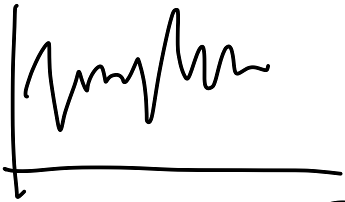
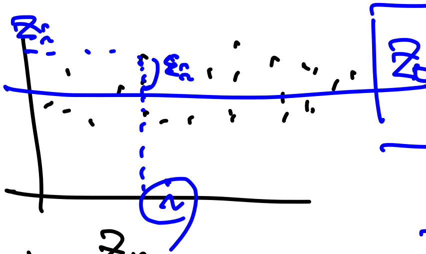
<최소제곱법> → 행렬 / 편미분.





$$Z_t = \beta_0 + \varepsilon_t$$

$$f(X_t ; \beta) = \beta_0$$

$z_1, \dots, z_n$

↳ $z_i = \beta_0 + \varepsilon_i$

↳ $\varepsilon_i = z_i - \beta_0$

$$S(\beta_0) = \sum_{i=1}^{n} (z_i - \beta_0)^2$$ ⇒ 최소가 되게하는 $\beta_0 = \hat{\beta}_0$ 찾으려!

$$= \sum_{i=1}^{n} (\beta_0^2 - 2\beta_0 z_i + z_i^2)$$

$$\frac{dS}{d\beta_0} = \sum_{i=1}^{n} (2\beta_0 - 2z_i)$$

$$= 2 \sum_{i=1}^{n} (\beta_0 - z_i)$$

$$= 2\left(\sum_{i=1}^{n} \beta_0 - \sum_{i=1}^{n} z_i \right)$$

$$= 2\left(n\beta_0 - \sum_{i=1}^{n} z_i \right)$$

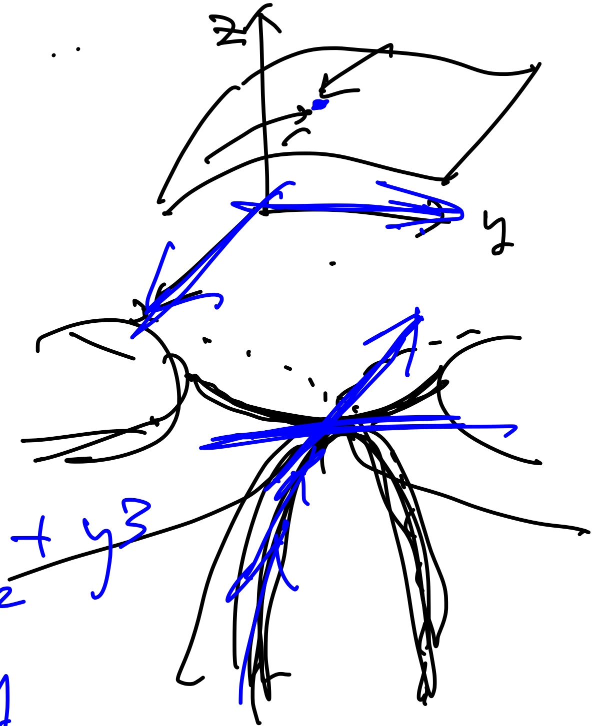
$$= 2n\left(\beta_0 - \frac{1}{n} \sum_{i=1}^{n} z_i \right)$$

$$= 2n(\beta_0 - \bar{z}) = 0$$

$\beta_0 = \hat{\beta}_0$

$$\boxed{\hat{\beta}_0 = \bar{z}}$$

$$f(x,y) = x^2 + y^2$$



$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

$$f(x,y) = x^3 + 2x^2y + y^3$$

$$(2y)x^2$$

$$\frac{\partial f}{\partial x} = 3x^2 + 4xy$$

$$\frac{\partial f}{\partial y} = 0 + 2x^2 + 3y^2$$

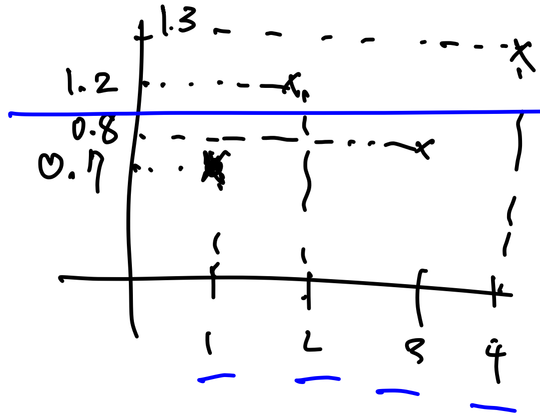
$$f(x,y) = x^2y - y^3 + 2xy$$

$$\frac{\partial f}{\partial x} = y \times 2x + 0 + 2y = 2xy + 2y$$

$$\frac{\partial f}{\partial y} = x^2 - 3y^2 + 2x$$

$$\beta_0 + \beta_1 x_i$$

①



$y = \beta_0$

$$\boxed{Z_t = \boxed{\beta_0} + \varepsilon_t}$$

$$S(\beta_0) = \sum_{t=1}^{4} (\beta_0 - Z_t)^2$$

Sol

$$S(\beta_0) = \sum_{t=1}^{4} (\beta_0 - Z_t)^2$$

$$= (\beta_0 - 0.7)^2 + (\beta_0 - 0.8)^2 + (\beta_0 - 1.2)^2 + (\beta_0 - 1.3)^2$$

