Using Sigmoid and Double-Sigmoid Functions for Earth-States Transitions.[1]

by

L. David Roper

roperld@vt.edu

ABSTRACT

The sigmoid and double-sigmoid functions are very useful for representing earth-states transitions in geological models. Simple hyperbolic-tangent versions are given for the functions. An example is given of fitting double-sigmoid functions to the transient state transitions of the Last Major Ice Age.

KEY WORDS: Sigmoid functions, Major Ice Ages, transient states, ocean currents

INTRODUCTION

Geology, as with all physical sciences, often involves transitions from one state to another, and perhaps back again to the first state (transient states). The sigmoid and double-sigmoid functions have the properties needed for developing models for state transitions in order to fit data.

When only a transition from one state to another is needed, the sigmoid function is used; when the second state is a transient state, the double-sigmoid function is used.

A physical transient-state situation is usually not symmetrical; i.e., the double sigmoid must have different transition times (width) for entering the state and for leaving the state.

The simplest and computationally better version of the sigmoid function and the double-sigmoid function involves the hyperbolic tangent function, rather than the exponential function. The hyperbolic-tangent version equations are given in this paper.

An example of applying the functions to fitting the Antarctica temperature data, determined from deuterium concentration in ice cores, for the Last Major Ice Age is given.

SIGMOID AND DOUBLE-SIGMOID FUNCTIONS

The sigmoid function is

$$y = \frac{1}{1 + \exp\left(-2\,\dfrac{x-c}{w}\right)} = \frac{1}{2}\left[1 + \tanh\left(\frac{x-c}{w}\right)\right] \text{ with constraint } w > 0.$$

Figure 1 shows a sigmoid, along with a symmetric double sigmoid described below.

The TableCurve software program (http://www.systat.com/products/TableCurve2D) gives the following equation for the asymmetric double sigmoid:

$$y = \frac{a}{1+\exp\left(-\frac{x-b+c/2}{d}\right)}\left[1-\frac{1}{1+\exp\left(-\frac{x-b-c/2}{e}\right)}\right]$$

with constraints $c > 0$, $d > 0$ and $e > 0$.

This is a cumbersome equation and computational difficulties can occur when using it because of the infinite asymptotic property of the exponential for large positive argument. (The symmetric double sigmoid given by TableCurve is even more cumbersome.)

A simpler and computationally better formulation for the double sigmoid is:

$$y = \frac{1}{2}\left[\tanh\left(\frac{x-c_1}{w_1}\right)-\tanh\left(\frac{x-c_2}{w_2}\right)\right] \text{ with constraints } w_1 > 0 \text{ and } w_2 > 0.$$

The symmetric version is simply the case where $w_1 = w_2 = w.$ Figure 1 shows a symmetric double sigmoid, along with a sigmoid described above.

Figure 2 shows some double sigmoids for different values of the second w_2 parameter; i.e. asymmetric double sigmoids. Note that, when the second width (w_2) becomes comparable to the difference between c_1 and c_2, the second sigmoid causes the function to dip down below the zero axis before the rise due to the first sigmoid. This effect is shown more clearly in Figure 3 for larger asymmetry. The first sigmoid of the double sigmoid, with its much smaller width, starts reversing the second sigmoid of the double sigmoid before the second sigmoid has reached its asymptote. Eventually, the first sigmoid of the double sigmoid asserts itself and causes a turning on of the transition state.

Finally, the long tail of the second sigmoid takes over for large times. As shown in the bottom half of Figure 3, for a negative strength factor the event is reversed in sign.

Thus, the double-sigmoid function can not only represent a physical event in which a transient state is simply turned off/on and then turned on/off, but it can also represent an event in which there is a dip followed by a rise into a transient state or a rise followed by a dip into a transient state. All four of these types of events are present in the example fit given below.

