MATH312: Homework 4 (due Oct. 15)

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1 Chapter 8 #12

1.1 Solution for (a)

Let c_n be the Fourier coefficients of f. For $n \neq 0$, we can write

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{inx}dx = \frac{1}{2\pi}\left[\frac{i}{n}e^{-inx}\right]_{-\delta}^{\delta} \\
&= \frac{i}{2n\pi}(e^{-in\delta} - e^{in\delta}) = \frac{i}{2n\pi}\cdot 2i\sin(-n\delta) = \frac{\sin(n\delta)}{n\pi}
\end{aligned}$$

For $n = 0$, $c_n = (1/2\pi)\int_{-\pi}^{\pi} f(x)dx = \delta/\pi$.

1.2 Solution for (b)

Let $S_N[f]$ as follows:

$$S_N[f](x) = \sum_{n=-N}^{N} c_n e^{inx}$$

For $|t| < \delta/2$, $|f(t) - f(0)| = 0$, so by theorem 8.14, we can write

$$\lim_{N\to\infty} S_N[f](0) = \lim_{N\to\infty}\sum_{n=-N}^{N} c_n = \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n\pi} = 1$$

Then, we can write

$$\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n} = \frac{\pi-\delta}{2}$$

1.3 Solution for (c)

As f is Riemann-integrable, we can apply theorem 8.16. Then,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2$$

We can also write

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} 1\, dx = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty}\left(\frac{\sin(n\delta)}{n\pi}\right)^2 = \frac{\delta}{\pi}$$

In conclusion,

$$\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi-\delta}{2}$$

1.4 Solution for (d)

Let $g(x) = (\sin^2 x)/x^2$ for $x > 0$, and h be a function defined as follows:

$$h(x) = \begin{cases} \left(\frac{\sin x}{x}\right)^2 & (x \neq 0) \\ 1 & (x = 0) \end{cases}$$

Using the L'Hôpital's rule, we know that $\lim_{x\to 0}(\sin x)/x = 1$. Thus, $h(x)$ is continuous at $x = 0$. Then for all $A > 0$, we can write

$$\lim_{c\to 0}\int_c^A g(x)dx = \int_0^A h(x)dx - \lim_{c\to 0}\int_0^c h(x)dx$$

Then, as $\int_0^t h(x)dx$ is continuous on $[0, A]$, we can conclude that

$$\int_0^A g(x)dx = \lim_{c\to 0}\int_c^A g(x)dx = \int_0^A h(x)dx$$

Also, for $t > A > 0$, we can write

$$\int_A^t g(x)dx \leq \int_A^t \frac{dx}{x^2} \leq \frac{1}{A} - \frac{1}{t} < \frac{1}{A}$$

As $g(x) \geq 0$ for all x, the improper integral $\int_A^\infty g(x)dx$ converges. Thus, the integral $\int_0^\infty g(x)dx$ exists.

Fix $\epsilon > 0$. Let A be a positive number. Then, h is uniformly continuous on $[0, A]$, so there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, for all $x \in [0, A]$ and $y \in [0, A]$. Now, take a partition $P = \{x_0, x_1, \ldots, x_n\} = \{0, \delta, 2\delta, \ldots, (n-1)\delta, A\}$. The lower and upper sum can be written as

$$U(P, h) = \sum_{i=1}^{n}\Delta x_i \sup_{x\in[x_{i-1},x_i]} h(x), \quad L(P, h) = \sum_{i=1}^{n}\Delta x_i \inf_{x\in[x_{i-1},x_i]} h(x)$$

Then,

$$\begin{aligned} U(P, h) - L(P, h) &= \sum_{i=1}^{n}\Delta x_i\left(\sup_{x\in[x_{i-1},x_i]} h(x) - \inf_{x\in[x_{i-1},x_i]} h(x)\right) \\ &< \epsilon\sum_{i=1}^{n}\Delta x_i = A\epsilon \end{aligned}$$

From the definition of lower and upper sums, we can also write

$$L(P,h) \le \sum_{i=1}^{n-1} h(i\delta)\delta + h(A)(A-(m-1)\delta) \le U(P,h)$$

Thus, we can write

$$\left| \int_0^A h(x)dx - \sum_{i=1}^{n-1} \frac{\sin^2(i\delta)}{i^2\delta} - h(A)(A-(m-1)\delta) \right| < A\epsilon$$

and

$$\left| \int_0^A h(x)dx - \sum_{i=1}^{n-1} h(i\delta)\delta \right| < A\epsilon + |h(A)\delta|$$

Thus, we can write

$$\int_0^A g(x)dx = \lim_{\delta\to 0} \sum_{i=1}^{n-1} \frac{\sin^2(i\delta)}{i^2\delta}$$

