MATH210: Homework 3 (due Mar. 21)

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1 Section 29 #5

Assume that $f(x)$ is pure imaginary at each point x on the segment of real axis lying in D. Let $F(z) = -\overline{f(\overline{z})}$. The components of the functions can be written as

$$f(z) = u(x, y) + iv(x, y), \quad F(z) = U(x, y) + iV(x, y)$$

where $z = x + iy$. By definition of $F(z)$, we can write

$$F(z) = -\overline{f(\overline{z})} = -u(x, -y) + iv(x, -y)$$

So $U(x, y)$ and $V(x, y)$ can be written as follows:

$$U(x, y) = -u(x, t), \quad V(x, y) = v(x, t) \tag{1}$$

Where $t = -y$. Since $f(x + it)$ is an analytic function of $x + it$, the first order partial derivatives of the functions $u(x, t)$ and $v(x, t)$ are continuous throughout D and satisfies the Cauchy-Riemann equations.

$$u_x = v_t, \quad u_t = -v_x \tag{2}$$

Also, we can write

$$U_x(x, y) = -u_x(x, t), \quad V_y(x, y) = v_t(x, t)\frac{dt}{dy} = -v_t(x, t)$$
$$U_y(x, y) = -u_t(x, t)\frac{dt}{dy} = u_t(x, t), \quad V_x(x, y) = v_x(x, t)$$

By (2), we can see that

$$U_x = V_y, \quad U_y = -V_x$$

By the sufficient conditions of differentiability, F is differentiable throughout D, so it is also analytic. Moreover, since $f(x)$ is pure imaginary on the segment of the imaginary axis lying in D, $u(x, 0) = 0$ holds on that segment. Using (1), we can write

$$F(x) = U(x, 0) + iV(x, 0) = -u(x, 0) + iv(x, 0) = iv(x, 0) = f(x)$$

Thus, we can conclude that $F(z) = f(z)$ at each point in the segment. By the theorem in section 28, $F(z) = f(z)$ holds throughout D, so $f(z) = -\overline{f(\overline{z})}$.

Let's prove the converse. Assume that $\overline{f(z)} = -f(\overline{z})$, then it can be written as:

$$u(x, y) - iv(x, y) = -u(x, -y) - iv(x, -y)$$

If $(x, 0)$ is a point on the segment of the real axis lying on D,

$$u(x, 0) - iv(x, 0) = -u(x, 0) - iv(x, 0)$$

We see that $u(x, 0) = 0$. Hence $f(x)$ is pure imaginary on the segment of the imaginary axis lying on D.

2 Section 30 #9

Assume that $z = n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$. We can write

$$\begin{aligned}\overline{\exp(iz)} &= \overline{\cos z + i \sin z} = \overline{\cos(n\pi) + i \sin(n\pi)} = \cos(n\pi) - i \sin(n\pi) = \cos(n\pi) \\ &= \exp(iz) = \exp(i\overline{z})\end{aligned}$$

and get the desired result.

Let's prove the converse. Assume that $\overline{\exp(iz)} = \exp(i\overline{z})$ holds. The following holds for $z = x + iy$.

$$\begin{aligned}\overline{\exp(iz)} &= \overline{\exp(ix - y)} = \overline{\exp(-y)\exp(ix)} = e^{-y}\overline{\exp(ix)} = e^{-y}\overline{\cos x + i \sin x} \\ &= e^{-y}(\cos x - i \sin x) = e^{-y}(\cos(-x) + i \sin(-x)) = e^{-y}\exp(-ix) \qquad (3)\end{aligned}$$

We can also write

$$\exp(i\overline{z}) = \exp i(x - iy) = e^{y}\exp(ix) \qquad (4)$$

From (3) and (4), $\overline{\exp(iz)}\exp(i\overline{z}) = 1$ holds. For $\overline{\exp(iz)} = \exp(i\overline{z})$ to hold, from

$$|\overline{\exp(iz)}\exp(i\overline{z})| = 1$$

$|\overline{\exp(iz)}| = |\exp(i\overline{z})| = 1$ should hold. We can write

$$|\overline{\exp(iz)}| = |e^{-y}\exp(-ix)| = e^{-y} = 1, \; |\exp(i\overline{z})| = |e^{y}\exp(ix)| = e^{y} = 1$$

So we know that $y = 0$. Then, $\overline{\exp(iz)} = \exp(i\overline{z})$ so $\exp(-ix) = \exp(ix)$. We can write

$$\exp(-ix) = \cos(-x) + i \sin(-x) = \cos x - i \sin x, \quad \exp(ix) = \cos x + i \sin x$$

$\sin x = 0$ is satisfied for $x = 0, \pm\pi, \pm 2\pi, \ldots$, so the converse is also true.

