Homework 1 (due Mar. 7)

손량(20220323)

Last compiled on: Tuesday 7th March, 2023, 18:23

1 Section 20 #4

By definition, $f'(z_0), g'(z_0)$ can be written as follows:

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0},\ g'(z_0) = \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}$$

By properties of limits,

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}{\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}} = \lim_{z\to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \lim_{z\to z_0} \frac{f(z)}{g(z)}$$

2 Section 20 #9

Δw can be written as follows, if $z = 0$.

$$\Delta w = f(z+\Delta z) - f(z) = f(\Delta z) - f(0) = f(\Delta z) = \frac{\overline{\Delta z}^2}{\Delta z}$$

Then $\Delta w/\Delta z = \left(\overline{\Delta z}/\Delta z\right)^2$. If Δz is on the real axis, $\overline{\Delta z} = \Delta z$ and $\left(\overline{\Delta z}/\Delta z\right)^2 = (\Delta z/\Delta z)^2 = 1$. If Δz is on the imaginary axis, $\overline{\Delta z} = -\Delta z$ and $\Delta w/\Delta z = \left(\overline{\Delta z}/\Delta z\right)^2 = (-\Delta z/\Delta z)^2 = 1$. However, on each nonzero point on line $\Delta y = \Delta x$ where Δz is written as $\Delta x + i\Delta y$,

$$\frac{\Delta w}{\Delta z} = \left(\frac{\overline{\Delta z}}{\Delta z}\right)^2 = \left(\frac{\Delta x - i\Delta x}{\Delta x + i\Delta x}\right)^2 = \left(\frac{1-i}{1+i}\right)^2 = (-i)^2 = -1$$

If the limit of $\Delta w/\Delta z$ exists, it can be found by letting Δz approach the origin in any manner. However, the limit is 1 if Δz approached the origin along the real or imaginary axis, or -1 if Δz approached the origin along the line $\Delta y = \Delta x$. Since limits are unique, it is a contradiction and $f'(0)$ does not exist.

3 Section 24 #3 (c)

$f(z)$ can be written as follows:

$$f(z) = f(x+iy) = (x+iy)y = xy + iy^2$$

Let $u(x,y) := xy$ and $v(x,y) = y^2$, then $f(z) = u(x,y) + iv(x,y)$. If f is differentible at $z_0 = x_0 + iy_0$, the first-order partial derivatives of u and v must exist at (x_0, y_0), and

Cauchy-Riemann equation must be satisfied. Since the partial derivatives exist at every (x_0, y_0) since u and v are polynomial functions, let's check the Cauchy-Riemann equation.

$$\begin{aligned} u_x(x_0, y_0) = v_y(x_0, y_0) &\iff y_0 = 2y_0 \\ u_y(x_0, y_0) = -v_x(x_0, y_0) &\iff x_0 = 0 \end{aligned}$$

Only $x_0 = 0, y_0 = 0$ satistifes the equation. f cannot be differentible at points other than 0. Let's check if it is differentible at 0:

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z - 0} &= \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z \operatorname{Im} \Delta z}{\Delta z} \\ &= \lim_{\Delta z \to 0} \operatorname{Im} \Delta z = 0 \end{aligned}$$

f is only differentible at 0, and $f'(0) = 0$.

4 Section 24 #7

4.1 Solution for (a)

According to problem #6, the following holds for $f'(z_0)$ where $z_0 = r_0 \exp(i\theta_0)$ and f is differentible at z_0: (u_r and v_r are evaluated at (r_0, θ_0))

$$f'(z_0) = u_x + iv_x = e^{-i\theta_0}(u_r + iv_r)$$

By the polar form of Cauchy-Riemann equation, u_r and v_r can be written as follows (u_θ and v_θ are evaluated at (r_0, θ_0))

$$u_r = v_\theta / r, \quad v_r = -u_\theta / r$$

Substitution gives us the result:

$$f'(z_0) = \frac{1}{r_0 e^{i\theta_0}}(v_\theta - iu_\theta) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

4.2 Solution for (b)

$f(z)$ can be written as follows, where $z = r\exp(i\theta) \neq 0$:

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}(\cos\theta - i\sin\theta)$$

The component functions are

$$u(r, \theta) = \frac{\cos\theta}{r}, \quad v(r, \theta) = \frac{-\sin\theta}{r}$$

Since $f(z)$ is differentible in $\mathbb{C} \setminus \{0\}$, we can use the relation in (a). Plugging those to the expression derived in (a), we get the desired result.

$$
\begin{aligned}
f'(z) &= \frac{-i}{z}(u_\theta + iv_\theta) = \frac{-i}{z}\left(\frac{-\sin\theta}{r} + i\frac{-\cos\theta}{r}\right) \\
&= -\frac{i}{zr}\left(\sin(-\theta) - i\cos(-\theta)\right) = -\frac{1}{zr}(\cos(-\theta) + i\sin(-\theta)) \\
&= -\frac{1}{zre^{i\theta}} = -\frac{1}{z^2}
\end{aligned}
$$

5 Section 24 #8

5.1 Solution for (a)

Using the chain rule,

$$
\begin{aligned}
\frac{\partial F}{\partial \overline{z}} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{\partial F}{\partial x}\left(\frac{\partial}{\partial \overline{z}}\frac{z+\overline{z}}{2}\right) + \frac{\partial F}{\partial y}\left(\frac{\partial}{\partial \overline{z}}\frac{z-\overline{z}}{2i}\right) \\
&= \frac{1}{2}\frac{\partial F}{\partial x} - \frac{1}{2i}\frac{\partial F}{\partial y} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)
\end{aligned}
$$

5.2 Solution for (b)

Using the defined operator,

$$
\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left[(u_x + iv_x) + i(u_y + iv_y)\right] = \frac{1}{2}\left[(u_x - v_y) + i(v_x + u_y)\right]
$$

Since the first order derivatives of the real and imaginary components of f satisfy the Cauchy-Riemann equations,

$$
u_x = v_y, \quad u_y = -v_x
$$

Both $u_x - v_y$ and $v_x + u_y$ are zero, giving us the desired result, $\partial f/\partial \overline{z} = 0$.