MATH210: Homework 9 (due May. 9)

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1 Section 77 #1

1.1 Solution for (b)

Using the Maclaurin series expansion of $\cos z$, we can write

$$z\cos\left(\frac{1}{z}\right) = z\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(\frac{1}{z}\right)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{z^{2n-1}} \quad (0<|z|<\infty)$$

The coefficient of $1/z$ in this sequence occurs when $2n-1=1$, hence

$$\operatorname*{Res}_{z=0} z\cos\left(\frac{1}{z}\right) = \frac{(-1)^1}{2!} = -\frac{1}{2}$$

1.2 Solution for (d)

We can write

$$f(z) = \frac{\cot z}{z^4} = \frac{\cos z}{z^4\sin z} = \cos z\left(\frac{1}{z^4\sin z}\right)$$

where $0<|z|<\pi$. Then we can write

$$\frac{1}{z^4\sin z} = \frac{1}{z^4\left(z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dots\right)} = \frac{1}{z^5}\left(\frac{1}{1-\dfrac{z^2}{3!}+\dfrac{z^4}{5!}-\cdots}\right) \quad (0<|z|<\pi)$$

Since $1/(z^4\sin z)$ is analytic where $0<|z|<\pi$. Using division, we obtain

$$\frac{1}{1-\dfrac{z^2}{3!}+\dfrac{z^4}{5!}-\cdots} = 1+\frac{1}{6}z^2+\frac{7}{360}z^4+\frac{31}{15120}z^6+\cdots \quad (0<|z|<\pi)$$

so

$$\frac{1}{z^4\sin z} = \frac{1}{z^5}+\frac{1}{6}\cdot\frac{1}{z^3}+\frac{7}{360}\cdot\frac{1}{z}+\frac{31}{15120}z+\cdots \quad (0<|z|<\pi)$$

Then we can calculate the product and obtain

$$
\begin{aligned}
\cos z\left(\frac{1}{z^4\sin z}\right) &= \left(1-\frac{1}{2!}z^2+\frac{1}{4!}z^4-\cdots\right)\left(\frac{1}{z^5}+\frac{1}{6}\cdot\frac{1}{z^3}+\frac{7}{360}\cdot\frac{1}{z}+\frac{31}{15120}z+\cdots\right)\\
&= \frac{1}{z^5}\left(1-\frac{1}{2!}z^2+\frac{1}{4!}z^4-\cdots\right)+\frac{1}{6}\cdot\frac{1}{z^3}\left(1-\frac{1}{2!}z^2+\frac{1}{4!}z^4-\cdots\right)\\
&+\frac{7}{360}\cdot\frac{1}{z}\left(1-\frac{1}{2!}z^2+\frac{1}{4!}z^4-\cdots\right)+\cdots \quad (0<|z|<\pi)
\end{aligned}
$$

The coefficient of $1/z$ term is

$$\frac{1}{4!}-\frac{1}{6}\cdot\frac{1}{2!}+\frac{7}{360}\cdot 1=-\frac{1}{45}$$

By definition, we know that $\operatorname{Res}_{z=0} f(z)=-1/45$.

2 Section 77 #2

2.1 Solution for (b)

Let $f(z)=\exp(-z)/(z-1)^2$, then $f(z)$ has an isolated singularity, $z=1$ and it is interior to the circle. By Cauchy's residue theorem,

$$\int_{|z|=3} f(z)dz = 2\pi i \operatorname*{Res}_{z=1} f(z)$$

From the Maclaurin series expansion of e^z,

$$f(z)=\frac{\exp(-z)}{(z-1)^2}=\frac{\exp(-(z-1))}{e(z-1)^2}=\frac{1}{e(z-1)^2}\sum_{n=0}^{\infty}\frac{[-(z-1)]^n}{n!} \quad (0<|z-1|<\infty) \tag{1}$$

the coefficient of $1/(z-1)$ in the series (1) is $-1/e$, so $\operatorname{Res}_{z=1} f(z)=-1/e$. Thus, the integral evaluates to $-2\pi i/e$.

2.2 Solution for (d)

Let $f(z)=(z+1)/(z^2-2z)=(z+1)/[z(z-2)]$, then $f(z)$ has two isolated singularities $z=0$ and $z=2$ and they are interior to the circle. From the Maclaurin series expansion of $1/(1-z)$,

$$f(z)=-\frac{1}{2}\left(1+\frac{1}{z}\right)\frac{1}{1-z/2}=-\frac{1}{2}\left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad (0<|z|<2) \tag{2}$$

The coefficient of $1/z$ in the series (2) is $-1/2$, so $\operatorname{Res}_{z=0} f(z)=-1/2$. We can also write

$$
\begin{aligned}
f(z) &= \frac{z+1}{z-2}\cdot\frac{1}{2-(2-z)}=\frac{1}{2}\left(1+\frac{3}{z-2}\right)\frac{1}{1-(2-z)/2}\\
&= \frac{1}{2}\left(1+\frac{3}{z-2}\right)\sum_{n=0}^{\infty}\left(\frac{2-z}{2}\right)^n\\
&= \frac{1}{2}\left(1+\frac{3}{z-2}\right)\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(z-2)^n \quad (0<|z-2|<2)
\end{aligned} \tag{3}
$$

