MATH230: Homework 1 (due Sep. 11)

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1 Chapter 2 #12

1.1 Solution for (a)

The subjects' type can be encoded using a string of length 3, $A_1A_2A_3$ where $A_1 \in \{Z, W, S\}, A_2 \in \{Y, N\}, A_3 \in \{M, F\}$. Then the sample space can be written as follows:

$$S = \{ZYM, ZYF, ZNM, ZNF, WYM, WYF, WNM, WNF, SYM, SYF, SNM, SNF\}$$

1.2 Solution for (b)

A can we written as:

$$A = \{ZYF, ZNF, WYF, WNF, SYF, SNF\}$$

B can be written as:

$$B = \{WYM, WYF, WNM, WNF\}$$

So we can write

$$A \cup B = \{ZYF, ZNF, WYM, WYF, WNM, WNF, SYF, SNF\}$$

1.3 Solution for (c)

Using A and B listed above, we can write

$$A \cap B = \{WYF, WNF\}$$

2 Chapter 2 #32

2.1 Solution for (a)

The number of ways is equal to the number of permutations of 6 objects, which is $6! = 720$.

2.2 Solution for (b)

First, there are $3! = 6$ ways to line up the three stubborn people. Then, treating those people as one person, there are $4! = 24$ ways to line up 3 other people and the three-person-group. Using the rule 2.2 in the textbook, we know that there are $3! \times 4! = 144$ ways to line up.

2.3 Solution for (c)

First, there are $4! = 24$ ways to line up 4 people, except the two people who refuse to follow each other. Then, there are $\binom{5}{2}$ positions between, front, or rear of the already lined-up people. Also, There are 2! ways to determine one of the two people who comes first in line. Using the rule 2.2 in the textbook, we know that there are $4! \times \binom{5}{2} \times 2! = 480$ ways to line up.

3 Chapter 2 #38

3.1 Solution for (a)

The number of ways is equal to the number of permutations of 6 objects, which is $6! = 720$.

3.2 Solution for (b)

There are $3! = 6$ ways to seat three couples, and each couple has 2! ways to sit: male sits left or female sits left in row. By the rule 2.2 in the textbook, the number of ways is $3! \times (2!)^3 = 48$.

3.3 Solution for (c)

There are 3! ways to seat 3 women, and 3! ways to seat 3 men. By the rule 2.2 in the textbook, the number of ways is $3! \times 3! = 36$.

4 Chapter 2 #43

By the theorem 2.3, the number of ways is $(6-1)! = 120$.

5 Chapter 2 #58

5.1 Solution for (a)

Considering two dice distinct, let's denote outcome of the dice throw as (a, b), where a and b are the numbers from each dice. Then, the sample space can be written as follows:

$$S = \{(1,1), (1,2), \ldots, (1,6), (2,1), (2,2), \ldots, (2,6), \ldots, (6,1), (6,2), \ldots, (6,6)\}$$

If A represents the event of the total being 9, we can write

$$A = \{(3,6), (4,5), (5,4), (6,3)\}$$

Since the dice is fair, each of the outcomes in S is equally likely to occur. We assign a probability of ω to each sample point of S. Then $36\omega = 1$ so $\omega = 1/36$. Thus,

$$P(A) = 4 \times \frac{1}{36} = \frac{1}{9}$$

5.2 Solution for (b)

If B represents the event of the total being at most 3, we can write

$$B = \{(1,1), (1,2), (2,1)\}$$

Using the similar argument we made in (a),

$$P(B) = 3 \times \frac{1}{36} = \frac{1}{12}$$

6 Chapter 2 #72

Using de Morgan's law, we can write

$$P(A' \cap B') = P((A \cup B)')$$

Using theorem 2.9 in the textbook,

$$P((A \cup B)') = 1 - P(A \cup B)$$

By theorem 2.7 in the textbook,

$$1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B))$$

Then we get the desired result:

$$P(A' \cap B') = 1 + P(A \cap B) - P(A) - P(B)$$

7 Chapter 2 #76

Consider the events:

- S_0: The person is a nonsmoker.
- S_1: The person is a moderate smoker.
- S_2: The person is a heavy smoker.
- H: The person is experiencing hypertension.
- NH: The person is not experiencing hypertension.

7.1 Solution for (a)

The probability we are interested in is $P(H|S_2)$. Using the definition 2.10,

$$P(H|S_2) = \frac{P(H \cap S_2)}{P(S_2)} = \frac{30}{180} \times \left(\frac{49}{180}\right)^{-1} = \frac{30}{49}$$

7.2 Solution for (b)

The probability we are interested in is $P(S_0|NH)$. Using the definition 2.10,

$$P(S_0|NH) = \frac{P(S_0 \cap NH)}{P(NH)} = \frac{48}{180} \times \left(\frac{48 + 26 + 19}{180}\right)^{-1} = \frac{48}{93}$$

8 Chapter 2 #78

Consider the events:

- R_1: The batch is rejected by the first inspection.
- R_2: The batch is rejected by the second inspection.
- R_3: The batch is rejected by the third inspection.

