MATH210: Homework 4 (due Mar. 28)

손량(20220323)

Last compiled on: Sunday 26$^{\text{th}}$ March, 2023, 22:29

1 Section 33 #12

Let $z-1=(x-1)+iy=r\exp(i\theta)$. By the definition of logarithm in section 33, we can write

$$\log(z-1)=\ln(r)+i(\Theta+2n\pi)\quad(n=0,\pm1,\pm2,\dots)$$

Where θ has any one of values $\theta=\Theta+2n\pi$ $(n=0,\pm1,\pm2,\dots)$, and $\Theta=\operatorname{Arg}z$. Se we can write

$$\begin{aligned}\operatorname{Re}[\log(z-1)]&=\operatorname{Re}[\ln r+i(\Theta+2n\pi)]\quad(n=0,\pm1,\pm2,\dots)\\&=\ln r=\ln|z-1|=\ln\sqrt{(x-1)^2+y^2}=\frac{1}{2}\ln[(x-1)^2+y^2]\end{aligned}$$

We can also write

$$\log z=\ln r+i\theta\quad(r>0,\alpha<\theta<\alpha+2\pi)\tag{1}$$

where α is some real number. Then, $\log z$ in (1) can be considered as a single-valued function. As shown in section 33, $\log z$ is analytic throughout the domain $r>0,\alpha<\theta<\alpha+2\pi$. Thus, $\log(z-1)$ is analytic throughout the domain $z\neq1$, and a component function, $\operatorname{Re}[\log(z-1)]$ is harmonic in $z\neq1$ by the theorem in section 27.

2 Section 36 #1 (b)

As shown in section 35, we can write

$$\begin{aligned}(1+i)^i=e^{i\log(1+i)}&=\exp(i[\ln|1+i|+i(\operatorname{Arg}(1+i)+2n\pi)])\\&=\exp\left(i^2\left(\frac{\pi}{4}+2n\pi\right)\right)\exp(i\ln\sqrt{2})\\&=\exp\left(-\frac{\pi}{4}-2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right)\end{aligned}\tag{2}$$

where $n=0,\pm1,\pm2,\dots$. Then, (2) can be written as $\exp(-\pi/4+2n\pi)\exp(i\ln2/2)$ and we get the desired result.

3 Section 36 #6

As shown in section 35, we can calculate $|z^a|$ by using the definition in section 33. $(n = 0, \pm 1, \pm 2, \ldots)$

$$\begin{aligned}|z^a| &= |\exp(a \log z)| = |\exp(a[\ln|z| + i(\operatorname{Arg} z + 2n\pi)])| \\ &= |\exp(a \ln|z|)||\exp(i(\operatorname{Arg} z + 2n\pi))| \\ &= |\exp(a \ln|z|)| \cdot 1 = |\exp(a \ln|z|)| = \exp(a \ln|z|)\end{aligned}$$

Also, we can calculate the principal value of $|z|^a$ as follows: (again, $n = 0, \pm 1, \pm 2, \ldots$)

$$\begin{aligned}|z|^a &= \exp(a \operatorname{Log}|z|) = \exp(a[\ln|z| + i(\operatorname{Arg}|z| + 2n\pi)]) \\ &= \exp(a(\ln|z| + 2ni\pi)) = \exp(a \ln|z|)e^{2ni\pi} = \exp(a \ln|z|)\end{aligned}$$

Then, we get the desired result.

4 Section 36 #7

As shown in section 35, we can write $(n = 0, \pm 1, \pm 2, \ldots)$

$$\begin{aligned}i^c &= \exp(c \log i) = \exp(c[\ln 1 + i(\operatorname{Arg} i + 2n\pi)]) = \exp\left(ic\left(\frac{\pi}{2} + 2n\pi\right)\right) \\ &= \exp\left(i(a+bi)\left(\frac{\pi}{2} + 2n\pi\right)\right) = \exp\left(ia\left(\frac{\pi}{2} + 2n\pi\right)\right)\exp\left(-b\left(\frac{\pi}{2} + 2n\pi\right)\right)\end{aligned}$$

We can see that i^c is multiple valued, as different values of n results in different values of i^c unless $c = 0, \pm 1, \pm 2, \ldots$.

