MATH311: Homework 6 (due Apr. 4)

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1 Section 3 #3

First, let's prove that $s_n < 2$ for $n = 1, 2, \ldots$. For $n = 1$, $s_n = \sqrt{2} < 2$. Suppose that $s_n < 2$ for $n = k$, where k is a positive integer. We can write

$$s_{k+1} = \sqrt{2 + \sqrt{s_k}} < \sqrt{2 + \sqrt{2}} < 2$$

By induction, we can know that $s_n < 2$ for $n = 1, 2, \ldots$.

Since $s_2 = \sqrt{2 + \sqrt{s_1}} = \sqrt{2 + \sqrt{2}}$, $s_2 > s_1$ holds. Suppose that $s_{n+1} > s_n$ holds for $n = k$, where k is a positive integer. We can write

$$s_{k+2} = \sqrt{2 + \sqrt{s_{k+1}}} > \sqrt{2 + \sqrt{s_k}} = s_{k+1}$$

By induction, we know that $s_{n+1} > s_n$ for $n = 1, 2, \ldots$. Since s_n is monotonically increasing but bounded above, s_n converges.

2 Section 3 #4

From the recurrence relations, we can write

$$s_{2m+1} = \frac{1}{2} + s_{2m} = \frac{1}{2} + \frac{s_{2m-1}}{2}$$
$$s_{2m+2} = \frac{s_{2m+1}}{2} = \frac{1}{2}\left(\frac{1}{2} + s_{2m}\right) = \frac{1}{4} + \frac{s_{2m}}{2}$$

for $m = 1, 2, \ldots$. Let $x_n = s_{2n-1} - 1$, then

$$x_{n+1} = s_{2n+1} - 1 = \frac{1}{2} + \frac{s_{2m-1}}{2} - 1 = \frac{s_{2m-1} - 1}{2} = \frac{x_n}{2}$$

so $x_n = -1/2^{n-1}$. Let $y_n = s_{2n} - 1/2$, then

$$y_{n+1} = s_{2n+2} - \frac{1}{2} = \frac{1}{4} + \frac{s_{2n}}{2} - \frac{1}{2} = \frac{s_{2n}}{2} - \frac{1}{4} = \frac{y_n}{2}$$

so $y_n = -1/2^n$. Fix $\epsilon > 0$, and let N be an integer such that $N > \log_2(1/\epsilon) + 1$. For $n \geq N$,

$$|x_n - 0| = |x_n| = \frac{1}{2^{n-1}} \leq \frac{1}{2^{N-1}} < \epsilon$$
$$|y_n - 0| = |y_n| = \frac{1}{2^n} \leq \frac{1}{2^N} < \frac{\epsilon}{2} < \epsilon$$

and we can conclude that x_n, y_n both converges to 0, so s_{2n-1}, s_{2n} converges to $1, 1/2$, respectively by properties of limits. Suppose that a subsequence of $\{s_{n_k}\}$ of $\{s_n\}$ converges to L, where $L \notin \{1/2, 1\}$. Fix $0 < \epsilon < \min\{|L - 1/2|, |L - 1|\}/2$, there exists an integer N such that $k \geq N$ implies $|s_{n_k} - L| < \epsilon$, $|s_{2k-1} - 1| < \epsilon$, and $|s_{2k} - 1/2| < \epsilon$. Since n_k is a strictly increasing sequence of integers, there exists $k_0 \geq N$ such that $m := n_{k_0} \geq 2N$. If m is even, $|s_m - 1/2| < \epsilon$. By triangular inequality,

$$\left|L - \frac{1}{2}\right| \leq \left|s_m - \frac{1}{2}\right| + |s_m - L| < 2\epsilon$$

and this contradicts with $\epsilon < \min\{|L-1/2|, |L-1|\}/2$. Similarly, if m is odd, $|s_m - 1| < \epsilon$ and

$$|L - 1| \leq |s_m - 1| + |s_m - L| < 2\epsilon$$

and this also contradicts with $\epsilon < \min\{|L - 1/2|, |L - 1|\}/2$. Thus, such L cannot exist and the set of subsequential limits is $E = \{1/2, 1\}$. Then, the upper limit is $\sup E = 1$, and the lower limit is $\inf E = 1/2$.

