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WHAT IS APPLIED CATEGORY THEORY?

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What is applied category theory?

Upon hearing the phrase "applied category theory," you *might* be thinking either one of two thoughts:

#1 *Applied category theory? Isn't that an oxymoron?*

#2 *Applied category theory? What's the hoopla? Hasn't category theory always been applied?*

For those thinking thought #1, I hope to convince you that the answer is *No way*! It's true that category theory sometimes goes by the name of *general abstract nonsense*, which might incline you to think that category theory is too pie-in-the-sky to have any impact on the "real world." My hope is that these notes will convince you that that's *far* from the truth!

For those thinking thought #2, *yes* it's true that ideas and results from category theory have found applications in computer science and quantum physics (not to mention pure mathematics itself), but these are not the only applications to which the word *applied* in *applied category theory* is being applied. So what *is* applied category theory?

Read on.

A quick note to the reader

Before we get started, I'll mention that this document is a collection of notes I amassed while participating in the 2018 Applied Category Theory Adjoint School—a wonderful online seminar that ran from January - April 2018 and culminated in a two-week workshop at the Lorentz Center in May 2018. I had a blast learning from the folks there, and I want to share some of the things I learned with anyone who's interested. (So thanks for being interested!) Later, I'll describe a couple of the research projects discussed during the workshop. Much of the information in this PDF can be found in various journal articles, blog posts, and videos of conference talks, most of which are freely available online. I've provided citations to these throughout. Here are a few other things to know:

- I'll assume the reader is comfortable with the basics of category theory: *categories*, *functors* and *natural transformations*. For a friendly introduction to these topics, feel free to browse through the articles from my blog Math3ma listed in the margin.
- I'll make heavy use of hyperlinks, as I have already, and I'll also incorporate the occasional use of color throughout the text. For these reasons, it's probably best to read this PDF on a computer rather than in print form.
- Finally, a fair warning: I use italics *a lot* (along with frequent parenthetical remarks). I also like exclamation points! And many of my sentences begin with a conjunction.

For a gentle introduction to (pure) category theory, here are a few places to start:

- What is Category Theory, Anyway?
- What is a Category?
- What is a Functor?
- What is a Natural Transformation?

At the first link, you'll find a list of other recommended resources for learning about category theory.

My gratitude goes to the participants and mentors of the 2018 ACT Workshop from whom I learned a great deal. I also thank John Baez, Joseph Hirsh, and Maximilien Péroux for providing valuable feedback on a first draft of these notes.

Introduction

One of the great features of category theory, birthed in the 1940s, is that its organizing principles have been used to reshape and reformulate problems within pure mathematics, including topology, homotopy theory and algebraic geometry. Category theory has light on those problems, making them easier to solve and opening doors for new avenues of research. Historically, then, category theory has found immense application within mathematics. As John Baez recently noted, "[category theory] *was meant to be applied.*"

More recently, however, category theory has found applications in a wide range of disciplines outside of pure mathematics—even beyond the closely related fields of computer science and quantum physics. These disciplines include chemistry, neuroscience, systems biology, natural language processing, causality, network theory, dynamical systems, and database theory to name a few. And what do they all have in common? That's precisely what applied category theory seeks to discover. In other words, the techniques, tools, and ideas of category theory are being used to identify recurring themes across these various disciplines with the purpose of making them a little more formal. This is what the phrase **applied category theory** (ACT) is meant to describe. As explained on the ACT 2018 workshop webpage,

> ...we should treat the use of categorical concepts as a natural part of transferring and integrating knowledge across disciplines. The restructuring employed in applied category theory cuts through jargon, helping to elucidate common themes across disciplines. Indeed, the drive for a common language and comparison of similar structures in algebra and topology is what led to the development category theory in the first place, and recent hints show that this approach is not only useful between mathematical disciplines, but between scientific ones as well.

Of course, one of the challenges of using category theory to transfer and integrate knowledge across disciplines is making category theory itself accessible to the broader scientific audience. John Baez and Brendan Fong address this very point in their 2016 paper on electrical circuit diagrams[1]:

> While diagrams of networks have been independently introduced in many disciplines, we do not expect formalizing these diagrams to immediately help the practitioners of these disciplines. At first the flow of information will mainly go in the other direction: by translating ideas from these disciplines into the language of modern mathematics, we can provide mathematicians with food for thought and interesting new problems to solve. We hope that in the long run mathematicians can return the favor by bringing new insights to the table.

[1] *A Compositional Framework for Passive Linear Networks*, https://arxiv.org/pdf/1504.05625.pdf

Although their comments refer to a particular project, they can apply to the field at large, too.

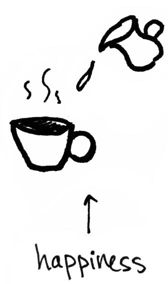
The goal of this document is to give a taste of applied category from a graduate student's perspective. In doing so, I'll share **two themes** and **two constructions** that appeared frequently during the ACT 2018 workshop. The math underlying these themes and constructions is not new. The newness, rather, is in how they are being *applied*. To illustrate the themes and constructions, I'll also share **two examples**—two research projects in the field of ACT. The first project relates to chemistry and the second to natural language processing, though the expositions are weighted unevenly. I'll devote considerably more time on the second example since that's where my own research interests lie. And that's what's on the carte du jour! **Two themes** and **two constructions** and **two examples**, along with a few crumbs (i.e. digressions) in between. Here's the menu in more detail:

Contents

Although the items are listed *linearly*, they are very much intertwined. The themes motivate the constructions; the constructions embody the themes, and both the themes and the constructions come to life in the examples.

I LIKE TO IMAGINE that category theory is like a cup of black coffee, while fields outside of pure mathematics are like fresh cream. Both are lovely on their own, but blending them makes for a beverage *par excellence*.

I hope you'll enjoy it as much as I do!



1 Two Themes

Two themes that appear over and over (and *over and over and over*) in applied category theory are **functorial semantics** and **compositionality.** Let's talk about the first one first.

1.1 Functorial Semantics

Functorial semantics relates to the idea that a structure-preserving functor between categories

$$\mathsf{C} \to \mathsf{D}$$

can be viewed as an *interpretation* of C within D. It's often helpful to think of C as somehow encoding for *syntax* while D provides *semantics*. Syntax refers to rules for putting things together and semantics refers to the meaning of those things. A functor

$$\mathsf{syntax} \to \mathsf{semantics}$$

provides a way to bring the syntax to *life*.

The phrase "functorial semantics" was coined by William Lawvere.

← This is how Lawvere *defines* the word "functor" in his book with Stephen Schanuel, Conceptual Mathematics! It's a nice introductory text, by the way.

To GET A BETTER IDEA of syntax vs. semantics, think of the English language where two important features of communication are 1) grammar, which provides rules for combining words to form sentences, and 2) the actual *meaning* conveyed by those words and sentences. Grammar is the syntax, and the meaning is the semantics.

$$\text{grammar} \rightsquigarrow \mathsf{syntax} \qquad\qquad \text{meaning} \rightsquigarrow \mathsf{semantics}.$$

Of course, neither is useful on their own. For instance, it's easy to come up with a sentence that is grammatically correct and yet has no meaning. That's the whole point behind MadLibs! As another example, here's a sentence attributed to linguist Noam Chomsky:

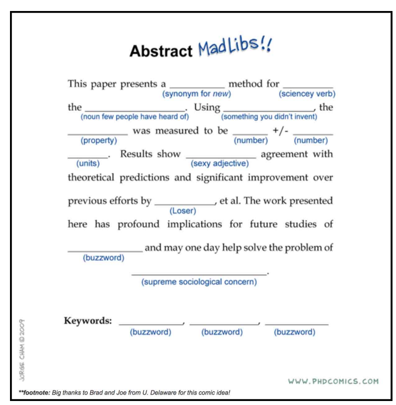
Colorless green ideas sleep furiously.

It is grammatically correct, yet it has no meaning. The point here is that *syntax vs. semantics* is nothing new. So when an applied category theorist wants to model some phenomena in the "real world," don't be surprised if their model is ultimately a functor from a syntax category to a semantics category!

I'm using English language as an analogy to illustrate syntax vs. semantics, but it's **more** than an analogy! As we'll see in Section 3.2, the pairings

$$\text{"grammar} \rightsquigarrow \mathsf{syntax}\text{"}$$
$$\text{"meanings of words} \rightsquigarrow \mathsf{semantics}\text{"}$$

become quite literal in applied category theory!



A small-ish digression...

Even though the idea goes by the fancy name of *functorial semantics*, it is *not* just a "category theory thing." Mind if I digress for a while to elaborate on this?

I'll take your silence as a No.

IF YOU KNOW A LITTLE BIT about *groups*, then you've seen functorial semantics in action before! How so? A group is a set endowed with some extra structure, though that tells us nothing about why groups are useful. It's better to think of a group as encoding for some kind of action or transformation. And this is why group representations are so great! A *group representation* provides a way to view your *abstract* group elements as *concrete* linear transformations of some vector space. Explicitly, given a vector space V, a **group representation** is a group homomorphism from G to $\mathrm{Aut}(V)$, the group of all automorphisms of V

$$G \to \mathrm{Aut}(V)$$

It assigns to each group element a linear isomorphism $V \to V$.

Group elements are like verbs. They DO stuff! For more on this notion from a categorical perspective, check out the article Group Elements, Categorically on Math3ma.

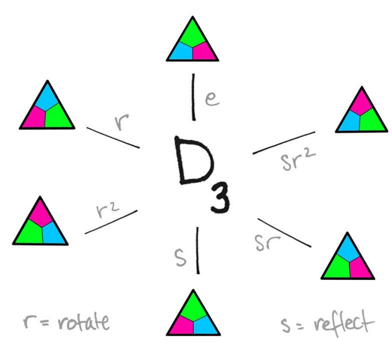
If we replace $\mathrm{Aut}(V)$ by $\mathrm{Aut}(X)$ for some set X (i.e. the group of automorphisms, i.e. bijections, on X), then a group homomorphism $G \to \mathrm{Aut}(X)$ is precisely a group action on X.

As a quick example, suppose our group is D_3, the dihedral group of order 6, which is the group of symmetries of an equilateral triangle. If we were to look at a presentation of the group,

$$D_3 = \langle r, s \mid r^3 = s^2 = rsrs = 1 \rangle$$

it might not seem to have anything to do with triangles. Fortunately, a representation of D_3 makes the connection clearer by assigning to each group element r and s a linear transformation of the real plane. Specifically, the standard representation of $D_3 \to \mathrm{Aut}(\mathbb{R}^2)$ assigns to each of r and s an invertible 2×2 matrix with real entries:

$$r \mapsto R = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} \qquad s \mapsto S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Here R is a rotation by 60° while S is reflection across the x-axis. Moreover R^3 and S^2 and $RSRS$ each are equal to the 2×2 identity matrix, which is exactly what we would expect: rotating an equilateral triangle by one full revolution leaves it unchanged, as does reflecting it twice in a row, and so on.

More generally then, we can think of a group G as providing the *syntax* while automorphisms $\mathrm{Aut}(V)$ provide the *semantics*

$$G \rightsquigarrow \text{syntax} \qquad \mathrm{Aut}(V) \rightsquigarrow \text{semantics}$$

So a group representation is like a (structure-preserving) morphism

$$\text{syntax} \to \text{semantics}$$

In fact... it's not *like* that. It **IS** that. If we view both the groups G and $\mathrm{Aut}(V)$ as one-object[2] categories, then a group representation

$$G \to \mathrm{Aut}(V)$$

IS a functor from syntax to semantics. That's because every group homomorphism *is* a functor when the groups are viewed as one-object categories! So although ***functor**ial semantics* has the word "functor" in it, don't think that the idea behind it is unique to category theory. Indeed, representation theory capitalizes on the relationship between syntax and semantics: a *representation* assigns to an abstract algebraic gadget (the syntax) some concrete meaning (the semantics).

[2] Every group G gives rise to a category having a single object $\bullet$ (the group itself) and a morphism $\bullet \xrightarrow{g} \bullet$ for each group element $g \in G$. Composition is given by the group operation.

Here's another example I can't resist sharing: **operads!** If you're not familiar with operads, just know that this is a souped-up version of the group theory example. If you *are* familiar with operads, then you know this is the souped-up version of the group theory example.

An operad is an example of syntax, while an algebra over that operad provides the semantics. For example, given a vector space V, an operad homomorphism from the [commutative, associative, Lie, Poisson,...] operad to the endomorphism operad on V **IS** a [commutative, associative, Lie, Poisson,...]-algebra! That is, the structure-preserving homomorphism provides an *interpretation* of each abstract n-ary operation as a actual, concrete operation $V^{\otimes n} \to V$.

I COULD END OUR DIGRESSION here, but I'd like to share one more instance of functorial semantics at work in pure mathematics. The next few examples involve monoids and monoidal categories, so I'll assume you are familiar with those words. If you are *not* familiar with those words, don't fret—you're in luck! Section 2.1 is all about monoids and monoidal categories, so feel free to read that section first then come back here. In either case, let's proceed with another neat example of functorial semantics in action:

This next comment is really digressing from the digression, but: I also like to think of simplicial sets as an instance of functorial semantics. A simplicial set X is a bit like syntax, while a topological space is like semantics. Geometric realization $X \mapsto |X|$ provides a map from one to the other.

Example: *a **monoid** is the image of a functor from a certain syntax category to a certain semantics category.*

More specifically,[3]

a **monoid** is a lax monoidal functor

$$\mathbb{1} \longrightarrow \mathsf{Set}$$

the "trivial category" the category of sets

[3] A functor $F\colon \mathsf{C} \to \mathsf{D}$ between monoidal categories is called **lax monoidal** if for every pair of objects c, c' in C there is a morphism

$$Fc \otimes Fc' \to F(c \otimes c')$$

(which assembles into a natural transformation.) It's called **strong monoidal** if $Fc \otimes Fc' \cong F(c \otimes c')$, and it's called **strict monoidal** if $Fc \otimes Fc' = F(c \otimes c')$.

Here I'm viewing both $\mathbb{1}$ and Set as monoidal categories. The symbol $\mathbb{1}$ is meant to represent the category with one object and only one morphism (the identity), which we can view as a monoidal category $(\mathbb{1}, \otimes, \mathbb{1})$ in exactly one way. The category of sets has a monoidal structure given by the Cartesian product with the set containing one element, denoted $\{*\}$, as the monoidal unit. *Technically* then, a **monoid** is a lax monoidal functor

$$(\mathbb{1}, \otimes, \mathbb{1}) \longrightarrow (\mathsf{Set}, \times, \{*\})$$

the category with one object and no non-identity morphisms

(a monoidal category)

the category of sets with Cartesian product & the one-point set as monoidal unit

(a monoidal category)

Why is this true? First observe that a functor $F\colon \mathbb{1} \to \mathsf{Set}$ picks out a set, $F\mathbb{1} := M$. And the data of a *lax monoidal* functor $F\colon (\mathbb{1}, \otimes, \mathbb{1}) \to (\mathsf{Set}, \times, \{*\})$ consists of a morphism

$$\bullet\colon F\mathbb{1} \times F\mathbb{1} \to F(\mathbb{1} \otimes \mathbb{1}) \qquad \text{i.e.} \qquad \bullet\colon M \times M \to M$$

along with a morphism

$$1\colon \{*\} \to F\mathbb{1} \qquad \text{i.e.} \qquad 1\colon \{*\} \to M$$

both of which are required to fit into some commuting diagrams. I won't write them here, but one diagram says "$\bullet$ is associative" and the other diagram says, "1 serves as an identity for $\bullet$." In summary, the data of a lax monoidal functor $F\colon (\mathbb{1}, \otimes, \mathbb{1}) \to (\mathsf{Set}, \times, \{*\})$ are

i) a set M

ii) an associative binary operation $\bullet\colon M \times M \to M$

iii) a special element $1 := 1(\{*\}) \in M$ that serves as a "multiplicative identity" for $\bullet$.

This triple $(M, \bullet, 1)$ is *precisely* a monoid! Or to borrow from Lawvere's terminology, the functor (equivalently, the monoid) is one *interpretation* of the category $\mathbb{1}$ in the Set. Interestingly, $\mathbb{1}$ may be interpreted in other categories as well. This leads to other familiar monoidal structures. Indeed, if we replace $(\mathsf{Set}, \times, \{*\})$ by any monoidal category $(\mathsf{C}, \otimes, 1)$, then a lax monoidal functor

$$(\mathbb{1}, \otimes, \mathbb{1}) \to (\mathsf{C}, \otimes, 1)$$

is a **monoid in the category** C. Sometimes this monoid goes by a familiar name. Here are some examples.

