# MATH210: Homework 7 (due Apr. 25)

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## 1 Section 65 #2

### 1.1 Solution for (a)

We can write

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = \sum_{n=0}^{\infty} \frac{e}{n!}(z-1)^n = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \quad (|z-1| < \infty)$$

### 1.2 Solution for (b)

Using Maclaurin series expansion of $e^z$, we can write

$$e^z = e \cdot e^{z-1} = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \quad (|z-1| < \infty)$$

## 2 Section 65 #3

Using the Maclaurin series expansion of $1/(1-z)$, we can write

$$\begin{aligned} f(z) = \frac{z}{z^4+4} &= \frac{z}{4} \cdot \frac{1}{1+(z^4/4)} = \frac{z}{4}\sum_{n=0}^{\infty}\left(-\frac{z^4}{4}\right)^n \\ &= \frac{z}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} z^{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2}} z^{4n+1} \quad (|z| < \sqrt{2}) \end{aligned}$$

## 3 Section 65 #4

Using the provided identity and Maclaurin series expansion of $\sin z$,

$$\begin{aligned} \cos z &= -\sin\left(z - \frac{\pi}{2}\right) \\ &= -\sum_{n=0}^{\infty}(-1)^n \frac{(z-\pi/2)^{2n+1}}{(2n+1)!} \quad (|z-\pi/2| < \infty) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(z - \frac{\pi}{2}\right)^{2n+1} \quad (|z-\pi/2| < \infty) \end{aligned}$$

## 4 Section 65 #9

Using the Maclaurin series expansion of $\sin z$,

$$\begin{aligned} f(z) = \sin(z^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2(2n+1)}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} \quad (|z^2| < \infty) \qquad (1) \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad (|z| < \infty) \qquad (2) \end{aligned}$$

From equality of (1) and (2), we know that

$$f^{(n)}(0) = \begin{cases} (-1)^k \frac{(4k+2)!}{(2n+1)!} & (n = 4k+2) \\ 0 & (n \neq 4k+2) \end{cases}$$

where $k$ is an integer. Then, there is no integer $n, k$ such that $4n = 4k+2$ or $2n+1 = 4k+2$, so we get the desired result.

$$f^{(4n)}(0) = 0, \quad f^{(2n+1)}(0) = 0$$

## 5 Section 72 #1

Differentiating $1/(1-z)$, we get $1/(1-z)^2$ and term by term differentiation of the given series gives

$$\frac{d}{dz} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} (n+1) z^n \quad (|z| < 1) \qquad (3)$$

and the radius of convergence does not change by theorem 2 in section 71. Differentiating $1/(1-z)^2$ again, we get $2/(1-z)^3$ and term by term differentiation of (3) gives

$$\frac{d}{dz} \sum_{n=0}^{\infty} (n+1) z^n = \sum_{n=0}^{\infty} (n+1)(n+2) z^n \quad (|z| < 1)$$

again, by theorem 2 in section 71 the radius of convergence does not change.

## 6 Section 72 #3

Using the Maclaurin series expansion of $1/(1-z)$,

$$\begin{aligned} \frac{1}{z} &= \frac{1}{2 + (z-2)} = \frac{1}{2} \cdot \frac{1}{1 + (z-2)/2} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left( -\frac{z-2}{2} \right)^n \quad \left( \left| -\frac{z-2}{2} \right| < 1 \right) \qquad (4) \end{aligned}$$

Differentiating $1/z$ yields $-1/z^2$, and by differentiating (4), we get

$$\begin{aligned}
-\frac{1}{z^2} = \frac{1}{2}\frac{d}{dz}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n &= \frac{1}{2}\frac{d}{dz}\sum_{n=0}^{\infty}\left[\left(-\frac{1}{2}\right)^n (z-2)^n\right] \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\left[n\left(-\frac{1}{2}\right)^n (z-2)^{n-1}\right] \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left[(n+1)\left(-\frac{1}{2}\right)^{n+1} (z-2)^{n}\right] \\
&= -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad \left(\left|-\frac{z-2}{2}\right| < 1\right)
\end{aligned}$$

The radius of convergence does not change after differentiation by theorem 2 in section 71. Since $|-(z-2)/2| < 1$ is equivalent to $|z-2| < 2$, we get the desired result.

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2| < 2)$$

# 7 Section 72 #6

We know that $\operatorname{Log} z$ is an antiderivative of $1/z$ on the domain $D := \{z \in \mathbb{C};\ |z-1| < 1\}$ since

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z} \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

holds for all $z$ such that $|z-1| < 1$. By the theorem in section 48, for all contours lying in $D$ extending from 1 to $z$, we can write

$$\int_C \frac{1}{w}dw = \int_1^z \frac{1}{w}dw = \operatorname{Log} z - \operatorname{Log} 1 = \operatorname{Log} z$$

Also, using theorem 1 in section 71, we can write

$$\int_C \sum_{n=0}^{\infty}(-1)^n(w-1)^n dw = \sum_{n=0}^{\infty}(-1)^n\int_C (w-1)^n dw \quad (|z-1| < 1) \tag{5}$$

for all contour $C$ interior to the circle $|z-1| < 1$. Since $(w-1)^n$ has its antiderivative $(n+1)^{-1}(w-1)^{n+1}$ for all integer $n \geq 0$, for all contour $C$ in the circle $|z-1| < 1$ extending from 1 to $z$, we can write (5) as

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\int_C (w-1)^n dw &= \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \quad (|z-1| < 1)
\end{aligned}$$

and get the desired result.

$$\operatorname{Log} z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \quad (|z-1| < 1) \tag{6}$$

## 8 Section 72 #7

Using (6), we can write

$$\frac{\operatorname{Log} z}{z-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(z-1)^n \quad (0 < |z-1| < 1)$$

Since the series on the right hand side evaluates to 1 where $z = 1$, we can write

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(z-1)^n \quad (|z-1| < 1) \tag{7}$$

Using ratio test,

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n+2}(z-1)^{n+1} \right| \left| \frac{(-1)^n}{n+1}(z-1)^n \right|^{-1} &= \lim_{n\to\infty} \left| \frac{n+1}{n+2}(z-1) \right| \\ &= |z-1| < 1 \end{aligned}$$

so (7) converges absolutely if $|z-1| < 1$. By theorem 1 in section 72, (7) is the Taylor series expansion for $f$ in powers of $z-1$, so $f$ is analytic throughout a disk $|z-1| < 1$. Also, $\operatorname{Log} z$ is analytic throughout $\{z \in \mathbb{C};\ |z| > 0 \text{ and } -\pi < \operatorname{Arg} z < \pi\}$ so $f$ is also analytic throughout $D := \{z \in \mathbb{C};\ |z| > 0 \text{ and } z \neq 1 \text{ and } -\pi < \operatorname{Arg} z < \pi\}$. Since $f$ is analytic throughout both $D$ and disk $|z-1| < 1$, $f$ is analytic throughout the domain $0 < |z| < \infty, -\pi < \operatorname{Arg} z < \pi$.