Homework 1 (due Feb. 28)

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1 Problem 1

1.1 Proof for $r + x$

We will use proof by contradiction here. Suppose that $r + x$ is a rational number. By the definition of rational numbers, $r + x$ can be written as m/n where $m, n \in \mathbb{Z}$ and $n \neq 0$. Then, x can be written as $m/n - r$. Since the set of rational number is a field, the additive inverse of r, $-r$ is a rational number. By this, $x = m/n - r$ is a rational number because of axioms for additions, which is a contradiction. Thus, $r + x$ is not a rational number. $r + x$ is irrational since it is a real number but not a rational number.

1.2 Proof for rx

Use the similar argument as $r + x$ case. Suppose that rx is a rational number. By the definition of rational numbers, rx can be written as m/n where $m, n \in \mathbb{Z}$ and $n \neq 0$. Then, x can be written as $(m/n)(1/r)$. Since the set of rational number is a field, the multiplicative inverse of r, $1/r$ is a rational number. By this, $x = (m/n)(1/r)$ is a rational number because of axioms for multiplications, which is a contradiction. Thus, rx is not a rational number. rx is irrational since it is a real number but not a rational number.

2 Problem 2

Let x be a real number where $x^2 = 12$, and y be a real number where $2y = x$. Then, $y^2 = 3$. Let's prove that y cannot be a rational number using proof by contradiction. Suppose that there exists $p \in \mathbb{Q}$ where $p^2 = 3$. p can be written as m/n where $m, n \in \mathbb{Z}$ are not both multiples of 3. Let's assume that this is done. Then the following holds:

$$3n^2 = m^2$$

This shows that m is a multiple of 3, and m^2 is divisible by 9. By this, the left side $3n^2$ is divisible by 9. Thus, n^2 is divisible by 3 and it implies that n is divisible by 3. This leads to the conclusion that m and n are both multiples of 3, which is a contradiction. Since y cannot be a rational number, it can be shown that $x = 2y$ also cannot be rational using the result proven in problem 1.

3 Problem 3

First, prove (a) using the axioms for multiplications.

$$\begin{aligned} y &= 1 \cdot y = ((1/x)x)y = (1/x)(xy) \\ &= (1/x)(xz) = ((1/x)x)z = 1 \cdot z = z \end{aligned}$$

Take $z = 1$ in (a) to obtain (b), and take $z = 1/x$ in (a) to obtain (c). Since $x(1/x) = 1$, (c) (with $1/x$, x in place of x and y, respectively) gives (d).

4 Problem 4

Suppose that there exists a nonempty subset E' where $\alpha' > \beta'$ holds for its lower bound α' and upper bound β'. By definition, $x \geq \alpha'\ \forall x \in E'$ and $x \leq \beta'\ \forall x \in E'$, so there exists an element $z \in E'$ such that $z \geq \alpha'$ and $z \leq \beta'$. There are three possibilities:

1. $z = \alpha'$
2. $z = \beta'$
3. $z \neq \alpha'$ and $z \neq \beta'$

For $z = \alpha'$ case, $z > \beta'$ because we assumed that $\alpha' > \beta'$, but it contradicts with $z \leq \beta'$, so it is impossible. $z = \beta'$ case is similarly impossible because it implies $\alpha' > z$, but it contradicts with $z \geq \alpha'$. For the final case, $z \neq \alpha'$ and $z \neq \beta'$ implies $z > \alpha'$ and $z < \beta'$, so $\alpha' < \beta'$ according to the axioms of ordered sets, which is contrary to the assumption. In conclusion, such subset E' cannot exist, meaning that for all subset E with lower bound α and upper bound β, $\alpha \leq \beta$ holds.

5 Problem 5

Let $\alpha := \inf A$. By the definition of lower bound, $x \geq \alpha\ \forall x \in A$. According to axioms of ordered fields, $-x \leq -\alpha\ \forall x \in A$. By the definition of $-A$, $y \leq -\alpha\ \forall y \in -A$. This implies that $-A$ is bounded above, and $-\alpha$ is an upper bound of $-A$. Since A is a set of real numbers, $-A$ is also a set of real numbers and it has the least upper bound, $\beta := \sup(-A)$. Suppose that $\beta < -\alpha$. By the definition of upper bound, $y \leq \beta\ \forall y \in -A$. Using axioms of ordered fields again, $-y \geq -\beta\ \forall y \in -A$, and the proposition 1.14 in the book implies that $x \geq -\beta\ \forall x \in A$, so $-\beta$ is a lower bound of A. However the assumption we made earlier implies that $-\beta > \alpha$, and it contradicts with the definition of α since lower bound greater than the greatest lower bound must not exist. Thus, $\beta \geq -\alpha$ holds, and since $-\alpha$ is an upper bound of $-A$, it becomes the least upper bound of $-A$. This means that $\alpha = \inf A = -\beta = -\sup(-A)$.