MATH210: Homework 6 (due Apr. 11)

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1 Section 53 #3

Let $z_0 = 2+i, R < 1$, then C_0 is interior to C and we can apply the corollary in section 53.

$$\int_C (z-2-i)^{n-1} dz = \int_{C_0} (z-z_0)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots \\ 2\pi i & \text{when } n = 0 \end{cases}$$

2 Section 53 #5

Let $C_a = C_1 - C_3$, then C_a is a simple closed contour since C_1 and C_3 are smooth arcs. Applying Cauchy-Goursat theorem, we get

$$\int_{C_a} f(z)dz = \int_{C_1 - C_3} f(z)dz = 0$$

so

$$\int_{C_1} f(z)dz = \int_{C_3} f(z)dz$$

Let $C_b = C_2 + C_3$, then C_b is also a simple closed contour since C_2 is a smooth arc. Again from Cauchy-Goursat theorem,

$$\int_{C_b} f(z)dz = \int_{C_2 + C_3} f(z)dz = 0$$

so

$$\int_{C_2} f(z)dz = \int_{-C_3} f(z)dz = -\int_{C_3} f(z)dz$$

then we can write

$$\int_{C_1} f(z)dz = -\int_{C_2} f(z)dz$$

and

$$\int_C f(z)dz = \int_{C_1 + C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 0$$

3 Section 57 #4

If z is outside of C, then $s - z \neq 0$ and the integrand is analytic at all points interior to and on C, so we can apply Cauchy-Goursat theorem.

$$\int_C \frac{s^3 + 2s}{(s-z)^3} ds = 0$$

Suppose that z is inside C. Let $f(z) = z^3 + 2z$, and using Cauchy integral formula we get

$$f^{(2)}(z) = \frac{2!}{2\pi i} \int_C \frac{f(s)ds}{(s-z)^3}$$

so

$$6\pi iz = \int_C \frac{f(s)ds}{(s-z)^3} = \int_C \frac{s^3 + 2s}{(s-z)^3} ds$$

and we get the desired result.

4 Section 57 #6

Let d be the smallest distance from z to points s on C and assume that $0 < |\Delta z| < d$. Since $|s - z| \geq d$ and $|\Delta z| < d$

$$|s - z - \Delta z| = |(s-z) - \Delta z| \geq ||s - z| - |\Delta z|| \geq d - |\Delta z| > 0 \tag{1}$$

we can write

$$\begin{aligned}
|g(z + \Delta z) - g(z)| &= \left| \frac{1}{2\pi i} \int_C \left(\frac{1}{s - z - \Delta z} - \frac{1}{s - z} \right) f(s)ds \right| \\
&= \left| \frac{1}{2\pi i} \int_C \frac{\Delta z}{(s - z - \Delta z)(s - z)} f(s)ds \right| \\
&\leq \frac{1}{2\pi} \left| \int_C \frac{\Delta z}{(s - z - \Delta z)(s - z)} f(s)ds \right| \\
&\leq \frac{1}{2\pi} \frac{|\Delta z| M}{(d - |\Delta z|)d} L
\end{aligned} \tag{2}$$

where L is the length of C and M is the maximum value of $|f(s)|$ on C. If we let Δz tend to zero, (2) goes to zero and we can conclude that $g(z)$ is continuous. Let γ be an arbitrary simple closed contour interior to C, then

$$\begin{aligned}
\int_\gamma g(z)dz &= \int_\gamma \left(\frac{1}{2\pi i} \int_C \frac{f(s)ds}{s - z} \right) dz \\
&= \frac{1}{2\pi i} \int_C \left(\int_\gamma \frac{f(s)}{s - z} dz \right) ds \\
&= \frac{1}{2\pi i} \int_C 0 ds = 0
\end{aligned}$$

since z is in the interior of C, so $z \notin C$. Using theorem 2 in section 57, we can conclude that $g(z)$ is analytic throughout the interior of C.

We can also write

$$\begin{aligned}
\frac{g(z+\Delta z)-g(z)}{\Delta z} &= \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z}-\frac{1}{s-z}\right)\frac{f(s)}{\Delta z}ds \\
&= \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z-\Delta z)(s-z)}
\end{aligned}$$

From

$$\frac{1}{(s-z-\Delta z)(s-z)} = \frac{1}{(s-z)^2}+\frac{\Delta z}{(s-z-\Delta z)(s-z)}$$

using (1) again,

$$\left|\int_C \frac{\Delta z f(s)ds}{(s-z-\Delta z)(s-z)^2}\right| \le \frac{|\Delta z|M}{(d-|\Delta z|)d^2}L \tag{3}$$

where L is the length of C and M is the maximum value of $|f(s)|$ on C. So

$$\frac{g(z+\Delta z)-g(z)}{\Delta z}-\frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2} = \frac{1}{2\pi i}\int_C \frac{\Delta z f(s)ds}{(s-z-\Delta z)(s-z)^2}$$

From (3) goes to 0 as $|\Delta z|$ tends to 0, so we can write

$$\lim_{\Delta z\to 0}\left|\frac{g(z+\Delta z)-g(z)}{\Delta z}-\frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}\right| = 0$$

and we get the desired result.

