MATH 2568

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homework #9

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Contents

10.1: exercise 1

Find an orthonormal basis for the solutions to the linear equation

$$2x_1 - x_2 + x_3 = 0.$$

Solution:

An orthonormal basis is a set of vectors that are orthogonal to each other, and each of unit length.

To solve this, we first find any vector that satisfies the equation. Consider the vector $(1, 1, -1)$. We can check that this vector is indeed a solution:

$$2(1) - (1) + (-1) = 2 - 1 - 1 = 0.$$

Thus, $(1, 1, -1)$ is a solution. Next, we normalize this vector to obtain a unit vector. The length of $(1, 1, -1)$ is calculated as

$$\sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}.$$

Dividing each component of $(1, 1, -1)$ by $\sqrt{3}$ gives the unit vector

$$w_1 = \frac{1}{\sqrt{3}}(1, 1, -1).$$

Next, we need to find another vector that is orthogonal to w_1 and also a solution to the equation. By inspection or systematic search, we consider the vector $(0, 1, 1)$. This vector also satisfies the given linear equation:

$$2(0) - (1) + (1) = 0 - 1 + 1 = 0.$$

To verify orthogonality with w_1, we compute the dot product:

$$w_1 \cdot (0, 1, 1) = \frac{1}{\sqrt{3}}(1 \cdot 0 + 1 \cdot 1 + (-1) \cdot 1) = \frac{1}{\sqrt{3}}(0 + 1 - 1) = 0.$$

Since the dot product is zero, $(0, 1, 1)$ is orthogonal to w_1. Normalizing this vector gives

$$||(0, 1, 1)|| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2},$$

and thus

$$w_2 = \frac{1}{\sqrt{2}}(0, 1, 1).$$

10.1: exercise 10

Prove that the rows of an $n \times n$ orthogonal matrix form an orthonormal basis for $\mathbb{R}^n$.

Solution: Consider an $n \times n$ orthogonal matrix, denoted by A. By definition, an orthogonal matrix has the property that its columns form an orthonormal basis for $\mathbb{R}^n$. Our goal is to prove that the rows of A also constitute an orthonormal basis for $\mathbb{R}^n$.

According to properties of orthogonal matrices, the transpose of A, denoted as A^T, is equal to its inverse, A^{-1}. This is a key property of orthogonal matrices and can be stated as:

$$A^T = A^{-1}.$$

Using this property, we can demonstrate that multiplying A by its transpose results in the identity matrix:

$$I_n = AA^{-1} = AA^T = (A^T)^T(A^T).$$

The equality $AA^T = I_n$ confirms that A^T itself is an orthogonal matrix because the product of an orthogonal matrix and its transpose results in the identity matrix. This implies that the columns of A^T must be orthonormal.

Since the columns of A^T are indeed the rows of A, this concludes that the rows of A form an orthonormal basis for $\mathbb{R}^n$, just as the columns do. Therefore, the rows of an orthogonal matrix A are confirmed to be an orthonormal basis for $\mathbb{R}^n$, completing the proof.

10.1: exercise 11

Show that if P and Q are $n \times n$ orthogonal matrices, then PQ is an $n \times n$ orthogonal matrix.

Solution:

To prove that the product of two $n \times n$ orthogonal matrices P and Q is also an orthogonal matrix, we start by recalling the definition of an orthogonal matrix. An $n \times n$ matrix P is orthogonal if it satisfies the condition:

$$P^T P = I_n,$$

where P^T is the transpose of P, and I_n is the $n \times n$ identity matrix. This property implies that the transpose of the matrix is also its inverse, i.e., $P^T = P^{-1}$.

Applying this definition to the matrices P and Q, which are both orthogonal, we proceed by examining the transpose of the product PQ:

$$(PQ)^T = Q^T P^T.$$

Given that P and Q are orthogonal, we know from their definitions that $P^T = P^{-1}$ and $Q^T = Q^{-1}$. Substituting these into the equation, we get:

$$(PQ)^T = Q^T P^T = Q^{-1} P^{-1}.$$

The property of inverses for a product of two matrices states that the inverse of a product is the product of the inverses in reverse order. Therefore, the expression $Q^{-1}P^{-1}$ can be rewritten as:

$$Q^{-1} P^{-1} = (PQ)^{-1}.$$

Thus, we have shown that:

$$(PQ)^T = (PQ)^{-1}.$$

This equality indicates that the product PQ satisfies the definition of an orthogonal matrix, because its transpose is equal to its inverse. Hence, PQ is indeed an orthogonal matrix, confirming our assertion.

10.2: exercise 2

Find an orthonormal basis of the plane $W \subset \mathbb{R}^3$ spanned by the vectors $w_1 = (1, 2, 3)$ and $w_2 = (2, 5, -1)$ by applying the Gram-Schmidt orthonormalization process.

Solution:

Apply the Gram-Schmidt process to these vectors.

