MATH 15910 - Problem Set 8

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1.1

The accumulation points for $(0, 1)$ are $x\epsilon\mathbb{R}$ such that $\boxed{0 \leq x \leq 1}$.

1.2

The accumulation points for $[-1, 0]$ are $x\epsilon\mathbb{R}$ such that $\boxed{-1 \leq x \leq 0}$.

1.3

The accumulation points for $\mathbb{Q}$ are $\boxed{\mathbb{R}}$ because the rational numbers are dense in the real numbers. From class, we know that for any two real numbers a and b such that $a < b$, there exists a rational number r such that $a < r < b$. Now, we need to show that there are infinitely many rational numbers between two real numbers. By the Archimedean property, we know there exists a positive integer N such that $\frac{1}{N} < \epsilon$. Choosing $b - r$ as an arbitrary ϵ, we know there exists a positive integer N such that $\frac{1}{N} < b-r$. We also know $\frac{1}{N+1} < \frac{1}{N}$, so $\frac{1}{N+1} < b-r$. Thus, we get an infinite set of rational numbers $X = \{\frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, ...\}$ such that for any $x\epsilon X$, $x < b - r$ or $a < r + x < b$.

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2.1

We claim that $\lim_{k\to\infty} 2^{-k} = 0$. We need to show that for any $\epsilon > 0$, there exists $N\epsilon\mathbb{Z}_+$ such that for all $k > N$, $|2^{-k} - 0| < \epsilon$. We know that $|2^{-k} - 0| = |2^{-k}| = 2^{-k}$. By the Archimedean property, we know that for any $\epsilon > 0$, there exists $N\epsilon\mathbb{Z}_+$ such that $\frac{1}{N} < \epsilon$. Since $2^N > N$, $\frac{1}{2^N} < \frac{1}{N}$ or $2^{-N} < \frac{1}{N}$. Also, for $k > N$, $2^{-k} < 2^{-N}$. This means $2^{-k} < 2^{-N} < \frac{1}{N} < \epsilon$, so $2^{-k} < \epsilon$. Thus, we have shown that for any $\epsilon > 0$, there exists $N\epsilon\mathbb{Z}_+$ such that for all $k > N$, $|a_k - L| < \epsilon$, where $a_k = 2^{-k}$ and $L = 0$. This means the sequence 2^{-k} $\boxed{\text{converges}}$, and the limit is $\boxed{0}$.

2.2

We claim that $\lim_{k\to\infty} \frac{k}{k+1} = 0$. We can rewrite $\frac{k}{k+1} = \frac{k+1-1}{k+1} = \frac{k+1}{k+1} - \frac{1}{k+1} = 1 - \frac{1}{k+1}$. We know $\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty}(1 - \frac{1}{k+1}) = \lim_{k\to\infty} 1 - \lim_{k\to\infty} \frac{1}{k+1} = 1 - \lim_{k\to\infty} \frac{1}{k+1}$.

We claim that $\lim_{k\to\infty} \frac{1}{k+1} = 0$. We need to show that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that for all $k > N$, $|\frac{1}{k+1} - 0| < \epsilon$. We know that $|\frac{1}{k+1} - 0| = |\frac{1}{k+1}| = \frac{1}{k+1}$. By the Archimedean property, we know that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that $\frac{1}{N} < \epsilon$. For $k > N$, $\frac{1}{k+1} < \frac{1}{N+1} < \frac{1}{N} < \epsilon$, so $\frac{1}{k+1} < \epsilon$. Thus, we have shown that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that for all $k > N$, $|a_k - L| < \epsilon$, where $a_k = \frac{1}{k+1}$ and $L = 0$. This means the sequence $\frac{1}{k+1}$ converges, and the limit is 0. Subsequently, $\lim_{k\to\infty} \frac{k}{k+1} = 1 - 0 = 1$. This means the sequence $\frac{k}{k+1}$ $\boxed{\text{converges}}$ and the limit is $\boxed{1}$.

2.3

We claim that $\lim_{k\to\infty} \frac{k}{k^2+1} = 0$. We need to show that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that for all $k > N$, $|\frac{k}{k^2+1} - 0| < \epsilon$. We know that $|\frac{k}{k^2+1} - 0| = |\frac{k}{k^2+1}| = \frac{k}{k^2+1}$. By the Archimedean property, we know that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that $\frac{1}{N} < \epsilon$. For $k > N$, $\frac{k}{k^2+1} < \frac{N}{N^2+1} < \frac{N}{N^2} = \frac{1}{N} < \epsilon$, so $\frac{k}{k^2+1} < \epsilon$. Thus, we have shown that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that for all $k > N$, $|a_k - L| < \epsilon$, where $a_k = \frac{k}{k^2+1}$ and $L = 0$. This means the sequence $\frac{k}{k^2+1}$ $\boxed{\text{converges}}$, and the limit is $\boxed{0}$.

