MATH311: Homework 10 (due May. 16)

손량(20220323)

Last compiled on: Tuesday 16^{th} May, 2023, 17:22

1 Section 5 #19

1.1 Proof for (a)

Fix $\epsilon > 0$. Since $f'(0)$ exists, there exists some positive integer N such that $n \geq N$ implies

$$\left|\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)\right| < \frac{\epsilon}{2}, \left|\frac{f(0) - f(\alpha_n)}{-\alpha_n} - f'(0)\right| < \frac{\epsilon}{2}$$

Since $\alpha_n < 0 < \beta_n$, we can write

$$\frac{\beta_n}{\beta_n - \alpha_n}\left|\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)\right| < \frac{\epsilon}{2}, \frac{-\alpha_n}{\beta_n - \alpha_n}\left|\frac{f(0) - f(\alpha_n)}{-\alpha_n} - f'(0)\right| < \frac{\epsilon}{2}$$

By triangular inequality,

$$\begin{aligned}
\left|D_n - f'(0)\right| &= \left|\frac{\beta_n}{\beta_n - \alpha_n}\frac{f(\beta_n) - f(0)}{\beta_n} + \frac{-\alpha_n}{\beta_n - \alpha_n}\frac{f(0) - f(\alpha_n)}{-\alpha_n} - f'(0)\right| \\
&\leq \frac{\beta_n}{\beta_n - \alpha_n}\left|\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)\right| + \frac{-\alpha_n}{\beta_n - \alpha_n}\left|\frac{f(0) - f(\alpha_n)}{-\alpha_n} - f'(0)\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

so $\lim_{n\to\infty} D_n = f'(0)$.

1.2 Proof for (b)

We can write

$$\begin{aligned}
D_n &= \frac{f(\beta_n) - f(0)}{\beta_n - \alpha_n} - \frac{f(\alpha_n) - f(0)}{\beta_n - \alpha_n} \\
&= \frac{f(\beta_n) - f(0)}{\beta_n} \cdot \frac{\beta_n}{\beta_n - \alpha_n} - \frac{f(\alpha_n) - f(0)}{\alpha_n} \cdot \frac{\alpha_n}{\beta_n - \alpha_n} \\
&= \frac{f(\beta_n) - f(0)}{\beta_n} \cdot \frac{\beta_n}{\beta_n - \alpha_n} - \frac{f(\alpha_n) - f(0)}{\alpha_n}\left(\frac{\beta}{\beta_n - \alpha_n} - 1\right) \\
&= \frac{f(\alpha_n) - f(0)}{\alpha_n} + \frac{\beta_n}{\beta_n - \alpha_n}\left(\frac{f(\beta_n) - f(0)}{\beta_n} - \frac{f(\alpha_n) - f(0)}{\alpha_n}\right) \qquad (1)
\end{aligned}$$

Since $\beta_n/(\beta_n - \alpha_n)$ is bounded, there exists some positive real number M such that $\beta_n/(\beta_n - \alpha_n) \leq M$ and $\beta_n > \alpha_n > 0$ so $\beta_n/(\beta_n - \alpha_n) > 0$. Using (1), we can write

$$\frac{f(\alpha_n) - f(0)}{\alpha_n} < D_n \leq \frac{f(\alpha_n) - f(0)}{\alpha_n} + M\left(\frac{f(\beta_n) - f(0)}{\beta_n} - \frac{f(\alpha_n) - f(0)}{\alpha_n}\right) \qquad (2)$$

Since $f'(0)$ exists and $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$, we can take the limit of all sides of (2) and obtain

$$f'(0) \leq \lim_{n\to\infty} D_n \leq f'(0) + M(f'(0) - f'(0)) = f'(0)$$

Thus, $\lim_{n\to\infty} D_n = f'(0)$.

