MATH230: Homework 2 (due Sep. 18)

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1 Chapter 3 #3

We can denote an outcome of a coin toss by length 3 string, $X_1X_2X_3$, where $X_1 \in \{H, T\}, X_2 \in \{H, T\}, X_3 \in \{H, T\}$. Then the sample space S can be written as follows:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

The values w of the random variable W can be assigned as follows:

Sample Space	w
HHH	3
HHT	1
HTH	1
HTT	-1
THH	1
THT	-1
TTH	-1
TTT	-3

2 Chapter 3 #8

By the defintion of probability distribution, as w can have one of $\{-3, -1, 1, 3\}$, we can write

$$f(-3) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}, \quad f(-1) = 3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = \frac{2}{9}$$
$$f(1) = 3\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{9}, \quad f(3) = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

Thus, the probability distribution can be written as

w	-3	-1	1	3
$f(w)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

3 Chapter 3 #9

3.1 Solution for (a)

Using the defintion of probability density function,

$$P(0 < X < 1) = \int_0^1 f(x)dx = \int_0^1 \frac{2(x+2)}{5}dx = \left[\frac{x^2}{5} + \frac{4x}{5}\right]_0^1 = 1$$

Thus, $P(0 < X < 1) = 1$.

3.2 Solution for (b)

Again using the defintion of probability density function,

$$P(1/4 < X < 3/4) = \int_{\frac{1}{4}}^{\frac{3}{4}} f(x)dx = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{2(x+2)}{5}dx = \left[\frac{x^2}{5} + \frac{4x}{5}\right]_{\frac{1}{4}}^{\frac{3}{4}} = \frac{1}{2}$$

4 Chapter 3 #30

4.1 Solution for (a)

By the defintion of probability density function,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{1} k(3-x^2)dx = \left[3kx - \frac{kx^3}{3}\right]_{-1}^{1} = 6k - \frac{2k}{3} = \frac{16k}{3} = 1$$

Thus, k should be $3/16$ if f is a valid probability density function.

4.2 Solution for (b)

Using the defintion of probability density function,

$$P(x < 1/2) = \int_{-\infty}^{\frac{1}{2}} f(x)dx = \int_{-1}^{\frac{1}{2}} f(x)dx = \left[3kx - \frac{kx^3}{3}\right]_{-1}^{\frac{1}{2}} = \frac{99}{128}$$

4.3 Solution for (c)

The magnitude of error is greater than 0.8 if and only if $x > 0.8$ or $x < -0.8$. Using the defintion of probability density function,

$$\begin{aligned}
P(|X| > 0.8) &= P(X < -0.8) + P(X > 0.8) = P(-1 \leq X < -0.8) + P(0.8 < X \leq 1) \\
&= P(-1 < X < -0.8) + P(0.8 < X < 1) = \int_{-1}^{-0.8} f(x)dx + \int_{0.8}^{1} f(x)dx \\
&= \frac{41}{500} + \frac{41}{500} = \frac{41}{250}
\end{aligned}$$

5 Chapter 3 #39

5.1 Solution for (a)

The possible pairs of values of X and Y, (x, y) are

$$(x, y) \in \{(0,1), (1,0), (0,2), (1,1), (2,0), (1,2), (2,1), (3,0), (2,2), (3,1)\}$$

Let $f(x, y)$ be the joint probability distribution. Then, $f(x, y)$ denote the probability of picking x oranges, y apples and $(4 - x - y)$ bananas. In total, there are $\binom{8}{4}$ ways of picking fruits, so we can write

$$f(x, y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{\binom{8}{4}}$$

for $x \in \{0, 1, 2, 3\}, y \in \{0, 1, 2\}, 1 \le x + y \le 4$.

5.2 Solution for (b)

We can write

$$\begin{aligned} P[(X, Y) \in A] = P(X + Y \le 2) &= f(0,1) + f(1,0) + f(0,2) + f(1,1) + f(2,0) \\ &= \frac{1}{35} + \frac{3}{70} + \frac{3}{70} + \frac{9}{35} + \frac{9}{70} = \frac{1}{2} \end{aligned}$$

6 Chapter 3 #40

6.1 Solution for (a)

Let $g(x)$ be the marginal density of X. By the defintion of marginal density,

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 f(x, y) dy = \int_0^1 \frac{2}{3}(x + 2y) dy = \left[\frac{2xy}{3} + \frac{2y^2}{3}\right]_{y=0}^1 = \frac{2x + 2}{3}$$

where $x \in [0, 1]$.

6.2 Solution for (b)

Let $h(y)$ be the marginal density of Y. By the defintion of marginal density,

$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^1 f(x, y) dx = \int_0^1 \frac{2}{3}(x + 2y) dx = \left[\frac{x^2}{3} + \frac{4xy}{3}\right]_{x=0}^1 = \frac{4y + 1}{3}$$

where $y \in [0, 1]$.

6.3 Solution for (c)

Using the defintion of marginal density, we can write

$$\begin{aligned} P(X \le 1/2) = P(X \le 1/2, -\infty < Y < \infty) &= \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^{\infty} f(x, y) dy dx \\ &= \int_0^{\frac{1}{2}} g(x) dx = \frac{5}{12} \end{aligned}$$

7 Chapter 3 #47

7.1 Solution for (a)

Let $g(x), h(y)$ be the marginal deinsities of X and Y. By the defintion of marginal density,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{x}^{1} 2dy = 2(1-x)$$
$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{y} 2dx = 2y$$

where $x \in (0,1)$, $y \in (0,1)$. By plugging $x = y = 1/2$, we obtain

$$f(1/2, 1/2) = 2 \neq g(1/2)h(1/2) = 1$$

Thus, X and Y are dependent.

7.2 Solution for (b)

We can write

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{1}{y}$$

Then,

$$P(1/4 < X < 1/2 | Y = 3/4) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x|y = 3/4)dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{4}{3} dx = \frac{1}{3}$$