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$$

5 Section 57 #7

Let $f(z) = \exp(az)$, then using the Cauchy integral formula,

$$f(0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z} = \frac{1}{2\pi i}\int_C \frac{e^{az}}{z}dz = 1$$

so we get

$$\int_C \frac{e^{az}}{z}dz = 2\pi i$$

Writing the contour integral explicitly,

$$\begin{aligned}
\int_C \frac{e^{az}}{z}dz = \int_{-\pi}^{\pi}\frac{e^{a\exp(i\theta)}}{\exp(i\theta)}ie^{i\theta}d\theta &= i\int_{-\pi}^{\pi}\exp(a(\cos\theta+i\sin\theta))d\theta \\
&= i\int_{-\pi}^{\pi} e^{a\cos\theta}\exp(ia\sin\theta)d\theta \\
&= i\int_{-\pi}^{\pi} e^{a\cos\theta}(\cos(a\sin\theta)+i\sin(a\sin\theta))d\theta \\
&= i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta - \int_{-\pi}^{\pi}\sin(a\sin\theta)d\theta
\end{aligned}$$

Since $\sin(a\sin(-\theta)) = -\sin(a\sin\theta)$ and $e^{a\cos(-\theta)}\cos(a\sin(-\theta)) = e^{a\cos\theta}\cos(a\sin\theta)$,

$$\begin{aligned}
&i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta - \int_{-\pi}^{\pi}\sin(a\sin\theta)d\theta \\
&= i\left(\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta + \int_{-\pi}^{0} e^{a\cos\theta}\cos(a\sin\theta)d\theta\right) \\
&\quad - \left(\int_0^{\pi}\sin(a\sin\theta)d\theta + \int_{-\pi}^{0}\sin(a\sin\theta)d\theta\right) \\
&= i\left(\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta + \int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta\right) \\
&\quad - \left(\int_0^{\pi}\sin(a\sin\theta)d\theta - \int_0^{\pi}\sin(a\sin\theta)d\theta\right) \\
&= 2i\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = 2\pi i
\end{aligned}$$

so

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi$$

6 Section 57 #10

Consider a positively oriented circle C_R with radius R, centered at z_0. Let M_R be the maximum of $f(z)$ on C_R, then $M_R \le A(|z_0| + R)$ since for all z on C_R, $A|z| \le A(|z_0| + |z - z_0|) = A(|z_0| + R)$ by triangluar inequlity. We can write

$$|f''(z_0)| \le \frac{2!M_R}{R^2} = \frac{2A(|z_0| + R)}{R^2}$$

and it implies $|f''(z_0)| = 0$, since this inequlity should hold for all R. Then, $f''(z) = 0$ everywhere, so $f'(z)$ is constant and $|f(0)| \le 0$, so $f(0) = 0$. Thus, if we write $f'(z) = a_1$

$$f(w) = \int_0^w f'(z)dz = [a_1 z]_0^w = a_1 w$$

7 Section 59 #4

We can write

$$\begin{aligned}
|f(z)|^2 &= |\sin z|^2 = |\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y \\
&= \sin^2 x + \left(\frac{e^y - e^{-y}}{2}\right)^2 = \sin^2 x + \left(\frac{e^{2y} - 2 + e^{-2y}}{4}\right)
\end{aligned}$$

We know that $\sin^2 x \le 1$ and $\sin^2(\pi/2) = 1$. Also,

$$\frac{d}{dy}\frac{e^{2y} - 2 + e^{-2y}}{4} = \frac{e^{2y} - e^{-2y}}{2} \ge 0$$

for $y \ge 0$, so $\sinh^2 y$ is monotonically increasing in $y \ge 0$, so $\sinh^2 y$ has maximum on $y = 1$ in the given region. Thus, $|f(z)|^2$ is the largest on $(\pi/2) + i$, so $|f(z)|$ is also the largest there.

8 Section 59 #7

Let $g(z) = \exp(-if(z)) = \exp(-if(x+iy)) = \exp(v(x,y) - iu(x,y))$, then $g(z)$ is continuous on R and analytic, nonconstant in the interior of R. Then $|g(z)| = |\exp(v(x,y) - iu(x,y))| = |\exp(v(x,y))||\exp(-iu(x,y))| = |\exp(v(x,y))|$ has its maximum on the boundary of R by the corollary in section 59, and since e^x is monotonically increasing for real number x, $v(x,y)$ is maximum there.

Considering $h(z) = 1/g(z)$, $h(z)$ is also continuous on R and analytic, nonconstant in the interior of R since $g(z) \neq 0$. Then $|h(z)| = |\exp(-v(x,y))|$ has its maximum on the boundary of R by the corollary in section 59, and since e^x is monotonically increasing for real number x, $-v(x,y)$ is maximum there and $v(x,y)$ is minimum there.