1.3 Proof for (c)

Fix $\epsilon > 0$. From continuity of $f'(x)$, there exists some $\delta > 0$ such that $|x| < \delta$ implies $|f'(x) - f'(0)| < \epsilon$. Since $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$, there exists some positive integer N such that $n \geq N$ implies $|\alpha_n| < \delta$ and $|\beta_n| < \delta$. By mean value theorem, for all n there exists γ_n between α_n and β_n such that $f'(\gamma_n) = D_n$. Then, $n \geq N$ implies $|\gamma_n| < \delta$, hence $|f'(\gamma_n) - f'(0)| < \epsilon$. Thus, we can conclude that $\lim_{n\to\infty} D_n = f'(0)$.

2 Section 5 #22

2.1 Proof for (a)

Suppose that $f(x)$ has two fixed points, α and β. By mean value theorem, there exists γ between α and β such that

$$f'(\gamma) = \frac{f(\alpha) - f(\beta)}{\alpha - \beta} = \frac{\alpha - \beta}{\alpha - \beta} = 1$$

which is a contradiction.

2.2 Proof for (b)

We can write

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2}$$

Since $e^t > 0$ for all t, $e^t < 1 + e^t < (1 + e^t)^2$, so $0 < e^t/(1 + e^t)^2 < 1$ and $0 < f'(t) < 1$. Also, $(1 + e^t)^{-1} > 0$ for all t and $f(t) > t$ and f does not have any fixed point.

2.3 Proof for (c)

Let $g(t) = f(t) - t$. Suppose that $f(t)$ does not have any fixed point. Then, Since f is differentiable, f is continuous and g is also continuous, and intermediate value theorem tells us that either $g(t) > 0$ or $g(t) < 0$ holds as $g(t) \neq 0$ for all t. For all real t, $|f'(t)| = |g'(t) + 1| \leq A$ so $-A - 1 \leq g'(t) \leq A - 1 < 0$.

Let's consider the case where $g(t) > 0$ for all t. By mean value theorem, for all $t > s$, we can write

$$\frac{g(t) - g(s)}{t - s} \leq A - 1 \tag{3}$$
$$g(t) \leq (t - s)(A - 1) + g(s)$$

We can take $t = s - g(s)/(A - 1)$, then

$$g(t) \leq \left(s - \frac{g(s)}{A - 1} - s\right)(A - 1) + g(s) = 0$$

which is a contradiction.

Now, let's consider the case where $g(t) < 0$ for all t. Using the mean value theorem again, (3) holds for all $t > s$. Then we can write

$$g(s) \geq g(t) - (t-s)(A-1)$$

Taking $s = t - g(t)/(A-1)$,

$$g(s) \geq g(t) - \left(t - t + \frac{g(t)}{A-1}\right)(A-1) = 0$$

which is a contradiction. In conclusion, $f(t)$ has at least one fixed point.

Since $x_{n+1} = f(x_n)$, $x_{n+1} - x = f(x_n) - x = f(x_n) - f(x)$. By mean value theorem, there exists c_n between x and x_n such that

$$x_{n+1} - x = f(x_n) - f(x) = (x_n - x)f'(c_n)$$

so

$$|x_{n+1} - x| \leq A|x_n - x|$$

by induction, we obtain

$$|x_{n+1} - x| \leq A^n|x_1 - x|$$

and since $0 \geq A < 1$, taking the limit of both sides gives us

$$\lim_{n\to\infty} |x_{n+1} - x| \leq \lim_{n\to\infty} A^n|x_1 - x| = 0$$

so $\lim_{n\to\infty} x_n = x$ by sandwich theorem and we get the desired result.

2.4 Solution for (d)

As $x_{n+1} = f(x_n)$, $(x_n, x_{n+1}) = (x_n, f(x_n))$ is a point on $y = f(x)$. Thus, the process in (c) can be visualized by the path in the problem statement.

3 Section 5 #25

3.1 Solution for (a)

A line tangent to the graph of $f(x)$ at the point $(x_n, f(x_n))$ is $y = f'(x_n)(x - x_n) + f(x_n)$. From this, we know that the tangent line passes $(x_{n+1}, 0)$. Thus, we can interpret x_{n+1} as the x-intercept of the tangent line of $f(x)$ at the point $(x_n, f(x_n))$.

