MATH210 Homework 2 (due Mar. 14)

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Last compiled on: Tuesday 14$^{\text{th}}$ March, 2023, 19:44

1 Section 26 #2 (b)

Let's check for differentiability. We can write the component functions as follows:

$$u(x,y) := 2xy, \quad v(x,y) := x^2 - y^2$$

By applying the Cauchy-Riemann equation,

$$u_x = v_y \Longrightarrow 2y = -2y, \quad u_y = -v_x \Longrightarrow 2x = -2x$$

We can know that the function is only differentiable at 0, using the sufficient condition for differentiability. If f is analytic at z_0, it should be differentiable in some neighborhood of z_0. Thus, f cannot be analytic in nonzero z_0 since f is not differentiable in z_0, let alone a neighborhood of z_0. For $z_0 = 0$, there exists some $z \in D(0,r)$ for all $r > 0$ where f is not differentiable at z. In other words, f cannot be differentiable in any open set containing 0, so f is not analytic at $z_0 = 0$. In conclusion, f is nowhere analytic.

2 Section 26 #6

For all points in its domain, the component functions $u(r,\theta) = \ln r, v(r,\theta) = \theta$ have the first-order partial derivatives with respect to r and θ. By applying the polar form of the Cauchy-Riemann equation,

$$ru_r = v_\theta \Longrightarrow 1 = 1, \quad u_\theta = -rv_r \Longrightarrow 0 = 0$$

We can see that the equation holds for every point in the domain. From the theorem in section 24, it follows that f is differentiable over its whole domain. The derivative can be written as the following:

$$f'(z) = f'(e^{i\theta}) = e^{-i\theta}(u_r + iv_r) = \frac{1}{r}e^{-i\theta} = \frac{1}{z}$$

Let $h(z) = z^2 + 1$. For all z_0, we can write

$$\begin{aligned} h'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z_0 \Delta z + \Delta z^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0 \end{aligned}$$

and conclude that h is analytic. Furthermore, we can observe that

$$h(z) = h(x+iy) = (x+iy)^2 + 1 = (x^2 - y^2 + 1) + 2xyi$$

and $\operatorname{Im} h(z) = 2xy > 0$ where $x > 0, y > 0$. For $r\exp(i\theta) = h(x+iy)$, $\operatorname{Im} h(z) \neq 0$ implies $h(z) \neq 0$, so $r > 0$. Also, $\operatorname{Im} h(z) \neq 0$ implies $h(z) \notin \mathbb{R}$, so $0 < \theta < 2\pi$ holds. This means that $G(z)$ is defined for all $z = x + iy$ where $x > 0, y > 0$ and thus analytic, using the chain rule. Also, we can know that the following holds in the quadrant $x > 0, y > 0$.

$$G'(z) = \frac{d}{dz} g(h(z)) = g'(h(z))h'(z) = \frac{2z}{z^2+1}$$

3 Section 26 #7

By the definition of analytic functions, f is differentiable everywhere in D. We can write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, and $v(x,y) = 0$ everywhere in D, so v_x, v_y are also 0 in D. By the Cauchy-Riemann equation,

$$u_x = v_y = 0, \quad u_y = -v_x = 0$$

This means that $f'(z) = u_x(x,y) + iv_x(x,y) = 0$ everywhere in D, so $f(z)$ is constant throughout D by the theorem in section 25.

4 Section 27 #1

Using the polar form of the Cauchy-Riemann equation, the following holds.

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

Since f is analytic in D, the first-order partial derivatives of u and v with respect to r and θ exists in D. By the assumption of continuity of partial derivatives, we can differentiate both sides of $ru_r = v_\theta$ with respect to r.

$$u_r + ru_{rr} = v_{\theta r}$$

Differentiating both sides of $u_\theta = -rv_r$ with respect to θ, we get

$$u_{\theta\theta} = -rv_{r\theta}$$

By the continuity of partial derivatives, $v_{\theta r} = v_{r\theta}$ holds. We can write

$$-rv_{r\theta} = u_{\theta\theta} = -r(u_r + ru_{rr})$$

Rearranging the equation, we get the desired result.

$$r^2 u_{rr}(r,\theta) + ru_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0$$

5 Section 27 #2

By implicit function theorem, the following holds for u and v along the points of the curves $u(x,y) = c_1, v(x,y) = c_2$.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0, \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

Since $f'(z_0) = f_x(z_0) \neq 0$, at least one of $u_x(x_0,y_0) \neq 0$ and $v_x(x_0,y_0) \neq 0$ holds.

