On the Statistical Theory of Isotropic Turbulence

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Introduction and summary

G. I. Taylor (1935), in a paper of fundamental importance, introduced the conception of isotropic turbulence and applied it, with interesting results, to the problem of the decay of turbulence in a windstream.

In this paper we develop a general theory of isotropic turbulence. The correlation coefficients between two arbitrary velocity components at two arbitrary points form a tensor called the correlation tensor. Due to isotropy the tensor corresponding to one fixed and one variable point has spherical symmetry; due to the condition of continuity it is completely determined by one scalar function. The mean products of the derivatives of the velocity fluctuations are expressed by the derivatives of the tensor components; in this way laborious calculations for obtaining such mean values are eliminated. The correlation between three components (triple correlation) is discussed.

After developing the kinematics of isotropic turbulence the dynamical problem of the change of the various mean values with time is considered. It is shown that using the equations of motion a partial differential equation connecting the double and triple correlation functions can be established. The solution of this equation is investigated first in the case when the triple correlation is neglected and the shape of the double correlation function remains similar and only its scale changes. Equations for the dissipation of energy and vorticity are deduced.

Finally, a possible solution for large Reynolds numbers is given and applied to Taylor's problem of the decay of turbulence behind a grid.

Taylor's fundamental relation between the width of the correlation function and the size of the small (dissipative) eddies is confirmed. However, it is believed that the linear law for the reciprocal of the root mean square of

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the velocity fluctuations as a function of the distance from the grid is a special case; in the last section of the present paper a more general functional relation is suggested.

§§ 6, 8 and 9 have been rewritten and § 11 inserted by the senior author in September 1937.

A—The kinematics of isotropic turbulence

1—*The definition of isotropic turbulence and immediate deductions*

Isotropic turbulence may be defined by the condition that the average value of any function of the velocity components and their derivatives at a particular point, defined in relation to a particular set of axes, is unaltered if the axes of reference are rotated in any manner and if the co-ordinate system is reflected in any plane through the origin. We consider average values with regard to the time and suppose that the fluctuations are so rapid that the variation of the average value is negligible throughout the period of time required for averaging. Thus, in fact, we shall consider the average values to be slowly varying functions of the time.

Consider a particular system of co-ordinate axes Ox_1, Ox_2 and Ox_3 and take two points $P(x_1, 0, 0)$ and $P'(x_1', 0, 0)$ on Ox_1. Denote by u_1, u_2, u_3 and u_1', u_2', u_3' the velocity components at P and P' respectively. Let us now suppose that $\overline{u_1^2}$, $\overline{u_2^2}$, $\overline{u_3^2}$ are all independent of position. (By isotropy, of course, $\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$ $(=\overline{u^2}$, say).)

The correlation coefficients $\overline{u_1u_1'}/\overline{u^2}$ and $\overline{u_ju_j'}/\overline{u^2}$ for $j = 2$ or 3 will be particular functions $f(r, t)$ and $g(r, t)$, say, of the distance r between P and P'* and of the time. It seems to be fairly evident physically that the mean value $\overline{u_iu_j'} = 0$ when $i \neq j$ in isotropic turbulence, since it appears to be equally probable that u_1u_2', for example, will be positive or negative. It is an easy matter to prove these results analytically. The mean values $\overline{u_1u_j'}$ and $\overline{u_1'u_j}$ for $j = 2$ or 3 can be shown to vanish by a rotation of the axes of x_2 and x_3 through 180° about the x_1-axis. Denoting transformed values by capital letters we see that $U_1 = u_1$, $U_1' = u_1'$, $U_j = -u_j$ and $U_j' = -u_j'$ for $j = 2$ or 3 so that, for example,

$$\overline{U_1U_j'} = -\overline{u_1u_j'}.$$

But by the isotropic property

$$\overline{U_1U_j'} = \overline{u_1u_j'},$$

* It is assumed that the turbulence is statistically uniform as well as isotropic so that correlations do not depend on the position or orientation of the line PP' but only on its length.

so that $\overline{u_1 u_j'}$ is zero for $j = 2$ or 3. Similarly $\overline{u_j u_1'}$ is also zero for $j = 2$ or 3. The mean values $\overline{u_2 u_3'}$ and $\overline{u_3 u_2'}$ can be shown to vanish by reflexion in the x_1, x_3 plane. Denoting, in this case, reflected values by capital letters we see that $U_2 = -u_2$, $U_2' = -u_2'$, $U_3 = u_3$, $U_3' = u_3'$, so that, for example,

$$\overline{U_2 U_3'} = -\overline{u_2 u_3'}.$$

But by the isotropic property

$$\overline{U_2 U_3'} = \overline{u_2 u_3'},$$

and hence $\overline{u_2 u_3'}$ vanishes. Similarly $\overline{u_3 u_2'}$ is zero.

2—*Lemma*

Consider now any two points P and Q in the fluid. Denote by p the velocity component in a particular direction PP' at P and by q the velocity component in the direction QQ' at Q. We shall determine an expression for the correlation coefficient $\overline{pq}/\overline{u^2}$.

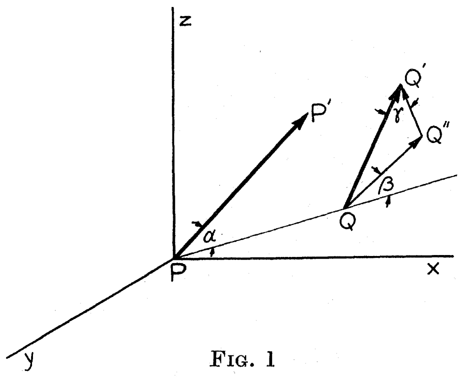

FIG. 1

The intersecting lines PP' and PQ (see fig. 1) determine a plane. Denote by QQ'' the orthogonal projection of QQ' on this plane; $Q''Q'$ will then be normal to the plane. Denote by α, β and γ the angles $P'\hat{P}Q$, $\pi - P\hat{Q}Q''$, $Q\hat{Q}'Q''$, and by p_1, p_2, p_3 and q_1, q_2, q_3, respectively, the velocity components at P and Q in the direction PQ, the direction normal to PQ and $Q''Q'$, and the direction $Q''Q'$.* Then

$$p = p_1 \cos\alpha + p_2 \sin\alpha,$$
$$q = q_1 \cos\beta \sin\gamma + q_2 \sin\beta \sin\gamma + q_3 \cos\gamma.$$

* For simplicity the figure is described as it is drawn in fig. 1. When P' and Q'' are on opposite sides of PQ the sign of β must be changed.

