MATH311: Homework 9 (due May. 9)

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1 Section 5 #1

For all $x \in \mathbb{R}$ and $h > 0$, we can write

$$|f(x+h) - f(x)| \leq h^2$$

Then

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq h$$

taking the limit $h \to 0$ to the both sides, by sandwich theorem we know that $f'(x) = 0$ for all $x \in \mathbb{R}$. For all $x \neq 0$, there exists some $c \in (\min\{0, x\}, \max\{0, x\})$ such that $xf'(c) = f(x) - f(0)$ by mean value theorem. Since $f'(c) = 0$, $f(x) = f(0)$ for all $x \neq 0$, so $f(x)$ is constant.

2 Section 5 #2

Suppose that f is not strictly increasing in (a, b). There exists some x_1, x_2 such that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$. Then, we can write

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq 0$$

By mean value theorem, there exists some $c \in (x_1, x_2)$ such that $f'(c) = (f(x_1) - f(x_2))/(x_1 - x_2) \leq 0$, which is a contradiction. Thus, f is strictly increasing in (a, b).

Fix $x \in f((a, b))$, and let $y = f(x)$. For sufficiently small $\delta > 0$, there exists some $x_1, x_2 \in (a, b)$ such that $f(x_1) = y - \delta$ and $f(x_2) = y + \delta$ since f is a strictly increasing function defined on an open set. Let $h(k) = g(y+k) - g(y)$ where $k \in (-\delta, \delta)$. Since f is a strictly increasing function, g is also a strictly increasing function so $x + h(k) = g(y+k) \in (g(y-\delta), g(y+\delta)) \subset (a, b)$ holds. Then we can write

$$\lim_{k \to 0} \frac{g(y+k) - g(y)}{k} = \lim_{k \to 0} \frac{h(k)}{f(x + h(k)) - f(x)}$$

Since f is a strictly increasing function on (a, b), f maps all open subsets of (a, b) to open set and g is continuous. Then $h(k) \to 0$ as $k \to 0$, so we can write

$$\lim_{k \to 0} \frac{h(k)}{f(x + h(k)) - f(x)} = \lim_{h \to 0} \frac{h}{f(x+h) - f(x)} = \frac{1}{f'(x)}$$

Let $h(t) = g(f(t))$, then since g is differentiable at $f(x)$, h is differentiable at x and $h'(x) = g'(f(x))f'(x)$. Since $h(t) = g(f(t)) = t$, we can write

$$g'(f(x)) = \frac{1}{f'(x)}$$

3 Section 5 #5

Fix $\epsilon > 0$. Since $f'(x) \to 0$ as $x \to \infty$, there exists $M \in \mathbb{R}$ such that $x > M$ implies $|f'(x)| < \epsilon$. By mean value theorem, there exists $z \in (x, x+1)$ such that $f'(z) = f(x+1) - f(x) = g(x)$. Then, $x > M$ implies $|g(x)| = |f(x+1) - f(x)| = |f'(z)| < \epsilon$ as $z > M$ so $g(x) \to 0$ as $x \to \infty$.

4 Section 5 #6

By mean value theorem, there exists $z \in (0, x)$ such that $f'(z) = f(x)/x = (f(x) - f(0))/(x - 0)$ for all $x > 0$. Since f' is monotonically increasing, $f'(x) \geq f'(z)$ so $f'(x) - f(x)/x \geq 0$ for all $x \geq 0$. For $x > 0$, g is differentiable and $g'(x) = (xf'(x) - f(x))/x^2 = (f'(x) - f(x)/x)/x \geq 0$, so g is monotonically increasing in $(0, \infty)$.

5 Section 5 #9

Fix $\epsilon > 0$. There exists $\delta > 0$ such that $|x| < \delta$ implies $|f'(x) - 3| < \epsilon$. For all $h > 0$, by mean value theorem there exists $x \in (0, h)$ such that $f'(x) = (f(h) - f(0))/h$. Likewise, for all $h < 0$ again by mean value theorem there exists $x \in (h, 0)$ such that $f'(x) = (f(h) - f(0))/h$. Thus, if $|h| < \delta$, there exists some $0 < |x| < |h|$ such that $f'(x) = (f(h) - f(0))/h$, so we can write

$$\left| \frac{f(h) - f(0)}{h} - 3 \right| = \left| f'(x) - 3 \right| < \epsilon$$

and know that $\lim_{h \to 0}(f(h) - f(0))/h = 3$, so $f'(0) = 3$.

