MATH312: Homework 6 (due Nov. 15)

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1 Chapter 9 #16

Using the given example, we can write

$$f'(t) = 1 + 4t\sin\left(\frac{1}{t}\right) - 2\cos\left(\frac{1}{t}\right)$$

for $t \neq 0$. Also,

$$\begin{aligned} f'(0) &= \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{1}{h}\left(h + 2h^2\sin\left(\frac{1}{h}\right)\right) \\ &= \lim_{h\to 0}\left(1 + 2h\sin\left(\frac{1}{h}\right)\right) \end{aligned}$$

As $|h\sin(1/h)| \le |h|$ for all h, $\lim_{h\to 0} h\sin(1/h) = 0$ by sandwich theorem. Thus, $f'(0) = 1$. By using triangle inequality, for $t \neq 0$,

$$|f'(t)| = \left|t + 2t^2\sin\left(\frac{1}{t}\right)\right| \le |t| + 2t^2\left|\sin\left(\frac{1}{t}\right)\right| \le |t| + 2t^2$$

Since $|t| + 2t^2$ is a continuous function, it is bounded in $[-1,1]$ so it is also bounded in $(-1,1)\setminus\{0\}$. Since $f'(0) = 1$, we know that f' is bounded in $(-1,1)$. Now, consider two sequences, $\{a_n\}$ and $\{b_n\}$ such that $a_n = 1/(2n\pi)$ and $b_n = 2/[(1+4n)\pi]$. Then, for $n \ge 1$,

$$\begin{aligned} f'(a_n) &= 1 + \frac{2}{n\pi}\sin(2n\pi) - 2\cos(2n\pi) = -1 \\ f'(b_n) &= 1 + \frac{8}{(4n+1)\pi}\sin\left(\frac{(4n+1)\pi}{2}\right) - 2\cos\left(\frac{(4n+1)\pi}{2}\right) = 1 + \frac{8}{(4n+1)\pi} \end{aligned}$$

Then, $\lim_{n\to\infty} f'(a_n) = -1$ and $\lim_{n\to\infty} f'(b_n) = 1$. However, since a_n and b_n converges to zero as $n \to \infty$, $f'(t)$ is not continuous at $t = 0$. Suppose that there exists an open set $U \subset \mathbb{R}$ such that $0 \in U$, $f : U \to V$ is one-to-one and $V = f(U)$. Then, f must be strictly increasing or decreasing. Since U is open, there exists $r > 0$ such that $(-r, r) \subset U$. Fix $x \in (0, r)$. Then, as f is one-to-one, $f(x) \neq f(0)$ holds and one of $f(x) > f(0)$ and $f(x) < f(0)$ holds. Consider the $f(x) > f(0)$ case. As $a_n \to 0$ as $n \to \infty$, there exists $k \in \mathbb{N}$ such that $0 < a_k < x$. As f' is continuous on $(0, r)$, there exists some open ball $B(a_k, s) \subset (0, r)$ with radius $s > 0$ which is centered at a_k such that for all $z \in B(a_k, s)$, $f'(z) < 0$ holds. Then, f is strictly decreasing in $B(a_k, s)$, so f is not strictly monotonic in $(-r, r)$, which is a contradiction. For $f(x) < f(0)$ case, there exists $B(b_k, s) \subset (0, r)$ such that $f'(z) > 0$ holds for all $z \in B(b_k, s)$, so f is not strictly monotonic in $(-r, r)$, which is also a contradiction. Thus, f cannot be one-to-one in U so such U cannot exist.

From this, we can conclude that the continuity of f' at one point is needed for inverse function theorem to hold.

2 Chapter 9 #17

2.1 Solution for (a)

Take $(a,b)\neq(0,0)$, let $r=\sqrt{a^2+b^2}$. Then, for $a/r, b/r$, $(a/r)^2+(b/r)^2=1$ so $(a/r,b/r)$ is a point on unit circle centered at $(0,0)$. Then, there exists y such that $(\cos y,\sin y)=(a/r,b/r)$. Also, as $r>0$, by taking $x=\log r$, $e^x=r$ and $f_1(x,y)=a, f_2(x,y)=b$ holds. Thus, (a,b) is in the range of $\mathbf{f}$. Since the choice of (a,b) was arbitrary, the range of $\mathbf{f}$ contains all $(a,b)\neq(0,0)$. If $f_1(x,y)=0$, $e^x>0$ so $\cos y=0$, then $\sin y\neq 0$ so $f_2(x,y)\neq 0$. From this, we know that there is no x,y such that $\mathbf{f}(x,y)=(0,0)$. In conclusion, the range of $\mathbf{f}$ is $\mathbb{R}^2\setminus\{(0,0)\}$.