3.2 Proof for (b)

Suppose that $x_k \in (\xi, b)$. Mean value theorem tells us that there exists some $c \in (\xi, x_n)$ such that

$$f(\xi) - f(x_k) = f'(c)(\xi - x_k) \tag{4}$$

Since $0 \leq f''(x) \leq M$ for all $x \in [a, b]$, f' is a monotonically increasing function on $[a, b]$, so (4) can be written as

$$-f(x_k) \geq f'(x_k)(\xi - x_k)$$

as $f'(x) \le f'(x_k)$. Let $l(x) = f'(x_k)(x - x_k) + f(x_k)$. Then we can write

$$l(\xi) = f'(x_k)(\xi - x_k) + f(x_k) \le -f(x_k) + f(x_k) = 0$$

If $l(\xi) = 0$, $x_{k+1} = \xi$ and $x_{k+1} < x_k$. If $l(\xi) < 0$, since $l(x)$ is continuous on $[\xi, x_k]$ intermediate value theorem tells us that there exists $\alpha \in (\xi, x_k)$ such that $l(\alpha) = 0$. Then, $x_{k+1} = \alpha$ and $x_{k+1} < x_k$ holds. Furthermore, $\xi = \xi - f(\xi)/f'(\xi)$, so by induction $x_n \in [\xi, b)$ implies $x_{n+1} \in [\xi, x_n)$ for all $n = 1, 2, \ldots$ and $x_{n+1} < x_n$.

Since $\{x_n\}$ is monotonically decreasing and bounded, it converges to some value L by monotone convergence theorem. If we take the limit of both sides of $x_{n+1} = x_n - f(x_n)/f'(x_n)$, we obtain $L = L - f(L)/f'(L)$ and since $L \in [\xi, x_1]$, $f'(L) > 0$ so $f(L) = 0$. By definition, ξ is the only point in (a, b) at which $f(\xi) = 0$, so $L = \xi$ and we get the desired result.

3.3 Proof for (c)

By Taylor's theorem, there exists t_n between ξ and x_n such that

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

then we can write

$$\begin{aligned}
-\frac{f(x_n)}{f'(x_n)} &= \xi - x_n + \frac{f''(t_n)}{2f'(x_n)}(\xi - x_n)^2 \\
x_n - \frac{f(x_n)}{f'(x_n)} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(\xi - x_n)^2 \\
x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2
\end{aligned}$$

and we get the desired result.

3.4 Proof for (d)

As we have proved in (b), $x_n \in [\xi, b)$ for all $n = 1, 2, \ldots$ so $x_{n+1} - \xi \ge 0$. Since $f'(x) \ge \delta$ and $f''(x) \le M$ for all $x \in [a, b]$, $f''(t_n)/(2f'(x_n)) \le M/(2\delta)$ for all $n = 1, 2, \ldots$. Thus, we can write

$$x_{n+1} - \xi \le \frac{M}{2\delta}(x_n - \xi)^2 = A(x_n - \xi)^2$$

Let $y_n = x_n - \xi$, then $y_{n+1} \le Ay_n^2 \le A(Ay_{n-1}^2)^2 \le \cdots \le A \cdot A^2 \cdot \cdots \cdot A^{2^{n-1}} y_1^{2^n} = A^{2^n - 1} y_1^{2^n} = [A(x_1 - \xi)]^{2^n}/A$ and we get the desired result.

3.5 Proof for (e)

Since $f'(x) > 0$ for all $x \in [a, b]$, $g(x) - x = 0$ if and only if $f(x) = 0$ and ξ is the only point in (a, b) at which $f(\xi) = 0$, so ξ is the only fixed point of $g(x)$ in $[a, b]$. Thus, Newton's method amounts to finding a fixed point of g in $[a, b]$.

We can write

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = 1 - \left(1 - \frac{f(x)f''(x)}{(f'(x))^2}\right) = \frac{f(x)f''(x)}{(f'(x))^2}$$

so $g'(x)$ tends to 0 as x approaches ξ.