6 Section 5 #11

Since $f''(x)$ exists, f' exists in a neighborhood of x. Applying theorem 5.13,

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} \\
&= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{2h} + \lim_{h \to 0} \frac{f'(x) - f'(x-h)}{2h} \\
&= \frac{f''(x)}{2} + \frac{f''(x)}{2} = f''(x)
\end{aligned}$$

Let $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x^2 & (x \geq 0) \\ -x^2 & (x < 0) \end{cases}$$

Then we can write

$$\lim_{h \to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = \lim_{h \to 0} \frac{h^2 + (-h^2)}{h^2} = 0$$

However, since $f'(x) = 2|x|$, f is not twice differentiable at 0.

7 Section 5 #14

Suppose that f is convex, then for all $s, t, u \in (a, b)$ such that $s < t < u$,

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

Then we can write

$$f'(s) = \lim_{t \to s} \frac{f(t) - f(s)}{t - s} \leq \lim_{t \to u} \frac{f(u) - f(t)}{u - t} = f'(u)$$

so $a < s < u < b$ implies $f'(s) \leq f'(u)$ by sandwich theorem, and f' is monotonically increasing.

Now, suppose that f' is monotonically increasing. For $x, y \in (a, b)$ and $\lambda \in (0, 1)$, if $x < y$ then by mean value theorem there exists $x_1 \in (x, \lambda x + (1-\lambda)y)$ and $x_2 \in (\lambda x + (1-\lambda)y, y)$ such that

$$\frac{f(\lambda x + (1-\lambda)y) - f(x)}{[\lambda x + (1-\lambda)y] - x} = f'(x_1) \leq f'(x_2) = \frac{f(y) - f(\lambda x + (1-\lambda)y)}{y - [\lambda x + (1-\lambda)y]}$$

Then we can write

$$\begin{aligned}
\frac{f(\lambda x + (1-\lambda)y) - f(x)}{(1-\lambda)(y - x)} &\leq \frac{f(y) - f(\lambda x + (1-\lambda)y)}{\lambda(y - x)} \\
\lambda(f(\lambda x + (1-\lambda)y) - f(x)) &\leq (1-\lambda)(f(y) - f(\lambda x + (1-\lambda)y)) \\
f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda)f(y)
\end{aligned}$$

If $x > y$, there exists $x_1 \in (y, \lambda x + (1-\lambda)y)$ and $x_2 \in (\lambda x + (1-\lambda)y, x)$ such that

$$\frac{f(\lambda x + (1-\lambda)y) - f(y)}{[\lambda x + (1-\lambda)y] - y} = f'(x_1) \leq f'(x_2) = \frac{f(x) - f(\lambda x + (1-\lambda)y)}{x - [\lambda x + (1-\lambda)y]}$$

Then we can also write

$$\begin{aligned}
\frac{f(\lambda x + (1-\lambda)y) - f(y)}{\lambda(x - y)} &\leq \frac{f(x) - f(\lambda x + (1-\lambda)y)}{(1-\lambda)(x - y)} \\
(1-\lambda)(f(\lambda x + (1-\lambda)y) - f(y)) &\leq \lambda(f(x) - f(\lambda x + (1-\lambda)y)) \\
f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda)f(y)
\end{aligned}$$

If $x = y$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ obviously holds, so f is convex.

Suppose that $f''(x) \geq 0$ for all $x \in (a, b)$. Then, f' is a monotonically increasing function so f is convex. If f is convex, f' is a monotonically increasing differentiable function. Then, for all $x \in (a, b)$ and $h > 0$ such that $x + h \in (a, b)$, we can write

$$\frac{f'(x + h) - f'(x)}{h} \geq 0$$

as f' is monotonically increasing. By sandwich theorem, $f''(x) = \lim_{h \to 0}(f'(x + h) - f(x))/h \geq 0$ and we get the desired result.