Homework 1 (due Mar. 7)

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1 Section 20 #4

By definition, $f'(z_0), g'(z_0)$ can be written as follows:

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0},\; g'(z_0) = \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}$$

By properties of limits,

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}{\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}} = \lim_{z\to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \lim_{z\to z_0} \frac{f(z)}{g(z)}$$

2 Section 20 #9

Δw can be written as follows, if $z = 0$.

$$\Delta w = f(z+\Delta z) - f(z) = f(\Delta z) - f(0) = f(\Delta z) = \frac{\overline{\Delta z}^2}{\Delta z}$$

Then $\Delta w/\Delta z = \left(\overline{\Delta z}/\Delta z\right)^2$. If Δz is on the real axis, $\overline{\Delta z} = \Delta z$ and $\left(\overline{\Delta z}/\Delta z\right)^2 = (\Delta z/\Delta z)^2 = 1$. If Δz is on the imaginary axis, $\overline{\Delta z} = -\Delta z$ and $\Delta w/\Delta z = \left(\overline{\Delta z}/\Delta z\right) = (-\Delta z/\Delta z)^2 = 1$. However, on each nonzero point on line $\Delta y = \Delta x$,

$$\frac{\Delta w}{\Delta z} = \left(\frac{\overline{\Delta z}}{\Delta z}\right)^2 = \left(\frac{\Delta x - i\Delta x}{\Delta x + i\Delta x}\right)^2 = \left(\frac{1-i}{1+i}\right)^2 = (-i)^2 = -1$$

If the limit of $\Delta w/\Delta z$ exists, it can be found by letting Δz approach the origin in any manner. However, the limit is 1 if Δz approached the origin along the real or imaginary axis, or -1 if Δz approached the origin along the line $\Delta y = \Delta x$. Hence, the limit does not exist and $f'(0)$ also does not exist.

3 Section 24 #3 (c)

$f(z)$ can be written as follows:

$$f(z) = f(x+iy) = (x+iy)y = xy + iy^2$$

Let $u(x,y) := xy$ and $v(x,y) = y^2$. If f is differentible at $z_0 = x_0 + iy_0$, the first-order partial derivatives of u and v must exist at (x_0, y_0), and Cauchy-Riemann equation must be satisfied. Since the partial derivatives exist at every (x_0, y_0), let's check the Cauchy-Riemann equation.

$$\begin{aligned} u_x(x_0, y_0) = v_y(x_0, y_0) &\iff y_0 = 2y_0 \\ u_y(x_0, y_0) = -v_x(x_0, y_0) &\iff x_0 = 0 \end{aligned}$$

Only $(0, 0)$ satistifes the equation. f cannot be differentible at points other than $(0, 0)$. Let's check if it is differentible at $(0, 0)$:

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z - 0} &= \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z \operatorname{Im} \Delta z}{\Delta z} \\ &= \lim_{\Delta z \to 0} \operatorname{Im} \Delta z = 0 \end{aligned}$$

f is only differentible at $(0, 0)$, and $f'(0) = 0$.

4 Section 24 #7

4.1 Solution for (a)

According to problem #6, the following holds for $f'(z)$:

$$f'(z) = u_x + iv_x = e^{-i\theta}(u_r + iv_r)$$

By the polar form (6) in section 24, u_r and v_r can be written as follows:

$$u_r = v_\theta / r, \quad v_r = -u_\theta / r$$

Substitution gives us the result:

$$f'(z) = \frac{1}{re^{i\theta}}(v_\theta - iu_\theta) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

4.2 Solution for (b)

$f(z)$ can be written as follows, where $z \neq 0$:

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}(\cos\theta - i\sin\theta)$$

The component functions are

$$u(r, \theta) = \frac{\cos\theta}{r}, \quad v(r, \theta) = \frac{-\sin\theta}{r}$$

Plugging those to the expression derived in (a), we get the desired result.

$$\begin{aligned} f'(z) = \frac{-i}{z}(u_\theta + iv_\theta) &= \frac{-i}{z}\left(\frac{-\sin\theta}{r} + i\frac{-\cos\theta}{r}\right) \\ &= -\frac{i}{zr}(\sin(-\theta) - i\cos(-\theta)) = -\frac{1}{zr}(\cos(-\theta) + i\sin(-\theta)) \\ &= -\frac{1}{zre^{i\theta}} = -\frac{1}{z^2} \end{aligned}$$

5 Section 24 #8

5.1 Solution for (a)

Using the chain rule,

$$\begin{aligned}\frac{\partial F}{\partial \overline{z}} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{\partial F}{\partial x}\left(\frac{\partial}{\partial \overline{z}}\frac{z+\overline{z}}{2}\right) + \frac{\partial F}{\partial y}\left(\frac{\partial}{\partial \overline{z}}\frac{z-\overline{z}}{2i}\right)\\ &= \frac{1}{2}\frac{\partial F}{\partial x} - \frac{1}{2i}\frac{\partial F}{\partial y} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)\end{aligned}$$

5.2 Solution for (b)

Using the defined operator,

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left[(u_x + iv_x) + i(u_y + iv_y)\right] = \frac{1}{2}\left[(u_x - v_y) + i(v_x + u_y)\right]$$

Since the first order derivatives of f satisfy the Cauchy-Riemann equations,

$$u_x = v_y, \quad u_y = -v_x$$

Both $u_x - v_y$ and $v_x + u_y$ are zero, giving us the desired result, $\partial f/\partial \overline{z} = 0$.