MATH210: Homework 5 (due Apr. 4)

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1 Section 43 #2

If $-\pi/2 < \theta < \pi/2$, $\cos\theta > 0$, so we can write

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta = \cos\theta\,(1 + i\tan\theta) \\ &= \frac{1 + i\tan\theta}{\sec\theta} = \frac{1 + i\tan\theta}{\sqrt{\sec^2\theta}} = \frac{1 + i\tan\theta}{\sqrt{1 + \tan^2\theta}} \end{aligned}$$

Then $z[\phi(y)]$ can be written as follows since $-\pi/2 < \phi(y) < \pi/2$.

$$\begin{aligned} z[\phi(y)] &= 2 \cdot \frac{1 + i\tan\phi(y)}{\sqrt{1 + \tan^2\phi(y)}} = 2 \cdot \frac{1 + i\frac{y}{\sqrt{4-y^2}}}{\sqrt{1 + \left(\frac{y}{\sqrt{4-y^2}}\right)^2}} \\ &= 2 \cdot \frac{1 + i\frac{y}{\sqrt{4-y^2}}}{\sqrt{\frac{4}{4-y^2}}} = 2 \cdot \frac{\sqrt{4 - y^2} + iy}{2} = \sqrt{4 - y^2} + iy \end{aligned}$$

Thus, we can conclude that $Z(y) = z[\phi(y)]$ for $-2 < y < 2$. From equality 7 in section 40 and chain rule, the derivative of $\phi(y)$ can be calculated as follows:

$$\begin{aligned} \frac{d}{dy}\phi(y) &= \frac{d}{dy}\left(\frac{y}{\sqrt{4 - y^2}}\right)\frac{1}{1 + \left(\frac{y}{\sqrt{4-y^2}}\right)^2} = \frac{\frac{\sqrt{4-y^2} - y\frac{-2y}{2\sqrt{4-y^2}}}{4-y^2}}{\frac{4}{4-y^2}} \\ &= \frac{\sqrt{4 - y^2} - y\frac{-y}{\sqrt{4-y^2}}}{4} = \frac{1}{\sqrt{4 - y^2}} \end{aligned}$$

from this, we know that $\phi(y)$ has a positive derivative.

2 Section 43 #6

2.1 Solution for (a)

By definition, $y(x) = 0$ if $x = 0$. From equation 13 in section 38, $\sin z = \sin x \cosh y + i\cos x \sinh y$ for $z = x + iy$, and since $\cosh y = (e^y + e^{-y})/2 > 0$ for all y, if $\sin z = 0$, then $x = n\pi$ where $n = 0, \pm1, \pm2, \ldots$. Thus, the arc intersects the real axis only at points $z = 1/n$ and $z = 0$.

2.2 Solution for (b)

For $z'(x) = 1 + iy'(x)$, $y'(x)$ is continuous in $0 < x \le 1$ since y $x^3, 1/x$ are continuously differentiable in $0 < x \le 1$, and $\sin x$ is continuously differentiable in $x \ge \pi$. We can write

$$y'(0) = \lim_{h\to 0} \frac{y(h) - y(0)}{h} = \lim_{h\to 0} \frac{h^3 \sin(\pi/h)}{h} = \lim_{h\to 0} h^2 \sin(\pi/h)$$

Also, from $|h^2 \sin(\pi/h)| \le h^2$ we can conclude that $y'(0) = 0$ using sandwich theorem. For $0 < x \le 1$, we can write

$$y'(x) = 3x^2 \sin\left(\frac{\pi}{x}\right) - \frac{\pi}{x^2} \cdot x^3 \cos\left(\frac{\pi}{x}\right) = 3x^2 \sin\left(\frac{\pi}{x}\right) - \pi x \cos\left(\frac{\pi}{x}\right)$$

Using triangular inequality,

$$\begin{aligned} |y'(x)| &= \left|3x^2 \sin\left(\frac{\pi}{x}\right) - \pi x \cos\left(\frac{\pi}{x}\right)\right| \\ &\le \left|2x^2 \sin\left(\frac{\pi}{x}\right)\right| + \left|-\pi x \cos\left(\frac{\pi}{x}\right)\right| \le 2x^2 + \pi|x| \end{aligned}$$