In footnote 3 you'll notice I dropped parentheses and wrote Fc rather than $F(c)$. The reason for my preference is *categorical!* Let me explain by saying a few words about sets: Did you know that an element x in a set X is the same thing as a function $\{*\} \to X$, where $\{*\}$ denotes the one-element set? It's true. The function $\{*\} \to X$ is uniquely determined by where it sends that one point $*$. So since an element x is the same thing as an arrow $\{*\} \to X$, we might as well label *that* arrow by x,

$$x\colon \{*\} \to X$$

Now if we have another function $f\colon X \to Y$, then an element $f(x) \in Y$ is *precisely* the composition

$$\{*\} \xrightarrow{x} X \xrightarrow{f} Y$$

That is,

$$f(x) = f \circ x = fx$$

where on the right hand side, I've omitted the composition symbol $\circ$ because it's cleaner. So there you have it! An element $f(x) \in Y$ is the same as a function fx. And since categorically-minded folks (such as you and I) prefer arrows over elements (Because of Yoneda. Also, we might be rethinking set theory.), the notation fx—and more generally, Fc as above—is preferred. By the way, this is all related to Lawvere's **philosophy of generalized elements**, which is the idea that a morphism $A \to B$ is really an "A-shaped element in B." For some examples, check out the articles "A Diagram is a Functor" as well as "The Yoneda Embedding" on Math3ma. Generalized elements are closely related to functorial semantics, so both links are worth a read!

1. **Topological Monoid.** Let $(\mathsf{Top}, \times, *)$ denote the category of topological spaces and continuous functions, viewed as a monoidal category with the Cartesian product $\times$, with the one-point space $*$ as monoidal unit. A lax monoidal functor

$$(\mathbb{1}, \otimes, \mathbb{1}) \to (\mathsf{Top}, \times, *)$$

is a topological monoid. That is, **a topological monoid is a monoid in the category of topological spaces.**

2. **Ring.** Let $(\mathsf{AbGroup}, \otimes, \mathbb{Z})$ denote the category of abelian groups (viewed as $\mathbb{Z}$-modules) and abelian group homomorphisms, viewed as a monoidal category with the tensor product $\otimes$, with the integers $\mathbb{Z}$ as monoidal unit. A lax monoidal functor

$$(\mathbb{1}, \otimes, \mathbb{1}) \to (\mathsf{AbGroup}, \otimes, \mathbb{Z})$$

is a ring (with unit). That is, **a ring is a monoid in the category of abelian groups.**

3. **Algebra.** Let $(\mathsf{FVect}, \otimes, \mathbb{k})$ denote the category of finite-dimensional vector spaces over a field $\mathbb{k}$ and linear maps, viewed as a monoidal category with the tensor product $\otimes$, with $\mathbb{k}$ as monoidal unit. A lax monoidal functor

$$(\mathbb{1}, \otimes, \mathbb{1}) \to (\mathsf{FVect}, \otimes, \mathbb{k})$$

is an algebra (with unit). That is, **an algebra is a monoid in the category of vector spaces.**

4. **Monad.** Let C be a category and let $\mathsf{End}_{\mathsf{C}}$ denote the category whose objects are functors $\mathsf{C} \to \mathsf{C}$ and whose morphisms are natural transformations. (So $\mathsf{End}_{\mathsf{C}}$ is the category of *endofunctors* on C.) Note that $\mathsf{End}_{\mathsf{C}}$ can be given the structure of a monoidal category: the monoidal product is composition of functors (i.e. if F, G are objects in $\mathsf{End}_{\mathsf{C}}$, then the monoidal product of F and G is $F \circ G$), and the monoidal unit is the identity functor 1_{C} on C (i.e. 1_{C} assigns each object and morphism in C to itself). Then a lax monoidal functor

$$(\mathbb{1}, \otimes, \mathbb{1}) \to (\mathsf{End}_{\mathsf{C}}, \circ, 1_{\mathsf{C}})$$

is a monad. That is, **a monad is a monoid in the category of endofunctors on C.**

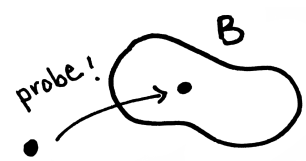
You'll notice that in each of these examples, a change in the semantics category C gives rise to a different interpretation of $\mathbb{1}$, which served as our syntax category. Pretty neat, right? For more details on the examples, see Emily Riehl's *Category Theory in Context* Definitions 1.6.3 and 5.1.1.

The idea that "a monoid in C is a lax monoidal functor $\mathbb{1} \to \mathsf{C}$" is **completely analogous** to claim that "an element in X is a function $\{*\} \to X$" made in the margin on the previous page. In both cases, with have two objects A and B of the same kind (monoidal categories on this page; sets on the previous page) together with a structure-preserving map $A \to B$.

(Caveat: a lax monoidal functor is only *somewhat* structure-preserving. That's why it's called *lax*. And a function is vacuously structure-preserving since sets don't have any structure! But I digress...)

In both cases the object A is trivial (technically, terminal)—it's just a *point*, so to speak. And in both cases the arrow provides an interpretation of that point within the context of B.

To phrase it another way, we are *probing* B with a point-shaped object. In the case when B is a set, probing it with a point will pick out an element. In the case when B is a monoidal category, probing it with a point will pick out a monoid!



THAT'S THE IDEA BEHIND functorial semantics. Now, how is it used in applied category theory? We'll see the answer when we look at the two examples—two research projects from the field—one from chemistry and one from natural language processing. In both examples, the *key* is the existence of a (structure-preserving) functor from a syntax category to a semantics category. Here's a sneak preview:

	syntax	semantics
1st example: (chemistry)	Petri nets	dynamical systems
2nd example: (natural language)	grammar	meanings of words

In Section 3.1 we'll see how the behavior of a chemical reaction network is modeled by a functor

$$\text{Petri nets} \to \text{dynamical systems}$$

as shown in "A Compositional Framework for Reaction Networks" by John Baez and Blake Pollard. In Section 3.2, we'll see how a model for natural language can be described by a functor

$$\text{grammar} \to \text{meanings of words}$$

via the work of Bob Coecke, Mehrnoosh Sadrzadeh, and Stephen Clark in "Mathematical Foundations for a Compositional Distributional Model of Meaning". And perhaps you're wondering, "How do Petri nets, dynamical systems, grammar, and meanings of words form categories? And what's a Petri net, anyway?" We'll answer these questions in the pages to come, but first I'd like to introduce another important theme in applied category theory: compositionality.

1.2 Compositionality

Compositionality, also known as the principal of compositionality, also known as Frege's principle, is the idea that the meaning of a complex expression is determined by

Frege as in *Gottlob Frege.*

1. the meanings of its constituent parts, and
2. the rules for how those parts are combined.

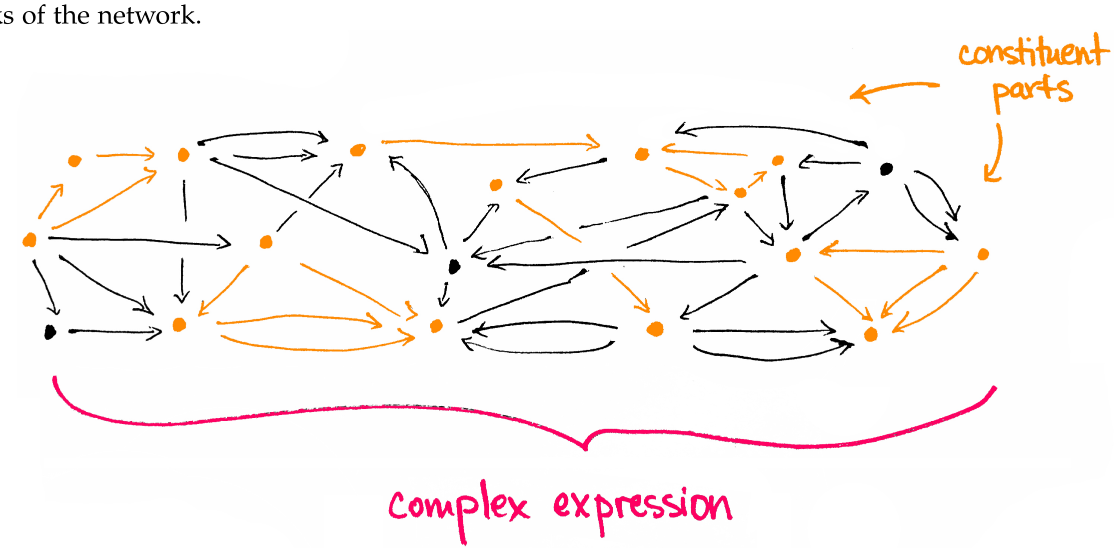
Or, as succinctly stated on the homepage of the brand new journal of applied-category-theory-and-related-fields,

> **compositionality** describes and quantifies how complex things can be assembled out of simpler parts.

As it turns out, the name of that journal is itself *Compositionality*,[4] which hints at the importance of this concept within the field.

[4] A contending title was *Applied Category Theory*, but in the end *Compositionality* had the most votes.

In Section 3.1, which contains our example from chemistry, the complex expression will be a network—a big complicated directed multigraph, if you like. Its constituent parts are simply smaller chunks of the network.



In Section 3.2, our example from natural language, the complex expression will be a sentence; its constituent parts are the words that comprise the sentence.

In both examples, functorial semantics and the principle of compositionality will go hand-in-hand. The former prompts us to model behavior using a functor between syntax and semantics categories. The latter encourages us to take things one at a time: *To model a huge system,* compositionality tells us, *it's enough to model smaller pieces of it and then stick those pieces together.* Simple enough. But what does it mean to "stick pieces together" *mathematically*? The answer is provided by the structure of a **monoidal category**. And *that* is the first of our two main constructions in ACT.

Matrix factorization provides another illustration of compositionality in mathematics. As an example, every $n \times m$ matrix M has a *singular value decomposition*, which means it can be written as a product of three matrices $M = UDV^\dagger$ where U and V are unitary square matrices (here $V^\dagger$ denotes the conjugate transpose of V) and D is a rectangular diagonal matrix. Intuitively then, the linear transformation M can be broken down into a rotation followed by a shear followed by another rotation. So you can analyze your transformation (or your data set, if that's what M is encoding) by understanding its constituent pieces—the factors—and how they compose together. More generally, I like to think that tensor networks are a good example of compositionality, but such a discussion might take us too far off course. Perhaps another day!

1.3 Further Reading

For more on functorial semantics and compositionality:

- Take a look at (this small notice on) William Lawvere's 1963 PhD thesis "Functorial Semantics of Algebraic Theories" for the formal foundations for functorial semantics.

- You might also enjoy this discussion on *doctrines* over at the n-Category Café: https://golem.ph.utexas.edu/category/2006/09/doctrines.html

- The preface to Brendan Fong's PhD thesis, "The Algebra of Open and Interconnected Systems", has a nice discussion on the principal of compositionality and includes various references. And while you're at it, take a look at the entire thesis, which is wonderfully written and provides the backbone of much of John Baez's current research in network theory, which we'll talk a little bit about in Section 3.1.

- Applied category theorist Jules Hedges has also written a nice exposition on compositionality, appropriately entitled "On Compositionality." In the article, you'll find a link to the Stanford Encyclopedia on Philosophy's entry on compositionality, which gives a thorough overview of the topic.

2 Two Constructions

Two constructions that appear over and over (and *over and over and over*) in (some projects in) applied category theory are **monoidal categories** and **decorated cospans**. Let's talk about the first one first.

2.1 Monoidal Categories

Monoids and monoidal categories were the main focus in the digression on page 7, but now I'll proceed as if they are new to the reader.

Actually, before we talk about monoidal categories, let's talk about *monoids*. Here are three examples of monoids: the *integers* $\mathbb{Z}$, the *rational numbers* $\mathbb{Q}$, and the set of all $n \times n$ *matrices* with real-number entries $M_n(\mathbb{R})$. Well, technically *these* are monoids:

$(\mathbb{Z}, +, 0)$	$(\mathbb{Q}, \cdot, 1)$	$(M_n(\mathbb{R}), \cdot, \mathbb{1})$
the set of integers with addition (which is associative!) Note that 0 is the additive identity.	the set of rational numbers with multiplication (which is associative!) Note that 1 is the multiplicative identity.	the set of matrices with matrix multiplication (which is associative!) Note that the identity matrix $\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}$ is the multiplicative identity.

Each example consists of a **set** X equipped with an **associative binary operation**, which I'll denote by $\bullet$. Moreover, there is a special element in the set, let's call it 1, that serves as an **identity** for the operation. *Those three things*—a set, an associative binary operation, an identity—comprise a **monoid**. Usually, we write this triple as

$$(X, \bullet, 1)$$

A "*binary operation* on X" is just the fancy name for a function $X \times X \to X$. So $\bullet$ is a function $\bullet\colon X \times X \to X$. It assigns to a pair $(x, y) \in X \times X$ an element $x \bullet y \in X$. To say that an element $1 \in X$ serves as an *identity* for $\bullet$ means it satisfies $1 \bullet x = x \bullet 1 = x$ for all $x \in X$.

Not too bad, right?

Great. Now imagine replacing the set X by a *category* C, and replacing the binary operation $\bullet\colon X \times X \to X$ by a *functor* $\bullet\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, and replacing the identity element $1 \in X$ by an *object* 1 in C. The resulting triad

$$(\mathsf{C}, \bullet, 1)$$

is called a **monoidal category**. The object 1 is often called the *monoidal unit*, and people usually prefer to write $\otimes$ (and call it the *monoidal product*) instead of $\bullet$ so let's do that too:

a **monoidal category** is a triple $(C, \otimes, 1)$ where

C is a category

$\otimes$ is a functor $C \times C \to C$

1 is an object in C that acts like an "identity" for $\otimes$

There's a little more to the story here since we want $\otimes$ to be associative: for any objects A, B, C in the category C, we want $A \otimes (B \otimes C) = (A \otimes B) \otimes C$. Alas, things are rarely equal on the nose. To compensate for this, we ask instead that there exist an *isomorphism* $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, which should behave nicely. I won't go into the details here, but of course you can find more on the Wikipedia page on monoidal categories. For a delightful exposition on the richness of monoidal categories, I strongly recommend "A Rosetta Stone" by John Baez and Mike Stay. It is a *gem*.

In short, a monoidal category is a category in which it makes sense to "combine" objects and morphisms.[5] As we'll see in Sections 3.1 and 3.2, each of the four categories mentioned on page 10—Petri nets, dynamical systems, grammar, and meanings of words,—are monoidal categories! Here are some more examples.

$(\mathsf{Set}, \times, \{*\})$

↑

the category of sets with the Cartesian product. The monoidal unit is the set with one element, $\{*\}$.

Note: if A & B are sets, then so is $A \times B$! Also,

$A \cong A \times \{*\} \cong \{*\} \times A \cong A$.

↑ "iso to" ↑ "iso to" ↑ "iso to"

$(\mathsf{Top}, \sqcup, \varnothing)$

↑

the category of topological spaces with disjoint union. The monoidal unit is the empty set.

Note: if A & B are spaces, then so is $A \sqcup B$! Also,

$A \cong A \sqcup \phi \cong \phi \sqcup A \cong A$.

$(\mathsf{FVect}, \otimes, \Bbbk)$

↑

the category of finite-dimensional vector spaces over a field $\Bbbk$ with tensor product. The monoidal unit is $\Bbbk$.

Note: if A & B are vector spaces, then so is $A \otimes B$! Also,

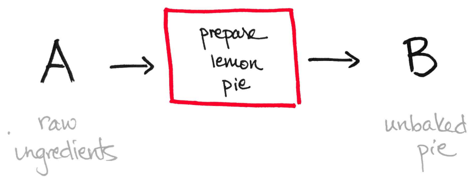
$A \cong A \otimes \Bbbk \cong \Bbbk \otimes A \cong A$.

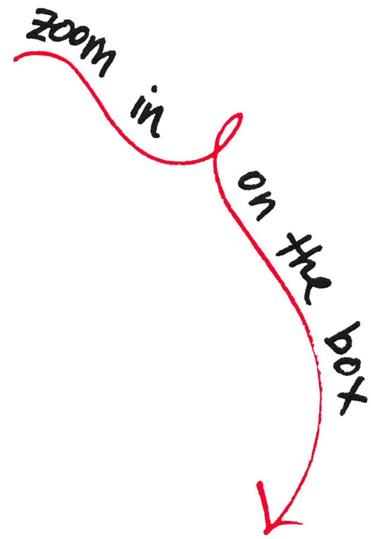
By the way, if there is an isomorphism $A \otimes B \cong B \otimes A$ for all objects A and B that behaves nicely in a sense that can be made precise, then we say that $(\mathsf{C}, \otimes, 1)$ is a **symmetric monoidal category**. Each of the three examples above are symmetric monoidal. Monoidal categories come in other flavors too (braided, Cartesian, closed, Cartesian closed, closed braided,...), depending on which properties are satisfied.