$$= 4\beta_0^2 - (1.4 + 1.6 + 2.4 + 2.6)\beta_0 + C$$

$$= 4\beta_0^2 - 8\beta_0 + C$$

$$\frac{dS}{d\beta_0} = 8\beta_0 - 8 = 0 \Rightarrow \boxed{\hat{\beta}_0 = 1}$$

② $f(x,y) = x^3 + x^2y^3 - 2y^2 = z$

$\frac{\partial f}{\partial x} = f_x$? $\frac{\partial f}{\partial y} = f_y$?

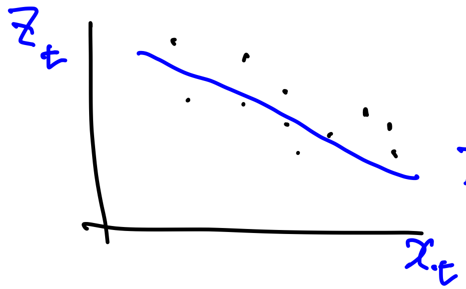
$f_x(2,1) =$? $f_y(2,1) =$?

Sol $f_x(x,y) = 3x^2 + 2xy^3 + 0.$

$f_y(x,y) = 3x^2y^2 - 4y$

$f_x(2,1) = 12 + 4 = 16$

$f_y(2,1) = 12 - 4 = 8.$ //



(β_0, β_1)

$f(x; \beta)$

$Z_t = \beta_0 + \beta_1 x_t + \varepsilon_t.$

$$S(\beta_0, \beta_1) := \sum_{t=1}^{n} \varepsilon_t^2$$

$$= \sum_{t=1}^{n} (Z_t - \beta_0 - \beta_1 x_t)^2$$

$$= \sum_{t=1}^{n} \left(Z_t^2 + \beta_0^2 + \beta_1^2 x_t^2 - 2\beta_0 Z_t - 2\beta_1 x_t Z_t + 2\beta_0\beta_1 x_t \right)$$

$$= n\beta_0^2 + \sum_{t=1}^{n} Z_t^2 + \beta_1^2 \sum_{t=1}^{n} x_t^2 - 2\beta_0 \sum_{t=1}^{n} Z_t - 2\beta_1 \sum_{t=1}^{n} x_t Z_t + 2\beta_0\beta_1 \sum_{t=1}^{n} x_t$$

$$S(\beta_0, \beta_1)$$

$$= n\beta_0^2 + \sum_{t=1}^{n} z_t^2 + \beta_1^2 \sum_{t=1}^{n} x_t^2 - 2\beta_0 \sum_{t=1}^{n} z_t - 2\beta_1 \sum_{t=1}^{n} x_t z_t + 2\beta_0\beta_1 \sum_{t=1}^{n} x_t$$

$$\frac{\partial S}{\partial \beta_0} = 2n\beta_0 - 2\sum_{t=1}^{n} z_t + 2\beta_1 \sum_{t=1}^{n} x_t = 0$$

$$\boxed{\begin{aligned} \frac{1}{n}\sum_{t=1}^{n} z_t &= \bar{z} \\ \frac{1}{n}\sum_{t=1}^{n} x_t &= \bar{x} \end{aligned}} \longrightarrow$$

$$\cancel{2n}\hat{\beta}_0 = 2\sum_{t=1}^{n} z_t - 2\beta_1 \sum_{t=1}^{n} x_t$$

$$= \cancel{2n}\,\bar{z} - \cancel{2n}\hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_0 = \bar{z} - \hat{\beta}_1 \bar{x}$$

$$\frac{\partial S}{\partial \beta_1} = 2\beta_1 \sum_{t=1}^{n} x_t^2 - 2\sum_{t=1}^{n} x_t Z_t + 2\beta_0 \sum_{t=1}^{n} x_t = 0$$

$$\Rightarrow \hat{\beta}_1 \sum_{t=1}^{n} x_t^2 = \sum_{t=1}^{n} x_t Z_t - \hat{\beta}_0 \sum_{t=1}^{n} x_t$$

$$\hat{\beta}_0 = \bar{Z} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 \sum_{t=1}^{n} x_t^2 = \sum_{t=1}^{n} (x_t Z_t) - \sum_{t=1}^{n} x_t (\bar{Z} - \hat{\beta}_1 \bar{x})$$

