

Homework 1 (due Mar. 7)

손량(20220323)

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1 Section 20 #4

By definition, $f'(z_0), g'(z_0)$ can be written as follows:

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}, g'(z_0) = \lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0}$$

By properties of limits,

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}}{\lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0}} = \lim_{z \rightarrow z_0} \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}} = \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)}$$

2 Section 20 #9

Δw can be written as follows, if $z = 0$.

$$\Delta w = f(z + \Delta z) - f(z) = f(\Delta z) - f(0) = f(\Delta z) = \frac{\overline{\Delta z}^2}{\Delta z}$$

Then $\Delta w / \Delta z = (\overline{\Delta z} / \Delta z)^2$. If Δz is on the real axis, $\overline{\Delta z} = \Delta z$ and $(\overline{\Delta z} / \Delta z)^2 = (\Delta z / \Delta z)^2 = 1$. If Δz is on the imaginary axis, $\overline{\Delta z} = -\Delta z$ and $\Delta w / \Delta z = (\overline{\Delta z} / \Delta z)^2 = (-\Delta z / \Delta z)^2 = 1$. However, on each nonzero point on line $\Delta y = \Delta x$ where Δz is written as $\Delta x + i\Delta y$,

$$\frac{\Delta w}{\Delta z} = \left(\frac{\overline{\Delta z}}{\Delta z} \right)^2 = \left(\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \right)^2 = \left(\frac{1 - i}{1 + i} \right)^2 = (-i)^2 = -1$$

If the limit of $\Delta w / \Delta z$ exists, it can be found by letting Δz approach the origin in any manner. However, the limit is 1 if Δz approached the origin along the real or imaginary axis, or -1 if Δz approached the origin along the line $\Delta y = \Delta x$. Since limits are unique, it is a contradiction and $f'(0)$ does not exist.

3 Section 24 #3 (c)

$f(z)$ can be written as follows:

$$f(z) = f(x + iy) = (x + iy)y = xy + iy^2$$

Let $u(x, y) := xy$ and $v(x, y) = y^2$, then $f(z) = u(x, y) + iv(x, y)$. If f is differentiable at $z_0 = x_0 + iy_0$, the first-order partial derivatives of u and v must exist at (x_0, y_0) , and

Cauchy-Riemann equation must be satisfied. Since the partial derivatives exist at every (x_0, y_0) since u and v are polynomial functions, let's check the Cauchy-Riemann equation.

$$\begin{aligned}u_x(x_0, y_0) &= v_y(x_0, y_0) \iff y_0 = 2y_0 \\u_y(x_0, y_0) &= -v_x(x_0, y_0) \iff x_0 = 0\end{aligned}$$

Only $x_0 = 0, y_0 = 0$ satisfies the equation. f cannot be differentiable at points other than 0. Let's check if it is differentiable at 0:

$$\begin{aligned}\lim_{\Delta z \rightarrow 0} \frac{f(\Delta z) - f(0)}{\Delta z - 0} &= \lim_{\Delta z \rightarrow 0} \frac{f(\Delta z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta z \operatorname{Im} \Delta z}{\Delta z} \\&= \lim_{\Delta z \rightarrow 0} \operatorname{Im} \Delta z = 0\end{aligned}$$

f is only differentiable at 0, and $f'(0) = 0$.

4 Section 24 #7

4.1 Solution for (a)

According to problem #6, the following holds for $f'(z_0)$ where $z_0 = r_0 \exp(i\theta_0)$ and f is differentiable at z_0 : (u_r and v_r are evaluated at (r_0, θ_0))

$$f'(z_0) = u_x + iv_x = e^{-i\theta_0}(u_r + iv_r)$$

By the polar form of Cauchy-Riemann equation, u_r and v_r can be written as follows (u_θ and v_θ are evaluated at (r_0, θ_0))

$$u_r = v_\theta/r, \quad v_r = -u_\theta/r$$

Substitution gives us the result:

$$f'(z_0) = \frac{1}{r_0 e^{i\theta_0}}(v_\theta - iu_\theta) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

4.2 Solution for (b)

$f(z)$ can be written as follows, where $z = r \exp(i\theta) \neq 0$:

$$f(z) = \frac{1}{z} = \frac{1}{r e^{i\theta}} = \frac{1}{r}(\cos(-\theta) + i \sin(-\theta)) = \frac{1}{r}(\cos \theta - i \sin \theta)$$

The component functions are

$$u(r, \theta) = \frac{\cos \theta}{r}, \quad v(r, \theta) = \frac{-\sin \theta}{r}$$

Since $f(z)$ is differentiable in $\mathbb{C} \setminus \{0\}$, we can use the relation in (a). Plugging those to the expression derived in (a), we get the desired result.

$$\begin{aligned}
f'(z) &= \frac{-i}{z}(u_\theta + iv_\theta) = \frac{-i}{z} \left(\frac{-\sin \theta}{r} + i \frac{-\cos \theta}{r} \right) \\
&= -\frac{i}{zr} (\sin(-\theta) - i \cos(-\theta)) = -\frac{1}{zr} (\cos(-\theta) + i \sin(-\theta)) \\
&= -\frac{1}{zre^{i\theta}} = -\frac{1}{z^2}
\end{aligned}$$

5 Section 24 #8

5.1 Solution for (a)

Using the chain rule,

$$\begin{aligned}
\frac{\partial F}{\partial \bar{z}} &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{\partial F}{\partial x} \left(\frac{\partial}{\partial \bar{z}} \frac{z + \bar{z}}{2} \right) + \frac{\partial F}{\partial y} \left(\frac{\partial}{\partial \bar{z}} \frac{z - \bar{z}}{2i} \right) \\
&= \frac{1}{2} \frac{\partial F}{\partial x} - \frac{1}{2i} \frac{\partial F}{\partial y} = \frac{1}{2} \left(\frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right)
\end{aligned}$$

5.2 Solution for (b)

Using the defined operator,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} [(u_x + iv_x) + i(u_y + iv_y)] = \frac{1}{2} [(u_x - v_y) + i(v_x + u_y)]$$

Since the first order derivatives of the real and imaginary components of f satisfy the Cauchy-Riemann equations,

$$u_x = v_y, \quad u_y = -v_x$$

Both $u_x - v_y$ and $v_x + u_y$ are zero, giving us the desired result, $\partial f / \partial \bar{z} = 0$.