THE MAIN TAKEAWAY here is that monoidal categories are the bread

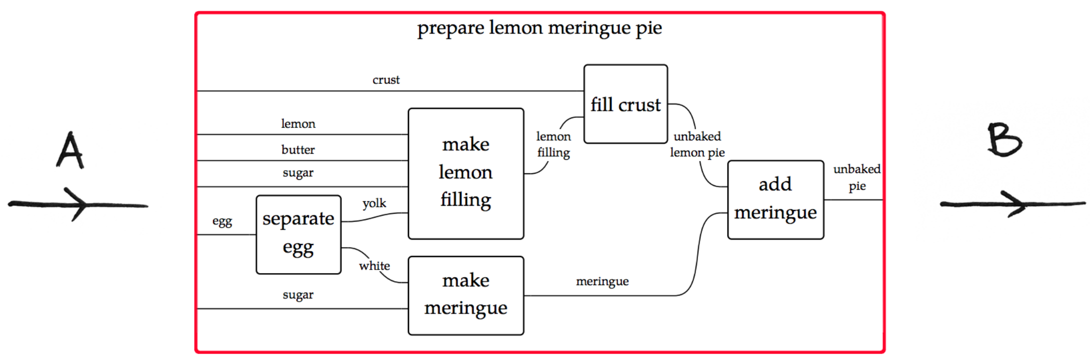
[5] Allow me to explain the "and morphisms" part. First remember, $\otimes$ is a *functor*! That means it's an assignment on objects *and* on morphisms. Consider $(\mathsf{Set}, \times, \{*\})$, for example, where $\times$ assigns to a pair of sets (A, B) their Cartesian product $A \times B$. And given two functions $f\colon A \to B$ and $g\colon A' \to B'$, it assigns to the pair (f, g) the function $f \times g\colon A \times B \to A' \times B'$, which is defined by: $(f \times g)(a, b) := (fa, gb)$. This is one example of the action of the monoidal product on morphisms. More generally, what $f \otimes g$ *is* depends on the explicit definition of $\otimes$.

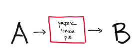
and butter of many applied category theorists. One reason for this is that monoidal categories provide a good setting in which to view morphisms $\rightarrow$ as *physical processes* and objects $A, B, \ldots$ as *states*. As a non-technical example, let's suppose A is a bunch of lemon meringue pie ingredients while B is a fully-assembled-yet-unbaked lemon meringue pie. We might view a morphism $A \rightarrow B$ as the process of mixing the raw ingredients together and then pouring the resulting concoction into a pre-baked crust.



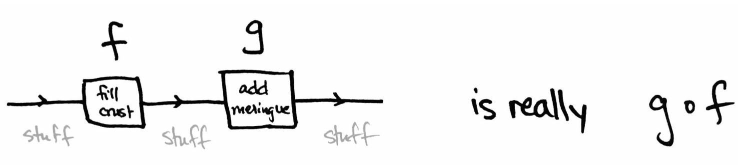


As it turns out, this pie example isn't so silly after all. It's one of the motivating examples that Brendan Fong and David Spivak use in their **excellent** book *Seven Sketches in Compositionality: An Invitation to Applied Category Theory* to illustrate both the ubiquity and the simplicity of monoidal categories. (If you haven't read *Seven Sketches* yet, you really **must**.) Below is a copy of their lemon meringue pie diagram, where I've drawn our A and B on the left as input and right as output.

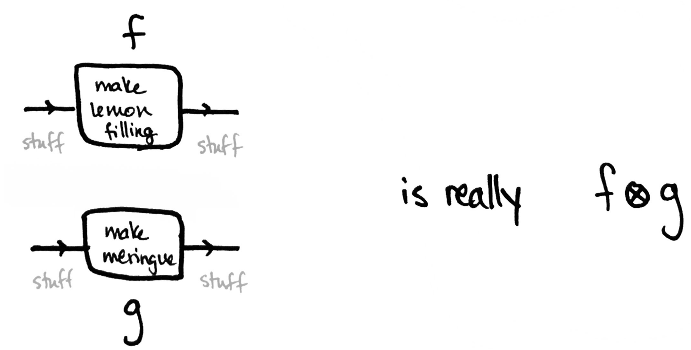


Now that we've zoomed in, we can see that our process  is actually made up of a bunch of other processes! This isn't too surprising as there are several steps that go into preparing a lemon pie: *separating the eggs*, *making the lemon filling*, *filling the crust*, and so on. Fong and Spivak's diagram illustrates just how those those individual steps combine to form the single process *prepare lemon meringue pie*. What's neat is that we can describe these steps using the language of monoidal categories! We'll go into more detail later in this section, but here's a quick preview:

- The category's composition $\circ$ corresponds to using one box's output wire as another box's input wire. For example,



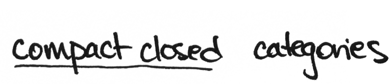
- The monoidal product $\otimes$ corresponds to stacking boxes on top of each other. For example,



In other words, $\circ$ means "do the processes in series" while the monoidal product $\otimes$ means "do the processes in parallel." The resulting picture is called a **string diagram**—a graphical representation of a process (or equation of processes) in a monoidal category. I'll give more detail on how string diagrams work in a second. But first, I'm reminded of something else about monoidal categories that I want to tell you! So let me tell you this new bit of information, then we'll come back to string diagrams. This small digression will, in fact, tie things together quite nicely. Bear with me.

EARLIER, I MENTIONED that the word "symmetric" can be used as an adjective for "monoidal categories":

Similarly, there is another flavor of monoidal categories that we should know about. This one will provide the main setting for our example in Section 3.2:



I'll explain.

Interlude: Compact Closed Categories and String Diagrams

Do you know what a finite dimensional vector space over $\mathbb{R}$ is? Then you know what a compact closed category is! Or rather, you know an example of one:

$$\mathsf{FVect} \textit{ is a compact closed category.}$$

What makes that sentence true? Answer: every finite dimensional vector space V has a *dual* space $V^* = \hom(V, \mathbb{R})$.

That's it.

A **compact closed category** just the name for a monoidal category in which every object has a dual![6] But what does "has a dual" mean? In other words, what makes a dual *dual*? Before I ~~tell you the answer~~ cite the definition, let's think back to the category FVect, by way of motivation, and let's assume each vector space comes with an inner product. In this case, there are two very important linear maps between the ground field—let's say it's $\mathbb{R}$ for now—and a vector space V tensored with its dual:

$$\eta_V : \mathbb{R} \to V \otimes V^* \qquad \epsilon_V : V^* \otimes V \to \mathbb{R}$$

[6] Technically, every object must have a *left* dual and a *right* dual. We need the distinction because not all monoidal categories are *symmetric* monoidal.

In fact, there's a nice fact from linear algebra, namely that once we fix a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V then there is an isomorphism $V \cong V^*$. So let's fix that basis (the standard one) and write V instead of V^*. Also, the subscript is a little cumbersome, so let's drop it for now. So we have two maps

We'll need η and ϵ for a computation in Section 3.2, so it's good to see what they look like explicitly.

$$\eta : \mathbb{R} \to V \otimes V \qquad \epsilon : V \otimes V \to \mathbb{R} \tag{1}$$

The map η is called the **unit** [7], and it assigns to every real number a vector in $V \otimes V$, namely:

$$\eta(1) = \sum_{i=1}^{n} \mathbf{e}_i \otimes \mathbf{e}_i \qquad \text{(and extend linearly)}$$

[7] Note: this unit is not to be confused with "monoidal unit"!

The map ϵ is called the **counit**[8], and it assigns to every vector in $V \otimes V$ a real number, namely:

$$\epsilon\left(\sum_{i=1}^{n} c_{ij}\mathbf{v}_i \otimes \mathbf{w}_j\right) = \sum_{i=1}^{n} c_{ij}(\mathbf{v}_i \cdot \mathbf{w}_j) \qquad \text{where } \cdot \text{ is the inner product}$$

[8] Note: *counit* is pronounced "coh-yew-nit" not "cow-nit." This is important.

Intuitively, we can think of ϵ as an *evaluation* map. That's because there is always a map $V^* \otimes V \to \mathbb{R}$ given by evaluation. Indeed, if $\mathbf{v} \in V$ and $f \in V^*$, then we can pair the two together to obtain $f\mathbf{v} \in \mathbb{R}$.

And if we view f as a $1 \times n$ matrix and $\mathbf{v}$ as an $n \times 1$ matrix, then $f\mathbf{v}$ is their inner product:

$$\begin{bmatrix} \cdots f \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{v} \\ \vdots \end{bmatrix} = \text{ a number}$$

The ϵ map just extends this linearly. That is, if we now have *any* vector $\sum_{ij} c_{ij}\mathbf{v}_i \otimes \mathbf{w}_j$ in $V \otimes V$, then $\epsilon\colon V \otimes V \to \mathbb{R}$ is given by

$$\epsilon\left(\sum_{ij} c_{ij}\mathbf{v}_i \otimes \mathbf{w}_i\right) = \sum_{ij} c_{ij}(\mathbf{v}_i \cdot \mathbf{w}_j)$$

as above.

Finally, the unit η and counit ϵ interact nicely with each other because they satisfy some equations called the *yanking equations*, which I'll explain shortly. The bottom line is that all the above—the maps η and ϵ and the equations they satisfy—makes V^* into a bona fide *dual* for V. The upshot is that compact closed categories generalize these notions.

Definition 2.1. A **compact closed category** is a monoidal category $(\mathsf{C}, \otimes, 1)$ where for every object c in C there exists objects c^l and c^r and morphisms

The η maps are called the left and right **units**, and the ϵ maps are called the left and right **counits**.

$$\begin{aligned} \eta^l_c &\colon 1 \to c \otimes c^l & \qquad \epsilon^l_c &\colon c^l \otimes c \to 1 \\ \eta^r_c &\colon 1 \to c^r \otimes c & \qquad \epsilon^r_c &\colon c \otimes c^r \to 1 \end{aligned}$$

that satisfy the "yanking (or snake) equations"

Here id_c denotes the identity morphism $\mathrm{id}_c\colon c \to c$.

$$\begin{aligned} (\mathrm{id}_c \otimes \epsilon^l) \circ (\eta^l \otimes \mathrm{id}_c) &= \mathrm{id}_c & \qquad (\epsilon^r \otimes \mathrm{id}_c) \circ (\mathrm{id}_c \otimes \eta^r) &= \mathrm{id}_c \\ (\epsilon^l \otimes \mathrm{id}_{c^l}) \circ (\mathrm{id}_{c^l} \otimes \eta^l) &= \mathrm{id}_{c^l} & \qquad (\mathrm{id}_{c^r} \otimes \epsilon^r) \circ (\eta^r \otimes \mathrm{id}_{c^r}) &= \mathrm{id}_{c^r} \end{aligned} \tag{2}$$

Yikes. What do these equations MEAN?

I'm glad you asked.

To answer, it's time to revisit our previous discussion on **string diagrams!** String diagrams are loved by applied category theorists far and wide because *they make life SO much easier.* As we saw earlier, a string diagram is a picture that represents morphisms in a monoidal category C. For now let's take that category to be FVect so that our objects are vector spaces $V, W, \ldots$. In this case, the left and right dual of space V is its vector space dual

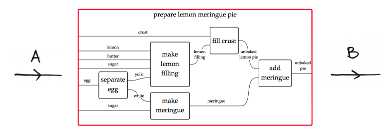
← In that sentence, I meant to convey that *string diagrams* make life so much easier, but one may argue that *applied category theorists* also (are working to) make life so much easier.

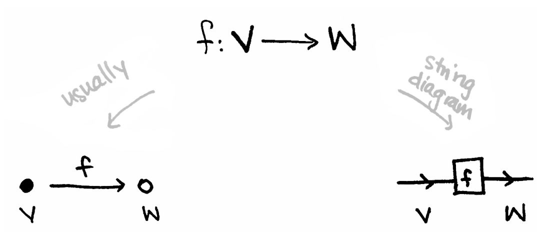
$$V^* = V^r = V^l$$

We'll get to the yanking equations shortly, but first: If this document has been your first introduction into string diagrams, then here is THE KEY thing to know:

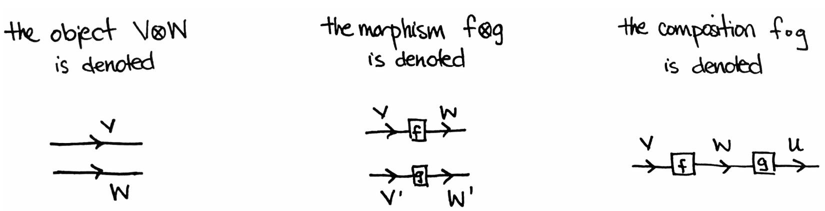
> In category theory, we often draw an object as a dot • and a morphism as an arrow • → ∘. To draw a string diagram, just **do the opposite!** (This goes back to Poincaré duality in topology.) To draw a string diagram, draw an object as an arrow and a morphism as a dot or, even better, a box.

The lemon pie diagram that we saw on page 15 is an example of a string diagram!

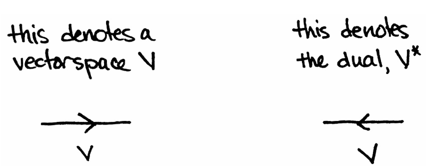




With this small artistic adjustment, we can represent the monoidal product $\otimes$ pictorially as well. The product of two spaces $V \otimes W$ is drawn as two lines, side-by-side. A similar picture holds for the product of two morphisms. Composition $\circ$ is represented by gluing strings together.



And as we saw above, every object in a compact closed category such as FVect has a dual. Its picture is given by an arrow pointing in the *opposite* direction.

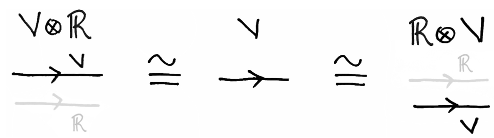


Another special object in a compact closed category is the monoidal unit, for instance $\mathbb{R}$ in FVect. Because the unit is an *object*, it's depicted as an arrow, too. People like to draw this arrow in a special way, namely as the "empty" arrow. In other words, people don't like to draw an arrow. That's because the monoidal unit $\mathbb{R}$ satisfies[9]

$$V \otimes \mathbb{R} \cong V \cong \mathbb{R} \otimes V \qquad \text{for all } V,$$

[9] More generally, the monoidal unit 1 in a monoidal category $(\mathsf{C}, \otimes, 1)$ satisfies $1 \otimes c \cong c \cong c \otimes 1$ for all objects c in C.

which suggests that the unit is "invisible." But I like to draw it anyway, shaded:

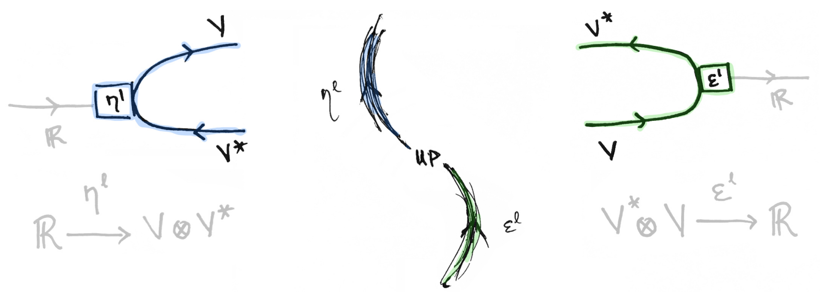


Now we are ready to get back to the yanking equations. Remember, part of the data of a compact closed category is that each object V has left and right duals V^r, V^l *together with* morphisms

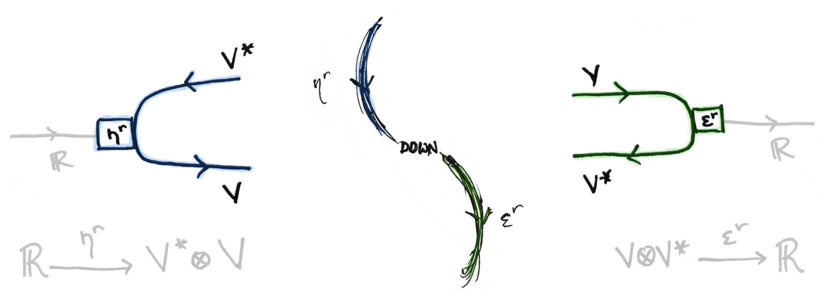
$$\eta^l_V : \mathbb{R} \to V \otimes V^l \qquad \epsilon^l_V : V^l \otimes V \to \mathbb{R}$$
$$\eta^r_V : \mathbb{R} \to V^r \otimes V \qquad \epsilon^r_V : V \otimes V^r \to \mathbb{R}$$

Again, to simplify the notation we'll use the fact that that for vector spaces, $V^* = V^r = V^l$. I'll also drop the subscripts to keep things clean.

