MATH210: Homework 10 (due May. 16)

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1 Section 83 #3 (b)

Let $p(z) = \exp(zt)$, $q(z) = \sinh z$ and $f(z) = p(z)/q(z)$. Since $p(\pi i) \neq 0$, $q(\pi i) = 0$ and $q'(\pi i) \neq 0$, we can apply theorem 2 in section 83 so

$$\operatorname*{Res}_{z=\pi i} f(z) = \frac{p(\pi i)}{q'(\pi i)} = \frac{\exp(\pi i t)}{\cosh(\pi i)} = -e^{\pi i t} = -\cos(\pi t) - i\sin(\pi t)$$

Also, $p(-\pi i) \neq 0$, $q(-\pi i) = 0$ and $q'(-\pi i) \neq 0$ we can apply theorem 2 in section 83 again,

$$\operatorname*{Res}_{z=-\pi i} f(z) = \frac{p(-\pi i)}{q'(-\pi i)} = \frac{\exp(-\pi i t)}{\cosh(-\pi i)} = -e^{\pi i t} = -\cos(\pi t) + i\sin(\pi t)$$

and get the desired result.

$$\operatorname*{Res}_{z=\pi i} f(z) + \operatorname*{Res}_{z=-\pi i} f(z) = -2\cos(\pi t)$$

2 Section 83 #4 (b)

Let $p(z) = \sinh z$, $q(z) = \cosh z$ and $f(z) = \tanh z = p(z)/q(z)$. Since $p(z_n) \neq 0$, $q(z_n) = 0$ and $q(z_n) \neq 0 \quad (n = 0, \pm 1, \pm 2, \dots)$, we can apply theorem 2 in section 83 so

$$\operatorname*{Res}_{z=z_n} f(z) = \frac{p(z_n)}{q'(z_n)} = \frac{\sinh z_n}{\sinh z_n} = 1 \quad (n = 0, \pm 1, \pm 2, \dots)$$

and get the desired result.

3 Section 83 #5 (a)

Let $p(z) = \sin z$, $q(z) = \cos z$ and $f(z) = \tan z = p(z)/q(z)$ where $z \neq (n + 1/2)\pi \quad (n = 0, \pm 1, \pm 2, \dots)$. Since $q(z) = 0$ where $z = z_n := (n + 1/2)\pi \quad (n = 0, \pm 1, \pm 2, \dots)$, $f(z)$ has isolated singularities at $z_n \quad (n = 0, \pm 1, \pm 2, \dots)$. Only $-\pi/2, \pi/2$ are isolated singularities inside C, so by using Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i \left(\operatorname*{Res}_{z=-\pi/2} f(z) + \operatorname*{Res}_{z=\pi/2} f(z) \right)$$

Since $p(-\pi/2) \neq 0$, $p(\pi/2) \neq 0\, q(-\pi/2) = q(\pi/2) = 0$, $q'(-\pi/2) \neq 0$ and $q'(\pi/2) \neq 0$, we can apply theorem 2 in section 83.

$$\operatorname*{Res}_{z=-\pi/2} f(z) = \frac{p(-\pi/2)}{q'(-\pi/2)} = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$$
$$\operatorname*{Res}_{z=\pi/2} f(z) = \frac{p(\pi/2)}{q'(\pi/2)} = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1$$

Then

$$\int_C \tan z dz = 2\pi i \left(\operatorname*{Res}_{z=-\pi/2} f(z) + \operatorname*{Res}_{z=\pi/2} f(z) \right) = -4\pi i$$

4 Section 83 #6

Let $q(z) = z^2 \sin z$ and $f(z) = 1/q(z)$. Since $q(z)$ is analytic and $q(n\pi) = 0$ where $n = 0, \pm 1, \pm 2, \ldots$, $f(z)$ has isolated singularities at $0, \pm\pi, \pm 2\pi, \ldots$. The isolated singularities inside C_N are $-N\pi, -(N-1)\pi, \ldots, -\pi, 0, \pi, \ldots, (N-1)\pi, N\pi$. By Cauchy's residue theorem, we can write

$$\int_{C_N} f(z)dz = 2\pi i \sum_{n=-N}^{N} \operatorname*{Res}_{z=n\pi} f(z) \tag{1}$$

