MATH312: Homework 1 (due Sep. 18)

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1 Chapter 7 #2

Let $f(x) := \lim_{n\to\infty} f_n(x), g(x) := \lim_{n\to\infty} g_n$. Fix $\epsilon > 0$. As both f_n and g_n converge uniformly, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies the following, for all $x \in E$.

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \quad |g_n(x) - g(x)| < \frac{\epsilon}{2}$$

Using triangle inequality, we can write

$$\begin{aligned} |(f_n(x) + g_n(x)) - (f(x) - g(x))| &= |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon \end{aligned}$$

Since the choice of ϵ here is arbitrary, there exists N such that $n \geq N$ implies $|(f_n(x) + g_n(x)) - (f(x) - g(x))| < \epsilon$. Thus, $\{f_n + g_n\}$ converges uniformly.

As $\{f_n\}$ is bounded for all n, there exists $\{A_n\} \subset \mathbb{R}$ such that $|f_n| \leq A_n$ for all n. As $f_n(x)$ converges uniformly to $f(x)$, three exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. Then by triangle inequality, $|f(x)| - |f_n(x)| \leq |f_n(x) - f(x)| < \epsilon$ holds for all $n \geq N$ and $x \in E$, so we can write

$$|f(x)| \leq |f_N(x)| + \epsilon \leq A_N + \epsilon$$

and f is a bounded function. We can also write

$$|f_n(x)| \leq |f(x)| + \epsilon$$

for all $n \geq N$. As f is a bounded function, the following holds:

$$|f_n(x)| \leq \max\{A_1, A_2, \ldots, A_N, A_N + 2\epsilon\}$$

Thus, $\{f_n\}$ can be bounded by the same constant.

Let A, B be real numbers such that $|f_n(x)| \leq A, |g_n(x)| \leq B$ for all n and $x \in E$. This can be done using the result we proved earlier. We can write

$$\begin{aligned} f_n(x)g_n(x) - f(x)g(x) &= \frac{1}{2}(f_n(x) - f(x))(g_n(x) - g(x)) \\ &\quad + \frac{1}{2}(f_n(x) + f(x))(g_n(x) - g(x)) \end{aligned}$$

By triangle inequality,

$$\begin{aligned}
&|f_n(x)g_n(x) - f(x)g(x)| \\
&\leq \left|\frac{1}{2}(f_n(x) - f(x))(g_n(x) - g(x))\right| + \left|\frac{1}{2}(f_n(x) + f(x))(g_n(x) - g(x))\right| \\
&\leq \frac{1}{2}|g_n(x) - g(x)|(|f_n(x)| + |f(x)|) + \frac{1}{2}|f_n(x) - f(x)|(|g_n(x)| + |g(x)|) \\
&\leq A|f_n(x) - f(x)| + B|g_n(x) - g(x)|
\end{aligned}$$

As $\{f_n\}$ and $\{g_n\}$ converge uniformly, there exists N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon/(2A)$ and $|g_n(x) - g(x)| < \epsilon/(2B)$. Thus, $\{f_n g_n\}$ converges uniformly.

2 Chapter 7 #3

Consider $f_n(x) = g_n(x) = x + (1/n)$ on $\mathbb{R}$. Fix $\epsilon > 0$. Let $N = \lceil 1/\epsilon \rceil$. For all $n \geq N$, we can write

$$|f_n(x) - x| = \left|\frac{1}{n}\right| = \left|\left\lceil\frac{1}{\epsilon}\right\rceil^{-1}\right| \leq \epsilon$$

and $\{f_n\}$ and $\{g_n\}$ converges uniformly to x.

As $1/n$ and $1/n^2$ converges to zero as n tends to infinity, $f_n(x)g_n(x) = x^2 + 2x/n + 1/n^2$ converges pointwisely to x^2. Consider a sequence $x_n = n$. We can write

$$|f_n(x_n)g_n(x_n) - x_n^2| = \left|\frac{2x_n}{n} + \frac{1}{n^2}\right| = \left|2 + \frac{1}{n^2}\right| \geq 2$$

Thus, $\{f_n g_n\}$ does not converge uniformly.

3 Chapter 7 #5

For $x \in \mathbb{R}$, let N as follows:

$$N = \begin{cases} 1 & (x \leq 0) \\ \lceil \frac{1}{x} \rceil & (x > 0) \end{cases}$$

Fix ϵ. For $n \geq N$, if $x \leq 0$ then $|f_n(x)| = 0 < \epsilon$, and if $x > 0$,

$$x = \left(\frac{1}{x}\right)^{-1} \geq \left\lceil\frac{1}{x}\right\rceil^{-1} = \frac{1}{N} \geq \frac{1}{n}$$

so $|f_n(x)| = 0 < \epsilon$ and $\{f_n\}$ converges to zero pointwisely, which is a constant function, hence continuous.

