MATH210: Homework 11 (due May. 23)

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1 Section 88 #2

Let $q(z) = z^2 + 1$, $f(z) = 1/q(z)$ and $g(z) = f(z)e^{iaz}$. Since $q(z)$ is analytic everywhere and $q(\pm i) = 0$, $f(z)$ and $g(z)$ has isolated singularities at $\pm i$. i is inside a semicircular region bounded by $z = x \quad (x \in [-R, R])$ and upper half C_R of the circle $|z| = R$, where $R > 1$. By Cauchy's residue theorem,

$$\int_{-R}^{R} g(x)dx + \int_{C_R} g(z)dz = 2\pi i \operatorname*{Res}_{z=i} g(z) \tag{1}$$

Since $q(i) = 0$ and $q'(i) \neq 0$ we can apply theorem 2 in section 83 and write

$$\operatorname*{Res}_{z=i} g(z) = \frac{e^{-a}}{q'(i)} = \frac{e^{-a}}{2i}$$

For all $z \in C_R$, we can write

$$|f(z)| = \left|\frac{1}{z^2+1}\right| = \frac{1}{|z^2+1|} \leq \frac{1}{|z^2|-1} = \frac{1}{R^2-1}$$

$\lim_{R\to\infty} 1/(R^2-1) = 0$, and $f(z)$ is analytic at all points in the upper half plane that are exterior to $|z| = 1$. Then we can use Jordan's lemma and write

$$\lim_{R\to\infty} \int_{C_R} g(z)dz = \lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0$$

and write (1) as

$$\lim_{R\to\infty} \int_{-R}^{R} g(x)dx = 2\pi i \operatorname*{Res}_{z=i} g(z) - \lim_{R\to\infty} \int_{C_R} g(z)dz = \pi e^{-a}$$

so

$$\int_{-\infty}^{\infty} g(x)dx = \pi e^{-a}$$

Since $\operatorname{Re} g(z) = \cos(az)/(z^2+1)$, we can write

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}dx = \operatorname{Re} \int_{-\infty}^{\infty} g(x)dx = \pi e^{-a}$$

and $\cos(ax)$ and $x^2 + 1$ are both odd functions,

$$\int_{0}^{\infty} \frac{\cos(ax)}{x^2+1}dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}dx = \frac{\pi}{2}e^{-a}$$

2 Section 88 #5

Let $q(z) = z^4 + 4$, $f(z) = z^3/q(z)$ and $g(z) = f(z)e^{iaz}$. $f(z)$ and $g(z)$ have isolated singularities at the zeros of $q(z)$, which are fourth roots of -4 and they are analytic anywhere else. The complex roots of -4 are $c_k = \sqrt{2}\exp(i(\pi + 2k\pi)/4) \quad (k = 0, 1, 2, 3)$ and none of them lies on the real axis. c_0, c_1 lie in the upper half plane, so they are inside a semicircular region bounded by $z = x \quad (x \in [-R, R])$ and upper half C_R of the circle $|z| = R$, where $R > \sqrt{2}$. By Cauchy's residue theorem,

$$\int_{-R}^{R} g(x)dx + \int_{C_R} g(z)dz = 2\pi i \left(\operatorname*{Res}_{z=c_0} g(z) + \operatorname*{Res}_{z=c_1} g(z) \right) \tag{2}$$

Since $q(c_k) = 0$ and $q'(c_k) \neq 0$ for $k = 0, 1$, we can apply theorem 2 in section 83 and write

$$\operatorname*{Res}_{z=c_k} g(z) = \frac{c_k^3 e^{iac_k}}{q'(c_k)} = \frac{c_k^3 e^{iac_k}}{4c_k^3} = \frac{e^{iac_k}}{4}$$

for $k = 0, 1$. For all $z \in C_R$, we can write

$$|f(z)| = \left| \frac{z^3}{z^4 + 4} \right| = \frac{R^3}{|z^4 + 4|} \leq \frac{R^3}{|z^4| - 4} = \frac{R^3}{R^4 - 4}$$