The coefficient of $1/(z-2)$ in the series (3) is $3/2$, so $\text{Res}_{z=2} f(z) = 3/2$. By Cauchy's residue theorem,

$$\int_{|z|=3} f(z)dz = 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=2} f(z) \right) = 2\pi i$$

3 Section 77 #4 (b)

Let $f(z) = 1/(1+z^2)$. Using the Maclaurin series expansion of $1/(1-z)$, we can write

$$f(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} \quad (|z^2| < 1)$$

then, $\text{Res}_{z=0} f(z) = 0$. Using the theorem in section 77, we can write

$$\int_{|z|=2} f(z)dz = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{1+z^2} = 2\pi i \operatorname*{Res}_{z=0} f(z) = 0$$

4 Section 79 #1 (a)

The function has an isolated singularity at $z = 0$. Using the Maclaurin series expansion of e^z, we can write

$$\begin{aligned} z \exp\left(\frac{1}{z}\right) &= z \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^{n-1}} \\ &= z + 1 + \sum_{n=2}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^{n-1}} \quad (0 < |z| < \infty) \end{aligned}$$

The principal part of the series at $z = 0$ can be written as

$$\sum_{n=2}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^{n-1}} = \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{3!} \cdot \frac{1}{z^2} + \cdots$$

If we write the principal part as $\sum_{n=1}^{\infty} b_n z^{-n}$, $b_n = (n+1)!$ so $b_n \neq 0$ for all $n = 1, 2, \ldots$. Thus, the point $z = 0$ is an essential singular point.

5 Section 79 #3

Since $f(z)$ is analytic, there exists some $R > 0$ such that f is analytic throught a disk $|z - z_0| < R$. Then, $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, \ldots)$$

by Taylor's theorem. $g(z)$ has an isolated singularity at z_0. We can write

$$\begin{aligned} g(z) &= \frac{f(z)}{z-z_0} = \frac{1}{z-z_0}\sum_{n=0}^{\infty} a_n(z-z_0)^n \\ &= \frac{a_0}{z-z_0} + \sum_{n=1}^{\infty} a_n(z-z_0)^{n-1} \quad (0<|z-z_0|<R) \end{aligned}$$

5.1 Solution for (a)

The principal part of the series is $a_0/(z-z_0)$. If we write the principal part as $\sum_{n=1}^{\infty} b_n z^{-n}$, $b_1 = a_0 = f(z_0) \neq 0$ and $b_n = 0 \quad (n = 2, 3, \dots)$. Thus, z_0 is a pole of order 1, so it is a simple pole of g. By definition, $\operatorname{Res}_{z=z_0} f(z) = b_1 = f(z_0)$.

5.2 Solution for (b)

Since $a_0 = f(z_0) = 0$, if we write the principal part as $\sum_{n=1}^{\infty} b_n z^{-n}$, $b_n = 0 \quad (n = 1, 2, \dots)$. Thus, z_0 is a removable singular point of g.

6 Section 81 #1 (d)

Let $f(z) = e^z/(z^2+\pi^2)$. Since $e^z, z^2+\pi^2$ are entire and $z^2+\pi^2 = 0$ at $z = \pm i\pi$, $f(z)$ has isolated singularities at $z = \pm i\pi$. Then, we can write

$$f(z) = \frac{\phi(z)}{z-i\pi}, \quad \phi(z) = \frac{e^z}{z+i\pi}$$

Here, $\phi(z)$ is analytic at $i\pi$ and $\phi(i\pi) \neq 0$. By the theorem in section 80, $i\pi$ is a pole of order $m = 1$ of f and $B = \operatorname{Res}_{z=i\pi} f(z) = \phi(i\pi) = i/2\pi$. We can also write

$$f(z) = \frac{\psi(z)}{z+i\pi}, \quad \psi(z) = \frac{e^z}{z-i\pi}$$

Then, $\psi(z)$ is analytic at $-i\pi$ and $\psi(-i\pi) \neq 0$. Using the theorem in section 80 again, $-i\pi$ is a pole of order $m = 1$ of f and $B = \operatorname{Res}_{z=-i\pi} f(z) = \psi(-i\pi) = -i/2\pi$.

7 Section 81 #2 (b)

Let $f(z) = \operatorname{Log} z/(z^2+1)^2$. Since $\operatorname{Log} z$ is analytic if $-\pi < \theta < \pi$ where $z = re^{i\theta}$, and $(z^2+1)^2$ is entire and $(z^2+1)^2 = 0$ at $z = \pm i$, $f(z)$ has isolated singularities at $z = \pm i$. Then, we can write

$$f(z) = \frac{\phi(z)}{(z-i)^2}, \quad \phi(z) = \frac{\operatorname{Log} z}{(z+i)^2}$$

Here, $\phi(z)$ is analytic at i and $\phi(i) \neq 0$. By the theorem in section 80,

$$\operatorname*{Res}_{z=i} f(z) = \frac{\phi'(i)}{1!} = \frac{(1/i)(i+i)^2 - 2(i+i)\operatorname{Log} i}{(i+i)^4} = \frac{4i+2\pi}{16} = \frac{\pi+2i}{8}$$