2.2 Solution for (b)

Since the partial derivatives of $\mathbf{f}$ exist and are continuous, $\mathbf{f}$ is continuously differentiable. Then for $\mathbf{x}=(x_1,x_2)$, we can write

$$\mathbf{f}'(\mathbf{x})=\begin{pmatrix}(D_1f_1)(\mathbf{x}) & (D_2f_1)(\mathbf{x})\\ (D_1f_2)(\mathbf{x}) & (D_2f_2)(\mathbf{x})\end{pmatrix}=\begin{pmatrix}e^{x_1}\cos x_2 & -e^{x_1}\sin x_2\\ e^{x_1}\sin x_2 & e^{x_1}\cos x_2\end{pmatrix}$$

Then the Jacobian can be written as

$$J_{\mathbf{f}}(\mathbf{x})=\det\mathbf{f}'(\mathbf{x})=\det\begin{pmatrix}e^{x_1}\cos x_2 & -e^{x_1}\sin x_2\\ e^{x_1}\sin x_2 & e^{x_1}\cos x_2\end{pmatrix}=e^{2x_1}$$

Since $e^{x_1}>0$ for all $x_1\in\mathbb{R}$, the Jacobian is not zero at any point of $\mathbb{R}^2$. This implies that $\mathbf{f}'(\mathbf{x})$ is invertible everywhere. By inverse function theorem, for all $\mathbf{x}\in\mathbb{R}^2$, there exists a neighborhood of $\mathbf{x}$ in which $\mathbf{f}$ is one-to-one. However, as $\mathbf{f}(0,0)=\mathbf{f}(0,2\pi)=(1,0)$, $\mathbf{f}$ is not one-to-one on $\mathbb{R}^2$.

2.3 Solution for (c)

Write $\mathbf{g}$ as follows:

$$\mathbf{g}(\mathbf{y})=(g_1(\mathbf{y}),g_2(\mathbf{y}))=\left(\log\sqrt{y_1^2+y_2^2},\arctan\left(\frac{y_2}{y_1}\right)\right)$$

where $\mathbf{y}=(y_1,y_2)\in(0,\infty)^2$, and $\arctan x$ is the inverse of $\tan:(-\pi/2,\pi/2)\to\mathbb{R}$. Then, for all $\mathbf{x}=(x_1,x_2)\in\mathbb{R}\times(0,\pi/2)$, we can write

$$\mathbf{g}(\mathbf{f}(\mathbf{x}))=\mathbf{g}(e^{x_1}\cos x_2,e^{x_1}\sin x_2)=\left(\log\sqrt{e^{2x_1}},\arctan\left(\frac{\sin x_2}{\cos x_2}\right)\right)=(x_1,x_2)$$

so $\mathbf{g}$ is an inverse of $\mathbf{f}:\mathbb{R}\times(0,\pi/2)\to(0,\infty)^2$. Since $\mathbf{b}=\mathbf{f}(\mathbf{a})=(1/2,\sqrt{3}/2)\in(0,\infty)^2$ and $(0,\infty)^2$ is open, $\mathbf{g}$ is defined in a neighborhood of $\mathbf{b}$. Let $\mathbf{b}=(b_1,b_2)$ then

$$\mathbf{f}'(\mathbf{a})=\begin{pmatrix}e^0\cos(\pi/3) & -e^0\sin(\pi/3)\\ e^0\sin(\pi/3) & e^0\cos(\pi/3)\end{pmatrix}=\begin{pmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix}$$

$$\mathbf{g}'(\mathbf{b})=\begin{pmatrix}(D_1g_1)(\mathbf{b}) & (D_2g_1)(\mathbf{b})\\ (D_1g_2)(\mathbf{b}) & (D_2g_2)(\mathbf{b})\end{pmatrix}=\begin{pmatrix}b_1/(b_1^2+b_2^2) & b_2/(b_1^2+b_2^2)\\ -b_2/(b_1^2+b_2^2) & b_1/(b_1^2+b_2^2)\end{pmatrix}=\begin{pmatrix}1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{pmatrix}$$

Since we can write

$$\mathbf{g}'(\mathbf{b})\mathbf{f}'(\mathbf{a})=\begin{pmatrix}1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{pmatrix}\begin{pmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

so the formula is verified.

2.4 Solution for (d)

For constant value of x, $\mathbf{f}(x,y)$ gives a circle centered on $(0,0)$ with radius e^x, parametrized by y. Thus, the images of lines parallel to y axis are circles. For constant value of y, $\mathbf{f}(x,y)$ gives a ray with its initial point at $(0,0)$ removed, forming an angle of y. Thus, the image of lines parallel to x axis are rays with with its initial point removed.

3 Chapter 9 #18

Let $\mathbf{f}(x,y) = (u(x,y), v(x,y))$.

