

# MATH210: Homework 6 (due Apr. 11)

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## 1 Section 53 #3

Let  $z_0 = 2 + i$ ,  $R < 1$ , then  $C_0$  is interior to  $C$  and we can apply the corollary in section 53.

$$\int_C (z - 2 - i)^{n-1} dz = \int_{C_0} (z - z_0)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \dots \\ 2\pi i & \text{when } n = 0 \end{cases}$$

## 2 Section 53 #5

Let  $C_a = C_1 - C_3$ , then  $C_a$  is a simple closed contour since  $C_1$  and  $C_3$  are smooth arcs. Applying Cauchy-Goursat theorem, we get

$$\int_{C_a} f(z) dz = \int_{C_1 - C_3} f(z) dz = 0$$

so

$$\int_{C_1} f(z) dz = \int_{C_3} f(z) dz$$

Let  $C_b = C_2 + C_3$ , then  $C_b$  is also a simple closed contour since  $C_2$  is a smooth arc. Again from Cauchy-Goursat theorem,

$$\int_{C_b} f(z) dz = \int_{C_2 + C_3} f(z) dz = 0$$

so

$$\int_{C_2} f(z) dz = \int_{-C_3} f(z) dz = - \int_{C_3} f(z) dz$$

then we can write

$$\int_{C_1} f(z) dz = - \int_{C_2} f(z) dz$$

and

$$\int_C f(z) dz = \int_{C_1 + C_2} f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz = 0$$

### 3 Section 57 #4

If  $z$  is outside of  $C$ , then  $s - z \neq 0$  and the integrand is analytic at all points interior to and on  $C$ , so we can apply Cauchy-Goursat theorem.

$$\int_C \frac{s^3 + 2s}{(s - z)^3} ds = 0$$

Suppose that  $z$  is inside  $C$ . Let  $f(z) = z^3 + 2z$ , and using Cauchy integral formula we get

$$f^{(2)}(z) = \frac{2!}{2\pi i} \int_C \frac{f(s)ds}{(s - z)^3}$$

so

$$6\pi iz = \int_C \frac{f(s)ds}{(s - z)^3} = \int_C \frac{s^3 + 2s}{(s - z)^3} ds$$

and we get the desired result.

### 4 Section 57 #6

Let  $d$  be the smallest distance from  $z$  to points  $s$  on  $C$  and assume that  $0 < |\Delta z| < d$ . We can write

$$\begin{aligned} |g(z + \Delta z) - g(z)| &= \left| \frac{1}{2\pi i} \int_C \left( \frac{1}{s - z - \Delta z} - \frac{1}{s - z} \right) f(s) ds \right| \\ &= \left| \frac{1}{2\pi i} \int_C \frac{\Delta z}{(s - z - \Delta z)(s - z)} f(s) ds \right| \\ &\leq \frac{1}{2\pi} \left| \int_C \frac{\Delta z}{(s - z - \Delta z)(s - z)} f(s) ds \right| \\ &\leq \frac{1}{2\pi} \frac{|\Delta z| M}{(d - |\Delta z|) d} L \end{aligned} \tag{1}$$

where  $L$  is the length of  $C$  and  $M$  is the maximum value of  $|f(s)|$  on  $C$ . If we let  $\Delta z$  tend to zero, (1) goes to zero and we can conclude that  $g(z)$  is continuous. Let  $\gamma$  be an arbitrary simple closed contour interior to  $C$ , then

$$\begin{aligned} \int_{\gamma} g(z) dz &= \int_{\gamma} \left( \frac{1}{2\pi i} \int_C \frac{f(s)ds}{s - z} \right) dz \\ &= \frac{1}{2\pi i} \int_C \left( \int_{\gamma} \frac{f(s)}{s - z} dz \right) ds \\ &= \frac{1}{2\pi i} \int_C 0 ds = 0 \end{aligned}$$

since  $z$  is in the interior of  $C$ , so  $z \notin C$ . Using theorem 2 in section 57, we can conclude that  $g(z)$  is analytic throughout the interior of  $C$ .

