MATH311: Homework 11 (due May. 23)

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1 Section 6 #5

Let $f : [a, b] \to \mathbb{R}$ as

$$f(x) = \begin{cases} 1 & (x \in \mathbb{Q} \cap [a, b]) \\ -1 & (x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [a, b]) \end{cases}$$

Then, $|f(x)| \leq 1$ for all $x \in [a, b]$ so f is bounded and $(f(x))^2 = 1$ for all $x \in [a, b]$, thus $f^2 \in \mathscr{R}$. However, for all partition $P = \{p_0, \ldots, p_n\}$ of $[a, b]$, by Archimedean property there exists some rational number in $[p_{i-1}, p_i]$ for $i = 1, 2, \ldots, n$. Thus, $U(P, f) = b - a$. On the other hand, by Archimedean property there exists some rational number q_i in $(p_{i-1}/\sqrt{2}, p_i/\sqrt{2})$ for $i = 1, 2, \ldots, n$, so there exists some irrational number $q_i\sqrt{2} \in (p_{i-1}, p_i)$ for $i = 1, 2, \ldots, n$. Thus, $L(P, f) = a - b$. For all partition P of $[a, b]$, $U(P, f) - L(P, f) = 2(b - a)$ so $f \notin \mathscr{R}$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ as

$$\phi(x) = \begin{cases} \sqrt[3]{x} & (x \geq 0) \\ -\sqrt[3]{-x} & (x < 0) \end{cases}$$

and ϕ is continuous on $\mathbb{R}$. Then, by theorem 6.11 $f = \phi(f^3)$ is integrable on $[a, b]$.

2 Section 6 #6

Recall the construction of Cantor's set we made earlier. By construction, $P \subset E_n$ where E_n consists of 2^n closed intervals $[u_{n,i}, v_{n,i}]$ $(i = 1, 2, \ldots, 2^n)$, and $v_{n,i} - u_{n,i} = 3^{-n}$. Also, the definition of E_n implies $u_{n,i+1} - v_{n,i} \geq 3^{-n-1}$ $(i = 1, 2, \ldots, 2^n - 1)$ as E_n is created by removing middle thirds of intervals in E_{n-1}. Let $E_n^* = \cup_{i=1}^{2^n}(u_{n,i}^*, v_{n,i}^*)$ where $u_{n,i}^* = u_{n,i} - 3^{-n-1}$, $v_{n,i}^* = v_{n,i} + 3^{-n-1}$. Then, $u_{n,i+1}^* - v_{n,i}^* = u_{n,i+1} - v_{n,i} - 2 \cdot 3^{-n-1} \geq 3^{-n-2}$ so $(u_{n,i}^*, v_{n,i}^*)$ are disjoint and $E_n \subset E_n^*$. Let $K = [0, 1] - E_k^*$, then K is compact since it is closed and bounded in $\mathbb{R}$. Fix $\epsilon > 0$. Since f is continuous on $[0, 1] - P$, so it is continuous on K, which implies uniform continuity on K. Thus, there exists $\delta > 0$ such that $|f(s) - f(t)| < \epsilon$ if $s, t \in K$ and $|s - t| < \delta$. Let $Q = \{x_0, \ldots, x_n\}$ be a partition of $[0, 1]$, which satisfies the following conditions:

1. Each $u_{k,i}^*$ and $v_{k,i}^*$ which is in $[0, 1]$ occurs in Q.

2. No point in E_k^* does not occur in Q.

3. If x_{i-1} is not one of the $u_{k,j}^*$ and zero, $\Delta x_i < \delta$.

Let $M = \sup|f(x)|$. Let $A = \{i \in \{1, 2, \ldots, n\};\ x_{i-1} \in \{0, u^*_{k,1}, \ldots, u^*_{k,2^k}\}\}$ and $B = \{1, 2, \ldots, n\} \setminus A$. For all $i = 1, 2, \ldots, n$, $M_i - m_i \le 2M$. If $i \in A$, $\Delta x_i < \delta$ so $M_i - m_i \le \epsilon$. If $i \in B$, $x_i \in \{v^*_{k,1}, \ldots, v^*_{k,2^k}, 1\}$ so $\Delta x_i \le 3^{-k} + 2 \cdot 3^{-k-1}$. Then we can write

$$\begin{aligned} U(Q, f) - L(Q, f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i \in A} (M_i - m_i)\Delta x_i + \sum_{i \in B} (M_i - m_i)\Delta x_i \\ &\le \sum_{i \in A} \epsilon \Delta x_i + \sum_{i \in B} 2M \cdot \frac{5}{3^{k+1}} \le \epsilon \sum_{i=1}^{n} \Delta x_i + 2^k \cdot 2M \cdot \frac{5}{3^{k+1}} < \epsilon + 5M\epsilon \end{aligned}$$

Since the choice of ϵ is arbitrary, $U(Q, f) - L(Q, f)$ can be made small as we like, and $f \in \mathscr{R}$ on $[0, 1]$.

