MATH311: Homework 3 (due Mar. 14)

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1 Section 1 #9

For all z, w in $\mathbb{C}$, if $z = w$ then $z > w$ or $z < w$ is impossible because $a < c$, $a > c$, $b < d$, $b > d$ are all false as the set of reals is an ordered set. Likewise, if $z < w$ then $z = w$ is impossible because if $a < c$, then $a = c$ is false by orderedness of the set of reals. If $a = c$ but $b < d$, then $b = d$ is false by similar reason. $z > w$ is also impossible because $a > c$ is false as one of $a < c$ or $a = c$ is true. By similar argument, it can be proved that $z = w$ and $z < w$ is false if $z > w$.

Let $z = a + bi, w = c + di, v = e + fi$ where $z < w$, and $w < v$. By the definition of the lexicographic order, $a \leq c$ and $c \leq e$ holds. If $a = c = e$, then $b < d$ and $d < f$ holds by the definition of the order, and $b < f$ is true since the set of reals is an ordered set, so $z < v$ holds by the definition. Otherwise, $a < e$ holds since the set of reals is an ordered set, and $z < v$ also holds by the definition. Thus, the lexicographic order turns $\mathbb{C}$ into an ordered set, since the order satisfies all conditions of definition 1.5.

Let A be a set of complex numbers $z = x + yi$ where $x < 0$. Since $x < 0$, $z < 0$ holds for all $z \in A$, by the definition of lexicographic order. Thus, A is bounded above since it has an upper bound. Let L be a set of upper bounds of A. For all $B = a + bi$ in L, if $a < 0$ then $a/2 \in A$, and $a/2 > a + bi$ by the definition of lexicographic order. Thus, $a \geq 0$ should hold for all $a + bi \in L$, so $L \subset \{z \in \mathbb{C};\ \operatorname{Re} z \geq 0\}$ holds. On the other hand, for all $a' + b'i$ where $a' \geq 0 > x$, $a' + b'i > z$ by the definition of lexicographic order. so $\{z \in \mathbb{C};\ \operatorname{Re} z \geq 0\} \subset L$ holds. This implies that $L = \{z \in \mathbb{C};\ \operatorname{Re} z \geq 0\}$. Suppose that there exists a $u + vi \in L$ such that $u + vi \leq B$ for all $B \in L$. However, $u + (v-1)i < u + vi$ and $u + (v-1)i \in L$, so this is a contradiction. Since a bounded subset $A \subset \mathbb{C}$ does not have the least upper bound, the ordered set does not have the least upper bound property.

2 Section 1 #15

Let $A = \sum |a_j|^2, B = \sum |b_j|^2, C = \sum a_j \overline{b_j}$. Then we can write the following:

$$\sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 - \left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 = AB - |C|^2 \geq 0$$

If $b_1 = \cdots = b_n = 0$, then $B = 0$ and the equality holds. Assume that $B > 0$, by theorem 1.31 we can write

$$
\begin{aligned}
AB - |C|^2 = \frac{B(AB - |C|^2)}{B} &= \frac{B^2A - B|C|^2}{B} \\
&= \frac{1}{B}\left(B^2\sum_{j=1}^{n}|a_j|^2 - B\overline{C}\sum_{j=1}^{n}a_j\overline{b_j} - BC\sum_{j=1}^{n}\overline{a_j}b_j + |C|^2\sum_{j=1}^{n}|b_j|^2\right) \\
&= \frac{1}{B}\sum_{j=1}^{n}(Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) = \frac{1}{B}\sum_{j=1}^{n}|Ba_j - Cb_j|^2
\end{aligned}
$$

If the equality holds, then $|Ba_j - Cb_j|^2 = 0$ for $j = 1, 2, \dots, n$, thus $Ba_j - Cb_j = 0$ by theorem 1.33. In other words, there exists some ratio $r \in \mathbb{C}$ where $a_j - rb_j = 0$ if the equality holds. We can prove the converse: suppose that there exists some $r \in \mathbb{C}$ where $a_j - rb_j = 0$ for $j = 1, 2, \dots, n$. Then we can write

$$
\begin{aligned}
A &= \sum_{j=1}^{n}|a_j|^2 = \sum_{j=1}^{n}|rb_j|^2 = \sum_{j=1}^{n}r^2|b_j|^2 = r^2\sum_{j=1}^{n}|b_j|^2 = r^2B \\
C &= \sum_{j=1}^{n}a_j\overline{b_j} = \sum_{j=1}^{n}rb_j\overline{b_j} = r\sum_{j=1}^{n}|b_j|^2 = rB
\end{aligned}
$$

and $AB - |C|^2 = r^2B^2 - |rB|^2 = 0$ by theorem 1.33, so the equality holds.