From $\lim_{R\to\infty} R^3/(R^4 - 4) = 0$ and since $f(z)$ is analytic at all points in the upper half plane that are exterior to $|z| = \sqrt{2}$, we can apply Jordan's lemma and obtain

$$\lim_{R\to\infty} \int_{C_R} g(z)dz = \lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0$$

We can write (2) as

$$\lim_{R\to\infty} \int_{-R}^{R} g(x)dx = 2\pi i \left(\operatorname*{Res}_{z=c_0} g(z) + \operatorname*{Res}_{z=c_1} g(z) \right) - \lim_{R\to\infty} \int_{C_R} g(z)dz$$

so

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)dx &= 2\pi i \left(\operatorname*{Res}_{z=c_0} g(z) + \operatorname*{Res}_{z=c_1} g(z) \right) = 2\pi i \left(\frac{e^{iac_0}}{4} + \frac{e^{iac_1}}{4} \right) \\
&= 2\pi i \left(\frac{e^{ia-a}}{4} + \frac{e^{-ia-a}}{4} \right) = i\pi e^{-a} \frac{e^{ia} + e^{-ia}}{2} = i\pi e^{-a} \cos a
\end{aligned}$$

Since $\operatorname{Im} g(z) = z^3 \sin(az)/(z^4 + 4)$,

$$\int_{-\infty}^{\infty} \frac{x^3 \sin(ax)}{x^4 + 4} dx = \operatorname{Im} \int_{-\infty}^{\infty} g(x)dx = \pi e^{-a} \cos a$$

3 Section 88 #9

Let $q(z) = z^2 + 2z + 2$, $f(z) = z/q(z)$ and $g(z) = f(z)e^{iz}$. $f(z)$ and $g(z)$ have isolated singularities at the zeros of $q(z)$, and they are analytic anywhere else. $q(z)$ has zeros at $-1 \pm i$ and none of them lies on the real axis. $-1 + i$ lie in the upper half plane, so it is inside a semicircular region bounded by $z = x \quad (x \in [-R, R])$ and upper half C_R of the circle $|z| = R$, where $R > 3$. By Cauchy's residue theorem,

$$\int_{-R}^{R} g(x)dx + \int_{C_R} g(z)dz = 2\pi i \operatorname*{Res}_{z=-1+i} g(z)$$

Since $q(-1+i) = 0$ and $q'(-1+i) \neq 0$, we can apply theorem 2 in section 83 and write

$$\operatorname*{Res}_{z=-1+i} g(z) = \frac{(-1+i)e^{i(-1+i)}}{q'(-1+i)} = \frac{(-1+i)e^{-1-i}}{2i} = \frac{(-1+i)(\cos(-1) + i\sin(-1))}{2ie}$$

For all $z \in C_R$, we can write

$$\begin{aligned} |f(z)| &= \left| \frac{z}{z^2+2z+2} \right| = \frac{R}{|z^2+2z+2|} \leq \frac{R}{|z^2+2z| - 2} = \frac{R}{|z||z+2| - 2} \\ &\leq \frac{R}{|z|(|z|-2) - 2} = \frac{R}{R^2 - 2R - 2} \end{aligned}$$

We can see that $\lim_{R\to\infty} R/(R^2 - 2R - 2) = 0$ and since $f(z)$ is analytic at all points in the upper half plane that are exterior to $|z| = 3$, we can apply Jordan's lemma.

$$\lim_{R\to\infty} \int_{C_R} g(z)dz = \lim_{R\to\infty} \int_{C_R} f(z)e^{iz} dz = 0$$

