MATH312: Homework 2 (due Sep. 28)

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1 Chapter 7 #13

1.1 Solution for (a)

By theorem 7.23, there exists a subsequence $\{f_{n'_k}\}$ of $\{f_n\}$ such that $\{f_{n'_k}(x)\}$ converges for every $x \in \mathbb{Q}$. Take such n'_k. Let $g : \mathbb{R} \to \mathbb{R}$ defined as follows: (the existence of supremum is guaranteed by the least upper bound principle.)

$$g(x) = \sup_{r \in (-\infty, x] \cap \mathbb{Q}} \lim_{k \to \infty} f_{n'_k}(r)$$

Consider reals x and y such that $x < y$. Then, as $((-\infty, x] \cap \mathbb{Q}) \subset ((-\infty, y] \cap \mathbb{Q})$, $g(x) \leq g(y)$. In other words, g is an monotonically increasing function. Now, consider a rational q. Since f_n is monotonically increasing for all n, we know that $f_{n'_k}(q) \geq f_{n'_k}(r)$ for all rational r such that $r \leq q$, for all k. Then, $\lim_{k\to\infty} f_{n'_k}(q) \geq \lim_{k\to\infty} f_{n'_k}(r)$ for all rational $r \leq q$. Thus, $g(q) = \lim_{k\to\infty} f_{n'_k}(q)$. Consider $p \in \mathbb{R}$. Suppose that g is continuous at p. Fix some $\epsilon > 0$. By the continuity of g at p, there exists some $\delta > 0$ such that $|x - p| < \delta$ implies $|g(x) - g(p)| < \epsilon$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists rationals q, r such that $p - \delta < q < p < r < p + \delta$. Then we can write

$$g(p) - \epsilon < g(q) \leq g(p) \leq g(r) < g(p) + \epsilon \tag{1}$$

as g is monotonically increasing. Also, as f_n is monotonically increasing for all n, for all k, we can write

$$f_{n'_k}(q) \leq f_{n'_k}(p) \leq f_{n'_k}(r) \tag{2}$$

Using (1) and (2), we can write

$$g(q) - f_{n'_k}(r) \leq g(p) - f_{n'_k}(p) \leq g(r) - f_{n'_k}(q)$$

As $g(q) = \lim_{k\to\infty} f_{n'_k}(q)$ and $g(r) = \lim_{k\to\infty} f_{n'_k}(r)$, there exists $N \in \mathbb{N}$ such that $k \geq N$ implies $|f_{n'_k}(q) - g(q)| < \epsilon$ and $|f_{n'_k}(r) - g(r)| < \epsilon$. Then, by triangle inequality,

$$|g(q) - f_{n'_k}(r)| \leq |g(q) - g(p)| + |g(p) - g(r)| + |g(r) - f_{n'_k}(r)| < \epsilon + \epsilon + \epsilon = 3\epsilon$$
$$|g(r) - f_{n'_k}(q)| \leq |g(r) - g(p)| + |g(p) - g(q)| + |g(q) - f_{n'_k}(q)| < \epsilon + \epsilon + \epsilon = 3\epsilon$$

Thus, $|g(p) - f_{n'_k}(p)| < 3\epsilon$. Since the choice of ϵ was arbitrary, $f_{n'_k}(p) \to g(p)$ as $k \to \infty$. Let E be a set of points of $\mathbb{R}$ at which f is discontinuous. Then, by theorem 4.30, E is at most countable. Applying theorem 7.22 again, there exists a subsequence $\{f_{n'_{k_l}}\}$ of $\{f_{n'_k}\}$ such that $f_{n'_{k_l}}(x)$ converges for every $x \in E$ as $l \to \infty$. Since $f_{n'_k}(x)$ converges for every $x \in E^C$, the subsequence $\{f_{n_k}\}$ we are looking for is $\{f_{n'_{k_l}}\}$, and f can be constructed by taking $l \to \infty$ limit to $f_{n'_{k_l}}(x)$ for all $x \in \mathbb{R}$.