In conclusion, the equality in the Schwarz inequality holds iff $b_1 = \cdots = b_n = 0$ or there exists some $r \in \mathbb{C}$ such that $a_j = rb_j$ for $j = 1, 2, \dots, n$.[1]

3 Section 1 #17

By the definition of norm, we can write

$$
\begin{aligned}
|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\
&= |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\mathbf{x}\cdot\mathbf{y} \\
&= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2
\end{aligned}
$$

Consider $\mathbf{x}, \mathbf{y}$ as the adjacent sides of a parallelogram. Then $\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}$ can be considered as two diagonals of the parallelogram. From the equation, we can know that the sum of squares of the length of sides of parallelogram is equal to the sum of square of the length of diagonals.

4 Section 2 #2

Let A_m be a set of equations $a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0$, where $a_0, a_1, \dots, a_n$ are integers and not all zero, and $n + |a_0| + |a_1| + \cdots + |a_n| = m$. The set A_m is finite since there are only finitely many equations satisfying this condition, according to the hint. For all $a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0$, there exists a set A_m which has $a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0$ as an element. Let B_m be a set of solutions of equations in A_m. Since A_m is a finite set, and every equation in A_m has finite number of solutions by the fundamental theorem of algebra, B_m is also a finite set. So the union $\bigcup_{i=1}^{\infty} B_i$ is at most countable by the theorem 2.12. Since all rational numbers are algebraic, the set of algebraic numbers is countable.

[1] Alternatively, we can state that the equality holds iff $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ are linearly dependent.

5 Section 2 #9

5.1 Proof for (a)

Suppose that there exists a point $q \in E^\circ$ such that for all of its neighborhood N there exists a point r which is not in E°. Since q is an interrior point of E, there exists a neighborhood N of q such that $N \subset E$. By the assumption, there exists a point $r \in N$ such that $r \notin E^\circ$. By the theorem 2.19, N is an open set so r is an interrior point of N. Then, there exists a neighborhood M of r such that $M \subset N$. Since $N \subset E$, $M \subset E$ holds and r is also an interrior point of E, which is a contradiction. Thus, such q cannot exist, so for all $p \in E^\circ$, there exists a neighborhood N of p such that $N \subset E^\circ$, which means that p is an interrior point of E°. In conclusion, every point of E° is an interrior point of E°, so E° is an open set by definition.

5.2 Proof for (b)

By definition, E is open iff its points is an interrior point, so it is same as saying that $E = E^\circ$.

5.3 Proof for (c)

Suppose that there exist a point $q \in G$ such that $q \notin E^\circ$. Since G is an open set, q is an interrior point of G, so there exists a neighborhood N of q such that $N \subset G$. However, $G \subset E$ implies that $N \subset E$, thus q is an interrior point of E by definition, so $q \in E^\circ$ which is a contradiction. In conclusion, such q does not exist, and for all $p \in G$, $p \in E^\circ$, so $G \subset E^\circ$.

6 Section 2 #10

6.1 Proof for metric

For all $x, y, z \in X$, $d(x, x) = 0$ and $d(x, y) = d(y, x)$ is evident from its definition. Also, if $x \neq y$, then $d(x, y) = 1 > 0$. Let's show that the triangle inequality $d(x, z) + d(z, y) \geq d(x, y)$ holds. Since the right hand side is at most 1 and it can be 1 only if $x \neq y$, z cannot be equal to both x and y. Thus, at least one of $d(x, z)$ and $d(z, y)$ is 1, so the left hand side is at least 1 and the equality holds. In conclusion, d is a metric since it satisfies all the conditions of definition 2.15.

6.2 Open and closed subsets

Theorem 1. *All subset of X are open set.*

Proof. Let A be a subset of X. For all $p \in A$, consider a neighborhood $N = \{q \in X;\ d(p, q) < 1\}$ of p. Since $d(p, q)$ is 0 or 1 for all $p, q \in X$, $d(p, q) < 1$ is equivalent to $d(p, q) = 0$. Thus, $N = \{q \in X;\ p = q\}$, so $N \subset A$ since $p \in A$. So, p is interrior point of A. In conclusion, since every point in A is interrior point, A is an open set. □

Theorem 2. *All subset of X are closed set.*

Proof. Let A be as subset of X. As we proved earlier, $A^C \subset X$ is open. By the corollary of theorem 2.23, A is closed since its complement is open. □