Therefore

$$\overline{pq} = \overline{p_1 q_1}\cos\alpha\,\cos\beta\,\sin\gamma + \overline{p_1 q_2}\cos\alpha\,\sin\beta\,\sin\gamma + \overline{p_1 q_3}\cos\alpha\,\cos\gamma$$
$$+\overline{p_2 q_1}\cos\beta\,\sin\gamma\,\sin\alpha + \overline{p_2 q_2}\sin\beta\,\sin\alpha\,\sin\gamma + \overline{p_2 q_3}\sin\alpha\,\cos\gamma.$$

We have already proved that all the mean values here except $\overline{p_1 q_1}$ and $\overline{p_2 q_2}$ vanish, and we have denoted by $f(r,t)$ and $g(r,t)$ the correlation coefficients $\overline{p_1 q_1}/\overline{u^2}$ and $\overline{p_2 q_2}/\overline{u^2}$, where PQ is supposed to be of length r. Thus

$$\frac{\overline{pq}}{\overline{u^2}} = [f(r,t)\cos\alpha\cos\beta + g(r,t)\sin\alpha\sin\beta]\sin\gamma. \tag{1}$$

3—*The correlation tensor*

Consider now any particular co-ordinate system and suppose the velocity components at $P(x_1, x_2, x_3)$ and $P'(x_1', x_2', x_3')$ are (u_1, u_2, u_3) and (u_1', u_2', u_3') respectively. The nine quantities $\overline{u_i u_j'}$ for $i, j = 1, 2$ or 3 may be shown to be the components of a second rank tensor. The transformation law can readily be seen to be satisfied since the dyadic product of two vectors is a tensor so that each of the contributions $u_i u_j'$ to the mean values form a tensor. Clearly the operation of taking a mean value will not alter the transformation law satisfied. Hence we may consider the "correlation tensor" **R** defined by

$$\overline{u^2}\mathbf{R} = \overline{u^2}\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} \overline{u_1 u_1'} & \overline{u_1 u_2'} & \overline{u_1 u_3'} \\ \overline{u_2 u_1'} & \overline{u_2 u_2'} & \overline{u_2 u_3'} \\ \overline{u_3 u_1'} & \overline{u_3 u_2'} & \overline{u_3 u_3'} \end{pmatrix}$$

or

$$\overline{u^2}\mathbf{R} = \overline{u^2}R_{ij} = \overline{u_i u_j'} \qquad i,j = 1,\ 2 \text{ or } 3. \tag{2}$$

By means of the lemma which has been established each component of **R** can be evaluated in terms of the functions f and g and the vector **r** whose components are $\xi_1 = x_1' - x_1$, $\xi_2 = x_2' - x_2$, $\xi_3 = x_3' - x_3$. In this way we obtain

$$\mathbf{R} = \frac{\{f(r,t) - g(r,t)\}}{r^2}\mathbf{rr} + g(r,t)\,\mathbf{I}, \tag{3}$$

where $r = |\mathbf{r}|$ and **I** is the unit tensor $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

By expressing the fact that the velocity fluctuations satisfy the equation of continuity we can now obtain a relation between $f(r, t)$ and $g(r, t)$. Since the velocity fluctuations at P' satisfy the equation of continuity

$$\frac{\partial u'_i}{\partial x'_i} = 0,$$

where the summation convention is in operation. Therefore

$$\frac{\partial}{\partial x'_i}(u'_i u_j) = 0, \tag{4}$$

since u_j is independent of x'_1, x'_2 and x'_3. Thus we obtain the result

$$\frac{\partial R_{ji}}{\partial x'_i} = 0, \tag{5}$$

for $j = 1$, 2 or 3 by dividing equation (4) by $\overline{u^2}$ (which is independent of position), by taking mean values and using the definition of R_{ji} (see equation (2)). By the definition of ξ_1, ξ_2, ξ_3 we may write equation (5) as

$$\frac{\partial R_{ji}}{\partial \xi_i} = 0. \tag{6}$$

Using the values of the components of $\mathbf{R}$ given by equation (3), and also the fact that equation (6) is valid for all values of ξ_1, ξ_2 and ξ_3, we find

$$2f(r, t) - 2g(r, t) = -r\frac{\partial f(r, t)}{\partial r}. \tag{7}$$

Since both $f(r, t)$ and $g(r, t)$ are even functions of r

$$f(r, t) = 1 + f''_0 \frac{r^2}{2!} + f^{\mathrm{iv}}_0 \frac{r^4}{4!} + \dots, \tag{8}$$

$$g(r, t) = 1 + g''_0 \frac{r^2}{2!} + g^{\mathrm{iv}}_0 \frac{r^4}{4!} + \dots. \tag{9}$$

From equation (7) we then find by equating coefficients of r^2 and r^4

$$2f''_0 = g''_0, \tag{10}$$

$$3f^{\mathrm{iv}}_0 = g^{\mathrm{iv}}_0. \tag{11}$$

Further, substituting these expansions for f and g in equation (3) we find, for small values of r, that

$$\mathbf{R} = \left(1 + \frac{g''_0}{2} r^2\right)\mathbf{I} + \left(\frac{f''_0 - g''_0}{2}\right)\mathbf{rr}, \tag{12}$$

neglecting fourth and higher powers of r. Using equation (10) we may write equation (12) in the form

$$\mathbf{R} = (1 + f_0'' r^2)\,\mathbf{I} - \tfrac{1}{2} f_0'' \mathbf{rr}. \tag{13}$$

We notice, in passing, that

$$\left.\begin{aligned}\left(\frac{\partial^2 R_{kl}}{\partial\xi_i\,\partial\xi_j}\right)_{\xi_1=\xi_2=\xi_3=0} &= f_0'' \quad \text{when} \quad k = l = i = j,\\ &= 2f_0'' \quad \text{when} \quad k = l \neq i = j,\\ &= -\tfrac{1}{2} f_0'' \quad \text{when} \quad k = i \neq l = j \quad \text{or} \quad k = j \neq l = i,\\ &= 0 \text{ otherwise;}\end{aligned}\right\} \tag{14}$$

we shall require these results later.

We may, at this stage, point out an analogy which is helpful in creating a physical picture. The expression (3) for the correlation tensor is of exactly the same form as that for the stress tensor for a continuous medium when there is spherical symmetry. In the analogy $f(r)$ is the principal radial stress at any point, $g(r)$ is any of the principal transverse stresses, and R_{ik} is the component in the k direction of the stress over a plane whose normal is in the i direction. Further, the relation between f and g given by continuity in our problem corresponds to the condition for equilibrium in the stress analogy.

4—*The correlation coefficients between derivatives of the velocities*

Taylor (1935) had occasion to calculate the various correlation coefficients between first derivatives of the velocity components. These calculations were made by transforming axes, using the equation of continuity and applying the definition of isotropy. Even for the first derivatives this method is fairly laborious, whilst for higher order derivatives the work involved makes it almost prohibitive; it is hardly possible to decide beforehand whether a particular transformation will lead to a new relation or will merely give one that has already been obtained.

The determination of the correlations both quickly and automatically is an interesting application of the correlation tensor. Let us take, as an example, the determination of $\overline{\dfrac{\partial u_k}{\partial x_i}\dfrac{\partial u_l}{\partial x_j}}$. Now

$$\frac{\partial}{\partial x_i}\overline{(u_k u_l')} = \overline{u^2}\frac{\partial}{\partial x_i} R_{kl} = -\overline{u^2}\frac{\partial R_{kl}}{\partial \xi_i}, \tag{15}$$

i.e. since the velocity fluctuations at P are all differentiable functions of x_i,

$$\overline{\frac{\partial u_k}{\partial x_i} u_l'} = -\overline{u^2}\frac{\partial R_{kl}}{\partial \xi_i}. \tag{16}$$

Further
$$\frac{\partial}{\partial x_j'}\left(\overline{\frac{\partial u_k}{\partial x_i} u_l'}\right) = -\overline{u^2}\frac{\partial}{\partial x_j'}\left(\frac{\partial R_{kl}}{\partial \xi_i}\right) = -\overline{u^2}\frac{\partial^2 R_{kl}}{\partial \xi_j \partial \xi_i},$$

i.e.
$$\overline{\frac{\partial u_k}{\partial x_i}\frac{\partial u_l'}{\partial x_j'}} = -\overline{u^2}\frac{\partial^2 R_{kl}}{\partial \xi_i \partial \xi_j}. \tag{17}$$

If, now, we make P and P' coincide we obtain the result

$$\overline{\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_j}} = -\overline{u^2}\left(\frac{\partial^2 R_{kl}}{\partial \xi_i \partial \xi_j}\right)_{\xi_1=\xi_2=\xi_3=0} \tag{18}$$

We obtain the values of $\dfrac{\partial^2 R_{kl}}{\partial \xi_i \partial \xi_j}$ when $\xi_1 = \xi_2 = \xi_3 = 0$ immediately from equations (14). Hence, for example, we see that

$$\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} = -\overline{u^2}f_0'', \tag{19}$$

$$\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2} = -2\overline{u^2}f_0'', \tag{20}$$

and that
$$\overline{\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}} = \overline{\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}} = \tfrac{1}{2}\overline{u^2}f_0''$$
$$= -\frac{1}{2}\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}, \tag{21}$$

or, alternatively,
$$= -\frac{1}{4}\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2}, \tag{22}$$

by equations (19) and (20) respectively.