For $n\pi$ where $n \neq 0$, $q(n\pi) = 0$ and $q'(n\pi) \neq 0$ so by applying theorem 2 in section 83,

$$\operatorname*{Res}_{z=n\pi} f(z) = \frac{1}{q'(n\pi)} = \frac{1}{2n\pi \sin(n\pi) + (n\pi)^2 \cos(n\pi)} = \frac{(-1)^n}{n^2\pi^2}$$

From the Maclaurin series expansion of $\sin z$,

$$\begin{aligned} q(z) &= z^2 \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+3}}{(2n+1)!} \\ &= z^3 - \frac{z^5}{3!} + \frac{z^7}{5!} - \cdots \quad (|z| < \infty) \end{aligned}$$

$q(z)$ has zero of order 3 at 0. By theorem 1 in section 83, $f(z)$ has a pole of order 3 at 0. Applying the theorem in section 80, $f(z)$ can be written as $\phi(z)/z^3$ where $\phi(z)$ is analytic and nonzero at 0. For $0 < |z| < \pi$, $\phi(z)$ can be written as $z/\sin z$ so for $0 < |z| < \pi$,

$$\phi'(z) = \frac{\sin z - z\cos z}{\sin^2 z} = (1 - z\cot z)\csc z$$
$$\phi''(0) = \lim_{h \to 0} \frac{\phi'(h) - \phi'(0)}{h} \tag{2}$$

Since $\phi'(z)$ is continuous function, $\lim_{z\to 0} \phi'(z) = \phi'(0)$ so by L'Hôspital's rule,

$$\lim_{z\to 0} \frac{\sin z - z\cos z}{\sin^2 z} = \lim_{z\to 0} \frac{z \sin z}{2\cos z \sin z} = \lim_{z \to 0} \frac{z}{2\cos z} = 0$$

Then the limit in (2) can be caculated as

$$\begin{aligned} \phi''(0) &= \lim_{h\to 0} \frac{\phi'(h) - \phi'(0)}{h} = \lim_{h\to 0} \frac{\phi'(h)}{h} = \lim_{h \to 0} \frac{\sin h - h\cos h}{h \sin^2 h} \\ &= \lim_{h\to 0} \frac{h \sin h}{\sin^2 h + 2h\cos h \sin h} = \lim_{h \to 0} \frac{1}{\frac{\sin h}{h} + 2\cos h} = \frac{1}{3} \end{aligned}$$

Applying the theorem in section 80,

$$\operatorname*{Res}_{z=0} f(z) = \frac{\phi''(0)}{2!} = \frac{1}{6}$$

and (1) can be written as

$$\begin{aligned}\int_{C_N} f(z)dz &= 2\pi i\left[\frac{1}{6} + \sum_{n=-N,n\neq 0}^{N} \frac{(-1)^n}{n^2\pi^2}\right]\\ &= 2\pi i\left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right]\end{aligned}$$

From $\sin z = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$,

$$|\sin z|^2 = (\sin x\cosh y)^2 + (\cos x\sinh y)^2 = \sin^2 x + \sinh^2 y$$

so $|\sin z| \geq |\sin x\cosh y| \geq |\sin x|$ as $\cosh y \geq 1$ for all real y and $|\sin z|^2 \geq \sinh^2 y$ as $\sin^2 x \geq 0$ for all real x. Then for all z on contour C_N, one of $|\sin z| \geq |\sin x| = |\sin(\pm(N+1/2)\pi)| = 1$ and $|\sin z| \geq |\sinh y| = |\sinh(3\pi/2)| > 1$ holds. Thus, $|\sin z| \geq 1$ so

$$|f(z)| = \left|\frac{1}{z^2\sin z}\right| \leq \left|\frac{1}{z^2}\right| \leq \frac{1}{[(N+1/2)\pi]^2}$$

and we can write

$$\int_{C_N} f(z)dz \leq \frac{1}{[(N+1/2)\pi]^2}\cdot 8\left(N+\frac{1}{2}\right)\pi \leq \frac{16}{(2N+1)\pi}$$