In conclusion,

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \lim_{\delta\to 0} \sum_{i=1}^{\infty} \frac{\sin^2(i\delta)}{i^2\delta} = \lim_{\delta\to 0} \frac{\pi-\delta}{2} = \frac{\pi}{2}$$

1.5 Solution for (e)

We obtain

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\pi/2)}{\pi n^2/2} = \sum_{n=1}^{\infty} \frac{2}{\pi(2n-1)^2} = \frac{\pi}{4}$$

which gives us

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

2 Chapter 8 #13

First, let's calculate the Fourier coefficients. For $n = 0$, as $\int_{-\pi}^{\pi} x\ dx = 0$, $c_0 = 0$. For $n \neq 0$, we can write

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi}\left[\frac{i}{n}xe^{-inx} + \frac{1}{n^2}e^{-inx}\right]_{-\pi}^{\pi} = \frac{i}{n}(-1)^n$$

By Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} c_n = 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|^2dx = \frac{1}{2\pi}\cdot\frac{2\pi^3}{3} = \frac{\pi^2}{3}$$

and we get the desired result:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

3 Chapter 8 #14

First, let's calculate the Fourier coefficients. For $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - |x|)^2 dx = \frac{1}{2\pi}\cdot\frac{2\pi^3}{3} = \frac{\pi^2}{3}$$

For $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - |x|)^2 e^{-inx}dx = \frac{1}{2\pi}\cdot\frac{2(2\pi n - ie^{-i\pi n} + ie^{i\pi n})}{n^3} \\ &= \frac{1}{2\pi}\cdot\frac{2(2\pi n - i(-1)^{-n} + i(-1)^n)}{n^3} = \frac{2}{n^2} \end{aligned}$$

For $x \in [-\pi, \pi]$ and $x + t \in [-\pi, \pi]$, we can write

$$\begin{aligned} |f(x+t) - f(x)| &= |(\pi - |x+t|)^2 - (\pi - |x|)^2| = |2\pi - |x+t| - |x|||x| - |x+t|| \\ &\leq 2\pi|t| \end{aligned}$$

By theorem 8.14, for $x \in [-\pi, \pi]$, we can write

$$\lim_{N\to\infty}\sum_{n=-N}^{N} c_n e^{inx} = f(x)$$

Then we can write

$$\begin{aligned} f(x) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(c_n e^{inx} + c_{-n}e^{-inx}) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2}{n^2}(e^{inx} + e^{-inx}) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx) \end{aligned}$$

Plugging $x = 0$, we immediately obtain

$$f(0) = \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

so

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

By Parseval's theorem, we can write

$$\begin{aligned} \sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty}\frac{4}{n^4} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - |x|)^4 dx \\ &= \frac{1}{\pi}\int_0^{\pi}(\pi - x)^4 dx = \frac{1}{\pi}\left[-\frac{1}{5}(\pi - x)^5\right]_0^{\pi} = \frac{\pi^4}{5} \end{aligned}$$

so

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

4 Chapter 8 #15

We can write

$$\begin{aligned}
K_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N} D_n(x) = \frac{1}{N+1}\frac{1}{\sin(\pi/2)}\frac{e^{ix/2}}{2i}\sum_{n=0}^{N}(\exp(inx) - \exp(-i(n+1)x)) \\
&= \frac{1}{N+1}\frac{1}{\sin(\pi/2)}\frac{e^{ix/2}}{2i}\left(\frac{(e^{ix})^{N+1}-1}{e^{ix}-1} - \frac{1}{e^{ix}}\frac{(e^{-ix})^{N+1}-1}{e^{-ix}-1}\right) \\
&= \frac{1}{N+1}\frac{1}{\sin(\pi/2)}\frac{e^{ix/2}}{2i}\frac{2\cos((N+1)x)-2}{e^{ix}-1} = \frac{1}{N+1}\frac{2-2\cos((N+1)x)}{(e^{ix/2}-e^{-ix/2})^2} \\
&= \frac{1}{N+1}\frac{1-\cos((N+1)x)}{1-\cos x}
\end{aligned}$$

If $x = 0$, then $D_n(x) = 1$ for all n so $K_N(x) = 1$. Otherwise, since $|\cos x| \leq 1$ and $\cos x \neq 1$, it is evident that $K_N(x) \geq 0$. Also, as $\int_{-\pi}^{\pi} D_n(x)dx = 2\pi$ for all n, we can write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)dx = \frac{1}{N+1}\cdot(N+1) = 1$$

