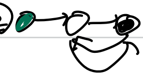
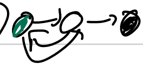
Reading Random Walks

1. Why $\frac{N!}{n_1! n_2!}$?

Consider  with $N=4$ and $m=2$ where $m = n_1 - n_2$

Possible paths: (1)  $n_1 = 3, n_2 = 1$

(2)  $n_1 = 3, n_2 = 1$

(3)  $n_1 = 3, n_2 = 1$

(4)  $n_1 = 3, n_2 = 1$

$\therefore \frac{N!}{n_1! n_2!} = \frac{4!}{3!1!} = 4$ different unique paths of reaching m

and n_1 & n_2 are fixed $\rightarrow$ each has the same prob $p^{n_1} p^{n_2}$

$\therefore$ Total probability to reach m: $\frac{N!}{n_1! n_2!} p^{n_1} p^{n_2}$

Since $N = n_1 + n_2$ and $m = n_1 - n_2$

$n_1 = \frac{N+m}{2}$ $\quad n_2 = \frac{N-m}{2}$

$$\Rightarrow P(m, N) = \frac{N!}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!} p^{\frac{1}{2}(N+m)} q^{\frac{1}{2}(N-m)} \qquad (1)$$

c.f. binomial distribution

$$P(k, N) = \binom{N}{k} p^k (1-p)^{N-k}$$

if $k = \frac{N+m}{2}$, $N-k = N - \frac{N+m}{2} = \frac{2N-N-m}{2} = \frac{N-m}{2}$

$\therefore (1)$ becomes $\frac{N!}{k!(N-k)!} p^k (1-p)^{N-k} = \binom{N}{k} p^k (1-p)^{N-k}$

Then mean : $Np \xrightarrow{\qquad} N/2$

variance : $Np(1-p) \xrightarrow{p=q=\frac{1}{2}} N/4$

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

μ : mean

σ^2 : Variance

As $N \to \infty$, $P(k, N) \to$ $\boxed{\text{Normal Dist.}}$ by CLT

$$\to \frac{1}{\sqrt{2\pi N/4}} e^{-\frac{(k-N/2)^2}{2N/4}}$$

Since $k = \frac{N+m}{2} \to m = 2k - N$ and $\frac{m}{2} = k - N/2$

$$P(m, N) = \frac{2}{\sqrt{2\pi N}} e^{-\frac{m^2}{2N}}$$

and $t = N\Delta t \to N = t/\Delta t$