Then we can conclude that $\lim_{x\to 0+} y'(x) = 0$ using sandwich theorem, and $y'(x)$ is continuous in $[0, 1]$. We write the tangent vector $\mathbf{T}$ as follows:

$$\mathbf{T} = \frac{z'(x)}{|z'(x)|} = \frac{1 + iy'(x)}{\sqrt{1 + (y'(x))^2}} = \frac{1}{\sqrt{1 + (y'(x))^2}} + i\frac{y'(x)}{\sqrt{1 + (y'(x))^2}}$$

Since $y'(x)$ is continuous in $[0, 1]$ and $\sqrt{1 + (y'(x))^2} > 0$, $\mathbf{T}$ is continuous in $[0, 1]$ and the arc is smooth.

3 Section 46 #9

3.1 Solution for (a)

Let C_1 be the top half of C, and represent it as follows:

$$z = e^{i\theta} \quad (0 \le \theta \le \pi)$$

We can write

$$f(z(\theta)) = \exp\left(-\frac{3}{4} \operatorname{Log} z(\theta)\right) = \exp\left(-\frac{3}{4}(\ln 1 + i\theta)\right) = \exp\left(-\frac{3i\theta}{4}\right)$$

so

$$f(z(\theta))z'(\theta) = \exp\left(-\frac{3i\theta}{4}\right) ie^{i\theta} = ie^{i\theta/4} = -\sin\frac{\theta}{4} + i\cos\frac{\theta}{4} \tag{1}$$

where $0 \le \theta < \pi$. Then, the left hand limits of real and imaginary components at $\theta = \pi$ exist

$$\lim_{\theta\to\pi-} \left(-\sin\frac{\theta}{4}\right) = -\frac{1}{\sqrt{2}}, \quad \lim_{\theta\to\pi-} \cos\frac{\theta}{4} = \frac{1}{\sqrt{2}}$$

and $f(z(\theta))z'(\theta)$ is continuous on the closed interval $[0, \pi]$ when we define $f(\pi)$ as $(-1 + i)/\sqrt{2}$.

Let C_2 be the bottom half of C, and write it as follows:

$$z = e^{i\theta} \quad (-\pi \le \theta \le 0)$$

We can write (1) in the same way as we did on C_1, where $-\pi < \theta \le 0$. The right hand limits of real and imaginary components at $\theta = -\pi$ exist

$$\lim_{\theta \to -\pi+} \left(-\sin \frac{\theta}{4} \right) = \frac{1}{\sqrt{2}}, \quad \lim_{\theta \to -\pi+} \cos \frac{\theta}{4} = \frac{1}{\sqrt{2}}$$

and $f(z(\theta))z'(\theta)$ is continuous on the closed interval $[-\pi, 0]$ when we define $f(-\pi)$ as $(1+i)/\sqrt{2}$. Then contour integral of f over C can be written as

$$\begin{aligned}
\int_C f(z)dz &= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = i\int_0^{\pi} e^{i\theta/4} d\theta + i\int_{-\pi}^{0} e^{i\theta/4} d\theta \\
&= i\int_{-\pi}^{\pi} e^{i\theta/4} d\theta = i\left[\frac{4}{i} e^{i\theta/4} \right]_{-\pi}^{\pi} = 4(e^{i\pi/4} - e^{-i\pi/4}) = 4\sqrt{2}
\end{aligned}$$

3.2 Solution for (b)

Let C_1 be the top half of C, and represent it as follows:

$$z = e^{i\theta} \quad (0 \le \theta \le \pi)$$

We can write

$$f(z(\theta)) = \exp\left(-\frac{3}{4} \log z(\theta) \right) = \exp\left(-\frac{3}{4}(\ln 1 + i\theta) \right) = \exp\left(-\frac{3i\theta}{4} \right)$$

so

$$f(z(\theta))z'(\theta) = \exp\left(-\frac{3i\theta}{4} \right) ie^{i\theta} = ie^{i\theta/4} = -\sin\frac{\theta}{4} + i\cos\frac{\theta}{4} \tag{2}$$

where $0 < \theta \le \pi$. Then, the right hand limits of real and imaginary components at $\theta = 0$ exist

$$\lim_{\theta \to 0+} \left(-\sin \frac{\theta}{4} \right) = 0, \quad \lim_{\theta \to 0+} \cos \frac{\theta}{4} = 1$$

and $f(z(\theta))z'(\theta)$ is continuous on the closed interval $[0, \pi]$ when we define $f(0)$ as i.