We can also write

$$K_N(x) \leq \frac{1}{N+1}\frac{2}{1-\cos x}$$

and for $0 < \delta \leq |x| \leq \pi$, $\cos\delta \geq \cos x$, so

$$K_N(x) \leq \frac{1}{N+1}\frac{2}{1-\cos\delta}$$

We can write

$$\begin{aligned}
\sigma_N(f;x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\cdot\frac{1}{N+1}\sum_{n=0}^{N} D_n(t)dt \\
&= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt = \frac{1}{N+1}\sum_{n=0}^{N} s_n
\end{aligned}$$

Now let's prove Fejér's theorem. Since f is continuous on $[-\pi,\pi]$, it is uniformly continuous on the interval. Fix $\epsilon > 0$. There exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$ for all $x \in [-\pi,\pi]$ and $y \in [-\pi,\pi]$. Then using the properties we have proven earlier, we can write

$$\begin{aligned}
|\sigma_N(f;x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt - f(x)\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(f(x-t)-f(x))K_N(t)dt\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-t)-f(x)|K_N(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{-\delta}|f(x-t)-f(x)|K_N(t)dt + \frac{1}{2\pi}\int_{-\delta}^{\delta}|f(x-t)-f(x)|K_N(t)dt \\
&+ \frac{1}{2\pi}\int_{\delta}^{\pi}|f(x-t)-f(x)|K_N(t)dt
\end{aligned}$$

Let $M = \sup|f(x)|$. Since $K_N(x)$ is even function we can write

$$\begin{aligned}|\sigma_N(f;x) - f(x)| &< \frac{1}{2\pi}\left(\epsilon\int_{-\delta}^{\delta} K_N(t)dt + 4M\int_{\delta}^{\pi} K_N(t)dt\right)\\ &\leq \frac{1}{2\pi}\left(4M\cdot\frac{\pi}{N+1}\cdot\frac{2}{1-\cos\delta} + \epsilon\cdot 2\pi\right) = \frac{4M}{N+1}\cdot\frac{1}{1-\cos\delta} + \epsilon\end{aligned}$$

By taking sufficiently large N, we can make $4M/(N+1)\cdot(1-\cos\delta)^{-1}$ as small as we like, preferrably less than ϵ. Since our choice of ϵ was arbitrary, $\sigma_N(f;x)$ converges to f uniformly.

5 Chapter 8 #17

5.1 Solution for (a)

Suppose that f is a monotonically increasing function. Then, by the result from the exercise 17 of chapter 6, we can write

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\left(\frac{i}{n}e^{-in\pi}f(\pi) - \frac{i}{n}e^{in\pi}f(-\pi) - \int_{-\pi}^{\pi}\frac{i}{n}e^{-inx}df\right)$$

from this,

$$nc_n = \frac{i}{2\pi}\left(e^{-in\pi}f(\pi) - e^{in\pi}f(-\pi) - \int_{-\pi}^{\pi} e^{-inx}df\right)$$

and we can write

$$\begin{aligned}|nc_n| &\leq \frac{1}{2\pi}\left(|e^{-inx}||f(\pi) - e^{2in\pi}f(-\pi)| + \left|\int_{-\pi}^{\pi} e^{-inx}df\right|\right)\\ &\leq \frac{1}{2\pi}\left(|f(\pi) - f(-\pi)| + \left|\int_{-\pi}^{\pi} df\right|\right) = \frac{1}{\pi}|f(\pi) - f(-\pi)|\end{aligned}$$

If f is monotonically decreasing, we can write

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\left(-\frac{i}{n}e^{-in\pi}(-f(\pi)) + \frac{i}{n}e^{in\pi}(-f(-\pi)) - \int_{-\pi}^{\pi} -\frac{i}{n}e^{-inx}d(-f)\right)$$

and

$$nc_n = -\frac{i}{2\pi}\left(e^{-in\pi}f(\pi) - e^{in\pi}f(-\pi) - \int_{-\pi}^{\pi} e^{-inx}d(-f)\right)$$

so

$$\begin{aligned}|nc_n| &= \frac{1}{2\pi}\left(|e^{-in\pi}||f(\pi) - e^{2in\pi}f(-\pi)| + \left|\int_{-\pi}^{\pi} e^{-inx}d(-f)\right|\right)\\ &\leq \frac{1}{2\pi}\left(|f(\pi) - f(-\pi)| + \left|\int_{-\pi}^{\pi} d(-f)\right|\right) = \frac{1}{\pi}|f(\pi) - f(-\pi)|\end{aligned}$$

so $\{nc_n\}$ is bounded in both cases.

