# MATH311: Homework 8 (due May. 2)

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Last compiled on: Tuesday 2$^{\text{nd}}$ May, 2023, 14:22

## 1 Section 4 #11

Fix $\epsilon > 0$. By uniform continuity of $f$, there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ for all $p, q \in X$ for which $d_X(p, q) < \delta$. Since $\{x_n\}$ is a Cauchy sequence, there exists an integer $N > 0$ such that $d_X(x_n, x_m) < \delta$ if $n, m \geq N$. Thus, $n, m \geq N$ implies that $d_X(x_n, x_m) < \delta$ so $d_Y(f(x_n), f(x_m)) < \epsilon$. In conclusion, $\{f(x_n)\}$ is a Cauchy sequence.

Define $g : X \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x) & (x \in E) \\ \lim_{n\to\infty} f(x_n) & (x \notin E) \end{cases}$$

where $\{x_n\}$ is a sequence in $E$ such that $\lim_{n\to\infty} x_n = x$. Then, $g$ is well defined on $X$ since for all $\{x_n\} \subset E$ such that $\lim_{n\to\infty} x_n = x$, $\{x_n\}$ is Cauchy sequence so $\{f(x_n)\}$ is also Cauchy, and from $\{f(x_n)\} \subset \mathbb{R}$, they converge to the same value. Fix $\epsilon > 0$. If $a, b \in E$, there exists some $\delta > 0$ such that $|g(a) - g(b)| < \epsilon/3$ since $f$ is uniformly continuous. If $a \in E, b \in X \setminus E$, there exists some $c \in E$ such that $d_X(c, b) < \delta - d_X(a, b)$ since $b \in X \setminus E$ is a limit point of $E$. Then, $|g(c) - g(b)| < \epsilon/3$ and we can write

$$d_X(a, c) \leq d_X(a, b) + d_X(c, b) < \delta$$

so $|g(a) - g(c)| < \epsilon/3$, and

$$|g(a) - g(b)| \leq |g(a) - g(c)| + |g(c) - g(b)| < \frac{2\epsilon}{3} < \epsilon$$

If $a, b \in X \setminus E$, there exists some $p, q \in E$ such that

$$d_X(a, p) < \frac{1}{2}(\delta - d_X(a, b)), \quad d_X(b, q) < \frac{1}{2}(\delta - d_X(a, b))$$

since $a, b$ are limit points of $E$. Then, $|g(a) - g(p)| < \epsilon/3$ and $|g(b) - g(q)| < \epsilon/3$ holds, and

$$d_X(p, q) \leq d_X(a, p) + d_X(a, b) + d_X(b, q) < \delta$$

so $|g(p) - g(q)| < \epsilon/3$. Thus,

$$|g(a) - g(b)| \leq |g(a) - g(p)| + |g(p) - g(q)| + |g(b) - g(q)| < \epsilon$$

So $g$ is uniformly continuous on $X$, then $g$ is continuous on $X$ and $f$ has a continuous extension from $E$ to $X$.

## 2 Section 4 #14

Suppose that there exists some continuous mapping $f : I \to I$ such that $f(x) \neq x$ for all $x \in I$. Let $g : I \to I, x \mapsto f(x) - x$. Then $g(x) \neq 0$ for all $x \in I$. In particular, $g(0) \neq 0$ so $g(0) > 0$. If there exists some $a \in I$ such that $g(a) < 0$, there exists $x \in (0, a)$ such that $g(x) = 0$ by intermediate value theorem, and it contradicts with the assumption we made earlier. Thus, $g(x) > 0$ for all $x \in I$. However, $g(1) = f(1) - 1$ and $f(1) \in I$ so $-1 \leq g(1) \leq 0$, which is a contradiction. Thus, such $f$ cannot exists and we get the desired result.