Graphically, the ηs and ϵs are drawn as below. The reason we have *two* versions of each map is because the "information flow" can either flow up or it can flow down.



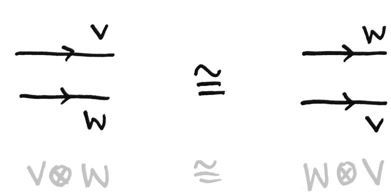
Alternatively, some folks will rotate the ϵ and η diagrams by 90° clockwise and counterclockwise, respectively, which is the reason for their common nickname of "cups and caps."

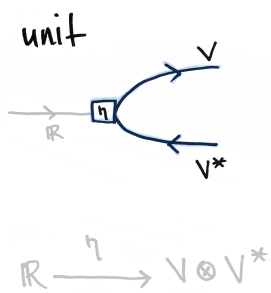


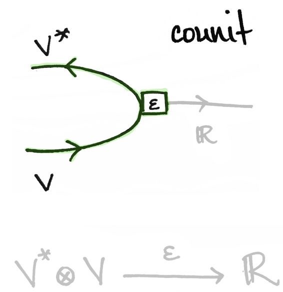
Note: the direction of the (invisible) arrow for the unit $\mathbb{R}$ can go either way. The monoidal unit is always self dual!

And since FVect is a *symmetric* monoidal category, and since the left and right duals are both V^*, there is really only *one* unit and *one* counit for vector spaces.[10]

[10] Remember, the sentence "FVect is symmetric monoidal" means there is an isomorphism $V \otimes W \cong W \otimes V$ for every pair of vector spaces V and W. In string diagram calculus, this means that the order in which we draw our arrows doesn't matter:







That is, in FVect

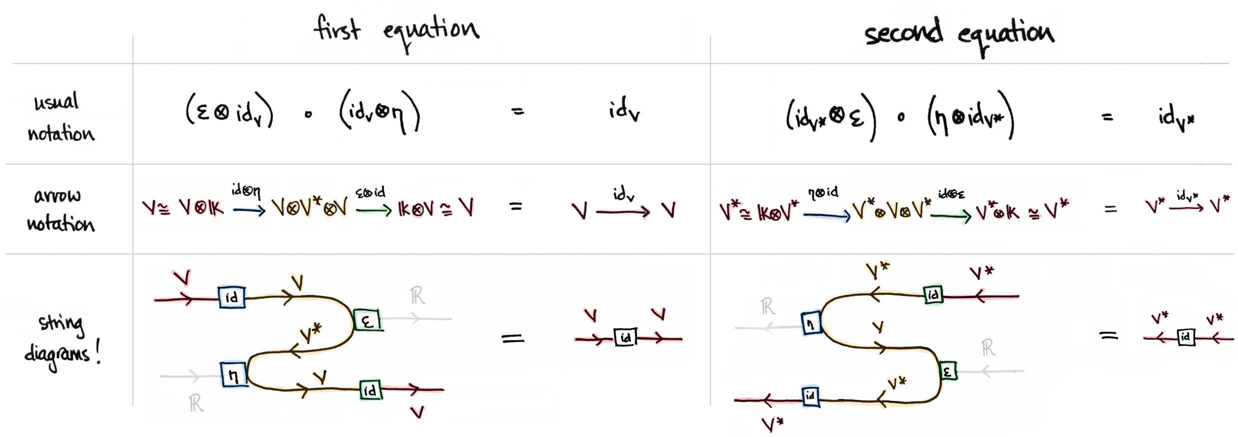
$$\eta = \eta^r = \eta^l \qquad \text{and} \qquad \epsilon = \epsilon^r = \epsilon^l,$$

and these are *precisely* the η and ϵ defined on page 17! So in this example, the four yanking equations of (2) reduce down to just *two*:

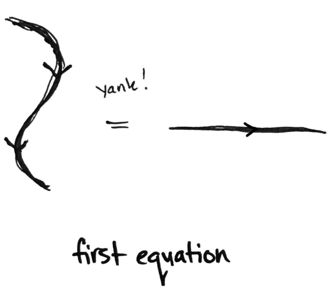
For fun, verify that the unit and counit maps on page 17 do indeed satisfy these two equations.

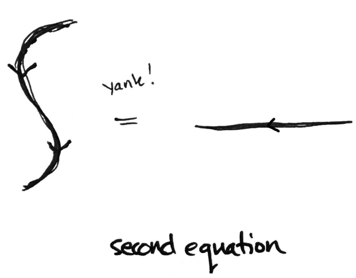
$$\begin{aligned} (\epsilon \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \eta) &= \mathrm{id}_V \\ (\mathrm{id}_{V^*} \otimes \epsilon) \circ (\eta \otimes \mathrm{id}_{V^*}) &= \mathrm{id}_{V^*} \end{aligned} \tag{3}$$

Graphically, these equations can be represented as follows:

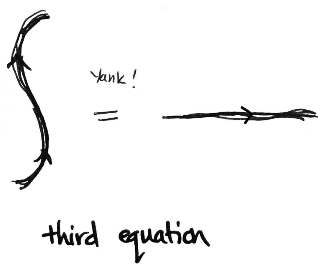


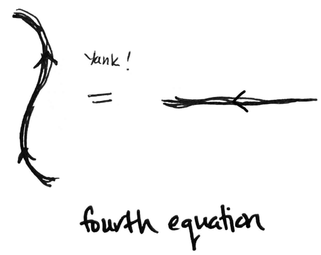
After yanking the strings taut, you'll notice that information flows *rightwards* in the first equation, while it flows *leftwards* in the second equation.





Since we're in a *symmetric* monoidal category, nothing changes if we reverse the arrows in the pre-yanked strings. If, however, the monoidal product $\otimes$ is not symmetric, then we obtain two more diagrams.





This gives a grand total of four equations—the four displayed in (2). And that's what gives us a compact closed category. The category FVect together with the unit and counit defined in (1) will make another appearance in Section 3.2. In that same section, we'll also seen an example of a compact closed category that is *not* symmetric.

By the way, a key feature of FVect (and more generally, all symmetric compact closed categories) is that processes, i.e. morphisms, $V \to W$ are in bijection with *states* $\mathbb{R} \to W \otimes V^* \cong V^* \otimes W$, which is the special name given to morphisms whose domain is the monoidal unit. This bijection is sometimes called **process-state duality**, and in the context of FVect, it means we can view linear maps as vectors in a tensor product[11] and vice versa!

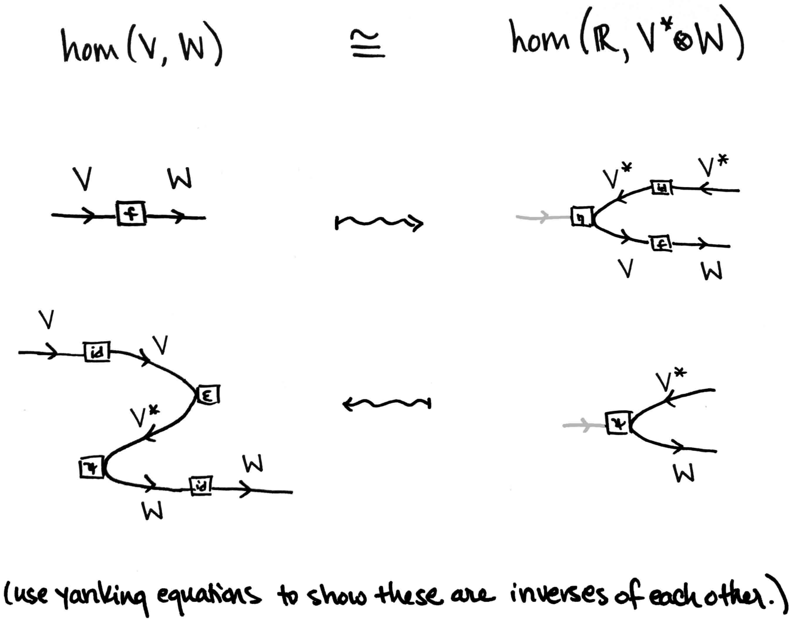
[11] While a linear map $\mathbb{R} \to V^* \otimes W$ is not itself a vector in $V^* \otimes W$, it can be identified with one, namely with the image of 1 in $\mathbb{R}$! More generally, for any finite-dimensional vector space A over $\mathbb{R}$, you can always think of $\hom(\mathbb{R}, A)$ as A itself, at least at the set level. That's because the forgetful functor $U\colon \mathsf{FVect} \to \mathsf{Set}$ is representable with representing object $\mathbb{R}$. In other words, linear maps $\mathbb{R} \to A$ are in one-to-one correspondence with the vectors in A, viewed as elements of its underlying set,

$$\hom(\mathbb{R}, A) \cong UA$$

This is completely analogous to how functions $\{*\} \to X$ from the one-point set to a set X are in one-to-one correspondence with the elements in X,

$$\hom(\{*\}, X) \cong X$$

and is another manifestation of the "probing" idea we saw in the margin on page 9.



I like to think of it this way: when V and W are $\mathbb{R}^n$ and $\mathbb{R}^m$ with the standard bases, process-state duality—taken together with the margin note on the previous page—is akin to the observation that matrices can be viewed as vectors and vice versa. That is, a linear map $\mathbb{R}^n \to \mathbb{R}^m$ has an $m \times n$ matrix representation which can be reshaped into an $nm \times 1$ column vector and then identified with a vector in $\mathbb{R}^n \otimes \mathbb{R}^m$. Conversely, there's a way to identify a vector in $\mathbb{R}^n \otimes \mathbb{R}^m$ with an $nm \times 1$ column vector that can be reshaped into an $n \times m$ matrix, which gives rise to a linear map $\mathbb{R}^n \to \mathbb{R}^m$.

Here we're making the mental identification

$$\text{linear map} \rightsquigarrow \text{process}$$
$$\text{vector} \rightsquigarrow \text{state}$$

which is closely related to the process-state duality seen in quantum physics. It's not quite the same, though—we'd need to replace $\mathsf{Mat}(\mathbb{R})$ by the category of completely positive maps, another compact closed category! For more, see Example 2.4 of "A categorical semantics for causal structure" by Aleks Kissinger and Sander Uijlen, as well as section 4.1.2 and chapter 6 of *Picturing Quantum Processes* by Bob Coecke and Aleks Kissinger.

Aside: There is, I think, a nice categorical way to piece this together. First note that there is a category $\mathsf{Mat}(\mathbb{R})$ whose objects are natural numbers $n, m, \ldots$ and whose morphisms $f\colon n \to m$ are $m \times n$ matrices with real entries. The identity $n \to n$ is the $n \times n$ identity matrix and composition is given by matrix multiplication. This category is actually a compact closed category! The monoidal product on objects is given by multiplication $n \otimes m := nm$ and on morphisms is given by the Kronecker product of matrices. The monoidal unit is $1 \in \mathbb{N}$. For the compact closed structure, each object is self-dual, $n^* := n$, and for each $n \in \mathbb{N}$ the unit map $\eta_n\colon 1 \to n^2$ is the $n^2 \times 1$ column vector obtained by stacking the standard bases vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ on top of each other. In other words, η is given by the Kronecker delta function $\eta_{ij} := \delta_{ij}$. The counit map $\epsilon_n\colon n^2 \to 1$ is the $1 \times n^2$ row vector obtained by taking the transpose of η_n. For example, if $n = 3$ then $\eta_3 = \begin{bmatrix}1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{bmatrix}^\top$ and $\epsilon_3 = \eta_3^\top$. Since $\mathsf{Mat}(\mathbb{R})$ is compact closed, it exhibits process-state duality, too:

$$\hom(n, m) \cong \hom(1, nm)$$

So the entries of η_3 are $\begin{bmatrix}\delta_{11} & \delta_{12} & \delta_{13} & \delta_{21} & \delta_{22} & \delta_{23} & \delta_{31} & \delta_{32} & \delta_{33}\end{bmatrix}$.

This correspondence is precisely the reshaping of $n \times m$ matrices into $nm \times 1$ column vectors and vice versa, which can be verified by using the unit and counit maps in a way analogous to the string diagrams shown at the bottom of the previous page. To tie this in to the remark about $\mathbb{R}^n$ and $\mathbb{R}^m$ in the previous paragraph, note that there is a functor $\mathsf{FVect} \xrightarrow{\cong} \mathsf{Mat}(\mathbb{R})$ sending a vector space V to its dimension $\dim(V)$ and a linear map $f\colon V \to W$ to its corresponding $\dim(W) \times \dim(V)$ matrix representation, and it defines an equivalence of categories! For details, see the discussion on page 30 as well as Corollary 1.5.11 of Emily Riehl's *Category Theory in Context*.

As we'll see in Section 3.2, process-state duality pairs very nicely with our intuition about language. There we'll discover that a *verb* can either be represented as a vector in a tensor product of vector spaces *or* as a linear map, i.e. a process. In other words, a verb is an action in the eyes of both grammar and mathematics!

Another digression: A conjunction with adjunctions?

If you're familiar with adjunctions in category theory, then you might wonder about this choice of naming and notation:

$$\eta \rightsquigarrow \textbf{“unit”} \qquad\qquad \epsilon \rightsquigarrow \textbf{“counit”}$$

Is it a coincidence that these two words are *also* used in the definition of an *adjunction*? NOPE. They are closely related. Specifically, the data V, V^*, η, and ϵ together with the yanking equations *are* an instance of a categorical adjunction! I think this is a neat fact,[12] so let's take yet another digression. Happily, it will tie in quite nicely with our discussion on string diagrams. We'll begin by recalling the definition of an adjunction.

[12] which appears on the first page of "Coherence for Compact Closed Categories" by Kelley and LaPlaza.

Definition 2.2. An **adjunction** between categories C and D is a pair of functors

$$L\colon \mathsf{C} \rightleftarrows \mathsf{D} \colon R$$

Equivalently, L and R form an adjunction if for all objects $c \in \mathsf{C}$, $d \in \mathsf{D}$ there is an isomorphism

$$\hom_{\mathsf{D}}(Lc, d) \xleftrightarrow{\cong} \hom_{\mathsf{C}}(c, Rd)$$

that's natural in both c and d.

and a pair of natural transformations

$$\eta\colon \mathrm{id}_{\mathsf{C}} \Longrightarrow RL \qquad\qquad \epsilon\colon LR \Longrightarrow \mathrm{id}_{\mathsf{D}}$$

There, id_{C} denotes the identity functor on C. It assigns each object and morphism in C to itself.

called the **unit** and **counit** respectively, such that these two triangles commute:

Here, $L \circ \eta$ denotes the natural transformation whose components are of the form $L\eta_c\colon c \to LRLc$, while $\epsilon \circ L$ is the natural transformation with components $e_{Lc}\colon LRLc \to Lc$. A similar story holds for $\eta \circ R$ and $R \circ \epsilon$. (As per the margin comment on page 7, I'd prefer to omit the composition symbol $\circ$, but I'm writing it now for good reason, as we'll soon see!)

$$\begin{tikzcd} L \arrow[r, Rightarrow, "L\circ\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_L"'] & LRL \arrow[d, Rightarrow, "\epsilon\circ L"] \\ & L \end{tikzcd} \qquad\qquad \begin{tikzcd} R \arrow[r, Rightarrow, "\eta\circ R"] \arrow[dr, Rightarrow, "\mathrm{id}_R"'] & RLR \arrow[d, Rightarrow, "R\circ\epsilon"] \\ & R \end{tikzcd}$$

The adjunction is denoted $L \dashv R$, and L is said to be **left adjoint** to R while R is said to be **right adjoint** to L.

Believe it or not, those commuting triangles—often called the *triangle identities*—are closely related to the yanking equations in (3)! Indeed, "these triangles commute" means that these two equations hold:

$$\begin{aligned} (\epsilon \circ L) \circ (L \circ \eta) &= \mathrm{id}_L \\ (R \circ \epsilon) \circ (\eta \circ R) &= \mathrm{id}_R \end{aligned} \tag{4}$$

Now lets compare them to (3):

$$(\epsilon \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \eta) = \mathrm{id}_V$$

$$(\mathrm{id}_{V^*} \otimes \epsilon) \circ (\eta \otimes \mathrm{id}_{V^*}) = \mathrm{id}_{V^*}$$

Why are (3) and (4) so similar? *What's going on here?* Is there a sense in which a vector space and its dual form an *adjunction*?

Is $V \dashv V^*$ a thing?

Yes!