3 Section 6 #7

3.1 Proof for (a)

We can write

$$L(\{0, c\}, f) \le \int_0^c f(x)dx \le U(\{0, c\}, f) \tag{1}$$

Since $f \in \mathscr{R}$ on $[0, 1]$, f is bounded on $[0, 1]$ and there exists some $M > 0$ such that $\sup_{x \in [0,c]} |f(x)| \le M$. Then we can write (1) as

$$-cM \le c \inf_{x \in [0,c]} f(x) \le \int_0^c f(x)dx \le c \sup_{x \in [0,c]} f(x) \le cM$$

by sandwich theorem, we know that $\lim_{c \to 0} \int_0^c f(x)dx = 0$, so we can write

$$\begin{aligned} \lim_{c \to 0} \int_c^1 f(x)dx &= \lim_{c \to 0} \left(\int_0^1 f(x)dx - \int_0^c f(x)dx \right) = \int_0^1 f(x)dx - \lim_{c \to 0} \int_0^c f(x)dx \\ &= \int_0^1 f(x)dx \end{aligned}$$

3.2 Solution for (b)

Let $f(x) = (-1)^n(n + 1)$, where n is the maximum integer $n \le 1/x$ then $f \in \mathscr{R}$ on $[c, 1]$ for all $c > 0$. If $N \le 1/c \le N + 1$ for $c > 0$, we can write

$$\begin{aligned} \int_c^1 f(x)dx &= (-1)^N(N + 1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} (-1)^k(k + 1)\left(\frac{1}{k} - \frac{1}{k + 1}\right) \\ &= (-1)^N(N + 1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{(-1)^k}{k} \end{aligned}$$

and

$$0 \le (N + 1)\left(\frac{1}{N} - c\right) \le \frac{1}{N}$$

since $N \to \infty$ as $c \to 0$, the first term converges to zero and by alternating series theorem $\sum_{k=1}^{N-1} (-1)^k/k$ converges as $N \to \infty$, so the integral converges. However, we can write

$$\int_c^1 |f(x)|dx = (N+1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{1}{k}$$

and

$$\sum_{k=1}^{N-1} \frac{1}{k} \le \int_c^1 |f(x)|dx \le \sum_{k=1}^{N} \frac{1}{k}$$

since harmonic series diverges, the integral also diverges as $c \to 0$.

4 Section 6 #8

Suppose that $\sum_{n=1}^{\infty} f(n)$ converges. Let $P = \{x_0, \ldots, x_n\} = \{1, 2, \ldots, n+1\}$. Then we can write

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} f(x_{i-1}) \Delta x_i = \sum_{i=1}^{n} f(i)$$
$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} f(x_i) \Delta x_i = \sum_{i=2}^{n+1} f(i)$$

and

$$L(P, f) = \sum_{i=2}^{n+1} f(i) \le \int_1^{n+1} f(x)dx \le \sum_{i=1}^{n} f(i) = U(P, f) \tag{2}$$

Using sandwich theorem,

$$\lim_{n\to\infty} \sum_{i=2}^{n+1} f(i) \le \lim_{n\to\infty} \int_1^{n+1} f(x)dx \le \lim_{n\to\infty} \sum_{i=1}^{n} f(i)$$

implies the existence $\int_1^{\infty} f(x)dx$.

Suppose that $\int_1^{\infty} f(x)dx$ converges. From (2), we obtain

$$\sum_{i=1}^{n} f(i) \le f(1) + \int_1^{n} f(x)dx \le f(1) + \int_1^{\infty} f(x)dx$$

as f is nonnegative, so $\int_1^n f(x)dx$ is monotonically increasing. Since $\sum_{k=1}^{n} f(k)$ is a monotonically increasing sequence which is bounded above, it converges by monotone convergence theorem.

