# MATH312: Homework 3 (due Oct. 4)

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## 1 Chapter 7 #20

By the theorem 7.26 there exists a sequence of polynomial $\{P_n\}$ such that $P_n$ converges to $f$ uniformly on $[0,1]$. As $P_n$ is polynomial on $[0,1]$, which is compact, so $P_n$ is bounded for all $n$. As $\{P_n\}$ converges uniformly, it is uniformly bounded. Thus, $\{fP_n\}$ converges uniformly to $f^2$ on $[0,1]$. Then, by the theorem 7.16, we can write

$$\int_0^1 f^2(x)dx = \lim_{n\to\infty}\int_0^1 f(x)P_n(x)dx$$

We can write $P_n(x)$ as $a_{nk}x^k + \cdots + a_{n1}x + a_{n0}$ for all $n$ where $a_{nk}, a_{n,k-1}, \ldots, a_{n1}, a_{n0}$ are the coefficients. Then, we can write

$$\begin{aligned}
\int_0^1 f(x)P_n(x)dx &= \int_0^1 f(x)(a_{nk}x^k + a_{n,k-1}x^{k-1} + \cdots + a_{n1}x + a_{n0})dx \\
&= a_{nk}\int_0^1 f(x)x^k dx + a_{n,k-1}\int_0^1 f(x)x^{k-1}dx + \cdots + a_{n0}\int_0^1 f(x)dx \\
&= 0
\end{aligned}$$

for all $n$. Thus, we know that $\int_0^1 f^2(x)dx = 0$. Then, as $f$ is continuous, $f^2$ is alsp continuous and $f^2 \geq 0$, so $f(x) = 0$ for all $x \in [0,1]$.

## 2 Chapter 7 #21

Consider $f \in \mathscr{A}$ such that $f(e^{i\theta}) = e^{i\theta}$. Then, $f(x_1) \neq f(x_2)$ for $x_1 \in K$ and $x_2 \in K$ such that $x_1 \neq x_2$, so $\mathscr{A}$ separates points on $K$. Now consider $g \in \mathscr{A}$ such that $g(e^{i\theta}) = 1$. Then, $g(x) \neq 0$ for all $x \in K$, so $\mathscr{A}$ vanishes at no point of $K$.

Now, suppose that there exists $h \in \bar{\mathscr{A}}$ such that $h(e^{i\theta}) = e^{-i\theta}$ where $\bar{\mathscr{A}}$ is a uniform closure of $\mathscr{A}$. Fix $\epsilon > 0$. Let $p = e^{i\phi} \in K$. Then we can write

$$\begin{aligned}
|h(e^{i\theta}) - h(p)| &= |e^{-i\theta} - e^{-i\phi}| = |(\cos(-\theta) - \cos(-\phi)) + i(\sin(-\theta) - \sin(-\phi))| \\
&= |(\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)| = \sqrt{(\cos\theta - \cos\phi)^2 + (-\sin\theta + \sin\phi)^2} \\
&= |(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)| = |e^{i\theta} - p|
\end{aligned}$$

Thus, if we take $\delta > 0$ with $\delta < \epsilon$, $|e^{i\theta} - p| < \delta$ implies $|h(e^{i\theta}) - h(p)| < \epsilon$ for all $e^{i\theta} \in K$. Since the choice of $\epsilon$ and $p$ was arbitrary, $h$ is continuous on $K$.

On the other hand, for all $f \in \mathscr{A}$, we can write

$$\begin{aligned}
\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta &= \int_0^{2\pi} e^{i\theta}\sum_{n=0}^{N} c_n e^{in\theta}\, d\theta = \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{i(n+1)\theta}d\theta \\
&= \sum_{n=0}^{N} c_n \int_0^{2\pi} (\cos((n+1)\theta) + i\sin((n+1)\theta))\, d\theta \\
&= \sum_{n=0}^{N} c_n \left[\frac{1}{n+1}\sin((n+1)\theta) + \frac{-i}{n+1}\cos((n+1)\theta)\right]_0^{2\pi} = 0
\end{aligned}$$

