# MATH311: Homework 4 (due Mar. 21)

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## 1 Problem #6

First, let's prove that $E'$ is closed. Consider a limit point $p$ of $E'$. We need to show that $p \in E'$. Let $r > 0$, then we need to find a point $q \in E$ such that $p \neq q$ and $d(p,q) < r$. Since $p$ is a limit point of $E'$, there exists a point $t \in E'$ such that $p \neq t$ and $d(p,t) < r$. There exists $q \in E$ with $q \neq t$ and $d(t,q) < s$, where $s = \min(r - d(p,t), d(p,t))$ since $t \in E'$. By triangular inequality, $d(p,q) \leq d(p,t)+d(t,q) < d(p,t)+s \leq r$ and $d(p,q) \geq d(p,t)-d(t,q)$. Since $s \leq d(p,t)$ by definition, $d(p,t) - d(t,q) > d(p,t) - s \geq 0$, so $d(p,q) > 0$. Thus, by definition of limit points, every limit point $p$ of $E'$ is a limit point of $E$, so $(E')' \subset E'$ and $E'$ is closed.

Let's show that $E' = (\bar{E})'$. Consider $p \in E'$. For all $r > 0$, there exists a point $q \in E$ such that $p \neq q$ and $d(p,q) < r$ since $p \in E'$. Since $\bar{E} = E \cup E'$, $q \in \bar{E}$ and it is evident that $p \in (\bar{E})'$, by the definition of limit points. Thus, $E' \subset (\bar{E})'$ holds. Now consider $s \in (\bar{E})'$. Let $r' > 0$, we need to find a point $t \in E$ such that $s \neq t$ and $d(s,t) < r'$ to show that $s$ is a limit point of $E$, There exists a point $u \in \bar{E}$ such that $s \neq u$ and $d(s,u) < r'$ since $s \in (\bar{E})'$. Since $\bar{E} = E \cup E'$, $u \in E$ or $u \in E'$. If $u \in E$, we can take $t = u$ and done. If $u \notin E$, then $u \in E'$ and there exists $t \in E$ with $t \neq u$ and $d(t,u) < r''$, where $r'' = \min(r' - d(s,u), d(s,u))$. By triangular inequality, $d(s,t) \leq d(s,u)+d(t,u) < d(s,u)+r'' < r'$ and $d(s,t) \geq d(s,u)-d(t,u)$. Since $r'' \leq d(s,u)$ by definition, $d(s,u) - d(t,u) > d(s,u) - r'' \geq 0$, so $d(s,t) > 0$, so we get the desired $t$. Thus, $(\bar{E})' \subset E'$ also holds, and $E$ and $\bar{E}$ has the same limit points.

$E$ and $E'$ generally does not share the same limit points. Consider $E = \{0\} \subset \mathbb{R}$, then $E' = \varnothing$ since for all $x \in \mathbb{R}$, $B_r(x) \cap E = \varnothing$ if $0 < r < d(0,x)$ and $x$ cannot be a limit point of $E$.

## 2 Problem #9

### 2.1 Proof for (d)

From (c), we can see that $E^\circ$ is the union of all open sets contained in $E$. Thus, its complement is the intersection of all closed sets contained in $E^C$, by theorem 2.22. Then, by theorem 2.27, $(E^\circ)^C$ is the closure of $E^C$.

### 2.2 Solution for (e)

If $E$ is $\mathbb{Q}$ in $\mathbb{R}$, for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and $r > 0$, there exists $y \in \mathbb{Q}$ such that $x - r < y < x$ because $\mathbb{Q}$ is dense in $\mathbb{R}$. Thus, $x \in E'$ so $\mathbb{R} \setminus \mathbb{Q} \subset E'$, and $\bar{E} = E \cup E' \supset \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$. Since $\bar{E} \subset \mathbb{R}$, $\bar{E} = \mathbb{R}$ holds. Then, $\bar{E}$ is an open set, so $x$ is an interrior point of $\bar{E}$ but it isn't for $E$ since $x \notin E$, and we can conclude that it is generally false.

## 2.3 Solution for (f)

If $E = \{0\}$ in $\mathbb{R}$, then $\bar{E} = \{0\}$ since $E' = \varnothing$, but $E^\circ = \varnothing$ so $\bar{E}^\circ = \varnothing$. Thus, it is generally false.

# 3 Problem #16

$E^C$ can be written as follows:

$$\begin{aligned} E^C &= \{p \in \mathbb{Q};\ p^2 \leq 2\} \cup \{p \in \mathbb{Q};\ p^2 \geq 3\} \\ &= \{p \in \mathbb{Q};\ -\sqrt{2} \leq p \leq \sqrt{2}\} \cup \{p \in \mathbb{Q};\ p \geq \sqrt{3}\} \cup \{p \in \mathbb{Q};\ p \leq -\sqrt{3}\} \end{aligned}$$

