Linear Methods for Classification

4.4 Logistic Regression
4.5 Separating Hyperplanes

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01. Regularized Logistic Regression

$$\max_{\beta_0,\beta}\left\{\sum_{i=1}^{N}\left[y_i(\beta_0+\beta^T x_i)-\log(1+e^{\beta_0+\beta^T x_i})\right]-\lambda\sum_{j=1}^{p}|\beta_j|\right\}$$

- L1 penalty used in the lasso can be used for variable selection and shrinkage with any linear regression model
- do not penalize the intercept term, and standardize the predictors for the penalty to be meaningful

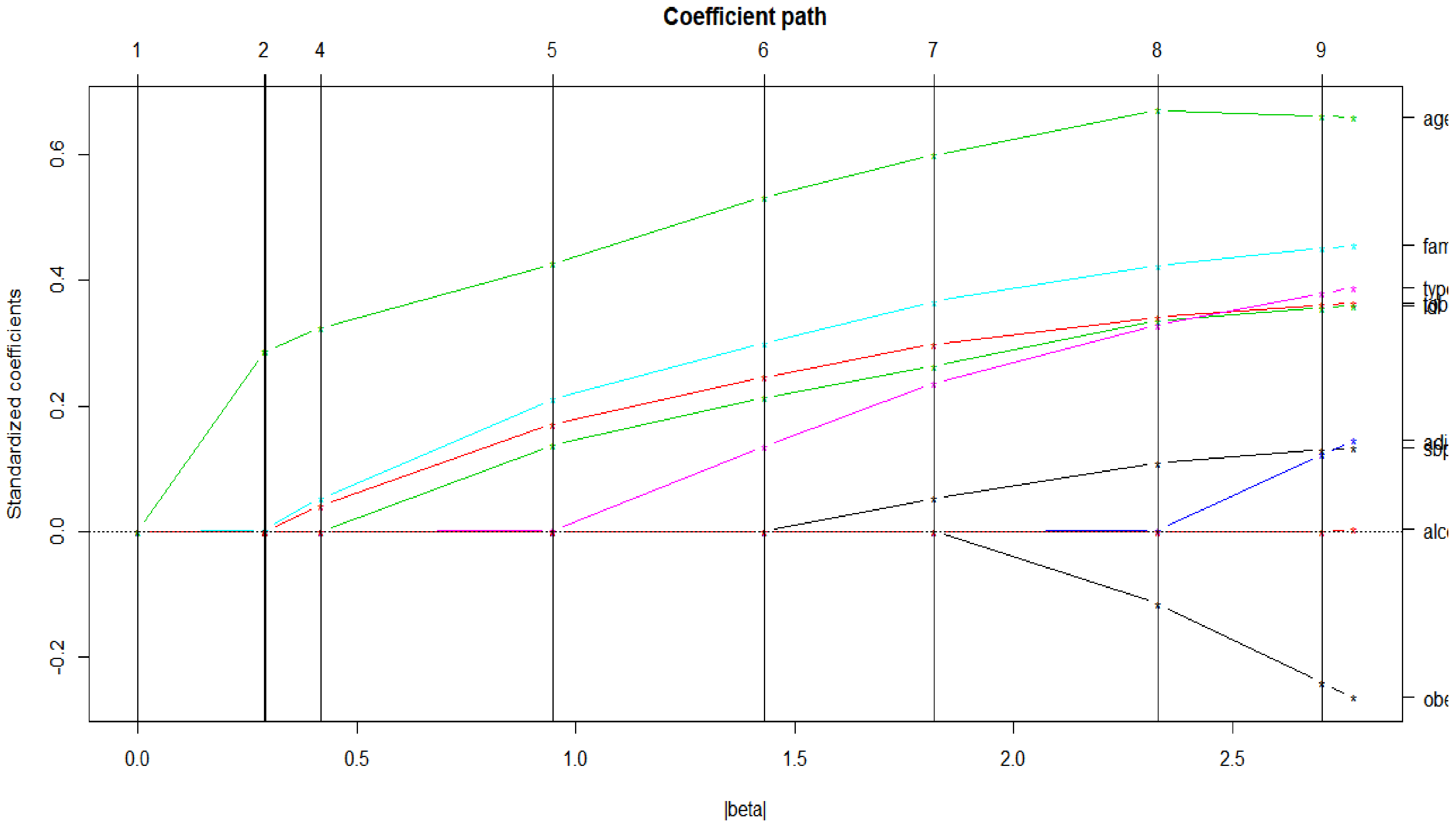
02. the South African heart disease data

- A total of 462 samples are included in this data set
- Adiposity is a measure of % bodyfat, whereas obesity measures weight-to-height rations (body-mass-index, bmi). Type-A behaviour pattern is characterised by an excessive competitive drive, impatience and anger/hostility.