3.6 Solution for (f)

We can write

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{x_n^{-2/3}/3} = -2x_n$$

so $x_n = (-2)^{n-1}x_1$ and $\{x_n\}$ does not converge as it is not bounded.

4 Section 6 #1

Since f is bounded on $[a, b]$ and discontinuous only on x_0 and α is continuous there, we can use theorem 6.10 and know that $f \in \mathscr{R}(\alpha)$.

Since f is zero except only one point, x_0, for all partition $P = \{p_0, \dots, p_n\}$ of $[a, b]$, $m_i = \inf_{p_{i-1} \le x \le p_i} f(x) = 0$ so $L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i = 0$. Thus, $\sup L(P, f, \alpha) = \underline{\int_a^b} f d\alpha = 0$. Since $f \in \mathscr{R}(\alpha)$, $\int_a^b f d\alpha = \underline{\int_a^b} f d\alpha = 0$.

5 Section 6 #2

Suppose that $f(\alpha) > 0$. Since $\int_a^b f dx = 0$, $\underline{\int_a^b} f dx = \sup L(P, f) = 0$. By definition, $f(x) \ge 0$ so for all partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$, $L(P, f) \ge 0$ as $m_i = \inf_{x_{i-1} \le x \le x_i} f(x) \ge 0$, which implies $L(P, f) = \sum_{i=1}^n m_i \Delta x_i \ge 0$. Then, since $\sup L(P, f) = 0$, $L(P, f) = 0$ for all partition P of $[a, b]$. Since f is continuous at α, there exists some $\delta > 0$ such that $(\alpha - \delta, \alpha + \delta) \in [a, b]$ and $|x - \alpha| < \delta$ implies $|f(x) - f(\alpha)| < f(\alpha)/2$. Then $f(x) > f(\alpha)/2$ for all $x \in (\alpha - \delta, \alpha + \delta)$. Consider a partition $P = \{a, \alpha - \delta/2, \alpha + \delta/2, b\}$. We can write

$$L(P, f) \ge \delta \inf_{\alpha - \frac{\delta}{2} \le x \le \alpha + \frac{\delta}{2}} f(x) \ge \frac{\delta f(\alpha)}{2} > 0$$

and it is a contradiction.

6 Section 6 #3

6.1 Proof for (a)

Suppose that $f \in \mathscr{R}(\beta_1)$. Fix $\epsilon > 0$. There exists some partition $P = \{p_0, \dots, p_n\}$ of $[-1, 1]$ such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Let $P' = P \cup \{0\}$, then P' is a refinement of P so $U(P, f, \beta_1) \ge U(P', f, \beta_1) \ge L(P', f, \beta_1) \ge L(P, f, \beta_1)$ and $U(P', f, \beta_1) - L(P', f, \beta_1) < \epsilon$. Suppose that $p_k \le 0 < p_{k+1}$, then

$$\begin{aligned} L(P', f, \beta_1) &= \sum_{i=0}^{k-1} (\beta_1(p_{i+1}) - \beta_1(p_i)) \inf_{x \in [p_i, p_{i+1}]} f(x) \\ &+ \sum_{i=k+1}^{n-1} (\beta_1(p_{i+1}) - \beta_1(p_i)) \inf_{x \in [p_i, p_{i+1}]} f(x) \\ &+ (\beta_1(0) - \beta_1(p_k)) \inf_{x \in [p_k, 0]} f(x) + (\beta_1(p_{k+1}) - \beta_1(0)) \inf_{x \in [0, p_{k+1}]} f(x) \\ &= \inf_{x \in [0, p_{k+1}]} f(x) \end{aligned}$$

and we can also write

$$
\begin{aligned}
U(P', f, \beta_1) &= \sum_{i=0}^{k-1} (\beta_1(p_{i+1}) - \beta_1(p_i)) \sup_{x \in [p_i, p_{i+1}]} f(x) \\
&+ \sum_{i=k+1}^{n-1} (\beta_1(p_{i+1}) - \beta_1(p_i)) \sup_{x \in [p_i, p_{i+1}]} f(x) \\
&+ (\beta_1(0) - \beta_1(p_k)) \sup_{x \in [p_k, 0]} f(x) + (\beta_1(p_{k+1}) - \beta_1(0)) \sup_{x \in [0, p_{k+1}]} f(x) \\
&= \sup_{x \in [0, p_{k+1}]} f(x)
\end{aligned}
$$

so

$$
U(P', f, \beta_1) - L(P', f, \beta_1) = \sup_{x \in [0, p_{k+1}]} f(x) - \inf_{x \in [0, p_{k+1}]} f(x) < \epsilon
$$