5 Section 6 #10

5.1 Proof for (a)

Fix v. Let $\phi(u) = u^p/p + v^q/q - uv$. Then $\phi(u) = u^{p-1} - v$. For $0 \le u < v^{1/(p-1)}$, $\phi'(u) < 0$ so $\phi(u)$ decreases there. For $u > v^{1/(p-1)}$, $\phi'(u) > 0$ so $\phi(u)$ increases there. Thus, $\phi(v^{1/(p-1)})$ is the minimum of $\phi(u)$ and we can write

$$\phi(v^{\frac{1}{p-1}}) = \frac{v^{\frac{p}{p-1}}}{p} + \frac{v^q}{q} - v^{\frac{p}{p-1}} = \left(\frac{1}{p} + \frac{1}{q}\right) v^q - v^q = 0$$

for all $u \geq 0$, $\phi(u) \geq \phi(v^{1/(p-1)}) = 0$ so the inequality holds. If $u^p = v^q$, we can write

$$\frac{u^p}{p} + \frac{v^q}{q} = \frac{u^p}{p} + \frac{u^p}{q} = u^p$$
$$uv = u \cdot u^{p/q} = u^{\frac{p+q}{q}} = u^{1+p/q} = u^{p\left(\frac{1}{p}+\frac{1}{q}\right)} = u^p$$

Thus the equality holds.

5.2 Proof for (b)

From the inequality we proved in (a) and properties of integral,

$$\int_a^b fg d\alpha \leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q}\right) d\alpha = \frac{1}{p}\int_a^b f^p d\alpha + \frac{1}{q}\int_a^b g^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

5.3 Proof for (c)

Let $A = \int_a^b |f|^p d\alpha$, $B = \int_a^b |g|^q d\alpha$ and $\hat{f}(x) = f(x)/A^{1/p}$, $\hat{g}(x) = g(x)/B^{1/q}$. Assume that $A > 0$ and $B > 0$. Then we can write

$$\int_a^b |\hat{f}|^p d\alpha = \int_a^b \left|\frac{f}{A^{\frac{1}{p}}}\right|^p d\alpha = \frac{1}{A}\int_a^b |f|^p d\alpha = 1$$
$$\int_a^b |\hat{g}|^q d\alpha = \int_a^b \left|\frac{g}{B^{\frac{1}{q}}}\right|^q d\alpha = \frac{1}{B}\int_a^b |g|^q d\alpha = 1$$

By the result proved in (b),

$$\int_a^b |\hat{f}\hat{g}| d\alpha \leq 1 \tag{3}$$

Suppose that the following holds:

$$\int_a^b \hat{f}\hat{g} d\alpha = r_0 e^{i\theta_0}$$

then we can write

$$\int_a^b e^{-i\theta_0}\hat{f}\hat{g} d\alpha = \mathrm{Re}\int_a^b e^{-i\theta_0}\hat{f}\hat{g} d\alpha = \int_a^b \mathrm{Re}(e^{-i\theta_0}\hat{f}\hat{g}) d\alpha = r_0$$

and $\mathrm{Re}(e^{-i\theta_0}\hat{f}\hat{g}) \leq |e^{-i\theta_0}\hat{f}\hat{g}| = |\hat{f}\hat{g}|$, so we can write

$$\left|\int_a^b \hat{f}\hat{g} d\alpha\right| \leq \int_a^b |\hat{f}\hat{g}| d\alpha$$

By (3),

$$\left|\int_a^b \hat{f}\hat{g} d\alpha\right| \leq \int_a^b |\hat{f}\hat{g}| d\alpha \leq 1 = \left(\int_a^b |\hat{f}|^p d\alpha\right)^{\frac{1}{p}} \left(\int_a^b |\hat{g}|^q d\alpha\right)^{\frac{1}{q}}$$
$$= \frac{1}{A^{1/p}B^{1/q}} \left(\int_a^b |f|^p d\alpha\right)^{\frac{1}{p}} \left(\int_a^b |g|^q d\alpha\right)^{\frac{1}{q}}$$

Multiplying both sides with $A^{1/p}B^{1/q}$ gives

$$\left|\int_a^b fg d\alpha\right| \leq \left(\int_a^b |f|^p d\alpha\right)^{\frac{1}{p}} \left(\int_a^b |g|^q d\alpha\right)^{\frac{1}{q}}$$

and we get the desired result.

5.4 Proof for (d)

Suppose that $f, g \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Then we have

$$\left|\int_c^1 fg dx\right| \leq \left(\int_c^1 |f|^p dx\right)^{\frac{1}{p}} \left(\int_c^1 |g|^q dx\right)^{\frac{1}{q}}$$

Since $|f| \geq 0$ and $|g| \geq 0$, the right hand side increases as $c \to 0$ and we can write

$$\left|\int_c^1 fg dx\right| \leq \left(\int_0^1 |f|^p dx\right)^{\frac{1}{p}} \left(\int_0^1 |g|^q dx\right)^{\frac{1}{q}}$$

Since $|z|$ is continuous for all $z \in \mathbb{C}$ and by properties of limits, we can write

$$\lim_{c\to 0}\left|\int_c^1 fg dx\right| = \left|\lim_{c\to 0}\int_c^1 fg dx\right| = \left|\int_0^1 fg dx\right| \leq \left(\int_0^1 |f|^p dx\right)^{\frac{1}{p}} \left(\int_0^1 |g|^q dx\right)^{\frac{1}{q}}$$