But to make sense of $V \dashv V^*$ we'll need to venture into the world of *2-categories*. A 2-category is an appropriate setting in which to talk about adjunctions, among other things. Here's why. As we know from the definition above, an adjunction consists of

i. some objects (categories)
ii. some arrows (functors)

} **These form a category!**

iii. some arrows *between* the arrows (natural transformations)

Recall: The data of an adjunction are functors between categories

$$L\colon \mathsf{C} \rightleftarrows \mathsf{D} \colon R$$

and natural transformations

$$\eta\colon \mathrm{id}_{\mathsf{C}} \Longrightarrow RL \qquad \epsilon\colon LR \Longrightarrow \mathrm{id}_{\mathsf{D}}$$

You'll notice that the objects and the arrows *themselves* form a category, namely Cat, the category of all categories. The objects of Cat are categories and the morphisms are functors.

Nice.

It'd be even nicer, though, if the natural transformations were also part of the data. That is, it'd be *super nice* if the threesome itself constituted a known categorical construction. But as it stands, it doesn't. There is no room for a notion of "arrows between arrows" in the definition of a category.

So what do we do?

We expand the definition. *Literally.* We add an extra dimension, which results in a 2-category. That is, a 2-category consists of

i. objects, now called **0-cells** $\bullet, \circ, \ldots$

ii. morphisms that go between objects, which are the usual arrows, but now we'll call them **1-cells** $\bullet \longrightarrow \circ$

iii. morphisms that go between 1-cells, which are not surprisingly called **2-cells** $\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \circ \end{tikzcd}$

What's more, in any 2-category there is a composition rule for 2-morphisms just like there is for 1-morphisms in an ordinary category! In fact, in a 2-category we require that the set $\hom(\bullet, \circ)$ be *more* than a set. We ask that it be a *category* itself! Its objects are 1-cells $\bullet \to \circ$, and its morphisms are 2-cells $\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \circ \end{tikzcd}$ There is also an identity 1-cell $\mathrm{id}_\bullet$ for each 0-cell $\bullet$ *and* there is an identity 2-cell $\mathrm{id}_{\mathrm{id}_\bullet}$ for each $\mathrm{id}_\bullet$. Confusingly, both of these identity morphisms are sometimes denoted as $\bullet$. And of course, there are the usual identity and associativity axioms, though I won't write them here.

As you might guess, the quintessential example of a 2-category is Cat, where the

i. 0-cells are *categories* $\mathsf{C}, \mathsf{D}, \ldots$

ii. 1-cells are *functors* $\mathsf{C} \xrightarrow{F} \mathsf{D}$

iii. 2-cells are *natural transformations* $\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathsf{D} \end{tikzcd}$

So a 2-category is a good generalization of the relationship we see exhibited among categories, functors, and natural transformations. Having generalized this trio, it becomes very easy to talk about "adjunctions" in any 2-category. Parallel to Definition 2.2, we might lay down the following proposed definition:

> **Definition (proposed).** An **adjunction** between 0-cells $\bullet$ and $\circ$ is a pair of 1-cells l and r,
>
> $$l\colon \bullet \rightleftarrows \circ : r$$
>
> and a pair of 2-cells η and ϵ, called the **unit** and **counit** respectively,
>
> $$\begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{id}_\bullet"{name=U}] & \bullet \arrow[l, bend left, "r\circ l"{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\eta"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \circ \arrow[r, bend left, "\mathrm{id}_\circ"{name=U}] & \circ \arrow[l, bend left, "l\circ r"{name=D}] \arrow[from=U, to=D, phantom, "\Uparrow\epsilon"] \end{tikzcd}$$
>
> such that these two triangles commute
>
> $$\begin{tikzcd} l \arrow[r, Rightarrow, "l\circ\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_l"'] & l\circ r\circ l \arrow[d, Rightarrow, "\epsilon\circ l"] \\ & l \end{tikzcd} \qquad \begin{tikzcd} r \arrow[r, Rightarrow, "\eta\circ r"] \arrow[dr, Rightarrow, "\mathrm{id}_r"'] & r\circ l\circ r \arrow[d, Rightarrow, "r\circ\epsilon"] \\ & r \end{tikzcd}$$
>
> i.e. such that the following equations hold[13]
>
> $$(\epsilon\circ l)\circ(l\circ\eta) = \mathrm{id}_l \qquad \text{and} \qquad (r\circ\epsilon)\circ(\eta\circ r) = \mathrm{id}_r$$
>
> where $l\circ\eta := \mathrm{id}_l \circ \eta$, and similarly for $\epsilon\circ l$ and so on. We'll say l is a **left adjoint** of r, and r is a **right adjoint** of l, and we'll denote the adjunction by $l \dashv r$.

[13] If you do a Google search for "definition of 2-category" you'll soon find that 2-cells can be composed in two ways: "vertically" and "horizontally." I didn't mention this earlier, but now's a good time to do so. Suppose we have three 1-cells from $\bullet$ to $\circ$ and 2-cells η and ϵ as shown below on the left,

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=A}] \arrow[r, ""{name=B}] \arrow[r, bend right=60, ""{name=C}] \arrow[from=A, to=B, phantom, "\Downarrow\eta"] \arrow[from=B, to=C, phantom, "\Downarrow\epsilon"] & \circ \end{tikzcd} \rightsquigarrow \begin{tikzcd} \bullet \arrow[r, bend left, ""{name=A}] \arrow[r, bend right, ""{name=C}] \arrow[from=A, to=C, phantom, "\Downarrow\epsilon\diamond\eta"] & \circ \end{tikzcd}$$

then *vertical* composition $\diamond$ gives a 2-cell $\epsilon\diamond\eta$ as shown above on the right. This is *composition along a common 1-cell* $\to$. On the other hand, given four 1-cells as shown below left,

$$\begin{tikzcd} \bullet \arrow[r, bend left, ""{name=A}] \arrow[r, bend right, ""{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow\eta"] & \star \arrow[r, bend left, ""{name=C}] \arrow[r, bend right, ""{name=D}] \arrow[from=C, to=D, phantom, "\Downarrow\epsilon"] & \circ \end{tikzcd} \rightsquigarrow \begin{tikzcd} \bullet \arrow[r, bend left, ""{name=A}] \arrow[r, bend right, ""{name=C}] \arrow[from=A, to=C, phantom, "\Downarrow\epsilon\circ\eta"] & \circ \end{tikzcd}$$

horizontal composition $\circ$ gives a 2-cell $\epsilon\circ\eta$ as shown above right. This is *composition along a common 0-cell* $\star$. Moreover, the triangle identities involve both compositions. That is, the *actual* equations are

$$(\epsilon\diamond l)\circ(l\diamond\eta) = \mathrm{id}_l \quad \text{and} \quad (r\diamond\epsilon)\circ(\eta\diamond r) = \mathrm{id}_r$$

Take note of the diamonds vs. the circles!

Alright, fine. *But what does this have to do with vector spaces?*

The answer lies in the following neat fact.

Neat Fact: Every monoidal category $(\mathsf{C}, \otimes, 1)$ *can be viewed as a 2-category!*

Er, actually, I shouldn't spread rumors.

~~*Neat Fact: Every monoidal category* $(\mathsf{C}, \otimes, 1)$ *can be viewed as a 2-category!*~~

Here's the *correct* statement:

Neat Fact: Every monoidal category $(\mathsf{C}, \otimes, 1)$ can be viewed as a bicategory!

A bicategory is basically a 2-category—the data is completely the same. There are 0-cells, 1-cells, and 2-cells. The only difference is what's in the margin.[14] So any monoidal category $(\mathsf{C}, \otimes, 1)$ *gives rise to* a bicategory $\mathcal{C}$ where the

[14] In a 2-category, the composition of 1-cells is associative, i.e. $f(gh) = (fg)h$ for any composable triple of 1-cells f, g, h. In a *bicategory*, however, we weaken this. Instead of asking for equality, we ask for the existence of an invertible 2-cell $f(gh) \Longleftrightarrow (fg)h$. As we'll see below, the category FVect gives rise to a bicategory rather than a 2-category because the two vector spaces $V\otimes(W\otimes U)$ and $(V\otimes W)\otimes U$ are not *equal*, but there certainly is a linear isomorphism

$$V\otimes(W\otimes U) \xleftrightarrow{\cong} (V\otimes W)\otimes U!$$

i. only 0-cell is the *category* C

ii. 1-cells are the *objects* of C; composition is $\otimes$

iii. 2-cells are the *morphisms* of C; composition is composition $\circ$ in C

Therefore it makes sense to talk about 1-cells in $\mathcal{C}$ (i.e. *objects* in C) having adjoints! And it makes sense to talk about 2-cells in $\mathcal{C}$ (i.e. *morphisms* in C) being units and counits of the adjunction, vis-a-vis our Proposed Definition! In particular, this is true of the symmetric monoidal category $(\mathsf{FVect}, \otimes, \mathbb{R})$. It gives rise to a bicategory $\mathcal{F}\mathsf{Vect}$ where the

i. only 0-cell is the category FVect

ii. 1-cells are *vector spaces*; composition is the tensor product $\otimes$

iii. 2-cells are *linear maps*; composition is the usual composition $\circ$

So there is an adjunction of vector spaces $V \dashv W$ whenever the conditions of our Proposed Definition hold. Of course, those conditions hold *precisely* when $W = V^*$ and η and ϵ are defined as in (1). Explicitly:

You'll notice that the monoidal unit $\mathbb{R}$ is taking the place of $\mathrm{id}_\bullet$ in the Proposed Definition. Indeed, $\mathrm{id}_\bullet$ and $\mathbb{R}$ are comparable since both are 1-cells that act as an identity on other 1-cells: For all 1-cells $\bullet \xrightarrow{f} \circ$ in $\mathcal{C}$

$$f \circ \mathrm{id}_\bullet = f = \mathrm{id}_\circ \circ f$$

and for all vector spaces V in $\mathcal{F}\mathsf{Vect}$,

$$V \otimes \mathbb{R} \cong V \cong \mathbb{R} \otimes V.$$

There is an **adjunction** $V \dashv V^*$ in the bicategory $\mathcal{F}\mathsf{Vect}$ since there are linear maps

$$\mathbb{R} \xrightarrow{\eta} V^* \otimes V \qquad \text{and} \qquad V \otimes V^* \xrightarrow{\epsilon} \mathbb{R}$$

so that the following triangles commute

$$\begin{tikzcd} V \cong V \otimes \mathbb{R} \arrow[r, Rightarrow, "V\otimes\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_V"'] & V \otimes V^* \otimes V \arrow[d, Rightarrow, "\epsilon\otimes V"] \\ & \mathbb{R} \otimes V \cong V \end{tikzcd} \qquad \begin{tikzcd} V^* \cong \mathbb{R} \otimes V^* \arrow[r, Rightarrow, "\eta\otimes V^*"] \arrow[dr, Rightarrow, "\mathrm{id}_{V^*}"'] & V^* \otimes V \otimes V^* \arrow[d, Rightarrow, "V^*\otimes\epsilon"] \\ & V^* \otimes \mathbb{R} \cong V^* \end{tikzcd}$$

i.e. so that the following equations hold

$$(\epsilon \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \eta) = \mathrm{id}_V \qquad \text{and} \qquad (\mathrm{id}_{V^*} \otimes \epsilon) \circ (\eta \otimes \mathrm{id}_{V^*}) = \mathrm{id}_{V^*}$$

and these are precisely the string diagram equations shown in the chart on page 21.

On the leftmost triangle, the notation $V \otimes \eta$ denotes the linear map

$$\mathrm{id}_V \otimes \eta \colon V \otimes \mathbb{R} \to V \otimes V^* \otimes V$$

that appears in the first equation. A similar statement holds for $\epsilon \otimes V$, etc. Also, take note of the different symbols $\otimes$ and $\circ$ and compare them with the diamond $\diamond$ and circle $\circ$ in the margin on the previous page.

Voila!

Finally, notice that the above holds for *every* vector space V in FVect. On the other hand, there are certainly 2-categories in which not every 1-cell is *dualizable*, i.e. has an adjoint. Take Cat for instance! Not every functor is part of an adjunction. There *is*, however, a special name given to those bicategories $\mathcal{C}$ that do arise from a monoidal category C and in which every 1-cell has an adjoint.

That name is **compact closed.**

The punchline for this section is that monoidal categories are an appropriate framework for *stacking things together,* and the calculus of string diagrams allows us to replace complicated, messy equations by simple, neat pictures. In Section 3, we'll see two examples of how this can be put into practice.

2.2 Decorated Cospans

A second construction that appears in some work within applied category is the *decorated cospan.* In any category, a diagram that looks like

$$A \to C \leftarrow B.$$

is called a **cospan**. In the next section, we'll only consider the case when A, B, and C are finite sets and the arrows are functions between them. A **decorated cospan** is a cospan where the middle set C has been endowed with some extra structure. That's the intuitive definition, though I'd like to postpone a more precise definition until the next section.

Now you might think it strange to give a name to a simple diagram like $A \to C \leftarrow B$, but cospans come in handy quite often! For instance, if for some reason you can't possibly hope to find a morphism between objects A and B, a common technique[15] is to instead look for a "larger" object C that "contains" both A and B. Then although you don't have maps between A and B, you *do* have maps $A \to C \leftarrow B$. In that case, your cospan is the next best thing.

[15] I learned this from Brendan during the 2018 ACT workshop. Thanks, Brendan!

ADMITTEDLY, this section is *bite-sized* compared to the behemoth on monoidal categories that we just finished, but that's not because decorated cospans are any less important! In fact, Brendan Fong developed the theory of decorated cospans as part of his PhD thesis "The Algebra of Open and Interconnected Systems", which has served as the foundation for *much* progress in applied category theory, as I mentioned earlier. But in these notes, we'll only use the cospan construction in our brief discussion on chemical reaction networks in Section 3.1. On the other hand, we will need the language of monoidal categories in both Sections 3.1 and 3.2. In fact, as we'll soon see, decorated cospans *themselves* form a monoidal category!

2.3 Further Reading

For more on monoidal categories and string diagrams:

- Read Chapters 3 and 4 of *Picturing Quantum Processes* by Bob Coecke and Aleks Kissinger. There you'll also find more information on the interpretation of morphisms in a monoidal category as *processes* and objects as *systems*.
- Take a look at TheCatsters videos on string diagrams, by Eugenia Cheng and Simon Willerton. On second thought, their entire collection of videos is great. Go watch them all!
- If you like ∞-categories, you'll be delighted to know that a version of string diagrams (affectionately called "strictly undulating squiggles") and the yanking equations (!) make an appearance in chapter 8 of *Elements of ∞-Category Theory*, a new book on model-independent ∞-category theory by Emily Riehl and Dominic Verity.

For more on decorated cospans:

- Read "Decorated Cospans" a blog post by John Baez on the *n*-Category Café.
- Read Chapter 6 of *Seven Sketches in Compositionality* by Brendan Fong and David Spivak. In Section 6.1, the authors give the following bit of motivation:

  > ...we produce a certain monoidal category—namely that of *cospans in* [a category] C, denoted $\mathbf{Cospan}_{\mathsf{C}}$—that can conveniently package C's colimits in terms of its own basic operations: composition and monoidal structure. In summary, the first part of this chapter is devoted to the slogan 'colimits model connection.' (emphasis theirs)

  As we'll see in Section 3.1, objects in $\mathbf{Cospan}_{\mathsf{C}}$ are cospans in C and a morphism between two cospans is given by a construction called a *colimit*. Like the composition $\circ$ and product $\otimes$ in a general monoidal category, a colimit is a categorical construction that allows you to connect things together. But for the sake of "time" (i.e. so that this document doesn't accidentally turn into a *book*...), I'll assume familiarity with colimits. But if you'd like to an intuitive introduction of colimits, as well as their dual construction, limits, I recommend that you

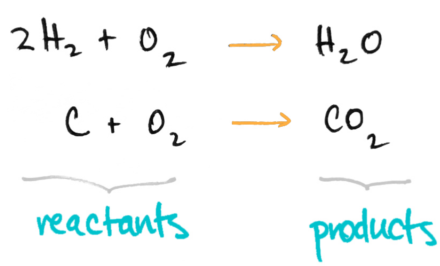
- Take a look at "Limits and Colimits (Part 1)" a blog post on Math3ma. Also see chapters 3 and 6 of *Seven Sketches* and chapter 3 of *Category Theory in Context* by Emily Riehl.

3 Two Examples

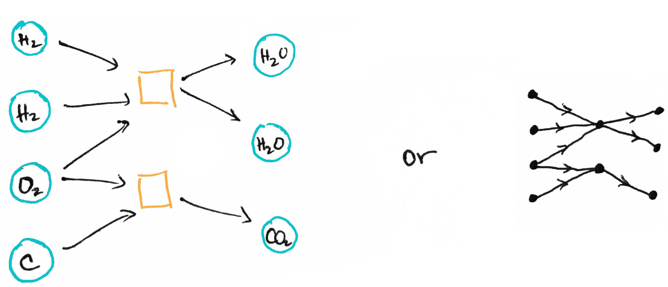
Having taken a leisurely stroll through two themes (functorial semantics and compositionality) and two constructions (monoidal categories and decorated cospans) within applied category theory, it's time to see them come to life in two examples. As mentioned in the introduction, we'll walk through the first example—chemical reaction networks—relatively quickly. There are several excellent resources available online, including John Baez's expositions on the *n*-Category Café as well as on his personal webpage. (I've included a few links to these in Section 3.3.) Afterwards we'll take a longer stroll through the second example—natural language processing—in Section 3.2.

