MATH230: Homework 3 (due Sep. 25)

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1 Chapter 4 #17

Let $S_x = \{-3, 6, 9\}$. Using the definition of expected value, we can write

$$\mu_{g(X)} = E[g(X)] = \sum_{x \in S_x} g(x)f(x) = \sum_{x \in S_x} (2x+1)^2 f(x) = 209$$

2 Chapter 4 #23

2.1 Solution for (a)

Let $S_x = \{2, 4\}$ and $S_y = \{1, 3, 5\}$. Using the definition of expected value, we can write

$$E[g(X,Y)] = \sum_{x \in S_x} \sum_{y \in S_y} g(x,y)f(x,y) = \sum_{x \in S_x} \sum_{y \in S_y} xy^2 f(x,y) = \frac{317}{10}$$

2.2 Solution for (b)

Using the definition of expected value, we can write

$$\mu_X = E(X) = \sum_{x \in S_x} \sum_{y \in S_y} xf(x,y) = \frac{29}{10}$$
$$\mu_Y = E(Y) = \sum_{x \in S_x} \sum_{y \in S_y} yf(x,y) = 3$$

3 Chapter 4 #26

Using the definition of expected value, we can write

$$\begin{aligned}
E(Z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2+y^2} f(x,y) dx dy = \int_0^1 \int_0^1 \sqrt{x^2+y^2} \cdot 4xy \; dx dy \\
&= \int_0^1 \left[\frac{4}{3} y (x^2+y^2)^{3/2} \right]_0^1 dy = \frac{4}{3} \int_0^1 \left[y(1+y^2)^{3/2} - y^4 \right] dy \\
&= \frac{4}{3} \int_0^1 y(1+y^2)^{3/2} dy - \frac{4}{15}
\end{aligned}$$

Using the substituion of $y = \tan u$, we can write

$$E(Z) = \frac{4}{3}\int_0^{\frac{\pi}{4}} (\tan u)(1+\tan^2 u)^{3/2}(\sec^2 u)du - \frac{4}{15} = \frac{4}{3}\int_0^{\frac{\pi}{4}} \tan u \sec^5 u \; du - \frac{4}{15}$$
$$= \frac{4}{3}\left[\frac{1}{5}\sec^5 u\right]_0^{\frac{\pi}{4}} - \frac{4}{15} = \frac{16\sqrt{2}-8}{15}$$

4 Chapter 4 #43

Using the definition and property of expected value, we can write

$$\mu_Y = E(Y) = E(3X-2) = 3E(X) - 2 = 3\int_{-\infty}^{\infty} xf(x)dx - 2$$
$$= 3\int_0^{\infty} x \cdot \frac{1}{4}e^{-x/4}dx - 2 = 3\left(\lim_{t\to\infty}\left[(-x-4)e^{-x/4}\right]_0^t\right) - 2 = 10$$

As we can show that $\lim_{t\to\infty} te^{-t} = 0$ using L'Hôpital's rule. Also, by the definition and property of variance, we can write

$$\sigma_Y^2 = \sigma_{3X-2}^2 = 9\sigma_X^2 = 9[E(X^2) - (E(X))^2] = 9\left[\int_{-\infty}^{\infty} x^2 f(x)dx - \left(\int_{-\infty}^{\infty} xf(x)dx\right)^2\right]$$
$$= 9\left[\int_0^{\infty} x^2 \cdot \frac{1}{4}e^{-x/4}dx - \left(\int_0^{\infty} x \cdot \frac{1}{4}e^{-x/4}dx\right)^2\right]$$
$$= 9\left[\lim_{t\to\infty}\left[(-x^2-8x-32)e^{-x/4}\right]_0^t - \left(\lim_{t\to\infty}\left[(-x-4)e^{-x/4}\right]_0^t\right)^2\right]$$
$$= 9(32-16) = 144$$