We can calculate $|i^c|$ as follows:

$$\begin{aligned}|i^c| &= \left|\exp\left(ia\left(\frac{\pi}{2} + 2n\pi\right)\right)\exp\left(-b\left(\frac{\pi}{2} + 2n\pi\right)\right)\right| \\ &= \left|\exp\left(ia\left(\frac{\pi}{2} + 2n\pi\right)\right)\right|\left|\exp\left(-b\left(\frac{\pi}{2} + 2n\pi\right)\right)\right| \\ &= \left|\cos\left(a\left(\frac{\pi}{2} + 2n\pi\right)\right) + i\sin\left(a\left(\frac{\pi}{2} + 2n\pi\right)\right)\right|\left|\exp\left(-b\left(\frac{\pi}{2} + 2n\pi\right)\right)\right| \\ &= \left|\exp\left(-b\left(\frac{\pi}{2} + 2n\pi\right)\right)\right|\end{aligned}$$

Since $f(x) = e^x$ is an one-to-one function, $b = 0$ has to hold if the values of $|i^c|$ are all the same. Thus, c has to be real.

5 Section 38 #2

5.1 Solution for (a)

Using expression 4 in section 37, we can write

$$\begin{aligned}e^{iz_1}e^{iz_2} &= (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2) \\ &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) \qquad (3)\end{aligned}$$

Using relation 3 in section 37, we can plug $z_1 = -z_1, z_2 = -z_2$ to (3) and obtain

$$\begin{aligned}e^{-iz_1}e^{-iz_2} &= \cos(-z_1)\cos(-z_2) - \sin(-z_1)\sin(-z_2) \\ &\quad + i(\sin(-z_1)\cos(-z_2) + \cos(-z_1)\sin(-z_2)) \\ &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)\end{aligned}$$

5.2 Solution for (b)

Using the provided fact, we can write

$$\begin{aligned}\sin(z_1+z_2) =& \frac{1}{2i}\left(e^{iz_1}e^{iz_2} = e^{-iz_1}e^{-iz_2}\right)\\ =& \frac{1}{2i}\{[\cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)]\\ &- [\cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)]\}\\ =& \frac{1}{2i}\cdot 2i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2\end{aligned}$$

6 Section 38 #12 (a)

Let $z = x+iy$. The equality 13 in section 37 shows that the following holds:

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

From this, we can know that $\sin z \in \mathbb{R}$ if $y = 0$. Using the reflection principle from section 29, we can conclude that $\overline{\sin z} = \sin \overline{z}$ since $\sin z$ is an entire function.

7 Section 38 #14 (a)

Let $z = x+iy$. Using equality 14 in section 37, the following holds:

$$\cos(iz) = \cos(-y+ix) = \cos(-y)\cosh x - i\sin(-y)\sinh x = \cos y \cosh x + i \sin y \sinh x$$

From this, we can know that $\cos(iz) \in \mathbb{R}$ if $y = 0$. Using the reflection principle from section 29, we can conclude that $\overline{\cos(iz)} = \cos(i\overline{z})$ since $\cos(iz)$ is entire function.