Let's show that $E^C$ is open. Suppose $x \in \mathbb{Q} \setminus E$. If $x^2 \leq 2$, then $x^2 < 2$, so $-\sqrt{2} < x < \sqrt{2}$ since there is no rational number whose square is 2. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $y, z \in \mathbb{Q}$ such that $-\sqrt{2} < y < x$ and $x < z < \sqrt{2}$. Let $r = \min(x - y, z - x)$, and consider $w \in (x - r, x + r)$. If $r = x - y$, then $w \in (y, 2x - y)$, and $2x - y \leq z$ so $w \in (y, z)$, thus $-\sqrt{2} < w < \sqrt{2}$ and $w^2 < 2$ holds. Otherwise, then $w \in (2x - z, z)$, and $2x - z \geq y$ so $w \in (y, z)$, thus $-\sqrt{2} < w < \sqrt{2}$ and $w^2 < 2$ also holds. If $x^2 \geq 3$, then $x^2 > 3$, so $x > \sqrt{3}$ or $x < -\sqrt{3}$ since there is no rational number whose square is 3. If $x > \sqrt{3}$ then there exists $t \in \mathbb{Q}$ such that $\sqrt{3} < t < x$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Let $s = x - t$, and consider $w \in (x - s, x + s)$. Since $x - s = t > \sqrt{3}$, $w^2 > t^2 > 3$ holds. If $x < -\sqrt{3}$ then there exists $u \in \mathbb{Q}$ such that $x < u < -\sqrt{3}$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Let $v = u - x$, and consider $w \in (x - v, x + v)$. Since $x + v = u < -\sqrt{3}$, $w^2 > u^2 > 3$ holds. Thus, $E^C$ is open in $\mathbb{Q}$, so $E$ is a closed in $\mathbb{Q}$.

For all $p \in E$, $d(p, 0) = |p| < 2$ so $E$ is bounded.

Consider a collection of sets $\{G_n\} = \{p \in \mathbb{Q};\ 2 < p^2 < 3 - 1/n\}$. We can write

$$G_n = [(\sqrt{2}, \sqrt{3 - 1/n}) \cap \mathbb{Q}] \cup [(-\sqrt{3 - 1/n}, -\sqrt{2}) \cap \mathbb{Q}]$$

and since $\mathbb{Q}$ is a open set, $G_n$ is open by (a) and (c) of theorem 2.24. Thus, $\{G_n\}$ is an open cover of $E$. $G_n$ cover $E$ since $\mathbb{Q}$ is dense in $\mathbb{R}$, but no finite collection of $\{G_n\}$ covers $E$. Thus, $E$ is not compact. Since $E$ can be written as

$$E = [(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}] \cup [(-\sqrt{3}, -\sqrt{2}) \cap \mathbb{Q}]$$

and by (a) and (c) of theorem 2.24, $E$ is also open.

# 4 Problem #22

By corollary of theorem 2.13, $\mathbb{Q}$ is countable and $\mathbb{Q}^k$ is also countable by theorem 2.13. For all $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, $x \in \mathbb{Q}^k$ or $x \notin \mathbb{Q}^k$. Let $r > 0$. If $x \notin \mathbb{Q}^k$, then there exists some $y = (y_1, y_2, \ldots, y_k) \in \mathbb{Q}^k$ such that $x_i < y_i < x_i + \sqrt{r^2/k}$, $(i = 1, 2, \ldots, k)$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Then, $0 < y_i - x_i < \sqrt{r^2/k}, (i = 1, 2, \ldots, k)$ holds. We can write

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_k - y_k)^2} < \sqrt{k \cdot \left(\sqrt{\frac{r^2}{k}}\right)^2} = r$$

From this, we can conclude that $x$ is a limit point of $\mathbb{Q}^k$, so $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$ by the definition of dense sets. Thus, $\mathbb{R}^k$ contains a countable dense subset $\mathbb{Q}^k$, so it is separable.

## 5 Problem #26

Using the hint, let $E$ be a set of $x_n$ where $x_n \notin G_1 \cup G_2 \cup \cdots \cup G_n$. There are infinitely many finite unions of $G_1, \ldots, G_n$ and every point is in some set of $G_1, \ldots, G_n$ by definition, so $E$ cannot be finite. Now consider a limit point $z$ of $E$, there exists $G_n$ where $z \in G_n$ by definition. Since $G_n$ is open, there exists $r > 0$ where $B_r(z) \subset G_n$. Then, $x_m \notin B_r(z)$ if $m \geq n$, since $x_m \notin G_1 \cup \cdots \cup G_m$, so $z$ cannot be a limit point of $E$, which is a contradiction.

## 6 Problem #29

Let $E \subset \mathbb{R}$ be such open subset. Consider a collection $\{G_\alpha\}$ consisting of $(\alpha - r, \alpha + r)$, where $\alpha, r \in \mathbb{Q}$ and $r > 0$. For all $x \in E$, there exists $r > 0$ such that $(x - r, x + r) \subset E$ since $E$ is open. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $s \in \mathbb{Q}$ such that $0 < s < r$, and $\beta \in \mathbb{Q}$ such that $x < \beta < x + s/2$. Then, $(\beta - s/2, \beta + s/2) \subset (x - s, x + s) \subset (x - r, x + r)$ and $x \in (\beta - s/2, \beta + s/2)$ holds, and $(\beta - s/2, \beta + s/2) \subset \{G_\alpha\}$ so $\{G_\alpha\}$ is a base of $\mathbb{R}$. By definition, every open set in $\mathbb{R}$ is the union of a subcollection of $\{G_\alpha\}$. Fix a subcollection $\{V_\alpha\}$ of $\{G_\alpha\}$, where $E = \cup_\alpha V_\alpha$. Let $I_x$ be the union of sets $A \in \{V_\alpha\}$ such that $A$ intersects an open interval in $\{V_\alpha\}$ that contains $x$. $I_x$ is also a segment and $I_x \subset E$. If $y \in E$ then $I_x = I_y$ or they are disjoint. The collection $\{I_x\}$ where $x \in E$ covers $E$. Since $\mathbb{R}$ is separable, it can be reduced to a countable subcover and we get the desired result.