MATH 15910 - Problem Set 6

Sohini Banerjee

November 8, 2023

1

If $sup(A) = L$, then by definition, we know that for all $a \epsilon A$, $a \leq L$. From here, we know that:

- $a \leq L$
- $(-L) + a + (-a) \leq (-L) + L + (-a)$ (addition property for order)
- $-L + (a + (-a)) \leq ((-L) + L) + (-a)$ (associativity for addition)
- $-L + 0 \leq 0 + (-a)$ (additive inverse for addition)
- $-L \leq -a$ (identity element for addition)

$-L \leq -a$ means $-L$ is a lower bound for $-A$, by definition. We need to show that $-L$ is the greatest lower bound for $-A$. Assume for contradiction that there exists some L' such that $L' > -L$ and L' is a lower bound for $-A$. This means for all $-a \epsilon A$, $-a \geq L'$. From here, we know that:

- $-a \geq L'$
- $(-L') + (-a) + a \geq (-L') + L' + a$ (addition property for order)
- $-L' + ((-a) + a) \geq ((-L') + L') + a$ (associativity for addition)
- $-L' + 0 \geq 0 + a$ (additive inverse for addition)
- $-L' \geq a$ (identity element for addition)

$-L' \geq a$ means that $-L'$ is an upper bound for A, by definition. However, we know that $L' > -L$. From here, we know that:

- $L' > -L$
- $(-L') + L' + L > (-L') + (-L) + L$ (addition property for order)
- $((-L') + L') + L > -L' + ((-L) + L)$ (associativity for addition)
- $0 + L > -L' + 0$ (additive inverse for addition)

- $L > -L'$ (identity element for addition)

$L > -L'$ contradicts L being the least upper bound for A since $-L'$ is also an upper bound for A, but $-L' < L$. So, there cannot exists an $L' > -L$ such that L' is the greatest lower bound for $-A$. This means that $-L$ must be the greatest lower bound for $-A$. Thus, we have shown that if $sup(A) = L$, then $inf(-A) = -L$.

2

We know that the infinite decimal $0.3333... = sup\{0.3, 0.33, 0.333, ...\}$. This is equivalent to $sup\{\sum_{k=1}^{\infty} \frac{3}{10^k}\} = sup\{3 \cdot \sum_{k=1}^{\infty}(\frac{1}{10})^k\} = 3 \cdot sup\{\sum_{k=1}^{\infty}(\frac{1}{10})^k\}$. To calculate the infinite sum, we need to find the understand the partial sum. Using the definition of geometric series, we know the partial sum $S_n = \frac{a_1 \cdot (1-x^n)}{1-x}$, which is the sum of the first n terms where $n \epsilon \mathbb{Z}_+$. Plugging in $a_1 = \frac{1}{10}$ and $x = \frac{1}{10}$, we get that $S_n = \frac{\frac{1}{10} \cdot (1-(\frac{1}{10})^n)}{1-\frac{1}{10}}$. This simplifies to $S_n = \frac{1}{9} \cdot (1 - \frac{1}{10^n})$. We need to find the supremum of the set of partial sums S_n, or the least upper bound of this set. We know that the supremum is achieved through increasing n because a larger n makes 10^n larger, so $\frac{1}{10^n}$ becomes smaller, so S_n is an increasing function for n. As n increases, the $\frac{1}{10^n}$ term approaches 0, so S_n approaches $\frac{1}{9} \cdot (1 - 0)$. This means that $sup\{\frac{1}{9} \cdot (1 - \frac{1}{10^n})\} = \frac{1}{9}$, or that $\sum_{k=1}^{\infty}(\frac{1}{10})^k = \frac{1}{9}$. Thus, $3 \cdot sup\{\sum_{k=1}^{\infty}(\frac{1}{10})^k\} = 3 \cdot \frac{1}{9} = \frac{1}{3}$. So, the infinite decimal $0.3333... = \frac{1}{3}$.

3

3.1

We must find $sup\{S_n\}$ where $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$ or $S_n = \sum_{k=1}^{n}(\frac{1}{k} - \frac{1}{k+1})$. If we enumerate S_n for some arbitrary n, we get that $S_n = (\frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}) - (\frac{1}{2} + ... + \frac{1}{n} + \frac{1}{n+1})$. It is evident that after the intermediate terms cancel out, we get $S_n = 1 - \frac{1}{n+1}$. We need to find the supremum of the set of partial sums S_n, or the least upper bound of this set. With increasing n, the $\frac{1}{n+1}$ term becomes smaller, so S_n becomes larger. This means the supremum is achieved through increasing n. As n increases, the $\frac{1}{n+1}$ term approaches 0, so S_n approaches $1 - 0 = 1$. This means that $sup\{S_n\} = 1$. Thus, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1$.