## 3 Section 4 #18

Fix $\epsilon > 0$. Let $p$ a real number, $M$ be an integer where $0 < 1/M < \epsilon$, and $m_n$ be the largest integer such that $m_n \leq np$ for positive integer $n$. Then, we can write

$$m_n \leq np \leq m_n + 1 \iff \frac{m_n}{n} \leq p \leq \frac{m_n + 1}{n}$$

Let $\delta_1 = \min\{(m_n+1)/n - p\}$ for $n = 1, 2, \ldots, M$. Suppose that there exists some rational $q = s/t \in (p, p + \delta_1)$ where $s, t \in \mathbb{Z}$ are coprime and $0 < t \leq M$. If $q \leq m_t/t$, $q \leq m_t/t \leq p$ so it is impossible. If $q \geq (m_t + 1)/t$, the following holds:

$$q - p \geq \frac{m_t + 1}{t} - p \geq \delta_1$$

and it is a contradiction. Thus, such $q$ cannot exist and we can conclude that $x \in (p, p+\delta_1)$ implies $f(x) < 1/M < \epsilon$. Let $\delta_2 = \min\{p - m_n/n\}$ for $n = 1, 2, \ldots, M$. Suppose that there exists some rational $q = s/t \in (p - \delta_2, p)$ where $s, t \in \mathbb{Z}$ are coprime and $0 < t \leq M$. If $q \geq (m_t + 1)/t$, $q \geq (m_t + 1)/t \geq p$ so it is impossible. If $q \leq m_t/t$, the following holds:

$$p - q \geq p - \frac{m_t}{t} \geq \delta_2$$

and it is a contradiction. Thus, such $q$ cannot exist and we can conclude that $x \in (p-\delta_2, p)$ implies $f(x) < 1/M < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$, then for all $x \in (p - \delta, p + \delta)$ such that $x \neq p$, $0 \geq f(x) < 1/M < \epsilon$ so $|f(x) - 0| < \epsilon$. Them we can conclude that $\lim_{x \to p} f(x) = 0$ for all $p \in \mathbb{R}$. If $p$ is rational, $f(p) \neq 0$ so $f(x)$ is not continuous at every rational point, and the discontinuities are simple since $f(p+) = f(p-) = 0$. If $p$ is irrational, $f(p) = 0$ so $f(x)$ is continuous at every irrational point.

## 4 Section 4 #20

### 4.1 Proof for (a)

Suppose that $\rho_E(x) = \inf_{z \in E} d(x, z) = 0$ and $x \notin \overline{E}$. Then, $x \in (\overline{E})^C$ and $(\overline{E})^C$ is an open set, so there exists some $r > 0$ such that for all $y \in X$, $d(x, y) < r$ implies $y \in (\overline{E})^C$ so $y \notin \overline{E}$. Thus, for all $z \in \overline{E}$, $d(x, z) \geq r$ so $d(x, z) \geq r$ for all $z \in E$, which is a contradiction since $\inf_{z \in E} d(x, z) = 0$. We can conclude that $\rho_E(x) = 0$ implies $x \in \overline{E}$.

Suppose that $x \in \overline{E}$. If $x \in E$, we can take $z = x$ and get $d(x, z) = 0$. If $x \in E'$, for all $r > 0$ there exists $z \neq x$ such that $d(x, z) < r$, so $\inf_{z \in E} d(x, z) > 0$ is impossible. Thus, $\inf_{z \in E} d(x, z) = 0$ also holds for this case and we can conclude that $x \in \overline{E}$ implies $\rho_E(x) = 0$.

In conclusion, $\rho_E(x) = 0$ holds if and only if $x \in \overline{E}$.

## 4.2 Proof for (b)

By triangular inequality, $d(x,z) \le d(x,y)+d(y,z)$ for all $z \in E$ and $x, y \in X$. We can take the infinimum of both sides and get $\inf_{z\in E} d(x,z) \le d(x,y) + \inf_{z\in E} d(y,z)$. (if otherwise, there exists some $z \in E$ such that $d(x,y) + d(y,z) < \inf_{t\in E} d(x,t) \le d(x,z)$, which is a contradiction) Thus, we can write $\rho_E(x) \le d(x,y) + \rho_E(y)$ so $|\rho_E(x) - \rho_E(y)| \le d(x,y)$ holds. For all $\epsilon > 0$, we can take $\delta = \epsilon$ then for all $x, y \in X$, $d(x,y) < \delta$ implies $|\rho_E(x) - \rho_E(y)| \le d(x,y) < \delta = \epsilon$, so $\rho_E$ is a uniformly continuous function on $X$.