From this, we know that $0 < x < p_{k+1}$ implies $|f(x) - f(0)| < \epsilon$ so $f(0+) = f(0)$. Also, we know that

$$
L(P, f, \beta_1) \le L(P', f, \beta_1) \le f(0) \le U(P', f, \beta_1) \le U(P, f, \beta_1)
$$

for all partition P of $[-1, 1]$ and $P' = P \cup \{0\}$, so

$$
\underline{\int_{-1}^{1}} f d\beta_1 \le f(0) \le \overline{\int_{-1}^{1}} f d\beta_1
$$

and since $f \in \mathscr{R}(\beta_1)$, $\underline{\int_{-1}^{1}} f d\beta_1 = \overline{\int_{-1}^{1}} f d\beta_1 = \int_{-1}^{1} f d\beta_1 = f(0)$.

Suppose that $f(0+) = f(0)$. Fix $\epsilon > 0$. There exists some $\delta > 0$ such that $0 < x < \delta$ implies $|f(x) - f(0)| < \epsilon/4$. Consider $P = \{-1, 0, \delta/2, 1\}$, a partition of $[-1, 1]$. Then we can write

$$
\begin{aligned}
L(P, f, \beta_1) &= (\beta_1(0) - \beta_1(-1)) \inf_{x \in [-1, 0]} f(x) + (\beta_1(\delta/2) - \beta_1(0)) \inf_{x \in [0, \delta/2]} f(x) \\
&+ (\beta_1(1) - \beta_1(\delta/2)) \inf_{x \in [\delta/2, 1]} f(x) \\
&= (\beta_1(\delta/2) - \beta_1(0)) \inf_{x \in [0, \delta/2]} f(x) = \inf_{x \in [0, \delta/2]} f(x) \\
U(P, f, \beta_1) &= (\beta_1(0) - \beta_1(-1)) \sup_{x \in [-1, 0]} f(x) + (\beta_1(\delta/2) - \beta_1(0)) \sup_{x \in [0, \delta/2]} f(x) \\
&+ (\beta_1(1) - \beta_1(\delta/2)) \sup_{x \in [\delta/2, 1]} f(x) \\
&= (\beta_1(\delta/2) - \beta_1(0)) \sup_{x \in [0, \delta/2]} f(x) = \sup_{x \in [0, \delta/2]} f(x)
\end{aligned}
$$

and

$$
\begin{aligned}
U(P, f, \beta_1) - L(P, f, \beta_1) &= \sup_{x \in [0, \delta/2]} f(x) - \inf_{x \in [0, \delta/2]} f(x) \\
&\le \left(f(0) + \frac{\epsilon}{4}\right) - \left(f(0) - \frac{\epsilon}{4}\right) = \frac{\epsilon}{2} < \epsilon
\end{aligned}
$$

Since the choice of ϵ is arbitrary, we can conclude that $f \in \mathscr{R}(\beta_1)$.

6.2 Proof for (b)

Let's prove that $f \in \mathscr{R}(\beta_2)$ if and only if $f(0-) = f(0)$.