3 Section 3 #5

Let $x_n = \sup_{k\geq n} a_k, y_n = \sup_{k\geq n} b_k, z_n = \sup_{k\geq n}(a_k + b_k)$ and $A_n = \{s + t;\ s \in \{a_k, a_{k+1}, \dots\}, t \in \{b_k, b_{k+1}, \dots\}\}$. Suppose that there exists an upper bound α of A_n, such that $\alpha < x_n + y_n$. Since α is an upper bound of A_n, $s+t \leq \alpha$ for all $s \in \{a_k, a_{k+1}, \dots\}$ and $t \in \{b_k, b_{k+1}, \dots\}$. Also, $s \leq x_n$ holds since $x_n = \sup_{k\geq n} a_k = \sup\{a_k, a_{k+1}, \dots\}$. Thus, $t \leq \alpha - x_n < y_n$ holds for all $t \in \{b_k, b_{k+1}, \dots\}$, then it is a contradiction and such α does not exist. Since $s + t \leq x_n + y_n$ for all $s \in \{a_k, a_{k+1}, \dots\}$ and $t \in \{b_k, b_{k+1}, \dots\}$, $\sup A_n = x_n + y_n$. From $\{a_k + b_k, a_{k+1} + b_{k+1}, \dots\} \subset A_n$, $z_n = \sup_{k\geq n}(a_k + b_k) \leq \sup A_n = x_n + y_n$. Then, we can write

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} x_n, \limsup_{n\to\infty} b_n = \lim_{n\to\infty} y_n, \limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} z_n$$

so

$$\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} z_n \leq \lim_{n\to\infty}(x_n + y_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

4 Section 3 #8

Since b_n is monotonic and bounded, it converges to some real number b. We can write

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} a_n(b_n - b) + \sum_{n=p}^{q} a_n b$$

For all sufficiently small $\epsilon > 0$, there exists some integer N such that $n \geq N$ implies $|b_n - b| < \epsilon$. For $q \geq p \geq N$, we can write

$$-\epsilon \sum_{n=p}^{q} a_n < \sum_{n=p}^{q} a_n(b_n - b) < \epsilon \sum_{n=p}^{q} a_n$$

so

$$(b-\epsilon)\sum_{n=p}^{q} a_n < \sum_{n=p}^{q} a_n b_n < (b+\epsilon)\sum_{n=p}^{q} a_n$$

Fix $\epsilon' > 0$, then there exists some $N' \geq N$ such that $n \geq N'$ implies

$$\left|\sum_{n=p}^{q} a_n\right| < \frac{\epsilon'}{\min\{|b-\epsilon|, |b+\epsilon|\}}$$

by Cauchy's criterion. Then, if $n \geq N'$ and $q \geq p \geq N'$

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &< \min\left\{\left|(b-\epsilon)\sum_{n=p}^{q} a_n\right|, \left|(b+\epsilon)\sum_{n=p}^{q} a_n\right|\right\} \\ &< \frac{\epsilon'}{\min\{|b-\epsilon|, |b+\epsilon|\}} \cdot \min\{|b-\epsilon|, |b+\epsilon|\} = \epsilon'\end{aligned}$$

so $\sum a_n b_n$ also satisfies Cauchy's criterion, thus it converges.

5 Section 3 #16

5.1 Solution for (a)

By AM-GM inequality,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \geq \frac{1}{2} \cdot 2\sqrt{\alpha} = \sqrt{\alpha} \tag{1}$$

for $n = 1, 2, \ldots$ so $x_n \geq \sqrt{\alpha}$ for $n = 1, 2, \ldots$. Then,

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - x_n = \frac{1}{2}\left(\frac{\alpha}{x_n} - x_n\right) \leq \frac{1}{2}(\sqrt{\alpha} - \sqrt{\alpha}) = 0$$

we get $x_{n+1} \leq x_n$, so x_n is monotonically decreasing and it converges to some value x. Then, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x$ and by using the properties of limits we get

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right)$$

Solving the equation gives $x = \sqrt{\alpha}$ or $x = -\sqrt{\alpha}$. Since $x_n \geq \sqrt{\alpha}$ for all n, $x \geq \sqrt{\alpha}$ and we obtain $x = \sqrt{\alpha}$. THus, x_n converges to $\sqrt{\alpha}$.