5 Chapter 4 #52

First, we can write

$$\mu_X = E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy = \int_0^1\int_0^y xf(x,y)dxdy$$
$$= \int_0^1 \left[x^2\right]_0^y dy = \int_0^1 y^2 \; dy = \frac{1}{3}$$
$$\mu_Y = E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy = \int_0^1\int_0^y 2y \; dxdy$$
$$= \int_0^1 2y^2 \; dy = \left[\frac{2}{3}y^3\right]_0^1 = \frac{2}{3}$$

Using the definition of variance, we can write

$$\begin{aligned}
\sigma_X^2 &= E(X^2) - \mu_X^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)dxdy - \frac{1}{9} = \int_0^1\int_0^y 2x^2 dxdy - \frac{1}{9} \\
&= \int_0^1 \left[\frac{2}{3}x^3\right]_0^y dy - \frac{1}{9} = \int_0^1 \frac{2}{3}y^3 dy - \frac{1}{9} = \frac{1}{18} \\
\sigma_Y^2 &= E(Y^2) - \mu_Y^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)dxdy - \frac{4}{9} = \int_0^1\int_0^y 2y^2 dxdy - \frac{4}{9} \\
&= \int_0^1 \left[2y^2 x\right]_0^y dy - \frac{4}{9} = \int_0^1 2y^3 dy - \frac{4}{9} = \frac{1}{18}
\end{aligned}$$

Using the definition of covariance, we can write

$$\begin{aligned}
\sigma_{XY} &= E(XY) - \mu_X\mu_Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy - \frac{2}{9} \\
&= \int_0^1\int_0^y 2xy\; dxdy - \frac{2}{9} = \int_0^1 \left[x^2 y\right]_0^y dy - \frac{2}{9} = \int_0^1 y^3\; dy - \frac{2}{9} = \frac{1}{36}
\end{aligned}$$

By the definition of correlation,

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{1}{2}$$

6 Chapter 4 #70

6.1 Solution for (a)

Considering the marginal densities $g(x)$ and $h(y)$ for $0 \le x \le 1$ and $0 \le y \le 1$,

$$\begin{aligned}
g(x) &= \int_{-\infty}^{\infty} f(x,y)dy = \frac{3}{2}\int_0^1 (x^2+y^2)dy = \frac{3}{2}\left[x^2 y + \frac{1}{3}y^3\right]_0^1 = \frac{3}{2}\left(x^2 + \frac{1}{3}\right) \\
h(y) &= \int_{-\infty}^{\infty} f(x,y)dx = \frac{3}{2}\int_0^1 (x^2+y^2)dx = \frac{3}{2}\left[\frac{1}{3}x^3 + xy^2\right]_0^1 = \frac{3}{2}\left(y^2 + \frac{1}{3}\right)
\end{aligned}$$

Take $x = 1$ and $y = 1$, then $f(x,y) = 3 \neq g(1)h(1) = 4$. Thus, X and Y are dependent.

6.2 Solution for (b)

By property of expected value, we can write

$$\begin{aligned}
E(X+Y) = E(X) + E(Y) &= \int_{-\infty}^{\infty} xg(x)dx + \int_{-\infty}^{\infty} yh(y)dy = \int_0^1 xg(x)dx + \int_0^1 yh(y)dy \\
&= \int_0^1 \frac{3}{2}x\left(x^2+\frac{1}{3}\right)dx + \int_0^1 \frac{3}{2}y\left(y^2+\frac{1}{3}\right)dy = \left[\frac{3}{8}x^4 + \frac{1}{4}x^2\right]_0^1 + \left[\frac{3}{8}y^4 + \frac{1}{4}y^2\right]_0^1 \\
&= \frac{5}{4}
\end{aligned}$$