Before we begin, let's prove this statement:

Theorem 1. *Let A, B be independent events of a sample space S, where $0 < P(A) < 1$ and $0 < P(B) < 1$. Then A' and B are independent.*

Proof. A and B are independent, so by the theorem 2.11, we can write

$$\begin{aligned}P(B) &= P(S \cap B) = P((A \cup A') \cap B) = P((A \cap B) \cup (A' \cap B)) \\ &= P(A \cap B) + P(A' \cap B) - P((A \cap B) \cap (A' \cap B)) \\ &= P(A \cap B) + P(A' \cap B) - P(\varnothing) = P(A)P(B) + P(A')P(B)\end{aligned}$$

Then,

$$P(A')P(B) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(A')P(B)$$

Again, by the theorem 2.11, A' and B are independent. $\square$

8.1 Solution for (a)

The probability we are interested in is $P(R_1' \cap R_2)$. As we proved earlier, R_1' and R_2 are independent as R_1 and R_2 are independent. Then we can write

$$P(R_1' \cap R_2) = P(R_1')P(R_2) = (1 - 0.10) \times 0.05 = 0.045$$

8.2 Solution for (b)

The probability we are interested in is $P(R_1' \cap R_2' \cap R_3)$. As we proved earlier, R_1' and R_2' are independent as R_1 and R_2 are independent, so $P(R_2'|R_1') = P(R_2')$. In a similar fashion, R_2' and R_3, R_1' and R_3 are independent. By theorem 2.12, we can write

$$P(R_1' \cap R_2' \cap R_3) = P(R_1')P(R_2')P(R_3) = (1 - 0.10) \times (1 - 0.05) \times 0.15 = 0.12825$$

9 Chapter 2 #87

Consider the events:

- U: The home is one of the homes that are left unlocked.
- K: The home can be opened by one of the three master keys.

The probability we are interested in is $P(U \cup K)$, and U and K can be considered independent as the key was randomly selected. We can write

$$P(U \cup K) = P(U) + P(K) - P(U)P(K) = 0.3 + \frac{\binom{7}{2}}{\binom{8}{3}} - 0.3 \times \frac{\binom{7}{2}}{\binom{8}{3}} = \frac{9}{16}$$

10 Chapter 2 #93

Let A, B, C, D, E be events where the component A, B, C, D, E works, respectively. Then we can write

$$P(A) = P(B) = 0.7, P(C) = P(D) = P(E) = 0.8$$

10.1 Solution for (a)

The system works if A, B works, or C, D, E works. Thus, the probability we are interested in is $P((A \cap B) \cup (C \cap D \cap E))$. Since the components fail independently,

$$\begin{aligned} P((A \cap B) \cup (C \cap D \cap E)) &= P(A \cap B) + P(C \cap D \cap E) - P(A \cap B \cap C \cap D \cap E) \\ &= P(A)P(B) + P(C)P(D)P(E) - P(A)P(B)P(C)P(C)P(E) \\ &= 0.75112 \end{aligned}$$

10.2 Solution for (b)

Let S be an event where the system works. The probability we are interested in is $P(A'|S)$. We can write

$$\begin{aligned} P(A'|S) &= \frac{P(A' \cap S)}{P(S)} = \frac{P(A' \cap [(A \cap B) \cup (C \cap D \cap E)])}{P((A \cap B) \cup (C \cap D \cap E))} \\ &= \frac{P(A' \cap C \cap D \cap E)}{P((A \cap B) \cup (C \cap D \cap E))} = \frac{1920}{9389} \end{aligned}$$

11 Chapter 2 #96

Let $A_i \quad (i = 1, 2, 3, 4)$ be events where the driver is passing L_i, respectively. Then,

$$P(A_1) = 0.2, P(A_2) = 0.1, P(A_3) = 0.5, P(A_4) = 0.2$$

Let B the event where radar traps resulting in speeding ticket. Then, the probability we are interested in is $P(B)$. We can write

$$P(B|A_1) = 0.4, P(B|A_2) = 0.3, P(B|A_3) = 0.2, P(B|A_4) = 0.3$$

Using the theorem 2.13, we can write

$$P(B) = \sum_{k=1}^{4} P(B|A_k)P(A_k) = 0.27$$

12 Chapter 2 #98

The probability we are interested in is $P(A_2|B)$. Using the Bayes' rule,

$$P(A_2|B) = \frac{P(A_2 \cap B)}{\sum_{k=1}^{4} P(A_k \cap B)} = \frac{P(B|A_2)P(A_2)}{\sum_{k=1}^{4} P(A_k)P(B|A_k)} = \frac{1}{9}$$

13 Chapter 2 #102

Let H be the event where you picked A and the host opened B revealing no prize. Since the door is chosen randomly, $P(A) = P(B) = P(C) = 1/3$. Assuming that the host does not open the chosen door or the door which has the prize, we know that

$$P(H|A) = \frac{1}{2}, P(H|B) = 0, P(H|C) = 1$$

Using the Bayes' rule, the probability of getting the prize if you don't switch is

$$\begin{aligned} P(A|H) &= \frac{P(A \cap H)}{P(A \cap H) + P(B \cap H) + P(C \cap H)} \\ &= \frac{P(H|A)P(A)}{P(H|A)P(A) + P(H|B)P(B) + P(H|C)P(C)} = \frac{1}{3} \end{aligned}$$

However, if you switch,

$$\begin{aligned} P(C|H) &= \frac{P(C \cap H)}{P(A \cap H) + P(B \cap H) + P(C \cap H)} \\ &= \frac{P(H|C)P(C)}{P(H|A)P(A) + P(H|B)P(B) + P(H|C)P(C)} = \frac{2}{3} \end{aligned}$$

Thus, switching door is better.