MATH311: Homework 5 (due Mar. 28)

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1 Section 2 #14

Let $\{G_n\} = (0, 1 - 1/n)$ where $n = 2, 3, \ldots$. For all $x \in (0, 1)$, there exists an integer $p > 1$ such that $p(1 - x) > 1$ by the theorem 1.20 in the book. Then, we get $x < 1 - 1/p$ so $x \in G_p$ and for all $G_k \in G_n$ $G_k \cap ((-\infty, 0] \cup [1, \infty)) = \varnothing$, so $\{G_n\}$ is an open cover of $(0, 1)$. Thus, for all $x \in (0, 1)$ there exists G_p such that $x \in G_p$. Since G_n is open for all $n = 2, 3, \ldots$, $\{G_n\}$ is an open cover. Suppose that there exists a finite subcover $\{G_{n_k}\}$ of $\{G_n\}$. Let M be the maximum element of the finite set of n_k and $y = 1 - 1/M$. For all $(a_{n_k}, b_{n_k}) \in \{G_{n_k}\}$, $b_{n_k} = 1 - 1/n_k \leq 1 - 1/M$. Then, we can conclude that the open cover $\{G_n\}$ has no finite subcover.

2 Section 2 #17

Let $E_0 = [0, 1]$. Let E_n be a set of real numbers where first n digits of one of its decimal expansion are 4 or 7. From this definition, we can write

$$E_n = \bigcap_{m=1}^{n-1} \bigcup_{k=0}^{10^m - 1} \left(\left[\frac{10k + 4}{10^m}, \frac{10k + 5}{10^m} \right] \cup \left[\frac{10k + 7}{10^m}, \frac{10k + 8}{10^m} \right] \right)$$

Let $E = \bigcap_{n=0}^{\infty} E_n$. Since E is an intersection of unions of finite number of closed sets, E is a closed set by theorem 2.24. Also, since E is an intersection of subsets of $[0, 1]$ which is a bounded set, E is also bounded. Then, by theorem 2.41, E is compact. Let $x \in E$, and let S be any open interval containing x. Let I_n be a closed interval of E_n which contains x. For sufficiently large n, $I_n \subset S$. Let x_n be an endpoint of I_n, such that $x \neq x_n$. By the definition of E, $x_n \in E$, and x is a limit point of E so E is a perfect set. By theorem 2.43, E is an uncountable set. Since decimal expansion of every x in E has only digits 4 and 7, so $x \leq 0.4$. Then 0 cannot be a limit point of E because $d(x, 0) = |x - 0| = (x - 0.4) + 0.4 > 0.4$. $0 \notin E$, so E is not dense.

3 Section 2 #19

3.1 Proof for (a)

By definition, $\bar{A} = A \cup A', \bar{B} = B \cup B'$. Since A and B are closed sets, $A' \subset A, B' \subset B$. Thus, $\bar{A} = A, \bar{B} = B$ and $\bar{A} \cap B = A \cap \bar{B} = A \cap B = \varnothing$ since A and B are disjoint and we get the desired result.

3.2 Proof for (b)

We need to prove that $A' \cap B = A \cap B' = \varnothing$. Let $p \in X$ be a limit point of A and suppose that $p \in B$. Since $p \in B$, there exists some $r > 0$ such that $N_r(p) \subset B$, and there exists $q \in N_r(p) \cap A$ such that $q \neq p$ since p is a limit point of A. Then, $q \in A$ and $q \in B$ both holds, and it is a contradiction since A and B are disjoint. Thus, such p does not exist and $A' \cap B = \varnothing$. By using the same argument, we can see that $A \cap B' = \varnothing$ is also true. In conclusion, $\bar{A} \cap B = (A \cup A') \cap B = (A \cap B) \cup (A' \cap B) = \varnothing \cup \varnothing = \varnothing$ and $A \cap \bar{B} = A \cap (B \cup B') = (A \cap B) \cup (A \cap B') = \varnothing \cup \varnothing = \varnothing$, so A and B are separate.

3.3 Proof for (c)

Let's prove that for all $q \in A$, $N_{\delta - d(p,q)}(q) \subset A$. Let $r \in N_{\delta - d(p,q)}$. By triangular inequality, $d(p,r) \leq d(p,q) + d(q,r) < d(p,q) + (\delta - d(p,q)) = \delta$. So $r \in A$, and we know that q is an interior point of A. Thus, A is open.

Now, let's prove that for all $q \in B$, $N_{d(p,q)-\delta}(q) \subset B$. Let $r \in N_{d(p,q)-\delta}(q)$. By triangular inequality, $d(p,r) \geq d(p,q) - d(q,r) > d(p,q) - (d(p,q) - \delta) = \delta$. So $r \in B$, and we know that q is an interior point of B. Thus, B is open.

For all $q \in A$, $d(p,q) < \delta$, so $d(p,q) > \delta$ is false and $q \notin B$. Thus, $A \cap B = \varnothing$ so A and B are separate by the result we proved in (b).