So the integral on the left hand side tends to zero as $N\to\infty$. Thus, it follows that

$$2\pi i\left[\frac{1}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2}\right] = 0$$

so

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} = -\frac{1}{12}$$

and get the desired result.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

5 Section 86 #4

Let $p(z) = x^2$, $q(z) = x^6+1$ and $f(z) = x^2/(x^6+1) = p(z)/q(z)$. $f(z)$ has isolated singularities at the zeros of $q(z)$, which are sixth roots of -1 and $f(z)$ is analytic anywhere else. The complex roots of -1 are $c_k = \exp(i(\pi+2k\pi)/6) \quad (k = 0, 1, \ldots, 6)$ and none of them lies on the real axis. c_0, c_1, c_2 lie in the upper half plane, so they are inside a semicircular region bounded by $z = x \quad (x \in [-R, R])$ and upper half C_R of the circle $|z| = R$, where $R > 1$. By Cauchy's residue theorem,

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \left(\operatorname*{Res}_{z=c_0} f(z) + \operatorname*{Res}_{z=c_1} f(z) + \operatorname*{Res}_{z=c_2} f(z) \right)$$

We can apply theorem 2 in section 83.

$$\operatorname*{Res}_{z=c_k} f(z) = \frac{p(c_k)}{q'(c_k)} = \frac{c_k^2}{6c_k^5} = \frac{c_k^3}{6c_k^6} = -\frac{c_k^3}{6}$$

Thus,

$$\begin{aligned}\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz &= -\frac{\pi i}{3}(c_0^3 + c_1^3 + c_2^3) \\ &= -\frac{\pi i}{3}(e^{i\pi/2} + e^{3i\pi/2} + e^{5i\pi/2}) \\ &= -\frac{\pi i}{3}(i - i + i) = \frac{\pi}{3}\end{aligned}$$

For z on C_R, we can write

$$\left| \frac{z^2}{z^6 + 1} \right| = \frac{R^2}{|z^6 + 1|} \le \frac{R^2}{|z^6| - 1} = \frac{R^2}{R^6 - 1}$$

so

$$\left| \int_{C_R} f(z)dz \right| \le \frac{R^2}{R^6 - 1} \cdot \pi R = \frac{\pi R^3}{R^6 - 1}$$

and the contour integral over C_R tends to zero as $R \to \infty$. Since $f(-z) = f(z)$ for all z, we can write

$$\int_0^\infty \frac{x^2 dx}{x^6 + 1} = \lim_{R\to\infty} \int_0^R \frac{x^2 dx}{x^6 + 1} = \lim_{R\to\infty} \frac{1}{2} \left(\frac{\pi}{3} - \int_{C_R} f(z)dz \right) = \frac{\pi}{6}$$

6 Section 86 #6

Let $p(z) = x^2$, $q(z) = (x^2+9)(x^2+4)^2$ and $f(z) = p(z)/q(z)$. $f(z)$ has isolated singularities at the zeros of $q(z)$, which are $\pm 3i, \pm 2i$ and $f(z)$ is analytic anywhere else. None of the isolated singularities lies on the real axis. $3i$ and $2i$ lie in the upper half plane, so they are inside a semicircular region bounded by $z = x \quad (x \in [-R, R])$ and upper half C_R of the circle $|z| = R$, where $R > 3$. By Cauchy's residue theorem,

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \left(\operatorname*{Res}_{z=3i} f(z) + \operatorname*{Res}_{z=2i} f(z) \right)$$

Since $q(3i) = 0$ and $q'(3i) \ne 0$, we can apply theorem 2 in section 83.

$$\operatorname*{Res}_{z=3i} f(z) = \frac{p(3i)}{q'(3i)} = \frac{-9}{6i(-9+4)^2} = \frac{3i}{50}$$

Since $q(2i) = q'(2i) = 0$ and $q''(2i) \ne 0$, $2i$ is a zero of order 2 of q, so $2i$ is a pole of order 2 of f by theorem 1 in section 83. Applying the theorem in section 80, $f(z)$ can be written as $\phi(z)/(z-2i)^2$ where $\phi(z)$ is analytic and nonzero at $2i$. For $0 < |z-2i| < \infty$, we can write

$$\phi(z) = \frac{z^2}{(z^2+9)(z+2i)^2}$$

and

$$\operatorname*{Res}_{z=2i} f(z) = \frac{\phi'(2i)}{1!} = -\frac{13i}{200}$$

So we can write

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i\left(\frac{3i}{50} - \frac{13i}{200}\right) = \frac{\pi}{100}$$