USING DOUBLE SIGMOIDS FOR TRANSIENT STATES' TRANSITIONS DURING THE LAST MAJOR ICE AGE

Data for Antarctica yearly mean temperatures relative to the current mean temperature for the last 425 kiloyears (the period of the last four Major Ice Ages) have been calculated from measurements of deuterium concentration in ice cores (Petit, 1999). For the fitting of these data the minima and maxima of the temperature data were tuned to the minima and maxima of the summer North-Pole solar insolation calculated by the Berger code (Berger, 1991). The tuned data were then fitted to the equation:

$$T_I\left(t\right) = C + F \cdot I\left(t\right) + \sum_{i=1}^{N} s_i \, \frac{1}{2}\left[\tanh\left(\frac{t - c_{1i}}{w_{1i}}\right) - \tanh\left(\frac{t - c_{2i}}{w_{2i}}\right)\right]$$

= temperature in degrees Celsius,

where $I\left(t\right) =$ calculated summer North-Pole insolation at time t.

Double-sigmoid parameters: $s_i =$ strength, $c_{ni} =$ position, $w_{ni} =$ width.

$N =$ number of double sigmoids used in the fit.

There is a factor F multiplying the summer North-Pole insolation that provides correlation with the summer North-Pole insolation and there is a constant C. The last sum-of-double-sigmoids term provides several transient states.

Fits were done for various assumptions for transient states that appear possible from viewing the data for the last four Major Ice Ages back to -425 kiloyears. The detailed results can be seen at http://www.roperld.com/Science/TransitionsModelMIA.pdf . One such fit for the Last Major Ice Age and the rises into the Last Major Interglacial and the Current Major Interglacial is shown in Figure 4 and its fit parameters are given in Table 1. (Data for the last 1000 years were excluded in the fit.)

No claim is made that the fit given here is unique. It is the best fit found in several tries.

It appears that there are some patterns to the transient events in the fit, which will not be discussed here except to note that there are two major temperature turn-on transient states associated with the Last Major Interglacial and two associated with the Current Major Interglacial. One of those two states is much shorter in duration than the other for the Last Major Interglacial; the comparison of their durations for the Current Major Interglacial will not be known for tens of thousands of years into the future. However, the two states' durations for the Current Major Interglacial were allowed to vary in the fit, the pattern came out the same as for the Last Major Interglacial.

Obvious candidates for some of the transient states in this fit are components of the Great Ocean Conveyor Belt (Colling, 2002), especially for the major short-term and long-term transient states associated with the Last and Current Major Interglacials. Other candidates for the minor transient states could be volcanic eruptions.

It should be pointed out again that the major maxima and minima of the Antarctica temperature data were tuned to the maxima and minima of the summer North-Pole insolation before the fit was made. Therefore, any lag of temperature relative to

North-Pole insolation was eliminated. Also, some judgments were necessary in identifying the major maxima and minima of the data.

CONCLUSIONS

Simple equations for sigmoid and double-sigmoid functions involving hyperbolic tangents are presented for use in geological models that require transitions between different Earth states (sigmoid) and into and out of transient earth states (double sigmoid).

Double-sigmoid functions can represent simple turn on/off followed by turn off/on of transient states and of transient events that involve a rise/dip followed by a dip/rise.

Using double-sigmoid functions in a fit to the Antarctica temperature data for the Last Major Ice Age yields a very good fit and some possible interesting patterns about the transient states throughout its existence.

REFERENCES

Berger, A. and Loutre, M. F., 1991, Insolation values for the climate of the last 10 million years: Quaternary Science Reviews, v. 10, p. 297-317. This code was used because it was possible to create a spreadsheet version of it, which made it much easier to calculate insolation values at the times of the many Antarctica temperatures.

Colling, A., 2002. Ocean Circulation, 2nd Ed., Butterworth Heinemann, Milton Keynes, England, 286 p.

Petit, J. R., Jouzel, J., Raynaud, D., Barkov, N. I., Barnola, J.-M., Basile, I., Bender, M., Chappellaz, J., Davis, M., Delaygue, G.,Delmotte, M., Kotlyakov, V. M., Legrand, M., Lipenkov, V. Y., Lorius, C., Pépin, L., Ritz, C., Saltzman, E. and Stievanard, M., 1999, Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica: Nature, v.399, p. 429-436. (1999). A shorter on-line version is available at http://cdiac.esd.ornl.gov/trends/tem/vostok/jouz_tem.htm . The data can be downloaded from the Internet at http://www.ngdc.noaa.gov/paleo/icecore/antarctica/vostok/vostok_isotope.html .