Let C_2 be the bottom half of C, and write it as follows:

$$z = e^{i\theta} \quad (\pi \le \theta \le 2\pi)$$

We can write (2) in the same way as we did on C_1, where $\pi \le \theta < 2\pi$. The left hand limits of real and imaginary components at $\theta = 2\pi$ exist

$$\lim_{\theta \to 2\pi-} \left(-\sin \frac{\theta}{4} \right) = -1, \quad \lim_{\theta \to 2\pi-} \cos \frac{\theta}{4} = 0$$

and $f(z(\theta))z'(\theta)$ is continuous on the closed interval $[\pi, 2\pi]$ when we define $f(2\pi)$ as -1. Then contour integral of f over C can be written as

$$\begin{aligned}
\int_C f(z)dz &= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = i\int_0^{\pi} e^{i\theta/4} d\theta + i\int_{\pi}^{2\pi} e^{i\theta/4} d\theta \\
&= i\int_{0}^{2\pi} e^{i\theta/4} d\theta = i\left[\frac{4}{i} e^{i\theta/4} \right]_{0}^{2\pi} = 4(e^{i\pi/2} - e^{0}) = -4 + 4i
\end{aligned}$$

4 Section 46 #13

If $n = 0$,

$$\int_{C_0} \frac{1}{z - z_0} dz = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} Rie^{i\theta} d\theta = 2\pi i$$

If $n = \pm 1, \pm 2, \ldots$,

$$\begin{aligned}\int_{C_0} (Re^{i\theta})^{n-1} dz &= \int_{-\pi}^{\pi} (Re^{i\theta})^{n-1} Rie^{i\theta} d\theta = iR^n \int_{-\pi}^{\pi} e^{in\theta} d\theta \\ &= \frac{R^n}{n}(e^{in\pi} - e^{-in\pi}) = \frac{2iR^n}{n} \sin n\pi = 0\end{aligned}$$

5 Section 47 #5

Let $f(z) = \operatorname{Log} z/z^2$. From the theorem in section 47, we can write

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2} dz \right| \leq ML$$

where M is a nonnegative constant such that $|f(z)| \leq M$ for all points z in C_R at which $f(z)$ is defined, and L is the length of C_R. Since z is a point in C_R, $|z| = R$ and we can represent C_R as $z = Re^{i\theta}$ for $-\pi \leq \theta \leq \pi$. Then, $\operatorname{Log} z = \ln R + i\theta$ for $-\pi < \theta < \pi$. Using triangular inequality,

$$|f(z)| = \left| \frac{\operatorname{Log} z}{z^2} \right| = \frac{|\operatorname{Log} z|}{|z^2|} = \frac{|\ln R + i\theta|}{|z|^2} \leq \frac{|\ln R| + |i\theta|}{|z|^2} < \frac{|\ln R| + \pi}{R^2} \tag{3}$$

The length of contour C_R is $2\pi R$, so we can write

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2} dz \right| \leq 2\pi \left(\frac{\pi + \ln R}{R} \right)$$

Since $|f(z)|$ is strictly less than $(|\ln R| + \pi)/R^2$, the equality cannot hold and we get the desired result.

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2} dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right)$$

Using l'Hospital's rule,

$$\lim_{R \to \infty} 2\pi \left(\frac{\pi + \ln R}{R} \right) = \lim_{R \to \infty} \frac{2\pi}{R} = 0$$

From sandwich theorem, we can conclude that the value of the contour integral tends to zero as $R \to \infty$.

6 Section 47 #6

We can write

$$z^{-1/2} = \exp\left(-\frac{1}{2} \log z\right) = \exp\left(-\frac{1}{2}(\ln|z| + i\theta)\right) = \exp\left(-\frac{1}{2}(\ln\rho + i\theta)\right)$$

where $\alpha < \theta < \alpha + 2\pi$ for some real number α. Then

$$|z^{-1/2}| = \left|\exp\left(-\frac{1}{2}(\ln\rho + i\theta)\right)\right| = \left|\exp\left(-\frac{1}{2}\ln\rho\right)\right|\left|e^{-i\theta/2}\right| = \left|\exp\left(-\frac{1}{2}\ln\rho\right)\right| \leq \frac{1}{\sqrt{\rho}}$$