5.2 Solution for (b)

We can write

$$\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n - 2\sqrt{\alpha} + \frac{\alpha}{x_n}\right) = \frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{2x_n} = \frac{\epsilon_n^2}{2x_n}$$

Suppose that $x_n > \sqrt{\alpha}$. The equality in (1) holds if and only if $x_n = \alpha/x_n$, but $x_n > \sqrt{\alpha}$ so the equality does not hold. Thus, $x_{n+1} > \sqrt{\alpha}$, and $x_n > \sqrt{\alpha}$ for $n = 1, 2, \ldots$ by induction. So

$$\frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

By setting $\beta = 2\sqrt{\alpha}$,

$$\epsilon_2 < \frac{\epsilon_1^2}{2\sqrt{\alpha}} = \frac{\epsilon_1^2}{\beta} = \beta\left(\frac{\epsilon_1}{\beta}\right)^2$$

Suppose that the inequality we want to prove holds for $n = k$. Then

$$\epsilon_{k+2} < \frac{\epsilon_{k+1}^2}{2\sqrt{\alpha}} = \beta\left(\frac{\epsilon_{k+1}}{\beta}\right)^2 < \frac{1}{\beta}\epsilon_{k+1}^2 < \frac{1}{\beta}\left[\beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}\right]^2 = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^{n+1}}$$

and we get the desired result.

5.3 Solution for (c)

From the given values, $\epsilon_1 = x_1 - \sqrt{\alpha} = 2 - \sqrt{3}$ so $\epsilon_1/\beta = (2-\sqrt{3})/(2\sqrt{3})$. We can write

$$\frac{\epsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{1}{4\sqrt{3}+6} < \frac{1}{10}$$

Thus,

$$\begin{aligned}
\epsilon_5 &= \beta\left(\frac{\epsilon_1}{\beta}\right)^{16} = 2\sqrt{3}\left(\frac{1}{10}\right)^{16} < 4 \cdot 10^{-16} \\
\epsilon_6 &= \beta\left(\frac{\epsilon_1}{\beta}\right)^{3}2 = 2\sqrt{3}\left(\frac{1}{10}\right)^{3}2 < 4 \cdot 10^{-32}
\end{aligned}$$

6 Section 3 #17

If $x_n > \sqrt{\alpha}$, then

$$\begin{aligned}
\alpha(\alpha-1) < x_n^2(\alpha-1) &\implies \alpha^2 + x_n^2 < \alpha x_n^2 + \alpha \\
&\implies \alpha^2 + 2\alpha x_n + x_n^2 < \alpha x_n^2 + 2\alpha x_n + \alpha \\
&\implies (\alpha + x_n)^2 < \alpha(x_n+1)^2
\end{aligned}$$

so

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} < \sqrt{\alpha}$$

If $x_n < \alpha$, then

$$\begin{aligned}
\alpha(\alpha-1) > x_n^2(\alpha-1) &\implies \alpha^2 + x_n^2 > \alpha x_n^2 + \alpha \\
&\implies \alpha^2 + 2\alpha x_n + x_n^2 > \alpha x_n^2 + 2\alpha x_n + \alpha \\
&\implies (\alpha + x_n)^2 > \alpha(x_n+1)^2
\end{aligned}$$

so

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} > \sqrt{\alpha}$$

Since $x_1 > \sqrt{\alpha}$, it can be shown that $x_{2n-1} > \sqrt{\alpha}, x_{2n} < \sqrt{\alpha}$ for $n = 1, 2, \ldots$ using induction.

6.1 Solution for (a)

First, let's prove $x_1 > x_3$.

$$x_3 = \frac{\alpha + x_2}{1 + x_2} = \frac{\alpha + \frac{\alpha + x_1}{1 + x_1}}{1 + \frac{\alpha + x_1}{1 + x_1}} = \frac{2\alpha + (\alpha + 1)x_1}{\alpha + 1 + 2x_1} = x_1 + \frac{2\alpha - 2x_1^2}{\alpha + 1 + 2x_1}$$

since $x_1 > \sqrt{\alpha}$, $2\alpha - 2x_1^2 < 0$ and $x_1 > x_3$. Suppose that $x_{2n-1} > x_{2n+1}$ for $n = k$. We can write

$$x_{2k+1} - x_{2k+3} = x_{2k+1} - \left(x_{2k+1} + \frac{2\alpha - 2x_{2k+1}^2}{\alpha + 1 + 2x_{2k+1}} \right) = \frac{2x_{2k+1}^2 - 2\alpha}{\alpha + 1 + 2x_{2k+1}} > 0$$

so $x_{2k+1} > x_{2k+3}$. By induction, $x_{2n-1} > x_{2n+1}$ holds for $n = 1, 2, \ldots$ so $x_1 > x_3 > x_5 > \ldots$.