Similarly, all other correlations of first derivatives can be obtained.

This method is immediately applicable to higher order derivatives. Exactly as for the first derivatives we obtain, for example, the result

$$\overline{\frac{\partial^2 u_k}{\partial x_i \partial x_j}\frac{\partial^2 u_l}{\partial x_r \partial x_s}} = \overline{u^2}\left(\frac{\partial^4 R_{kl}}{\partial \xi_i \partial \xi_j \partial \xi_r \partial \xi_s}\right)_{\xi_1=\xi_2=\xi_3=0} \tag{23}$$

The fourth derivative of the components of $\mathbf{R}$ can be obtained from equation (3), and using the expansions (8) and (9) we find, for example,

$$\overline{\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2} = \overline{u^2}f_0^{\text{iv}}, \tag{24}$$

$$\overline{\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2} = 3\overline{u^2} f_0^{\text{iv}}, \tag{25}$$

and
$$\overline{u^2}\left(\frac{\partial^4 R_{11}}{\partial \xi_1^2 \, \partial \xi_2^2}\right)_{\xi_1=\xi_2=\xi_3=0} = \tfrac{2}{3}\overline{u^2} f_0^{\text{iv}},$$

so that
$$\overline{\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_1}{\partial x_2^2}} = \overline{\left(\frac{\partial^2 u_1}{\partial x_1 \, \partial x_2}\right)^2} = \frac{2}{9}\overline{\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2} = \frac{2}{3}\overline{\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2}, \tag{26}$$

and so on.

For completeness we include the results for first and second derivatives, writing (u, v, w) and (x, y, z) for (u_1, u_2, u_3) and (x_1, x_2, x_3). For the first derivatives*

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \frac{1}{2}\overline{\left(\frac{\partial u}{\partial y}\right)^2}, \quad \overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} = \overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} = -\frac{1}{4}\overline{\left(\frac{\partial u}{\partial y}\right)^2}. \tag{27}$$

For the second derivatives

$$\left.\begin{aligned}
&\overline{\left(\frac{\partial^2 u}{\partial x^2}\right)^2} = \overline{\left(\frac{\partial^2 u}{\partial y \, \partial z}\right)^2} = \overline{\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial z^2}} = \frac{1}{3}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2}, \\
&\overline{\left(\frac{\partial^2 u}{\partial x \, \partial y}\right)^2} = \overline{\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}} = \frac{2}{9}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2}, \\
&\overline{\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x \, \partial y}} = \overline{\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x \, \partial y}} = -\frac{1}{6}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2}, \\
&\overline{\frac{\partial^2 u}{\partial x \, \partial z}\frac{\partial^2 v}{\partial y \, \partial z}} = \overline{\frac{\partial^2 u}{\partial z^2}\frac{\partial^2 v}{\partial x \, \partial y}} = \overline{\frac{\partial^2 u}{\partial y \, \partial z}\frac{\partial^2 v}{\partial z \, \partial x}} = -\frac{1}{18}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2}.
\end{aligned}\right\} \tag{28}$$

All mean values which cannot be brought into one of these forms by cyclic permutation of the letters vanish.

It is obvious that the process could be continued and the higher derivatives of R_{ik} give analogous expressions for the mean products of the higher derivatives of the velocity components. Taylor's development formula for his correlation function R_y (1935, equation (46)) is a special case of the general formulae obtained in this way. His equation, which he effectively found as early as 1921, would be expressed in our notation

$$g(r, t) = 1 - \frac{\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2}}{\overline{u^2}}\frac{r^2}{2!} + \frac{\overline{\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2}}{\overline{u^2}}\frac{r^4}{4!} - \dots.$$

An analogous expression can be written for $f(r, t)$.

* These results were first given by Taylor (1935, p. 436).

5—*Expression of mean values by integrals*

It is known that the velocity components in an incompressible fluid can be expressed by a vector and a scalar potential. In the present case (indefinitely extended fluid without sources throughout the whole fluid) the scalar potential vanishes and the velocity components can be written (it is convenient to use x, y, z, u, v, w for the co-ordinates and the velocity components in this section)

$$u = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \quad v = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \quad w = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}, \tag{29}^*$$

where

$$F = \frac{1}{4\pi}\iiint \frac{\omega'_x}{r}\,dx'\,dy'\,dz', \quad G = \frac{1}{4\pi}\iiint \frac{\omega'_y}{r}\,dx'\,dy'\,dz',$$
$$H = \frac{1}{4\pi}\iiint \frac{\omega'_z}{r}\,dx'\,dy'\,dz', \tag{30}$$

ω'_x, ω'_y, ω'_z are the components of vorticity at the point (x', y', z'),

$$r^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2,$$

and the integration extends throughout the fluid. These results enable us to express correlations between vorticity components and velocity derivatives conveniently as integrals. Consider, for example, $\overline{\omega_x \frac{\partial w}{\partial y}}$. Now

$$\frac{\partial w}{\partial y} = \frac{1}{4\pi}\left\{\frac{\partial^2}{\partial y\,\partial x}\iiint \frac{\omega'_y}{r}\,dx'\,dy'\,dz' - \frac{\partial^2}{\partial y^2}\iiint \frac{\omega'_x}{r}\,dx'\,dy'\,dz'\right\}$$
$$= \frac{1}{4\pi}\iiint\left[\frac{3(x'-x)(y'-y)}{r^5}\,\omega'_y - \left\{\frac{3(y'-y)^2}{r^5} - \frac{1}{r^3}\right\}\omega'_x\right]dx'\,dy'\,dz', \tag{31}$$

so that

$$\overline{\omega_x \frac{\partial w}{\partial y}} = \frac{1}{4\pi}\iiint\left[\frac{3(x'-x)(y'-y)}{r^5}\,\overline{\omega_x\,\omega'_y} - \left\{\frac{3(y'-y)^2}{r^5} - \frac{1}{r^3}\right\}\overline{\omega_x\,\omega'_x}\right]dx'\,dy'\,dz'. \tag{32}$$

Now we can define a correlation tensor **V** for the vorticity components in precisely the same way as **R** was defined for the velocity components. Hence, using an obvious notation,

$$\overline{\omega_x\,\omega'_y} = \overline{\omega_x^2}\,V_{xy}, \quad \overline{\omega_x\,\omega'_x} = \overline{\omega_x^2}\,V_{xx}$$

* See Lamb (1932). These results are purely kinematic.

and

$$\overline{\omega_x \frac{\partial w}{\partial y}} = \frac{\overline{\omega_x^2}}{4\pi}\iiint\left[\frac{3(x'-x)(y'-y)}{r^5}V_{xy} - \left\{\frac{3(y'-y)^2}{r^5}-\frac{1}{r^3}\right\}V_{xx}\right]dx'\,dy'\,dz'. \tag{33}$$

When the turbulence is isotropic this method of approach has no special merit since the required results may be obtained more easily by other methods. It is clear that the form for **V** is exactly similar to that for **R** (including the condition arising from continuity) in isotropic flow; using this fact the integral in equation (33) can be evaluated and the result compared with the one that can be deduced immediately from equations (27).

