MATH210: Homework 8 (due May. 2)

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1 Section 68 #6

By Maclaurin series expansion of $1/(1-z)$, we can write

$$\frac{1}{1-\frac{z-1}{2}} = \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^n} \quad (|z-1| < 2) \tag{1}$$

By observing

$$\frac{z}{(z-1)(z-3)} + \frac{1}{2(z-1)} = \frac{z + \frac{z-3}{2}}{(z-1)(z-3)} = \frac{3}{2(z-3)} = -\frac{3}{4} \cdot \frac{1}{1-\frac{z-1}{2}}$$

where $0 < |z-1| < 2$ and using (1), we obtain

$$\frac{z}{(z-1)(z-3)} + \frac{1}{2(z-1)} = -\frac{3}{4} \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^n} = -3 \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} \quad (0 < |z-1| < 2)$$

and we get the desired result.

$$\frac{z}{(z-1)(z-3)} = -3 \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)} \quad (0 < |z-1| < 2)$$

2 Section 68 #7

2.1 Solution for (a)

By Maclaurin series expansion of $1/(1-z)$ and since $|a| < |z| < \infty$ implies $|a/z| < 1$, we get

$$\frac{a}{z-a} = \frac{a}{z} \cdot \frac{1}{1-a/z} = \frac{a}{z} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n+1} = \sum_{n=1}^{\infty} \left(\frac{a}{z}\right)^n \quad (|a| < |z| < \infty)$$

2.2 Solution for (b)

If we write $z = e^{i\theta}$, $|z| = 1$ so from $-1 < a < 1$, $|a| < |z| < \infty$ holds. Then we can write

$$\frac{a}{z-a} = \frac{a}{e^{i\theta} - a} = \sum_{n=1}^{\infty} \left(\frac{a}{e^{i\theta}}\right)^n = \sum_{n=1}^{\infty} a^n e^{-in\theta} \quad (-1 < a < 1)$$

We can also write

$$
\begin{aligned}
\frac{a}{z-a} &= \frac{a(\overline{z}-a)}{(z-a)(\overline{z}-a)} = \frac{a}{|z|^2 - a(z+\overline{z}) + a^2}(\overline{z}-a) \\
&= \frac{a(\overline{z}-a)}{|z|^2 - 2a\operatorname{Re} z + a^2} = \frac{a(e^{-i\theta}-a)}{1-2a\cos\theta + a^2} = \frac{a(\cos\theta - i\sin\theta) - a^2}{1-2a\cos\theta+a^2} \\
&= \frac{a\cos\theta - a^2}{1-2a\cos\theta+a^2} - \frac{ia\sin\theta}{1-2a\cos\theta+a^2} \quad (-1<a<1)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=1}^{\infty} a^n e^{-in\theta} &= \sum_{n=1}^{\infty} a^n(\cos(-n\theta) + i\sin(-n\theta)) \\
&= \sum_{n=1}^{\infty} a^n(\cos(n\theta) - i\sin(n\theta)) \\
&= \sum_{n=1}^{\infty} a^n\cos(n\theta) - i\sum_{n=1}^{\infty} a^n\sin(n\theta) \quad (-1<a<1)
\end{aligned}
$$

so we get

$$
\begin{aligned}
\operatorname{Re}\sum_{n=1}^{\infty} a^n e^{-in\theta} &= \sum_{n=1}^{\infty} a^n\cos(n\theta) = \operatorname{Re}\frac{a}{z-a} = \frac{a\cos\theta - a^2}{1-2a\cos\theta+a^2} \quad (-1<a<1) \\
\operatorname{Im}\sum_{n=1}^{\infty} a^n e^{-in\theta} &= -\sum_{n=1}^{\infty} a^n\sin(n\theta) = \operatorname{Im}\frac{a}{z-a} = -\frac{a\sin\theta}{1-2a\cos\theta+a^2} \quad (-1<a<1)
\end{aligned}
$$

and the desired result.

$$
\begin{aligned}
\sum_{n=1}^{\infty} a^n\cos(n\theta) &= \frac{a\cos\theta - a^2}{1-2a\cos\theta+a^2} \\
\sum_{n=1}^{\infty} a^n\sin(n\theta) &= \frac{a\sin\theta}{1-2a\cos\theta+a^2}
\end{aligned}
$$

where $-1 < a < 1$.