Table 1. Fit parameters for the Last Major Ice (including the rise into and out of it, but excluding the last 1000 years): using 18 double sigmoids.

Double Sigmoid	S (degrees C)	c_1 (kiloyears)	w_1 (kiloyears)	c_2 (kiloyears)	w_2 (kiloyears)	Comments	F (degC)/(W/m^2):
DS1	3.885	-134.94	2.106	-65.13	52.68	Long-term DS for Last Major Interglacial	0.04143
DS2	3.077	-130.61	1.449	-117.72	1.209	Short-term DS for Last Major Interglacial	C (degrees C):
DS3	-1.464	-126.51	0.05651	-123.00	2.927	Dip after peak of Last Major Interglacial	-29.07
DS4	-12.65	-111.14	1.395	-110.73	1.134	Sharp dip at first minimum after Last Major Interglacial	
DS5	-1.730	-108.61	0.4738	-105.57	0.5374	Square dip	
DS6	-1.351	-103.91	0.4680	-100.01	0.3387	Square dip	
DS7	-3.693	-79.20	2.074	-79.05	0.3716	Dip-rise	
DS8	1.616	-76.57	1.19E-08	-72.23	6.647	Dip-rise	
DS9	1.452	-68.97	0.9193	-67.43	0.02173	Rise at minimum at about -67 kiloyears	
DS10	-46.13	-63.01	15.35	-61.94	14.18	Broad dip from -60 to -80 kiloyears	
DS11	-13.94	-55.70	1.696	-55.54	2.450	Rise-dip	
DS12	-57.65	-44.77	2.909	-44.58	2.741	Deep dip at about -46 kiloyears	
DS13	-1.278	-40.79	1.86E-06	-35.88	0.1177	Square dip	No. of data:
DS14	-1.094	-34.19	0.2150	-26.64	0.02544	Square dip	1377
DS15	-3.330	-17.93	6.368	-15.88	0.8503	Large dip-rise	No. of parameters:
DS16	4.139	-13.88	0.4074	39.64	11.98	Long-term DS for Current Major Interglacial	92
DS17	-3.496	-13.60	0.1376	-11.64	0.3663	Younger-Dryas dip	Fit Chi Square:
DS18	2.225	-6.596	2.046	11.95	1.643	Short-term DS for Current Major Interglacial	414

