MATH312: Homework 9 (due Dec. 6)

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1 Problem #1

As B is Lebesgue measurable, $m^*(E) = m^*(E \cap B) + m^*(E \cap B^C)$ holds. $E \cap B \subset B$ implies $m^*(E \cap B) \leq m^*(B) = 0$ by monotonicity, so $m^*(E) = m^*(E \cap B^C)$. As $A^C \supset B^C$, $m^*(E \cap A^C) \geq m^*(E \cap B^C)$, so $m^*(E) = m^*(E \cap B^C) \leq m^*(E \cap A^C) \leq m^*(E)$ and $m^*(E) = m^*(E \cap A^C)$. Since $A \subset B$, $m^*(A) \leq m^*(B) = 0$ and $m^*(E \cap A) \leq m^*(A) = 0$. In conclusion,

$$m^*(E) = m^*(E \cap A^C) = m^*(E \cap A) + m^*(E \cap A^C)$$

and A is measurable. Since $m^*(A) = 0$, $m(A) = 0$.

2 Problem #2

Like Cantor set, A can be constructed by removing $(4/10, 5/10), (4/100, 5/100), (1/10 + 4/100, 1/10+5/100), \dots, (3/10+4/100, 3/10+5/100), (5/10+4/100, 5/10+5/100), \dots, (9/10+4/100, 9/10 + 5/100), \dots$ in series. As A is a set constructed by removing disjoint open intervals from $[0, 1]$, A is a countable intersection of closed sets, and as closed sets are Lebesgue measurable, A is also Lebesgue measurable. We can write

$$\begin{aligned} m(A) &= m([0,1]) - m((4/10, 5/10)) - \sum_{n=0, n\neq 4}^{9} m((n/10 + 4/100, n/10 + 5/100)) - \dots \\ &= 1 - \frac{1}{10} - \frac{9 \times 1}{100} - \dots = 1 - \sum_{n=0}^{\infty} \frac{1}{10} \left(\frac{9}{10}\right)^n = 1 - \frac{1}{10} \times \frac{1}{1 - \frac{9}{10}} = 0 \end{aligned}$$

3 Problem #3

3.1 Solution for (i)

Suppose that A is Lebesgue measurable. Then, A^C is also Lebesgue measurable, so there exists an open set $O \supset A^C$ such that $m^*(O \setminus A^C) \leq \epsilon$. Since $O \setminus A^C = O \cap (A^C)^C = O \cap A = A \setminus O^C$, we can take $C = O^C$ and $m^*(A \setminus C) \leq \epsilon$ holds.

Now, Suppose that there exists a closed set $C \subset \mathbb{R}$ such that $m^*(A \setminus C) \leq \epsilon$, for all $\epsilon > 0$. As $A \setminus C = A \cap C^C = C^C \cap A = C^C \setminus A$, $m^*(C^C \setminus A^C) \leq \epsilon$, and C^C is open, thus A^C is Lebesgue measurable. Thus, A is also Lebesgue measurable.

3.2 Solution for (ii)

Suppose that A is Lebesgue measurable. Fix $\epsilon > 0$. There exists open set $O \supset A$ and closed set $C \subset A$ such that $m^*(O \setminus A) \leq \epsilon/2, m^*(A \setminus C) \leq \epsilon/2$. Then, we can write

$$m^*(O \setminus C) \leq m^*(O \setminus A) + m^*(A \setminus C) \leq \epsilon$$

and we get the desired result.