Suppose that $f \in \mathscr{R}(\beta_2)$. Fix $\epsilon > 0$. There exists some partition $P = \{p_0, \dots, p_n\}$ of $[-1, 1]$ such that $U(P, f, \beta_2) - L(P, f, \beta_2) < \epsilon$. Let $P' = P \cup \{0\}$, then P' is a refinement of P so $U(P, f, \beta_2) \geq U(P', f, \beta_2) \geq L(P', f, \beta_2) \geq L(P, f, \beta_2)$ and $U(P', f, \beta_2) - L(P', f, \beta_2) < \epsilon$. Suppose that $p_k < 0 \leq p_{k+1}$, then

$$\begin{aligned}
L(P', f, \beta_2) &= \sum_{i=0}^{k-1} (\beta_2(p_{i+1}) - \beta_2(p_i)) \inf_{x \in [p_i, p_{i+1}]} f(x) \\
&+ \sum_{i=k+1}^{n-1} (\beta_2(p_{i+1}) - \beta_2(p_i)) \inf_{x \in [p_i, p_{i+1}]} f(x) \\
&+ (\beta_2(0) - \beta_2(p_k)) \inf_{x \in [p_k, 0]} f(x) + (\beta_2(p_{k+1}) - \beta_2(0)) \inf_{x \in [0, p_{k+1}]} f(x) \\
&= \inf_{x \in [p_k, 0]} f(x)
\end{aligned}$$

and we can also write

$$\begin{aligned}
U(P', f, \beta_2) &= \sum_{i=0}^{k-1} (\beta_2(p_{i+1}) - \beta_2(p_i)) \sup_{x \in [p_i, p_{i+1}]} f(x) \\
&+ \sum_{i=k+1}^{n-1} (\beta_2(p_{i+1}) - \beta_2(p_i)) \sup_{x \in [p_i, p_{i+1}]} f(x) \\
&+ (\beta_2(0) - \beta_2(p_k)) \sup_{x \in [p_k, 0]} f(x) + (\beta_2(p_{k+1}) - \beta_2(0)) \sup_{x \in [0, p_{k+1}]} f(x) \\
&= \sup_{x \in [p_k, 0]} f(x)
\end{aligned}$$

so

$$U(P', f, \beta_2) - L(P', f, \beta_2) = \sup_{x \in [p_k, 0]} f(x) - \inf_{x \in [p_k, 0]} f(x) < \epsilon$$

From this, we know that $p_k < x < 0$ implies $|f(x) - f(0)| < \epsilon$ so $f(0-) = f(0)$. Also, we know that

$$L(P, f, \beta_2) \leq L(P', f, \beta_2) \leq f(0) \leq U(P', f, \beta_2) \leq U(P, f, \beta_2)$$

for all partition P of $[-1, 1]$ and $P' = P \cup \{0\}$, so

$$\underline{\int_{-1}^{1}} f d\beta_2 \leq f(0) \leq \overline{\int_{-1}^{1}} f d\beta_2$$

and since $f \in \mathscr{R}(\beta_2)$, $\underline{\int_{-1}^{1}} f d\beta_2 = \overline{\int_{-1}^{1}} f d\beta_2 = \int_{-1}^{1} f d\beta_2 = f(0)$.

Suppose that $f(0-) = f(0)$. Fix $\epsilon > 0$. There exists some $\delta > 0$ such that $-\delta < x < 0$ implies $|f(x) - f(0)| < \epsilon/4$. Consider $P = \{-1, -\delta/2, 0, 1\}$, a partition of $[-1, 1]$. Then we can write

$$
\begin{aligned}
L(P,f,\beta_2) &= (\beta_2(-\delta/2)-\beta_2(-1))\inf_{x\in[-1,-\delta/2]} f(x) + (\beta_2(0)-\beta_2(-\delta/2))\inf_{x\in[-\delta/2,0]} f(x)\\
&\quad + (\beta_2(1)-\beta_2(0))\inf_{x\in[0,1]} f(x)\\
&= (\beta_2(0)-\beta_2(-\delta/2))\inf_{x\in[-\delta/2,0]} f(x) = \inf_{x\in[-\delta/2,0]} f(x)\\
U(P,f,\beta_2) &= (\beta_2(-\delta/2)-\beta_2(-1))\sup_{x\in[-1,-\delta/2]} f(x) + (\beta_2(0)-\beta_2(-\delta/2))\sup_{x\in[-\delta/2,0]} f(x)\\
&\quad + (\beta_2(1)-\beta_2(0))\sup_{x\in[0,1]} f(x)\\
&= (\beta_2(0)-\beta_2(-\delta/2))\sup_{x\in[-\delta/2,0]} f(x) = \sup_{x\in[-\delta/2,0]} f(x)
\end{aligned}
$$