### 5 Section 57 #7

Let  $f(z) = \exp(az)$ , then using the Cauchy integral formula,

$$f(0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z} = \frac{1}{2\pi i} \int_C \frac{e^{az}}{z} dz = 1$$

so we get

$$\int_C \frac{e^{az}}{z} dz = 2\pi i$$

Writing the contour integral explicitly,

$$\begin{aligned} \int_C \frac{e^{az}}{z} dz &= \int_{-\pi}^{\pi} \frac{e^{a \exp(i\theta)}}{\exp(i\theta)} i e^{i\theta} d\theta = i \int_{-\pi}^{\pi} \exp(a(\cos \theta + i \sin \theta)) d\theta \\ &= i \int_{-\pi}^{\pi} e^{a \cos \theta} \exp(ia \sin \theta) d\theta \\ &= i \int_{-\pi}^{\pi} e^{a \cos \theta} (\cos(a \sin \theta) + i \sin(a \sin \theta)) d\theta \\ &= i \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta - \int_{-\pi}^{\pi} \sin(a \sin \theta) d\theta \end{aligned}$$

Since  $\sin(a \sin(-\theta)) = -\sin(a \sin \theta)$  and  $e^{a \cos(-\theta)} \cos(a \sin(-\theta)) = e^{a \cos \theta} \cos(a \sin \theta)$ ,

$$\begin{aligned} &i \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta - \int_{-\pi}^{\pi} \sin(a \sin \theta) d\theta \\ &= i \left( \int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta + \int_{-\pi}^0 e^{a \cos \theta} \cos(a \sin \theta) d\theta \right) \\ &\quad - \left( \int_0^{\pi} \sin(a \sin \theta) d\theta + \int_{-\pi}^0 \sin(a \sin \theta) d\theta \right) \\ &= i \left( \int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta + \int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta \right) \\ &\quad - \left( \int_0^{\pi} \sin(a \sin \theta) d\theta - \int_0^{\pi} \sin(a \sin \theta) d\theta \right) \\ &= 2i \int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = 2\pi i \end{aligned}$$

so

$$\int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = \pi$$

## 6 Section 57 #10

Consider a positively oriented circle  $C_R$  with radius  $R$ , centered at  $z_0$ . Let  $M_R$  be the maximum of  $f(z)$  on  $C_R$ , then  $M_R \leq A(|z_0| + R)$  since for all  $z$  on  $C_R$ ,  $A|z| \leq A(|z_0| + |z - z_0|) = A(|z_0| + R)$  by triangular inequality. We can write

$$|f''(z_0)| \leq \frac{2!M_R}{R^2} = \frac{2A(|z_0| + R)}{R^2}$$

and it implies  $|f''(z_0)| = 0$ , since this inequality should hold for all  $R$ . Then,  $f''(z) = 0$  everywhere, so  $f'(z)$  is constant and  $|f'(0)| \leq 0$ , so  $f'(0) = 0$ . Thus, if we write  $f'(z) = a_1$

$$f(w) = \int_0^w f'(z) dz = [a_1 z]_0^w = a_1 w$$

## 7 Section 59 #4

We can write

$$\begin{aligned}|f(z)|^2 &= |\sin z|^2 = |\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y \\ &= \sin^2 x + \left(\frac{e^y - e^{-y}}{2}\right)^2 = \sin^2 x + \left(\frac{e^{2y} - 2 + e^{-2y}}{4}\right)\end{aligned}$$

We know that  $\sin^2 x \leq 1$  and  $\sin^2(\pi/2) = 1$ . Also,

$$\frac{d}{dy} \frac{e^{2y} - 2 + e^{-2y}}{4} = \frac{e^{2y} - e^{-2y}}{2} \geq 0$$

for  $y \geq 0$ , so  $\sinh^2 y$  is monotonically increasing in  $y \geq 0$ , so  $\sinh^2 y$  has maximum on  $y = 1$  in the given region. Thus,  $|f(z)|^2$  is the largest on  $(\pi/2) + i$ , so  $|f(z)|$  is also the largest there.

## 8 Section 59 #7

Let  $g(z) = \exp(-if(z)) = \exp(-if(x + iy)) = \exp(v(x, y) - iu(x, y))$ , then  $g(z)$  is continuous on  $R$  and analytic, nonconstant in the interior of  $R$ . Then  $|g(z)| = |\exp(v(x, y) - iu(x, y))| = |\exp(v(x, y))| |\exp(-iu(x, y))| = |\exp(v(x, y))|$  has its maximum on the boundary of  $R$  by the corollary in section 59, and since  $e^x$  is monotonically increasing for real number  $x$ ,  $v(x, y)$  is maximum there.

Considering  $h(z) = 1/g(z)$ ,  $h(z)$  is also continuous on  $R$  and analytic, nonconstant in the interior of  $R$  since  $g(z) \neq 0$ . Then  $|h(z)| = |\exp(-v(x, y))|$  has its maximum on the boundary of  $R$  by the corollary in section 59, and since  $e^x$  is monotonically increasing for real number  $x$ ,  $-v(x, y)$  is maximum there and  $v(x, y)$  is minimum there.