

MATH230: Homework 3 (due Sep. 25)

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Last compiled on: Monday 25th September, 2023, 17:19

1 Chapter 4 #17

Let $S_x = \{-3, 6, 9\}$. Using the definition of expected value, we can write

$$\mu_{g(X)} = E[g(X)] = \sum_{x \in S_x} g(x)f(x) = \sum_{x \in S_x} (2x+1)^2 f(x) = \frac{131}{3}$$

2 Chapter 4 #23

2.1 Solution for (a)

Let $S_x = \{2, 4\}$ and $S_y = \{1, 3, 5\}$. Using the definition of expected value, we can write

$$E[g(X, Y)] = \sum_{x \in S_x} \sum_{y \in S_y} g(x, y)f(x, y) = \sum_{x \in S_x} \sum_{y \in S_y} xy^2 f(x, y) = \frac{317}{10}$$

2.2 Solution for (b)

Using the definition of expected value, we can write

$$\begin{aligned}\mu_X = E(X) &= \sum_{x \in S_x} \sum_{y \in S_y} xf(x, y) = \frac{29}{10} \\ \mu_Y = E(Y) &= \sum_{x \in S_x} \sum_{y \in S_y} yf(x, y) = 3\end{aligned}$$

3 Chapter 4 #26

Using the definition of expected value, we can write

$$\begin{aligned}E(Z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2 + y^2} f(x, y) dx dy = \int_0^1 \int_0^1 \sqrt{x^2 + y^2} \cdot 4xy \, dx dy \\ &= \int_0^1 \left[\frac{4}{3} y(x^2 + y^2)^{3/2} \right]_0^1 dy = \frac{4}{3} \int_0^1 \left[y(1 + y^2)^{3/2} - y^4 \right] dy \\ &= \frac{4}{3} \int_0^1 y(1 + y^2)^{3/2} dy - \frac{4}{15}\end{aligned}$$

Using the substitution of $y = \tan u$, we can write

$$\begin{aligned} E(Z) &= \frac{4}{3} \int_0^{\frac{\pi}{4}} (\tan u)(1 + \tan^2 u)^{3/2} (\sec^2 u) du = \frac{4}{3} \int_0^{\frac{\pi}{4}} \tan u \sec^5 u \, du \\ &= \frac{4}{3} \left[\frac{1}{5} \sec^5 u \right]_0^{\frac{\pi}{4}} = \frac{4(\sqrt{2} - 1)}{15} \end{aligned}$$

4 Chapter 4 #43

Using the definition and property of expected value, we can write

$$\begin{aligned} \mu_Y &= E(Y) = E(3X - 2) = 3E(X) - 2 = 3 \int_{-\infty}^{\infty} xf(x)dx - 2 \\ &= 3 \int_0^{\infty} x \cdot \frac{1}{4} e^{-x/4} dx - 2 = 3 \left(\lim_{t \rightarrow \infty} \left[(-x - 4)e^{-x/4} \right]_0^t \right) - 2 = 10 \end{aligned}$$

As we can show that $\lim_{t \rightarrow \infty} te^{-t} = 0$ using L'Hôpital's rule. Also, by the definition and property of variance, we can write

$$\begin{aligned} \sigma_Y^2 &= \sigma_{3X-2}^2 = 9\sigma_X^2 = 9[E(X^2) - (E(X))^2] = 9 \left[\int_{-\infty}^{\infty} x^2 f(x) dx - \left(\int_{-\infty}^{\infty} xf(x) dx \right)^2 \right] \\ &= 9 \left[\int_0^{\infty} x^2 \cdot \frac{1}{4} e^{-x/4} dx - \left(\int_0^{\infty} x \cdot \frac{1}{4} e^{-x/4} dx \right)^2 \right] \\ &= 9 \left[\lim_{t \rightarrow \infty} \left[(-x^2 - 8x - 32)e^{-x/4} \right]_0^t - \left(\lim_{t \rightarrow \infty} \left[(-x - 4)e^{-x/4} \right]_0^t \right)^2 \right] \\ &= 9(32 - 16) = 144 \end{aligned}$$