6—*Triple correlations**

We shall now consider the mean values of the product of three-velocity components u_i, u_j, u'_k, where u_i and u_j are the instantaneous values of the i and j components of the velocity observed at an arbitrary point P and u'_k is the k-component of the velocity observed at another arbitrary point P'. Let us consider again homogeneous isotropic turbulence. Then

$$\overline{u_i^2} = \overline{u_j^2} = \overline{u_k'^2} = \overline{u^2},$$

and we write

$$\overline{u_i u_j u'_k} = (\overline{u^2})^{\frac{3}{2}}\, T_{ijk},$$

where T_{ijk} is a tensor of third rank. The tensor T_{ijk} will be a function of ξ_1, ξ_2, ξ_3 where $\xi_1 = x'_1 - x_1$, $\xi_2 = x'_2 - x_2$, $\xi_3 = x'_3 - x_3$; we call it the tensor of the triple correlations. In order to find the expression for T_{ijk} as a function of the ξ's, let us assume first that both points P and P' lie on the x-axis. Then it is obvious that all quantities $\overline{u_i u_j u'_k}$ belong to one of the following six groups, $\overline{u_1^2 u'_1}$, $\overline{u_1^2 u'_k}$, $\overline{u_1 u_i u'_1}$, $\overline{u_1 u_j u'_k}$, $\overline{u_i u_j u'_1}$ and $\overline{u_i u_j u'_k}$, where i, j, k can be equal to 2 or 3. Due to the assumption of isotropy $\overline{u_1^2 u'_k} = 0$, $\overline{u_1 u_i u'_1} = 0$, and $\overline{u_i u_j u'_k} = 0$, because by reflexion of at least one of the axes x_2 or x_3 these expressions certainly change their sign. Furthermore, $\overline{u_1 u_j u'_k}$ and $\overline{u_i u_j u'_1}$ vanish for the same reason unless $j = k$ or $i = j$. Hence only the following mean values remain as possibly different from zero: $\overline{u_1^2 u'_1}$, $\overline{u_1 u_2 u'_2}$, $\overline{u_1 u_3 u'_3}$, $\overline{u_2^2 u'_1}$, $\overline{u_3^2 u'_1}$. Obviously because of isotropy $\overline{u_1 u_2 u'_2} = \overline{u_1 u_3 u'_3}$ and $\overline{u_2^2 u'_1} = \overline{u_3^2 u'_1}$. Hence three independent quantities remain different from zero; we put $\overline{u_1^2 u'_1} = (\overline{u^2})^{\frac{3}{2}}\, k(r)$, $\overline{u_1 u_2 u'_2} = \overline{u_1 u_3 u'_3} = (\overline{u^2})^{\frac{3}{2}}\, q(r)$ and $\overline{u_2^2 u'_1} = \overline{u_3^2 u'_1} = (\overline{u^2})^{\frac{3}{2}}\, h(r)$.

* In order to distinguish between correlations connecting two and three velocity components the correlations treated in the previous sections will be referred to in the following sections as "double correlations".

(Fig. 2 represents the double and triple correlation functions which do not vanish because of isotropy.)

It will be seen that the series development of the function $k(r)$ for small values of r starts with the term in r^3. For, developing u_1' in a series of powers of the variable $\xi_1 = x_1' - x_1 = r$, we obtain

$$\overline{u_1^2 u_1'} = \overline{u_1^3} + \overline{u_1^2 \frac{\partial u_1}{\partial \xi_1}}\, \xi_1 + \tfrac{1}{2}\, \overline{u_1^2 \frac{\partial^2 u_1}{\partial \xi_1^2}}\, \frac{\xi_1^2}{2} + \dots.$$

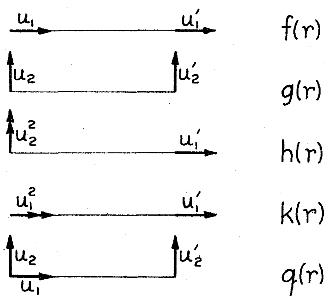

FIG. 2

Now, reflecting the x-axis all the coefficients of even powers of ξ_1 change their sign, and consequently these coefficients must vanish in the case of isotropic turbulence. Because of the homogeneity of turbulence

$$\overline{u_1^2 \frac{\partial u_1}{\partial \xi_1}} = \frac{1}{3} \frac{\partial \overline{u_1^3}}{\partial \xi_1},$$

also vanishes. Hence, the development of $\overline{u_1^2 u_1'}$ starts with the term

$$\tfrac{1}{6}\, \overline{u_1^2 \frac{\partial^3 u_1}{\partial \xi_1^3}}\, \xi_1^3$$

and correspondingly

$$k(r) = k'''(0) \frac{r^3}{6} + k^{\mathrm{v}}(0) \frac{r^5}{120} + \dots.$$

Until now it has been assumed that P and P' are situated on the x-axis. In order to obtain the general expression for $\overline{u_i u_j u_k'}$ we take $\overline{PP'}$ to be an arbitrary direction and denote by (p_1, p_2, p_3) and (p_1', p_2', p_3') the velocity components at P and P' along three mutually perpendicular lines whose direction cosines are (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3). In particular, we suppose that $l_1 = \xi_1/r$, $m_1 = \xi_2/r$, $n_1 = \xi_3/r$, so that the first line is in the direction $\overline{PP'}$.

We then obtain for the velocity components along the x, y, z axes:

$$\left.\begin{aligned} u_1 &= l_i p_i, & u_1' &= l_i p_i', \\ u_2 &= m_i p_i, & u_2' &= m_i p_i', \\ u_3 &= n_i p_i, & u_3' &= n_i p_i'. \end{aligned}\right\} \tag{34}$$

On the other hand, because of isotropy

$$\overline{p_1^2 p_1'} = (\overline{u^2})^{\frac{3}{2}}\, k(r),\ \overline{p_1 p_2 p_2'} = \overline{p_1 p_3 p_3'} = (\overline{u^2})^{\frac{3}{2}}\, q(r) \text{ and } \overline{p_2^2 p_1'} = \overline{p_3^2 p_1'} = (\overline{u^2})^{\frac{3}{2}}\, h(r),$$

all other combinations being equal to zero.

Using the expressions (34) and substituting for the mean values of the products of the p's the functions $k(r)$, $q(r)$ and $h(r)$ we find after some analysis that

$$\overline{u_i u_j u_k'} = (\overline{u^2})^{\frac{3}{2}}\, T_{ijk} = (\overline{u^2})^{\frac{3}{2}}\left\{\xi_i \xi_j \xi_k \frac{k-h-2q}{r^3} + \delta_{ij}\xi_k \frac{h}{r} + \delta_{ik}\xi_j \frac{q}{r} + \delta_{jk}\xi_i \frac{q}{r}\right\}, \tag{35}$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$).

It is seen that T_{ijk} is an odd function of the variables ξ_1, ξ_2, ξ_3. By interchanging P and P' we change ξ_1, ξ_2 and ξ_3 into $-\xi_1$, $-\xi_2$ and $-\xi_3$ respectively. Hence the mean values of triple products composed of one component measured at P and two components measured at P' can be expressed by the T_{ijk}'s. For instance

$$\overline{u_i u_j' u_k'} = -\overline{u_i' u_j u_k} = -\overline{u_k u_j u_i'} = -(\overline{u^2})^{\frac{3}{2}}\, T_{kji}.$$

This relation will be used in § 8.