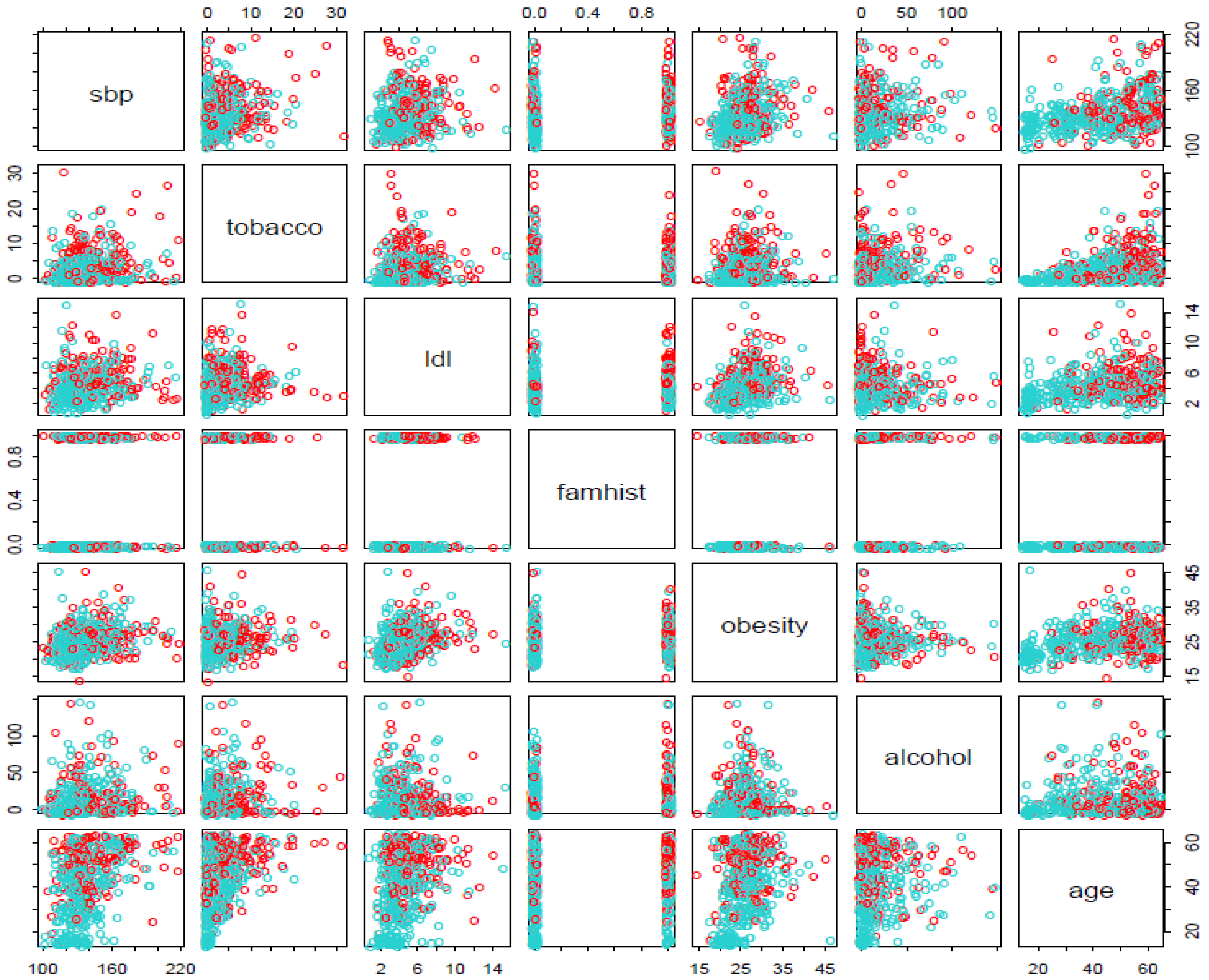
- systolic blood pressure (**sbp**)
- cumulative tobacco (**tobacco**)
- low density lipoprotein cholesterol (**ldl**)
- **Adiposity**
- family history of heart disease (**famhist**)
- type-A behavior (**typea**)
- **Obesity**
- **alcohol**
- **Age**

02.