5 Chapter 4 #52

First, we can write

$$\begin{aligned} \mu_X &= E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x, y) dx dy = \int_0^1 \int_0^y xf(x, y) dx dy \\ &= \int_0^1 [x^2]_0^y dy = \int_0^1 y^2 dy = \frac{1}{3} \\ \mu_Y &= E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x, y) dx dy = \int_0^1 \int_0^y 2y \, dx dy \\ &= \int_0^1 2y^2 \, dy = \left[\frac{2}{3} y^3 \right]_0^1 = \frac{2}{3} \end{aligned}$$

Using the definition of variance, we can write

$$\begin{aligned}
\sigma_X^2 &= E(X^2) - \mu_X^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f(x, y) dx dy - \frac{1}{9} = \int_0^1 \int_0^y 2x^2 dx dy - \frac{1}{9} \\
&= \int_0^1 \left[\frac{2}{3} x^3 \right]_0^y dy - \frac{1}{9} = \int_0^1 \frac{2}{3} y^3 dy - \frac{1}{9} = \frac{1}{18} \\
\sigma_Y^2 &= E(Y^2) - \mu_Y^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 f(x, y) dx dy - \frac{4}{9} = \int_0^1 \int_0^y 2y^2 dx dy - \frac{4}{9} \\
&= \int_0^1 [2y^2 x]_0^y dy - \frac{4}{9} = \int_0^1 2y^3 dy - \frac{4}{9} = \frac{1}{18}
\end{aligned}$$

Using the definition of covariance, we can write

$$\begin{aligned}
\sigma_{XY} &= E(XY) - \mu_X \mu_Y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x, y) dx dy - \frac{2}{9} \\
&= \int_0^1 \int_0^y 2xy dx dy - \frac{2}{9} = \int_0^1 [x^2 y]_0^y dy - \frac{2}{9} = \int_0^1 y^3 dy - \frac{2}{9} = \frac{1}{36}
\end{aligned}$$

By the definition of correlation,

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{1}{2}$$

6 Chapter 4 #70

6.1 Solution for (a)

Considering the marginal densities $g(x)$ and $h(y)$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$,

$$\begin{aligned}
g(x) &= \int_{-\infty}^{\infty} f(x, y) dy = \frac{3}{2} \int_0^1 (x^2 + y^2) dy = \frac{3}{2} \left[x^2 y + \frac{1}{3} y^3 \right]_0^1 = \frac{3}{2} \left(x^2 + \frac{1}{3} \right) \\
h(y) &= \int_{-\infty}^{\infty} f(x, y) dx = \frac{3}{2} \int_0^1 (x^2 + y^2) dx = \frac{3}{2} \left[\frac{1}{3} x^3 + xy^2 \right]_0^1 = \frac{3}{2} \left(y^2 + \frac{1}{3} \right)
\end{aligned}$$

Take $x = 1$ and $y = 1$, then $f(x, y) = 3 \neq g(1)h(1) = 4$. Thus, X and Y are dependent.

6.2 Solution for (b)

By property of expected value, we can write

$$\begin{aligned}
E(X + Y) &= E(X) + E(Y) = \int_{-\infty}^{\infty} xg(x) dx + \int_{-\infty}^{\infty} yh(y) dy = \int_0^1 xg(x) dx + \int_0^1 yh(y) dy \\
&= \int_0^1 \frac{3}{2} x \left(x^2 + \frac{1}{3} \right) dx + \int_0^1 \frac{3}{2} y \left(y^2 + \frac{1}{3} \right) dy = \left[\frac{3}{8} x^4 + \frac{1}{4} x^2 \right]_0^1 + \left[\frac{3}{8} y^4 + \frac{1}{4} y^2 \right]_0^1 \\
&= \frac{5}{4}
\end{aligned}$$

