MATH 15910 - Problem Set 3

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1

We can show $a < b$ implies $a \leq b - 1$ by proving the contrapositive. Negate $a \leq b - 1$, so take $a > b - 1$. Assume for contradiction that $a < b$. This means that $b - 1 < a < b$ (transitivity for order), or that $b < a + 1 < b + 1$ (addition property for order). Since a is an integer, $a + 1$ is also an integer, so there must be an integer between b and $b+1$. Denote this as integer x such that $b < x < b+1$, which is equivalent to $b+(-b) < x+(-b) < b+1+(-b)$ (addition property for order), or $b + (-b) < x + (-b) < b + (-b) + 1$ (commutativity for addition). This means $(b + (-b)) < x + (-b) < (b + (-b)) + 1$ (associativity for addition), so $0 < x + (-b) < 1$ (additive inverse for addition). $x + (-b)$ is an integer (addition is $\mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$), so this means there must exist an integer between 0 and 1. This is a contradiction (by proposition shown in class that for any positive integer n, $n \geq 1$). Therefore, the contrapositive of if $a < b$, then $a \leq b - 1$ is true. Thus, if $a < b$, then $a \leq b - 1$.

2

2.1

If $a > 0$ and $b < 0$, then $a \cdot b < a \cdot 0$ (multiplicative property for order). This means $a \cdot b < 0$ (by proposition shown in class that $x \cdot 0 = 0$). Thus, if $a > 0$ and $b < 0$, then $a \cdot b < 0$.

2.2

If $a < 0$ and $b < 0$, then $(-a) > 0$ and $(-b) > 0$ (by proposition shown in class that $x < 0$ if and only if $(-x) > 0$). This means $(-a) \cdot (-b) > (-a) \cdot 0$ (multiplicative property for order), or $(-a) \cdot (-b) > 0$ (by proposition shown in class that $x \cdot 0 = 0$). We know that $(-b) = -(b)$ (by proposition shown in class that $x \cdot (-1) = (-x)$). This is equivalent to $-(-a) \cdot b > 0$ (associativity for multiplication). We must show that $-(-a) = a$. We know that $-a + (-(-a)) = 0$ (additive inverse for addition), so $a + (-a) + (-(-a)) = a + 0$ (cancellation for addition). This means that $0 + (-(-a)) = a + 0$ (additive inverse for addition), which means $-(-a) = a$ (identity element for addition). Since $-(-a) = a$, we know that $-(-a) \cdot b > 0$ means $a \cdot b > 0$. Thus, if $a < 0$ and $b < 0$, then $a \cdot b > 0$.

3

3.1

Let $a = 1$. We know that for all $a \epsilon \mathbb{Z}$, there exists an element $(-a)$ such that $a+(-a) = 0$. We can write $1+1 = 0$ as $a+a = 0$, or $a+(-a) = 0$ (additive inverse for addition). This means $a + a = a + (-a)$, so $(-a) + a + a = (-a) + a + (-a)$ (cancellation for addition). This means $((-a) + a) + a = ((-a) + a) + (-a)$ (associativity for addition) and that $0 + a = 0 + (-a)$ (additive inverse for addition). This is equivalent to $a = (-a)$ (identity element for addition). If $a > 0$, then $(-a) < 0$ and if $a < 0$, then $(-a) > 0$ (by proposition shown in class). Since $a = -a$, an element cannot be both greater or less than 0, so $a = 0$ (trichotomy for order). However, this is a contradiction because we assumed $a = 1$. Thus, $1 + 1 \neq 0$.

