \leq n\}$. From here, we know that $A = \bigcup_{n=0}^{\infty} A(n)$, which is true because $A(n)$ represents every subset of A. Each $S\epsilon A(n) \subset \{-n, -(n-1), ..., 0, ..., n-1, n\}$, which is a finite set, and any subset of a finite set is also finite, meaning $S\epsilon A(n)$ is finite. Since the subsets are finite, they are countable by definition because we can enumerate the elements from $x_1, ..., x_m$ where m is the cardinality of the finite set. From class, we know that the union of countable sets is countable, so since $A(n)$ is countable, then $\bigcup_{n=0}^{\infty} A(n)$ is countable. Thus, we have shown that A is countable, or that the set of all finite subsets of $\mathbb{Z}$ is countable.