MATH 15910 - Bonus Problems

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1.1

To show $\mathbb{Q}$ was countable, we used Cantor's diagonal path. Each element in $\mathbb{Q}$ was written in the form of $\frac{a}{b}$, where a denoted the row number and b denoted the column number. We also know that both sets A and B in the Cartesian product are countable, so we can label the row numbers as $a_1, a_2, ...$ and column numbers as $b_1, b_2, ...$ We can apply the same argument to the Cartesian product of two sets, where we replace $\frac{a}{b}$ with (a, b) and follow the same Cantor's diagonal path, where each element (a, b) is some (a_i, b_j) for the row and column number, which correspond to elements of A and B. Thus, the Cartesian products of two countable sets is also countable.

1.2

First, consider the set of degree n polynomials. From 1.1, we know that the product of two Cartesian sets is countable. The set of degree n polynomials can be represented as the Cartesian product between $\mathbb{Q}$ and $\mathbb{Q}$, the Cartesian product of that result with $\mathbb{Q}$, and so we get $(\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q}$, until we have taken the Cartesian product of n $\mathbb{Q}$ sets. Denote the set of all degree n polynomial as X_n. We know that this set is also countable. Now, we enumerate all the degree n polynmials as $Y = \bigcup\limits_{n=0}^{\infty} X_n$. We know that X_n is countable from above, and the union of countable sets is also countable. Thus, Y is countable. This means the set of polynomials with rational coefficients is countable.

1.3

From 1.2, we know that the set of all rational polynomials is countable. Let X_n denote the set of all rational polynomials of degree n. Each element in X_n can have at most n zeros. We can write $X_n = \{x_1, x_2, ...\}$ wher each x_i is a rational polynomial. Each x_i has at most n zeros, we can list the zeros of each x_i as $\{x_{11}, x_{12}, ...\}$. This is a finite set, so we take the union of the sets of x_i's zeros, to get Y_n, the set of all zeros for a polynomial of degree n. Like before, we can let $Z = Y = \bigcup_{n=0}^{\infty} Y_n$, where Z is the set of zeros for all rational polynomials. Since Y_n is countable, the countable union of countable sets Z is also countable.

2

2.1

Let $(a_n) = \frac{1}{n}$ and $(b_n) = (-1)^n$. We know that $A = \lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty}(-1)^n \frac{1}{n} = 0$ (class notes). However, $\lim_{n\to\infty}(-1)^n$ does not exist because 1 and -1 are both accumulation points of the sequence. This proves the statement false.

2.2

We know that $\lim_{n\to\infty}(a_n) = A$. Let $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} c_n = C$, which we know exists. The sequence (c_n) has a limit, so it is bounded. Therefore, (c_n) is bounded, so there exists an M such that $|c_n| \leq M$ for all n. We can define $b_n = \frac{c_n}{a_n}$ as a_n approaches A. We know that a_n does not approach 0 since $A \neq 0$, which means b_n is well-defined. We let $|b_n| = \frac{|c_n|}{|a_n|} \leq \frac{M}{|a_n|}$. We also know that $\frac{M}{|a_n|}$ is bounded as a_n approaches A. Thus, $|b_n| \leq \frac{|c_n|}{|a_n|} \leq \frac{M}{|A|}$, or that $|b_n| \leq \frac{M}{|A|}$. This means $|b_n|$ is bounded from above, so $|b_n|$ must have a supremum, which is the limit. By definition, we know a sequence if Cauchy if $\forall \epsilon > 0$, there exists an $N \epsilon \mathbb{Z}_+$ such that for all $k, m > N$, $|b_k - b_m| < \epsilon$. We know that $|b_k - b_m| \leq |b_k| + |-b_m| = |b_k| + |b_m|$. We know that $|b_n| \leq \frac{M}{|A|}$, so the sequence is Cauchy. Since $|b_n|$ is a Cauchy sequence, it converges, meaning the limit exists. This proves the statement true.

3

3.1

We are given that $a > 0$, $b > 0$, and $a > b$.

- $a > b$
  - $\implies a^2 > ab$ (multiplication property for order)
    - $\implies a^3 > a^2 b$ (multiplication property for order)
    - $\implies a^2 b > ab^2$ (multiplication property for order)
  - $\implies ab > b^2$ (multiplication property for order)
    - $\implies a^2 b > ab^2$ (multiplication property for order)
    - $\implies ab^2 > b^3$ (multiplication property for order)

Since $a > b \implies a^3 > a^2 b > ab^2 > b^3$, we know that $a > b \implies a^3 > b^3$.