FIGURES

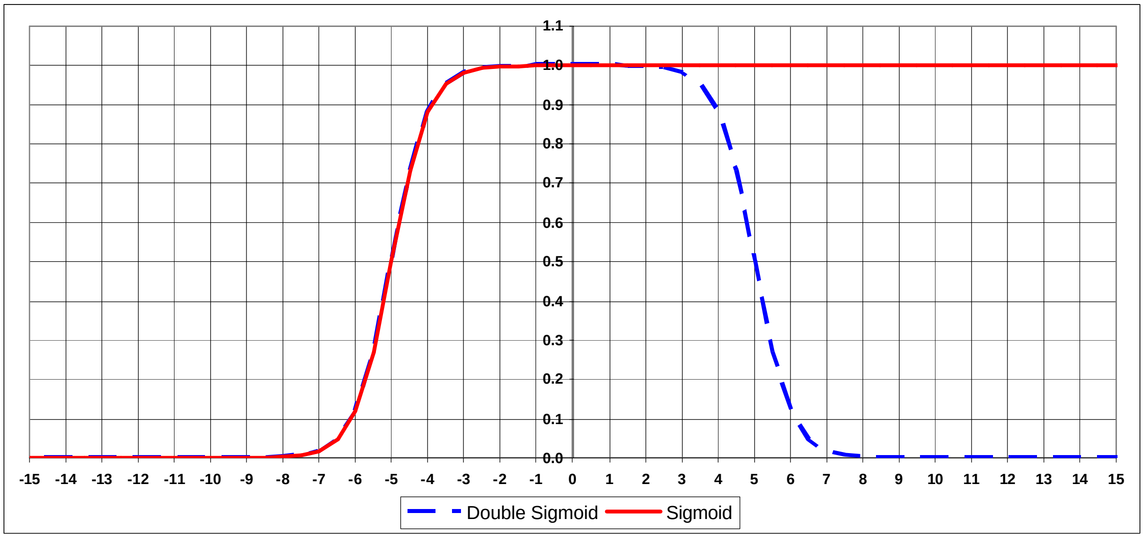


Figure 1. A single sigmoid and a symmetrical double sigmoid for parameters $c_1 = -5$, $c_2 = 5$ and $w = 1$. Of course, the curve could be downward instead of upward by changing the sign of a factor multiplying the function.

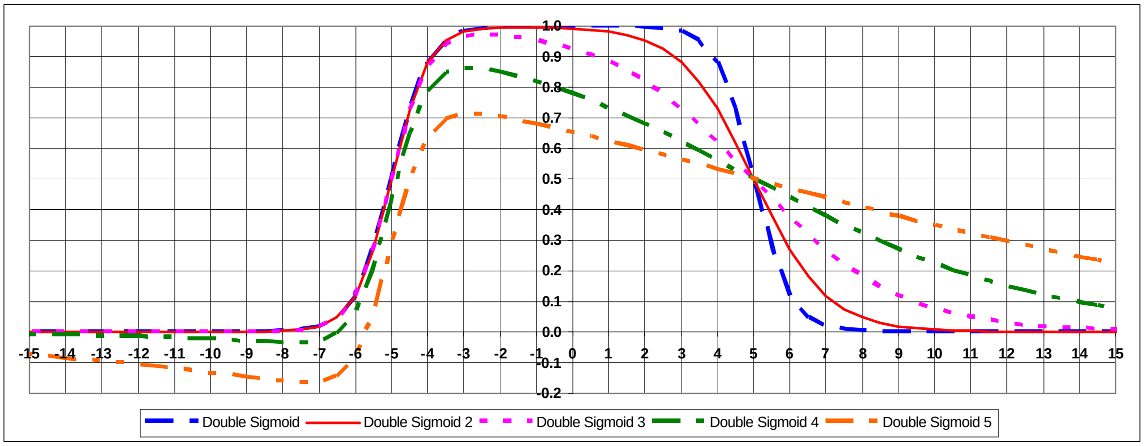


Figure 2. Double sigmoids for different amounts of asymmetry. The parameters are $c_1 = -5$, $c_2 = 5$, $w_1 = 1$ and $w_2 = (1, 2, 4, 8, 16)$. Of course, the curve could be downward instead of upward by changing the sign of a factor multiplying the function.