First, we compute the norm of w_1:

$$||w_1|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

We then normalize w_1 to find v_1:

$$v_1 = \frac{1}{||w_1||} w_1 = \frac{1}{\sqrt{14}}(1, 2, 3).$$

Next, we need to orthogonalize w_2 with respect to v_1. We start by calculating the dot product of w_2 with v_1:

$$w_2 \cdot v_1 = 2 \cdot \frac{1}{\sqrt{14}} + 5 \cdot \frac{2}{\sqrt{14}} - 1 \cdot \frac{3}{\sqrt{14}} = \frac{9}{\sqrt{14}}.$$

The projection of w_2 onto v_1 is:

$$(w_2 \cdot v_1)v_1 = \frac{9}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}}(1, 2, 3) = \frac{9}{14}(1, 2, 3).$$

We subtract this projection from w_2 to get the vector orthogonal to v_1:

$$v_2' = w_2 - (w_2 \cdot v_1)v_1 = (2, 5, -1) - \frac{9}{14}(1, 2, 3) = \frac{1}{14}(19, 52, -41).$$

Finally, we normalize v_2' to obtain v_2:

$$||v_2'|| = \sqrt{\left(\frac{19}{14}\right)^2 + \left(\frac{52}{14}\right)^2 + \left(\frac{-41}{14}\right)^2} = \frac{\sqrt{4746}}{14},$$

and thus

$$v_2 = \frac{1}{||v_2'||}v_2' = \frac{14}{\sqrt{4746}}\left(\frac{1}{14}(19, 52, -41)\right) = \frac{1}{\sqrt{4746}}(19, 52, -41).$$

Thus, the vectors $v_1 = \frac{1}{\sqrt{14}}(1, 2, 3)$ and $v_2 = \frac{1}{\sqrt{4746}}(19, 52, -41)$ form an orthonormal basis for W.

10.3: exercise 2

Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A and verify that the eigenvectors are orthogonal.

Solution:

First, we calculate the eigenvalues of the matrix A. The matrix A is given by:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}.$$

To find the eigenvalues, we solve the characteristic equation $\det(A - \lambda I) = 0$, where I is the identity matrix and λ represents the eigenvalues. This leads to:

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda) - 2 \cdot 2 = \lambda^2 + \lambda - 6.$$

Factoring this quadratic equation gives:

$$\lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0,$$

which implies that the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$.

Next, we find the eigenvectors corresponding to each eigenvalue:

For $\lambda_1 = 2$:

$$(A - 2I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

solving which gives the eigenvector $v_1 = (2, 1)$.

For $\lambda_2 = -3$:

$$(A + 3I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

solving which gives the eigenvector $v_2 = (1, -2)$.

Finally, to verify that these eigenvectors are orthogonal, we calculate their dot product:

$$v_1 \cdot v_2 = (2, 1) \cdot (1, -2) = 2 \times 1 + 1 \times (-2) = 2 - 2 = 0.$$

Since the dot product is zero, the eigenvectors v_1 and v_2 are indeed orthogonal.

10.3: exercise 4

Let $\mathcal{S}_2$ be the set of real 2×2 symmetric matrices.

(a) Verify that $\mathcal{S}_2$ is a vector space and that

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

is a basis of $\mathcal{S}_2$. Hence $\mathcal{S}_2$ is 3-dimensional.

(b) Let P be a 2×2 orthogonal matrix. Verify that the map $M_P : \mathcal{S}_2 \to \mathcal{S}_2$ defined by

$$M_P(A) = P^T A P$$

is linear.

(c) Let P be the matrix

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

Verify that P is an orthogonal matrix and compute the eigenvalues and eigenvectors of M_P.

Solution:

(a) Symmetric matrices are closed under addition and scalar multiplication. Therefore, the set $\mathcal{S}_2$ of real 2×2 symmetric matrices is a vector space. Every matrix in $\mathcal{S}_2$ can be expressed as:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = aE_1 + bE_2 + cE_3.$$

This expression shows that the vectors $\mathcal{E} = \{E_1, E_2, E_3\}$ span $\mathcal{S}_2$ and are linearly independent, thus forming a basis for $\mathcal{S}_2$. Therefore, the dimension of $\mathcal{S}_2$ is 3.

(b) To verify the linearity of the map $M_P : \mathcal{S}_2 \to \mathcal{S}_2$ defined by $M_P(A) = P^T A P$, we perform the following calculations:

$$\begin{aligned} M_P(A+B) &= P^T(A+B)P = P^T AP + P^T BP = M_P(A) + M_P(B), \\ M_P(cA) &= P^T(cA)P = cP^T AP = cM_P(A), \end{aligned}$$

for all $A, B \in \mathcal{S}_2$ and $c \in \mathbb{R}$. These properties confirm the linearity of M_P.

(c) To compute the matrix of M_P in the basis given in (1), where P is a specific orthogonal matrix, we calculate:

$$\begin{aligned} &M_P(E_1) = P^T E_1 P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_3, \\ &M_P(E_2) = -E_2, \\ &M_P(E_3) = E_1. \end{aligned}$$

Thus, the matrix of M_P in the basis $\mathcal{E}$ is:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of this matrix is calculated as:

$$p(\lambda) = \det\left(\begin{pmatrix} -\lambda & 0 & 1 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} \right) = (1-\lambda^2)(1+\lambda),$$

giving eigenvalues $-1, -1, 1$. The eigenvectors in $\mathbb{R}^3$ corresponding to -1 are $(1, 0, -1)^T$ and $(0, 1, 0)^T$, which correspond to the symmetric matrices $E_1 - E_3$ and E_2. The eigenvector for 1 is $(1, 0, 1)^T$ corresponding to the symmetric matrix $E_1 + E_3$.