We can write

$$\begin{aligned} \text{P.V.} \int_{-\infty}^{\infty} g(z)dz &= \lim_{R\to\infty} \int_{-R}^{R} g(z)dz = 2\pi i \operatorname*{Res}_{z=-1+i} g(z) - \lim_{R\to\infty} \int_{C_R} g(z)dz \\ &= \frac{\pi}{e}(-1+i)(\cos 1 - i\sin 1) \end{aligned}$$

so

$$\begin{aligned} \text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin x dx}{x^2+2x+2} &= \operatorname{Im} \text{P.V.} \int_{-\infty}^{\infty} g(z)dz = \operatorname{Im} \frac{\pi}{e}(-1+i)(\cos 1 - i\sin 1) \\ &= \frac{\pi}{e}(\sin 1 + \cos 1) \end{aligned}$$

4 Section 91 #1

If $a = b$, then $\int_0^\infty (\cos(ax) - \cos(bx))/x^2 dx = \int_0^\infty 0 dx = 0$ and there is nothing to prove.

Let's consider the $a \neq b$ case. Using the contour in figure 108 of section 90, we can write

$$\int_{L_1} f(z)dz + \int_{C_R} f(z)dz + \int_{L_2} f(z)dz + \int_{C_\rho} f(z)dz = 0$$

by the Cauchy-Goursat theorem. Then,

$$\begin{aligned} \int_{L_1} f(z)dz + \int_{L_2} f(z)dz &= \int_{L_1} f(z)dz - \int_{-L_2} f(z)dz \\ &= \int_\rho^R f(x)dx + \int_\rho^R f(-x)dx \\ &= \int_\rho^R \frac{(e^{iax} + e^{-iax}) - (e^{ibx} + e^{-ibx})}{x^2} dx \\ &= \int_\rho^R \frac{2(\cos(ax) - \cos(bx))}{x^2} dx \\ &= -\int_{C_R} f(z)dz - \int_{C_\rho} f(z)dz \qquad (3) \end{aligned}$$

Let $g(z) = 1/z^2$. For all $z \in C_R$, we can write

$$|g(z)| = \left| \frac{1}{z^2} \right| = \frac{1}{R^2}$$

We can see that $\lim_{R\to\infty} 1/R^2 = 0$ and since $g(z)$ is analytic at all points in the upper half plane that are exterior to $|z| = R_0$ for some $R_0 > 0$, we can apply Jordan's lemma.

$$\lim_{R\to\infty} \int_{C_R} g(z) e^{iaz} dz = 0$$

for $a > 0$. Since $\int_{C_R} g(z)dz \leq \pi R/R^2 = \pi/R$, $\lim_{R\to\infty} \int_{C_R} g(z)e^{iaz}dz = 0$ for $a \geq 0$. Thus, we can write

$$\begin{aligned}
\lim_{R\to\infty} \int_{C_R} f(z)dz &= \lim_{R\to\infty} \int_{C_R} g(z)(e^{iaz} - e^{ibz})dz \\
&= \lim_{R\to\infty} \left(\int_{C_R} g(z)e^{iaz}dz - \int_{C_R} g(z)e^{ibz}dz \right) = 0
\end{aligned}$$

for $a \geq 0$, $b \geq 0$. Using the Maclaurin series expansion of e^{iaz}, we can write

$$\begin{aligned}
f(z) &= \frac{1}{z^2} \left[\left(1 + iaz + \frac{(iaz)^2}{2!} + \cdots \right) - \left(1 + ibz + \frac{(ibz)^2}{2!} + \cdots \right) \right] \\
&= \frac{1}{z^2} \sum_{n=1}^{\infty} \frac{(ia)^n - (ib)^n}{n!} z^n = \sum_{n=1}^{\infty} \frac{(ia)^n - (ib)^n}{n!} z^{n-2} \quad (0 < |z| < \infty)
\end{aligned}$$

and we know that $f(z)$ has a simple pole at $z = 0$, with residue $B_0 = \operatorname{Res}_{z=0} f(z) = i(a-b)$. Using the theorem in section 89, we can write

$$\lim_{\rho\to 0} \int_{C_\rho} f(z)dz = -B_0 \pi i = \pi(a-b)$$

From (3),

$$\begin{aligned}
\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2} dx &= \lim_{R\to\infty} \lim_{\rho\to 0} \int_\rho^R \frac{\cos(ax) - \cos(bx)}{x^2} dx \\
&= -\frac{1}{2} \lim_{R\to\infty} \lim_{\rho\to 0} \left(\int_{C_R} f(z)dz + \int_{C_\rho} f(z)dz \right) \\
&= \frac{\pi}{2}(b-a)
\end{aligned}$$

and we get the desired result.

