MATH312: Homework 7 (due Nov. 22)

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Last compiled on: Tuesday 21st November, 2023, 13:07

1 Chapter 9 #22

1.1 Solution for (a)

We can write

$$\nabla f(x,y) = (6x^2 + 6y^2 - 6x, 12xy + 6y)$$

Solving $\nabla f(x,y) = (0,0)$, we can find solutions $(0,0), (1,0)$. From the two-variable version of Hessian matrix we discussed in class, we can write

$$[(D_{12}f)(0,0)]^2 - [(D_{11}f)(0,0)][(D_{22}f)(0,0)] = 6 > 0$$
$$[(D_{12}f)(1,0)]^2 - [(D_{11}f)(1,0)][(D_{22}f)(1,0)] = -72 < 0, \quad (D_{11}f)(1,0) = 12 > 0$$

and conclude that f has a local minimum at $(1,0)$ and a saddle point at $(0,0)$.

1.2 Solution for (b)

By the implicit function theorem, $(D_1f)(x,y)$ needs to be invertible in order to write $f(x,y) = 0$ as $f(g(y),y) = 0$, when g is defined on some open subset. Thus, for points that $f(x,y) = 0$ cannot be solved for x in terms of y in all neighborhood, $(D_1f)(x,y) = 0$ should hold. By calculation, the points satisfying $f(x,y) = 0$ and $(D_1f)(x,y) = 0$ are $(0,0)$ and $(\sqrt{3}/2, \pm\sqrt{2\sqrt{3} - 3}/2)$. By similar argument, for points that $f(x,y) = 0$ cannot be solved for y in terms of x in all neighborhood, $(D_2f)(x,y) = 0$ should hold. By calculation, the points satisfying $f(x,y) = 0$ and $(D_2f)(x,y) = 0$ are $(0,0)$ and $(3/2,0)$. S draws an α-shaped curve which is reflected with respect to y axis, having an asymptote $x = -1/2$.

2 Chapter 9 #27

2.1 Solution for (a)

For $(x,y) \neq (0,0)$, since f is a polynomial divided by nonzero polynomial, the function is continuous and differentiable. By applying the usual rules of differentiation, we can conclude that the partial derivatives are also differentiable there. Fix $\epsilon > 0$. by taking $\delta = \sqrt{\epsilon}$, $0 < |(x,y)| < \delta$ implies

$$|f(x,y) - 0| = \left|\frac{xy(x^2 - y^2)}{x^2 + y^2}\right| \leq \frac{|xy|(x^2 + y^2)}{x^2 + y^2} = |xy| < \delta^2 < \epsilon$$

and we can conclude that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ and $f(x,y)$ is continuous at $(0,0)$. We can write

$$\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)}{h} = 0, \quad \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h)}{h} = 0$$

so $(D_1 f)(0,0) = (D_2 f)(0,0) = 0$ holds. For $(x,y) \neq (0,0)$, by usual rules of differentiation, we obtain

$$(D_1 f)(x,y) = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}, \quad (D_2 f)(x,y) = \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}$$

By taking $\delta = \epsilon/3$, $0 < |(x,y)| < \delta$ implies

$$\begin{aligned}
|(D_1 f)(x,y) - 0| &= \left|\frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}\right| \leq \frac{|y|(|x^4|+|4x^2y^2|+|y^4|)}{(x^2+y^2)^2} \\
&\leq |y|\left[1+\frac{2x^2y^2}{(x^2+y^2)^2}\right] \leq |y|\left[1+\frac{2(x^2+y^2)^2}{(x^2+y^2)^2}\right] = 3|y| < 3\delta = \epsilon \\
|(D_2 f)(x,y) - 0| &= \left|\frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}\right| \leq \frac{|x|(|x^4|+|4x^2y^2|+|y^4|)}{(x^2+y^2)^2} \\
&\leq |x|\left[1+\frac{2x^2y^2}{(x^2+y^2)^2}\right] \leq |x|\left[1+\frac{2(x^2+y^2)^2}{(x^2+y^2)^2}\right] = 3|x| < 3\delta = \epsilon
\end{aligned}$$

so $\lim_{(x,y)\to(0,0)}(D_1 f)(x,y) = \lim_{(x,y)\to(0,0)}(D_2 f)(x,y) = 0$, and the partial derivatives are continuous at $(0,0)$. Thus, $f, D_1 f, D_2 f$ are all continuous in $\mathbb{R}^2$.

