MATH 2568

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homework #3

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Contents

3.1: exercise 1

Let

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1}$$

(1) Compute Ax

Solution:

(1)

$$Ax = \left(\begin{array}{cc} 2 & 1 \\ -1 & 4 \end{array} \right) \left(\begin{array}{c} 3 \\ -2 \end{array} \right) = \left(\begin{array}{c} 6-2 \\ -3-8 \end{array} \right) = \boxed{\left(\begin{array}{c} 4 \\ -11 \end{array} \right)}$$

3.1: exercise 4

In exercise 4 decide whether or not the matrix vector product Ax can be computed; if it can, compute the product.

(4)

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \tag{2}$$

Solution:

(4) The matrix-vector product Ax $\boxed{cannot}$ be computed. This is because the number of columns in the matrix A (2) does not match the number of elements in the vector x (3). For matrix-vector multiplication to be defined, these dimensions must be equal.

3.2: exercise 9

(9) Find a 2×2 matrix that reflects vectors in the (x, y) plane across the line $x = y$.

Solution:

(9) Consider a map, denoted by L_A, that reflects vectors across the line $y = x$. We can represent this transformation as $(x, y) \mapsto (y, x)$, where (x, y) is the original vector and (y, x) is its reflected counterpart.

This reflection can be conveniently described using a linear transformation matrix, denoted by A. In this case, the appropriate matrix is:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

3.2: exercise 10

Suppose the mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ is linear and satisfies the following equations:

$$L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

(10) What is the 2×3 matrix A such that $L = L_A$?

Solution:

(10) Because $(0, 1, 0) = (0, 1, 1) - (0, 0, 1)$, the second column must be:

$$L\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} - \begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}3\\-4\end{pmatrix}$$

$$A = \boxed{\begin{pmatrix}1 & 3 & -1\\2 & -4 & 4\end{pmatrix}}$$

3.3: exercise 10

Determine which of the following maps are linear maps. If the map is linear give the matrix associated to the linear map. Explain your reasoning.

1. $L_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ maps $\begin{pmatrix}x\\y\end{pmatrix}$ to $\begin{pmatrix}x+y+3\\2y+1\end{pmatrix}$.
2. $L_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ maps $\begin{pmatrix}x\\y\end{pmatrix}$ to $\begin{pmatrix}\sin x\\x+y\\2y\end{pmatrix}$.
3. $L_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ maps $\begin{pmatrix}x\\y\end{pmatrix}$ to $x + y$.

Solution:

1. A linear map sends the origin to the origin. However, $L_1\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$, so L_1 is not linear.
2. Linear maps L satisfy $L(cX) = cL(X)$. In this case,

$$cL_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}c\sin x\\cx+cy\\2cy\end{pmatrix} \quad \text{and} \quad L_2\left(\begin{pmatrix}cx\\cy\end{pmatrix}\right) = \begin{pmatrix}\sin(cx)\\cx+cy\\2cy\end{pmatrix}$$

   Since $\sin(cx) \neq c\sin(x)$, L_2 is not linear.

3. All matrix mappings are linear. We can write

$$L_3\begin{pmatrix}x\\y\end{pmatrix} = x + y = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

   Therefore, L_3 is linear with the 1×2 matrix $A = \begin{pmatrix}1 & 1\end{pmatrix}$.

3.4: exercise 2

Write all solutions to the homogeneous system of linear equations as the general superposition of three vectors.

(2)

$$\begin{aligned} x_1 + 2x_2 + x_4 - x_5 &= 0\\ x_3 - 2x_4 + x_5 &= 0 \end{aligned}$$

Solution:

(2) Given the vectors:

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

We want to write the matrix of the corresponding homogeneous system and find its solutions.

The homogeneous system consists of equations where each vector $\mathbf{v}_i$ forms a separate row. Therefore, the matrix of the system is:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

We cannot simplify this matrix further using row reduction. Therefore, any solution can be written as a linear combination of the original vectors:

$$\mathbf{x} = x_2\mathbf{v}_2 + x_4\mathbf{v}_4 + x_5\mathbf{v}_5 = \begin{pmatrix} x_5 - x_4 - 2x_2 \\ x_2 \\ -x_5 + 2x_4 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

This expresses the solution space as the span of three linearly independent vectors, confirming that the system has infinitely many solutions.

3.4: exercise 8(a)

Let A be a 3×3 matrix. Suppose

$$A \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Find a solution to the inhomogeneous system

$$Ax = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Solution:

(a)

$$A\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}$$
$$A\begin{pmatrix}0\\4\\0\end{pmatrix}=\begin{pmatrix}-2\\0\\1\end{pmatrix}$$
$$A\begin{pmatrix}-1&0\\2&4\\1&0\end{pmatrix}=\begin{pmatrix}3&-2\\1&0\\1&1\end{pmatrix}$$
$$\mathbf{x}=t\begin{pmatrix}-1\\2\\1\end{pmatrix}+s\begin{pmatrix}0\\4\\0\end{pmatrix}=\begin{pmatrix}-t\\2t+4s\\t\end{pmatrix}$$
$$A\begin{pmatrix}-t\\2t+4s\\t\end{pmatrix}=\begin{pmatrix}3&-2\\1&0\\1&1\end{pmatrix}$$
$$\Rightarrow\begin{pmatrix}-at & 2at+4as & at\end{pmatrix}=\begin{pmatrix}3&-2\\1&0\\1&1\end{pmatrix}$$
$$\Rightarrow t=1,\ s=-\frac{1}{2}$$
$$\mathbf{x}=\begin{pmatrix}-1\\2+4\cdot(-\frac{1}{2})\\1\end{pmatrix}$$
$$=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Therefore, $\mathbf{x}=\boxed{\begin{pmatrix}-1\\0\\1\end{pmatrix}}$ is a solution.

3.5: exercise 3

In this exercise determine whether or not the matrix products AB or BA can be computed for each given pair of matrices A and B. If the product is possible, perform the computation.

(3) $A=\begin{pmatrix}8&0&2&3\\-3&0&-10&3\end{pmatrix}$ and $B=\begin{pmatrix}0&2&5\\-1&3&-1\\0&1&-5\end{pmatrix}$.

Solution:

(3) The number of columns in A (4) is not equal to the number of rows in B (3), thus AB is not defined.
The number of columns in B (3) is not equal to the number of rows in A (2), thus BA is also not defined.

Therefore, neither AB nor BA can be computed.

3.6: exercise 6 (MATLAB)

In exercise 6 use MATLAB to verify that $(A+B)C=AC+BC$ for the given matrices.

(6)
$$A = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ 1 & 5 \end{pmatrix}$$

Solution:

(6)

Listing 1: MATLAB code

```
A = [0 2; 2 1];
B = [-2 1; 3 0];
C = [2 -1; 1 5];

leftSide = (A + B) * C

rightSide = A * C + B * C



>>>3.6ex6

leftSide =

    -1    17
    11     0


rightSide =

    -1    17
    11     0
```

3.6: exercise 10 (MATLAB)

Experimentally, find two symmetric 2×2 matrices A and B for which the matrix product AB is *not* symmetric.
Solution:

Listing 2: MATLAB code

```
A = randi(10, 2);
B = A';


C = A * B;
isSymmetric = isequal(C, C');
while issymmetric
    A = randi(10, 2);
    B = A'
    C = A * B;
    isSymmetric = isequal(C, C');
end
fprintf('Non-symmetric product:\n');
A
B
C
```

```
>> 3.6ex10
Non-symmetric product C:

A =

     2     2
     3    10


B =

     2     3
     2    10


C =

     8    26
    26   109
```