As $h \in \bar{\mathscr{A}}$, there exists $\{h_n\} \subset \mathscr{A}$ such that $\{h_n\}$ converges to $h$ uniformly. Then, as $h_n \in \mathscr{A}$ for all $n$,

$$\int_0^{2\pi} h_n(e^{i\theta})e^{i\theta}d\theta = 0$$

However, for $h$, we can write

$$\int_0^{2\pi} h(e^{i\theta})e^{i\theta}d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi$$

Thus, we can write

$$\lim_{n\to\infty}\int_0^{2\pi} h_n(e^{i\theta})e^{i\theta}d\theta = 0 \neq 2\pi$$

so from the contrapositive of theorem 7.16, the sequence of function of $\theta$ on $[0, 2\pi]$, $\{h_n(e^{i\theta})\}$ does not converge uniformly to $h(e^{i\theta})$. Then, there exists $\epsilon_0 \geq 0$, sequence $\{n_k\}$ and $\{x_k\} \subset [0, 2\pi]$ such that $|h_{n_k}(e^{ix_k}) - h(e^{ix_k})| \geq \epsilon_0$ holds for sufficiently large $k$. Then, we can write $y_k = e^{ix_k}$ and $|h_{n_k}(y_k) - h(y_k)| \geq \epsilon_0$. In conclusion, $\{h_n\}$ does not converge uniformly to $h$, which is a contradiction and $h \notin \bar{\mathscr{A}}$.

# 3 Chapter 7 #22

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Let $g$ be a function on $[a, b]$ which is defined as follows:

$$g(t) = \frac{x_i - t}{\Delta x_i}f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i}f(x_i)$$

where $t \in [x_{i-1}, x_i]$. Fix $\epsilon \in (0, 1)$. As $f \in \mathscr{R}(\alpha)$, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Also, there exists a partition $P$ such that

$$\sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \frac{\epsilon^2}{2M}$$

Then we can write

$$\int_a^b |f - g|^2 d\alpha \leq \sum_{i=1}^{n}(M_i - m_i)^2\Delta\alpha_i \leq \sum_{i=1}^{n} 2M(M_i - m_i)\Delta\alpha_i < \epsilon^2 < \epsilon$$

Thus, there exists continuous $g$ such that

$$\int_a^b |f-g|^2 d\alpha < \epsilon^2 < \epsilon$$

As $g(t)$ is continuous on $[a,b]$, by theorem 7.26 there exists a sequence of polynomials $P_n$ such that $\{P_n\}$ converges uniformly to $g$ on $[a,b]$. Thus, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|g(x) - P_n(x)| < \epsilon$ for all $x \in [a,b]$. Then we can write

$$\int_a^b |g-P_n|^2 d\alpha < \int_a^b \epsilon^2 d\alpha = \epsilon^2(\alpha(b)-\alpha(a)) < \epsilon(\alpha(b)-\alpha(a))$$

Also, by the Schwarz inequality, we can write

$$\begin{aligned}
\int_a^b |f-P_n|^2 d\alpha &= \int_a^b (f-g+g-P_n)\overline{(f-g+g-P_n)} d\alpha \\
&\leq \int_a^b \left[ (|f-g|^2 + |g-P_n|^2) + (g-P_n)\overline{(f-g)} + (f-g)\overline{(g-P_n)} \right] d\alpha \\
&\leq \int_a^b |f-g|^2 d\alpha + \int_a^b |g-P_n|^2 d\alpha + 2\sqrt{\int_a^b |g-P_n|^2 d\alpha}\sqrt{\int_a^b |f-g|^2 d\alpha} \\
&\leq \left( \sqrt{\int_a^b |f-g|^2 d\alpha} + \sqrt{\int_b^a |g-P_n|^2 d\alpha} \right)^2 \\
&= (\sqrt{\epsilon} + \sqrt{\epsilon(\alpha(b)-\alpha(a))})^2 = (1+\sqrt{\alpha(b)-\alpha(a)})^2 \epsilon
\end{aligned}$$

Since the choice of $\epsilon$ is arbitrary, we get the desired result.