Setting $a = 0$, $b = 2$ gives us

$$\int_0^\infty \frac{1 - \cos(2x)}{x^2} dx = \int_0^\infty \frac{2\sin^2 x}{x^2} dx = \pi$$

so

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

5 Section 91 #4

Let $q(z) = (z+a)(z+b)$ and $f(z) = \sqrt[3]{z}/q(z)$ where $|z| > 0$, $0 < \arg z < 2\pi$. Then

$$f(z) = \frac{\exp\left(\frac{1}{3}\log z\right)}{(z+a)(z+b)} = \frac{\exp\left(\frac{1}{3}(\ln r + i\theta)\right)}{(re^{i\theta}+a)(re^{i\theta}+b)}$$

where $z = re^{i\theta}$. Then we can write

$$\lim_{\theta\to 0+} f(re^{i\theta}) = \lim_{\theta\to 0+} \frac{\exp\left(\frac{1}{3}(\ln r + i\theta)\right)}{(re^{i\theta}+a)(re^{i\theta}+b)} = \frac{\exp\left(\frac{1}{3}\ln r\right)}{(r+a)(r+b)} = \frac{\sqrt[3]{r}}{(r+a)(r+b)}$$

and

$$\begin{aligned}\lim_{\theta\to 2\pi-} f(re^{i\theta}) &= \lim_{\theta\to 2\pi-} \frac{\exp\left(\frac{1}{3}(\ln r + i\theta)\right)}{(re^{i\theta}+a)(re^{i\theta}+b)} \\ &= \frac{\exp\left(\frac{1}{3}(\ln r + 2\pi i)\right)}{(re^{2\pi i}+a)(re^{2\pi i}+b)} = \frac{\exp\left(\frac{1}{3}(\ln r + 2\pi i)\right)}{(r+a)(r+b)} = \frac{\sqrt[3]{r}e^{2\pi i/3}}{(r+a)(r+b)}\end{aligned}$$

Since $f(z)$ has isolated singularities at a and b and analytic anywhere else, we can use Cauchy's residue theorem and obtain

$$\begin{aligned}&\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr + \int_{C_R} f(z)dz - \int_\rho^R \frac{\sqrt[3]{r}e^{2\pi i/3}}{(r+a)(r+b)}dr + \int_{C_\rho} f(z)dz \\ &= 2\pi i\left(\operatorname*{Res}_{z=-a} f(z) + \operatorname*{Res}_{z=-b} f(z)\right)\end{aligned} \tag{4}$$

Since $q(-a) = q(-b) = 0$, $q'(-a) \neq 0$ and $q'(-b) \neq 0$, we can apply theorem 2 in section 83 and write

$$\operatorname*{Res}_{z=-a} f(z) = \frac{\sqrt[3]{-a}}{q'(-a)} = \frac{\exp\left(\frac{1}{3}(\ln a + i\pi)\right)}{-a+b} = \frac{\sqrt[3]{a}e^{i\pi/3}}{-a+b}$$
$$\operatorname*{Res}_{z=-b} f(z) = \frac{\sqrt[3]{-b}}{q'(-b)} = \frac{\exp\left(\frac{1}{3}(\ln b + i\pi)\right)}{a-b} = \frac{\sqrt[3]{b}e^{i\pi/3}}{a-b}$$

so we can write (4) as

$$(1-e^{2\pi i/3})\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr = 2\pi i(-e^{i\pi/3})\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} - \int_{C_R} f(z)dz - \int_{C_\rho} f(z)dz \tag{5}$$