Now, suppose that there exists open set $O_\epsilon \supset A$ and closed set $C_\epsilon \subset A$ such that $m^*(O_\epsilon \setminus C_\epsilon) \leq \epsilon$ for all $\epsilon > 0$. For all $E \subset \mathbb{R}$, as $E \setminus C_\epsilon \subset (E \setminus O_\epsilon) \cup (O_\epsilon \setminus C_\epsilon)$, we can write

$$m^*(E \setminus C_\epsilon) \leq m^*((E \setminus O_\epsilon) \cup (O_\epsilon \setminus C_\epsilon)) \leq m^*(E \setminus O_\epsilon) + m^*(O_\epsilon \setminus C_\epsilon)$$

Then,

$$\begin{aligned} m^*(E) &\leq m^*(E \cap A) + m^*(E \cap A^C) \leq m^*(E \cap O_\epsilon) + m^*(E \cap C_\epsilon^C) \\ &\leq m^*(E \cap O_\epsilon) + m^*(E \cap O_\epsilon^C) + m^*(O \setminus C_\epsilon) \leq m^*(E \cap O_\epsilon) + m^*(E \cap O_\epsilon^C) + \epsilon \\ &\leq m^*(E) + \epsilon \end{aligned}$$

Since our choice of ϵ was arbitrary, $m^*(E) = m^*(E \cap A) + m^*(E \cap A^C)$ and A is Lebesgue measurable.

4 Problem #4

Suppose that A is Lebesgue measurable. Fix $\epsilon > 0$. For $n \geq 0$, there exists open set $O_n \supset A$ such that $m^*(O_n \setminus A) \leq \epsilon/2^n$, and closed set $C_n \subset A$ such that $m^*(A \setminus C_n) \leq \epsilon/2^n$, then we can write

$$m^*(A \setminus C_n) + m^*(C_n) = m^*(A) = m^*(O_n) - m^*(O_n \setminus A)$$

so

$$m^*(O_n) - \frac{\epsilon}{2^n} \leq m^*(A) \leq m^*(C_n) + \frac{\epsilon}{2^n}$$

Let $O := \bigcap_{n=0}^\infty O_n, C := \bigcup_{n=0}^\infty C_n$, then $O \subset O_n, C \supset C_n$ holds for all n. From this, $m^*(O) \leq m^*(A) \leq m^*(C)$ holds, and $C \subset A \subset O$ is true, so $m^*(O) = m^*(A) = m^*(C)$. Let $N_1 := O \setminus A, N_2 := A \setminus C$. Then, $A = O \setminus N_1 = C \cup N_2$. As C_n are Lebesgue measurable for all $n \geq 0$, its countable union, C is also Lebesgue measurable. Likewise, O_n^C are Lebesgue measurable for all $n \geq 0$, its countable union, O^C, is also Lebesgue measurable. Then, its complement, O is also Lebesgue measurable. Then N_1 and N_2 are Lebesgue measurable,

$$\begin{aligned} m^*(O) &= m^*(O \cap A) + m^*(O \setminus A) = m^*(O \cap A) + m^*(N_1) = m^*(A) + m^*(N_1) \\ m^*(A) &= m^*(A \cap C) + m^*(A \setminus C) = m^*(A \cap C) + m^*(N_2) = m^*(C) + m^*(N_2) \end{aligned}$$

and $m^*(N_1) = m^*(N_2) = 0$. As O and C are G_δ and F_σ, respectively, we can conclude that (i) implies (ii) and (iii).

Now, suppose that $A = V \setminus N_1$, for G_δ set V and measure-zero set N_1. For all $E \subset \mathbb{R}$, we can write

$$\begin{aligned} m^*(E \cap A) &= m^*(E \cap (V \setminus N_1)) \leq m^*(E \cap V \cap N_1^C) \leq m^*(E \cap V) \\ m^*(E \cap A^C) &= m^*(E \cap (V \setminus N_1)^C) \leq m^*(E \cap (V^C \cup N_1)) \\ &\leq m^*(E \cap V^C) + m^*(E \cap N_1) \leq m^*(E \cap V^C) + m^*(N_1) = m^*(E \cap V^C) \end{aligned}$$

Then, as V^C is a countable union of closed sets, V^C and V are Lebesgue measurable, and we can write

$$m^*(E) \leq m^*(E \cap A) + m^*(E \cap A^C) \leq m^*(E \cap V) + m^*(E \cap V^C) = m^*(E)$$

Thus, A is Lebesgue measurable, and (ii) implies (i).