Also, by the definition of expected value,

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y)dx dy = \int_0^1 \int_0^1 \frac{3}{2}xy(x^2 + y^2)dx dy \\ &= \int_0^1 \left[\frac{3}{8}x^4y + \frac{3}{4}x^2y^3 \right]_{x=0}^1 dy = \int_0^1 \left(\frac{3}{8}y + \frac{3}{4}y^3 \right) dy \\ &= \left[\frac{3}{16}y^2 + \frac{3}{16}y^4 \right]_0^1 = \frac{3}{8} \end{aligned}$$

6.3 Solution for (c)

By a property of variance, we can write

$$\begin{aligned} \text{Var}(X) &= E(X^2) - \mu_X^2 = \int_{-\infty}^{\infty} x^2g(x)dx - \mu_X^2 = \int_0^1 \frac{3}{2}x^2 \left(x^2 + \frac{1}{3} \right) dx - \left(\frac{5}{8} \right)^2 \\ &= \left[\frac{3}{10}x^5 + \frac{1}{6}x^3 \right]_0^1 - \left(\frac{5}{8} \right)^2 = \frac{73}{960} \\ \text{Var}(Y) &= E(Y^2) - \mu_Y^2 = \int_{-\infty}^{\infty} y^2h(y)dy - \mu_Y^2 = \int_0^1 \frac{3}{2}y^2 \left(y^2 + \frac{1}{3} \right) dy - \left(\frac{5}{8} \right)^2 \\ &= \left[\frac{3}{10}y^5 + \frac{1}{6}y^3 \right]_0^1 - \left(\frac{5}{8} \right)^2 = \frac{73}{960} \end{aligned}$$

By a property of covariance,

$$\text{Cov}(X, Y) = E(XY) - \mu_X\mu_Y = \frac{3}{8} - \frac{5}{8} \cdot \frac{5}{8} = -\frac{1}{64}$$

6.4 Solution for (d)

By properties of variance and covariance,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) = \frac{73}{960} + \frac{73}{960} + 2 \cdot \left(-\frac{1}{64} \right) = \frac{29}{240}$$

7 Chapter 4 #78

By the definition of expected value, we can write

$$\begin{aligned} \mu &= E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 30x^3(1-x)^2dx = 30 \int_0^1 (x^3 - 2x^4 + x^5)dx \\ &= 30 \left[\frac{1}{4}x^4 - \frac{2}{5}x^5 + \frac{1}{6}x^6 \right]_0^1 = \frac{1}{2} \end{aligned}$$

By the definition of variance,

$$\begin{aligned} \sigma^2 &= E(X^2) - \mu^2 = \int_{-\infty}^{\infty} x^2f(x)dx - \mu^2 = \int_0^1 30x^4(1-x)^2dx - \mu^2 \\ &= 30 \int_0^1 (x^4 - 2x^5 + x^6)dx - \mu^2 = 30 \left[\frac{1}{5}x^5 - \frac{2}{6}x^6 + \frac{1}{7}x^7 \right]_0^1 - \mu^2 = \frac{2}{7} - \frac{1}{4} = \frac{1}{28} \end{aligned}$$

Thus, we can write

$$\begin{aligned} P(\mu - 2\sigma < X < \mu + 2\sigma) &= \int_{\mu-2\sigma}^{\mu+2\sigma} f(x)dx = \int_{\frac{1}{2}-\frac{1}{\sqrt{7}}}^{\frac{1}{2}+\frac{1}{\sqrt{7}}} 30x^2(1-x)^2 dx \\ &= 30 \left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_{\frac{1}{2}-\frac{1}{\sqrt{7}}}^{\frac{1}{2}+\frac{1}{\sqrt{7}}} = 0.96998 \end{aligned}$$

The lower bound given by Chebyshev's theorem is $1 - 2^{-2} = 3/4$.