the South African heart disease data



02. the South African heart disease data



02. the South African heart disease data

1) Logistic Regression

Coefficients:

	Estimate	Std. Error	z value	Pr(>|z|)	
(Intercept)	-6.1507209	1.3082600	-4.701	2.58e-06	***
sbp	0.0065040	0.0057304	1.135	0.256374	
tobacco	0.0793764	0.0266028	2.984	0.002847	**
ldl	0.1739239	0.0596617	2.915	0.003555	**
adiposity	0.0185866	0.0292894	0.635	0.525700	
famhist	0.9253704	0.2278940	4.061	4.90e-05	***
typea	0.0395950	0.0123202	3.214	0.001310	**
obesity	-0.0629099	0.0442477	-1.422	0.155095	
alcohol	0.0001217	0.0044832	0.027	0.978350	
age	0.0452253	0.0121298	3.728	0.000193	***

73.37%

2) Logistic Regression -> stepwise

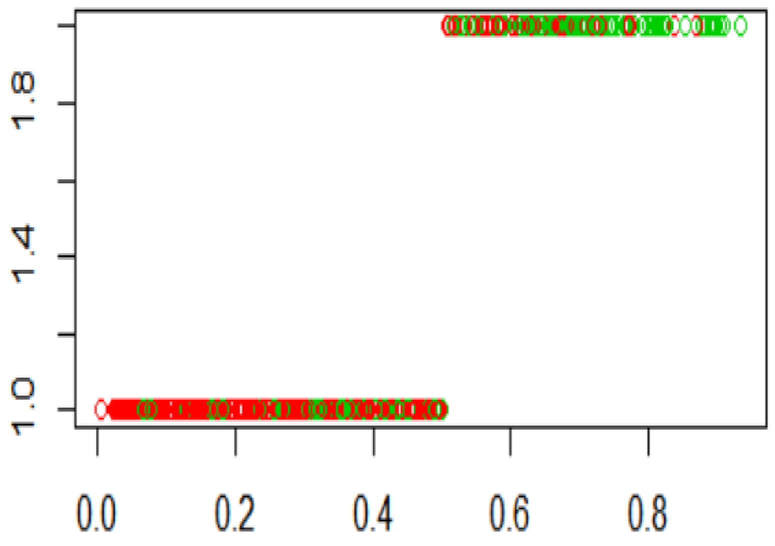
Coefficients:

	Estimate	Std. Error	z value	Pr(>|z|)	
(Intercept)	-6.44644	0.92087	-7.000	2.55e-12	***
tobacco	0.08038	0.02588	3.106	0.00190	**
ldl	0.16199	0.05497	2.947	0.00321	**
famhist	0.90818	0.22576	4.023	5.75e-05	***
typea	0.03712	0.01217	3.051	0.00228	**
age	0.05046	0.01021	4.944	7.65e-07	***

74.24%

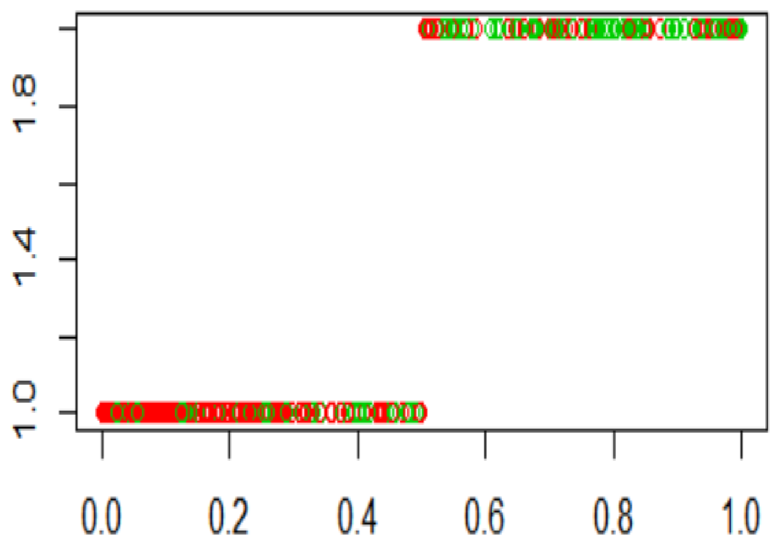
02. the South African heart disease data

