MATH230: Homework 6 (due Oct. 16)

손량(20220323)

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1 Chapter 6 #1

1.1 Solution for (a)

We can write

$$\mu = \int_{-\infty}^{\infty} x f(x; A, B) dx = \int_A^B \frac{x}{B-A} dx = \left[\frac{x^2}{2(B-A)} \right]_A^B = \frac{A+B}{2}$$

1.2 Solution for (b)

We can write

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x; A, B) dx = \int_A^B \left(x - \frac{A+B}{2} \right)^2 \frac{1}{B-A} dx \\ &= \frac{1}{B-A} \left[\frac{1}{3} \left(x - \frac{A+B}{2} \right)^3 \right]_A^B = \frac{1}{3(B-A)} \cdot 2 \left(\frac{B-A}{2} \right)^3 = \frac{(B-A)^2}{12} \end{aligned}$$

2 Chapter 6 #2

We can write

$$P(X < 3.5 \mid X \geq 1) = \frac{P(1 \leq X < 3.5)}{P(X \geq 1)} = \frac{\int_1^{3.5} f(x; 0, 5) dx}{\int_1^{\infty} f(x; 0, 5) dx} = \frac{\int_1^{3.5} \frac{1}{5} dx}{\int_1^5 \frac{1}{5} dx} = \frac{5}{8}$$

3 Chapter 6 #10

Let $\Phi(z)$ be distribution function of standard normal distribution. Then we can write

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = \Phi(2) - \Phi(-2)$$

According to the table A.3 in the textbook, we can conclude that the quantity in question is 0.9544.

4 Chapter 6 #49

4.1 Solution for (a)

We can write

$$\mu = \frac{\alpha}{\alpha + \beta} = \frac{1}{1+3} = \frac{1}{4}$$

For median, we can calculate the integral from 0 to t, where $0 < t < 1$.

$$\begin{aligned} F(t) &= \int_0^t \frac{1}{B(1,3)}(1-x)^2 dx = \frac{\Gamma(4)}{\Gamma(1)\Gamma(3)} \int_0^t (1-x)^2 dx = 3\left[-\frac{1}{3}(1-x)^3\right]_0^t \\ &= 1 - (1-t)^3 \end{aligned}$$

Then, the solution of $F(t) = 1/2$ is $t = 1 - \sqrt[3]{1/2}$ and it is the median.

4.2 Solution for (b)

According to the formula of variance of gamma distribution,

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{3}{4^2 \cdot 5} = \frac{3}{80}$$

5 Chapter 6 #53

5.1 Solution for (a)

According to the formula of mean of gamma distribution,

$$\mu = \frac{\alpha}{\alpha + \beta} = \frac{5}{5+10} = \frac{1}{3}$$

5.2 Solution for (b)

According to the formula of variance of gamma distribution,

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{50}{15^2 \cdot 16} = \frac{1}{72}$$

6 Chapter 6 #58

6.1 Solution for (a)

Let $N(t)$ be the number of cars passed in first t minutes. Then $N(t)$ follows a $p(x;5)$. The probability we are interested in is $P(N(t) > 10)$.

$$P(N(t) > 10) = 1 - \sum_{x=0}^{10} P(N(t) = x) = 1 - \sum_{x=0}^{10} \frac{e^{-5} \cdot 5^x}{x!} = 0.01370$$

6.2 Solution for (b)

Let X be the arrival time of the tenth car. Then the probability we are interested in is $P(X > 2)$. Using the relation of exponential distribution and Poisson distribution,

$$P(X > 2) = P(N(2) < 10) = \sum_{k=0}^{9} e^{-10} \cdot \frac{10^k}{k!} = 0.45793$$

7 Chapter 6 #59

7.1 Solution for (a)

By definition of Poisson process, the time T between events follows an exponential distribution with mean 1/5. The probability we are interested in is $P(T > 1)$.

$$P(T > 1) = \frac{e^{-5} \cdot (5)^0}{0!} = e^{-5} = 0.00674$$

7.2 Solution for (b)

As stated earlier, T follows an exponential distribution with mean 1/5.

8 Supplemental Exercises

8.1 Solution for #1

For mean, we can write

$$E(|Z|) = \int_{-\infty}^{\infty} \frac{|z|}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} z e^{-\frac{1}{2}z^2} dz = \frac{2}{\sqrt{2\pi}} \left[-e^{-\frac{1}{2}z^2} \right]_0^{\infty} = \sqrt{\frac{2}{\pi}}$$

For variance, as Z follows standard normal distribution,

$$\begin{aligned} \sigma_{|}^2 Z| &= E(|Z|^2) - (E(|Z|))^2 = E(Z^2) - (E(|Z|))^2 \\ &= E(Z^2) - (E(Z))^2 + (E(Z))^2 - (E(|Z|))^2 = 1 + 0 - \frac{2}{\pi} = 1 - \frac{2}{\pi} \end{aligned}$$

8.2 Solution for #2

We can write

$$\begin{aligned} E(Z^4) = \int_{-\infty}^{\infty} \frac{z^4}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz &= \frac{1}{\sqrt{2\pi}} \left(\left[-z^3 e^{-\frac{1}{2}z^2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-3z^2) e^{-\frac{1}{2}z^2} dz \right) \\ = 3 \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz &= 3E(Z^2) = 3[1 + (E(Z))^2] = 3 \end{aligned}$$

8.3 Solution for #3

Let $R(t)$ be the probability of the battery lasting more than t years. By the relationship of $R(t)$ and $k(t)$ outlined in the lecture note, we can write

$$\begin{aligned} R(4) &= \exp\left[-\int_0^4 k(s) ds \right] = \exp\left[-\int_0^2 0.05 ds - \int_2^4 (0.05t - 0.05) dt \right] \\ &= \exp\left(-0.1 - \left[0.025t^2 - 0.05t \right]_2^4 \right) = \exp(-0.3) \end{aligned}$$