MATH 2568

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homework #4

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Contents

3.7: exercise 2

Let $\alpha \neq 0$ be a real number and let A be an invertible matrix. Show that the inverse of the matrix αA is given by $\frac{1}{\alpha}A^{-1}$.
Solution:

(2)

$$\boxed{(\alpha A)\left(\frac{1}{\alpha}A^{-1}\right) = \left(\alpha \cdot \frac{1}{\alpha}\right)(AA^{-1}) = I} \tag{1}$$

3.7: exercise 10

For which values of a, b, c is the matrix

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

invertible? Find A^{-1} when it exists.
Solution:

(10) The matrix A is invertible for any choice of a, b, and c, and

$$A^{-1} = \begin{pmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

A matrix is invertible if it is row equivalent to I_n. We demonstrate this by performing row reduction on the augmented matrix $(A|I_3)$:

$$\left(\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_1]{-R_1+aR_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & -b+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Since the resulting matrix is in reduced row echelon form with a leading 1 in each row, A is indeed invertible for any values of a, b, and c. Consequently, the inverse of A is:

$$A^{-1} = \boxed{\begin{pmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}}.$$

3.7: exercise 14

Suppose the mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ is linear and satisfies the following equations: Let A and B be 3×3 invertible matrices so that

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Without computing A or B, determine the following:

(a) The rank(A)

(b) The solution to $Bx =$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) $(2BA)^{-1}$

(d) The matrix C so that $ACB + 3I_3 = 0$.

Solution:

(a) is an invertible 3×3 matrix, so rank$(A) = \boxed{3}$.

(b) The solution is

$$x = B^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \boxed{\begin{pmatrix}3\\2\\1\end{pmatrix}}$$

(c)

$$(2BA)^{-1} = \frac{1}{2}A^{-1}B^{-1} = \boxed{\frac{1}{2}\begin{pmatrix}0 & 1 & 1\\-2 & -2 & -1\\0 & 1 & 0\end{pmatrix}}$$

(d) Recall that multiplication on the left by a matrix is not the same as multiplication on the right. We have that

$$\begin{aligned}
ACB = -3I_3 &\implies A^{-1}ACB = -3A^{-1}I_3 && \text{multiplying on the left by } A^{-1}\\
&\implies CB = -3A^{-1}\\
&\implies CBB^{-1} = -3A^{-1}B^{-1} && \text{multiplying on the right by } B^{-1}\\
&\implies C = -3A^{-1}B^{-1}\\
&\implies C = \boxed{-3\begin{pmatrix}0 & 1 & 1\\-2 & -2 & -1\\0 & 1 & 0\end{pmatrix}}
\end{aligned}$$

3.8: exercise 4

Let A be a 2×2 matrix having integer entries. Find a condition on the entries of A that guarantees that A^{-1} has integer entries.

Solution:

(4) The formula for the inverse of a 2x2 matrix, A^{-1}, involves a term $\frac{1}{ad-bc}$, where a, b, c, and d are the elements of the matrix A. For this inverse to have integer entries (whole numbers), this term needs to be an integer itself.

Since a, b, c, and d are all integers, the only way for $\frac{1}{ad-bc}$ to be an integer is if the absolute value of the expression $\boxed{|ad - bc| = 1}$. In simpler terms, the product of the diagonal elements (a and d) minus the product of the off-diagonal elements (b and c) must be either 1 or -1.

3.8: exercise 6

Suppose a 2×2 matrix A satisfies the following equation:

$$\mathbf{A}\begin{pmatrix}0 & 2\\1 & 2\end{pmatrix} = \begin{pmatrix}-1 & 2\\1 & 4\end{pmatrix} \tag{2}$$

Without calculating the entries of $\mathbf{A}$, find $\det(\mathbf{A})$.

Solution:

(6)

$$\begin{aligned}
\det\left(A\begin{pmatrix}0 & 2\\ 1 & 2\end{pmatrix}\right) &= \det(A)\det\begin{pmatrix}0 & 2\\ 1 & 2\end{pmatrix}\\
&= -2\det(A)\\
&= \det\begin{pmatrix}-1 & 2\\ 1 & 4\end{pmatrix}\\
&= -6.
\end{aligned}$$

Therefore, $\boxed{\det(A) = 3}$.