3) LDA



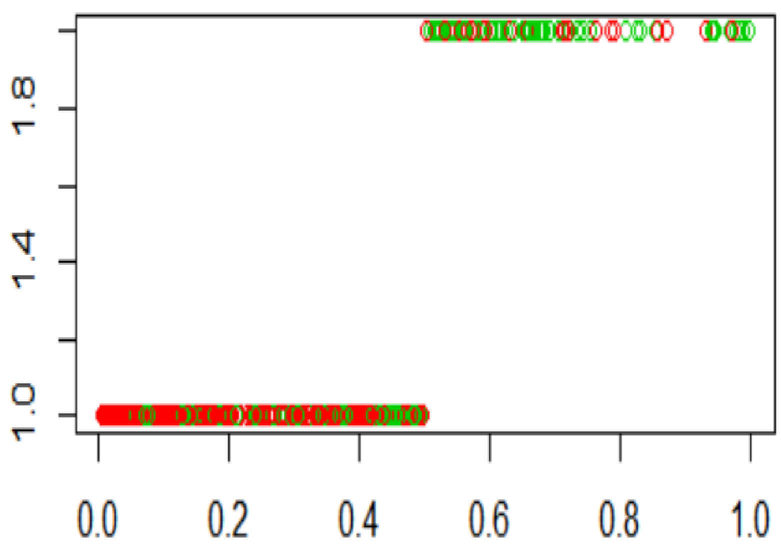
72.94%

4) QDA



75.75%

5) RDA



75.54%

03. Logistic Regression or LDA?

- the log-posterior odds between class k and K are linear functions of x

$$\begin{aligned}
\log \frac{\Pr(G=k|X=x)}{\Pr(G=K|X=x)} &= \log \frac{\pi_k}{\pi_K} - \frac{1}{2}(\mu_k + \mu_K)^T \Sigma^{-1}(\mu_k - \mu_K) \\
&\quad + x^T \Sigma^{-1}(\mu_k - \mu_K) \\
&= \alpha_{k0} + \alpha_k^T x.
\end{aligned}$$

- the linear logistic model by construction has linear logits

$$\log \frac{\Pr(G=k|X=x)}{\Pr(G=K|X=x)} = \beta_{k0} + \beta_k^T x$$

- seems that the models are the same
  - > the difference lies in the way the linear coefficients are estimated.

- The logistic regression model is more general, in that it makes less assumptions.

03. Logistic Regression or LDA?

- We can write the joint density of X and G as, (Pr(X) : the marginal density of the inputs X)

$$\Pr(X, G=k) = \Pr(X)\Pr(G=k|X)$$

- The logistic regression model leaves the marginal density of X as an arbitrary density function Pr(X), and fits the parameters of Pr(G|X) by maximizing the conditional likelihood

- with LDA we fit the parameters by maximizing the full log-likelihood, based on the joint density, where ϕ is the Gaussian density function.