3.1 Solution for (a)

We can write

$$u + iv = (x + iy)^2$$

Then, for all $(a,b) \in \mathbb{R}^2$, the fundamental theorem of algebra implies that there exists some $z \in \mathbb{C}$ such that $z^2 = a + bi$ holds. By taking $x = \operatorname{Re} z, y = \operatorname{Im} z$, $\mathbf{f}(x,y) = (a,b)$ so (a,b) is in the range of $\mathbf{f}$.

3.2 Solution for (b)

Since the partial derivatives of $\mathbf{f}$ exist and are continuous, $\mathbf{f}$ is continuously differentiable. For $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, we can write

$$\mathbf{f}'(\mathbf{x}) = \begin{pmatrix} (D_1u)(\mathbf{x}) & (D_2u)(\mathbf{x}) \\ (D_1v)(\mathbf{x}) & (D_2v)(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{pmatrix}$$

Then the Jacobian can be calculated as

$$J_{\mathbf{f}}(\mathbf{x}) = \det \mathbf{f}'(\mathbf{x}) = \det \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{pmatrix} = 4x_1^2 + 4x_2^2$$

From this, we can know that the Jacobian is nonzero everywhere except $(0,0)$. Thus, $\mathbf{f}'(\mathbf{x})$ is invertible everywhere except $(0,0)$. Thus, by inverse function theorem for all $\mathbf{x} \in \mathbb{R}^2 \setminus \{(0,0)\}$, there exists some neighborhood of $\mathbf{x}$ in which $\mathbf{f}$ is one-to-one. However, as $\mathbf{f}(1,0) = \mathbf{f}(-1,0) = 1$ so $\mathbf{f}$ is not one-to-one in $\mathbb{R}^2$.

3.3 Solution for (c)

Here, let's use $\mathbf{a} = (2,1)$ and $\mathbf{b} = \mathbf{f}(\mathbf{a}) = (3,4)$. Write $\mathbf{g}$ as follows:

$$\begin{aligned} \mathbf{g}(u,v) &= \left((u^2+v^2)^{1/4} \cos\left(\frac{1}{2} \arctan \frac{v}{u}\right), (u^2+v^2)^{1/4} \sin\left(\frac{1}{2} \arctan \frac{v}{u}\right) \right) \\ &= (g_1(u,v), g_2(u,v)) \end{aligned}$$

where $(u, v) \in (0, \infty)^2$ and $\arctan x$ is the inverse of $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$. Then, for all $\mathbf{x} = (x_1, x_2) \in U$, where $U = \{(x, y) \mid 0 < y < x, x \in (0, \infty)\}$, we can write

$$\begin{aligned}
\mathbf{g}(\mathbf{f}(\mathbf{x})) &= \sqrt{x_1^2 + x_2^2} \left(\sqrt{\frac{1 + \left(1 + \frac{v^2}{u^2}\right)^{-1/2}}{2}}, \sqrt{\frac{1 - \left(1 + \frac{v^2}{u^2}\right)^{-1/2}}{2}} \right) \\
&= \sqrt{x_1^2 + x_2^2} \left(\sqrt{\frac{x_1^2}{x_1^2 + x_2^2}}, \sqrt{\frac{x_2^2}{x_1^2 + x_2^2}} \right) = (x_1, x_2)
\end{aligned}$$

so $\mathbf{g}$ is indeed an inverse of $\mathbf{f} : U \to V$, where $V = (0, \infty)^2$. Then, we can write

$$\begin{aligned}
\mathbf{f}'(\mathbf{a}) &= \begin{pmatrix} 4 & -2 \\ 2 & 4 \end{pmatrix} \\
\mathbf{g}'(\mathbf{b}) &= \begin{pmatrix} (D_1 g_1)(\mathbf{b}) & (D_2 g_1)(\mathbf{b}) \\ (D_1 g_2)(\mathbf{b}) & (D_2 g_2)(\mathbf{b}) \end{pmatrix} = \begin{pmatrix} 1/5 & 1/10 \\ -1/10 & 1/5 \end{pmatrix}
\end{aligned}$$

Since we can write

$$\mathbf{g}'(\mathbf{b})\mathbf{f}'(\mathbf{a}) = \begin{pmatrix} 4 & -2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1/5 & 1/10 \\ -1/10 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the formula is verified.

3.4 Solution for (d)

For constant value of x, $\mathbf{f}(x, y)$ gives a parabola which is symmetric with respect to u axis, which stretches towards $-u$ direction. For constant value of y, $\mathbf{f}(x, y)$ gives a parabola which is also symmetric with respect to u axis, which stretches towards $+u$ direction. Thus, the images of lines parallel to axes are parabola.