1.2 Solution for (b)

Consider a compact subset $K \in \mathbb{R}$. As f is continuous on K, f is uniformly continuous on K. Fix $\epsilon > 0$. There exists some $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. As K is compact, there exists $t_1, t_2, \dots, t_N$ such that $\bigcup_{i=1}^{N}(t_i - \delta/2, t_i + \delta/2) \supset K$. As $f_{n_k}(x)$ converges to $f(x)$ for all x, there exists M_i where $i = 1, 2, \dots, N$ such that $k \geq M_i$ implies $|f_{n_k}(t_i) - f(t_i)| < \epsilon$, $f_{n_k}(t_i + \delta/2) - f_{n_k}(t_i) = f_{n_k}(t_i + \delta/2) - f(t_i + \delta/2) + f(t_i + \delta/2) - f(t_i) + f(t_i) - f_{n_k}(t_i) < 3\epsilon$ and $f_{n_k}(t_i) - f_{n_k}(t_i - \delta/2) < 3\epsilon$. Let $M = \max\{M_1, \dots, M_N\}$, then $k \geq M$ implies $|f_{n_k}(t_i) - f(t_i)| < \epsilon$, $f_{n_k}(t_i + \delta/2) - f_{n_k}(t_i) < 3\epsilon$ and $f_{n_k}(t_i) - f_{n_k}(t_i - \delta/2) < 3\epsilon$ for $i = 1, 2, \dots N$. Now, consider $x \in K$. There exists $t_m \in \{t_1, \dots, t_N\}$ such that $x \in (t_m - \delta/2, t_m + \delta/2)$. For $k \geq M$, we can write

$$|f_{n_k}(x) - f(x)| \leq |f_{n_k}(x) - f_{n_k}(t_m)| + |f_{n_k}(t_m) - f(t_m)| + |f(t_m) - f(x)|$$

Also,

$$|f_{n_k}(x) - f_{n_k}(t_m)| \leq \max\{f_{n_k}(t_m + \delta/2) - f_{n_k}(t_m), f_{n_k}(t_m) - f_{n_k}(t_m - \delta/2)\} < 3\epsilon$$

Since $k \geq M$, $|f_{n_k}(t_m) - f(t_m)| \leq \epsilon$, and $|t_m - x| < \delta$ implies $|f(t_m) - f(x)| < \epsilon$. Thus, $k \geq M$ implies $|f_{n_k}(x) - f(x)| < 5\epsilon$ for all $x \in K$. Since our choice of ϵ was arbitrary, $\{f_{n_k}\}$ uniformly converges to f on K.

2 Chapter 7 #14

Let $x_k(t)$ and $y_k(t)$ as follows:

$$x_k(t) = \sum_{n=1}^{k} 2^{-n} f(3^{2n-1}t), \quad y_k(t) = \sum_{n=1}^{k} 2^{-n} f(3^{2n}t)$$

Then, $x_k(t)$ and $y_k(t)$ are continuous for all k as they are linear combination of continuous functions. Let $M_n = 2^{-n}$. Then, as $\sum M_n$ converges and $|2^{-n} f(3^{2n-1}t)| \leq M_n$, $x_k(t)$ converges uniformly as $k \to \infty$ by theorem 7.10. For similar reason, $y_k(t)$ also converges uniformly. By theorem 7.12, $x(t)$ and $y(t)$ are continuous and so does $\Phi(t)$, by theorem 4.10.

Following the hint, each $(x_0, y_0) \in I^2$ has a binary expansion form:

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \quad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

where each a_i is 0 or 1. Let t_0 as

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$$

Then, t_0 is an element of the Cantor set as its ternary fraction does not have digit 1. Then, for $k > 1$, we can write

$$\begin{aligned} 3^k t_0 = 3^k \sum_{i=1}^{\infty} 3^{-i-1}(2a_i) &= 3^k \sum_{i=1}^{k-1} 3^{-i-1}(2a_i) + \frac{2a_k}{3} + 3^k \sum_{i=k+1}^{\infty} 3^{-i-1}(2a_i) \\ &= \sum_{i=1}^{k-1} 3^{k-i-1}(2a_i) + \frac{2a_k}{3} + 3^k \sum_{i=k+1}^{\infty} 3^{-i-1}(2a_i) \end{aligned}$$

Then, every term of $\sum_{i=1}^{k-1} 3^{k-i-1}(2a_i)$ is an even number, so $\sum_{i=1}^{k-1} 3^{k-i-1}(2a_i)$ is even. Then,

$$f(3^k t_0) = f\left(\frac{2a_k}{3} + 3^k \sum_{i=k+1}^{\infty} 3^{-i-1}(2a_i)\right) \tag{3}$$

as $f(t) = f(t+2)$ for all t. Also, we can write

$$0 \leq 3^k \sum_{i=k+1}^{\infty} 3^{-i-1}(2a_i) \leq 3^k \sum_{i=k+1}^{\infty} 2 \cdot 3^{-i-1} = \frac{1}{3}$$

If $a_k = 0$, then (3) evaluates to

$$f\left(3^k \sum_{i=k+1}^{\infty} 3^{-i-1}(2a_i)\right) = 0$$

If $a_k = 1$, then (3) evaluates to

$$f\left(\frac{2}{3} + 3^k \sum_{i=k+1}^{\infty} 3^{-i-1}(2a_i)\right) = 1$$

Thus, $f(3^k t_0) = a_k$ holds. Then we can write

$$x(t_0) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1} t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1} = x_0$$
$$y(t_0) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n} t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n} = y_0$$

In conclusion, $\Phi(t)$ maps the Cantor set onto I^2. Since the Cantor set is a subset of I, $\Phi(t)$ maps I onto I^2.

