MATH230: Homework 11 (due Nov. 20)

손량(20220323)

Last compiled on: Saturday 18th November, 2023, 22:37

1 Chapter 9 #3

By using the provided values, we can plug $\sigma = 0.0015, n = 75, \bar{x} = 0.310, \alpha = 0.05$ to the formula for confidence invterval, we get

$$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = 0.310 \pm 0.0003394757$$

which can be written as $(0.3096605, 0.3103395)$.

2 Chapter 9 #7

Let $d = 0.0005$, then a lower bound of sample size required can be calculated as

$$\left(\frac{z_{\frac{\alpha}{2}} \sigma}{d} \right)^2 = 34.57313$$

Thus, at least 35 samples are needed.

3 Chapter 9 #11

In this case, the variance of the distribution is not known, so we have to resort to the formula involving sample variance. Plugging in $n = 9, \bar{x} = 1.005556, s = 0.02455153, \alpha = 0.05$, we get

$$\bar{x} \pm t_{(n-1), \frac{\alpha}{2}} \frac{s}{\sqrt{n}} = 1.005556 \pm 0.01887198$$

which can be written as $(0.9866836, 1.024428)$.

4 Chapter 9 #30

4.1 Solution for (a)

Let μ be the mean of the original distribution. We can write

$$\begin{aligned}\sum_{i=1}^{n}(X_i-\mu)^2 &= \sum_{i=1}^{n}(X_i-\bar{X}+\bar{X}-\mu)^2\\ &= \sum_{i=1}^{n}(X_i-\bar{X})^2+2(\bar{X}-\mu)\sum_{i=1}^{n}(X_i-\bar{X})+n(\bar{X}-\mu)^2\\ &= \sum_{i=1}^{n}(X_i-\bar{X})^2+n(\bar{X}-\mu)^2\end{aligned}$$

By taking expectations of both sides, we get

$$n\sigma^2 = E\left(\sum_{i=1}^{n}(X_i-\bar{X})^2\right)+n\cdot\frac{\sigma^2}{n}$$

Thus, we can write

$$E(S'^2)=\frac{n-1}{n}\sigma^2$$

In conclusion, the bias is $E(S'^2-\sigma^2)=E(S'^2)-\sigma^2=-\sigma^2/n$.

4.2 Solution for (b)

It is well known that $1/n$ converges to zero as $n\to\infty$, so the bias $-\sigma^2/n$ also converges to zero as $n\to\infty$.

5 Chapter 9 #31

5.1 Solution for (a)

As $E(X)=np$, $E(\hat{P})=E(X/n)=E(X)/n=np/n=p$, so the bias is $E(\hat{P}-p)=E(\hat{P})-p=0$. Thus, $\hat{P}$ is an unbiased estimator of p.

5.2 Solution for (b)

We can write

$$E(P')=E\left(\frac{X+\sqrt{n}/2}{n+\sqrt{n}}\right)=\frac{E(X)+\sqrt{n}/2}{n+\sqrt{n}}=\frac{np+\sqrt{n}/2}{n+\sqrt{n}}$$

Then, the bias can be calculated as

$$E(P'-p)=E(P')-p=\frac{np+\sqrt{n}/2}{n+\sqrt{n}}-p=\frac{\sqrt{n}/2-p\sqrt{n}}{n+\sqrt{n}}\neq 0$$

Thus, P' is a biased estimator of p.