# 5 Section 4 #21

Using the result from #20 (b), $\rho_F$ is a uniformly continuous function on $X$. By extreme value theorem, there exists $x_0 \in K$ such that $\rho_F(x_0) = \inf_{p\in K} \rho_F(p)$. Then we can write

$$\rho_F(x_0) = \inf_{p\in K} \rho_F(p) = \inf_{p\in K, q\in F} d(p,q)$$

Since $K$ and $F$ are disjoint, $x \in K$ implies $x \notin F$, and since $F$ is closed $F = \overline{F}$ so $x \notin \overline{F}$. Using the result from #20 (a), $\rho_F(x) > 0$ for all $x \in K$ so $\rho_F(x_0) > 0$. Then, we can take $\delta \in (0, \rho_F(x_0))$ and $d(p,q) \ge \rho_F(x_0) > \delta$ for all $p \in K, q \in F$.

Consider the subsets of $\mathbb{R}^2$, $A = \{(x,0); x \in \mathbb{R}\}$ and $B = \{(0, 1/x); x \in \mathbb{R}, n > 0\}$. For all points $(a,b) \in \mathbb{R}^2$ where $b \ne 0$, all points $p \in \mathbb{R}^2$ such that $d(p,(a,b)) < b/2$ are not in $A$, so such $(a,b)$ is an interior point of $A^C$. Thus, $A^C$ is open so $A$ is closed, but not bounded so $A$ is not compact. Let $f : (0,\infty) \to \mathbb{R}^2, x \mapsto (x, 1/x)$. Then $f$ is a continuous function defined on $(0,\infty)$. For all converging sequence $\{(x_n, 1/x_n)\}$ on $B$, the point the sequence is converging to is also a point in $B$ since $B$ is a graph of $f$, which is continuous. Thus $B' \subset B$ and $B$ is closed, but not bounded so $B$ is not compact. Suppose that for all $p \in A, q \in B$, $d(p,q) > \delta$ for some $\delta > 0$. Since $(1/\delta, 0) \in A$ and $(1/\delta, \delta) \in B$, $d((1/\delta,0),(1/\delta,\delta)) = \delta$ and it is a contradiction, so such $\delta$ cannot exist.

# 6 Section 4 #22

Since $A$ and $B$ are disjoint nonempty closed sets, $\overline{A} \cap \overline{B} = A \cap B = \varnothing$, so there is no $p \in X$ such that $\rho_A(p) = \rho_B(p) = 0$. Thus, $f(p)$ is continuous on $X$ since continuous function on $X$ satisfies $\rho_A(p)+\rho_B(p) \ne 0$ for all $p \in X$, and $\rho_A(p)$ is also continuous, so $\rho_A(p)/(\rho_A(p)+\rho_B(p))$ is also continuous on $X$. Since $\rho_A(p)$ and $\rho_B(p)$ are both nonnegative, $f(p) \ge 0$ and $f(p) \le 1$ from $\rho_A(p) \le \rho_A(p) + \rho_B(p)$. If $p \in A$, then $\rho_A(p) = 0$ and $\rho_B(p) \ne 0$, so $f(p) = 0$. If $p \in B$, then $\rho_A(p) \ne 0$ and $\rho_B(p) = 0$, so $f(p) = 1$. By intermediate value theorem, for all $y \in (0,1)$ there exists $x \in (0,1)$ such that $f(x) = y$. Thus, the range of $f$ lies in $[0,1]$.

Let $g : X \to [0,1], p \mapsto f(p)$, then $g$ is a continuous function on $X$. Since $[0,1/2), (1/2,1]$ are both open in $[0,1]$, $V = g^{-1}([0,1/2)) = f^{-1}([0,1/2))$ and $W = g^{-1}((1/2,1]) = f^{-1}((1/2,1])$ are both open in $X$. By the result proved in #20 (a), $f(p) = 1$ if and only if $p \in A$, and $f(p) = 0$ if and only if $p \in B$ so $A = f^{-1}(\{1\}) \subset V, B = f^{-1}(\{0\}) \subset W$. Also, since $f$ cannot map a single value in $X$ to multiple values, $f^{-1}([0,1/2))$ and $f^{-1}((1/2,1])$ are disjoint because $[0,1/2), (1/2,1]$ are disjoint.