3.2

We must find $sup\{S_n(x)\}$ where $S_n(x) = \sum_{k=1}^{n} \frac{1}{2^k}$. Using the definition of geometric series, we know the partial sum $S_n = \frac{a_1 \cdot (1-x^n)}{1-x}$, which is the sum of the first n terms where $n \epsilon \mathbb{Z}_+$. Plugging in $a_1 = \frac{1}{2}$ and $x = \frac{1}{2}$, we get that $S_n = \frac{\frac{1}{2} \cdot (1-(\frac{1}{2})^n)}{1-\frac{1}{2}}$. This simplifies to $S_n = 1 - \frac{1}{2^n}$. We need to find the supremum of the set of partial sums S_n, or the least upper bound of this set. The supremum is achieved through increasing n because a larger n makes $\frac{1}{2^n}$ smaller and thus, S_n gets larger. As n increases, the $\frac{1}{2^n}$ term approaches 0, so S_n approaches $1 - 0 = 1$. This means that $sup\{S_n\} = 1$. Thus, $\sum_{k=1}^{n} \frac{1}{2^k} = 1$.

4

For any $a \epsilon A$, $a < 3$ because if $a \geq 3$, then $a^2 \geq 9 > 3$. This means 3 is an upper bound for A. So, by definition of real numbers, A has a least upper bound L, so $L = sup(A)$. We know that $L \geq 1$ because $1 \epsilon A$. We need to show that $L^2 = 3$. We can prove this by contradiction since by trichotomy, if $L^2 \neq 3$, then $L^2 < 3$ or $L^2 > 3$.

- Consider $L^2 > 3$. We know that 3 is an upper bound for A, so $L \leq 3$ because L is the least upper bound for A. From class, we know that for $N \geq 1$, $N \leq N^2$, so $\frac{1}{N^2} \leq \frac{1}{N}$. For any integer $N \geq 1$, we know that $(L - \frac{1}{N})^2 = L^2 - \frac{2L}{N} + \frac{1}{N^2} > L^2 - \frac{2L}{N} \geq L^2 - \frac{2(3)}{N} = L^2 - \frac{6}{N}$ since $L \leq 3$. By the Archimedean property, there exists $N \epsilon \mathbb{Z}_+$ such that $N(L^2 - 3) > 6$, so $L^2 - \frac{6}{N} > 3$. Since $(L - \frac{1}{N})^2 > L^2 - \frac{6}{N}$ and $L^2 - \frac{6}{N} > 3$, then $(L - \frac{1}{N})^2 > 3 > a^2$ for all $a \epsilon A$. Since $a^2 < (L - \frac{1}{N})^2$, then $a < L - \frac{1}{N}$, so $L - \frac{1}{N}$ is an upper bound for A. However, $L - \frac{1}{N} < L$, which contradicts L being the least upper bound for A.

- Consider $L^2 < 3$. We know that 3 is an upper bound for A, so $L \leq 3$ because L is the least upper bound for A. From class, we know that for $N \geq 1$, $N \leq N^2$, so $\frac{1}{N^2} \leq \frac{1}{N}$. For any integer $N \geq 1$, we know that $(L + \frac{1}{N})^2 = L^2 + \frac{2L}{N} + \frac{1}{N^2} \leq L^2 + \frac{2L}{N} + \frac{1}{N} \leq L^2 + \frac{2(3)}{N} + \frac{1}{N} = L^2 + \frac{7}{N}$ since $L \leq 3$. By the Archimedean property, there exists $N \epsilon \mathbb{Z}_+$ such that $N(3 - L^2) > 7$, so $L^2 + \frac{7}{N} < 3$. Since $(L + \frac{1}{N})^2 \leq L^2 + \frac{7}{N}$ and $L^2 + \frac{7}{N} < 3$, then $(L + \frac{1}{N})^2 < 3$, so $L + \frac{1}{N} \epsilon A$. However, $L + \frac{1}{N} > L$, so L is not an upper bound for A, which contradicts L being the least upper bound for A.

We found that there is a contradiction when $L^2 < 3$ and $L^2 > 3$, so by trichotomy, $L^2 = 3$. Since $L = sup(A)$, we know that $(sup(A))^2 = 3$.