3 Section 72 #3

Using the Maclaurin series expansion of $1/(1-z)$,

$$
\begin{aligned}
\frac{1}{z} &= \frac{1}{2+(z-2)} = \frac{1}{2}\cdot\frac{1}{1+(z-2)/2} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n \quad \left(\left|-\frac{z-2}{2}\right| < 1\right) \qquad (2)
\end{aligned}
$$

Differentiating $1/z$ yields $-1/z^2$, and by differentiating (2), we get

$$\begin{aligned}
-\frac{1}{z^2} = \frac{1}{2}\frac{d}{dz}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n &= \frac{1}{2}\frac{d}{dz}\sum_{n=0}^{\infty}\left[\left(-\frac{1}{2}\right)^n (z-2)^n\right] \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\left[n\left(-\frac{1}{2}\right)^n (z-2)^{n-1}\right] \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left[(n+1)\left(-\frac{1}{2}\right)^{n+1} (z-2)^n\right] \\
&= -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \quad \left(\left|-\frac{z-2}{2}\right| < 1\right)
\end{aligned}$$

The radius of convergence does not change after differentiation by theorem 2 in section 71. Since $|-(z-2)/2| < 1$ is equivalent to $|z-2| < 2$, we get the desired result.

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \quad (|z-2| < 2)$$

4 Section 72 #6

We know that $\operatorname{Log} z$ is an antiderivative, $1/z$ on the domain $D := \{z \in \mathbb{C};\ |z-1| < 1\}$ since

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z} \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

holds for all z such that $|z-1| < 1$. By the theorem in section 48, for all contours lying in D extending from 1 to z, we can write

$$\int_C \frac{1}{w}dw = \int_1^z \frac{1}{w}dw = \operatorname{Log} z - \operatorname{Log} 1 = \operatorname{Log} z$$

Also, using theorem 1 in section 71, we can write

$$\int_C \sum_{n=0}^{\infty}(-1)^n(w-1)^n dw = \sum_{n=0}^{\infty}(-1)^n\int_C (w-1)^n dw \quad (|z-1| < 1) \tag{3}$$

for all contour C interior to the circle $|z-1| < 1$. Since $(w-1)^n$ has its antiderivative $(n+1)^{-1}(w-1)^{n+1}$ for all integer $n \geq 0$, for all contour C in the circle $|z-1| < 1$ extending from 1 to z, we can write (3) as

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\int_C (w-1)^n dw &= \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \quad (|z-1| < 1)
\end{aligned}$$

and get the desired result.

$$\operatorname{Log} z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \quad (|z-1| < 1) \tag{4}$$

5 Section 72 #8

By definition, f is differentiable in some neighborhood of z_0, which means that there exists some $r > 0$ such that f is differentiable for all z such that $|z - z_0| < r$. Then we can write

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad (|z - z_0| < r) \tag{5}$$

Since $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, (5) can be written as

$$f(z) = \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad (|z - z_0| < r)$$

so

$$\begin{aligned} \frac{f(z)}{(z - z_0)^{m+1}} &= \frac{1}{(z - z_0)^{m+1}} \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \\ &= \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!} \frac{(z - z_0)^n}{(z - z_0)^{m+1}} \\ &= \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^{n-m-1} \quad (0 < |z - z_0| < r) \end{aligned}$$

Then, we can write $g(z)$ as a series.

$$g(z) = \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^{n-m-1} \quad (|z - z_0| < r) \tag{6}$$

By the corollary in section 71, we can conclude that $g(z)$ is analytic at z such that $|z - z_0| < r$, so $g(z)$ is analytic at z_0.

6 Section 73 #6

6.1 Proof for (a)

Let f be a function where

$$f(z) = \begin{cases} \frac{\sinh z}{z} & (z \neq 0) \\ 1 & (z = 0) \end{cases}$$

