

# Homework 1 (due Feb. 28)

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## 1 Problem 1

### 1.1 Proof for $r + x$

We will use proof by contradiction here. Suppose that  $r + x$  is a rational number. By the definition of rational numbers,  $r + x$  can be written as  $m/n$  where  $m, n \in \mathbb{Z}$  and  $n \neq 0$ . Then,  $x$  can be written as  $m/n - r$ . Since the set of rational number is a field, the additive inverse of  $r$ ,  $-r$  is a rational number. Also,  $m/n - r$  is a rational number because of axioms for addition, which is a contradiction. Thus,  $r + x$  is not a rational number. By the definition of irrational number,  $r + x$  is irrational since it is a real number but not a rational number.

### 1.2 Proof for $rx$

Use the similar argument as  $r + x$  case. Suppose that  $rx$  is a rational number. By the definition of rational numbers,  $rx$  can be written as  $m/n$  where  $m, n \in \mathbb{Z}$  and  $n \neq 0$ . Then,  $x$  can be written as  $(m/n)(1/r)$ . Since the set of rational number is a field, the multiplicative inverse of  $r$ ,  $1/r$  is a rational number. Also,  $(m/n)(1/r)$  is a rational number because of axioms for multiplication, which is a contradiction. Thus,  $rx$  is not a rational number. By the definition of irrational number,  $rx$  is irrational since it is a real number but not a rational number.

## 2 Problem 2

Let  $x$  be a real number where  $x^2 = 12$ , and  $y$  be a real number where  $2y = x$ . Then,  $y^2 = 3$ . Let's prove that  $y$  cannot be a rational number using proof by contradiction. Suppose that there exists  $p \in \mathbb{Q}$  where  $p^2 = 3$ . By the definition of rational numbers,  $p$  can be written as  $m/n$  where  $m, n \in \mathbb{Z}$  are not both multiples of 3. Let's assume that this is done. Then the following holds:

$$3n^2 = m^2$$

This shows that  $m$  is a multiple of 3, and  $m^2$  is divisible by 9. By this, the left side  $3n^2$  is divisible by 9. Thus,  $n^2$  is divisible by 3 and it implies that  $n$  is divisible by 3. This leads to the conclusion that  $m$  and  $n$  are both multiples of 3, which is a contradiction. Since  $y$  cannot be a rational number,  $x$  also cannot be rational using the result proven in problem 1, since  $x = 2y$ .

### 3 Problem 3

First, prove (a) using the axioms (M).

$$\begin{aligned} y &= 1 \cdot y = ((1/x)x)y = (1/x)(xy) \\ &= (1/x)(xz) = ((1/x)x)z = 1 \cdot z = z \end{aligned}$$

Take  $z = 1$  in (a) to obtain (b), and take  $z = 1/x$  in (a) to obtain (c). Since  $x(1/x) = 1$ , (c) (with  $1/x$ ,  $x$  in place of  $x$  and  $y$ , respectively) gives (d).

### 4 Problem 4

Suppose that there exists a subset  $E'$  where  $\alpha' > \beta'$  holds for its lower bound  $\alpha'$  and upper bound  $\beta'$ . By definition,  $x \geq \alpha' \forall x \in E'$  and  $x \leq \beta' \forall x \in E'$ , so there exists an element  $z$  such that  $z \geq \alpha'$  and  $z \leq \beta'$ . There are three possibilities:

1.  $z = \alpha'$
2.  $z = \beta'$
3.  $z \neq \alpha'$  and  $z \neq \beta'$

For  $z = \alpha'$  case,  $z > \beta'$  because we assumed that  $\alpha' > \beta'$ , but it contradicts with  $z \leq \beta'$ .  $z = \beta'$  case is similar because it implies  $\alpha' > z$ , but it contradicts with  $z \geq \alpha'$ . For the final case,  $z \neq \alpha'$  and  $z \neq \beta'$  implies  $z > \alpha'$  and  $z < \beta'$ , so  $\alpha' < \beta'$  according to the axioms of ordered sets. In conclusion, such subset  $E'$  cannot exist, meaning that for all subset  $E$  with lower bound  $\alpha$  and upper bound  $\beta$ ,  $\alpha \leq \beta$  holds.

### 5 Problem 5

Let  $\alpha := \inf A$ . By the definition of lower bound,  $x \geq \alpha \forall x \in A$ . According to axioms of ordered fields,  $-x \leq -\alpha \forall x \in A$ . By the definition of  $-A$ ,  $y \leq -\alpha \forall y \in -A$ . This implies that  $-A$  is bounded above, and  $-\alpha$  is an upper bound of  $-A$ . Since  $A$  is a set of real numbers, so  $-A$  is also a set of real numbers and it has the least upper bound,  $\beta := \sup(-A)$ . Suppose that  $\beta < -\alpha$ . By the definition of upper bound,  $y \leq \beta \forall y \in -A$ . Using axioms of ordered fields again,  $-y \geq -\beta \forall y \in -A$ , and the proposition 1.14 in the book implies that  $x \geq -\beta \forall x \in A$ , so  $-\beta$  is a lower bound of  $A$ . However the assumption we made earlier implies that  $-\beta > \alpha$ , and it contradicts with the definition of  $\alpha$  since lower bound greater than the greatest lower bound must not exist. Thus,  $\beta \geq -\alpha$  holds, and since  $-\alpha$  is an upper bound of  $-A$ , it becomes the least upper bound of  $-A$ . This means that  $\alpha = \inf A = -\beta = -\sup(-A)$ .