MATH230: Homework 7 (due Oct. 30)

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1 Chapter 7 #4

Let $Z_1 = X + Y$, $Z_2 = Y$ and $g(z_1, z_2) = P(Z_1 = z_1, Z_2 = z_2)$. Then we can write

$$g(z_1, z_2) = f(z_1 - z_2, z_2) = \begin{cases} \frac{z_1 + z_2}{27} & (z_2 \in \{0, 1, 2\}, z_1 - z_2 \in \{0, 1, 2\}) \\ 0 & (\text{otherwise}) \end{cases}$$

In conclusion, we can calculate the marginal distribution of $Z_1 = Z$.

$$\begin{aligned} &P(Z = 0) = g(0, 0) = 0 \\ &P(Z = 1) = g(1, 0) + g(1, 1) = \frac{1}{9} \\ &P(Z = 2) = g(2, 0) + g(2, 1) + g(2, 2) = \frac{1}{3} \\ &P(Z = 3) = g(3, 1) + g(3, 2) = \frac{1}{3} \\ &P(Z = 4) = g(4, 2) = \frac{2}{9} \end{aligned}$$

2 Chapter 7 #5

Let $w(y) = \exp(-y/2)$. Then $w(Y) = X$ holds, so we can write

$$g(y) = f(w(y))|w'(y)| = \begin{cases} \frac{\exp(-y/2)}{2} & (y > 0) \\ 0 & (y \leq 0) \end{cases}$$

and $g(y)$ is probability density function of Y. Thus, Y follows an exponential distribution with $\lambda = 1/2$.

3 Chapter 7 #7

Let $\phi(w) = \sqrt{2w/m}$. Then $\phi(W) = V$ holds, so we can write

$$g(w) = f(\phi(w))|\phi'(w)| = \begin{cases} \frac{k\sqrt{2w}}{m^{3/2}} \exp(-2bw/m) & (w > 0) \\ 0 & (w \leq 0) \end{cases}$$

which is a probability distribution of W.

4 Chapter 7 #12

Let $w_1(y_1, y_2) = y_1 y_2, w_2(y_1, y_2) = y_1 - y_1 y_2$. Then $X_1 = w_1(Y_1, Y_2), X_2 = w_2(Y_1, Y_2)$, so

$$J = \det \begin{pmatrix} \frac{\partial w_1}{\partial y_1} & \frac{\partial w_1}{\partial y_2} \\ \frac{\partial w_2}{\partial y_1} & \frac{\partial w_2}{\partial y_2} \end{pmatrix} = \det \begin{pmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{pmatrix} = -y_1$$

Then, we can write the joint probability density of Y_1 and Y_2 as

$$\begin{aligned} g(y_1, y_2) &= f(w_1(y_1, y_2)) f(w_2(y_1, y_2)) |J| \\ &= \begin{cases} e^{-y_1 y_2} e^{-y_1 + y_1 y_2} |y_1| & (y_1 y_2 > 0, y_1 - y_1 y_2 > 0) \\ 0 & (\text{elsewhere}) \end{cases} \\ &= \begin{cases} e^{-y_1} |y_1| & (y_1 y_2 > 0, y_1 - y_1 y_2 > 0) \\ 0 & (\text{elsewhere}) \end{cases} \end{aligned}$$

The marginal distribution of Y_1 can be obtained as

$$h_1(y_1) = \int_{-\infty}^{\infty} g(y_1, y_2) dy_2 = \int_0^1 e^{-y_1} |y_1| dy_2 = e^{-y_1} |y_1|$$

for $y_1 > 0$. For $y_1 \leq 0$, $g(y_1, y_2)$ is zero so $h_1(y_1) = 0$. Similarly, the marginal distribution of Y_2 can be obtained as

$$\begin{aligned} h_2(y_2) &= \int_{-\infty}^{\infty} g(y_1, y_2) dy_1 = \int_0^{\infty} e^{-y_1} |y_1| dy_1 = \int_0^{\infty} y_1 e^{-y_1} dy_1 \\ &= \left[(-y_1 - 1) e^{-y_1} \right]_0^{\infty} = 1 \end{aligned}$$

for $0 < y_2 < 1$. Otherwise, $g(y_1, y_2)$ is zero so $h_2(y_2) = 0$. Then,

$$h_1(y_1) h_2(y_2) = \begin{cases} e^{-y_1} |y_1| & (y_1 > 0, 0 < y_2 < 1) \\ 0 & (\text{elsewhere}) \end{cases}$$

As $y_1 y_2 > 0$ and $y_1 - y_1 y_2 > 0$ is equivalent to $y_1 > 0$ and $0 < y_2 < 1$, $g(y_1, y_2) = h_1(y_1) h_2(y_2)$ holds so Y_1 and Y_2 are independent.