3.1 Chemical Reaction Networks

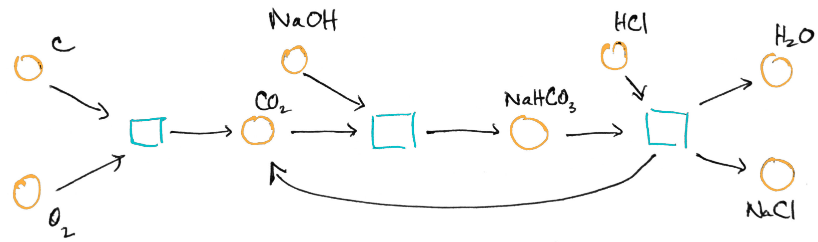
The first example comes from a paper by John Baez and Blake Pollard called "A Compositional Framework for Reaction Networks." Specifically, they provide a compositional framework for modeling *chemical* reaction networks. A **chemical reaction network** is, well, a networks of chemical reactions. And a **chemical reaction** is exactly what you think it is. It's what you learned back in high school: You start with some reactants and some products, and there's a chemical process that takes one to the other.



What's nice is that these reactions can be depicted graphically:

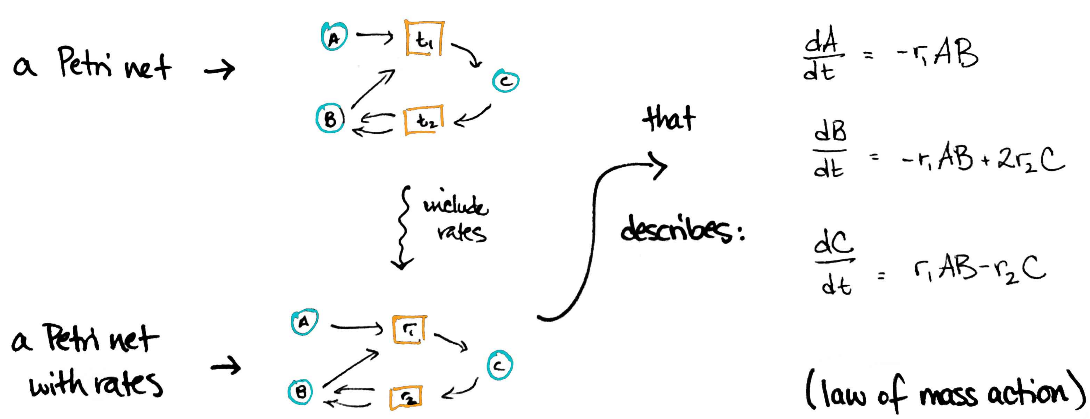


Of course, you can imagine that there might be *lots* of various reactants, products, and chemical processes. The corresponding *network* would then be a (possibly huge) collection of these graphs stacked side-by-side, perhaps with connecting edges and loops and so on. For instance, this chemical reaction network made a cameo appearance in Baez's 2016 talk "The Mathematics of Networks":



Graphs such as these are examples of Petri nets. A **Petri net** is essentially a bipartite directed (multi)graph that allows us to visually represent reactions, though they are used outside of chemistry as well.

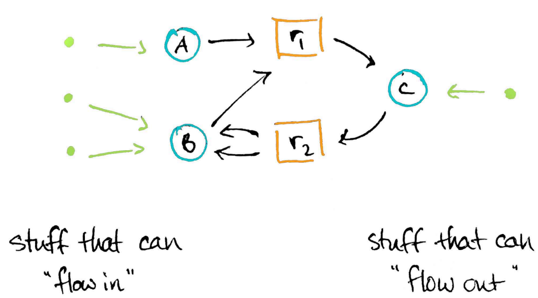
But if we do wish to model chemical reactions, then an important thing we'd like to account for is the *rate* at which one or more chemicals change over to another. A Petri net with rates included is called, appropriately, a **Petri net with rates**. More specifically, it's a bipartite directed graph whose two types of vertices are called places, which represent chemical species, and transitions, which represent chemical reactions. Moreover, each transition τ_i is assigned a *rate* r_i, a positive real number that describes how fast or how likely it is for τ_i to occur. These rates then allow us to write down differential equations that describe the system. A Petri net with rates is thus a pictorial representation of a set of differential equations that describe a system. So, for instance, if you *did* watch Baez's "The Mathematics of Networks" talk then this example will look familiar:



It tells us that, for example, substances with concentrations A and B combine and produce a substance with concentration C at a rate proportional to r_1. The differential equations you see are due to the

law of mass action, which says that the rate with which a chemical reaction will occur is equal to its rate constant r_i multiplied by the product of the concentration of the reactants, i.e. the concentration of the "inputs" of the reaction.

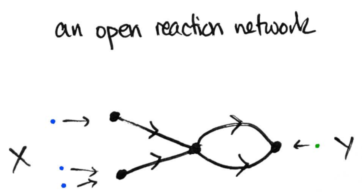
By the way, the rates themselves could change with time, which might suggest the presence of a **dynamical system.** What's more, a system such as the above could potentially interact with its environment, which is to say there might be some quantities that flow in and some quantities that flow out, resulting in an **open Petri net with rates**:

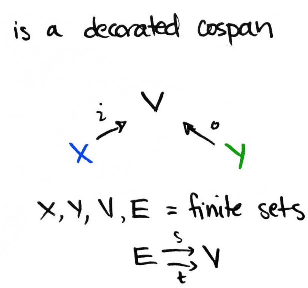


These quantities can be incorporated into the equations, too, resulting in an **open dynamical system.** As Baez and Pollard summarize, the goal is to use these observations "to build up a reaction network from smaller pieces, in such a way that its rate equation can be determined from those of the pieces." This is what is meant by a *compositional* framework, and is a prime is example of the principal of compositionality mentioned in Section 1.2. What's more, a key step towards achieving this goal is given by functorial semantics! That is, we start by thinking of a Petri net as **syntax** and a set of differential equations as **semantics**. And if we have a *collection* of Petri nets that model a very large network, then—guided by the principal of compositionality—we would like to compose them by gluing graphs together, and we would like to aggregate them by stacking graphs on top of each other. In other words, we hope that Petri nets form a monoidal category! Similarly, one would hope that there is a sense in which dynamical systems form a monoidal category so that differential equations can be "composed" and "aggregated" as well. One would *also* wish for a monoidal functor $\textsf{Petri nets} \to \textsf{dynamical systems}$.

That's a lot of wishes, but amazingly they all come true, for this is precisely what Baez and Pollard proved in their paper! But how exactly? How was it all formalized? The key is the decorated cospan construction of Brendan Fong that we mentioned in Section 2.2. (What's amazing is that Fong's construction is general enough to model other **open reaction networks**[16] as well! But more on that later.)

[16] This is the catch-all phrase for a network that interacts with its environment so that stuff can either flow in or flow out

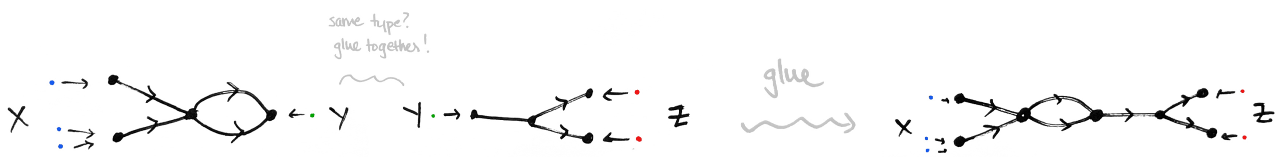




The reason the arrows from X and Y point *in* is that X and Y might be thought of as "leftputs and rightputs" rather than as inputs and outputs. In other words, you'd like the freedom to think of things as flowing either in *or* out of either end. For example, a physical pipe doesn't know the different between left and right. Water can flow in or out at either direction. As Baez notes, "The main reason for these designations is to remember that when we screw together two pipes, we attach the output of the first to the input of the second."

A **cospan** in the category of **finite sets**, for example, is just a diagram of the form $X \to V \leftarrow Y$, where we're meant to think of V as the set of places (i.e. chemical species) in the Petri net. To account for the edges in the graph, we ask that V is "decorated" with extra structure, namely source and target maps from the set of edges. This results in a **decorated cospan,** and Fong proved that these constructions form a category! That is, there is a category where objects are finite sets $X, Y, \ldots$, and a morphism $X \to Y$ is a decorated cospan whose feet are X and Y. Composition is given by the *pushout*[17], which amounts to gluing graphs together. (This composition is only associative up to isomorphism, so the morphisms are really *isomorphism classes* of cospans. Also, the identity $X \to X$ is the empty graph.)

[17] A pushout is a type of colimit, a major construction in category theory that's a bit like like mathematical glue. Anytime you mush two mathematical objects together—like the graphs in the picture—you've probably got a colimit construction. More intuition behind colimits and their dual construction, limits, can be found on Math3ma.



What's more, Fong showed that this category has a **symmetric monoidal** structure by stacking graphs on top of each other, i.e. by taking their disjoint union. In fact, it's **compact closed** and also a **hypergraph** category!

A hypergraph category is a symmetric monoidal category in which every object has a special commutative Frobenius structure. This allows more freedom (i.e. messiness) when composing morphisms, reflecting the messiness of most network diagrams!

In summary, Fong's constructions quickly give rise to key results, which I'll summarize here. The first is that the syntax category of Baez and Pollard is indeed a category:

Theorem 3.1 (Baez, Pollard)**.** *There is a symmetric monoidal category* Petri *where*

- ***objects** are finite sets $X, Y, \ldots$*
- *a **morphism** $X \to Y$ is a **open Petri net with rates**, i.e. a cospan*

$$\begin{tikzcd} & V & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

together with a Petri net with rates whose places are comprised of V.

Really, it's an isomorphism class of cospans. Also, you'll notice that in Theorem 12 of Baez and Pollard's "A Compositional Framework," their syntax category is something called RxNet. That stands for the category of *open reaction networks with rates*. An open reaction network is very nearly the same as an open Petri net, though I'm glossing over this a bit.

The next corollary provides the same statement for the semantics category:

Theorem 3.2 (Baez, Pollard)**.** *There is a symmetric monoidal category* Dynam *where*

- ***objects*** *are finite sets* $X, Y, \ldots$
- *a* ***morphism*** $X \to Y$ *is an* ***open dynamical system,*** *i.e. a cospan*

$$\begin{tikzcd} & V & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

together with a smooth vector field on $\mathbb{R}^V$*.*

Again, it's really an isomorphism class of cospans. And again we can think of V as the set of all places in a Petri net where, as before, there may be a real number r_i attached to each vertex than can vary with time. The description of *how* these things vary in time is precisely a vector field on $\mathbb{R}^V$.

Finally, another result of Baez and Pollard shows the existence of a symmetric monoidal functor $\square\colon \mathsf{Petri} \to \mathsf{Dynam}$ from the syntax to the semantics.

Theorem 3.3 (Baez, Pollard)**.** *There is a symmetric monoidal functor* $\square\colon \mathsf{Petri} \to \mathsf{Dynam}$ *sending any open Petri net with rates to the corresponding open dynamical system.*

But see Theorem 18 of Baez and Pollard where, since RxNet is used in lieu of Petri, the symmetric monoidal functor $\mathsf{RxNet} \to \mathsf{Dynam}$ is a slightly different *gray boxing functor*.

The upshot is that functoriality and monoidality

$$\square(f \circ g) = \square f \circ \square g$$
$$\square(f \otimes g) = \square f \otimes \square g$$

tell us that if you want to understand the open dynamical systems of the composite (or tensor product) of two open Petri nets, then you just have to find the open dynamical systems of each one and then compose (tensor). This is *exactly* the **principle of compositionality**: to determine the behavior of a big complicated thing, you need only understand the behaviors of its components, and then assemble them together. And by the way, this works for many other kinds of network graphs, not just Petri nets. It's all part of Baez's larger body of work on a general categorical framework for a theory of networks which encompasses electrical circuits, Markov processes, signal-flow graphs in control theory, and more!

THIS RAPID TOUR through chemical reaction networks is only *one* way that compositionality, functorial semantics, and monoidal categories are being used in applications. The next example gives a second way: natural language processing.

3.2 Natural Language Processing

At long last, we've made it to our second application of category theory—natural language processing! It is, simply put, a branch of artificial intelligence that aims to train computers to understand human language. What's nice is that computers can understand meanings of words (through models like Word2vec, for instance[18]) and computers can understand grammar (through parts of speech tagging, for instance[19]). But what's not-so-nice is that computers aren't too good at understanding meanings of *sentences* and longer bodies of text.

In 2010, Bob Coecke, Mehrnoosh Sadrzadeh, and Stephen Clark sought to address this problem in "Mathematical Foundations for a Compositional Distributional Model of Meaning." In this paper, the authors rely heavily on the principle of compositionality—the idea that the meaning of a sentence can be determined by the meanings of its individual words together with the grammatical rules for combining them. So if a computer can understand meanings of individual words and if it can understand grammatical rules, then then only thing it needs help with is knowing how to combine them to form a meaningful whole. And that's where the category theory comes in! Guided by functorial semantics, Coecke et. al. model natural language as a (monoidal) functor between *compact closed categories*

$$\mathsf{grammar} \to \mathsf{meanings\ of\ words}$$

This functor assigns a grammar type to a word, and the monoidal structures provide a way to combine the meanings of those words (and their grammar types) to form a sentence, whose meaning can be determined via the principal of compositionality.

In the remaining pages, we'll dive into the details by answering the following questions:

i. **(the syntax category)** How can we make sense of grammar, mathematically? Specifically, how does $\mathsf{grammar}$ form a compact closed category?

ii. **(the semantics category)** How can we make sense of meanings of words, mathematically? That is, how do $\mathsf{meanings\ of\ words}$ form a compact closed category?

iii. **(the functor)** How is the functor $\mathsf{grammar} \to \mathsf{meanings\ of\ words}$ defined and how does it allow one to determine the meaning of a full sentence?

Let's start by answering the first question.

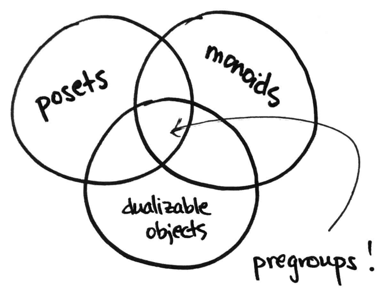
[18] In the literature, these models are often called *distributional models of meanings.*

[19] These are often called *symbolic* or *compositional models of meaning.*

The Syntax Category: Pregroup

Following the work of Coecke et. al., we can model grammar algebraically via a *pregroup*, a construction due to mathematician Joachim Lambek in the early 1990s. Informally, a pregroup is cooked up from the following recipe:

poset + **monoid** + **"duals"** = **pregroup**



If you were to develop your own categorical-compositional-distributional model (often called **DisCoCat** models) of meaning, you might wish to work with a construction *other* than pregroups. And that's fine. But the nice thing about a pregroup is that (as we'll soon see) it has the exact same categorical structure as our semantics category, namely compact closure. That means we consider a functor from a pregroup into the semantics category that *preserves* the compact-closed structure. More generally, then, you might define a DisCoCat-type language model to be any monoidal functor $\mathsf{C} \to \mathsf{meanings\ of\ words}$ where C is any compact closed category accounting for grammar.

In other words, a pregroup is

- a poset $(P, \leqslant)$
- that has a multiplication (we'll denote it by juxtaposition) that's compatible with the partial order, i.e. if $p \leqslant q$ then $ap \leqslant aq$ and $pa \leqslant qa$ for all $a \in P$,
- together with a unit 1 satisfying $1p = p1 = p$ for all $p \in P$,
- and moreover each element p has both a left dual p^l and a right dual p^r with maps

$$p^l p \overset{\epsilon^l}{\leqslant} 1 \overset{\eta^l}{\leqslant} pp^l \qquad \text{and} \qquad pp^r \overset{\epsilon^r}{\leqslant} 1 \overset{\eta^r}{\leqslant} p^r p.$$

  that are required to satisfy the yanking equations in (2).