3

3.1

By definition, $\lim_{k\to\infty} a_k = A$ means that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that for all $k > N$, $|a_k - A| < \epsilon$. We can choose $\frac{\epsilon}{|c|}$ where $c \neq 0$ as an arbitrary ϵ, so $|a_k - A| < \frac{\epsilon}{|c|}$. This is equivalent to $|c| \cdot |a_k - A| < \epsilon$. Furthermore, this means $|c(a_k - A)| < \epsilon$ or $|ca_k - cA| < \epsilon$. If $c = 0$, we have that $\lim_{k\to\infty} ca_k = \lim_{k\to\infty} 0 = 0$ and $cA = 0$, so our claim holds for $c = 0$. In conclusion, we know that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that for all $k > N$, $|ca_k - cA| < \epsilon$, or that $\boxed{\lim\limits_{k\to\infty} ca_k = cA}$.

3.2

By definition, $\lim_{k\to\infty} a_k = A$ and $\lim_{k\to\infty} b_k = B$ mean that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that for all $k > N$, $|a_k - A| < \epsilon$ and $|b_k - B| < \epsilon$. We can choose $\frac{\epsilon}{2}$ as an arbitrary ϵ, so $|a_k - A| < \frac{\epsilon}{2}$ and $|b_k - B| < \frac{\epsilon}{2}$. Adding these inequalities together, we get that $|a_k - A| + |b_k - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. By the triangle inequality, we know that $|a_k - A + b_k - B| \leq |a_k - A| + |b_k - B|$, so $|a_k - A + b_k - B| \leq |a_k - A| + |b_k - B| < \epsilon$ or $|a_k - A + b_k - B| < \epsilon$. Rearranging this, we get $|(a_k + b_k) - (A + B)| < \epsilon$. By definition, this means that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that for all $k > N$, $|(a_k + b_k) - (A + B)| < \epsilon$, or that $\boxed{\lim_{k \to \infty} (a_k + b_k) = A + B}$.

4

We need to show that given that a_k has a lower bound and $a_{k+1} \leq a_k$, the sequence converges. We know that if a sequence has a limit, then it converges. Since the sequence has a lower bound, it must have a greatest lower bound, or infimum. We will use the $\epsilon - N$ definition to check the infimum L of a_k is the limit. By the property of infimum, we know that for any $\epsilon > 0$, there exists a_N in a_k such that $0 \leq a_N - L < \epsilon$. Monotonicity of a_k tells us that for any $n > N$, $a_n \leq a_{n-1} \leq ... \leq a_N$, so $0 \leq a_n - L < \epsilon$ for all $n \geq N$. By the $\epsilon - N$ definition, this shows that $\lim_{k \to \infty} a_k = L$, so the sequence $\boxed{\text{converges}}$.

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Assume for contradiction that $A > B$. From the problem, we are given that $a_k \leq b_k$. By definition, $\lim_{k \to \infty} a_k = A$ and $\lim_{k \to \infty} b_k = B$ mean that for any $\epsilon > 0$, there exists $N \epsilon \mathbb{Z}_+$ such that for all $k > N$, $|a_k - A| < \epsilon$ and $|b_k - B| < \epsilon$. We can choose $\epsilon = \frac{A-B}{2}$. Since $|a_k - A| < \epsilon$, we know that $-\epsilon < a_k - A < \epsilon$, so $A - \epsilon < a_k < A + \epsilon$. This means $A - \frac{A-B}{2} < a_k < A + \frac{A-B}{2}$ or $\frac{A+B}{2} < a_k < \frac{3A-B}{2}$. Similarly, since $|b_k - B| < \epsilon$, we know that $-\epsilon < b_k - B < \epsilon$, so $B - \epsilon < b_k < B + \epsilon$. This means $B - \frac{A-B}{2} < b_k < B + \frac{A-B}{2}$ or $\frac{3B-A}{2} < b_k < \frac{A+B}{2}$. Combining these, we get that $\frac{3B-A}{2} < b_k < \frac{A+B}{2} < a_k < \frac{3A-B}{2}$, so $b_k < a_k$. This contradicts our initial assumption that $a_k \leq b_k$, so $\boxed{A \leq B}$ must hold.