Also, by the definition of expected value,

$$
\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \int_0^1\int_0^1 \frac{3}{2}xy(x^2+y^2)dxdy \\
&= \int_0^1 \left[\frac{3}{8}x^4y + \frac{3}{4}x^2y^3\right]_{x=0}^1 dy = \int_0^1 \left(\frac{3}{8}y + \frac{3}{4}y^3\right) dy \\
&= \left[\frac{3}{16}y^2 + \frac{3}{16}y^4\right]_0^1 = \frac{3}{8}
\end{aligned}
$$

6.3 Solution for (c)

By a property of variance, we can write

$$
\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - \mu_X^2 = \int_{-\infty}^{\infty} x^2 g(x)dx - \mu_X^2 = \int_0^1 \frac{3}{2}x^2\left(x^2+\frac{1}{3}\right)dx - \left(\frac{5}{8}\right)^2 \\
&= \left[\frac{3}{10}x^5 + \frac{1}{6}x^3\right]_0^1 - \left(\frac{5}{8}\right)^2 = \frac{73}{960} \\
\mathrm{Var}(Y) &= E(Y^2) - \mu_Y^2 = \int_{-\infty}^{\infty} y^2 h(y)dy - \mu_Y^2 = \int_0^1 \frac{3}{2}y^2\left(y^2+\frac{1}{3}\right)dy - \left(\frac{5}{8}\right)^2 \\
&= \left[\frac{3}{10}y^5 + \frac{1}{6}y^3\right]_0^1 - \left(\frac{5}{8}\right)^2 = \frac{73}{960}
\end{aligned}
$$

By a property of covariance,

$$
\mathrm{Cov}(X,Y) = E(XY) - \mu_X\mu_Y = \frac{3}{8} - \frac{5}{8}\cdot\frac{5}{8} = -\frac{1}{64}
$$

6.4 Solution for (d)

By properties of variance and covariance,

$$
\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) = \frac{73}{960} + \frac{73}{960} + 2\cdot\left(-\frac{1}{64}\right) = \frac{29}{240}
$$

7 Chapter 4 #78

By the definition of expected value, we can write

$$
\begin{aligned}
\mu = E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 30x^3(1-x)^2dx = 30\int_0^1 (x^3 - 2x^4 + x^5)dx \\
&= 30\left[\frac{1}{4}x^4 - \frac{2}{5}x^5 + \frac{1}{6}x^6\right]_0^1 = \frac{1}{2}
\end{aligned}
$$

By the definition of variance,

$$
\begin{aligned}
\sigma^2 = E(X^2) - \mu^2 &= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_0^1 30x^4(1-x)^2dx - \mu^2 \\
&= 30\int_0^1 (x^4 - 2x^5 + x^6)dx - \mu^2 = 30\left[\frac{1}{5}x^5 - \frac{1}{3}x^6 + \frac{1}{7}x^7\right]_0^1 - \mu^2 = \frac{2}{7} - \frac{1}{4} = \frac{1}{28}
\end{aligned}
$$

Thus, we can write

$$
\begin{aligned}
P(\mu-2\sigma<X<\mu+2\sigma) &= \int_{\mu-2\sigma}^{\mu+2\sigma} f(x)dx = \int_{\frac{1}{2}-\frac{1}{\sqrt{7}}}^{\frac{1}{2}+\frac{1}{\sqrt{7}}} 30x^2(1-x)^2dx \\
&= 30\left[\frac{1}{3}x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_{\frac{1}{2}-\frac{1}{\sqrt{7}}}^{\frac{1}{2}+\frac{1}{\sqrt{7}}} = 0.96998
\end{aligned}
$$

The lower bound given by Chebyshev's theorem is $1-2^{-2}=3/4$.