2.2 Solution for (b), (c)

For $(x,y) \neq (0,0)$, $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ are polynomial divided by nonzero polynomial, so they are differentiable with respect to x and y. Thus, $D_{12} f$ and $D_{21} f$ exist at every point in $\mathbb{R}^2$. Moreover, using the usual differentiation rules, we can write

$$(D_{12} f)(x,y) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}, \quad (D_{21} f)(x,y) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$

so they are again polynomial divided by nonzero polynomial, hence continuous at points except $(0,0)$. We can write

$$\begin{aligned}
(D_{12} f)(0,0) = [D_1(D_2 f)](0,0) &= \lim_{h\to 0} \frac{(D_2 f)(h,0) - (D_2 f)(0,0)}{h} \\
&= \lim_{h\to 0} \frac{(D_2 f)(h,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1 \\
(D_{21} f)(0,0) = [D_2(D_1 f)](0,0) &= \lim_{h\to 0} \frac{(D_1 f)(0,h) - (D_1 f)(0,0)}{h} \\
&= \lim_{h\to 0} \frac{(D_1 f)(0,h)}{h} = \lim_{h\to 0} \frac{-h}{h} - 1
\end{aligned}$$

Thus, $D_{12} f$ and $D_{21} f$ exist at every point in $\mathbb{R}^2$, and $(D_{12} f)(0,0) = 1, (D_{21} f)(0,0) = -1$.

3 Chapter 9 #28

First, $x, -x, -x+2\sqrt{t}, x-2\sqrt{|t|}, 0$ are all continuous on $\mathbb{R}^2$ since they are additions and compositions of continuous functions. Consider $A = \{(x,t) \mid t \geq 0, 0 \leq x \leq \sqrt{t}, (x,t) \in$

$\mathbb{R}^2\}, B = \{(x,t) \mid t \geq 0, \sqrt{t} \leq x \leq 2\sqrt{t}, (x,t) \in \mathbb{R}^2\}, C = \{(x,t) \mid t < 0, 0 \leq x \leq \sqrt{|t|}, (x,t) \in \mathbb{R}^2\}, D = \{(x,t) \mid t < 0, \sqrt{|t|} \leq x \leq 2\sqrt{|t|}\}, E = (A \cup B \cup C \cup D \cup E)^C$. As $\varphi(x,t)$ is continuous in A, B, C, D, E and the values of φ at limit points of A, B, C, D, E coincide, φ is continuous in $\mathbb{R}^2$. For all $x < 0$, $\varphi(x,t) = 0$ holds, so $(D_2\varphi)(x,0) = 0$. For $x = 0$, φ is also zero, so $(D_2\varphi)(x,0) = 0$ also holds. For $x > 0$, φ is zero for closed interval $[-x^2/4, x^2/4]$, so for $0 < |h| < t^2/4$,

$$\left| \frac{\varphi(x,h) - \varphi(x,0)}{h} - 0 \right| = 0$$

and we know that $(D_2\varphi)(x,0) = 0$. If $0 \leq t < 1/4$, then we can write

$$\begin{aligned} f(t) = \int_{-1}^{1} \varphi(x,t)dx &= \int_{0}^{2\sqrt{t}} \varphi(x,t)dx = \int_{0}^{\sqrt{t}} \varphi(x,t)dx + \int_{\sqrt{t}}^{2\sqrt{t}} \varphi(x,t)dx \\ &= \int_{0}^{\sqrt{t}} x\, dx + \int_{\sqrt{t}}^{2\sqrt{t}} (-x + 2\sqrt{t})dx = \frac{t}{2} - \frac{3t}{2} + 2t = t \end{aligned}$$

