# MATH312: Homework 5 (due Nov. 8)

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Last compiled on: Monday $6^{\text{th}}$ November, 2023, 10:10

## 1 Chapter 9 #6

Since polynomial functions are differentiable, $D_1 f$ and $D_2 f$ exists for all $(x, y)$ that is not $(0, 0)$ as $f$ is a quotient of two polynomials, and $x^2 + y^2 \neq 0$ for those points. We can also write

$$(D_1 f)(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$(D_2 f)(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

Thus, $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ exists for all $(x, y)$.

Now, suppose that $f$ is continuous at $(0, 0)$. Then for all sequences $\{x_n\}$ that converges to $(0, 0)$, $\{f(x_n)\}$ converges to $f(0, 0) = 0$ as $n \to \infty$. Let $a_n = (1/n, 0)$ and $b_n = (1/n, 1/n)$. They converge to $(0, 0)$, and we can write

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(1/n, 0) = \lim_{n\to\infty} 0 = 0$$
$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} f(1/n, 1/n) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

Which is a contradiction. $f$ is not continuous at $(0, 0)$.

## 2 Chapter 9 #7

Fix $\mathbf{p} \in E$. For all $\mathbf{x} \in E$, we can write

$$\mathbf{x} - \mathbf{p} = \sum_{i=1}^{n} h_i \mathbf{e}_i$$

where $\mathbf{e}_i$ $(i = 1, 2, \ldots, n)$ are standard bases of $\mathbb{R}^n$. Let $\mathbf{v}_0 = 0, \mathbf{v}_k = h_1\mathbf{e}_1 + \ldots h_k\mathbf{e}_k$. Then we can write

$$f(\mathbf{x}) - f(\mathbf{p}) = \sum_{i=1}^{n} (f(\mathbf{v}_i) - f(\mathbf{v}_{i-1}))$$

By triangle inequality,

$$|f(\mathbf{x}) - f(\mathbf{p})| \leq \sum_{i=1}^{n} |f(\mathbf{v}_i) - f(\mathbf{v}_{i-1})| \leq \sum_{i=1}^{n} |h_i| \left| \frac{f(\mathbf{v}_i) - f(\mathbf{v}_{i-1})}{h_i} \right|$$

By mean value theorem, there exists $c_i$ between 0 and $h_i$ such that

$$(D_i f)(\mathbf{v}_{i-1} + c_i \mathbf{e}_i) = \frac{f(\mathbf{v}_i) - f(\mathbf{v}_{i-1})}{h_i}$$

for all $i = 1, \ldots, n$. Since the partial derivatives are bounded, there exists $M_i \in \mathbb{R}$ such that $|(D_i f)(\mathbf{x})| \leq M_i$ for all $i = 1, \ldots, n$ and $\mathbf{x} \in E$. Then we can write

$$|f(\mathbf{x}) - f(\mathbf{p})| \leq \sum_{i=1}^{n} |h_i||(D_i f)(\mathbf{v}_{i-1} + c_i \mathbf{e}_i)| \leq \sum_{i=1}^{n} |h_i| M_i \leq |\mathbf{x} - \mathbf{p}| \sum_{i=1}^{n} M_i$$

For all $\epsilon > 0$, we can take $0 < \delta < \left(\sum_{i=1}^{n} M_i\right)^{-1}$. Then, $|\mathbf{x} - \mathbf{p}| < \delta$ implies $|f(\mathbf{x}) - f(\mathbf{p})| < \epsilon$. Since the choice of $\mathbf{p}$ was arbitrary, $f$ is continuous in $E$.

## 3 Chapter 9 #8

As $f$ is differentiable, by theorem 9.17, we can write

$$f'(\mathbf{x})\mathbf{e}_i = (D_i f)(\mathbf{x})$$

for all $i = 1, \ldots, n$ and $\mathbf{x} \in E$. Here, we denote the standard basis of $\mathbb{R}^n$. Suppose that $f$ has a local maximum at $\mathbf{p} = (p_1, \ldots, p_n) \in E$. Using the definition of local maximum, there exists an open ball $B(\mathbf{p}, r) \subset E$ which is centered at $\mathbf{p}$ with radius $r > 0$ such that for all $\mathbf{x} \in B(\mathbf{p}, r)$, $f(\mathbf{x}) \leq f(\mathbf{p})$. Let $g_i(t) := f(\mathbf{p} + t\mathbf{e}_i)$ for $i = 1, \ldots, n$. Then $g_i$ has a local maximum at 0 for all $i$, since for all $t \in (-r, r)$, $\mathbf{p} + t\mathbf{e}_i \in B(\mathbf{p}, r)$ so $g_i(t) \leq g_i(0)$. By theorem 5.8, $g_i'(0) = 0$ for all $i$. Then, for all $i$, we can write

$$g_i'(0) = \lim_{t \to 0} \frac{g_i(t) - g_i(0)}{t} = \lim_{t \to 0} \frac{f(\mathbf{p} + t\mathbf{e}_i) - f(\mathbf{p})}{t} = (D_i f)(\mathbf{p}) = 0$$

In conclusion, $f'(\mathbf{p}) = 0$ as all the columns of $[f'(\mathbf{p})]$ are zero.