8 Lecture Note Exercise #4.1

8.1 Solution for (a)

Let $S_x = \{0, 1, 2\}$ and $S_y = \{1, 2\}$. We can write

$$\begin{aligned} \mu_X &= \sum_{x \in S_x} xg(x) = 1 \\ \sigma_X^2 &= E(X^2) - \mu_X^2 = \sum_{x \in S_x} x^2g(x) - \mu_X^2 = \frac{14}{9} - 1 = \frac{5}{9} \end{aligned}$$

8.2 Solution for (b)

Let $m(y) = \sum_{x \in S_x} xf(x|y)$. Then we can write

$$\begin{aligned} m(1) &= \sum_{x \in S_x} xf(x|y=1) = \sum_{x \in S_x} x \frac{f(x,1)}{h(1)} = \frac{f(1,1)}{h(1)} + \frac{2f(2,1)}{h(1)} \\ &= \frac{4}{18} \cdot \left(\frac{1}{2}\right)^{-1} + 2 \left(\frac{1}{18}\right) \left(\frac{1}{2}\right)^{-1} = \frac{2}{3} \\ m(2) &= \sum_{x \in S_x} xf(x|y=2) = \sum_{x \in S_x} x \frac{f(x,2)}{h(2)} = \frac{f(1,2)}{h(2)} + \frac{2f(2,2)}{h(2)} \\ &= \frac{4}{18} \cdot \left(\frac{1}{2}\right)^{-1} + 2 \left(\frac{4}{18}\right) \left(\frac{1}{2}\right)^{-1} = \frac{4}{3} \end{aligned}$$

We can write

$$\mu_X = E[m(Y)] = \sum_{y \in S_y} m(y)h(y) = m(1)h(1) + m(2)h(2) = \frac{2}{3} \cdot \frac{1}{2} + \frac{4}{3} \cdot \frac{1}{2} = 1$$

This coincides with the result of (a).

8.3 Solution for (c)

Let $v(y) = E(X^2|y) - (m(y))^2$. Then

$$\begin{aligned}
 v(1) &= E(X^2|y=1) - (m(1))^2 = \sum_{x \in S_x} x^2 f(x|y=1) - (m(1))^2 \\
 &= \frac{f(1,1)}{h(1)} + \frac{4f(2,1)}{h(1)} - (m(1))^2 = \frac{4}{18} \cdot \left(\frac{1}{2}\right)^{-1} + 4 \left(\frac{1}{18}\right) \left(\frac{1}{2}\right)^{-1} - \left(\frac{2}{3}\right)^{-1} = \frac{4}{9} \\
 v(2) &= E(X^2|y=2) - (m(2))^2 = \sum_{x \in S_x} x^2 f(x|y=2) - (m(2))^2 \\
 &= \frac{f(1,2)}{h(2)} + \frac{4f(2,2)}{h(2)} - (m(2))^2 = \frac{4}{18} \cdot \left(\frac{1}{2}\right)^{-1} + 4 \left(\frac{4}{18}\right) \left(\frac{1}{2}\right)^{-1} - \left(\frac{4}{3}\right)^{-1} = \frac{4}{9}
 \end{aligned}$$

We can write

$$\begin{aligned}
 \sigma_X^2 &= E[v(Y)] + \text{Var}[m(Y)] = \sum_{y \in S_y} v(y)h(y) + \sum_{y \in S_y} (m(y))^2 h(y) - \left(\sum_{y \in S_y} m(y)h(y) \right)^2 \\
 &= v(1)h(1) + v(2)h(2) + (m(1))^2 h(1) + (m(2))^2 h(2) - (m(1)h(1) + m(2)h(2))^2 = \frac{5}{9}
 \end{aligned}$$

As $m(y)$ is the expected value of X when given a value of Y , the variation from randomness in coin selection is captured by $\text{Var}[m(Y)]$. The term can be calculated as

$$\begin{aligned}
 \text{Var}[m(Y)] &= \sum_{y \in S_y} (m(y))^2 h(y) - \left(\sum_{y \in S_y} m(y)h(y) \right)^2 \\
 &= (m(1))^2 h(1) + (m(2))^2 h(2) - (m(1)h(1) + m(2)h(2))^2 = \frac{1}{9}
 \end{aligned}$$