If $u_x(x_0, y_0) \neq 0$, $v_x(x_0, y_0) \neq 0$ both hold, Then the slope of the tangent line of $u(x, y) = c_1$ at (x_0, y_0) is $dy/dx = -u_x/u_y$, and the slope of the tangent line of $v(x, y) = c_2$ at (x_0, y_0) is $dy/dx = -v_x/v_y$. By the Cauchy-Riemann equation, $u_x = v_y, u_y = -v_x$ holds, so $-v_x/v_y = u_y/u_x$. The slope of $u(x, y) = c_1$ is $-u_x/u_y$ and the slope of $v(x, y) = c_2$ is u_y/u_x. Since $(-u_x/u_y)(u_y/u_x) = -1$, the tangent lines are perpendicular.

If $u_x(x_0, y_0) = 0$, then the slope of the tangent line of $u(x, y) = c_1$ at (x_0, y_0) is $dy/dx = -u_x/u_y = 0$. Using the implicit function theorem again, we get

$$\frac{\partial v}{\partial x}\frac{dx}{dy} + \frac{\partial v}{\partial y} = 0$$

along the points of the curve $v(x, y) = c_2$, and conclude that $dx/dy = -v_y/v_x = u_x/u_y = 0$ using the Cauchy-Riemann equation. This implies that the tangent line of $v(x, y) = c_2$ at (x_0, y_0) is perpendicular to the x-axis, so the tangent lines are also perpendicular in this case.

By using similar argument to $v_x(x_0, y_0) = 0$ case, the tangent line of $u(x, y) = c_1$ at (x_0, y_0) is perpendicular to the x-axis, and the tangent line of $v(x, y) = c_2$ at the same point has a slope of 0. In conclusion, the tangent lines of both curves at (x_0, y_0) are perpendicular.

6 Section 29 #1

Suppose that the $f(z)$ has a constant value w_0 throughout some neighborhood in D. A constant function $f(z) = w_o$ can be such function. By the theorem in section 28, such f is unique, so only the constant function $f(z) = w_0$ can be constant throughout the neighborhood in D, which is contradiction. Thus, the $f(z)$ which is analytic and not constant throughout D cannot be constant throughout any neighborhood contained in D.

7 Section 29 #2

Let D_1, D_2, D_3 be domains of f_1, f_2, f_3, respectively. Then there exists an open set contained in $D_1 \cap D_2$, and same holds for $D_2 \cap D_3$. Since $f_1(z) = f_2(z)$ for every point in $D_1 \cap D_2$, f_2 is an analytic continuation of f_1 by the theorem in section 28.

Likewise, $f_2(z) = f_3(z)$ for every point in $D_2 \cap D_3$ implies f_3 is an analytic continuation of f_2 the theorem in section 28.

For $z = x + iy$ where $x > 0, y > 0$, z can be written as $r\exp(i\theta) = r\exp(i(\theta + 2\pi)) = r(\cos\theta + i\sin\theta)$ where $r > 0, 0 < \theta < \pi/2$. $f_1(z)$ can be written as the following:

$$f_1(z) = f_1(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$$

$f_3(z)$ can be written as the following:

$$f_3(z) = f_3(re^{i(\theta+2\pi)}) = \sqrt{r}e^{i(\theta+2\pi)/2} = \sqrt{r}e^{i\theta/2}e^{i\pi} = -\sqrt{r}e^{i\theta/2}$$

Thus, $f_3(z) = -f_1(z)$ holds.