7 Section 2 #11

Theorem 3. $d_1(x,y) = (x-y)^2$ *is not a metric.*

Proof. For all $x, y, z \in \mathbb{R}$, $d_1(x,z) + d_1(z,y) - d_1(x,y)$ can be written as

$$\begin{aligned} d_1(x,z) + d_1(z,y) - d_1(x,y) &= (x-z)^2 + (z-y)^2 - (x-y)^2 \\ &= x^2 - 2xz + z^2 + z^2 - 2zy + y^2 - (x^2 - 2xy + y^2) \\ &= 2z^2 - 2xz - 2zy + 2xy = 2(z-x)(z-y) \end{aligned}$$

If $x < z < y$ then $d_1(x,z) + d_1(z,u) - d_1(x,y) = 2(z-x)(z-y) < 0$, so d_1 does not satisfy the condition (c) of definition 2.15. Thus, d_1 is not a metric. $\square$

Theorem 4. $d_2(x,y) = \sqrt{|x-y|}$ *is a metric.*

Proof. For all $x, y, z \in \mathbb{R}$, $d_2(x,x) = \sqrt{|x-x|} = 0$ and $d_2(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d_2(y,x)$ holds. If $x \neq y$, then $d_2(x,y) = \sqrt{|x-y|}$ and $|x-y| > 0$, so $\sqrt{|x-y|} > 0$. $(d_2(x,z) + d_2(z,y))^2 - (d_2(x,y))^2$ can be written as

$$\begin{aligned} (d_2(x,z) + d_2(z,y))^2 - (d_2(x,y))^2 &= \left(\sqrt{|x-z|} + \sqrt{|z-y|}\right)^2 - |x-y| \\ &= |x-z| + |z-y| + 2\sqrt{|x-z||z-y|} - |x-y| \end{aligned}$$

Since $|x-z| + |z-y| \geq |x-y|$ by theorem 1.37 and $\sqrt{|x-z||z-y|} \geq 0$, $(d_2(x,z) + d_2(z,y))^2 - (d_2(x,y))^2 \geq 0$ holds. From $d_2(x,y) \geq 0$, we can conclude that $d_2(x,z) + d_2(z,y) \geq d_2(x,y)$, thus d_2 is a metric since it satisfies all the conditions of definition 2.15. $\square$

Theorem 5. $d_3(x,y) = |x^2 - y^2|$ *is not a metric.*

Proof. Let x be a nonzero real number. Since $2x \neq 0$, we can say that $x \neq -x$. However, $d_3(x,-x) = |x^2 - (-x)^2| = |x^2 - x^2| = 0$ holds, so d_3 does not satisfy the condition (a) of definition 2.15. Thus, d_3 is not a metric. $\square$

Theorem 6. $d_4(x,y) = |x - 2y|$ *is not a metric.*

Proof. Let x be a nonzero real number. $d_4(x,x) = |x - 2x| = |-x| = |x| \neq 0$ holds, so d_4 does not satisfy the condition (a) of definition 2.15. Thus, d_4 is not a metric. $\square$

Theorem 7. $d_5(x,y) = |x-y|/(1+|x-y|)$ *is a metric.*

Proof. For all $x, y, z \in \mathbb{R}$, $d_5(x,x) = |x-x|/(1+|x-x|) = 0$ and $d_5(x,y) = |x-y|/(1+|x-y|) = |y-x|/(1+|y-x|) = d_5(y,x)$ holds. If $x \neq y$, then $d_5(x,y) = |x-y|/(1+|x-y|)$ and $|x-y| > 0$, so $|x-y|/(1+|x-y|) > 0$. We can write

$$\begin{aligned} d_5(x,z) + d_5(z,y) &= \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} \\ &= \frac{|x-z|(1+|z-y|) + |z-y|(1+|x-z|)}{(1+|x-z|)(1+|z-y|)} \\ &= \frac{|x-z| + |z-y| + 2|x-z||z-y|}{1+|x-z|+|z-y|+|x-z||z-y|} \\ &= 1 - \frac{1 - |x-z||z-y|}{1+|x-z|+|z-y|+|x-z||z-y|} \end{aligned}$$

We can also write

$$
\begin{aligned}
1 - d_5(x, y) &= \frac{1}{1 + |x - y|} \geq \frac{1}{1 + |x - z| + |z - y|} \geq \frac{1}{1 + |x - z| + |z - y| + |x - z||z - y|} \\
&\geq \frac{1 - |x - z||z - y|}{1 + |x - z| + |z - y| + |x - z||z - y|} = 1 - (d_5(x, z) + d_5(z, y))
\end{aligned}
$$

Thus, $d_5(x, z) + d_5(z, y) \geq d_5(x, y)$ holds and we can conclude that d_5 is a metric since it satisfies all the conditions of definition 2.15. $\square$