MATH 15910 - Problem Set 4

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1

We know that if $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$, then $ab' = a'b$ and $cd' = c'd$ (by definition of fractions). We need to show that $\frac{a'c'}{b'd'} = \frac{ac}{bd}$. Since $ab' = a'b$, then $ab'c'd = a'bc'd$. Similarly, since $cd' = c'd$, then $ab'c'd = ab'cd'$. Setting these two equal to each other, we get that $a'bc'd = ab'cd'$, so $\frac{a'bc'd}{bb'dd'} = \frac{ab'cd'}{bb'dd'}$. This means that $\frac{a'c'}{b'd'} = \frac{ac}{bd}$, or that $\frac{a'}{b'} \cdot \frac{c'}{d'} = \frac{a}{b} \cdot \frac{c}{d}$ (by definition of multiplication of the rational numbers). Thus, we have shown that multiplication of the rational numbers is well-defined.

2

We need to show that $(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f} = \frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f})$. We know that $(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f} = (\frac{a \cdot c}{b \cdot d}) \cdot \frac{e}{f}$ (by definition of multiplication of the rational numbers). Furthermore, we know that $(\frac{a \cdot c}{b \cdot d}) \cdot \frac{e}{f} = \frac{(a \cdot c) \cdot e}{(b \cdot d) \cdot f}$ (by definition of multiplication of the rational numbers). Since the numerator and denominator of rational numbers are integers, we can apply previous axioms of integers to the numerator and denominator separately. This means $\frac{(a \cdot c) \cdot e}{(b \cdot d) \cdot f} = \frac{a \cdot (c \cdot e)}{b \cdot (d \cdot f)}$ (by associativity of multiplication of integers). From here, we get that $\frac{a \cdot (c \cdot e)}{b \cdot (d \cdot f)} = \frac{a}{b} \cdot (\frac{c \cdot e}{d \cdot f})$ (by definition of multiplication of the rational numbers). Similarly, we get that $\frac{a}{b} \cdot (\frac{c \cdot e}{d \cdot f}) = \frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f})$ (by definition of multiplication of the rational numbers). Therefore, we have shown associativity of multiplication of the rational numbers because $(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f} = \frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f})$.

3

We need to show that $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$. We know that $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ (by definition of multiplication of the rational numbers). Since the numerator and denominator of rational numbers are integers, we can apply previous axioms of integers to the numerator and denominator separately. So, we know that $\frac{a \cdot c}{b \cdot d} = \frac{c \cdot a}{d \cdot b}$ (by commutativity of multiplication of integers). Furthermore, we know that $\frac{c \cdot a}{d \cdot b} = \frac{c}{d} \cdot \frac{a}{b}$ (by definition of multiplication of the rational numbers). Therefore, we have shown commutativity of multiplication of the rational numbers because $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$.

4

We need to show the existence of an inverse element for multiplication of the rational numbers, or that for any $a \epsilon \mathbb{Q}$ where $a \neq 0$, there exists $a^{-1} \epsilon \mathbb{Q}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. However, we first need to show that for $x \epsilon \mathbb{Z}$ that $\frac{x}{x} = 1$. We know that $x = x$, so $x \cdot 1 = x \cdot 1$ (by identity element of multiplication of integers). This is true if and only if $\frac{x}{x} = \frac{1}{1}$, since that is equivalent to the cross-multiplication equality of $x \cdot 1 = x \cdot 1$ (by definition of fractions). Furthermore, we know that for $y \epsilon \mathbb{Z}$, there exists $\frac{y}{1} \epsilon \mathbb{Q}$ such that $y = \frac{y}{1}$ (by proposition from class). Thus, this means that $1 = \frac{1}{1}$, so $\frac{x}{x} = \frac{1}{1} = 1$.

Since $a \epsilon \mathbb{Q}$, we know that $a = \frac{b}{c}$ where $b \epsilon \mathbb{Z}$ and $c \epsilon \mathbb{Z}$. Since $a \neq 0$, we know that $b \neq 0$ and since a is defined, $c \neq 0$. This means that $\frac{c}{b} \neq 0$. Since the numerator and denominator of rational numbers are integers, we can apply previous axioms of integers to the numerator and denominator separately. If we let $a^{-1} = \frac{c}{b}$, we know that $a \cdot a^{-1} = \frac{b}{c} \cdot \frac{c}{b}$. This is equal to $\frac{b \cdot c}{c \cdot b}$ (by definition of multiplication of the rational numbers). This is the same as $\frac{b \cdot c}{b \cdot c}$ (by commutativity of multiplication of integers). So, $a \cdot a^{-1} = \frac{b \cdot c}{b \cdot c} = 1$ (by proof above where $x = b \cdot c$ since integers are closed under multiplication). Similarly, $a^{-1} \cdot a = \frac{c}{b} \cdot \frac{b}{c}$. This is equal to $\frac{c \cdot b}{b \cdot c}$ (by definition of multiplication of the rational numbers). This is the same as $\frac{c \cdot b}{c \cdot b}$ (by commutativity of multiplication of integers). So, $a^{-1} \cdot a = \frac{c \cdot b}{c \cdot b} = 1$ (by proof above, where $x = c \cdot b$ since integers are closed under multiplication). Thus, we have shown that for any $a \epsilon \mathbb{Q}$, there exists a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.