# MATH 15910 - Problem Set 2

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October 11, 2023

## 1

### 1.1

We define $f : A \mapsto A$ where $f(x) = x^2$ and $A$ is the set of all integers. To determine if $f$ is injective, we must show that if $a \epsilon A$ and $b \epsilon A$ and $f(a) = f(b)$, then $a = b$. This means $f(a) = f(b)$ and so $a^2 = b^2$. This is not true if $a = 2$ and $b = -2$, for example since $f(2) = 2^2 = 4$ and $f(-2) = (-2)^2 = 4$. Thus, the function is $\boxed{\text{not injective}}$. To determine if $f$ is surjective, we must show that for all $y \epsilon A$, there is an $x$ such that $f(x) = y$. $f(x) = x^2 \geq 0$, so if $y = -1$, there is no such $x \epsilon A$ where $x^2 = -1$. Thus, the function is $\boxed{\text{not surjective}}$. Since the function is not both injective and surjective, it is $\boxed{\text{not bijective}}$.

### 1.2

We define $f : A \mapsto A$ where $f(x) = x^2$ and $A$ is the set of all positive integers. To determine if $f$ is injective, we must show that if $a \epsilon A$ and $b \epsilon A$ and $f(a) = f(b)$, then $a = b$. This means $f(a) = f(b)$ and so $a^2 = b^2$. $A$ is positive integers, so this means $a = b$ since the $\sqrt{a^2} = a$ and $\sqrt{b^2} = b$. Thus, the function is $\boxed{\text{injective}}$. To determine if $f$ is surjective, we must show that for all $y \epsilon A$, there is an $x$ such that $f(x) = y$. If $y = 2$, there is no such $x \epsilon A$ where $x^2 = 2$ because $x = \sqrt{2} \notin A$ as $A$ is the set of all positive integers. Thus, the function is $\boxed{\text{not surjective}}$. Since the function is not both injective and surjective, it is $\boxed{\text{not bijective}}$.

### 1.3

We define $f : A \mapsto B$ where $f(x) = 2x$ and $A$ is the set of all odd integers and $B$ is the set of all even integers. To determine if $f$ is injective, we must show that if $a \epsilon A$ and $b \epsilon A$ and $f(a) = f(b)$, then $a = b$. This means $f(a) = f(b)$ and so $2a = 2b$ and $a = b$. Thus, the function is $\boxed{\text{injective}}$. To determine if $f$ is surjective, we must show that for all $y \epsilon A$, there is an $x$ such that $f(x) = y$. $A$ consists of odd numbers only, so if $y$ =8, then $x = 4$ but $4 \notin A$. Thus, the function is $\boxed{\text{not surjective}}$. Since the function is not both injective and surjective, it is $\boxed{\text{not bijective}}$.

# 2

## 2.1

We must show that if $f : A \mapsto B$ is a bijection, there exists a function $g : B \mapsto A$ that is also a bijection. To do so, we can show $f : A \mapsto B$ has an inverse $f^{-1}$ and that $f^{-1}$ is bijective.

- $f$ has an inverse if and only if for all $a \epsilon A$, $f^{-1}(f(a)) = a$ and for all $b \epsilon B$, $f(f^{-1}(b)) = b$. For any $y \epsilon B$, we know that $f^{-1}(y)$ exists (because $f$ is surjective) and is unique (because $f$ is injective), so we can let $f^{-1}(y) = x$. Thus, $f^{-1}(f(x)) = f^{-1}(y) = x$ and $f(f^{-1}(y)) = f(x) = y$, so $f : A \mapsto B$ has an inverse $f^{-1} : B \mapsto A$. We can denote $g = f^{-1}$ where $g : B \mapsto A$.

- $g$ is bijective if and only if it is injective and surjective. For some $y_1 \epsilon B$ and $y_2 \epsilon B$, if $g(y_1) = g(y_2)$, we can let an $x \epsilon A$ such that $g(y_1) = x$ and $g(y_2) = x$, meaning $f(x) = y_1$ and $f(x) = y_2$ (since $f$ and $g$ are inverses of each other). Because $f$ is a function, a given $x$ cannot map to multiple $y$ values, so $y_1 = y_2$. Thus, we have shown that if $g(y_1) = g(y_2)$, then $y_1 = y_2$, so $g$ must be injective. For any $a \epsilon A$, we must show that there exists $b \epsilon B$ such that $g(b) = a$. Because $f$ is a function, all $a \epsilon A$ maps to some $b \epsilon B$, so there exists a $b \epsilon B$ such that $g(b) = a$. Thus, $g$, or $f^{-1}$ is also a bijective function because it is both injective and surjective.