If $-1/4 < t < 0$, then

$$\begin{aligned} f(t) &= \int_{-1}^{1} \varphi(x,t)dx = \int_{0}^{2\sqrt{|t|}} \varphi(x,t)dx = \int_{0}^{\sqrt{|t|}} \varphi(x,t)dx + \int_{\sqrt{|t|}}^{2\sqrt{|t|}} \varphi(x,t)dx \\ &= \int_{0}^{\sqrt{|t|}} -\varphi(x,|t|)dx + \int_{\sqrt{|t|}}^{2\sqrt{|t|}} -\varphi(x,|t|)dx = \int_{0}^{\sqrt{|t|}} -x\, dx + \int_{\sqrt{|t|}}^{2\sqrt{|t|}} (x - 2\sqrt{t})dx \\ &= -\frac{t}{2} + \frac{3t}{2} - 2t = -t \end{aligned}$$

so $f(t) = t$ for $|t| < 1/4$. From this, we can write $f'(0) = 1$, and conclude

$$f'(0) \neq \int_{-1}^{1} (D_2\varphi)(x,0)dx = 0$$

4 Chapter 9 #29

Consider $f \in C^k(E)$ such that $k \geq 2$. Fix $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in E$. Take a neighborhood $U \subset E$ of $\mathbf{p}$, and we can define $g_{\mathbf{p},ij} : V \to \mathbb{R}$ such that $g_{\mathbf{p},ij}(x,y) = f(\mathbf{p} + x\mathbf{e}_i + y\mathbf{e}_j)$ where $1 \leq i < j \leq n$ and $V = \{(x_i, x_j) \mid (x_1, \ldots, x_n) \in U\}$. Then, we can write

$$(D_{ij}f)(\mathbf{p}) = (D_{12}g_{\mathbf{p},ij})(0,0) = (D_{21}g_{\mathbf{p},ij})(0,0) = (D_{ji}f)(\mathbf{p})$$

Since the choice of $\mathbf{p}$ was arbitrary and $C^a(E) \subset C^b(E)$ if $a \geq b$, for a k-th order derivative of f, which can be written as $D_{i_1 i_2 \ldots i_k} f$, we can exchange adjacent indicies i_{l-1} and i_l, where $2 \leq l \leq n$, using the property we have proven earlier. Then, all of the permutations of $i_1, i_2, \ldots, i_k$ can be made by repeatedly exchanging the indicies in finite steps and we know that the derivative is constant for all permutations.

5 Chapter 9 #30

5.1 Solution for (a)

Consider $\varphi \in C^1(E)$. We can write

$$\frac{d}{dt}\varphi(\mathbf{p}(t)) = \nabla\varphi(\mathbf{p}(t)) \cdot \mathbf{x} = (D_1\varphi)(\mathbf{p}(t))x_1 + \cdots + (D_n\varphi)(\mathbf{p}(t))x_n$$

From this, we can observe that derivative of $\varphi(\mathbf{p}(t))$ is a linear combinations of functions mapping E to $\mathbb{R}$ composed with $\mathbf{p}(t)$. If we are to obtain k-th derivative of $h(t) = f(\mathbf{t})$, then we can use the formula repeatedly since $f \in C^m(E)$, for $1 \leq k \leq m$. Then, $h^{(k)}(t)$ can be written as

$$h^{(k)}(t) = \sum (D_{i_1 \dots i_k} f)(\mathbf{p}(t)) x_{i_1} \dots x_{i_k}$$

and the sum is over all ordered k-tuples $(i_1, \dots, i_k)$.

5.2 Solution for (b)

Using the result we have obtained in (a),

$$h^{(k)}(0) = \sum (D_{i_1 \dots i_k} f)(\mathbf{p}(0)) x_{i_1} \dots x_{i_k} = \sum (D_{i_1 \dots i_k} f)(\mathbf{a}) x_{i_1} \dots x_{i_k}$$

then we can write

$$\begin{aligned} f(\mathbf{a}+\mathbf{x}) = h(1) &= \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!} \\ &= \sum_{k=1}^{m-1} \left[\frac{1}{k!} \sum (D_{i_1 \dots i_k} f)(\mathbf{a}) x_{i_1} \dots x_{i_k} \right] + \frac{1}{m!} \sum (D_{i_1 \dots i_m} f)(\mathbf{a} + t\mathbf{x}) x_{i_1} \dots x_{i_m} \end{aligned}$$

so the remainder term can be written as

$$r(\mathbf{x}) = \frac{1}{m!} \sum (D_{i_1 \dots i_m} f)(\mathbf{a} + t\mathbf{x}) x_{i_1} \dots x_{i_m}$$