We shall now show that because of the continuity relation between the velocity components the functions k and q can be expressed by h. Since u_i and u_j are independent of the variation of x_k', it follows from the continuity equation that

$$\frac{\partial}{\partial x_k'}(\overline{u_i u_j u_k'}) = 0. \tag{36}$$

Hence
$$\frac{\partial}{\partial \xi_k}(\overline{u_i u_j u_k'}) = 0 \quad \text{or} \quad \frac{\partial}{\partial \xi_k} T_{ijk} = 0. \tag{37}$$

By substituting (35) in (37) and carrying out the differentiations, we find

$$\xi_i \xi_j \left[\frac{k'-h'}{r^2} + \frac{2k-2h-6q}{r^3}\right] + \delta_{ij}\left(\frac{2h}{r} + \frac{2q}{r} + h'\right) = 0. \tag{38}$$

The symbols k' and h' denote differential quotients with respect to r. The k, h and q being only functions of r, the expressions in the brackets must vanish separately. This leads to the results

$$\left.\begin{aligned} k &= -2h, \\ q &= -h - \frac{r}{2}\frac{dh}{dr}. \end{aligned}\right\} \tag{39}$$

It was seen above that the development of $k(r)$ for small values of r starts with the term containing r^3. Because of (39) the same is true for $h(r)$ and $q(r)$.

The expression (35) and the relations (39) carry the general analysis of the triple correlations as far as we did in § 3 in the case of the double correlations.

7—*The correlation between pressure and velocity*

We shall now prove that the mean values $\overline{\varpi u'_j}$ ($j = 1$, 2 or 3), where ϖ is the pressure at (x_1, x_2, x_3), all vanish. Again, denote by (p'_1, p'_2, p'_3) the velocity components at (x'_1, x'_2, x'_3) along the lines whose direction cosines are given by (l_i, m_i, n_i) ($i = 1$, 2 or 3) and defined in the preceding paragraph. It is clear, by symmetry, that $\overline{\varpi p'_j}$ ($j = 2$ or 3) vanishes. It is, however, necessary to introduce the equation of continuity to show that $\overline{\varpi p'_1} = 0$. Then let us write

$$\overline{\varpi p'_1} = \{\overline{\varpi^2}\}^{\frac{1}{2}}\{\overline{u^2}\}^{\frac{1}{2}}\, s(r).$$

We find immediately from (34) that

$$\left.\begin{aligned} \overline{\varpi u'_1} &= \{\overline{\varpi^2}\}^{\frac{1}{2}}\{\overline{u^2}\}^{\frac{1}{2}}\,[l_1 s(r)], \\ \overline{\varpi u'_2} &= \{\overline{\varpi^2}\}^{\frac{1}{2}}\{\overline{u^2}\}^{\frac{1}{2}}\,[m_1 s(r)], \\ \overline{\varpi u'_3} &= \{\overline{\varpi^2}\}^{\frac{1}{2}}\{\overline{u^2}\}^{\frac{1}{2}}\,[n_1 s(r)]. \end{aligned}\right\} \tag{40}$$

Here again, the equation of continuity leads to the condition

$$\frac{\partial}{\partial \xi_i}(\overline{\varpi u'_i}) = 0$$

for all values of ξ_1, ξ_2, ξ_3. Transforming to spherical polar co-ordinates we see that

$$\frac{\partial}{\partial r}[r^2 \sin\theta\, s(r)] = 0$$

for all values of r and θ, so that

$$\frac{d}{dr}s(r) + \frac{2s(r)}{r} = 0.$$

Again, since s is regular at the origin, the appropriate solution is

$$s = 0.$$

Hence, $\overline{\varpi u_j'} = 0$ for $j = 1$, 2 or 3. Similarly, $\overline{\varpi' u_j} = 0$ for $j = 1$, 2 or 3.

B—Dynamics of isotropic turbulence

8—*The equation for the propagation of the correlation*

The equations of motion at the point $P(x_1, x_2, x_3)$ can be written

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \varpi}{\partial x_i} + \nu \nabla_x^2 u_i \tag{41}$$

for $i = 1$, 2 or 3, where $\nabla_x^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ and ϖ denotes the pressure. Let us multiply equation (41) by the k-component u_k' of the velocity at $P'(x_1', x_2', x_3')$, so that

$$u_k' \frac{\partial u_i}{\partial t} + u_k' u_j \frac{\partial u_i}{\partial x_j} = \frac{-u_k'}{\rho}\frac{\partial \varpi}{\partial x_i} + \nu u_k' \nabla_x^2 u_i. \tag{42}$$

Consider now the expression $u_k' u_j \frac{\partial u_i}{\partial x_j}$. It will be seen that, in virtue of the equation of continuity,

$$u_k' u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i u_j u_k'). \tag{43}$$

Now $\overline{u_i u_j u_k'}$ is a function of ξ_1, ξ_2 and ξ_3 and obviously

$$\frac{\partial}{\partial x_j}(\overline{u_i u_j u_k'}) = -\frac{\partial}{\partial \xi_j}(\overline{u_i u_j u_k'}). \tag{44}$$

Furthermore,
$$\frac{-u_k'}{\rho}\frac{\partial \varpi}{\partial x_i} = \frac{1}{\rho}\frac{\partial}{\partial \xi_i}(\varpi u_k'), \tag{45}$$

and since $\overline{\varpi u_k'}$ vanishes everywhere, $\overline{\frac{-u_k'}{\rho}\frac{\partial \varpi}{\partial x_i}}$ also vanishes. Finally,

$$\nabla_x^2(\overline{u_i u_k'}) = \nabla^2(\overline{u_i u_k'}),$$

where
$$\nabla^2 = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2}.$$

Hence we obtain the equation

$$\overline{u_k' \frac{\partial u_i}{\partial t}} - \frac{\partial}{\partial \xi_j}(\overline{u_i u_j u_k'}) = \nu \nabla^2(\overline{u_i u_k'}). \tag{46}$$

In an analogous way we obtain

$$\overline{u_i \frac{\partial u'_k}{\partial t}} + \frac{\partial}{\partial \xi_j}(\overline{u_i u'_j u'_k}) = \nu \nabla^2 (\overline{u_i u'_k}). \tag{47}$$

We have shown in § 7 that

$$\overline{u_i u'_j u'_k} = -(\overline{u'_i u_j u_k}), \tag{48}$$

and substituting (48) in equation (47) we obtain

$$\overline{u_i \frac{\partial u'_k}{\partial t}} - \frac{\partial}{\partial \xi_j}(\overline{u_j u_k u'_i}) = \nu \nabla^2 (\overline{u_i u'_k}). \tag{49}$$

Adding the equations (46) and (49) and remembering that $\overline{u_i u'_k} = \overline{u^2} R_{ik}$ and $\overline{u_i u_j u'_k} = (\overline{u^2})^{\frac{3}{2}} T_{ijk}$, we obtain finally

$$\frac{\partial}{\partial t}(\overline{u^2} R_{ik}) - (\overline{u^2})^{\frac{3}{2}} \frac{\partial}{\partial \xi_j}(T_{ijk} + T_{kji}) = 2\nu \overline{u^2} \nabla^2 R_{ik}. \tag{50}$$

In view of the form given for the tensors R_{ik} and T_{ijk} in equations (3) and (35) respectively, equation (50) may clearly be reduced to a differential equation connecting the functions f, g, k, q, h. By using the equation of continuity in the form given by equations (7) and (39) we can eliminate g, k and q and obtain a partial differential equation connecting f and h. The analysis can be carried out by choosing any component of R_{ik}. Then equation (50) contains terms multiplied by $\xi_i \xi_k$ and terms which are functions of r alone. Equating the terms containing $\xi_i \xi_k$ we obtain a relation for $\partial/\partial t\,(f-g)$ and equating the terms which are functions of r a relation for $\partial g/\partial t$. Eliminating $\partial g/\partial t$ the following equation for f results

$$\frac{\partial (f\overline{u^2})}{\partial t} + 2(\overline{u^2})^{\frac{3}{2}}\left(\frac{\partial h}{\partial r} + \frac{4}{r}h\right) = 2\nu \overline{u^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right). \tag{51}$$

We call (51) the fundamental equation for the propagation of the correlation function $f(r)$.