Since $f(z)$ is analytic in the disk $|z| \leq 1$, it is bounded in the disk and there exists a nonnegative constant M_0 such that $|f(z)| \leq M_0$ for all z in the disk $|z| \leq 1$. From this, we know that $|z^{-1/2}f(z)| \leq M_0/\sqrt{\rho}$. Using the theorem in section 47, we can write

$$\left|\int_{C_\rho} z^{-1/2} f(z) dz\right| \leq 2\pi\rho \cdot \frac{M_0}{\sqrt{\rho}} = 2\pi M_0 \sqrt{\rho}$$

We can take $M = M_0$ and get the desired result. $2\pi M\sqrt{\rho}$ tends to 0 as $\rho \to 0$, so the value of the contour integral also tends to 0 by the sandwich theorem.

7 Section 49 #5

Let $f(z)$ be z^i where the branch $-\pi/2 < \arg z < 3\pi/2$ is taken, and $g(z)$ be z^i where the principal branch is taken, and

$$F(z) = \frac{\exp((i+1)\log z)}{i+1} = \frac{\exp((i+1)(\ln|z| + i\arg z))}{i+1}$$

where $-\pi/2 < \arg z < 3\pi/2$. Then

$$F'(z) = \exp((i+1)\log z) \cdot 1/z = \exp((i+1)\log z) \cdot \exp(-\log z) = \exp(i\log z) = z^i$$

So $F(z)$ is an antiderivative of $f(z)$. From

$$\begin{aligned} f(z) &= z^i = \exp(i\log z) = \exp(i(\ln|z| + i\arg z)) \quad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right) \\ g(z) &= z^i = \exp(i\log z) = \exp(i(\ln|z| + i\,\mathrm{Arg}\, z)) \quad (|z| > 0, -\pi < \mathrm{Arg}\, z < \pi) \end{aligned}$$

In any contour from $z = -1$ to $z = 1$, $f(z) = g(z)$, so we can write

$$\begin{aligned} \int_{-1}^{1} g(z)dz = \int_{-1}^{1} f(z)dz &= [F(z)]_{-1}^{1} = F(1) - F(-1) \\ &= \frac{1 - e^{(i+1)i\pi}}{i+1} = \frac{1+e^{-\pi}}{i+1} = \frac{1+e^{-\pi}}{2}(1-i) \end{aligned}$$

8 Section 53 #1

8.1 Solution for (b)

$f(z) = ze^{-z}$ is a product of two entire functions, z and e^{-z}, so it is anlaytic in the contour and its interior. Applying the Cauchy-Goursat theorem, we know that the integral is zero regardless of direction.

8.2 Solution for (f)

$f(z) = \mathrm{Log}(z+2)$ is analytic in the contour and its interior since the points in the unit disk with its center on $z = 2$ does not intersect with branch cut. Applying the Cauchy-Goursat theorem, we know that the integral is zero regardless of direction.

9 Section 53 #2

9.1 Solution for (b)

Since $\sin(z/2) = 0$ holds if and only if $z = 0, \pm 2\pi, \pm 4\pi, \dots$, $\sin(z/2) \neq 0$ in the closed region defined by C_1 and C_2, so $f(z)$ is analytic in the region. By the corollary in section 53, we get the desired result.

9.2 Solution for (c)

Since $1 - e^z = 0$ holds if and only if $z = 2in\pi, (n = 0, 1, 2, \dots)$, $1 - e^z \neq 0$ in the closed region defined by C_1 and C_2, so $f(z)$ is analytic in the region. By the corollary in section 53, we get the desired result.

10 Section 53 #7

Let $f(z) = f(x + iy) = u(x, y) + iv(x, y) = \overline{z}$ and $z(t) = x(t) + iy(t) \quad (a \leq t \leq b)$. We can write

$$\begin{aligned}
\int_C f(z)dz &= \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')dt \\
&= \int_C udx - vdy + i\int_C vdx + udy \\
&= \iint_R (-v_x - u_y)dA + i\iint_R (u_x - v_y)dA \\
&= \iint_R 0dA + i\iint [1 - (-1)]dA = 2i\iint dA
\end{aligned}$$

where R is the region enclosed by C. Let S be the area of the region, then

$$S = \iint_R dA = \frac{1}{2i}\int_C \overline{z}dz$$

and we get the desired result.