## 4 Chapter 9 #9

Let $E_{\mathbf{y}} = \{\mathbf{x} \mid \mathbf{f}(\mathbf{x}) = \mathbf{y}\}$. Consider a nonempty $E_{\mathbf{y}}$. For all $\mathbf{x} \in E_{\mathbf{y}}$, there exists some open ball centered at $\mathbf{x}$ of radius $r > 0$, $B(\mathbf{x}, r)$ contained in $E$. As open ball is convex, by theorem 9.19 $|\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{x})| \leq 0$ for all $\mathbf{z} \in B(\mathbf{x}, r)$. By property of norm, $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{z})$ so $\mathbf{z} \in E_{\mathbf{y}}$. Thus, since the open ball is contained in $E_{\mathbf{y}}$, $E_{\mathbf{y}}$ is open. For empty $E_{\mathbf{y}}$, empty set is open so $E_{\mathbf{y}}$ is open for all $\mathbf{y} \in \mathbb{R}^m$. Fix $\mathbf{p} \in E$. Then $E_{\mathbf{f}(\mathbf{p})}$ is open, and $E - E_{\mathbf{f}((p))} = \bigcup_{\mathbf{y} \in \mathbb{R}^m - \{\mathbf{f}(\mathbf{p})\}} E_{\mathbf{y}}$ is also open as it is a union of open sets. Since $E_{\mathbf{f}(\mathbf{p})}$ and $E - E_{\mathbf{f}(\mathbf{p})}$ are disjoint open sets in metric space $E$, they are separated. Sinec $E$ is connected, it cannot be a union of two nonempty separated sets, so one of $E_{\mathbf{f}(\mathbf{p})}$ and $E - E_{\mathbf{f}(\mathbf{p})}$ is empty. Since $E_{\mathbf{f}(\mathbf{p})}$ cannot be empty by definition, $E - E_{\mathbf{f}(\mathbf{p})}$ is empty, which means that $E = E_{\mathbf{f}(\mathbf{p})}$. In conclusion, $\mathbf{f}$ is constant.

## 5 Chapter 9 #13

Let $\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t))$ and $g(x_1, x_2, x_3) := x_1^2 + x_2^2 + x_3^2$. Then, we can write

$$\lim_{\mathbf{h} \to 0} \frac{|g(\mathbf{x} + \mathbf{h}) - g(\mathbf{x}) - (\nabla g(\mathbf{x})\mathbf{h})|}{|\mathbf{h}|} = \lim_{(h_1, h_2, h_3) \to (0,0,0)} \frac{|h_1^2 + h_2^2 + h_3^2|}{|\mathbf{h}|} = \lim_{\mathbf{h} \to 0} |\mathbf{h}| = 0$$

Thus, $g$ is differentiable. Then, as we can write $|\mathbf{f}(t)| = 1$ as $g(\mathbf{f}(t)) = 1$, we can write

$$\begin{aligned}
g'(\mathbf{f}(t))\mathbf{f}'(t) &= \begin{pmatrix} 2f_1(t) & 2f_2(t) & 2f_3(t) \end{pmatrix} \begin{pmatrix} (D_1 f)(t) \\ (D_2 f)(t) \\ (D_3 f)(t) \end{pmatrix} \\
&= 2 \begin{pmatrix} f_1(t) & f_2(t) & f_3(t) \end{pmatrix} \begin{pmatrix} (D_1 f)(t) \\ (D_2 f)(t) \\ (D_3 f)(t) \end{pmatrix} = 2\mathbf{f}(t) \cdot \mathbf{f}'(t) = 0
\end{aligned}$$

and get the desired result. Geometrically, $\mathbf{f}$ draws a curve embedded in a unit sphere, and the result implies that the curve's velocity vector is normal to the radial vector. In other words, the velocity vector is tanget to the spherical surface.