4 Chapter 9 #19

Let $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^3$ as follows:

$$\begin{aligned}
\mathbf{f}(x, y, u, z) &= (f_1(x, y, u, z), f_2(x, y, u, z), f_3(x, y, u, z)) \\
f_1(x, y, u, z) &= 3x + y - z + u^2 \\
f_2(x, y, u, z) &= x - y + 2z + u \\
f_3(x, y, u, z) &= 2x + 2y - 3z + 2u
\end{aligned}$$

Since $\mathbf{f}(0, 0, 0, 0) = (0, 0, 0)$ and,

$$\det \begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \neq 0$$

so by implicit function theorem, there exists some open sets $U \subset \mathbb{R}^4$ and $W \subset \mathbb{R}$ with $(0, 0, 0, 0) \in U$ and $0 \in W$, such that for every $z \in W$ there exists $\mathbf{F}(z)$ such that $\mathbf{F}(0) = (0, 0, 0)$ and $\mathbf{f}(\mathbf{F}(z), z) = 0$. Then, we can take $x = F_1(z), y = F_2(z), u = F_3(z)$ where

$\mathbf{F}(z) = (F_1(z), F_2(z), F_3(z))$ and get the desired solution. By similar logic, we can define $\mathbf{g}$ and $\mathbf{h}$ with same domain and codomain with $\mathbf{f}$ as follows:

$$\begin{aligned}
\mathbf{g}(x,z,u,y) &= (g_1(x,z,u,y), g_2(x,z,u,y), g_3(x,z,u,y)) \\
g_1(x,z,u,y) &= f_1(x,y,u,z), \quad g_2(x,z,u,y) = f_2(x,y,u,z), \quad g_3(x,z,u,y) = f_3(x,y,u,z) \\
\mathbf{h}(y,z,u,x) &= (h_1(y,z,u,x), h_2(y,z,u,x), h_3(y,z,u,x)) \\
h_1(y,z,u,x) &= f_1(x,y,u,z), \quad h_2(y,z,u,x) = f_2(x,y,u,z), \quad h_3(y,z,u,x) = f_3(x,y,u,z)
\end{aligned}$$

Since $\mathbf{g}(0,0,0,0) = \mathbf{h}(0,0,0,0) = (0,0,0)$ and,

$$\det \begin{pmatrix} 3 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix} \neq 0, \quad \det \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix} \neq 0$$

so similarly, there exists $W' \subset \mathbb{R}$ and $W'' \subset \mathbb{R}$ such that $0 \in W' \cap W''$, and for every $z \in W'$ there exists $\mathbf{G}(z)$ such that $\mathbf{G}(0) = (0,0,0)$ and $\mathbf{g}(\mathbf{G}(z), z) = 0$. Likewise, for every $z \in W''$ there exists $\mathbf{H}(z)$ such that $\mathbf{H}(0) = (0,0,0)$ and $\mathbf{h}(\mathbf{H}(z), z) = 0$. Then $\mathbf{G}$ and $\mathbf{H}$ are the solutions we are looking for.

On the other hand, we can write

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -u^2 \\ -u \\ -2u \end{pmatrix}$$

Since the determinant of the 3-by-3 matrix on the left hand side is zero, x, y, z cannot be solved in terms of u.

5 Chapter 9 #20

Take (a,b) such that $f(a,b) = 0$ and $(D_1 f)(a,b)$ is invertible, there exists $U \subset \mathbb{R}^2$ and $W \subset \mathbb{R}$ such that for all $y \in W$ there exists a unique x such that $(x,y) \in U$ and $f(x,y) = 0$. Such x can be defined to be $g(y)$ and $g : W \to \mathbb{R}$ is continuously differentiable. Then, $g(b) = a$ and $f(g(b), b) = 0$ holds.

Graphically, we can argue that if $f(a,b) = 0$ and $(D_1 f)(a,b)$ is invertible, that is, nonzero, then the curve from f does not have a vertical tanget at (a,b) and there exists a function $y = g(x)$ whose graph near $x = a$ is same as the graph of f.

6 Chapter 9 #23

We know that $f(0,1,-1) = 0$ by simple calculation, and

$$(D_1 f)(x, y_1, y_2) = 2xy_1 + e^x$$

so $(D_1 f)(0,1,-1) = 1 \neq 0$. Then, by the implicit function theorem, there exists open sets $U \subset \mathbb{R}^3$ and $W \subset \mathbb{R}^2$ with $(0,1,-1) \in U, (1,-1) \in W$, such that for every $(y_1,y_2) \in W$ there exists a unique x such that $(x, y_1, y_2) \in U$ and $f(x, y_1, y_2) = 0$. Let $g(y_1, y_2)$ be such x. Then g is a continuously differentiable mapping of W to $\mathbb{R}$, and $g(1,-1) = 0$. Then, by definition, $f(g(y_1,y_2), y_1, y_2) = 0$ holds for $(y_1, y_2) \in W$ and we get the desired result.