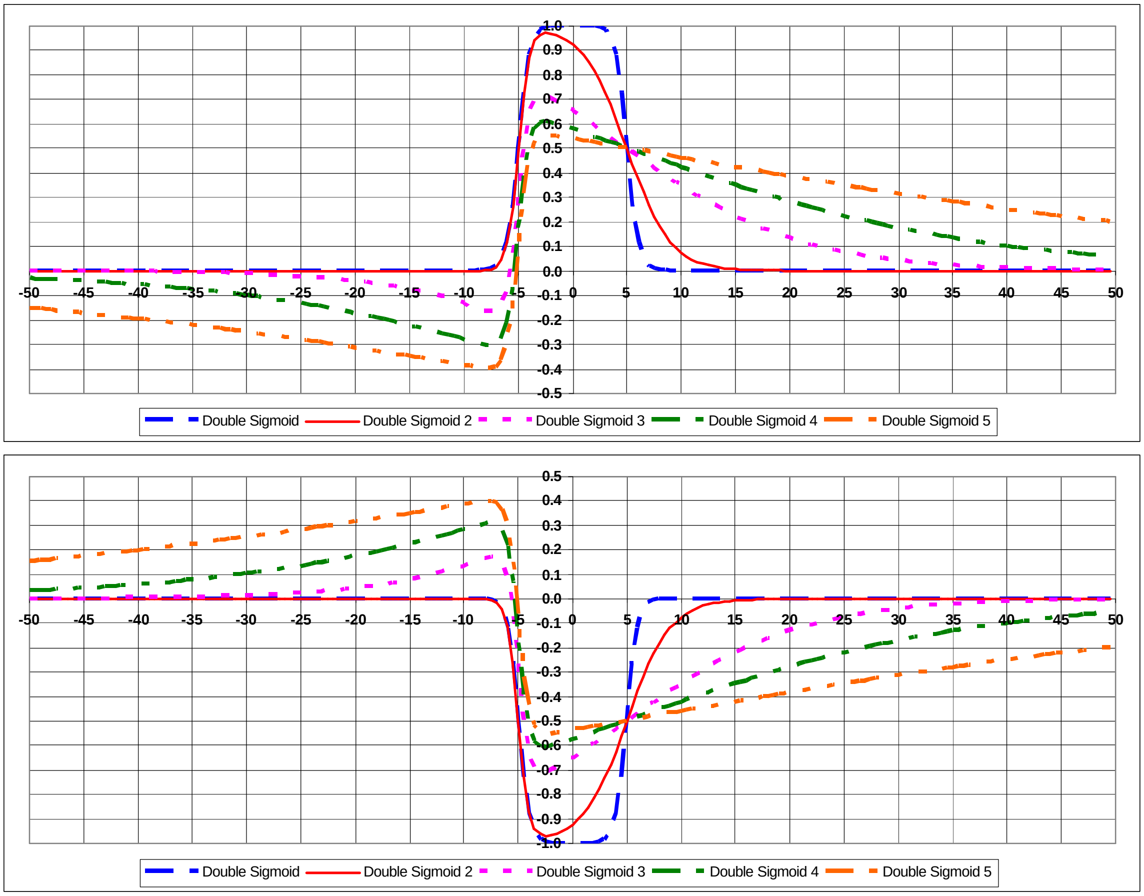
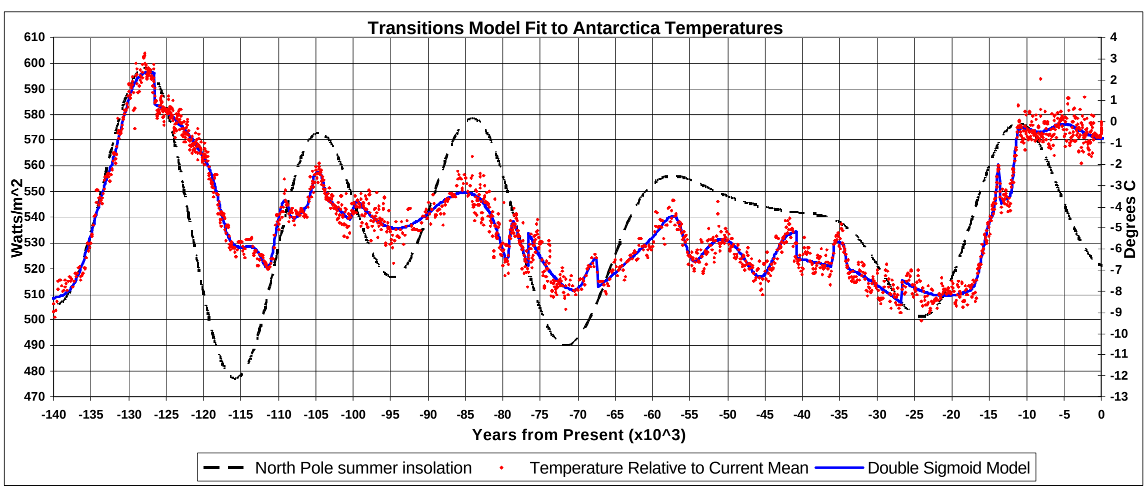


Figure 3. Double sigmoids for different amounts of asymmetry. The parameters are $c_1 = -5,\ c_2 = 5,\ w_1 = 1$ and $\mathrm{w}_2 = (1,\ 4,\ 16,\ 32,\ 64)$. The bottom graph is for a negative factor multiplying the function.