However, for $x_n = 1/(n + 1/2)$, $f_n(x_n) = 1 \geq 1$ for all n. Thus, $\{f_n\}$ does not converge uniformly.

We can write

$$\sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \sin^2\left(\frac{\pi}{x}\right) \mathbf{I}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}(x)$$

where $\mathbf{I}_A(x)$ is an inidcator function of set A. As $(1/(n+1), 1/n)$, $(1/(m+1), 1/m)$ are disjoint for $n \neq m$ and $f_n(1/k) = 0$ for all $n \in \mathbb{N}$, $k \in \mathbb{N}$, for all $x \in \mathbb{R}$, at most one of $f_1(x), f_2(x), \ldots$ is nonzero. Thus, for all x, only one of the terms of $\sum f_n(x)$ is nonzero, so $\sum f_n(x)$ converges pointwisely for all x, and since $f_n(x) \geq 0$ for all x, $\sum f_n(x)$ converges absolutely.

4 Chapter 7 #7

Using AM-GM inequality, for $x \neq 0$ we can write

$$|f_n(x)| = \left|\frac{x}{1+nx^2}\right| \leq \frac{|x|}{2|x|\sqrt{n}} = \frac{1}{2\sqrt{n}}$$

Since $f_n(0) = 0$, $f_n(x) \leq 1/(2\sqrt{n})$ holds for all n. For all $\epsilon > 0$, by taking $N \in \mathbb{N}$ with $N > 1/(4\epsilon^2)$, for all $n \geq N$ the following holds for all $x \in \mathbb{R}$:

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} < \epsilon$$

Thus $\{f_n\}$ converges uniformly to $f(x) = 0$.

Taking the derivative of $f_n(x)$,

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$$

For $x \neq 0$,

$$0 \leq f_n'(x) \leq \frac{\frac{1}{n} - x^2}{\frac{1}{n} + 2x^2 + nx^4} \leq \frac{\frac{1}{n}}{nx^4} < \frac{1}{n^2x^4}$$

By sandwich theorem, $\lim_{n\to\infty} f_n'(x) = 0$ as $1/(n^2x^4)$ converges to zero as n tends to infinity. Since $f'(x) = 0$, it is clear that $\lim_{n\to\infty} f'(x) = 0$ for $x \neq 0$. For $x = 0$, $f_n'(x) = 1$ so $f'(x) \neq \lim_{n\to\infty} f_n'(x)$.

5 Chapter 7 #9

By triangle inequality,

$$|f_n(x_n) - f(x)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$$

Fix $\epsilon > 0$. Since $\{f_n\}$ converges uniformly, there exists $N_1 \in N$ such that $n \geq N_1$ implies $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in E$. Thus, $|f_n(x_n) - f(x_n)| < \epsilon/2$. By theorem 7.12 in the book, f is continuous so by definition, for all sequence $\{x_n\} \subset E$ that converges to x, $f(x_n)$ converges to $f(x)$ as n tends to infinity. Thus, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|f(x_n) - f(x)| < \epsilon/2$. Then, $n \geq \max\{N_1, N_2\}$ implies $|f_n(x_n) - f(x)| < \epsilon$. Since the choice of ϵ here is arbitrary, we can conclude that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

Consider a sequence of continuous functions, $f_n : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f_n(x) = \begin{cases} \sin^2 \pi x & (x \in [n, n+1]) \\ 0 & (x \notin [n, n+1]) \end{cases}$$

Fix $\epsilon > 0$. Let $\{x_n\} \subset \mathbb{R}$ be a sequence that converges to $x \in \mathbb{R}$. For $x \leq 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| \leq |x|/2$ as $x_n \to x$. Then, $x_n \leq -|x|/2$ so $|f_n(x_n)| = 0 < \epsilon$. For $x > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \lceil x \rceil - x + 1/2$ as $x_n \to x$. Then, $n \geq \max\{N, \lceil x \rceil + 1/2\}$ implies $|f_n(x_n)| = 0 < \epsilon$ as $x_n < \lceil x \rceil + 1/2$ so $n \geq N \geq \lceil x \rceil + 1/2 > x_n$. Thus, $\lim_{n\to\infty} f_n(x_n) = 0$ for all $\{x_n\} \subset \mathbb{R}$ such that $x_n \to x \in \mathbb{R}$. However, $\{f_n\}$ does not converge to $f(x) = 0$ uniformly. Consider $x_n = n + 1/2$. Then, $f_n(x_n) = \sin^2(\pi(n + 1/2)) = 1$ so $|f_n(x_n)| \geq 1$. In conclusion, the converse is not true.