5.3 Solution for (c)

Letting $q = 1 - p$, We can write

$$\begin{aligned}
\mathrm{MSE}(\hat{P}) &= E((\hat{P}-p)^2) = E(\hat{P}^2) - 2pE(\hat{P}) + p^2 = \frac{E(X^2)}{n^2} - p^2 \\
&= \frac{\mathrm{Var}(X) + (E(X))^2}{n^2} - p^2 = \frac{npq + n^2p^2}{n^2} - p^2 = \frac{pq}{n} \\
\mathrm{MSE}(P') &= E((P'-p)^2) = E(P'^2) - 2pE(P') + p^2 \\
&= E\left(\left(\frac{X+\sqrt{n}/2}{n+\sqrt{n}}\right)^2\right) - 2p \cdot \frac{np+\sqrt{n}/2}{n+\sqrt{n}} + p^2 \\
&= \frac{E(X^2) + \sqrt{n}E(X) + n/4}{(n+\sqrt{n})^2} - 2p \cdot \frac{np+\sqrt{n}/2}{n+\sqrt{n}} + p^2 \\
&= \frac{\mathrm{Var}(X) + (E(X))^2 + \sqrt{n}E(X) + n/4}{(n+\sqrt{n})^2} - 2p \cdot \frac{np+\sqrt{n}/2}{n+\sqrt{n}} + p^2 \\
&= \frac{npq + (np)^2 + \sqrt{n}np + n/4}{(n+\sqrt{n})^2} - 2p \cdot \frac{np+\sqrt{n}/2}{n+\sqrt{n}} + p^2 \\
&= \frac{n+\sqrt{n}}{4[n^2 + (3n+1)\sqrt{n} + 3n]}
\end{aligned}$$

6 Chapter 9 #33

We can write

$$\begin{aligned}
\mathrm{Var}(S^2) &= \frac{\sigma^4}{(n-1)^2}\mathrm{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) = \frac{\sigma^4}{(n-1)^2}\mathrm{Var}(\chi^2_{n-1}) \\
&= \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{n-1} \\
\mathrm{Var}(S'^2) &= \mathrm{Var}\left(\frac{n-1}{n}S^2\right) = \frac{(n-1)^2}{n^2}\mathrm{Var}(S^2) = \frac{2\sigma^4(n-1)}{n^2}
\end{aligned}$$

As $(n-1)^{-2} > n^{-2}$, $(n-1)^{-1} > (n-1)/n^2$ holds, so $\mathrm{Var}(S^2) > \mathrm{Var}(S'^2)$, which means that S'^2 is more efficient, when considering only variance.

7 Chapter 9 #34

We can write

$$\begin{aligned}
\mathrm{MSE}(S^2) &= E((S^2-\sigma^2)^2) = E((S^2)^2) - 2\sigma^2E(S^2) + \sigma^4 \\
&= \mathrm{Var}(S^2) + (E(S^2))^2 - 2\sigma^2E(S^2) + \sigma^4 = \mathrm{Var}(S^2) + \sigma^4 - 2\sigma^4 + \sigma^4 \\
&= \mathrm{Var}(S^2) = \frac{2\sigma^4}{n-1} \\
\mathrm{MSE}(S'^2) &= E((S'^2-\sigma^2)^2) = E((S'^2)^2) - 2\sigma^2E(S'^2) + \sigma^4 \\
&= \mathrm{Var}(S'^2) + (E(S'^2))^2 - 2\sigma^2E(S'^2) + \sigma^4 \\
&= \frac{2\sigma^4(n-1)}{n^2} + \left(\frac{n-1}{n}\sigma^2\right)^2 - 2\sigma^2\left(\frac{n-1}{n}\sigma^2\right) + \sigma^4 = \frac{2n-1}{n^2}\sigma^4
\end{aligned}$$

From this,

$$\frac{\mathrm{MSE}(S^2)}{\mathrm{MSE}(S'^2)} = \frac{2\sigma^4}{n-1}\left(\frac{2n-1}{n^2}\sigma^4\right)^{-1} = \frac{2n^2}{(n-1)(2n-1)} = \frac{2n^2}{2n^2-3n+1}$$

For $n \geq 1$, $2n^2 > 2n^2 - 3n + 1$ holds, so $\mathrm{MSE}(S^2)/\mathrm{MSE}(S'^2) > 1$ and we can conclude that S'^2 is more efficient in terms of mean-squared-error.