I've referred to the inequalities a *maps* (and have labeled them as such) because they are actually *morphisms* in a category! Indeed, every poset $(P, \leqslant)$ can be viewed as a category: an object is an element in P and there is an arrow $p \to q$ if and only if $p \leqslant q$. (In particular, there is *at most* one arrow between any two elements in a poset.) Composition is given by transitivity: if $p \leqslant q$ and $q \leqslant r$ then $p \leqslant r$, and associativity is immediate. Also, every element has an identity arrow since the partial order is reflexive: $p \leqslant p$ for all $p \in P$. So because a pregroup is a poset, we may also view it as a category. Therefore I'll now draw an arrow $\to$ in lieu of the partial order $\leqslant$. Moreover, a pregroup is really a poset-with-extra-structure and therefore we may view it as a category-with-extra-structure. Not surprisingly, given the reappearance of the yanking equations, that extra structure is compact closure! In summary, a pregroup is an example of a **compact closed category**. It is, in particular, a compact closed category that is *not symmetric.* Indeed, those four inequalities above are really the unit and counit maps

$$p^l p \xrightarrow{\epsilon^l} 1 \xrightarrow{\eta^l} pp^l \qquad \text{and} \qquad pp^r \xrightarrow{\epsilon^r} 1 \xrightarrow{\eta^r} p^r p.$$

discussed in Section 2.1, and the yanking equations (2) amount to the following:

$$p = p \cdot 1 \xrightarrow{1 \cdot \eta^r} pp^r p \xrightarrow{\epsilon^r \cdot 1} 1 \cdot p = p$$

$$p = 1 \cdot p \xrightarrow{\eta^l \cdot 1} pp^l p \xrightarrow{1 \cdot \eta^l} p \cdot 1 = p$$

$$p = 1 \cdot p^r \xrightarrow{\eta^r \cdot 1} p^r pp^r \xrightarrow{1 \cdot \epsilon^r} p^r \cdot 1 = p^r$$

$$p = p^l \cdot 1 \xrightarrow{1 \cdot \eta^l} p^l pp^l \xrightarrow{\epsilon^l \cdot 1} 1 \cdot p^l = p^l$$

In first equality of the third line we're rewriting p^r as $1 \cdot p^r$ rather than $p^r \cdot 1$ because neither of the ηs nor ϵs provide a map $1 \to pp^r$. Similarly for the last line, write $p^l = p^l \cdot 1$ rather than $p^l = 1 \cdot p^l$ since there's no map $1 \to p^l p$.

LET'S LOOK AT TWO examples of pregroups. The first is an arithmetic example, which will help to get our feet wet. The second is a grammatical example, which is used in the DisCoCat model of Coecke, Sadrzadeh, and Clark.

Example 3.4. The set

$$\{f\colon \mathbb{Z} \to \mathbb{Z} \mid f \text{ is monotone and unbounded}\}$$

is a pregroup. The partial order is given pointwise: $f \leqslant g$ if and only if $fn \leqslant gn$ for all n. The monoid multiplication is given by function composition $f \cdot g := f \circ g$. The monoidal unit is $\mathrm{id}_{\mathbb{Z}}$. Given such a function f, its left and right duals are given by

$$f^l n := \min\{m \in \mathbb{Z} \mid n \leqslant fm\} \qquad f^r n = \max\{m \in \mathbb{Z} \mid fm \leqslant n\}.$$

For example, if $fm = 2m$, then

$$f^l n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases} \qquad f^r = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

In short, $f^l n = \lfloor \frac{n+1}{2} \rfloor$ and $f^r n = \lfloor \frac{n}{2} \rfloor$.

You can find this example in "Iterated Galois Connections in Arithmetic and Linguistics" by Lambek, which appears in the Springer book *Galois Connections and Applications*. You'll also find mention of it in the "Mathematical Foundations" paper of Coecke et. al.

Fun fact: the pair (f^l, f) forms a special kind of categorical adjunction called a **Galois connection** since it satisfies

$$f^l n \leqslant m \quad \Leftrightarrow \quad n \leqslant fm.$$

Indeed if n is even, then $n/2 \leqslant m \Leftrightarrow n \leqslant 2m$. And if n is odd, then $(n+1)/2 \leqslant m$ which means $n+1 \leqslant 2m$ which is true iff $n \leqslant 2m$. Similarly, the pair (f, f^r) forms a Galois connection since

$$fn \leqslant m \quad \Leftrightarrow n \leqslant f^r m.$$

Indeed, if n is even then $2n \leqslant m \Leftrightarrow n \leqslant m/2$. And if n is odd, then $2n \leqslant m$ means $n \leqslant m/2$ which is true iff $n \leqslant (m-1)/2$.

For a couple of great introductions to *Galois connections* (They are super cool and appear in lots of places in math!) take a look at Lecture 4 of John Baez's online course on applied category theory as well as Section 1.5 of *Seven Sketches* by Fong and Spivak.

While arithmetic is fun, this next example is the one we're most interested in.

Example 3.5. Given any finite poset X, we can construct the *free pregroup generated by* X, denoted $\mathsf{Preg}X$. For a simple example, suppose $X = \{n, s\}$ whose elements we'll think of as basic grammar types: n is the type of a *noun* and s is the type of a (declarative) *sentence*. Elements of $\mathsf{Preg}\{n, s\}$ are concatenations of the letters n and s and their left and right duals and iterations of those duals and so on. For example, some grammatical types in $\mathsf{Preg}\{n, s\}$ are:

n = noun

s = sentence

nn^l = adjective

$n^r s n^l$ = transitive verb

The strings of letters are called *compound types*, and there is a morphism $a \to b$ between compound types if and only if a can *reduce* to b by application of one or more of the counit maps ϵ^r and ϵ^l.

Consider a banana, for example. It has type n, of course, while the adjective yellow has type nn^l. The reason that adjectives have grammar type nn^l is that an adjective can always be paired on the left with a noun, resulting in a new noun—e.g. *yellow banana.*

$$\underset{nn^l}{\text{yellow}} \quad \underset{n}{\text{banana}}$$

Indeed, to verify that the grammar type of *yellow banana* is n, we start by concatenating the types of the individual words to obtain $nn^l n$. Then we apply the counit map $\epsilon^l : nn^l \to 1$ together with the identity map $1_n : n \to n$ (this is given to us by the reflexivity axiom of posets: $n \leqslant n$) to see that $nn^l n$ reduces down to n:

The dot $\cdot$ in $\epsilon^l \cdot 1_n$ is meant to suggest "apply ϵ^l to nn^l while simultaneously applying 1_n to n."

This tells us that the phrase yellow banana has grammar type n. That's good. A yellow banana is a noun!

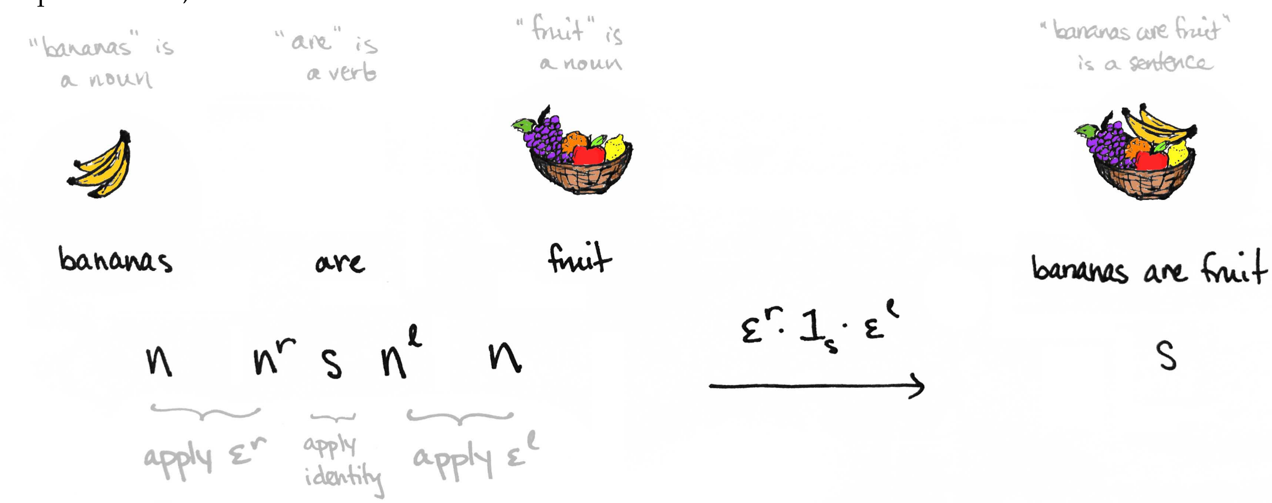
In light of this discussion on yellow bananas, you might enjoy taking a few seconds to think about why n^rsn^l represents the grammar type of *transitive verb*.

`<ponder> ... </ponder>`

A transitive verb is a word that accepts a noun on the right and another noun on the left such that the resulting phrase is a full sentence. Since we like bananas, here's another fruit-based example:

$$\underset{n}{\text{bananas}} \quad \underset{n^rsn^l}{\text{are}} \quad \underset{n}{\text{fruit}}$$

To determine the grammar type of this phrase, we concatenate the grammatical types of the individual words and then apply the counit maps to reduce, as before:



Here's that same reduction written out step-by-step. For clarity, I'll indicate the concatenation with a dot:

$$nn^rsn^ln = nn^r \cdot s \cdot n^ln \xrightarrow{\epsilon^r \cdot 1_s \cdot 1_n} 1 \cdot s \cdot n^ln = s \cdot n^ln \xrightarrow{1_s \cdot \epsilon^l} s \cdot 1 = s$$

In words, we've used the counit maps to reduce the concatenation of the grammar types for *bananas are fruit* to the letter s, which confirms that "bananas are fruit" is indeed a grammatically correct sentence. More generally, for any $a, b \in \mathsf{Preg}\{n, s\}$, we draw a morphism $a \to b$ if and only if a can be reduced to b in a similar fashion.

ALRIGHT, THAT'S (a very condensed version of) the pregroup story! To summarize, the language model of Coecke et. al. amounts to a structure-preserving functor

$$\textsf{syntax} \to \textsf{semantics}$$

In this section, we've just shown that the syntax category is taken to be a pregroup freely generated on a finite set of basic grammar types, i.e. $\mathsf{syntax} = \mathsf{Preg}X$. Let's move on to semantics now.

The Semantics Category: Vector Spaces

As stated in the introduction to Section 3.2, computers are able to understand meanings of individual words pretty well. That's because computers understand numbers! For example, a great way to inform a computer of the meaning of the word *banana* is to represent *banana* by a number and then give that number to the computer.

What number?

Well, it's not exactly a number. It's an array of numbers—a vector.

Okay, what vector?

The answer is simple, though I'd like to motivate it by sharing the following theorem.

Theorem (The Yoneda Lemma for Linguistics). *You shall know a word by the company it keeps.*

Proof. John Firth[20] □

[20] Firth, J. R. A synopsis of linguistic theory, 1930–1955. In *Selected Papers of JR Firth*, 1952–59 (ed. J. Firth and F. Palmer). Indiana University Press.

Okay, so it's not a theorem. But it is a great quote! Firth's idea is that *words that appear in similar contexts will have similar meaning.* In the linguistics community, this is referred to as the distributional hypothesis. So you might imagine that *apple* is more similar to *banana* than it is to *puppy* since *apples* and *bananas* often occur near words such as sweet, snack, green, eat, etc., whereas puppy occurs more often near words such as pet, cute, furry, bark, and so on. As another example, you might not know what the word *yegg* means (or perhaps you do), but you can probably infer it from this sentence:

Are you wondering why I've referred to Firth's idea as the Yoneda Lemma? To find out why, I recommend reading up on the Yoneda Perspective.

The cops grabbed him and another yegg for a Philadelphia store burglary.[21]

[21] James Lardner and Thomas Reppetto, *NYPD: A City and Its Police*, 2000

So we can represent the meaning of a word by a vector. This is often called a *distributional model of meaning.* But what, exactly, *is* the assignment *word* $\mapsto$ *vector*? Suppose we have a fixed corpus—your favorite book, say. Start by choosing a set of so-called **context words** $\{w_1, \ldots, w_n\}$. This can be every word in the corpus or some subset of it. By representing each w_i as the ith standard basis vector

$$\mathbf{w}_i = (0, \ldots, \overbrace{1}^{i^{\text{th}}\text{spot}}, \ldots, 0)$$

we obtain a basis $\{\mathbf{w_1}, \ldots, \mathbf{w_n}\}$ for a vector space V. Then any word w in the corpus has a vector representation given by a linear combination of the context words

$$\mathbf{w} = \sum_{i=1}^{n} c_i \mathbf{w_i}$$

The coefficients c_i are real numbers that indicate the number of times that w occurs near[22] w_i in the corpus.

[22] You can decide what "near" means. That is, the *context of w* is the set of words within k words of w, where $k = 1$ or 2 or 3 or whatever you like.

Here's an example. Suppose we're reading a book that contains the words

$$\{sweet, green, furry\}$$

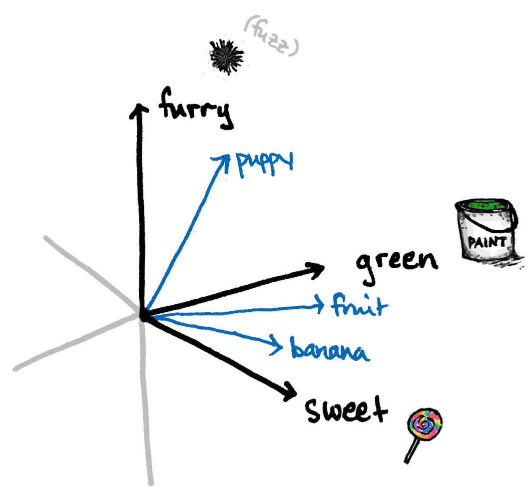
Let's choose them to be our context words and make the assignment so that

$$\mathbf{sweet} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \mathbf{green} = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \mathbf{furry} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Then if *banana, puppy* and *fruit* are also words in our book, we might have something like

$$\mathbf{banana} = \begin{bmatrix}21\\9\\0\end{bmatrix} \qquad \mathbf{puppy} = \begin{bmatrix}8\\1\\32\end{bmatrix} \qquad \mathbf{fruit} = \begin{bmatrix}43\\19\\0\end{bmatrix}$$

In other words, we've used data from the corpus to *embed* these words as vectors inside of a three-dimensional vector space. This prompts us to say that the *meaning* of the word banana *is* the vector $(21, 9, 0)$, the meaning of puppy is $(8, 1, 32)$, and the meaning of fruit is $(43, 19, 0)$.



And this works! That is, you can feed distributional models into your computer, and they'll ace the word-similarity portion of your SAT exam. Or your can compute the dot product between words, and you'll find that vectors are closer together precisely when the words they represent have the same meaning. It's all familiar territory for NLP practitioners. The semantics category for Coecke et. al. is thus the category of finite dimensional vector spaces over $\mathbb{R}$. That is,

$$\text{meanings of words} = \mathsf{FVect}$$

Unfortunately, the distributional model does *not* work for sentences. The same sentence rarely occurs twice in a given document, therefore we can't follow the same procedure above. This is where category theory can help lend a hand. In light of the principle of compositionality, the meaning of a sentence should be able to be computed given the meanings of its individual words and the rules of grammar for combining them. And we can pair meanings-of-words with grammatical types via a map from syntax (grammar) to semantics (meanings of words), i.e. via a functor

$$F\colon \mathsf{Preg}X \to \mathsf{FVect}$$

where X is a finite set of basic grammar types. In fact, as we know from Section 2.1, both FVect and Preg$X$ are compact closed categories, so we'll ask that F be a *strong monoidal functor*—one that preserves the compact closed structure. That's the gist behind the categorical compositional distributional model of Coecke et. al.

But how is F actually *defined*? Let's talk about that next.

The Functor: Syntax → Semantics

In this section, we'll give an explicit description of the functor

$$F\colon \mathsf{syntax} \to \mathsf{semantics}$$

or more specifically,

$$F\colon \mathsf{Preg}X \to \mathsf{FVect}$$

For simplicity, let's take $X = \{n, s\}$ as we did before. Now to define a functor, we need simply to say what it does on objects and morphisms. So let's do that. On objects,

- F assigns to the noun type n a vector space $N := Fn$, which we'll call a *noun space*
- F assigns to the sentence type s a vector space $S := Fs$, which we'll call a *sentence space*

and on morphisms

- F assigns to a type reduction $a \xrightarrow{r} b$ a linear map $Fa \xrightarrow{Fr} Fb$ that sends the vector corresponding to a word or phrase of type a in Fa to the vector corresponding to a word or phrase of type b in Fb.