3.4 Proof for (d)

Let $x, y \in X$ and $D = d(x,y)$. Suppose that X is connected and there exists $\delta \in [0, D]$ such that for all $p \in X$, $d(x,p) \neq \delta$. Then, we can partition X into two disjoint subsets $A = \{p \in X;\ d(x,p) < \delta\}$ and $B = \{p \in X;\ d(x,p) > \delta\}$, and $A \cup B = X$ holds. By the resule proven in (c), A and B are separate, which is a contradiction. In conclusion, such δ does not exist and there exists a 1-1 mapping of $[0, D]$ onto a subset of X, so that subset of X is uncountable. Thus, X is also uncountable since it has an uncountable subset.

4 Section 2 #20

Let $E \subset X$ be a connected set whose closure $\bar{E}$ is not connected where X is a metric space. Then, there exists $A, B \subset \bar{E}$ such that $A \cap \bar{B} = \bar{A} \cap B = \varnothing$ and $A \cup B = \bar{E}$. Since $\bar{E} = E \cup E'$, we can write

$$\begin{aligned} A \cap \bar{B} &= (A \cap \bar{E}) \cap (\bar{B} \cap \bar{E}) = A \cap \bar{B} \cap \bar{E} \\ &= A \cap \bar{B} \cap (E \cup E') = (A \cap \bar{B} \cap E) \cup (A \cap \bar{B} \cap E') \\ &\supset A \cap \bar{B} \cap E \end{aligned}$$

Let $C = E \setminus A$, then $C \subset B$ since $(\bar{E} \setminus A) \subset B$ and $E \subset \bar{E}$. Since $C \subset B \subset \bar{B}$, by theorem 2.27 $\bar{C} \subset \bar{B}$ holds. From connectedness of E, $(A \cap E) \cap \bar{C} \neq \varnothing$, so $A \cap \bar{B} \cap E \supset A \cap \bar{C} \cap E \neq \varnothing$, which is a contradiction. Thus, such connected set E does not exist and closures of all connected sets are connected.

Let $E = ([-1,1] \times \{0\}) \cup ((-\infty, -1] \times \mathbb{R}) \cup ([1, \infty) \times \mathbb{R})$. E is connected since it has no A, B such that $E = A \cup B$ where $\bar{A} \cap B = A \cap \bar{B} = \varnothing$. The interior of E is $((-\infty, -1) \times \mathbb{R}) \cup ((1, \infty) \times \mathbb{R})$. Let $A = (-\infty, -1) \times \mathbb{R}, B = (1, \infty) \times \mathbb{R}$ and $\bar{A} \cap B = A \cap \bar{B} = \varnothing$. So the interior of E is not connected.

5 Section 3 #1

Let $a_n = (-1)^n$. $|a_n| = 1$, so for all $\epsilon > 0$, $||a_n| - 1| = 0$ holds for $n \geq 1$ and we can know that $|a_n|$ converges to 1. Suppose that a_n has a limit x. $|a_n - x| + |x - 0| \geq |a_n| = 1$ holds by triangular inequality, so $|a_n - x| \geq 1 - |x|$. Similarly, $|a_n - x| + |0 - a_n| = |a_n - x| + 1 \geq |0 - x|$ holds, so $|a_n - x| \geq |x| - 1$, thus $|a_n - x| \geq ||x| - 1|$. Then, for all $x \in \mathbb{R}$, $|a_n - x| \geq ||x| - 1|$ for all $n \in \mathbb{N}$, and x is either 1 or -1. If $x = 1$, $|a_n - 1| = 2 \geq \epsilon$ for odd n and sufficiently small ϵ, and if $x = -1$, $|a_n - 1| = 2 \geq \epsilon$ for even n and sufficiently small ϵ. Thus, the limit of a_n does not exist, and we can see that the converse is not generally true.

6 Section 3 #2

Let $a_n = \sqrt{n^2 + n} - n - 1/2$. We can write

$$a_n = \sqrt{n^2 + n} - \left(n + \frac{1}{2}\right) = \frac{n^2 + n - (n + 1/2)^2}{\sqrt{n^2 + n} + n + 1/2} = -\frac{1}{4\sqrt{n^2 + n} + 4n + 2}$$

Then,

$$|a_n| = \frac{1}{4\sqrt{n^2 + n} + 4n + 2} \leq \frac{1}{4n} \leq \frac{1}{n}$$

For all $\epsilon > 0$, $|a_n - 0| \leq 1/n < \epsilon$ holds for $n \geq N$, where N is an integer which is $N > 1/\epsilon$. Thus, a_n converges to 0. Since $\lim_{n\to\infty}(1/2) = 1/2$, $\lim_{n\to\infty}(\sqrt{n^2 + n} - n) = \lim_{n\to\infty}(a_n + 1/2) = 1/2$.