9—*The decay of turbulence*

Before proceeding with the discussion of the solutions of equation (51) we shall show how Taylor's equation for the decay of energy and the equation for the decay of vorticity already given by the senior author (Kármán 1937*a*) can be obtained as deductions from equation (51).

$$\text{For } r = 0, f = 1 \text{ and } \frac{dh}{dr} + \frac{4}{r}h = 0 \quad \text{(cf. § 6)}.$$

Hence we obtain from (51)

$$\frac{\partial \overline{u^2}}{\partial t} = 2\nu \overline{u^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right)_{r=0}, \tag{52}$$

or with

$$f = 1 + \tfrac{1}{2} f_0'' r^2 + \tfrac{1}{24} f_0^{\mathrm{iv}} r^4 + \ldots, \tag{53}$$

$$\frac{d\overline{u^2}}{dt} = 10\nu f_0'' \overline{u^2}. \tag{54}$$

Now Taylor's definition of the length λ is given by the equation

$$\lambda^2 = -2/g_0''. \tag{54a}$$

We have shown (equation 10) that $f'' = \frac{1}{2} g_0''$. Hence

$$\frac{d\overline{u^2}}{dt} = -10\nu \frac{\overline{u^2}}{\lambda^2}. \tag{55}$$

Or if, considering Taylor's problem of the decay of turbulence behind a honeycomb, we substitute $\dfrac{d}{dt} = \dfrac{1}{U}\dfrac{d}{dx}$,

$$\frac{1}{U}\frac{d}{dx}\overline{u^2} = -10\nu \frac{\overline{u^2}}{\lambda^2}, \tag{56}$$

which is Taylor's equation for the decrease of the mean kinetic energy of turbulence.

Now we shall substitute the expansions given by equation (53) in equation (51) and equate the coefficients of the terms containing r^2. Concerning the triple correlation function $h(r)$ we notice that

$$h(r) = \tfrac{1}{6} h_0''' r^3 + \text{higher terms}. \tag{57}$$

Hence we find

$$\frac{d}{dt}[\tfrac{1}{2} f_0'' \overline{u^2}] + \tfrac{7}{3} h_0''' (\overline{u^2})^{\frac{3}{2}} = \tfrac{7}{3} \nu f_0^{\mathrm{iv}} \overline{u^2}. \tag{58}$$

Obviously $f_0'' \overline{u^2} = -\overline{u^2}/\lambda^2$. On the other hand, we find the mean square of the vorticity components using the formulae (19), (20) and (21)

$$\overline{\omega_1^2} = \overline{\omega_2^2} = \overline{\omega_3^2} = 5\overline{u^2}/\lambda.$$

Hence equation (58) can be interpreted as the equation for the decay of vorticity. Let us denote $\overline{\omega_1^2} + \overline{\omega_2^2} + \overline{\omega_3^2}$ by $\overline{\omega^2}$ then $\overline{\omega^2} = 15\overline{u^2}/\lambda^2$ and from (58) follows

$$\frac{d\overline{\omega^2}}{dt} - 70 h_0''' (\overline{u^2})^{\frac{3}{2}} = -\frac{14}{3}\nu \overline{\omega^2} \lambda^2 f_0^{\mathrm{iv}}.$$

$\lambda^2 f_0^{\mathrm{iv}}$ is the reciprocal value of the square of a length; in order to obtain conformity with the energy equation (55) we put $\frac{1}{\lambda_\omega^2} = \frac{7}{15}\lambda^2 f_0^{\mathrm{iv}}$. Then

$$\frac{d\overline{\omega^2}}{dt} - 70h_0'''(\overline{u^2})^{\frac{3}{2}} = -10\nu\frac{\overline{\omega^2}}{\lambda_\omega^2}. \tag{59}$$

From the equations of motion the senior author obtained the equation* (Kármán 1937*a*)

$$\frac{d\overline{\omega^2}}{dt} - 2\overline{\omega_i\,\omega_k\frac{\partial u_i}{\partial x_k}} = -10\nu\frac{\overline{\omega^2}}{\lambda_\omega^2}. \tag{60}$$

Equations (59) and (60) are identical. Because of the continuity equation

$$-2\,\overline{\omega_i\,\omega_k\frac{\partial u_i}{\partial x_k}} = \overline{\left(\frac{\partial^2(u_ju_i)}{\partial x_j\partial x_k} - \frac{\partial^2(u_ju_k)}{\partial x_j\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)}.$$

Now the mean values, which constitute the expression on the right side, can be calculated by differentiation of the equation (35). For instance,

$$\overline{\frac{\partial^2(u_iu_j)}{\partial x_j\partial x_k}\frac{\partial u_i}{\partial x_k}} = \left[\frac{\partial^3\overline{(u_iu_ju_i')}}{\partial\xi_j\partial\xi_k\partial\xi_k}\right]_{\xi_i=\xi_j=\xi_k=0},$$

and so on.

Carrying out the computations the identity of equations (59) and (60) can be shown without difficulty.

It is seen that the equation for the decay of turbulence and for the decay of vorticity follow from the general equation for the spread of correlations by development of the correlation functions in powers of the distance r. The expression for the decay of vorticity contains two terms: one corresponds to the change of vorticity by deformation of the vortex tubes, the other corresponds to the action of viscosity.

* As the senior author found the equation (60) he realized that the condition of isotropy alone does not lead to a further reduction of the equation. However, he thought that in a random isotropic turbulence the expression containing the triple correlations should vanish because the extension and the contraction of portions of the vortex tubes should be equally probable in an ideal fluid. As the authors carried out their first general analysis of correlations, it seemed that a general proof could be found for the vanishing of the triple correlations. Correspondingly, the senior author stated in his papers on the subject (1937 *a*, *b*) that the triple correlations and the term in question in equation (60) vanish because of isotropy. Closer analysis showed that this statement is erroneous. To make the triple correlations vanish, some further physical assumption on the random character of turbulence is necessary.

10—*Self-preserving correlation functions*

We write equation (51) in the form

$$\frac{\partial(f\overline{u^2})}{\partial t} = 2\nu\overline{u^2}\left(\frac{\partial}{\partial r}+\frac{4}{r}\right)\left[\frac{\partial f}{\partial r}-\frac{\sqrt{\overline{u^2}}}{\nu}h\right]. \tag{61}$$

Obviously, the influence of the triple correlations depend on the relative magnitude of the quantities $\frac{\partial f}{\partial r}$ and $\frac{\sqrt{\overline{u^2}}}{\nu}h$. Let us consider first the case of "small Reynolds number" of the turbulence, i.e. let us assume that $\frac{\partial f}{\partial r} \gg \frac{\sqrt{\overline{u^2}}h}{\nu}$. Then neglecting the triple correlation function h, we write

$$\frac{\partial(f\overline{u^2})}{\partial t} = 2\nu\overline{u^2}\left(\frac{\partial^2 f}{\partial r^2}+\frac{4}{r}\frac{\partial f}{\partial r}\right), \tag{62}$$

or eliminating $\overline{u^2}$ by use of equation (54)

$$\frac{\partial f}{\partial t} = 2\nu\left(\frac{\partial^2 f}{\partial r^2}+\frac{4}{r}\frac{\partial f}{\partial r}-5f''_0 f\right). \tag{63}$$

We notice that in this case $f(r,t)$ is determined by the values of $f(r,t_0)$, i.e. if the correlation function f is given for $t=t_0$ it is given for any arbitrary time $t>t_0$.