For z on C_R,

$$\begin{aligned}\left|\frac{z^2}{(z^2+9)(z^2+4)^2}\right| &= \frac{|z^2|}{|z^2+9||z^2+4|^2} \\ &\le \frac{|z|^2}{(|z|^2-9)(|z|^2-4)^2} = \frac{R^2}{(R^2-9)(R^2-4)^2}\end{aligned}$$

so

$$\left|\int_{C_R} f(z)dz\right| \le \frac{R^2}{(R^2-9)(R^2-4)^2}\cdot \pi R = \frac{\pi R^3}{(R^2-9)(R^2-4)^2}$$

and the contour integral over C_R tends to zero as $R \to \infty$. Since $f(-z) = f(z)$, we can write

$$\begin{aligned}\int_0^\infty \frac{x^2dx}{(x^2+9)(x^2+4)^2} &= \lim_{R\to\infty}\int_0^R \frac{x^2dx}{(x^2+9)(x^2+4)^2} \\ &= \lim_{R\to\infty}\frac{1}{2}\left(\frac{\pi}{100} - \int_{C_R} f(z)dz\right) = \frac{\pi}{200}\end{aligned}$$

7 Section 86 #9

Let $q(z) = z^3+1$, $f(z) = 1/q(z)$, C_R be the circular part of the contour given in the problem, and L_R be the line segment part connecting $Re^{i2\pi/3}$ to 0. $f(z)$ has isolated singularities at the zeros of $q(z)$, which are $\exp(i\pi/3), -1, \exp(5i\pi/3)$ and $f(z)$ is analytic anywhere else. None of the isolated singularities lies on the real axis. Only $\exp(i\pi/3)$ is inside C_R, so Cauchy's residue theorem gives

$$\int_0^R f(x)dx + \int_{C_R} f(z)dz + \int_{L_R} f(z)dz = 2\pi i \operatorname*{Res}_{z=\exp(i\pi/3)} f(z)$$

Since $q(\exp(i\pi/3)) = 0$ and $q'(\exp(i\pi/3)) \ne 0$, we can apply theorem 2 in section 83.

$$\operatorname*{Res}_{z=\exp(i\pi/3)} f(z) = \frac{1}{q'(\exp(i\pi/3))} = \frac{1}{3e^{2i\pi/3}} = \frac{e^{4i\pi/3}}{3} = -\frac{\sqrt{3}i}{6} - \frac{1}{6}$$

so

$$\int_0^R f(x)dx + \int_{C_R} f(z)dz + \int_{L_R} f(z)dz = 2\pi i\left(-\frac{\sqrt{3}i}{6} - \frac{1}{6}\right) = \frac{\sqrt{3}\pi}{3} - \frac{\pi i}{3}$$

We can write

$$\int_{L_R} f(z)dz = \int_R^0 f(te^{2\pi i/3})e^{2\pi i/3}dt = -e^{2\pi i/3}\int_0^R \frac{dt}{t^3 \cdot e^{2\pi i} + 1} = -e^{2\pi i/3}\int_0^R f(t)dt$$

Also, for all z on C_R,

$$|f(z)| = \left|\frac{1}{z^3+1}\right| = \frac{1}{|z^3+1|} \leq \frac{1}{|z^3|-1} = \frac{1}{R^3-1}$$

so

$$\left|\int_{C_R} f(z)dz\right| \leq \frac{1}{R^3-1} \cdot \frac{2\pi R}{3} = \frac{2\pi R}{3(R^3-1)}$$

the contour integral over C_R tends to zero as $R \to \infty$. We can write

$$\begin{aligned}\int_0^R f(x)dx + \int_{C_R} f(z)dz + \int_{L_R} f(z)dz &= (1 - e^{2\pi i/3})\int_0^R f(x)dx + \int_{C_R} f(z)dz \\ &= \frac{\sqrt{3}\pi}{3} - \frac{\pi i}{3}\end{aligned}$$

and get the desired result.

$$\int_0^\infty \frac{dx}{x^3+1} = \lim_{R\to\infty}\int_0^R \frac{dx}{x^3+1} = \lim_{R\to\infty}\frac{1}{1-e^{2\pi i/3}}\left(\frac{\sqrt{3}\pi}{3} - \frac{\pi i}{3} - \int_{C_R} f(z)dz\right) = \frac{2\pi}{3\sqrt{3}}$$