3.2

For the base case $n = 1$, we have $a^1 > b^1$ or $a > b$, which is given. Thus, the base case holds. For the induction hypothesis, assume that $a^k > b^k$ for all $k \epsilon \mathbb{Z}_+$. We know that $a > b$ means $\frac{a}{b} > 1$. Similarly, $a^k > b^k$ means $\frac{a^k}{b^k} > 1$ or $\frac{b^k}{a^k} < 1$. From here, we see that $\frac{b^k}{a^k} < 1 < \frac{a}{b}$, so $\frac{b^k}{a^k} < \frac{a}{b}$. This means $b(b^k) < a(a^k)$ (class notes) or $b^{k+1} < a^{k+1}$. So, we have shown that $a^k > b^k$ implies $a^{k+1} > b^{k+1}$. Thus, $a^n > b^n$ holds for all $n \epsilon \mathbb{Z}_+$ where $a > 0$ and $b > 0$.

4

Assume for contradiction that the set of sequences (a_n), denoted X, where each element a_n is 0 or 1 is countable. Then, we can enumerate the set as the following:

- $X_1 = (a_{11}, a_{12}, a_{13}, ...)$
- $X_2 = (a_{21}, a_{22}, a_{23}, ...)$
- $X_3 = (a_{31}, a_{32}, a_{33}, ...)$
- ...

We can construct a sequence where the ith element of the sequence is $1 - a_{ii}$. The first element of this sequence is different from the first element of X_1, so the sequence cannot be X_1. The second element of this sequence is different from the second element of X_2, so the sequence cannot be X_2. Continuing this, we are able to construct a sequence not in Cantor's diagonal path of the set of sequences where each element is a 0 or 1. Thus, we have shown that this set is uncountable.

5

If a set A is finite, we can denote its cardinality as $\#n$, where $A = \{x_1, ..., x_n\}$.

We claim that $sup(A) = max(A) = x_m$ for some $1 \leq m \leq n$. Assume for contradiction that $sup(A) = L_0 < max(A)$. This means $L_0 < x_m$, which is a contradiction since the L_0 must be an upper bound for elements in A. Thus, $sup(A) = max(A)$, so a finite set of real numbers contains a supremum.

We claim that $inf(A) = min(A) = x_m$ for some $1 \leq m \leq n$. Assume for contradiction that $inf(A) = L_0 > min(A)$. This means $L_0 > x_m$, which is a contradiction since the L_0 must be a lower bound for elements in A. Thus, $inf(A) = min(A)$, so a finite set of real numbers contains a infimum.

6

Assume for contradiction that $xy > \frac{x^2+y^2}{2}$. Then, $2xy > x^2 + y^2$, so $x^2 + y^2 - 2xy < 0$. This means that $(x-y)^2 > 0$, which is a contradiction because $(x-y)^2 \geq 0$. Thus, $xy \leq \frac{x^2+y^2}{2}$ must hold.

7

If $a\epsilon\mathbb{Q}$, then $a^2\epsilon\mathbb{Q}$ since $a = \frac{b}{c}$ and $a^2 = \frac{b^2}{c^2}$. Since the contrapositive is true, we know that if $a^2 \notin \mathbb{Q}$, then $a \notin \mathbb{Q}$. Let $a = \sqrt{2} + \sqrt{3}$. Then $a^2 = 5 + 2\sqrt{6}$. Assume for contradiction that $5 + 2\sqrt{6}\epsilon\mathbb{Q}$. Setting $5 + 2\sqrt{6} = \frac{x}{y}$ where $x, y\epsilon\mathbb{Z}$, we know that $\sqrt{6} = \frac{a}{5b} - \frac{2}{5}\epsilon\mathbb{Q}$. So, we let $\sqrt{6} = \frac{p}{q}$, where $p, q\epsilon\mathbb{Z}$ and p and q are coprime. This means $6 = \frac{p^2}{q^2}$, so $p^2 = 6q^2 = 2(3q^2)$, and $2 \mid p^2$. From here, we get that $2 \mid p$ since 2 is prime (class notes). Writing $p = 2m$ for $m\epsilon\mathbb{Z}$, we get $(2m)^2 = 6q^2$ or $2m^2 = 3q^2$. $3q^2$ is even, so q^2 is even and $2 \mid q^2$. Again, this means $2 \mid q$ since 2 is prime. Since $2 \mid p$ and $2 \mid q$, we get a contradiction since we assumed p and q were coprime.

8

- $\frac{a}{b} < \frac{c}{d} \implies ad < bc$ (class notes)
  - $ab + ad < ab + bc \iff a(b+d) < b(a+c) \iff \frac{a}{b} < \frac{a+c}{b+d}$
  - $ad + cd < bc + cd \iff d(a+c) < c(b+d) \iff \frac{a+c}{b+d} < \frac{c}{d}$

Since $\frac{a}{b} < \frac{c}{d}$ implies that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$, we have proven the stronger condition of $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$.