$$\Pr(X, G=k) = \phi(X; \mu_k, \Sigma)\pi_k$$

$$\Pr(X) = \sum_{k=1}^{K} \pi_k \phi(X; \mu_k, \Sigma).$$

unlike in the conditional case, the marginal density Pr(X) does play a role here

04. Separating Hyperplane

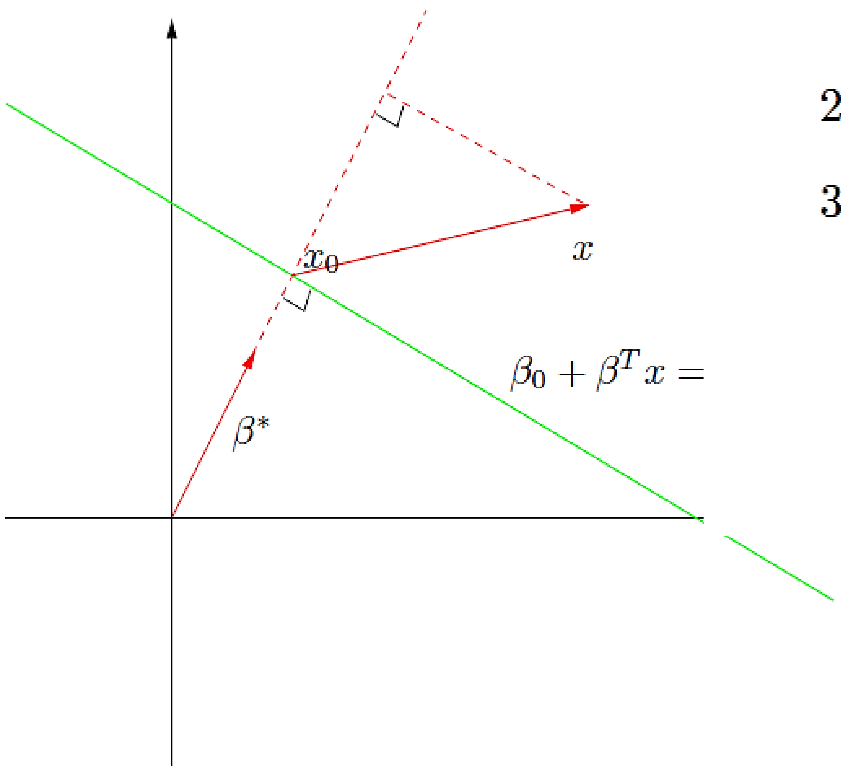


FIGURE 4.15. *The linear algebra of a hyperplane (affine set).*

1. For any two points x_1 and x_2 lying in L, $\beta^T(x_1 - x_2) = 0$, and hence $\beta^* = \beta/||\beta||$ is the vector normal to the surface of L.
2. For any point x_0 in L, $\beta^T x_0 = -\beta_0$.
3. The signed distance of any point x to L is given by

$$\begin{aligned} \beta^{*T}(x - x_0) &= \frac{1}{||\beta||}(\beta^T x + \beta_0) \\ &= \frac{1}{||f'(x)||} f(x). \end{aligned} \tag{4.40}$$

f(x) is proportional to the signed distance

from x to the hyperplane defined by f(x) = 0.

05. Rosenblatt's Perceptron Learning Algorithm

- compute a linear combination of the input features and return the sign
- obtained by regressing the −1/1 response Y on X
- to find a separating hyperplane by minimizing the distance of misclassified points

  to the decision boundary. M indexes the set of misclassified points

$$D(\beta, \beta_0) = -\sum_{i \in \mathcal{M}} y_i (x_i^T \beta + \beta_0)$$

- If x in K=1, $x_i^T \beta + \beta_0 > 0$ / If x in K=2, $x_i^T \beta + \beta_0 < 0$

05. Rosenblatt's Perceptron Learning Algorithm

- The gradient is given by

$$\partial\frac{D(\beta,\beta_0)}{\partial\beta} = -\sum_{i\in\mathcal{M}} y_i x_i,$$

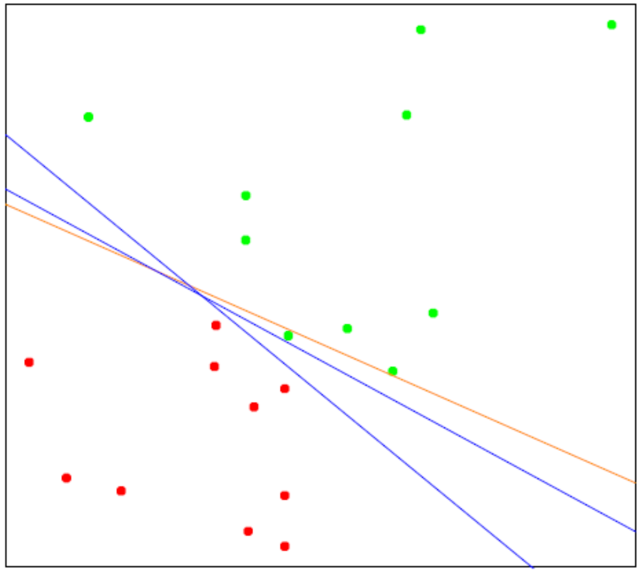
$$\partial\frac{D(\beta,\beta_0)}{\partial\beta_0} = -\sum_{i\in\mathcal{M}} y_i.$$

- The algorithm in fact uses *stochastic gradient descent* to minimize this piecewise linear criterion.
- the misclassified observations are visited in some sequence, and the β are updated via

$$\begin{pmatrix}\beta\\ \beta_0\end{pmatrix} \leftarrow \begin{pmatrix}\beta\\ \beta_0\end{pmatrix} + \rho\begin{pmatrix}y_i x_i\\ y_i\end{pmatrix}$$

, ρ is the learning rate, which in this case can be taken to be 1 without loss in generality

05. Rosenblatt's Perceptron Learning Algorithm



Problems with this algorithm

- When the data are separable, there are many solutions, and which one is found depends on the starting values.
- The "finite" number of steps can be very large. The smaller the gap, the longer the time to find it.
- When the data are not separable, the algorithm will not converge, and cycles develop. The cycles can be long and therefore hard to detect.

06. Optimal Separating Hyperplane

Thank you

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