Now, suppose that $f, g \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Then we have

$$\left|\int_a^b fg dx\right| \leq \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g|^q dx\right)^{\frac{1}{q}}$$

Since $|f| \geq 0$ and $|g| \geq 0$, the right hand side increases as $b \to \infty$ and we can write

$$\left|\int_a^b fg dx\right| \leq \left(\int_a^\infty |f|^p dx\right)^{\frac{1}{p}} \left(\int_a^\infty |g|^q dx\right)^{\frac{1}{q}}$$

Since $|z|$ is continuous for all $z \in \mathbb{C}$ and by properties of limits, we can write

$$\lim_{b\to\infty}\left|\int_a^b fg dx\right| = \left|\lim_{b\to\infty}\int_a^b fg dx\right| = \left|\int_a^\infty fg dx\right| \leq \left(\int_a^\infty |f|^p dx\right)^{\frac{1}{p}} \left(\int_a^\infty |g|^q dx\right)^{\frac{1}{q}}$$

and we get the desired result.

6 Section 6 #11

By Schwarz inequality,

$$\left|\int_a^b |f-g||g-h| d\alpha\right| \leq \left(\int_a^b |f-g|^2 d\alpha\right)^{\frac{1}{2}} \left(\int_a^b |g-h|^2 d\alpha\right)^{\frac{1}{2}}$$

$$\int_a^b (|f-g|^2 + 2|f-g||g-h| + |g-h|^2) d\alpha \leq \left[\left(\int_a^b |f-g|^2 d\alpha\right)^{\frac{1}{2}} + \left(\int_a^b |g-h|^2 d\alpha\right)^{\frac{1}{2}}\right]^2$$

$$\int_a^b (|f-g| + |g-h|)^2 d\alpha \leq \left[\left(\int_a^b |f-g|^2 d\alpha\right)^{\frac{1}{2}} + \left(\int_a^b |g-h|^2 d\alpha\right)^{\frac{1}{2}}\right]^2$$

By triangular inequality, $|f-h| \leq |f-g| + |g-h|$ and since $|f-g|$, $|g-h|, |f-h|$ are all nonnegative,

$$\int_a^b |f-h|^2 d\alpha \leq \left[\left(\int_a^b |f-g|^2 d\alpha\right)^{\frac{1}{2}} + \left(\int_a^b |g-h|^2 d\alpha\right)^{\frac{1}{2}}\right]^2$$

$$\left(\int_a^b |f-h|^2 d\alpha\right)^{\frac{1}{2}} \leq \left(\int_a^b |f-g|^2 d\alpha\right)^{\frac{1}{2}} + \left(\int_a^b |g-h|^2 d\alpha\right)^{\frac{1}{2}}$$

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2$$

and we obtain the desired result.

7 Extra Problem

Fix $\epsilon > 0$. There exists an integer $N > 0$ such that $1/(N+1) < \epsilon/2$. Let $A = \{1, 1/2, 1/3, 2/3, \dots, 1/N, \dots, (N-1)/N\}$. For all $x \in [0,1]$ such that $x \notin A$, $f(x) \leq 1/(N+1) < \epsilon/2$. Let m be the number of elements of A, $P = \{p_0, p_1, \dots, p_n\}$ which is a partition of $[0,1]$ where $n > m$, $\Delta x_i < \epsilon/4m$, and $I_1 = \{i = \{1, \dots, n\};\ A \cap [p_{i-1}, p_i] \neq \varnothing\}$ and $I_2 = \{1, \dots, n\} \setminus I_1$. Then we can write

$$\begin{aligned} U(P,f) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i \in I_1} M_i \Delta x_i + \sum_{i \in I_2} M_i \Delta x_i \\ &\leq \frac{k\epsilon}{4m} + \frac{\epsilon}{2} \sum_{i \in I_2} \Delta x_i \leq \frac{k\epsilon}{4m} + \frac{\epsilon}{2} \end{aligned}$$

where k is the number of elements in I_1, and there are at most $2m$ intervals $[p_{i-1}, p_i]$ such that $[p_{i-1}, p_i] \cap A$ is nonempty. Thus $k \leq 2m$ so $U(P,f) \leq \epsilon$. As stated in the solution for #5, for all (p_{i-1}, p_i) there exists at least one irrational number in the open interval, so $L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = 0$. Then, $U(P,f) - L(P,f) \leq \epsilon$ and since our choice of ϵ is arbitrary, $f \in \mathscr{R}$ on $[0,1]$.