# 6 Chapter 9 #14

## 6.1 Solution for (a)

For $(x, y) \neq (0, 0)$, we can write

$$\begin{aligned}
(D_1 f)(x, y) &= \frac{3x^2(x^2 + y^2) - 2x(x^3)}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2y^2}{(x^2 + y^2)^2} \\
(D_2 f)(x, y) &= \frac{-x^3(2y)}{(x^2 + y^2)^2} = \frac{-2x^3y}{(x^2 + y^2)^2}
\end{aligned}$$

Also,

$$\begin{aligned}
(D_1 f)(0, 0) &= \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1 \\
(D_2 f)(0, 0) &= \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} 0 = 0
\end{aligned}$$

For $(x, y) \neq (0, 0)$, we can write

$$\begin{aligned}
|(D_1 f)(x, y)| &= \left| \frac{(x^2 + y^2)^2 + y^2(x^2 - y^2)}{(x^2 + y^2)^2} \right| \leq 1 + \frac{|y^2(x^2 - y^2)|}{(x^2 + y^2)^2} \leq 1 + y^2 \cdot \frac{|x^2| + |y^2|}{(x^2 + y^2)^2} \\
&= 1 + \frac{y^2}{x^2 + y^2} \leq 2 \\
|(D_2 f)(x, y)| &= \left| \frac{-2x^3y}{(x^2 + y^2)^2} \right| = \left( \frac{x^2}{x^2 + y^2} \right) \left( \frac{|2xy|}{x^2 + y^2} \right) \leq \frac{|2xy|}{x^2 + y^2} \leq \frac{x^2 + y^2}{x^2 + y^2} = 1
\end{aligned}$$

From this, we can conclude that the partial derivatives are bounded.

## 6.2 Solution for (b)

Let $\mathbf{u} = (u_1, u_2)$. Then we can write

$$\begin{aligned}
(D_{\mathbf{u}} f)(0, 0) &= \lim_{t \to 0} \frac{f(t\mathbf{u}) - f(0, 0)}{t} = \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0, 0)}{t} = \lim_{t \to 0} \frac{1}{t} \frac{(tu_1)^3}{(tu_1)^2 + (tu_2)^2} \\
&= \lim_{t \to 0} \frac{(tu_1)^3}{t^3} = u_1^3
\end{aligned}$$

Since $|\mathbf{u}| = 1$, $|u_1| \leq 1$ so the absolute value of $(D_{\mathbf{u}} f)(0, 0)$ is at most 1.

### 6.3 Solution for (c)

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. As $\gamma$ is differentiable, $\gamma_1$ and $\gamma_2$ are also differentiable. For $t$ with $\gamma(t) \neq (0,0)$, $g(t)$ is differentiable as it is a quotient of two differentiable functions, $(\gamma_1(t))^3$ and $(\gamma_1(t))^2 + (\gamma_2(t))^2$. For $t$ with $\gamma(t) = (0,0)$, we can write

$$\begin{aligned}
&\lim_{h\to 0} \frac{g(t+h)-g(t)}{h} = \lim_{h\to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h} = \lim_{h\to 0} \frac{f(\gamma(t+h))}{h} \\
&= \lim_{h\to 0} \frac{(\gamma_1(t+h))^3}{h(\gamma_1(t+h))^2 + (\gamma_2(t+h))^2} \\
&= \lim_{h\to 0} \left(\frac{\gamma_1(t+h)-\gamma_1(t)}{h}\right)^3 \left[\left(\frac{\gamma_1(t+h)-\gamma_1(t)}{h}\right)^2 + \left(\frac{\gamma_2(t+h)-\gamma_2(t)}{h}\right)^2\right]^{-1} \\
&= \frac{(\gamma_1'(t))^3}{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}
\end{aligned}$$

Thus, $g(t)$ is differentiable for all $t$. Furthermore, we can write

$$g'(t) = \begin{cases} \dfrac{(\gamma_1'(t))^3}{(\gamma_1'(t))^2 + (\gamma_2'(t))^2} & (\gamma(t) = (0,0)) \\ \dfrac{(\gamma_1(t))^4\gamma_1'(t) + 3(\gamma_1(t))^2(\gamma_2(t))^2\gamma_1'(t) - 2(\gamma_1(t))^3(\gamma_2(t))\gamma_2'(t)}{[(\gamma_1(t))^2 + (\gamma_2(t))^2]^2} & (\gamma(t) \neq (0,0)) \end{cases}$$

Now suppose that $\gamma$ is continuously differentiable. For $u$ such that $\gamma(u) = (0,0)$, we can write

$$\begin{aligned}
\lim_{t\to u} g'(t) &= \lim_{t\to u} \frac{(\gamma_1(t))^4\gamma_1'(t) + 3(\gamma_1(t))^2(\gamma_2(t))^2\gamma_1'(t) - 2(\gamma_1(t))^3(\gamma_2(t))\gamma_2'(t)}{[(\gamma_1(t))^2 + (\gamma_2(t))^2]^2} \\
&= \frac{(\gamma_1'(u))^3}{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}
\end{aligned}$$

Thus, $g'$ is continuous at $u$. Since $g'$ is obviously continuous at points where $\gamma$ is not $(0,0)$, $g'$ is continuous function so $g$ is continuously differentiable.