and

$$
\begin{aligned}
U(P,f,\beta_2)-L(P,f,\beta_2) &= \sup_{x\in[-\delta/2,0]} f(x) - \inf_{x\in[-\delta/2,0]} f(x)\\
&\le \left(f(0)+\frac{\epsilon}{4}\right) - \left(f(0)-\frac{\epsilon}{4}\right) = \frac{\epsilon}{2} < \epsilon
\end{aligned}
$$

Since the choice of ϵ is arbitrary, we can conclude that $f \in \mathscr{R}(\beta_2)$.

6.3 Proof for (c)

Suppose that $f \in \mathscr{R}(\beta_3)$. Fix $\epsilon > 0$. There exists some partition $P = \{p_0, \ldots, p_{n-1}\}$ of $[-1,1]$ such that $U(P,f,\beta_3) - L(P,f,\beta_3) < \epsilon/2$. Let $P' = P \cup \{0\} = \{p'_0, \ldots, p'_n\}$, then P' is a refinement of P so $U(P,f,\beta_3) \ge U(P',f,\beta_3) \ge L(P',f,\beta_3) \ge L(P,f,\beta_3)$ and $U(P',f,\beta_3) - L(P',f,\beta_3) < \epsilon/2$. Suppose that $p'_k = 0$, then

$$
\begin{aligned}
L(P',f,\beta_3) &= \sum_{i=0}^{n-1}(\beta_3(p'_{i+1})-\beta_3(p'_i))\inf_{x\in[p'_i,p'_{i+1}]} f(x)\\
&= (\beta_3(0)-\beta_3(p'_{k-1}))\inf_{x\in[p'_{k-1},0]} f(x) + (\beta_3(p'_{k+1})-\beta_3(0))\inf_{x\in[0,p'_{k+1}]} f(x)\\
&= \frac{1}{2}\inf_{x\in[p'_{k-1},0]} f(x) + \frac{1}{2}\inf_{x\in[0,p'_{k+1}]} f(x)
\end{aligned}
$$

and we can also write

$$
\begin{aligned}
U(P',f,\beta_3) &= \sum_{i=0}^{n-1}(\beta_3(p'_{i+1})-\beta_3(p'_i))\sup_{x\in[p'_i,p'_{i+1}]} f(x)\\
&= (\beta_3(0)-\beta_3(p'_{k-1}))\sup_{x\in[p'_{k-1},0]} f(x) + (\beta_3(p'_{k+1})-\beta_3(0))\sup_{x\in[0,p'_{k+1}]} f(x)\\
&= \frac{1}{2}\sup_{x\in[p'_{k-1},0]} f(x) + \frac{1}{2}\sup_{x\in[0,p'_{k+1}]} f(x)
\end{aligned}
$$

so

$$
\begin{aligned}
U(P',f,\beta_3)-L(P',f,\beta_3) &= \frac{1}{2}\left(\sup_{x\in[p'_{k-1},0]} f(x) - \inf_{x\in[p'_{k-1},0]} f(x)\right)\\
&\quad + \frac{1}{2}\left(\sup_{x\in[0,p'_{k+1}]} f(x) - \inf_{x\in[0,p'_{k+1}]} f(x)\right) < \epsilon/2
\end{aligned}
$$

then we can write

$$\sup_{x\in[p'_{k-1},0]} f(x) - \inf_{x\in[p'_{k-1},0]} f(x) < \epsilon$$
$$\sup_{x\in[0,p'_{k+1}]} f(x) - \inf_{x\in[0,p'_{k+1}]} f(x) < \epsilon$$

We can take $\delta = \min\{|p'_{k-1}|, |p'_{k+1}|\}$ then $0 < |x| < \delta$ implies $|f(x) - f(0)| < \epsilon$ so $\lim_{x\to 0} f(x) = f(0)$ and f is continuous at $x = 0$.