9

We can write $0.73\overline{4} = 0.73 + 0.00\overline{4} = 0.73 + 4(0.00\overline{1}) = \frac{73}{100} + \frac{4}{100}(0.\overline{1})$. We know the following:

- $0.\overline{1} = \{\sum_{k=1}^{\infty}(\frac{1}{10})^k\}$
- $= sup\{0.1, 0.1 + 0.01, 0.1 + 0.01 + 0.001, ...\}$
- $= sup\{\sum_{k=1}^{n}(\frac{1}{10})^k\}$
- $= sup\{\sum_{k=0}^{n}(\frac{1}{10})^k - \frac{1}{10^0}\}$
- $= sup\{\sum_{k=0}^{n}(\frac{1}{10})^k\} - 1$
- $= \frac{1}{1-\frac{1}{10}} - 1$
- $= \frac{1}{9}$

So, $0.\overline{1} = \frac{1}{9}$. This means $0.73\overline{4} = \frac{73}{100} + \frac{4}{100} \cdot \frac{1}{9} = \frac{661}{900}$.

10

Assume for contradiction that $A \subset \mathbb{R}$ has two suprema, L_1 and L_2 such that $L_1 < L_2$. By definition of the supremum, for all $a \epsilon A$, $a \leq L_1$ and subsequently, $a \leq L_1 < L_2$. L_2 is an upper bound for A but not the least upper bound since L_1 is also an upper bound but $L_1 < L_2$. This means that if a set contains multiple suprema, all the supremuma greater than the minimum suprema cannot be a supremum since they are not the least upper bound.

11

For each positive integer n, we can define the sequence (a_k) to be one that cycles through the elements $1, ..., n$. This means there are n accumulation points. The following examples demonstrate this:

- $n = 1 : (a_k) = 1, ...$
- $n = 2 : (a_k) = 1, 2, 1, 2, ...$
- $n = 3 : (a_k) = 1, 2, 3, 1, 2, 3, ...$

12

The set of $\mathbb{Q}$ is countable, so we can use its elements to define a sequence. Let us define (a_k) where the kth element of the sequence is the kth term in the Cantor diagonal path of $\mathbb{Q}$. In problem set 8, we showed that between any two real numbers $a < b$, there exists a rational number such that $a < r < b$. If we let $b = a + \epsilon$ where $\epsilon > 0$, then for any $\epsilon > 0$ there exists a rational number r such that $a < r < a + \epsilon$. This means that there are infinitely many points in the interval $(a - \epsilon, a + \epsilon)$ that belong to (a_k), so every real number is an accumulation point for this sequence.

13

If the set X is countably infinite, then we can enumerate its elements as $X = \{x_1, x_2, ...\}$. In this case, we can let X_0 be all elements in X except the first element, so $X_0 = \{x_2, x_3, ...\}$. The elements of X_0 can also be enumerated, so the infinite set X_0 is also countable.

14

Since f is a function, we know that all $f^{-1}(\{y\})$ for $y \epsilon Y$ are disjoint. Otherwise, there would be an $x \epsilon X$ that maps to two different $y \epsilon Y$. Since Y is countable, we can write the elements of Y as $\{y_1, y_2, ...\}$. Since f is surjective, we know that $X = \bigcup\limits_{i=1} f^{-1}(\{y_i\})$. We are given that $f^{-1}(\{y_i\})$ is finite, so it is countable. Thus, X is the union of countable sets, so X is also countable.

15

Let Y be the set of isolated points for some X. We can define a map f that takes an isolated point in Y and maps it to its neighborhood. Thus, $f(y) = (y-\epsilon, y+\epsilon)$ for every $y \epsilon Y$. We also know that $\mathbb{Q}$ is dense in $\mathbb{R}$, which means the Cartesian product of $\mathbb{Q}^n$ is also dense in R since $\mathbb{Q}^n$ is countable as shown in 1.2, meaning it has the same properties as $\mathbb{Q}$. First, we show that f is injective. If $f(y_1) = f(y_2)$, then $y_1 = y_2$ since each interval can contain only 1 isolated point. Second, we have all the set of intervals, where we can make the endpoints rational (since $\mathbb{Q}$ is dense in $\mathbb{R}$), and so the intervals are all countable because it is a Cartesian product of all $\mathbb{Q}$. This means that f is an injection from isolated points to a countable set of intervals, so the set of isolated points must also be countable.