MATH311: Homework 7 (due Apr. 19)

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Last compiled on: Monday 24th April, 2023, 19:25

1 Section 4 #1

We write f as

$$f(x) = \begin{cases} 1 & (x = 0) \\ 0 & (x \neq 0) \end{cases}$$

Fix $\epsilon = 1/2$, then $|f(x) - f(0)| < \epsilon$ for only $x = 1$, so there is no $\delta > 0$ such that $|x - 0| < \delta$ implies $|f(x) - f(0)| < \epsilon$ and f is not continuous at $x = 0$. Fix $\epsilon > 0$. For all $x \in \mathbb{R}$, there exists $\delta > 0$ such that $0 < |h| < \delta$ implies $|f(x+h) - f(x-h)| < \epsilon$ since if $x \neq 0$, x is an element of either $(0, \infty)$ or $(-\infty, 0)$ which are both open sets. If $x = 0$, for all h such that $0 < |h|$, $|f(x+h) - f(x-h)| = |0 - 0| = 0$ so such δ exists and $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$. Thus, the condition does not imply that f is continuous.

2 Section 4 #2

Since f is continuous and $\overline{f(E)}$ is a closed subset of Y, $f^{-1}(\overline{f(E)})$ is also a closed subset of X. $E \subset f^{-1}(\overline{f(E)})$, so $\overline{E} \subset f^{-1}(\overline{f(E)})$ and $f(\overline{E}) \subset \overline{f(E)}$.

Consider $f : E \to \mathbb{R}, x \mapsto 1/(1 + e^{-x})$, where $E = \mathbb{R}$. Since $e^{-x} > 0$ for all $x \in E$, $f(E) = (0, 1)$. We can see that $f(\overline{E}) = f(E) = (0, 1)$, so $f(\overline{E}) \subset \overline{f(E)} = [0, 1]$ and $f(\overline{E})$ is a proper subset of $\overline{f(E)}$.

3 Section 4 #3

Let $E := \{0\}$. E is a closed set since $E' = \varnothing$. Then, $f^{-1}(E)$ is also closed since f is continuous. By definition, $f^{-1}(E)$ is the set of all $x \in X$ such that $f(x) \in E$, so $f^{-1}(E) = Z(f)$ and we get the desired result.

4 Section 4 #4

By the result we proved in #3, $f(E) \subset f(X) = f(\overline{E}) \subset \overline{f(E)}$ since E is dense in X. From $f(X) \subset \overline{f(E)}$, all points in $f(X)$ are either point or limit point of $f(E)$ so $f(E)$ is dense in $f(X)$.

Let $h : X \to Y, x \mapsto f(x) - g(x)$, then $h(E) = \{0\}$. By the result we proved earlier, $h(E)$ is dense in $h(X)$ so every point in $h(X)$ is either a limit point or point of $h(E)$. Since $[h(E)]' = \varnothing$, $h(X) \subset \{0\}$, thus $h(X) = \{0\}$ and $f(x) - g(x) = 0$ for all $x \in X$.

5 Section 4 #5

Using the resule proved in exercise 29 of chapter 2, the complement of E can be written union of at most countably many disjoint open intervals, $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n), \dots$ with possibly one or both of (a_0, ∞) and $(-\infty, b)$. Let $g(x)$ be a function defined on $\mathbb{R}$ such that

$$g(x) = \begin{cases} f(x) & (x \in E) \\ \frac{f(b_k)-f(a_k)}{b_k-a_k}(x-a_k)+f(a_k) & (x \in (a_k, b_k)) \\ f(a) & (x \geq a) \\ f(b) & (x \leq b) \end{cases}$$

For $p \in E^C$, g is continuous at $x = p$ since it is a point in the region where $g(x)$ is a polynomial function. For $p \in E^\circ$, g is continuous at $x = p$ since for all $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x \in E^\circ$ satisfying $|x - p| < \delta$ and $(x - \delta, x + \delta) \subset E^\circ$ as f is continuous at $x = p$. Consider the case where $p = a_k$, where $k \geq 0$. We can take $(p - r, p + r)$ such that $0 < r < b_k - a_k$ and $b_{k-1} < p - r$ if $k > 0$. (If $k = 0$, then we can simply ignore $b_{k-1} < p - r$ condition.) Then, $(p - r, a_k] \subset E$ and $(a_k, p + r) \subset E^C$. Fix $\epsilon > 0$, then there exists some $\delta_1 > 0$ such that $|g(x) - g(p)| < \epsilon$ for all $x \in (p - r, a_k]$ satisfying $|x - p| < \delta_1$ as g is continuous in $(p - r, a_k] \subset E$. Also, there exists some $\delta_2 > 0$ such that $|g(x) - g(p)| < \epsilon$ for all $x \in (a_k, p + r)$ satisfying $|x - p| < \delta_2$ as g is a polynomial function in $(a_k, p + r)$ and it's continuous. Let $\delta = \min\{\delta_1, \delta_2\}$ then $|g(x) - g(p)| < \epsilon$ for all $x \in (p - r, p + r)$ satisfying $|x - p| < \delta$ so g is continuous at $x = p$. We can prove that g is continuous at $x = b_k$ where $k \geq 0$ and $x = b$ using the similar logic.

Consider $f : (0, 1] \to \mathbb{R}, x \mapsto 1/x$. Suppose that a continuous extension g on $\mathbb{R}$ such that $f(x) = g(x)$ for all $x \in (0, 1)$. Since g is a continuous function on $\mathbb{R}$, $g(0) = \lim_{x\to0} g(x)$ should hold. However, for sequence $p_n = 1/n$, $\lim_{n\to\infty} p_n = 0$ but $\lim_{n\to\infty} g(p_n) = \lim_{n\to\infty} n \neq 0$ so it is a contradiction and the result is not generally true for all domains.

For continuous vector valued function $\mathbf{f} : E \to \mathbb{R}^k, x \mapsto (f_1(x), f_2(x), \dots, f_k(x))$ defined on closed sets, $f_1, f_2, \dots, f_k$ are continuous on E. Let $g_1, g_2, \dots, g_k$ be continuous extensions of $f_1, f_2, \dots, f_k$ respectively. Then, we can define $\mathbf{g} : \mathbb{R} \to \mathbb{R}^k, x \mapsto (g_1(x), g_2(x), \dots, g_k(x))$ and it is continuous on $\mathbb{R}$ since its components are continuous on $\mathbb{R}$.