MATH312: Homework 8 (due Nov. 29)

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1 Problem #1

Suppose that $\mathcal{A}$ is a σ-algebra. Then, by definition, $\mathcal{A}$ is closed under countable union, and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Now, suppose that $\mathcal{A}$ is closed under countable increasing union. That is, for all $A_1, A_2, \dots \in \mathcal{A}$ such that $A_1 \subset A_2 \subset \dots$, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ holds. For all $B_1, B_2, \dots \in \mathcal{A}$, let $C_k = \bigcup_{i=1}^{k} B_i$. Then, $C_1 \subset C_2 \subset \dots$ holds. As $\mathcal{A}$ is closed under countable increasing unions, $\bigcup_{i=1}^{\infty} C_i \in \mathcal{A}$. Since $\bigcup_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} A_i$, $\mathcal{A}$ is a σ-algebra.

2 Problem #2

As $A \setminus B$ is disjoint with B and $A \cap B$, we can write

$$\mu(A \cup B) = \mu(A \setminus B) + \mu(B), \quad \mu(A) = \mu(A \setminus B) + \mu(A \cap B)$$

From this, we obtain the desired result.

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

3 Problem #3

It is known that every open sets in $\mathbb{R}$ can be written as a countable union of open intervals. Consider an open set $U \subset \mathbb{R}$. U can be written as $U = \bigcup_{i=1}^{\infty} (a_i, b_i)$ where $a_i < b_i$ for all i, and since $\mathcal{S}$ is a σ-algebra and $(a_i, b_i) \in \mathcal{S}$ for all i, $U \in \mathcal{S}$ also holds. Since the choice of U was arbitrary, $\mathcal{S}$ contains all open subsets of $\mathbb{R}$. For all σ-algebra $\mathcal{A}$ containing all open sets, $\mathcal{A}$ contains all open intervals as open intervals are open in $\mathbb{R}$. Then, as $\mathcal{S}$ is the smallest σ-algebra containing all open intervals, $\mathcal{S} \subset \mathcal{A}$. Thus, $\mathcal{S}$ is the smallest σ-algebra containing all open subsets of $\mathbb{R}$. In other words, $\mathcal{S}$ is $\mathcal{B}(\mathbb{R})$.

4 Problem #4

Suppose that $m^*((a,b) \cup (c,d)) = m^*((a,b)) + m^*((c,d))$ and $(a,b) \cap (c,d) \neq \varnothing$. As $(a,b) \cup (c,d) = (\min\{a,c\}, \max\{b,d\})$ if (a,b) and c,d are not disjoint, we can write

$$\max\{b,d\} - \min\{a,c\} = (b-a) + (d-c)$$

If $(a,b) \subset (c,d)$, $(a,b) \cup (c,d) = (c,d)$ so $m^*((a,b) \cup (c,d)) = d - c$, and the equality above does not hold, and by similar argument, for $(c,d) \subset (a,b)$ the equality also does not hold. Otherwise, if $a < c < b < d$ then $(a,b) \cup (c,d) = (a,d)$ so $m^((a,b) \cup (c,d)) =$

$d-a < (d-a)+(b-c)$, and the equality does not hold. By similar argument the equality does not hold for $c < a < d < b$ case. Thus, $(a,b) \cap (c,d) \neq \varnothing$ is a contradiction, and $(a,b) \cap (c,d) = \varnothing$ should hold.

Now, suppose that $(a,b) \cap (c,d) = \varnothing$. By a property of outer measure, $m^*((a,b) \cup (c,d)) \leq m^*((a,b)) + m^*((c,d))$ holds. Consider a collection of open intervals, $\{I_n\}$ such that $\bigcup_{j=1}^{\infty} I_j \supset (a,b) \cup (c,d)$. Let $J_n := I_n \cap (-\infty, a), K_n := I_n \cap (a,b), L_n := I_n \cap (b, \infty)$. Then we can write

$$\sum_{i=1}^{\infty} l(I_i) \geq \sum_{i=1}^{\infty} (l(J_i) + l(L_i)) + \sum_{i=1}^{\infty} l(K_i) \geq m^*((a,b)) + m^*((c,d))$$

Taking infinimum of both sides, we obtain $m^*((a,b) \cup (c,d)) \geq m^*((a,b)) + m^*((c,d))$. In conclusion, $m^*((a,b) \cup (c,d)) = m^*((a,b)) + m^*((c,d))$.

5 Problem #5

If $c = 0$, then $cA \subset \{0\}$. Then, as $cA \subset \{0\} \subset (-r, r)$ for all $r > 0$, $\inf\{l((-r,r)) \mid r \in (0, \infty)\} = 0$ holds, so $m^*(cA) = 0 = |c| m^*(A)$. Now, suppose that $c \neq 0$. Consider a collection of open intervals, $\{I_n\}$ such that $\bigcup_{j=1}^{\infty} I_j \supset A$. Then, we can write

$$c \left(\bigcup_{j=1}^{\infty} I_j \right) = \bigcup_{j=1}^{\infty} cI_j \supset cA$$

and

$$m^*(cA) \leq \sum_{j=1}^{\infty} l(cI_j) = |c| \sum_{j=1}^{\infty} l(I_j)$$

Taking infinimum of both sides, we obtain $m^*(cA) \leq |c| m^*(A)$. Now, let $B := cA, b := 1/c$. Then, by the result we have proven earlier, $m^*(bB) \leq |b| m^*(B)$, so $m^*(cA) = m^*(B) \geq |b|^{-1} m^*(bB) = |c| m^*(A)$ holds. In conclusion, $m^*(cA) = |c| m^*(A)$.

6 Problem #6

Since $A \supset A \cap (-n, n)$, $m^*(A) \geq m^*(A \cap (-n,n))$ holds for all $n \geq 1$, so $m^*(A) \geq \lim_{n \to \infty} m^*(A \cap (-n,n))$. Let $B_n := A \cap ((-n, -n+1] \cup [n-1, n))$. Then, we can write

$$m^*(A \cap (-n,n)) = m^* \left(\bigcup_{i=1}^{n} B_i \right) \leq \sum_{i=1}^{n} m^*(B_i)$$

Now, consider a collection of open intervals, $\{I_n\}$ such that $\cup_{j=1}^{\infty} I_j \supset A \cap (-n,n)$. Let $J_{ij} = I_j \cap ((-i, -i+1] \cup [i-1, i))$, then we can write

$$\sum_{j=1}^{\infty} l(I_j) \geq \sum_{i=1}^{n} \sum_{j=1}^{\infty} l(J_{ij}) \geq \sum_{i=1}^{n} m^*(B_i)$$

By taking infinimum of both sides, we obtain

$$m^*(A \cap (-n,n)) \geq \sum_{i=1}^{n} m^*(B_i)$$

Thus, $m^*(A \cap (-n, n)) = \sum_{i=1}^{n} m^*(B_i)$. By taking limit of both sides,

$$\lim_{n\to\infty} m^*(A \cap (-n, n)) = \sum_{i=1}^{\infty} m^*(B_i) \geq m^* \left(\bigcup_{i=1}^{\infty} B_i \right) = m^*(A)$$

In conclusion, $m^*(A) = \lim_{n\to\infty} m^*(A \cap (-n, n))$ and we get the desired result.