Then we can write

$$f(z) = 1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots = \sum_{n=0}^{\infty} a_n z^n \quad (0 < |z| < \infty) \tag{7}$$

and

$$a_n = \begin{cases} \frac{1}{(n+1)!} & (n = 2k) \\ 0 & (n = 2k + 1) \end{cases}$$

where k is nonnegative integer. The series in the right hand side of (7) converges if $z = 0$, so we can write

$$f(z) = 1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < \infty)$$

By the corollary in section 71, f is analytic everywhere. From $f(0) \neq 0$ and $\sinh z/z \neq 0$ for $0 < |z| < \pi$, we can conclude that $f(z) \neq 0$ for all z such that $|z| < \pi$. So can write

$$\frac{1}{f(z)} = \frac{1}{1 + z^2/3! + z^4/5! + \ldots} = d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \ldots \quad (|z| < \pi)$$

and calculate the Cauchy product as follows:

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} d_n z^n\right) = \sum_{n=0}^{\infty} b_n z^n \quad (|z| < \pi)$$

where

$$b_n = \sum_{k=0}^{n} a_k d_{n-k} \quad (|z| < \pi)$$

so we can write

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} d_n z^n\right) - 1 &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k d_{n-k}\right) z^n - 1 \\
&= (a_0 d_0 - 1) + (a_0 d_1 + a_1 d_0) z \\
&+ (a_0 d_2 + a_1 d_1 + a_2 d_0) z^2 \\
&+ (a_0 d_3 + a_1 d_2 + a_2 d_1 + a_3 d_0) z^3 \\
&+ (a_0 d_4 + a_1 d_3 + a_2 d_2 + a_3 d_1 + a_4 d_0) z^4 + \ldots \\
&= (d_0 - 1) + d_1 z + \left(d_2 + \frac{1}{3!} d_0\right) z^2 + \left(d_3 + \frac{1}{3!} d_1\right) z^3 \\
&+ \left(d_4 + \frac{1}{3!} d_2 + \frac{1}{5!} d_0\right) z^4 + \ldots \qquad (8) \\
&= 0 \quad (|z| < \pi)
\end{aligned}$$

6.2 Solution for (b)

Solving (8), we obtain

$$d_0 = 1, d_1 = 0, d_2 = -\frac{1}{3!} = -\frac{1}{6}, d_3 = 0, d_4 = -\frac{1}{3!} d_2 - \frac{1}{5!} d_0 = \frac{7}{360}$$

7 Section 73 #8

7.1 Solution for (a)

Let $g(z) = f(f(z))$. We can write

$$\begin{aligned} g(0) &= f(f(0)) = 0 \\ \frac{d}{dz}\Big|_{z=0} g(z) &= f'(0)f'(f(0)) = 1 \\ \frac{d^2}{dz^2}\Big|_{z=0} g(z) &= f''(0)f'(f(0)) + (f'(0))^2 f''(f(0)) = 4a_2 \\ \frac{d^3}{dz^3}\Big|_{z=0} g(z) &= f^{(3)}(0)f'(f(0)) + 3f'(0)f''(0)f''(f(0)) \\ &\quad + (f'(0))^2 f'(0) f^{(3)}(f(0)) = 12a_2^2 + 12a_3 \end{aligned}$$

and

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n = g(0) + g'(0)z + \frac{g''(0)}{2!}z^2 + \frac{g^{(3)}(0)}{3!}z^3 + \dots \\ &= z + 2a_2 z^2 + 2(a_2^2 + a_3)z^3 + \dots \quad (|z| < \infty) \end{aligned} \tag{9}$$

7.2 Solution for (b)

We write

$$\begin{aligned} f(f(z)) &= f(z) + a_2(f(z))^2 + a_3(f(z))^3 + \dots \\ &= (z + a_2 z^2 + a_3 z^3 + \dots) + a_2(z + a_2 z^2 + a_3 z^3 + \dots)^2 \\ &+ a_3(z + a_2 z^2 + a_3 z^3 + \dots)^3 + \dots \\ &= (z + a_2 z^2 + a_3 z^3 + \dots) + (a_2 z^2 + 2a_2^2 z^3 + \dots) + (a_3 z^3 + \dots) + \dots \\ &= z + 2a_2 z^2 + 2(a_2^2 + a_3)z^3 + \dots \quad (|z| < \infty) \end{aligned}$$

7.3 Solution for (c)

From Maclaurin series expansion of $\sin z$, we can write

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

Using (9),

$$\sin(\sin z) = z + 2 \cdot 0 \cdot z^2 + 2\left(0^2 - \frac{1}{3!}\right) z^3 + \dots = z - \frac{1}{3}z^3 + \dots \quad (|z| < \infty)$$