Then,

$$\begin{aligned} \left| \frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}} - 0 \right| &= \left| \frac{1}{|\mathbf{x}|^{m-1} m!} \sum (D_{i_1 \dots i_m} f)(\mathbf{a} + t\mathbf{x}) x_{i_1} \dots x_{i_m} \right| \\ &\leq \frac{1}{|\mathbf{x}|^{m-1} m!} \sum |(D_{i_1 \dots i_m} f)(\mathbf{a} + t\mathbf{x}) x_{i_1} \dots x_{i_m}| \\ &\leq \frac{1}{|\mathbf{x}|^{m-1} m!} \sum |(D_{i_1 \dots i_m} f)(\mathbf{a} + t\mathbf{x})| |\mathbf{x}|^m \\ &\leq \frac{|\mathbf{x}|}{m!} \sum |(D_{i_1 \dots i_m} f)(\mathbf{a} + t\mathbf{x})| \end{aligned}$$

Since $f \in C^m(E)$, every k-th order partial derivatives are continuous, hence bounded. By sandwich theorem, we can conclude that $r(\mathbf{x})/|\mathbf{x}|^{m-1}$ converges to zero as $\mathbf{x} \to 0$.

5.3 Solution for (c)

As we have discussed in #29, the partial derivative is unchanged when we calculate the partial derivative in different order, as the function is in $C^m(E)$. Generally, a $(s_1 + \dots + s_n)$-th order partial derivative with indicies i occuring s_i times occur $(s_1 + \dots + s_n)!/(s_1! \dots s_n!)\cdot$

$\binom{n}{s_1+\cdots+s_n}$ times in the Taylor polynomial. Then the Taylor polynomial can be written as

$$\begin{aligned}
&\sum_{k=0}^{m-1}\left[\frac{1}{k!}\sum(D_{i_1\ldots i_k}f)(\mathbf{a})x_{i_1}\ldots x_{i_k}\right]\\
&=\sum_{k=0}^{m-1}\left[\frac{1}{k!}\sum_{(s_1,\ldots,s_n)\in S_k}\frac{(s_1+\cdots+s_n)!}{s_1!\ldots s_n!}\cdot\binom{n}{s_1+\cdots+s_n}(D_1^{s_1}\ldots D_n^{s_n}f)(\mathbf{a})x_1^{s_1}\ldots x_n^{s_n}\right]\\
&=\sum_{k=0}^{m-1}\left[\frac{1}{k!}\sum_{(s_1,\ldots,s_n)\in S_k}\frac{n!}{s_1!\ldots s_n!(n-s_1-\cdots-s_n)!}(D_1^{s_1}\ldots D_n^{s_n}f)(\mathbf{a})x_1^{s_1}\ldots x_n^{s_n}\right]\\
&=\sum_{k=0}^{m-1}\left[\sum_{(s_1,\ldots,s_n)\in S_k}\frac{(D_1^{s_1}\ldots D_n^{s_n}f)(\mathbf{a})}{s_1!\ldots s_n!}x_1^{s_1}\ldots x_n^{s_n}\right]
\end{aligned}$$

where $S_k=\{(s_1,\ldots,s_n)\mid s_1+\cdots+s_n\le k, s_1,\ldots,s_n\in(\mathbb{N}\cup\{0\})^n\}$. Then, we can just sum over ordered n-tuples of nonnegative integers $(s_1,\ldots,s_n)$ such that $s_1+\cdots+s_n\le m-1$ and write

$$\sum\frac{(D_1^{s_1}\ldots D_n^{s_n}f)(\mathbf{a})}{s_1!\ldots s_n!}x_1^{s_1}\ldots x_n^{s_n}$$