5.2 Solution for (b)

Since f is monotonic function, by theorem 4.29, $f(x+)$ and $f(x-)$ exists for all x. By the result of the exercise 16, for every x,

$$\lim_{N\to\infty} \sigma_N(f;x) = \frac{1}{2}(f(x+) + f(x-))$$

As $\{nc_n\}$ is bounded, $|n(s_n - s_{n-1})| = |nc_n|$ is bounded. Then, as $\sigma_N(f;x)$ converges to $(f(x+) + f(x-))/2$ for all x, by the result of the exercise 14(e) of chapter 3, $s_N(f;x)$ also converges to $(f(x+) + f(x-))/2$ for all x.

5.3 Solution for (c)

Let g be defined as follows:

$$g(x) = \begin{cases} f(\alpha) & (-\pi \le x \le \alpha) \\ f(x) & (\alpha < x < \beta) \\ f(\beta) & (\beta \le x < \pi) \end{cases}$$

Then g is bounded and monotonic on $[-\pi, \pi)$. Thus, for all x, we can write

$$\lim_{N\to\infty} s_N(g;x) = \frac{1}{2}(g(x+) + g(x-))$$

As $f(x) = g(x)$ for all $x \in (\alpha, \beta)$, by the localizztion theorem, we can write

$$\lim_{N\to\infty} s_N(f;x) = \lim_{N\to\infty} s_N(g;x) = \frac{1}{2}(g(x+) + g(x-)) = \frac{1}{2}(f(x+) + f(x-))$$

6 Chapter 8 #19

Consider $f(x) = e^{ikx}$, where k is integer. If $k = 0$, then $f(x) = 1$ so the equality holds. If $k \neq 0$, we can write

$$\frac{1}{N}\sum_{i=1}^{N} f(x+n\alpha) = \frac{1}{N}\sum_{i=1}^{N} e^{ik(x+n\alpha)} = \frac{e^{ikx}}{N}\sum_{i=1}^{N} e^{ikn\alpha} = \frac{e^{ik(x+\alpha)}}{N}\frac{(e^{ik\alpha})^N - 1}{e^{ik\alpha} - 1}$$

then,

$$\begin{aligned}\left|\frac{1}{N}\sum_{i=1}^{N} f(x+n\alpha)\right| &= \left|\frac{e^{ik(x+\alpha)}}{N}\frac{(e^{ik\alpha})^N - 1}{e^{ik\alpha} - 1}\right| = \left|\frac{1}{N}\right|\left|\frac{(e^{ik\alpha})^N - 1}{e^{ik\alpha} - 1}\right| \le \left|\frac{1}{N}\right| \frac{|(e^{ik\alpha})^N| + 1}{|e^{ik\alpha} - 1|} \\ &= \left|\frac{1}{N}\right| \frac{2}{|e^{ik\alpha} - 1|}\end{aligned}$$

As $k\alpha$ cannot be integer multiple of π, $e^{ik\alpha} \neq 1$. Thus, we can take $N \to \infty$ limit and by sandwich theorem,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} f(x+n\alpha) = 0$$

On the other hand, the integral evaluates to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}dt = \frac{1}{2\pi}\left[-\frac{i}{k}e^{ikt}\right]_{-\pi}^{\pi} = 0$$

thus the theorem holds for $f(x) = e^{ikx}$.

Fix $\epsilon > 0$. By theorem 8.15, there exists a trigonometric polynomial P such that $|P(x) - f(x)| < \epsilon$ for all x. Also, as $P(x)$ is a linear combination of e^{ikx} where k is some integer, there exists $M \in \mathbb{N}$ such that $N \geq M$ implies

$$\left|\frac{1}{N}\sum_{i=1}^{N} P(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)dt\right| < \epsilon$$

Then,

$$\begin{aligned}
\left|\frac{1}{N}\sum_{i=1}^{N} f(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt\right| &\leq \left|\frac{1}{N}\sum_{i=1}^{N}(f(x+n\alpha) - P(x+n\alpha)) - \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t)-P(t))dt\right| \\
&+ \left|\frac{1}{N}\sum_{i=1}^{N} P(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)dt\right| \\
&\leq \left|\frac{1}{N}\sum_{i=1}^{N}(f(x+n\alpha) - P(x+n\alpha))\right| \\
&+ \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t)-P(t))dt\right| \\
&+ \left|\frac{1}{N}\sum_{i=1}^{N} P(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)dt\right| \\
&\leq \frac{1}{N}\sum_{i=1}^{N}|f(x+n\alpha) - P(x+n\alpha)| \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)-P(t)|dt \\
&+ \left|\frac{1}{N}\sum_{i=1}^{N} P(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)dt\right| < 3\epsilon
\end{aligned}$$

Since the choice of ϵ was arbitrary, we can conclude that the equality holds for all continuous and 2π-periodic function f.