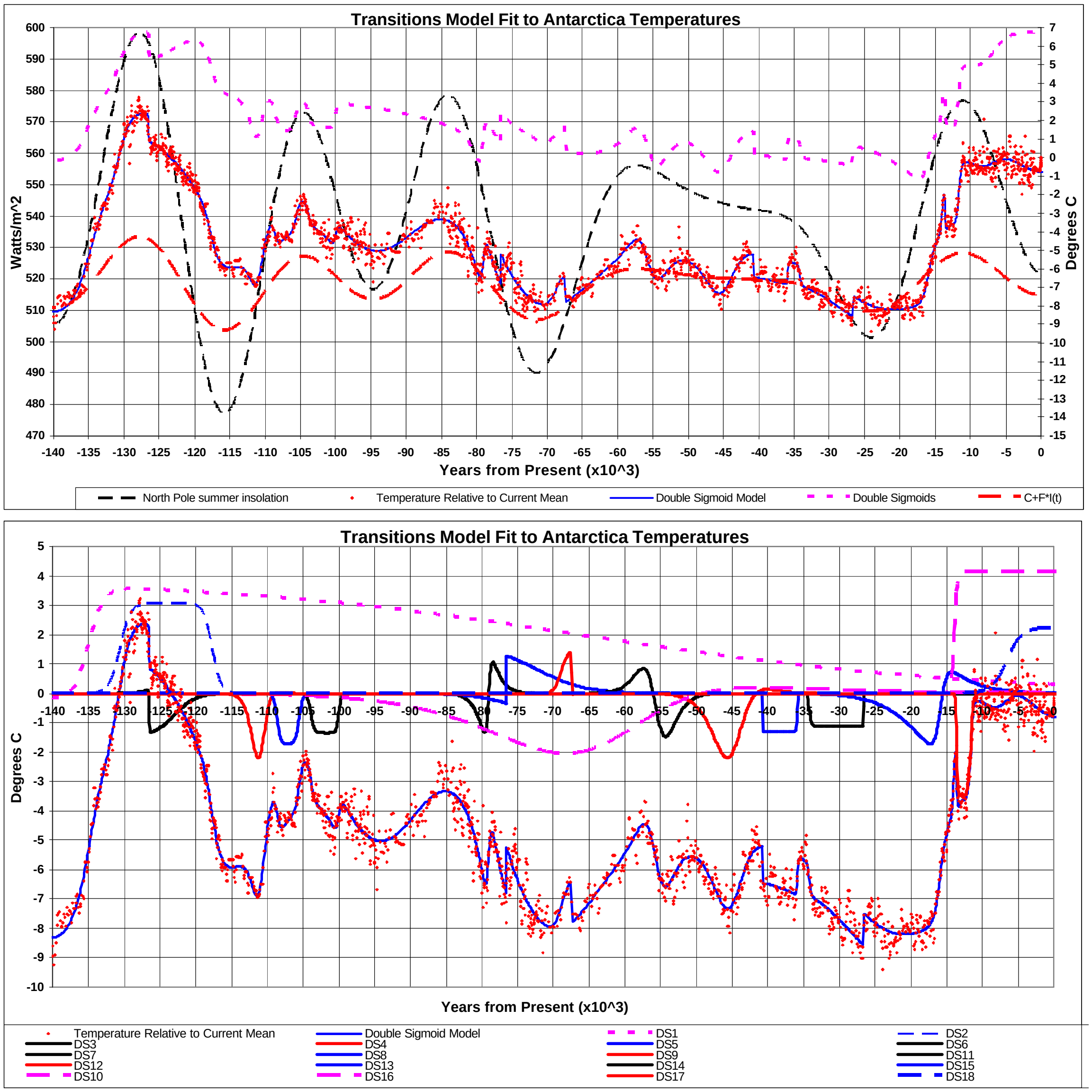


Figure 4. The top graph shows the measured Antarctica temperatures for the Last Major Ice Age, including the rise into it. The fit to the temperature data is shown and the North Pole solar insolation is shown as a dashed line. The middle graph also shows the contributions to the fit by the North Pole insolation term with factor F (plus the constant C) and the contributions of the sum of the 18 double sigmoids. The bottom graph breaks out the double sigmoids into the four major ones associated with the Last and Current Major Interglacials and the other 14 ones.