MATH 15910 - Problem Set 5

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1.1

Let $a = \frac{p}{q}$, $b = \frac{r}{s}$, and $c = \frac{x}{y}$ where $p\epsilon\mathbb{Z}$, $q\epsilon\mathbb{Z}$, $r\epsilon\mathbb{Z}$, $s\epsilon\mathbb{Z}$, $x\epsilon\mathbb{Z}$, and $y\epsilon\mathbb{Z}$. By definition, we know that $\frac{p}{q} < \frac{r}{s}$ if and only if $\frac{r}{s} - \frac{p}{q} > 0$. Since $a < b$, we know that $\frac{r}{s} - \frac{p}{q} > 0$ holds. This means $\frac{r}{s} - \frac{p}{q} + 0 > 0$ (identity element for addition). So, $\frac{r}{s} - \frac{p}{q} + (c + (-c)) > 0$ (additive inverse for addition) or $\frac{r}{s} - \frac{p}{q} + (\frac{x}{y} + (-\frac{x}{y})) > 0$. This means $(\frac{r}{s} + \frac{x}{y}) - (\frac{p}{q} + \frac{x}{y}) > 0$ (associativity for addition and distributivity), or $(b + c) - (a + c) > 0$. Thus, by definition, $a + c < b + c$.

1.2

Let $a = \frac{p}{q}$, $b = \frac{r}{s}$, and $c = \frac{x}{y}$ where $p\epsilon\mathbb{Z}$, $q\epsilon\mathbb{Z}$, $r\epsilon\mathbb{Z}$, $s\epsilon\mathbb{Z}$, $x\epsilon\mathbb{Z}$, and $y\epsilon\mathbb{Z}$. Also, assume $c > 0$. By definition, we know that $\frac{p}{q} < \frac{r}{s}$ if and only if $\frac{r}{s} - \frac{p}{q} > 0$. Since $a < b$, we know that $\frac{r}{s} - \frac{p}{q} > 0$ holds. Rational numbers are closed under addition, so $\frac{r}{s} - \frac{p}{q}\epsilon\mathbb{Q}$. Let us denote $\frac{r}{s} - \frac{p}{q}$ as $\frac{m}{n}$. Since $\frac{m}{n} > 0$, we know that $m \cdot n > 0$ (by proposition shown in class) because $\frac{r}{s} - \frac{p}{q} > 0$. Also, since $c > 0$, $x \cdot y > 0$. Thus, $\frac{x}{y} \cdot \frac{m}{n} = \frac{x \cdot m}{y \cdot n} > 0$ because $(x \cdot m) \cdot (y \cdot n) = (x \cdot y) \cdot (m \cdot n) > 0$ (associativity of multiplication). With $\frac{r}{s} - \frac{p}{q} > 0$ and $\frac{x}{y} > 0$, we know that $\frac{x}{y} \cdot (\frac{r}{s} - \frac{p}{q}) > \frac{x}{y} \cdot 0$, or $\frac{x}{y} \cdot (\frac{r}{s} - \frac{p}{q}) > 0$ (by proposition shown in class). This means $\frac{x}{y} \cdot \frac{r}{s} - \frac{x}{y} \cdot \frac{p}{q} > 0$ (distributivity). This means $c \cdot b - c \cdot a > 0$. Rational numbers are closed under multiplication, so $c \cdot b\epsilon\mathbb{Q}$ and $c \cdot a\epsilon\mathbb{Q}$. Also, $c \cdot a = a \cdot c$ and $c \cdot b = b \cdot c$ (commutativity of multiplication). Thus, by definition, $c \cdot b - c \cdot a > 0$ means $a \cdot c < b \cdot c$.

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We will prove this using contradiction. Since $a\epsilon\mathbb{Q}$, we can write $a = \frac{p}{q}$ where $p\epsilon\mathbb{Z}$ and $q\epsilon\mathbb{Z}$ such that p and q are co-prime. This means $a^2 = (\frac{p}{q})^2 = \frac{p^2}{q^2} = 5$. Rewriting this, we get $p^2 = 5q^2$. So, $5 \mid p^2$, or $5 \mid (p \cdot p)$. Since 5 is prime and 5 divides $p \cdot p$, we know $5 \mid p$ (by theorem in problem). Thus, we can write $p = 5n$ where $n \epsilon \mathbb{Z}$. Substituting into $p^2 = 5q^2$, we get $(5n)^2 = 5q^2$, or $25n^2 = 5q^2$. Simplifying, we get $5n^2 = q^2$. Since $n \epsilon \mathbb{Z}$, we know that then $5 \mid q^2$ or $5 \mid q$ (by theorem in problem). So, we have that $5 \mid p$ and $5 \mid q$, but this is a contradiction because we assumed that p and q are co-prime, so they should have no common factors other than 1. We have shown that the proposition is true by contradiction. Thus, there does not exist an $a \epsilon \mathbb{Q}$ such that $a^2 = 5$.