3 Section 30 #13

Since f is analytic in D, u and v have continuous partial derivatives in D and they satisfy the Cauchy-Riemann equations.

$$u_x = v_y, \quad u_y = -v_x \qquad (5)$$

Using rules for differentiation, we get

$$\begin{aligned}U_x(x,y) &= u_x(x,y)e^{u(x,y)}\cos v(x,y) - v_x(x,y)e^{u(x,y)}\sin v(x,y) \\ U_y(x,y) &= u_y(x,y)e^{u(x,y)}\cos v(x,y) - v_y(x,y)e^{u(x,y)}\sin v(x,y) \\ V_x(x,y) &= u_x(x,y)e^{u(x,y)}\sin v(x,y) + v_x(x,y)e^{u(x,y)}\cos v(x,y) \\ V_y(x,y) &= u_y(x,y)e^{u(x,y)}\sin v(x,y) + v_y(x,y)e^{u(x,y)}\cos v(x,y)\end{aligned}$$

Using (5), we can write

$$\begin{aligned}U_x(x,y) &= u_x(x,y)e^{u(x,y)}\cos v(x,y) + u_y(x,y)e^{u(x,y)}\sin v(x,y) \\ U_y(x,y) &= u_y(x,y)e^{u(x,y)}\cos v(x,y) - u_x(x,y)e^{u(x,y)}\sin v(x,y) \\ V_x(x,y) &= u_x(x,y)e^{u(x,y)}\sin v(x,y) - u_y(x,y)e^{u(x,y)}\cos v(x,y) \\ V_y(x,y) &= u_y(x,y)e^{u(x,y)}\sin v(x,y) + u_x(x,y)e^{u(x,y)}\cos v(x,y)\end{aligned}$$

From this, we know that

$$U_x = V_y, \quad U_y = -V_x$$

Let $F(z) = U(x,y) + iV(x,y)$. Then U and V have continuous partial derivatives in D, and they satisfy the Cauchy-Riemann equations. By the sufficient condition for differentiability, F is an analytic function. Using the theorem in section 27, the component functions U and V are harmonic in D.

4 Section 30 #14

4.1 Solution for (a)

For any nonzero $x \in \mathbb{C}$, $x^1 = x$ and we know that $x^0 = x^{1-1} = x/x = 1$.

Then $(e^z)^0 = e^{0 \cdot z} = 1$ since $e \neq 0$ and $e^z \neq 0$.

Suppose that $(e^z)^k = e^{kz}$ for $k \geq 0$. Using the rules of exponentation,

$$(e^z)^{k+1} = (e^z)^k (e^z)^1 = e^{kz} e^z = e^{(k+1)z}$$

By mathematical induction, we can see that $(e^z)^n = e^{nz}$ holds for all nonnegative integer n.

4.2 Solution for (b)

For $m = -n = 1, 2, \ldots$, $(e^z)^n = (1/e^z)^m$ holds by the definition of negative integer in section 8. By the result proved in (a), $(1/e^z)^m = 1/(e^z)^m = 1/e^{mz}$ holds. From property 6 in section 30, it follows that $1/e^{mz} = e^{-mz} = e^{nz}$. So, $(e^z)^n = e^{nz}$ also holds for negative integer n.

5 Section 33 #2 (c)

If $-1 + \sqrt{3}i$ then $r = 2$ and $\Theta = 2\pi/3$. Hence

$$\log(-1 + \sqrt{3}i) = \ln r + i(\Theta + 2n\pi) = \ln 2 + i\left(\frac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$$

where $n = 0, \pm 1, \pm 2, \ldots$.

6 Section 33 #7

Let $z = x + yi = re^{i\theta}$. As written in section 7, $r = |z| = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$ holds, so $\theta = \tan^{-1}(y/x)$ satisfies $x + yi = re^{i\theta}$. Using the definition in section 33, we can write

$$\begin{aligned} \log z &= \ln r + i\theta \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi) \\ &= \ln \sqrt{x^2 + y^2} + i \tan^{-1}\left(\frac{y}{x}\right) \\ &= \frac{1}{2}\ln(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

We can write the component functions as follows:

$$u(x,y) = \frac{1}{2}\ln(x^2 + y^2), \quad v(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

u and v has continuous partial derivatives in its domain, so we can differentiate them.

$$\begin{aligned} u_x(x,y) &= \frac{x}{x^2+y^2}, \quad u_y(x,y) = \frac{y}{x^2+y^2} \\ v_x(x,y) &= -\frac{y}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{x^2+y^2} \\ v_y(x,y) &= \frac{1}{x}\frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} \end{aligned}$$

Then $u_x = v_y, u_y = -v_x$ so the Cauchy-Riemann equation holds, and $\log z$ is analytic in its domain by the sufficient conditions of differentiability. Thus, $\log z$ is differentiable in its domain and we can write

$$\frac{d}{dz}\log z = u_x(x,y) + iv_x(x,y) = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = \frac{x-yi}{x^2+y^2} = \frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}$$

and get the desired result.

7 Section 33 #8

By the definition in section 31, we can write

$$\log z = \ln|z| + i\arg z$$

Since $\log z = i\pi/2$, it is clear that $|z| = 1$ and $\arg z = \pi/2 + 2n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$. Then z can be written as

$$z = 1 \cdot \exp\left(i\left(\frac{\pi}{2} + 2n\pi\right)\right) = e^{i\pi/2} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

and get the root.