6.2 Solution for (b)

First, let's prove $x_2 < x_4$.

$$x_4 = \frac{\alpha + x_3}{1 + x_3} = \frac{\alpha + \frac{\alpha + x_2}{1 + x_2}}{1 + \frac{\alpha + x_2}{1 + x_2}} = \frac{2\alpha + (\alpha + 1)x_2}{\alpha + 1 + 2x_2} = x_2 + \frac{2\alpha - 2x_2^2}{\alpha + 1 + 2x_2}$$

Since $x_2 < \sqrt{\alpha}$, $2\alpha - 2x_2^2 > 0$ and $x_2 < x_4$. Suppose that $x_{2n} < x_{2n+2}$ for $n = k$. We can write

$$x_{2k+2} - x_{2k+4} = x_{2k+2} - \left(x_{2k+2} + \frac{2\alpha - 2x_{2k+2}^2}{\alpha + 1 + 2x_{2k+2}} \right) = \frac{2x_{2k+2}^2 - 2\alpha}{\alpha + 1 + 2x_{2k+2}} < 0$$

so $x_{2k+2} < x_{2k+4}$. By induction, $x_{2n} < x_{2n+2}$ holds for $n = 1, 2, \ldots$ so $x_2 < x_4 < x_6 < \ldots$.

6.3 Solution for (c)

From (a) and (b), we can write

$$x_2 < x_4 < \cdots < \sqrt{\alpha} < \cdots < x_3 < x_1$$

Subsequence x_{2n} is monotonically increasing and bounded, so it converges to some value x. Subsequence x_{2n-1} is also monotonically increasing and bounded, so it converges to some value x'. Then, using properties of limits

$$x_{2n+2} - x_{2n} = \frac{2x_{2n}^2 - 2\alpha}{\alpha + 1 + 2x_{2n}} \implies \frac{2x^2 - 2\alpha}{\alpha + 1 + 2x} = 0$$

we can conclude that $x = \sqrt{\alpha}$ since $x_{2n} > 0$. Likewise,

$$x_{2n+1} - x_{2n-1} = \frac{2x_{2n-1}^2 - 2\alpha}{\alpha + 1 + 2x_{2n-1}} \implies \frac{2x'^2 - 2\alpha}{\alpha + 1 + 2x'} = 0$$

$x' = \sqrt{\alpha}$ since $x_{2n-1} > 0$. Since x_{2n} and x_{2n-1} both converge to $\sqrt{\alpha}$, for all $\epsilon > 0$ there exists some integer N such that $n \geq N$ implies $|x_{2n} - \sqrt{\alpha}|, |x_{2n-1} - \sqrt{\alpha}| < \epsilon$. Then, $n \geq 2N - 1$ implies that $|x_n - \sqrt{\alpha}| < \epsilon$, so x_n converges to $\sqrt{\alpha}$.

6.4 Solution for (d)

Let $\epsilon_n = x_n - \sqrt{\alpha}$. We can write

$$\begin{aligned}\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{\alpha + x_n}{1 + x_n} - \sqrt{\alpha} &= \frac{\alpha + x_n - \sqrt{\alpha} - \sqrt{\alpha}x_n}{1 + x_n} \\ = \frac{(\sqrt{\alpha} - 1)(\sqrt{\alpha} - x_n)}{1 + x_n} &= -\frac{(\sqrt{\alpha} - 1)\epsilon_n}{1 + x_n}\end{aligned}$$

then

$$|\epsilon_{n+1}| = \left|\frac{(\sqrt{\alpha} - 1)\epsilon_n}{1 + x_n}\right| = \left|\frac{\sqrt{\alpha} - 1}{1 + x_n}\right| |\epsilon_n| \geq \left|\frac{\sqrt{\alpha} - 1}{1 + x_1}\right| |\epsilon_n|$$

Using induction, we obtain

$$|\epsilon_{n+1}| \geq \left|\frac{\sqrt{\alpha} - 1}{1 + x_1}\right|^n |\epsilon_1|$$

From this, we can see that $|\epsilon_n|$ goes to zero no faster than geometric sequence. Let δ_n be the n-th error term we have written in (c). Then, δ_n goes to zero faster than doubly exponential sequence, so this progress converges slower than the process described in problem 16.

7 Section 3 #20

Fix $\epsilon > 0$. There exists some N such that $n, m \geq N$ implies $d(p_n, p_m) < \epsilon/2$, and $k \geq N$ implies $d(p_{n_k}, p) < \epsilon/2$. Since n_i is a strictly increasing sequence of integers, there exists some $k_0 \geq N$ such that $q := n_{k_0} \geq N$. By triangular inequality,

$$d(p_l, p) \leq d(p_l, p_q) + d(p_q, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

holds if $l \geq N$ and we know that p_n converges to p.