We shall consider a particular set of solutions of equation (63). We notice that this equation reduces to an ordinary differential equation if we suppose f is a function of $\chi = \frac{r}{\sqrt{(\nu t)}}$ only. When this substitution is made, we obtain the equation

$$f''+\left(\frac{4}{\chi}+\frac{\chi}{4}\right)f'-5f''(0)f = 0, \tag{64}$$

where dashes denote differentiations with regard to χ; we shall speak of the correlation functions given by the solution of equation (64) as "self-preserving", since the form of these curves is the same at all instants although the actual length scale varies. We shall now denote by α the arbitrary constant $-f''(0)$.

First of all it may be pointed out that equation (64) is related to the confluent hypergeometric equation. In fact, the solution we require (i.e. the one which satisfies $f = 1$ and $f' = 0$ when $\chi = 0$) is equal to

$$f(\chi) = 2^{15/4}\chi^{-5/2}e^{-\chi^2/16}M_{(10\alpha-\frac{5}{4}),\frac{3}{4}}\left(\frac{\chi^2}{8}\right), \tag{65}$$

where $M_{k,m}(z)$ is the same solution of the confluent hypergeometric equation as that defined by Whittaker and Watson (1927, p. 337, para. 16.1) and denoted by this symbol.

When $\alpha < \frac{1}{4}$ it will be seen, after some reduction, that (Whittaker and Watson 1927, p. 352, example 1)

$$f(\chi) = \frac{\Gamma(5/2)}{\Gamma(10\alpha)\,\Gamma(\frac{5}{2}-10\alpha)}\int_0^1 \tau^{(10\alpha-1)}\,(1-\tau)^{(\frac{3}{2}-10\alpha)}\,e^{-\frac{\chi^2}{8}\tau}\,d\tau. \tag{66}$$

When $\alpha = \frac{1}{4}$ we obtain the solution

$$f(\chi) = e^{-\chi^2/8}. \tag{67}$$

When $\alpha > \frac{1}{4}$ the solution is given by the integral of the same integrand round a particular contour (Whittaker and Watson, 1927, p. 257).

Let us now consider the decay of turbulence when the correlation function is one of these special types. Returning to equation (54) we see that

$$\frac{\partial \overline{u^2}}{\partial t} = -10\frac{\overline{u^2}}{t}\alpha, \tag{68}$$

and integrating we find

$$\frac{1}{\sqrt{\overline{u^2}}} = \frac{1}{\sqrt{\overline{u_0^2}}}\left(\frac{t}{t_0}\right)^{5\alpha}, \tag{69}$$

where the condition $\overline{u^2} = \overline{u_0^2}$ when $t = t_0$ has been applied.

Assuming, for the moment, that the turbulence produced by a particular grid or honeycomb is of the type which has a self-preserving correlation function and starting from some point outside the wind shadow, where the fluid passes at the time t_0, we write $t = t_0 + x/U$ (U denotes the velocity of the main flow and x is the distance measured downstream). Then we may write equation (69) in the form

$$\frac{1}{\sqrt{\overline{u^2}}} = \frac{1}{\sqrt{\overline{u_0^2}}}\left(1+\frac{x}{Ut_0}\right)^{5\alpha}, \tag{70}$$

where $\overline{u^2} = \overline{u_0^2}$, when $x = 0$.

The quantity Ut_0 is a length which we can relate to the value λ_0 of the length λ (defined in equation (54a) above) at the origin from which x is measured. For

$$\left(\frac{\partial^2 f}{\partial r^2}\right)_{r=0} = \frac{1}{\nu t}f''(0) = -\frac{\alpha}{\nu t},$$

so that, with $\left(\dfrac{\partial^2 f}{\partial r^2}\right)_{r=0} = -1/\lambda^2$

$$\lambda^2 = \frac{\nu t}{\alpha}, \tag{71}$$

and therefore
$$t_0 = \frac{\alpha\lambda_0^2}{\nu}.$$

Finally, from equation (70),

$$\frac{U}{\sqrt{\overline{u^2}}} = \frac{U}{\sqrt{\overline{u_0^2}}}\left(1+\frac{x\nu}{\alpha\lambda_0^2 U}\right)^{5\alpha}, \tag{72}$$

where $\overline{u^2} = \overline{u_0^2}$, $\lambda = \lambda_0$ when $x = 0$.

The results given by equations (71) and (72) would be formally equivalent with Taylor's results, when $\alpha = \frac{1}{5}$. However, the corresponding correlation function f which is given in fig. 3 as calculated by numerical integration from the integral obtained by integrating (66) by parts is very

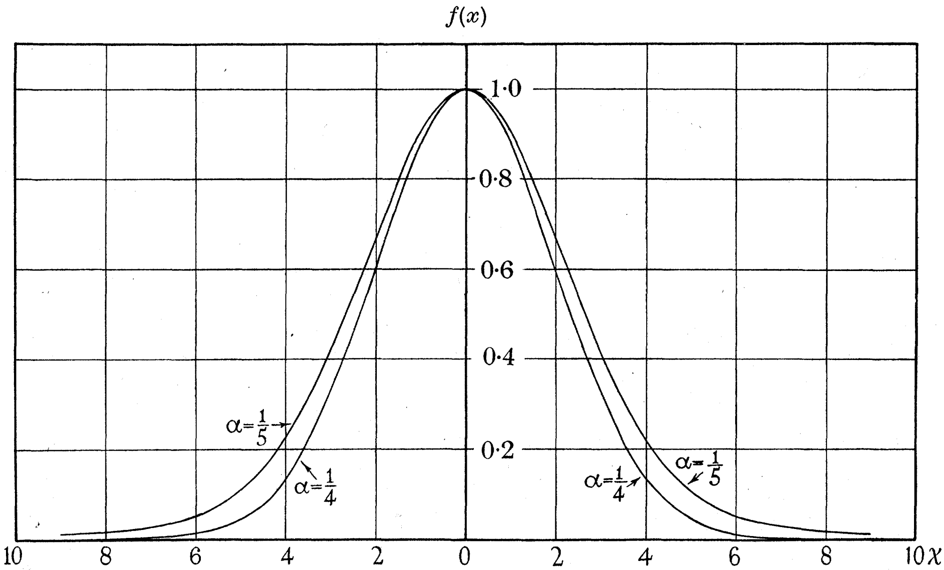

FIG. 3

different from those measured by Simmons in such cases, in which Taylor's linear law of the decay of turbulence apparently holds. Hence the coincidence between these equations and Taylor's result is rather formal. We come back to this question in the last section of this paper. The correlation function f corresponding to $\alpha = \frac{1}{4}$ (see equation (67)) is also included in fig. 3.

11—*Possible solution for large Reynolds numbers**

It was seen that if the triple correlations are neglected the double correlation function $f(r)$ is determined for all times $t > t_0$ provided $f(r)$ is known for $t = t_0$. In the general case $\partial f/\partial t$ depends also on the triple correlation function $h(r)$. Now a further equation can be obtained from the equations of motion for $\partial h/\partial t$; however, this new equation contains terms with the quadruple correlations and so on. Hence, for an arbitrary value of the Reynolds number the problem is too complicated for analytical treatment.