3.2

We must show that if $a > b$, then $a^2 > b^2$ and if $a^2 > b^2$, then $a > b$. Starting with the former, take $a > b$. Since $a > 0$ and $b > 0$, we know $a \cdot a > a \cdot b$ and $a \cdot b > b \cdot b$ (multiplicative property for order). Since $a \cdot a > a \cdot b$ and $a \cdot b > b \cdot b$, then $a \cdot a > b \cdot b$ (transitivity for order), so $a^2 > b^2$. For the latter, we can show that $a^2 > b^2$ implies $a > b$ by proving the contrapositive. The negation of $a > b$ has two cases. First, take $a = b$. Then, $a \cdot a = a \cdot b$ (multiplication is $\mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$), or $a^2 = a \cdot b$. Similarly, $b \cdot a = b \cdot b$ (multiplication is $\mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$), or $a \cdot b = b^2$. Since $a \cdot b = a^2$ and $a \cdot b = b^2$, then $a^2 = b^2$ (transitivity), so $a^2 > b^2$ is false. Second, take $a < b$. Then, $a \cdot a < a \cdot b$ (multiplicative property for order), or $a^2 < a \cdot b$. Similarly, $b \cdot a < b \cdot b$ (multiplicative property for order), or $a \cdot b < b^2$. Since $a^2 < a \cdot b$ and $a \cdot b < b^2$, then $a^2 < b^2$ (transitivity for order), so $a^2 > b^2$ is false. If $a = b$ or $a < b$, then $a^2 > b^2$ is false, proving the contrapositive. Thus, we have shown that if $a > b$, then $a^2 > b^2$ and if $a^2 > b^2$, then $a > b$, so $a > b$ if and only if $a^2 > b^2$.

4

4.1

Let $P(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. $P(1) = 1^2 = \frac{1(1+1)(2 \cdot 1+1)}{6} = 1$. Thus, the base case $n = 1$ is true. Assume $P(n)$ is true for $n \epsilon \mathbb{Z}_+$, meaning $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. We can add $(n+1)^2$ to both sides, giving us $\sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$. So, $\sum_{i=1}^{n+1} i^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$. We can simplify the RHS as follows:
$= \frac{n(n+1)(2n+1)+6(n+1)(n+1)}{6}$

$= \frac{(n+1)(n(2n+1)+6(n+1))}{6}$
$= \frac{(n+1)(2n^2+n+6n+6)}{6}$
$= \frac{(n+1)(2n^2+4n+3n+6)}{6}$
$= \frac{(n+1)(2n(n+2)+3(n+2))}{6}$
$= \frac{(n+1)(n+2)(2n+3)}{6}$
$= \frac{(n+1)(n+2)(2(n+1)+1)}{6}$

This implies that $P(n+1)$ is true. Thus, mathematical induction shows that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ holds for any $n\epsilon\mathbb{Z}_+$.

4.2

Let $P(n) = (1+a)^n \geq 1+n\cdot a$. $P(1) = (1+a)^1 \geq 1+1\cdot a$, which is true because $1+a \geq 1+a$. Thus, the base case $n=1$ is true. Assume $P(n)$ is true for $n\epsilon\mathbb{Z}_+$, meaning $P(n) = (1+a)^n \geq 1+n\cdot a$. We must show that $P(n+1)$ is also true. $P(n+1) = (1+a)^{n+1} \geq 1+(n+1)\cdot a$. We can rewrite $(1+a)^{n+1}$ as $(1+a)^n(1+a)$. Using the assumption that $P(n)$ is true, $(1+a)^n(1+a) \geq (1+n\cdot a)(1+a)$. $(1+n\cdot a)(1+a) = 1+a+n\cdot a+n\cdot a^2 = 1+(n+1)\cdot a+n\cdot a^2 \geq 1+(n+1)\cdot a$. So, $(1+a)^n(1+a) \geq (1+n\cdot a)(1+a) \geq 1+(n+1)\cdot a$, so $(1+a)^{n+1} \geq 1+(n+1)\cdot a$. Thus, $(1+a)^{n+1} \geq 1+(n+1)\cdot a$. Thus, mathematical induction shows that $(1+a)^n \geq 1+n\cdot a$ holds for any $n\epsilon\mathbb{Z}_+$.