Suppose that f is continuous at $x = 0$. Fix $\epsilon > 0$. There exists some $\delta > 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| < \epsilon/4$. Consider $P = \{-1, -\delta/2, 0, \delta/2, 1\}$, a partition of $[-1, 1]$. Then we can write

$$\begin{aligned}
L(P, f, \beta_3) &= (\beta_3(-\delta/2) - \beta_3(-1)) \inf_{x\in[-1,-\delta/2]} f(x) + (\beta_3(0) - \beta_3(-\delta/2)) \inf_{x\in[-\delta/2,0]} f(x) \\
&\quad + (\beta_3(\delta/2) - \beta_3(0)) \inf_{x\in[0,\delta/2]} f(x) + (\beta_3(1) - \beta_3(\delta/2)) \inf_{x\in[\delta/2,1]} f(x) \\
&= \frac{1}{2} \inf_{x\in[-\delta/2,0]} f(x) + \frac{1}{2} \inf_{x\in[0,\delta/2]} f(x) \\
U(P, f, \beta_3) &= (\beta_3(-\delta/2) - \beta_3(-1)) \sup_{x\in[-1,-\delta/2]} f(x) + (\beta_3(0) - \beta_3(-\delta/2)) \sup_{x\in[-\delta/2,0]} f(x) \\
&\quad + (\beta_3(\delta/2) - \beta_3(0)) \sup_{x\in[0,\delta/2]} f(x) + (\beta_3(1) - \beta_3(\delta/2)) \sup_{x\in[\delta/2,1]} f(x) \\
&= \frac{1}{2} \sup_{x\in[-\delta/2,0]} f(x) + \frac{1}{2} \sup_{x\in[0,\delta/2]} f(x)
\end{aligned}$$

and

$$\begin{aligned}
U(P, f, \beta_3) - L(P, f, \beta_3) &= \frac{1}{2}\left(\sup_{x\in[-\delta/2,0]} f(x) - \inf_{x\in[-\delta/2,0]} f(x)\right) \\
&\quad + \frac{1}{2}\left(\sup_{x\in[0,\delta/2]} f(x) - \inf_{x\in[0,\delta/2]} f(x)\right) \\
&\leq 2 \cdot \frac{1}{2}\left[\left(f(0) + \frac{\epsilon}{4}\right) - \left(f(0) - \frac{\epsilon}{4}\right)\right] = \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

Since the choice of ϵ is arbitrary, we can conclude that $f \in \mathscr{R}(\beta_3)$.

6.4 Proof for (d)

Since f is continuous at $x = 0$, $f(0-) = f(0) = f(0+)$ so by the result we have proven earlier,

$$\int f d\beta_2 = f(0)$$

Fix $\epsilon > 0$. $f \in \mathscr{R}(\beta_3)$ implies that there exists some partition $P = \{p_0, \ldots, p_{n-1}\}$ of $[-1, 1]$ such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon/2$. Let $P' = P \cup \{0\} = \{p'_0, \ldots, p'_n\}$ then P' is a refinement of P so $U(P, f, \beta_3) \geq U(P', f, \beta_3) \geq L(P', f, \beta_3) \geq L(P, f, \beta_3)$ and $U(P', f, \beta_3) - L(P', f, \beta_3) < \epsilon/2$. From the proof for (c), we know that

$$L(P, f, \beta_3) \leq L(P', f, \beta_3) \leq f(0) \leq U(P', f, \beta_3) \leq U(P, f, \beta_3)$$

for all partition P of $[-1,1]$ and $P' = P \cup \{0\}$, so

$$\underline{\int_{-1}^{1}} f d\beta_3 \leq f(0) \leq \overline{\int_{-1}^{1}} f d\beta_3$$

and since $f \in \mathscr{R}(\beta_3)$, $\underline{\int_{-1}^{1}} f d\beta_3 = \overline{\int_{-1}^{1}} f d\beta_3 = \int_{-1}^{1} f d\beta_3 = f(0)$ and we get the desired result.