Finally, suppose that $A = H \cup N_2$, for F_σ set H and measure-zero set N_2. For all $E \subset \mathbb{R}$, we can write

$$\begin{aligned} m^*(E \cap A) &= m^*(E \cap (H \cup N_2)) \leq m^*(E \cap H) + m^*(E \cap N_2) \\ &\leq m^*(E \cap H) + m^*(N_2) = m^*(E \cap H) \\ m^*(E \cap A^C) &= m^*(E \cap (H \cup N_2)^C) = m^*(E \cap H^C \cap N_2^C) \leq m^*(E \cap H^C) \end{aligned}$$

Then, as H is a countable union of closed sets, H is Lebesgue measurable, and we can write

$$m^*(E) \leq m^*(E \cap A) + m^*(E \cap A^C) \leq m^*(E \cap H) + m^*(E \cap H^C) = m^*(E)$$

Thus, A is Lebesgue measurable, and (iii) implies (i) and we know that (i), (ii), (iii) are equivalent.

5 Problem #5

Suppose that f is measurable. Then, as (a, ∞) is open, $f^{-1}((a, \infty))$ is measurable by the definition of measurable functions.

Now, suppose that $f^{-1}((a, \infty))$ is measurable for all $a \in \mathbb{R}$. Take $a, b \in \mathbb{R}$, with $a < b$. Pick r such that $0 < r < b-a$, and let $A_n := (a, b-r/n]$. Then, $A_n = (a, \infty) \setminus (b-r/n, \infty)$, and $f^{-1}(A_n) = f^{-1}((a, \infty)) \setminus f^{-1}((b-r/n, \infty))$. Then, as r/n converges to zero as $n \to \infty$, $(a, b) = \bigcup_{j=1}^{\infty} A_j$, so

$$f^{-1}((a, b)) = \bigcup_{j=1}^{\infty} f^{-1}(A_j) = \bigcup_{j=1}^{\infty} [f^{-1}((a, \infty)) \setminus f^{-1}((b - r/j, \infty))]$$

As $f^{-1}(A_j)$ is measurable and $f^{-1}((a, b))$ can be written as countable union of measurable sets, it is also measurable. Since the choice of a and b was arbitrary, f is measurable.

6 Problem #6

Let $A = \{x \mid f(x) \neq g(x)\}$, and $h : \mathbb{R} \to \mathbb{R}$ as $h(x) = f(x) - g(x)$. Define A_α as follows:

$$A_\alpha := \begin{cases} \{x \mid h(x) > \alpha\} & (\alpha \geq 0) \\ \{x \mid h(x) \leq \alpha\} & (\alpha < 0) \end{cases}$$

Then, for all $\alpha \in \mathbb{R}$, $A_\alpha \subset A$ holds. For all $E \subset \mathbb{R}$, we can write

$$m^*(E \cap A_\alpha) \leq m^*(A_\alpha) \leq m^*(A) = 0, \quad m^*(E \cap A_\alpha^C) \leq m^*(E)$$

so $m^*(E \cap A_\alpha) + m^*(E \cap A_\alpha^C) = m^*(E \cap A_\alpha^C) \leq m^*(E)$. As $m^*(E) \leq m^*(E \cap A_\alpha) + m^*(E \cap A_\alpha^C)$ also holds, $m^*(E) = m^*(E \cap A_\alpha) + m^*(E \cap A_\alpha^C)$ and A_α is also Lebesgue measurable, and the measure is zero. Then, $h^{-1}((a, b))$ can be constructed by at most countable unions and complements of A_α sets, so $h^{-1}((a, b))$ is Lebesgue measurable, and $g(x) = f(x) - h(x)$ is also Lebesgue measurable.