5 Chapter 7 #18

We can write

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p q^{x-1} = \sum_{x=0}^{\infty} e^{t(x+1)} p q^x = p e^t \sum_{x=0}^{\infty} (q e^t)^x = \frac{p e^t}{1 - q e^t}$$

using the Taylor series expansion of geometric series. Now, we can write

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} = \frac{p e^t}{(1 - q e^t)^2} \Big|_{t=0} = \frac{p}{(1 - q)^2} = \frac{1}{p}$$

So the mean of X is $1/p$. We can also write

$$E(X^2) = \left(\frac{d}{dt} \right)^2 M_X(t) \Big|_{t=0} = \frac{p e^t (1 + q e^t)}{(1 - q e^t)^3} \Big|_{t=0} = \frac{1 + q}{p^2}$$

Thus, the variance of X is $E(X^2) - (E(X))^2 = q/p^2$.

6 Chapter 7 #23

6.1 Solution for (a)

We can write

$$P(U=u)=\sum_{x=0}^{u}P(X=x)P(Y=u-x)=\sum_{x=0}^{u}(q^{x}p)(q^{u-x}p)=\sum_{x=0}^{u}q^{u}p^{2}=(u+1)q^{u}p^{2}$$

for $u=0,1,2,\ldots$. (otherwise the probability is zero)

6.2 Solution for (b)

For $x\le u$, we can write

$$\begin{aligned}P(X=x,X+Y=u)&=P(X=x,Y=u-x)=P(X=x)P(Y=u-x)\\&=(q^{x}p)(q^{u-x}p)=q^{u}p^{2}\end{aligned}$$

Using the definition of conditional probability,

$$P(X=x|X+Y=u)=\frac{P(X=x,X+Y=u)}{P(X+Y=u)}=\frac{q^{u}p^{2}}{(u+1)q^{u}p^{2}}=\frac{1}{u+1}$$

7 Supplemental Exercises #1

We can write

$$\begin{aligned}
P(D=0)&=\sum_{x=0}^{5}P(X_1=x)P(X_2=x)=6\cdot\frac{1}{6^2}=\frac{1}{6}\\
P(D=1)&=\sum_{x=0}^{4}P(X_1=x)P(X_2=x+1)+\sum_{x=0}^{4}P(X_1=x+1)P(X_2=x)\\
&=2\cdot5\cdot\frac{1}{6^2}=\frac{5}{18}\\
P(D=2)&=\sum_{x=0}^{3}P(X_1=x)P(X_2=x+2)+\sum_{x=0}^{3}P(X_1=x+2)P(X_2=x)\\
&=2\cdot4\cdot\frac{1}{6^2}=\frac{2}{9}\\
P(D=3)&=\sum_{x=0}^{2}P(X_1=x)P(X_2=x+3)+\sum_{x=0}^{2}P(X_1=x+3)P(X_2=x)\\
&=2\cdot3\cdot\frac{1}{6^2}=\frac{1}{6}\\
P(D=4)&=\sum_{x=0}^{1}P(X_1=x)P(X_2=x+4)+\sum_{x=0}^{1}P(X_1=x+4)P(X_2=x)\\
&=2\cdot2\cdot\frac{1}{6^2}=\frac{1}{9}\\
P(D=5)&=\sum_{x=0}^{0}P(X_1=x)P(X_2=x+5)+\sum_{x=0}^{0}P(X_1=x+5)P(X_2=x)\\
&=2\cdot1\cdot\frac{1}{6^2}=\frac{1}{18}
\end{aligned}$$

8 Supplemental Exercises #3 (a)

Let $\phi(x)$ be probability density function of standard normal distribution. Since $\mathrm{Cov}((X_1 + 1)^2, X_2) = 0$,

$$
\begin{aligned}
E(Y) = E((X_1+1)^2 + X_2) = E((X_1+1)^2) + E(X_2) &= \int_{-\infty}^{\infty} (x+1)^2 \phi(x) dx \\
&= \int_{-\infty}^{\infty} x^2 \phi(x) dx + \int_{-\infty}^{\infty} 2x\phi(x) dx + \int_{-\infty}^{\infty} \phi(x) dx = E(X_1^2) + E(X_1) + 1 \\
&= \mathrm{Var}(X_1) + (E(X_1))^2 + E(X_1) + 1 = 2
\end{aligned}
$$