8 Section 81 #5 (a)

Let $f(z) = 1/[z^3(z+4)]$. Since $z^3(z+4)$ is an entire function, $f(z)$ has isolated singularities at $z = 0, -4$. Using Cauchy's residue theorem, since only 0 is inside and on C

$$\int_C f(z)dz = \int_C \frac{dz}{z^3(z+4)} = 2\pi i \operatorname*{Res}_{z=0} f(z)$$

We can write

$$f(z) = \frac{\phi(z)}{z^3}, \quad \phi(z) = \frac{1}{z+4}$$

Then, $\phi(z)$ is analytic at 0 and $\phi(0) \neq 0$. Using the theorem in section 80, 0 is a pole of order 3 of f and $\mathrm{Res}_{z=0} f(z) = \phi''(0)/2! = 2 \cdot 4^{-3}/2! = 1/64$. In conclusion,

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \operatorname*{Res}_{z=0} f(z) = \pi i/32$$

9 Section 81 #6

Let $f(z) = \cosh(\pi z)/[z(z^2+1)]$. Since $\cosh(\pi z), z(z^2+1)$ are entire function, $f(z)$ has isolated singularities at $z = 0, \pm i$. Using Cauchy's residue theorem, since $0, \pm i$ are all inside and on C

$$\int_C f(z)dz = 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right)$$

We can write

$$f(z) = \frac{\phi(z)}{z}, \quad \phi(z) = \frac{\cosh(\pi z)}{z^2+1}$$

Then, $\phi(z)$ is analytic at 0 and $\phi(0) \neq 0$. Using the theorem in section 80, 0 is a pole of order 1 of f and $\mathrm{Res}_{z=0} f(z) = \phi(0) = 1$. We can also write

$$f(z) = \frac{\psi(z)}{z-i}, \quad \psi(z) = \frac{\cosh(\pi z)}{z(z+i)}$$

Then, $\psi(z)$ is analytic at i and $\psi(i) \neq 0$. Using the theorem in section 80, i is a pole of order 1 of f and $\mathrm{Res}_{z=i} f(z) = \psi(i) = 1/2$. We can also write

$$f(z) = \frac{\omega(z)}{z+i}, \quad \omega(z) = \frac{\cosh(\pi z)}{z(z-i)}$$

Then, $\omega(z)$ is analytic at $-i$ and $\omega(-i) \neq 0$. Using the theorem in section 80, $-i$ is a pole of order 1 of f and $\mathrm{Res}_{z=-i} f(z) = \omega(-i) = 1/2$. In conclusion,

$$\int_C \frac{\cosh \pi z}{z(z^2+1)} dz = 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right) = 4\pi i$$

10 Section 83 #8

By definition, z_0 is a zero of order 1 of $q(z)$. By theorem 1 in section 82, $q(z) = (z-z_0)g(z)$ where $g(z)$ is analytic and nonzero at z_0. Then, we can write

$$f(z) = \frac{\phi(z)}{(z-z_0)^2}, \quad \phi(z) = \frac{1}{(g(z))^2}$$

and $\phi(z)$ is analytic and nonzero at z_0 as g is analytic and nonzero at z_0. By the theorem in section 80, z_0 is a pole of order 2 of f, and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi'(z_0)}{1!} = -2g'(z_0)(g(z_0))^{-3}$$

From $q'(z) = g(z)+(z-z_0)g'(z)$ and $q''(z) = 2g'(z)+(z-z_0)g''(z)$, $q'(z_0) = g(z_0), q''(z_0) = 2g'(z_0)$. Thus, $\operatorname{Res}_{z=z_0} f(z) = -q''(z_0)/(q'(z_0))^3$ and we get the desired result.

11 Section 83 #10

By the theorem in section 80 we can write

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z-z_0)^m}$$

where $\phi(z)$ is analytic and nonzero at z_0. Then, since p is analytic and nonzero at z_0, we can write

$$\frac{1}{q(z)} = \frac{1}{(z-z_0)^m} \cdot \frac{\phi(z)}{p(z)}$$

and $\phi(z)/p(z)$ is also analytic and nonzero at z_0, so z_0 is a pole of order m of $1/q(z)$ by the theorem in section 80. Since q is analytic at z_0, there exists some $R > 0$ such that $1/q(z)$ is analytic on $0 < |z| < R$. Then we can write

$$\begin{aligned}
\frac{1}{q(z)} &= \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n \\
&= \frac{1}{(z-z_0)^m}\left[\sum_{n=0}^{m-1} b_{m-n}(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^{m+n}\right] = \frac{h(z)}{(z-z_0)^m}
\end{aligned}$$

and $h(z)$ is analytic at z_0, and $h(z_0) = b_m \neq 0$ so $1/h(z)$ is also analytic and nonzero at z_0. In conclusion, $q(z) = (z-z_0)^m/h(z)$ and by theorem 1 in section 82 q has a zero of order m at z_0.