For all z such that $|z| = \rho$,

$$|f(z)| = \left|\frac{\sqrt[3]{z}}{(z+a)(z+b)}\right| = \frac{|z|^{1/3}}{|z+a||z+b|} \leq \frac{\sqrt[3]{\rho}}{(a-\rho)(b-\rho)}$$

so

$$\left|\int_{C_\rho} f(z)dz\right| \leq \frac{\sqrt[3]{\rho}}{(a-\rho)(b-\rho)}\cdot \pi\rho$$

since $\lim_{\rho\to 0} \pi\rho\sqrt[3]{\rho}/[(a-\rho)(b-\rho)] = 0$, by sandwich theorem $\int_{C_\rho} f(z)dz$ tends to zero as $\rho \to 0$. For all z such that $|z| = R$,

$$|f(z)| = \left|\frac{\sqrt[3]{z}}{(z+a)(z+b)}\right| = \frac{|z|^{1/3}}{|z+a||z+b|} \leq \frac{\sqrt[3]{R}}{(R-a)(R-b)}$$

so

$$\left|\int_{C_R} f(z)dz\right| \leq \frac{\sqrt[3]{R}}{(R-a)(R-b)} \cdot \pi R$$

since $\lim_{R\to\infty} \pi R\sqrt[3]{R}/[(R-a)(R-b)] = 0$, by sandwich theorem $\int_{C_R} f(z)dz$ tends to zero as $R \to \infty$. Then, we can take $\rho \to 0$, $R \to \infty$ limits to (5) and write

$$(1-e^{2\pi i/3})\int_0^\infty \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr = 2\pi i(-e^{i\pi/3})\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}$$

and we obtain the desired result.

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}dx = \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}$$

6 Section 91 #6

6.1 Proof for (a)

Let Γ be a contour created by joining Γ_R, L, Γ_ρ and contour connecting ρ to R. Since $f_1(z)$ has an isolated singularity at -1 and analytic anywhere else inside and on Γ, we can apply Cauchy's residue theorem. The linear contour connecting ρ to R can be parametrized as $z = r \quad (\rho \leq r \leq R)$, so

$$\begin{aligned}\int_\Gamma f_1(z)dz &= \int_\rho^R f(r)dr + \int_{\Gamma_R} f_1(z)dz + \int_L f_1(z)dz + \int_{\Gamma_\rho} f_1(z)dz \\ &= \int_\rho^R \frac{r^{-a}}{r+1}dr + \int_{\Gamma_R} f_1(z)dz + \int_L f_1(z)dz + \int_{\Gamma_\rho} f_1(z)dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z) \quad (6)\end{aligned}$$

6.2 Proof for (b)

Let γ be a contour created by joining γ_R, $-L$, γ_ρ and contour connecting R to ρ. Since $f_2(z)$ has an isolated singularity at -1 and analytic anywhere else except the branch cut $\theta = \pi/2$, we can apply Cauchy-Goursat theorem. The linear contour connecting R to ρ can be parametrized as $z = R + \rho - r \quad (\rho \leq r \leq R)$, so

$$\begin{aligned}\int_\gamma f_2(z)dz &= -\int_\rho^R f_2(R+\rho-r)dr + \int_{\gamma_\rho} f_2(z)dz + \int_{-L} f_2(z)dz + \int_{\gamma_R} f_2(z)dz \\ &= -\int_\rho^R f_2(r)dr + \int_{\gamma_\rho} f_2(z)dz - \int_L f_2(z)dz + \int_{\gamma_R} f_2(z)dz \\ &= -\int_\rho^R \frac{e^{-a(\ln|r|+i\arg r)}}{r+1}dr + \int_{\gamma_\rho} f_2(z)dz - \int_L f_2(z)dz + \int_{\gamma_R} f_2(z)dz \\ &= -\int_\rho^R \frac{r^{-a}e^{-i2a\pi}}{r+1}dr + \int_{\gamma_\rho} f_2(z)dz - \int_L f_2(z)dz + \int_{\gamma_R} f_2(z)dz = 0 \quad (7)\end{aligned}$$