### 6.4 Solution for (d)

Using the hint, suppose that $f$ is differentiable. Then, using $\mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2})$, we can write

$$(D_{\mathbf{u}}f)(0,0) = \frac{1}{\sqrt{2}}(D_1 f)(0,0) + \frac{1}{\sqrt{2}}(D_2 f)(0,0) = \frac{1}{\sqrt{2}}$$

However, according to the result of (b), the value should be $1/(2\sqrt{2})$, which is a contradiction.

## 7 Chapter 9 #15

### 7.1 Solution for (a)

As $(x^4 - y^2)^2 \geq 0$ holds, $(x^4 + y^2)^2 = (x^4 - y^2)^2 + 4x^4y^2 \geq 4x^4y^2$ also holds, proving the inequality.

For $(x,y) \neq (0,0)$, since $f$ is a sum of polynomial and quotient of nonzero polynomials, $f$ is continuous. Now let's show that $f$ is continuous at $(0,0)$. Since polynomial is continuous everywhere, we only have to show that $4x^6y^2/(x^4+y^2)^2$ is continuous at $(0,0)$. We can write

$$\left|\frac{4x^6y^2}{(x^4+y^2)^2}\right| \leq \left|\frac{4x^6y^2}{4x^4y^2}\right| = x^2$$

using the result we have proven earlier. Fix $\epsilon > 0$. Then by taking $\delta = \sqrt{\epsilon}$, $|(x,y)| \leq \delta$ implies $x^2 \leq \epsilon$. Since the choice of $\epsilon$ was arbitrary, $4x^6y^2/(x^4+y^2)^2$ is continuous at $(0,0)$. Thus, $f$ is a continuous function.

## 7.2 Solution for (b)

By definition, $g_\theta(0) = f(0,0) = 0$. We can write

$$\begin{aligned}
g_\theta'(0) &= \lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t} \\
&= \lim_{t\to 0} \frac{1}{t}\left[t^2 - 2t^3\cos^2\theta\sin\theta - \frac{4t^4\cos^6\theta\sin^2\theta}{(t^2\cos^4\theta + \sin^2\theta)^2}\right] \\
&= \lim_{t\to 0} \left[t - 2t^2\cos^2\theta\sin\theta - \frac{4t^3\cos^6\theta\sin^2\theta}{(t^2\cos^4\theta + \sin^2\theta)^2}\right]
\end{aligned}$$

For $\theta$ with $\sin\theta \neq 0$, the denominator of the fraction term cannot be zero, so the limit is zero. For $\theta$ with $\sin\theta = 0$, the numerator is zero so the limit is zero again. Thus, $g_\theta'(0) = 0$. With calculation we can see that

$$g_\theta'(t) = 2t - 6t^2\cos^2\theta\sin\theta - \cos^6\theta\sin^2\theta\frac{16t^3\sin^2\theta}{(t^2\cos^4\theta + \sin^2\theta)^3}$$

and

$$\begin{aligned}
g_\theta''(0) &= \lim_{t\to 0} \frac{g_\theta'(t) - g_\theta'(0)}{t} = \lim_{t\to 0} \frac{g_\theta'(t)}{t} \\
&= \lim_{t\to 0} \frac{1}{t}\left[2t - 6t^2\cos^2\theta\sin\theta - \cos^6\theta\sin^2\theta\frac{16t^3\sin^2\theta}{(t^2\cos^4\theta + \sin^2\theta)^3}\right] \\
&= \lim_{t\to 0} \left[2 - 6t\cos^2\theta\sin\theta - \cos^6\theta\sin^2\theta\frac{16t^2\sin^2\theta}{(t^2\cos^4\theta + \sin^2\theta)^3}\right]
\end{aligned}$$

For $\theta$ with $\sin\theta \neq 0$, the denominator of the fraction term cannot be zero, so the limit is zero. If $\sin\theta = 0$, then the numerator is zero so the limit is also zero. Thus $g_\theta''(0) = 2$.

## 7.3 Solution for (c)

Plugging in the variables, we obtain $f(x,x^2) = -x^4$. For all $0 < r < 1$, we can take $x = r/\sqrt{2}$ then $|(x,x^2)| = \sqrt{x^2+x^4} < \sqrt{2x^2} = \sqrt{2}x < r$. As $r > 0$, $x > 0$ so $(x,x^2) \neq (0,0)$, and $f(x,x^2) < f(0,0) = 0$. In other words, for all $r > 0$, there exists a point $p$ in $B((0,0),r)$, which is an open ball centered at $(0,0)$ whose radius is $r$. This implies that $f$ does not have a local minimum at $(0,0)$.