3

We need to show that l is the greatest lower bound for A if and only if for any $\epsilon > 0$, there exists $a \epsilon A$ such that $0 \leq a - l < \epsilon$. We need to demonstrate both directions of the statement.

- First, we need to show that if l is the greatest lower bound for A, then for any $\epsilon > 0$, there exists $a \epsilon A$ such that $0 \leq a - l < \epsilon$. Assume that there exists $\epsilon' > 0$ such that no $a \epsilon A$ permits $0 \leq a - l < \epsilon'$ to be true. This means for any $a \epsilon A$, $a - l \geq \epsilon'$, or that $l + \epsilon' \leq a$ for any $a \epsilon A$. So, $l + \epsilon'$ is a lower bound for A, but we also know that $l + \epsilon' > l$, so l is not the greatest lower bound, proving the contrapositive.

- Next, we need to show that if for any $\epsilon > 0$, there exists $a \epsilon A$ such that $0 \leq a - l < \epsilon$, then l is the greatest lower bound for A. Assume that l is not the greatest lower bound of A. Then, let l' be the greatest lower bound for A, so $l' > l$. This means $l' - l > 0$, so there exists $a \epsilon A$ such that $0 \leq a - l < l' - l$. From $a - l < l' - l$, we get that $l' > a$, which means l' cannot be the lower bound of A, which is a contradiction to what we assumed.

By proving both directions of the statement, we can conclude that l is the greatest lower bound for A if and only if for any $\epsilon > 0$, there exists $a \epsilon A$ such that $0 \leq a - l < \epsilon$.

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We need to show this in two steps: first, $sup(C) \leq sup(A) + sup(B)$ and that $sup(A) + sup(B) \leq sup(C)$, which together would imply that $sup(C) = sup(A) + sup(B)$.

- For the first condition, we know that for all $a \epsilon A$, $a \leq sup(A)$ and for all $b \epsilon B$, $b \leq sup(B)$ by definition of supremum. Adding these inequalities together, we get that $a + b \leq sup(A) + sup(B)$. Since $a + b$ is an arbitrary element in C, we know that $sup(A) + sup(B)$ is an upper bound for C. Thus, we can conclude that $sup(C) \leq sup(A) + sup(B)$.

- For the second condition, let $a \epsilon A$ be an arbitrary element. We know for all $b \epsilon B$ that $a + b \leq sup(A + B)$. This is equivalent to $a \leq sup(A + B) - b$.

Since a is an arbitrary element of A, $sup(A+B)-b$ is an upper bound for A, or that $sup(A) \leq sup(A+B)-b$. Rearranging this, we get that $b \leq sup(A+B)-sup(A)$. Since b is an arbitrary element of B, we know that $sup(A+B)-sup(A)$ is an upper bound for B. This means $sup(B) \leq sup(A+B)-sup(A)$, or that $sup(A)+sup(B) \leq sup(A+B)$, which means $sup(A)+sup(B) \leq sup(C)$.

So, we know that $sup(C) \leq sup(A)+sup(B)$ and $sup(A)+sup(B) \leq sup(C)$, which means $sup(C) = sup(A)+sup(B)$.

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5.1

Assume for contradiction that $a > b$. By definition of order relation, this means $a-b > 0$. We know that since $\frac{1}{2} > 0$, $\frac{1}{2} \cdot (a-b) > \frac{1}{2} \cdot 0$ (multiplication property for order), so $\frac{1}{2} \cdot (a-b) > 0$ (by proposition shown in class), or that $\frac{a-b}{2} > 0$. Since the statement applies to all $\epsilon > 0$, let $\epsilon = \frac{a-b}{2}$. Then, $a \leq b+\epsilon$, so $a \leq b + \frac{a-b}{2}$. This means $2a \leq 2b+a-b$ (multiplication property of order), or that $a \leq b$, but this is a contradiction because we assumed that $a > b$. Thus, if for any $\epsilon > 0$, $a \leq b+\epsilon$, then $a \leq b$.

5.2

We know that any $b \epsilon B$ is an upper bound for A because for all $a \epsilon A$, every $b \epsilon B$ follows that $a \leq b$. If an element $b \epsilon B$ is an upper bound for A, we can say that $sup(A) \leq b$. We know that $sup(A) \leq b$ for all $b \epsilon B$, which means that $sup(A)$ is a lower bound for B. This is equivalent to noting that $sup(A) \leq inf(B)$. Thus, if $a \leq b$ for all $a \epsilon A$ and $b \epsilon B$, then $sup(A) \leq inf(B)$.