8 Lecture Note Exercise #4.1

8.1 Solution for (a)

Let $S_x=\{0,1,2\}$ and $S_y=\{1,2\}$. We can write

$$
\begin{aligned}
\mu_X &= \sum_{x\in S_x} xg(x) = 1 \\
\sigma_X^2 &= E(X^2)-\mu_X^2 = \sum_{x\in S_x} x^2g(x)-\mu_X^2 = \frac{14}{9}-1=\frac{5}{9}
\end{aligned}
$$

8.2 Solution for (b)

Let $m(y)=\sum_{x\in S_x} xf(x|y)$. Then we can write

$$
\begin{aligned}
m(1) &= \sum_{x\in S_x} xf(x|y=1) = \sum_{x\in S_x} x\frac{f(x,1)}{h(1)} = \frac{f(1,1)}{h(1)}+\frac{2f(2,1)}{h(1)} \\
&= \frac{4}{18}\cdot\left(\frac{1}{2}\right)^{-1}+2\left(\frac{1}{18}\right)\left(\frac{1}{2}\right)^{-1}=\frac{2}{3} \\
m(2) &= \sum_{x\in S_x} xf(x|y=1) = \sum_{x\in S_x} x\frac{f(x,2)}{h(2)} = \frac{f(1,2)}{h(2)}+\frac{2f(2,2)}{h(2)} \\
&= \frac{4}{18}\cdot\left(\frac{1}{2}\right)^{-1}+2\left(\frac{4}{18}\right)\left(\frac{1}{2}\right)^{-1}=\frac{4}{3}
\end{aligned}
$$

We can write

$$
\mu_X = E[m(Y)] = \sum_{y\in S_y} m(y)h(y) = m(1)h(1)+m(2)h(2) = \frac{2}{3}\cdot\frac{1}{2}+\frac{4}{3}\cdot\frac{1}{2}=1
$$

This conicides with the result of (a).

8.3 Solution for (c)

Let $v(y) = E(X^2|y) - (m(y))^2$. Then

$$
\begin{aligned}
v(1) &= E(X^2|y=1) - (m(1))^2 = \sum_{x \in S_x} x^2 f(x|y=1) - (m(1))^2 \\
&= \frac{f(1,1)}{h(1)} + \frac{4f(2,1)}{h(1)} - (m(1))^2 = \frac{4}{18} \cdot \left(\frac{1}{2}\right)^{-1} + 4\left(\frac{1}{18}\right)\left(\frac{1}{2}\right)^{-1} - \left(\frac{2}{3}\right)^{-1} = \frac{4}{9} \\
v(2) &= E(X^2|y=2) - (m(2))^2 = \sum_{x \in S_x} x^2 f(x|y=2) - (m(2))^2 \\
&= \frac{f(1,2)}{h(2)} + \frac{4f(2,2)}{h(2)} - (m(2))^2 = \frac{4}{18} \cdot \left(\frac{1}{2}\right)^{-1} + 4\left(\frac{4}{18}\right)\left(\frac{1}{2}\right)^{-1} - \left(\frac{4}{3}\right)^{-1} = \frac{4}{9}
\end{aligned}
$$

We can write

$$
\begin{aligned}
\sigma_X^2 &= E[v(Y)] + \mathrm{Var}[m(Y)] = \sum_{y \in S_y} v(y)h(y) + \sum_{y \in S_y} (m(y))^2 h(y) - \left(\sum_{y \in S_y} m(y)h(y)\right)^2 \\
&= v(1)h(1) + v(2)h(2) + (m(1))^2 h(1) + (m(2))^2 h(2) - (m(1)h(1) + m(2)h(2))^2 = \frac{5}{9}
\end{aligned}
$$

As $m(y)$ is the expected value of X when given a value of Y, the variation from randomness in coin selection is captured by $\mathrm{Var}[m(Y)]$. The term can be calculated as

$$
\begin{aligned}
\mathrm{Var}[m(Y)] &= \sum_{y \in S_y} (m(y))^2 h(y) - \left(\sum_{y \in S_y} m(y)h(y)\right)^2 \\
&= (m(1))^2 h(1) + (m(2))^2 h(2) - (m(1)h(1) + m(2)h(2))^2 = \frac{1}{9}
\end{aligned}
$$