8 Section 42 #2 (d)

Let $z = x+iy$, where $x > 0$. Then we can write

$$\begin{aligned}\int_0^\infty e^{-zt}dt = \int_0^\infty e^{-(x+iy)t}dt = \int_0^\infty e^{-xt}e^{-iyt}dt = \int_0^\infty e^{-xt}(\cos(-yt)+i\sin(-yt))dt\\ = \int_0^\infty e^{-xt}\cos(-yt)dt + i\int_0^\infty e^{-xt}\sin(-yt)dt\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}\int_0^u e^{-xt}\cos(-yt)dt &= \left[-\frac{1}{x}e^{-xt}\cos(-yt)\right]_0^u - \int_0^u \left(-\frac{1}{x}e^{-xt}\right)(y\sin(-yt))dt\\ &= \frac{1-e^{-xu}\cos(-yu)}{x} + \frac{y}{x}\int_0^u e^{-xt}\sin(-yt)dt \qquad (4)\end{aligned}$$

and we also get

$$\begin{aligned}\int_0^u e^{-xt}\sin(-yt)dt &= \left[-\frac{1}{x}e^{-xt}\sin(-yt)\right]_0^u - \int_0^u \left(-\frac{1}{x}e^{-xt}\right)(-y\cos(-yt))dt\\ &= -\frac{e^{-xu}\sin(-yu)}{x} - \frac{y}{x}\int_0^u e^{-xt}\cos(-yt)dt \qquad (5)\end{aligned}$$

Using (4) and (5), we get

$$\int_0^u e^{-xt}\cos(-yt)dt = -\frac{x\cos(uy) - y\sin(uy)}{x^2+y^2}e^{-ux} + \frac{x}{x^2+y^2}$$
$$\int_0^u e^{-xt}\sin(-yt)dt = \frac{y\cos(uy) + x\sin(uy)}{x^2+y^2}e^{-ux} - \frac{y}{x^2+y^2}$$

So we get

$$\begin{aligned}
\int_0^u e^{-zt} &= \int_0^u e^{-xt}\cos(-yt)dt + i\int_0^u e^{-xt}\sin(-yt)dt \\
&= \frac{(-x+iy)\cos(uy) + (y+ix)\sin(uy)}{x^2+y^2}e^{-ux} + \frac{x-iy}{x^2+y^2} \\
&= \frac{-\overline{z}\cos(uy) + i\overline{z}\sin(uy)}{z\overline{z}}e^{-ux} + \frac{\overline{z}}{z\overline{z}} \\
&= \frac{-\cos(uy) + i\sin(uy)}{z}e^{-ux} + \frac{1}{z}
\end{aligned}$$

Since $-e^{-ux} \le e^{-ux}\cos(uy) \le e^{-ux}$ and $\lim_{u\to\infty} e^{-ux} = 0$, $\lim_{u\to\infty} e^{-ux}\cos(uy) = 0$ holds by the sandwich theorem. Similarly, $-e^{-ux} \le e^{-ux}\sin(uy) \le e^{-ux}$ and $\lim_{u\to\infty} e^{-ux} = 0$ implies $\lim_{u\to\infty} e^{-ux}\sin(uy) = 0$ by sandwich theorem. Thus, we get the desired result.

$$\int_0^\infty e^{-zt} = \lim_{u\to\infty}\int_0^u e^{-zt} = \lim_{u\to\infty}\left(\frac{-\cos(uy) + i\sin(uy)}{z}e^{-ux} + \frac{1}{z}\right) = \frac{1}{z}$$

9 Section 42 #3

We can write

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta = \int_0^{2\pi} e^{i(m-n)\theta} \tag{6}$$

If $m = n$, then (6) can be written as

$$\int_0^{2\pi} 1d\theta = 2\pi$$

If $m \neq n$, then (6) can be written follows using the Euler's formula.

$$\begin{aligned}
&\int_0^{2\pi} (\cos((m-n)\theta) + i\sin((m-n)\theta))d\theta \\
&= \int_0^{2\pi} \cos((m-n)\theta)d\theta + i\int_0^{2\pi} \sin((m-n)\theta)d\theta \\
&= \left[\frac{1}{m-n}\sin((m-n)\theta)\right]_0^{2\pi} - i\left[\frac{1}{m-n}\cos((m-n)\theta)\right]_0^{2\pi} \qquad = 0
\end{aligned}$$