However, certain interesting results can be obtained for the limiting case of very large Reynolds numbers under the assumption that the correlation functions $f(r, t)$ and $h(r, t)$ are independent of viscosity with the exception of small values of the distance r. Let us consider the problem of turbulence created by passing a uniform windstream through a grid or similar turbulence-producing device and let us assume geometrically similar arrangements. Then, denoting the mean velocity of the windstream by U and a characteristic linear dimension of the device mentioned by M, it seems evident by dimensional considerations that f has the form

$$f\left(\frac{r}{\sqrt{(\nu t)}}, \frac{r}{M}, \frac{Ut}{M}\right).$$

If the correlation functions are—according to the above assumption—independent of viscosity, this means that $f(r, t)$ does not depend on the variable $\chi = \dfrac{r}{\sqrt{(\nu t)}}$ and is only a function of the variables r/M and Ut/M. Let us now assume in addition that the correlation functions preserve their shape and only their scale is changing. Obviously this assumption amounts to the statement that both $f(r, t)$ and $h(r, t)$ are functions of one variable $\psi = r/L$ only, where L is a function of M and Ut.

It must be noticed that both assumptions—namely that the correlation functions are independent of χ and that they preserve their shape—shall be made only for large values of $\chi = \dfrac{r}{\sqrt{(\nu t)}}$.

Then introducing $f(\psi)$ and $h(\psi)$ in equation (51), we obtain the following expression:

$$-\frac{df}{d\psi}\psi\frac{\overline{u^2}}{L}\frac{dL}{dt}+f\frac{d\overline{u^2}}{dt}+2\frac{(\overline{u^2})^{\frac{3}{2}}}{L}\left(\frac{dh}{d\psi}+\frac{4h}{\psi}\right)=2\nu\frac{\overline{u^2}}{L^2}\left(\frac{d^2f}{d\psi^2}+\frac{4}{\psi}\frac{df}{d\psi}\right). \tag{73}$$

* This section was inserted by the senior author in Sept. 1937, especially after reading G. I. Taylor's contribution (1937).

Let us consider the quantity $\dfrac{\sqrt{\overline{u^2}}L}{\nu}$ as the Reynolds number related to the problem, then for large values of this number it appears justified to neglect the term on the right side. Furthermore, according to (55)

$$\frac{d\overline{u^2}}{dt} = -10\nu\frac{\overline{u^2}}{\lambda^2}.$$

Hence, from (73) it follows that

$$-\frac{df}{d\psi}\psi\frac{\overline{u^2}}{L}\frac{dL}{dt}-f\frac{10\nu\overline{u^2}}{\lambda^2}+\left(\frac{dh}{d\psi}+\frac{4h}{\psi}\right)2\frac{(\overline{u^2})^{\frac{3}{2}}}{L} = 0. \tag{74}$$

Obviously equation (74) can be satisfied only when the coefficients which are functions of t without being functions of ψ, are proportional, i.e. their ratios are constants. Hence

$$\frac{\sqrt{\overline{u^2}}\lambda^2}{L\nu} = A, \tag{75}$$

$$\frac{dL}{dt} = B\sqrt{\overline{u^2}}, \tag{76}$$

where A and B are numerical constants.

We easily obtain from (75) and (76) a differential equation for $L(t)$. We substitute in (75) from (55) $\dfrac{\lambda^2}{\nu} = -10\overline{u^2}\Big/\dfrac{d}{dt}\overline{u^2}$ and eliminate $\overline{u^2}$ from (75) and (76). Thus we obtain

$$L\frac{d^2L}{dt^2} = -\frac{5}{AB}\left(\frac{dL}{dt}\right)^2. \tag{77}$$

The general solution of (77) is

$$L = L_0\left(1+\frac{t}{t_0}\right)^{\frac{5}{5+AB}}, \tag{78}$$

where L_0 and t_0 are arbitrary constants. The origin of the time t being arbitrary, we may write

$$L = L_0\left(\frac{t}{t_0}\right)^{\frac{5}{5+AB}} \tag{79}$$

Introducing the expression (79) in (76) we obtain by differentiation

$$\sqrt{\overline{u^2}} = \frac{5}{5+AB}L_0\left(\frac{t}{t_0}\right)^{\frac{5}{5+AB}}\frac{1}{t}, \tag{80}$$

or denoting the value of $\sqrt{\overline{u^2}}$ at $t = t_0$ by $\sqrt{\overline{u_0^2}}$

$$\sqrt{\overline{u^2}} = \sqrt{\overline{u_0^2}}\left(\frac{t}{t_0}\right)^{\frac{-AB}{5+AB}} \tag{81}$$

Finally from (55)
$$\lambda^2 = \left(5+\frac{25}{AB}\right)\nu t. \tag{82}$$

It is seen that equations (81) and (82) with $\dfrac{1}{\alpha} = 5+\dfrac{25}{AB}$ are identical with the corresponding equations (69) and (71) obtained in § 10 for the case of the small Reynolds number.

There is one case which is not included in equation (77), namely when $L =$ const. say $L = L_0$. Then from (75) with $\dfrac{\lambda^2}{\nu} = -5\sqrt{\overline{u^2}}\Big/\dfrac{d}{dt}\sqrt{\overline{u^2}}$ it follows that

$$\frac{1}{\sqrt{\overline{u^2}}} = -\frac{5}{AL_0}t \tag{83}$$

and
$$\lambda^2 = 5\nu t. \tag{84}$$

Let us compare the results obtained with the researches of Taylor and Dryden. Our last equations (83) and (84) are identical with Taylor's results; especially if we assume with Taylor that L_0 is proportional to M, which is in this case evident from dimensional consideration. However, Taylor does not have the solutions (81) and (82). The reason is that Taylor found the equation (75) with a remarkable vision for the relations between the quantities involved; however, instead of (76) he assumed L proportional to M, i.e. a fixed ratio between the scale of turbulence and the linear size of the turbulence-producing device.

In both cases—using Taylor's assumption or our broader relations (75) and (76)—the theory leads to the conclusion that the scale and distribution of turbulence in a windstream is independent of the speed of the windstream and so is the ratio $U/\sqrt{\overline{u^2}}$ measured at a certain distance x from the grid. The difference is that according to our theory the scale of turbulence may increase downstream, while according to Taylor's assumption, it remains unchanged. Instead of $L = \text{const.} \times M$, we obtain in general $L = M \times$ function of Ut/M, and in the case of "self-preserving" turbulence the function involved is proportional to a certain power of Ut/M. It appears that further experimental results will decide whether Taylor's assumptions are not too narrow and whether the equations (79) and (80) correspond more closely to the experimental facts.

It is interesting that as far as the laws for the decay of turbulence and for the spread of correlation curves are concerned, our analysis of correlations leads essentially to the same results as the recently published theory of Dryden's (1937) which is based on entirely different conceptions. It appears that the next step in the development of the theory should be to find the physical mechanism which is behind the mathematical relations (75) and (76), especially the mechanism which tends to increase the scale of turbulence without the action of the viscosity.

References

Dryden, H. L. 1937 *J. Aero. Sci.* **4**, 273.
Kármán, T. de 1937 *a* *Proc. Nat. Acad. Sci., Wash.*, **23**, 98.
— 1937 *b* *J. Aero. Sci.* **4**, 131.
Lamb 1932 "Hydrodynamics", 6th ed. pp. 208–9.
Taylor, G. I. 1935 *Proc. Roy. Soc.* A, **151**, 421–78.
— 1937 *J. Aero. Sci.* **4**, 311.
Whittaker, E. T. and Watson 1927 "Modern Analysis", 4th ed. p. 337.