3 Chapter 7 #15

Since $\{f_n\}$ is equicontinuous in $[0,1]$, for all $\epsilon > 0$ there exists $\delta > 0$ for all n, $|x-y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$ for $x \in [0,1]$ and $y \in [0,1]$. Since $|f_n(x) - f_n(y)| = |f(nx) - f(ny)|$, for all n, there exists $\delta' > 0$ such that $|t-u| < \delta' = n\delta$ implies $|f(t) - f(u)| < \epsilon$. In other words, f is uniformly continuous on $[0,n]$, for all n, although the δ' value is different for distinct n.

4 Chapter 7 #16

Fix $\epsilon > 0$. As $\{f_n\}$ is equicontinuous on K, for all n there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$ for $x \in K$ and $y \in K$. Since K is compact, there exists $\{x_1, x_2, \ldots, x_N\} \subset K$ such that $\bigcup_{i=1}^{N} B(x_i, \delta) \supset K$, where $B(p,r)$ is a open ball whose center is p and radius is r. Since $\{f_n(x)\}$ converges for all $x \in K$, there exists M_i such that $n \geq M_i$ and $m \geq M_i$ implies $|f_n(x_i) = f_m(x_i)| < \epsilon$, for $i = 1, 2, \ldots, N$. Consider $x \in K$. There exists $x_k \in \{x_1, \ldots, x_N\}$ such that $|x - x_k| < \delta$. Let $M = \max\{M_1, \ldots, M_N\}$. Then for $n \geq M$ and $m \geq M$, we can write

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(x)|$$

From $|x - x_k| < \delta$, $|f_n(x) - f_n(x_k)| < \epsilon$ and $|f_m(x_k) - f_m(x)| < \epsilon$ holds. From $n, m \geq M \geq M_k$, $|f_n(x_k) - f_m(x_k)| < \epsilon$ holds. Thus, $|f_n(x) - f_m(x)| < 3\epsilon$. Since our choice of ϵ and x were arbitrary, by theorem 7.8, $\{f_n\}$ converges uniformly on K.

5 Chapter 7 #18

Since $\{f_n\}$ is uniformly bounded, there exists $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $x \in [a, b]$ and n. Fix $\epsilon > 0$ and let $\delta = \epsilon/M$. Then we can write

$$|F_n(x) - F_n(y)| = \left| \int_x^y f_n(t)dt \right| \leq \left| \int_x^y |f_n(t)|dt \right| \leq M|x - y| < M\delta = \epsilon$$

for all $x \in [a, b]$, $y \in [a, b]$ and n. Thus, $\{F_n\}$ is equicontinuous on $[a, b]$. Furthermore, we can also write

$$|F_n(x)| = \left| \int_a^x f_n(t)dt \right| \leq \int_a^x |f_n(t)|dt \leq \int_a^x M\ dt = M(x - a)$$

for all $x \in [a, b]$. Thus, $\{F_n\}$ is pointwise bounded on $[a, b]$. By theorem 6.20, $F_n(x)$ is continuous on $[a, b]$ for all n. In conclusion, we can apply theorem 7.25 on $\{F_n\}$, so there exists a subsequence $\{F_{n_k}\}$ which is uniformly convergent on $[a, b]$.

6 Chapter 7 #25

6.1 Solution for (a)

Following the hint, we know from the definition of f_n that

$$|f_n'(t)| = |\phi(x_i, f_n(x_i))| \leq M$$

also, using the definition,

$$|\Delta_n(t)| = |f_n'(t) - \phi(t, f_n(t))| \leq |f_n'(t)| + |\phi(t, f_n(t))| \leq M + M = 2M$$

Since $f_n'(t)$ and $\phi(t, f_n(t))$ are continuous, $\Delta_n(t)$ is also continuous so $\Delta_n \in \mathscr{R}$. We can also write

$$\begin{aligned}
|f_n(x)| &= \left| c + \int_0^x (\phi(t, f_n(t)) + \Delta_n(t))dt \right| \leq |c| + \left| \int_0^x (\phi(t, f_n(t)) + \Delta_n(t))dt \right| \\
&\leq |c| + \int_0^x |\phi(t, f_n(t)) + \Delta_n(t)|dt = |c| + \int_0^x |f_n'(t)|dt \\
&\leq |c| + \int_0^1 |f_n'(t)|dt \leq |c| + M
\end{aligned}$$

6.2 Solution for (b)

Fix $\epsilon > 0$. For $x \in [0, 1]$ and $y \in [0, 1]$, we can write

$$\begin{aligned}
|f_n(x) - f_n(y)| &= \left| \int_x^y (\phi(t, f_n(t)) + \Delta_n(t))dt \right| = \left| \int_x^y f_n'(t)dt \right| \leq \left| \int_x^y |f_n'(t)|dt \right| \\
&\leq M|x - y|
\end{aligned}$$

Taking $\delta = \epsilon/M$, $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| \leq \epsilon$, for all $x \in [0, 1]$, $y \in [0, 1]$ and n. Thus, $\{f_n\}$ is equicontinuous on $[0, 1]$.