6.3 Proof for (c)

Let $z = re^{i\theta} \quad (r > 0)$. Then we can write

$$\lim_{\theta\to 0+} f(z) = \lim_{\theta\to 0+} \frac{\exp(-a(\ln r + i\theta))}{re^{i\theta}+1} = \frac{r^{-a}}{r+1} \tag{8}$$

$$\lim_{\theta\to 2\pi-} f(z) = \lim_{\theta\to 2\pi-} \frac{\exp(-a(\ln r + i\theta))}{re^{i\theta}+1} = \frac{r^{-a}e^{-i2a\pi}}{r+1} \tag{9}$$

Since these limits exist, $f(z)$ is continuous on Γ when its value at $\theta = 0$ is defined according to (8). Thus, we can replace f_1 in (6) to f and write

$$\int_\rho^R \frac{r^{-a}}{r+1}dr + \int_{\Gamma_R} f(z)dz + \int_L f(z)dz + \int_{\Gamma_\rho} f(z)dz = 2\pi i \operatorname*{Res}_{z=-1} f(z) \tag{10}$$

Also, $f(z)$ is continuous on γ when its value at $\theta = 2\pi$ is defined according to (9). Then we can also replace f_2 in (7) to f and write

$$-\int_\rho^R \frac{r^{-a}e^{-i2a\pi}}{r+1}dr + \int_{\gamma_\rho} f(z)dz - \int_L f(z)dz + \int_{\gamma_R} f(z)dz = 0 \tag{11}$$

Adding both sides of (10) and (11), we can write

$$(1-e^{-i2a\pi})\int_\rho^R \frac{r^{-a}}{r+1}dr + \int_{C_R} f(z)dz + \int_{C_\rho} f(z)dz = 2\pi i \operatorname*{Res}_{z=-1} f(z)$$

where C_R and C_ρ are circles $|z| = R$ and $|z| = \rho$, respectively.

7 Section 92 #4

Using the substitution introduced in section 92, we can write

$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \int_C \frac{1}{1+a\left(\frac{z+z^{-1}}{2}\right)}\frac{dz}{iz} = \int_C \frac{2/(ia)}{z^2+(2/a)z+1}dz$$

where C is the positively oriented circle $|z| = 1$. The denominator of the integrand has zeros at

$$z_1 = \frac{-1+\sqrt{1-a^2}}{a},\ z_2 = \frac{-1-\sqrt{1-a^2}}{a}$$

We can write

$$\begin{aligned}
|z_2| - 1 &= \left|\frac{-1-\sqrt{1-a^2}}{a}\right| - 1 = \frac{|-1-\sqrt{1-a^2}|}{|a|} - 1 \\
&= \frac{1+\sqrt{1-a^2}}{|a|} - 1 = \frac{1-|a|+\sqrt{1-|a|^2}}{|a|} \\
&= \frac{\sqrt{1-|a|}(\sqrt{1-|a|}+\sqrt{1+|a|})}{|a|} > 0
\end{aligned}$$

Since $z_1 z_2 = 1$, we know that $|z_1| < 1$ so there is no singular point on C and only z_1 is an isolated singularity of $f(z)$ and it is interior to C. By Cauchy's residue theorem,

$$\int_C \frac{2/(ia)}{z^2 + (2/a)z + 1} dz = 2\pi i \operatorname*{Res}_{z=z_1} \frac{2/(ia)}{z^2 + (2/a)z + 1} \tag{12}$$

Let $q(z) = z^2 + (2/a)z + 1$. Since $q(z_1) = 0$ and $q'(z_1) \neq 0$, by theorem 2 in section 83 we can write

$$\operatorname*{Res}_{z=z_1} \frac{2/(ia)}{z^2 + (2/a)z + 1} = \frac{2/(ia)}{q'(z_1)} = \frac{1}{i\sqrt{1-a^2}}$$

and (12) gives us

$$\int_C \frac{2/(ia)}{z^2 + (2/a)z + 1} dz = 2\pi i \cdot \frac{1}{i\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$$