Moreover, asking that F preserve the compact closed structure means that

- units and counits in Preg$\{n, s\}$ map to units and counits in FVect

  e.g. given $n \in \mathsf{Preg}\{n, s\}$,

  $$F(1 \xrightarrow{\eta_n^r} n^r n) = F(1 \xrightarrow{\eta_n^l} nn^l) = \eta_N \qquad \text{and} \qquad F(nn^r \xrightarrow{\epsilon_n^r} 1) = F(n^l n \xrightarrow{\epsilon_n^l} 1) = \epsilon_N$$

  where $\eta_N\colon \mathbb{R} \to N \otimes N$ and $\epsilon_N\colon N \otimes N \to \mathbb{R}$ are the linear maps that we defined on p. 17. A similar idea holds if we replace n by any element of Preg$\{n, s\}$.

- duals map to duals

  e.g. $Fn^r = Fn^l = N^*$. But our vector spaces are finite dimensional and so $N^* \cong N$ and therefore $Fn^r = Fn^l = N$.

- a compound type is assigned to a tensor product of vector spaces.

  e.g. $F(n^r s n^l) \cong Fn^r \otimes Fs \otimes Fn^l = N \otimes S \otimes N$

And that's it!

Except... this might not be very enlightening yet. It'll surely be helpful to look at a toy example. So in the next couple of pages, let's use the DisCoCat model to compute the meaning of the sentence

bananas are fruit

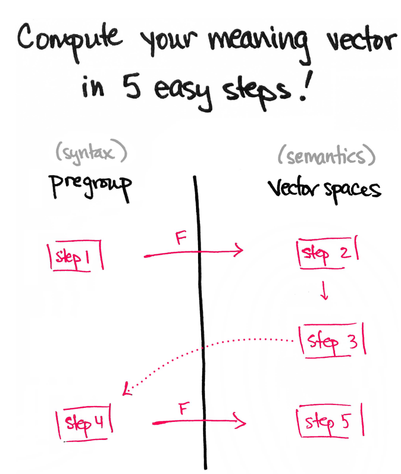
By "compute the meaning," I mean the following: we want to be able to view the sentence *bananas are fruit* as a vector, then feed that vector into the functor F and get an output vector that encodes for the meaning of the sentence.

$$F(bananas\ are\ fruit) = ??$$

That output vector will be the "meaning" of the sentence. Our goal is to find that meaning.

Goal: Compute the meaning of *bananas are fruit*.

Let's proceed systematically. I'll list the computations step-by-step, starting from the beginning.



Step 1: Assign each word a grammar type in $\mathsf{Preg}\{n, s\}$.

That's easy enough:

bananas $\rightsquigarrow n$
are $\rightsquigarrow n^r s n^l$
fruit $\rightsquigarrow n$

Step 2: Fix a noun space $Fn = N$ and a sentence space $Fs = S$.

Let's suppose N is the three-dimensional space spanned by the basis vectors

sweet, **green**, **furry**

which we can represent as column vectors

$$\mathbf{w_1} = \mathbf{sweet} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{w_2} = \mathbf{green} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{w_3} = \mathbf{furry} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as before. These basis vectors generate the noun space. But what about the sentence space S? For simplicity, let's define S to be a "true or false" space so that it's a one-dimensional vector space spanned by a single vector $\vec{1}$. The origin $0 \in S$ corresponds to "false" while $\vec{1}$ corresponds to "true." What about scalar multiplies of $\vec{1}$? If you like, you're more than welcome to think of a positive scalar multiple of $\vec{1}$ as the meaning vector for sentence that is *super* true. The larger the scalar, the more true the sentence!

Finally, note that once we've established N and S, the verb space comes for free:

$$F(n^r s n^l) = N \otimes S \otimes N$$

This is a nine-dimensional space spanned by vectors of the form $\mathbf{w_i} \otimes \vec{1} \otimes \mathbf{w_j}$ where i and j range between 1 and 3.

Step 3: Determine the vector representations of each word in the sentence.

We'll simply recycle the vectors we used earlier:

$$\mathbf{bananas} = \begin{bmatrix} 21 \\ 9 \\ 0 \end{bmatrix}, \qquad \mathbf{fruit} = \begin{bmatrix} 43 \\ 19 \\ 0 \end{bmatrix}$$

Note that both of these vectors live in the noun space N since each word has grammar type n. But what about the transitive verb *are*? By Step 1, we know that *are* has grammar type $n^r s n^l$ and is therefore a vector in the tensor product $N \otimes S \otimes N$. That is, there are coefficients $c_{ij} \in \mathbb{R}$ so that

$$are = c_{11}\mathbf{sweet}\otimes\vec{1}\otimes\mathbf{sweet} \;+\; c_{12}\mathbf{sweet}\otimes\vec{1}\otimes\mathbf{green} \;+\; \cdots \;+\; c_{33}\mathbf{furry}\otimes\vec{1}\otimes\mathbf{furry}.$$

Eek. That looks uncomely.

Fortunately, we learned in Section 2.1 that FVect is a compact closed category and therefore it exhibits process-state duality, which is the sophisticated way of saying

every vector in a tensor product can be identified with a linear map,

which is the long way of saying

every vector is really a matrix!

And that is excellent news, for if we know "what is what," i.e. if we know that $puppies$ are $furry$ but not $green$ and so on, then we can re-express the vector for *are* as a 3×3 matrix. The ijth entry of this matrix is the coefficient c_{ij} which is

$$c_{ij} = \begin{cases} 1, & \text{if } w_i \text{ is } w_j, \\ 0, & \text{otherwise} \end{cases}$$

The upshot is that the transitive verb *are* has matrix representation

It's no surprise that we get the identity matrix. *Being* is all about identity. That is, the verb *are* tells you when something IS something else.

$$\mathbf{are} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 4: Choose a type reduction in $\mathsf{Preg}\{n, s\}$

In this step, which takes place in the grammar category, we simply perform the type reduction already done on page 39. To recap, we know the grammar types of *bananas* and *are* and $fruit$, and so we concatenate those types to obtain $nn^r sn^l n$. Using the left and right counit maps, this string of letters reduces down to s, which confirms that the phrase *bananas are* $fruit$ is a tried-and-true sentence. In Step 4, we simply take that reduction morphism

$$nn^r sn^l n \xrightarrow{\epsilon_n^r \cdot 1_s \cdot \epsilon_n^l} n$$

and hold on to it. We'll need to use it in Step 5.

Aisde: You might wonder about the word "Choose" in "Step 4: Choose a type reduction." *What's up with that?* Incidentally, no choice was needed in this toy example of ours, so the purpose of this aside might be unclear. Indeed, there's only one way to parse the sentence *bananas are fruit*. But there exist sentences that can be parsed in *more than one* way. Consequently, the grammar type of such sentences may reduce down to type s via *more than one* reduction morphism. In Step 4, we are required to choose one. As an illustration, here is a nice sentence:

I saw a man with a telescope.

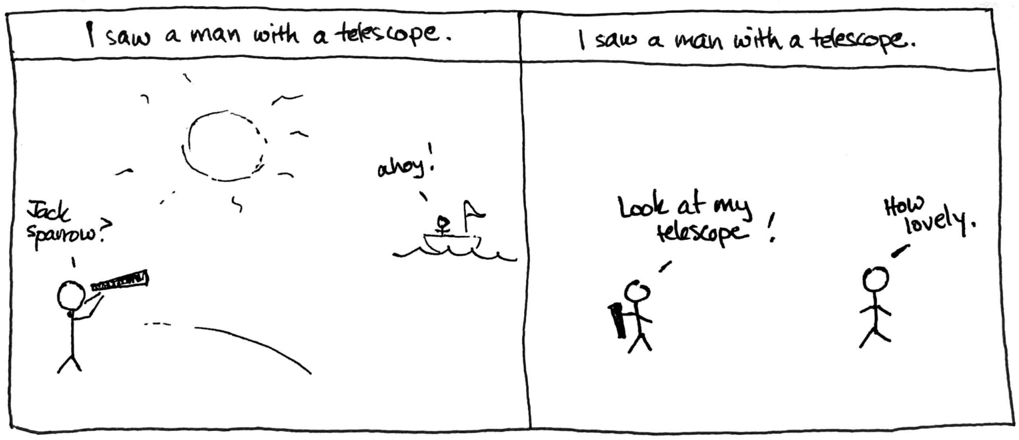
How did you parse it? Perhaps

I saw (a man with a telescope).

or perhaps

I saw (a man) with a telescope.

Those are two parsings of the same sentence, each of which corresponds to a different type reduction in the pregroup. In turn, this gives rise to different meaning vectors! And rightly so. Those two sentences have different meanings! Step 4 is simply reminding us of this fact.



Step 5: Apply F!

This is the fun part! We have a morphism in the pregroup

$$nn^r sn^l n \xrightarrow{\epsilon_n^r \cdot 1_s \cdot \epsilon_n^l} n$$

and we can apply F to get a linear map of vector spaces

$$N \otimes N \otimes S \otimes N \otimes N \xrightarrow{\epsilon_N \otimes 1_S \otimes \epsilon_N} S$$

where $\epsilon \colon N \otimes N \to \mathbb{R}$ is the linear map given on page 17 and $1_S \colon S \to S$ denotes the identity map on S. Finally, apply this linear map to the vector corresponding to the sentence

$$\epsilon_N \otimes 1_S \otimes \epsilon_N(\textbf{bananas} \otimes \textbf{are} \otimes \textbf{fruit})$$

which amounts to a simple matrix multiplication

$$\epsilon_N \otimes 1_S \otimes \epsilon_N(\textbf{bananas} \otimes \textbf{are} \otimes \textbf{fruit}) = \begin{bmatrix} 21 & 9 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 43 \\ 19 \\ 0 \end{bmatrix} = 1074$$

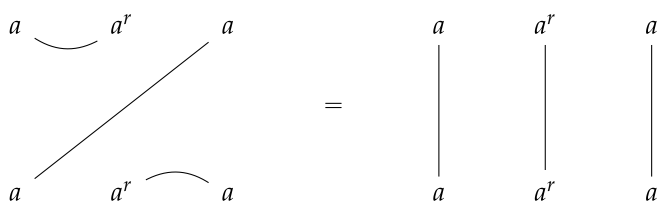
Conclusion? The meaning of the sentence *bananas are fruit* is

$$1074\vec{1}$$

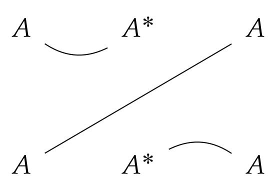
which is *super true*. Voila!

Some Closing Remarks

This functor described above is somewhat reminiscent of a *topological quantum field theory*, which is a functor from the category of cobordisms (another compact closed category) to the category of complex Hilbert spaces. But in 2014 Anne Preller showed that the only functors from a pregroup P freely generated on a finite set of basic types to FVect are those mapping to one-dimensional spaces. The key to her proof is the fact that P is a *poset* and hence there is at most *one* morphism between any two objects. In particular, any morphism from an object to itself must be the identity. As a consequence, if a is in P then the morphism $(1_a \cdot \eta_a) \circ (\epsilon_a \cdot 1_a)$ from $aa^r a \to aa^r a$ must equal the identity on $aa^r a$. Graphically:



Now consider a functor $P \to \mathsf{FVect}$. It assigns a in P to a vector space A in FVect, and it assigns $(1_a \cdot \eta_a) \circ (\epsilon_a \cdot 1_a)$ to the corresponding linear map, $(1_A \cdot \eta_A) \circ (\epsilon_A \cdot 1_A)\colon A \otimes A^* \otimes A \to A \otimes A^* \otimes A$, which we'll just denote by f,

$f =$ 

and which must be an isomorphism. Now if the dimension of A is at least 2, then we can choose orthogonal basis vectors e_1 and e_2 so that $e_1 \otimes e_2^* \otimes e_1 \in A \otimes A^* \otimes A$. And since ϵ_A computes the inner product between e_1 and e_2^*, we have $f(e_1 \otimes e_2^* \otimes e_1) = 0$. Therefore f is not injective, and so it cannot be an isomorphism.

The intuition is, perhaps, that pregroups have *too few* morphisms to capture the semantics. In particular, pregroups do not allow us to distinguish different parsings of strings of types. One string may reduce in several ways—e.g. (Men and women) whom I like vs. Men and (women whom I like)—and the morphisms in a pregroup do not account for this. So in some sense, there isn't enough "wiggle room" for meaning in pregroup syntax, so the output can only be a one-dimensional vector space. But all is not lost! As Preller showed, the problem can be fixed by replacing a free pregroup with a free compact closed category. For more details, see her paper "From Logical to Distributional Models."

3.3 Further Reading

For more on the work of Baez and Pollard:

- Certainly take a look at their paper, "A Compositional Framework for Reaction Networks". The *n*-Category Café also contains expositions, including a post by Baez with the same title and "Dynamical Systems and Their Steady States" by Maru Sarazola.

- The examples in Section 3.1 can be found in "The Mathematics of Networks", a wonderfully accessible talk by Baez on YouTube.

- All of the decorated cospan formalism can be found in Brendan Fong's thesis, "The Algebra of Open and Interconnected Systems." As part of the ACT workshop, Jonathan Lorand and Fabrizio Genovese wrote about Fong's thesis in an article titled "Hypergraph Categories of Cospans" on the *n*-Category Café.

- The framework of chemical reaction networks is also being used to model ATP coupling! This project was birthed during the ACT workshop. For more, take a look at "Coupling Through Emergent Conservation Laws " on Baez's Azimuth blog.

For more on the work of Coecke, Sadrzadeh, and Clark:

- There is a delightful 5-minute YouTube video (made in the style of Minute Physics!) called "How Quantum Theory Can Help Understanding Natural Language" that does an excellent job of explaining the ideas behind DisCoCat in a non-technical way.
- The *yellow banana* example of the previous section was just a toy example meant to showcase the functor of the DisCoCat model of meaning. But this is a document on *applied* category theory, and so you'd surely like to see some applications! For empirical data arising from actual implementations of the DisCoCat model, take a look at:
  - "Experimental support for a categorical compositional distributional model of meaning" by Edward Grefenstette and Mehrnoosh Sadrzadeh
  - "Prior disambiguation of word tensors for constructing sentence vectors" by Dimitri Kartsaklis and Mehrnoosh Sadrzadeh
  - "Quantization, Frobenius and bi-algebras from the categorical framework of quantum mechanics to natural language semantics" by Mehrnoosh Sadrzadeh
- The DisCoCat model was also featured on the *n*-Category Café as part of the Applied Category Theory Workshop. In fact, it was featured *twice*! The two blog posts are:
  - "Linguistics Using Category Theory" by Corey Griffith and Jade Master. This wonderfully written article gives another recap of the DisCoCat model, as well as another toy example. Check out the comment section, too. There are some nice discussion going on there!
  - "Cognition, Convexity, and Category Theory" by Brad Theilman and me. This blog post is a summary of "Interacting Conceptual Spaces I," a paper in which the authors change the semantics category of the DisCoCat model from FVect to something that attempts to model human cognition more closely, namely convex spaces! This is a great example of tweaking the *semantics* part of *functorial* ***semantics***.
- The last suggested resource is unrelated to DisCoCat, but it's still in the vein of machine learning + applied category theory and so I thought I'd share: Did you know that backpropagation is a functor?

4 But Wait! There's More...

There's much more to applied category theory—I've only presented a very tiny subset of hand-selected ideas. But there's so much more to see, learn, and do! So to close out these notes, I'll leave you with a few more links where you can discover other *themes, constructions,* and *examples* of applied category theory.

- To start, there's the main Applied Category Theory webpage, which has
  - a description of the 2018 workshop that took place at the Lorentz Center in Leiden, Netherlands
  - a call for the 2019 workshop, to be hosted by Bob Coecke in Oxford.
- Jelle Harold and the folks at Statebox filmed most of the 2018 workshop talks, and you can watch them here: https://statebox.org/events/act-leiden.html. Speakers include Samson Abramsky, John Baez, Bob Coecke, Kathryn Hess, Aleks Kissinger, Tom Leinster, David Spivak, and many more!
- Back in March 2018, there was an applied category theory workshop hosted at the National Institute of Standards and Technology. Slides and videos of the talk can be found here: http://www.appliedcategorytheory.org/nist-workshop-slides/
- There is also *Seven Sketches in Compositionality* (subtitle: "An Invitation to Applied Category Theory") by Brendan Fong and David Spivak. I've referenced this book several times already, but that's because it's such a gem! (I was sold just after reading the preface.) It's a delightful and insightful introduction to more themes, more constructions, and more examples within applied category theory. Even better, no prior knowledge of category theory is assumed. The book is based on a MIT course the authors taught. You can find videos of their lectures here: http://math.mit.edu/ dspivak/teaching/sp18/
- And as if all of these great resources weren't enough, John Baez is running a *free online course* on applied category theory. Participants have been working through the *Seven Sketches* book. The lectures and ensuing discussions are a treasure trove of exciting mathematics: https://forum.azimuthproject.org/categories/applied-category-theory-course. Also be sure to take a look at the "applied category" tag on his blog, Azimuth.