6.3 Solution for (c)

As $[0,1]$ is compact, f_n is continuous and bounded for all n, $\{f_n\}$ is pointwise bounded, and equicontinuous, by theorem 7.25, $\{f_n\}$ contains a subsequence $\{f_{n_k}\}$ which is uniformly convergent on $[0,1]$.

6.4 Solution for (d)

Fix $\epsilon > 0$. As ϕ is continuous on the rectangle $0 \leq x \leq 1$, $|y| \leq M_1$, ϕ is uniformly continuous on the rectangle as the rectangle is compact. Thus, there exists $\delta > 0$ such that $d((x_1, y_1), (x_2, y_2)) < \delta$ implies $|\phi(x_1, y_1) - \phi(x_2, y_2)| < \epsilon$ for (x_1, y_1) and (x_2, y_2) in the rectangle. (Here, d is a Euclidean norm.) Also, f_{n_k} converges uniformly to f on $[0,1]$. Then, there exists some $N \in \mathbb{N}$ such that $k \geq N$ implies $|f_{n_k}(t) - f(t)| < \delta$ for all $t \in [0,1]$, which is equivalent to $d((t, f_{n_k}(t)), (t, f(t))) < \delta$. Combining these results, there exists $N \in \mathbb{N}$ such that $k \geq N$ implies $|\phi(t, f_{n_k}(t)) - \phi(t, f_n(t))| < \epsilon$ for all $t \in [0,1]$. Thus, $\{\phi(t, f_{n_k}(t))\}$ converges uniformly to $\phi(t, f(t))$ on $[0,1]$.

6.5 Solution for (e)

Fix $\epsilon > 0$. As described in the solution for (d), ϕ is uniformly continuous on the rectangle. Thus, there exists $\delta > 0$ such that $d((x_1, y_1), (x_2, y_2)) < \delta$ implies $|\phi(x_1, y_1) - \phi(x_2, y_2)| < \epsilon$ for all (x_1, y_1) and (x_2, y_2) in the rectangle. Also, f_n is continuous on $[0,1]$ for all n, so f_n is uniformly continuous on $[0,1]$. By definition, $|x_i - t| \geq 1/n$ so there exists some $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x_i) - f_n(t)| < \delta/\sqrt{2}$ and $|x_i - t| < 1/n < \delta/\sqrt{2}$. Then, we can write

$$d((x_i, f_n(x_i)), (t, f_n(t))) = \sqrt{(x_i - t)^2 + (f_n(x_i) - f_n(t))^2} < \sqrt{\left(\frac{\delta}{\sqrt{2}}\right)^2 + \left(\frac{\delta}{\sqrt{2}}\right)^2} = \delta$$

Then, by the uniform continuity of ϕ, $n \geq N$ implies

$$|\Delta_n(t)| = |\phi(x_i, f_n(x_i)) - \phi(t, f_n(t))| < \epsilon$$

for all $t \in [0,1]$. In conclusion, $\{\Delta_n(t)\}$ converges uniformly to zero on $[0,1]$.

6.6 Solution for (f)

$\phi(t, f_{n_k}(t))$ and $\Delta_n(t)$ uniformly converge to $\phi(t, f(t))$ and zero respectively on $[0,1]$. Also, $\phi(t, f_{n_k}(t)) \in \mathscr{R}$ on $[0,1]$ as it is continuous on $[0,1]$ and we have proved that $\Delta_n \in \mathscr{R}$ on $[0,1]$ in (a). Then, by theorem 7.16, we can write

$$\int_0^x \phi(t, f(t))dt = \lim_{k \to \infty} \int_0^x \phi(t, f_{n_k}(t))dt$$

and

$$\lim_{k \to \infty} \int_0^x \Delta_{n_k}(t)dt = 0$$

Thus, we can write

$$f(x) = \lim_{k \to \infty} f_{n_k}(x) = \lim_{k \to \infty} \left(c + \int_0^x (\phi(t, f_{n_k}(t)) + \Delta_{n_k}(t))dt\right) = c + \int_0^x \phi(t, f(t))dt$$

Differentiating both sides reveals the equation,

$$f'(x) = \phi(x, f(x))$$

and $f(c) = 0$. Thus, f is a solution of the given problem.