# 7 Section 4 #23

Let $p \in (a,b)$, $h > 0$ such that $(p-h, p+h) \subset (a,b)$, and $M = \max\{f(p-h), f(p+h)\}$. Then, for all $\lambda \in (0,1)$, $f(\lambda(p-h) + (1-\lambda)(p+h)) \le \lambda f(p-h) + (1-\lambda) f(p+h) \le M$, so $f(x) \leq M$ for all $x \in (p-h, p+h)$. Let $t \in (-h, h)$. Then we can write

$$f(p) \leq \frac{1}{2}f(p+t) + \frac{1}{2}f(p-t)$$

so

$$f(p+t) \geq 2f(p) - f(p-t) \geq 2f(p) - M$$

Thus, $2f(p) - M \leq f(x) \leq M$ holds for all $x \in (p-h, p+h)$. Let $x, y \in (p-h, p+h)$ and $x < y$. Take $\epsilon > 0$ such that $y + \epsilon < p + h$, then $\lambda = |y-x|/(|y-x|+\epsilon)$. Let $z = x + (y-x)/\lambda = y + \epsilon$, then $z \in (p-h, p+h)$. Then we can write

$$f(y) \leq \lambda f(z) + (1-\lambda)f(x) = \lambda(f(z) - f(x)) + f(x)$$

so

$$f(y) - f(x) = \lambda(M - (2f(p) - M)) < \frac{|y-x|}{\epsilon}(2M - 2f(p))$$

By taking $\delta = \epsilon^2/(4M - 4f(p))$ for all $x, y \in (p-h, p+h)$ such that $|x-y| < \delta$, $|f(y) - f(x)| \leq \epsilon/2 < \epsilon$. In conclusion, $f$ is uniformly continuous on $(p-h, p+h)$, so it is continuous at $x = p$, so $f$ is continuous on $(a, b)$.

Suppose that $f : (a, b) \to \mathbb{R}$ is a convex function, and $g : f((a, b)) \to \mathbb{R}$ is an increasing convex function. For all $x, y \in (a, b)$ and $\lambda \in (0, 1)$, the following holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{1}$$

The following also holds:

$$g(\lambda f(x) + (1-\lambda)f(y)) \leq \lambda g(f(x)) + (1-\lambda)g(f(y))$$

Since $g$ is increasing function, from (1),

$$g(f(\lambda x + (1-\lambda)y)) \leq g(\lambda f(x) + (1-\lambda)f(y)) \leq \lambda g(f(x)) + (1-\lambda)g(f(y))$$

Thus $g(f(x))$ is also a convex function on $(a, b)$.

From

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \tag{2}$$

we can write the equivalent inequalities of (2).

$$\begin{aligned}
(u-s)(f(t) - f(s)) &\leq (t-s)(f(u) - f(s)) \\
(u-s)f(t) &\leq (t-s)f(u) + (u-t)f(s) \\
f(t) &\leq \frac{t-s}{u-s}f(u) + \frac{u-t}{u-s}f(s) \\
f\left(\frac{t-s}{u-s}u + \left(1 - \frac{t-s}{u-s}\right)s\right) &\leq \frac{t-s}{u-s}f(u) + \left(1 - \frac{t-s}{u-s}\right)f(s)
\end{aligned}$$

From the definition of convexity, (2) holds since $(t-s)/(u-s) \in (0, 1)$. From

$$\frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t} \tag{3}$$

we can again write the equivalent inequalities of (3).

$$
\begin{aligned}
(u-t)(f(u)-f(s)) &\leq (u-s)(f(u)-f(t)) \\
(s-t)f(u) &\leq (u-t)f(s)-(u-s)f(t) \\
f(u) &\leq \frac{u-t}{s-t}f(s)-\frac{u-s}{s-t}f(t) \\
f\left(\frac{u-t}{s-t}s+\left(1-\frac{u-t}{s-t}\right)t\right) &\leq \frac{u-t}{s-t}f(s)+\left(1-\frac{u-t}{s-t}\right)f(t)
\end{aligned}
$$

From the definition of convexivity, (3) holds since $(u-t)/(s-t) \in (0,1)$.