From above, we know that if $f$ is a bijective function, and we let $g = f^{-1}$, $g$ is $\boxed{\text{bijective}}$. Thus, there exists a bijective function $g : B \mapsto A$. Similarly, we have shown that for all $a \epsilon A$, $f^{-1}(f(a)) = a$ and for all $b \epsilon B$, $f(f^{-1}(b)) = b$ (from determining that the bijective function $f$ has an inverse). This is equivalent to noting that for all $a \epsilon A$, $g(f(a)) = a$ and for all $b \epsilon B$, $f(g(b)) = b$, which proves that both $f \circ g : B \mapsto B$ and $g \circ f : A \mapsto A$ are $\boxed{\text{identity maps}}$.

## 2.2

We must show that the composition of bijections is also a bijection, meaning is is injective and surjective. Let us assume for some $x_1 \epsilon A$ and $x_2 \epsilon B$, $(g \circ f)(x_1) = (g \circ f)(x_2)$. We must show that then $x_1 = x_2$. We can rewrite the equality as $g(f(x_1)) = g(f(x_2))$. Since $g$ is injective, we know that $f(x_1) = f(x_2)$, and since $f$ is injective, we know that $x_1 = x_2$. Thus, we have shown that $g \circ f$ is an injective function since if $g(f(x_1)) = g(f(x_2))$, then $x_1 = x_2$. Now, we must show that $g \circ f$ is surjective. For any $c \epsilon C$, there must be some $a \epsilon A$ such that $(g \circ f)(a) = c$, or $g(f(a)) = c$. We know that $g$ is surjective, so there exists some $b \epsilon B$ such that $g(b) = c$. Similarly, since $f$ is surjective, there exists some $a \epsilon A$ such that $f(a) = b$. Thus, we have shown that $g \circ f$ is a surjective function since there exists some $a \epsilon A$ such that $g(f(a)) = c$. Because $g \circ f$ is injective and surjective, it is also $\boxed{\text{bijective}}$.

## 2.3

For $R$ to be an equivalence relation, it must satisfy the reflexive, symmetric, and transitive properties.

- $\boxed{\text{Reflexive}}$: For any $x \epsilon X$, $(x, x) \epsilon R$ because there must exist a bijection between a set and itself. A set and itself have the same number of elements, so it is possible to produce a bijection by mapping each element of $x$ to itself, such as through the identity map $f : x \mapsto x$. For $a_1 \epsilon x$ and $a_2 \epsilon x$, where $f(a_1) = f(a_2)$, we know that $f(a_1) = a_1$ and $f(a_2) = a_2$, so $a_1 = a_2$, so $f$ must be injective. Similarly, for any $b \epsilon x$, there exists a $a \epsilon x$ such that $f(a) = b$, where $a = b$, so $f$ must be surjective. Since the identity map is injective and surjective, it must be bijective and thus, $(x, x) \epsilon R$.

- $\boxed{\text{Symmetric}}$: For $x \epsilon X$ and $y \epsilon X$, where $(x, y) \epsilon R$, then $(y, x) \epsilon R$. If there is a bijection $f : x \mapsto y$, we can produce a bijection $g : y \mapsto x$ where $g = f^{-1}$. In question 2.1, we have shown that if $f$ is bijective, there is a function $g$ where $g = f^{-1}$ and that $g$ is also bijective, so $(y, x) \epsilon R$.

- $\boxed{\text{Transitive}}$: For $x \epsilon X$, $y \epsilon Y$, and $z \epsilon Z$, where $(x, y) \epsilon R$ and $(y, z) \epsilon R$, then $(x, z) \epsilon R$. If there a bijection $f : x \mapsto y$ and $g : y \mapsto z$, then we must show that $g \circ f : x \mapsto z$ is also a bijection. In question 2.2, we have shown that the composition of bijective functions is also a bijection, so $(x, z) \epsilon R$.

# 3

The cardinality of $S$ is the number of subsets formed from a set $A$ where $\#A = n$. That means, $\#S$ is the sum of the number of subsets formed of size 0 to $n$. This is equivalent to $\sum_{k=0}^{n} \binom{n}{k}$, where $k$ is the number of elements in a particular subset and $\binom{n}{k}$ is the number of ways we can pick $k$ elements from $n$ elements. The binomial theorem states $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. To match the latter with the total number of subsets $\sum_{k=0}^{n} \binom{n}{k}$, $a^{n-k} b^k$ must always evaluate to 1, and this works if $a = 1$ and $b = 1$. Plugging in these into $(a + b)^n$, we get that the $(1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k}$, so $\boxed{\#S = 2^n}$.

# 4

The pigeonhole principle states that if set $\#A = m$ and $\#B = n$, where $A$ is the set of 29 candies and $B$ is the set of 4 people, and $f : A \mapsto B$ is a function, then for some $b \epsilon B$, or some person, $\#(f^{-1}(b)) \geq \lceil \frac{m}{n} \rceil$. Specifically, for $m$ candies and $n$ people, there exists a person who will be distributed at least $\lceil \frac{m}{n} \rceil$ candies. In this case, $m = 29$ and $n = 4$, so there exists a person who will get at least $\lceil \frac{29}{4} \rceil = \lceil 7.25 \rceil = \boxed{\text{8 candies}}$.