6 Chapter 7 #12

Let $h : (0, 1] \to \mathbb{R}$ and $h_n : (0, 1] \to \mathbb{R}$ be defined as follows:

$$h(t) = \int_t^1 f(x)dx, \quad h_n(t) = \int_t^1 f_n(x)dx$$

Fix $\epsilon > 0$. We can write

$$\begin{aligned}
|h_n(t) - h(t)| &= \left|\int_t^1 (f_n(x) - f(x))dx\right| \le \int_t^1 |f_n(x) - f(x)|dx \\
&\le (1-t) \sup_{t \le x \le 1} |f_n(x) - f(x)| \le (1-t) \sup_{t \in (0,\infty)} |f_n(x) - f(x)| \\
&\le \sup_{t \in (0,\infty)} |f_n(x) - f(x)|
\end{aligned}$$

By theorem 7.9 there exists $N \in \mathbb{N}$ such that $\sup_{t\in(0,\infty)} |f_n(x) - f(x)| < \epsilon$. Thus, $\{h_n\}$ converges uniformly to h. Using the theorem 7.11,

$$\lim_{t\to 0} \lim_{n\to\infty} h_n(t) = \lim_{n\to\infty} \lim_{t\to 0} h_n(t)$$

and this can be written as

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx \tag{1}$$

By sandwich theorem, for all $x \in (0, \infty)$, $\lim_{n\to\infty} |f_n(x)| = |f(x)| \le g(x)$. By the integral test of series, $\int_1^\infty f_n(x)dx$ converges if and only if $\sum_{k=1}^\infty f_n(k)$ converges. Since we know that $\int_0^\infty g(x)dx$ converges, $\int_1^\infty g(x)dx$ also converges as $g(x) \ge 0$, and $\sum_{n=1}^\infty g(n)$ also converges as a result of the integral test. By comparison test, we know that $\sum_{k=1}^\infty |f_n(k)|$ converges for all n, so $\sum_{k=1}^\infty f_n(k)$ converges absolutely. Thus, $\int_1^\infty f_n(x)dx$ converges for all n. The same logic can be applied for $f(x)$, and $\int_1^\infty f(x)dx$ also converges. Let $u \in (1, \infty)$. By triangle inequality,

$$\begin{aligned}
&\left|\int_1^\infty f_n(x)dx - \int_1^\infty f(x)dx\right| \\
&= \left|\int_1^\infty f_n(x)dx - \int_1^u f_n(x)dx + \int_1^u f_n(x)dx - \int_1^u f(x)dx + \int_1^u f(x)dx - \int_1^\infty f(x)dx\right| \\
&\le \left|\int_1^\infty f_n(x)dx - \int_1^u f_n(x)dx\right| + \left|\int_1^u f_n(x)dx - \int_1^u f(x)dx\right| + \left|\int_1^u f(x)dx - \int_1^\infty f(x)dx\right|
\end{aligned}$$

Fix $\epsilon > 0$. Since $|f_n(x)| \le g(x)$, we can write

$$\left|\int_1^\infty f_n(x)dx - \int_1^u f_n(x)dx\right| = \left|\int_u^\infty f_n(x)dx\right| \le \int_u^\infty |f_n(x)|dx \le \int_u^\infty g(x)dx$$

There exists some constant $M > 1$ such that $u \ge M$ implies $\int_u^\infty g(x)dx < \epsilon/3$ by the definition of improper integral. Also, we can write

$$\left|\int_1^\infty f(x)dx - \int_1^u f(x)dx\right| = \left|\int_u^\infty f(x)dx\right| \le \int_u^\infty |f(x)|dx \le \int_u^\infty g(x)dx$$

Then, $u \ge M$ implies $\left|\int_1^\infty f_n(x)dx - \int_1^u f_n(x)dx\right| < \epsilon/3$ and $\left|\int_1^\infty f_n(x)dx - \int_1^u f_n(x)dx\right| < \epsilon/3$. Fix u to some real number greater or equal to M. Then, by theorem 7.16, $\lim_{n\to\infty} \int_1^u f_n(x)dx =$

$\int_1^u f(x)dx$, so there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\int_1^u f_n(x)dx - \int_1^u f(x)dx\right| < \epsilon/3$. Thus, we can write

$$\left|\int_1^\infty f_n(x)dx - \int_1^\infty f(x)dx\right| < \epsilon$$

for $n \geq N$, so

$$\int_1^\infty f(x)dx = \lim_{n\to\infty} \int_1^\infty f_n(x)dx \tag{2}$$

Using (1) and (2), we get the desired result:

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$$