$$= \sum_{t=1}^{n} (x_t Z_t) - \bar{Z} \sum_{t=1}^{n} x_t + \hat{\beta}_1 \bar{x} \sum_{t=1}^{n} x_t$$

$$= \sum_{t=1}^{n} (x_t Z_t) - \bar{Z} \sum_{t=1}^{n} x_t + n \hat{\beta}_1 \bar{x}^2$$

$$\hat{\beta}_1 \left(\sum_{t=1}^{n} x_t^2 - n\bar{x}^2 \right) = \sum_{t=1}^{n} (x_t Z_t) - \bar{Z} \sum_{t=1}^{n} x_t$$

$$\hat{\beta}_1\left(\sum_{t=1}^{n} x_t^2 - n\bar{x}^2\right) = \sum_{t=1}^{n}(x_t z_t) - \bar{z}\sum_{t=1}^{n} x_t$$

$$\boxed{\hat{\beta}_1 = \frac{\sum_{t=1}^{n}(x_t-\bar{x})z_t}{\sum_{t=1}^{n}(x_t-\bar{x})^2}}$$

$$= \sum_{t=1}^{n}(x_t z_t) - \bar{x}\sum_{t=1}^{n} z_t$$

$$= \sum_{t=1}^{n}(x_t z_t) - \sum_{t=1}^{n}\bar{x} z_t$$

$$= \sum_{t=1}^{n}(x_t z_t - \bar{x} z_t)$$

$$\hat{\beta}_1\left(\sum (x_t-\bar{x})^2\right) = \sum_{t=1}^{n}(x_t-\bar{x}) z_t$$

※ $nE\left((x_t-\bar{x})^2\right) = n \times \frac{1}{n}\sum_{t=1}^{n}(x_t-\bar{x})^2$

$$= n\left(E(x_t^2) - \bar{x}^2\right)$$

$$= \sum_{t=1}^{n} x_t^2 - n\bar{x}^2$$

$$\bar{z}\sum_{t=1}^{n} x_t = \frac{1}{n}\sum_{t=1}^{n} z_t \sum_{t=1}^{n} x_t$$

$$= \bar{x}\sum_{t=1}^{n} z_t$$

$$\sum (x_t - \bar{x})^2 \qquad x_t = 1, 2, 3, 4$$

$$\bar{x} = 2.5$$

$$\sum_{t=1}^{n} (x_t - \bar{x}) z_t$$

$$\| x_t - \bar{x} \| = \left\| \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2.5 \\ 2.5 \\ 2.5 \\ 2.5 \end{pmatrix} \right\|$$

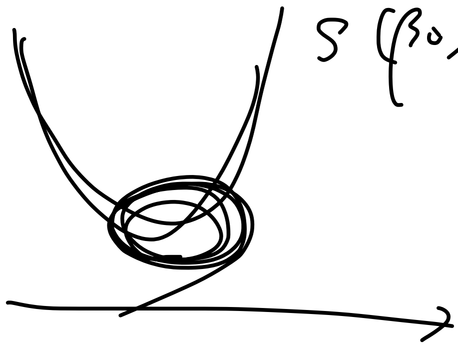
$$\left\| \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \right\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} = \left\| \begin{pmatrix} 1 - 2.5 \\ 2 - 2.5 \\ 3 - 2.5 \\ 4 - 2.5 \end{pmatrix} \right\|$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = x_1 z_1 + x_2 z_2 + x_3 z_3 + x_4 z_4$$

$$n E(x_t^2) = \frac{1}{\cancel{n}} \sum_{t=1}^{n} x_t^2$$

$\sum \varepsilon_t^2 = S(\beta_0) \to$ 이차함수

$S(\beta_0)$



β_0

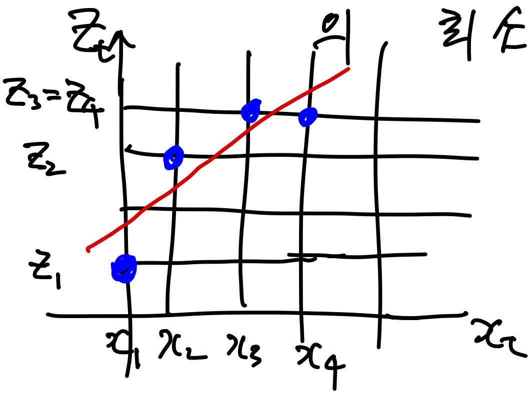
$x^2 + ax + b \to (x+\alpha)^2 + \beta$

$x = -\frac{a}{2}$

$2x + a$

$\left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4}$

예제3) $(0,1)$, $(1,3)$, $(2,4)$, $(3,4)$ 의 최소제곱 직선.



$$Z_t = \boxed{\beta_0 + \beta_1 x_t} + \varepsilon_t.$$

$$\sum_{t=1}^{4} (\varepsilon_t)^2 = \sum_{t=1}^{4} (Z_t - \beta_0 - \beta_1 x_t)^2$$

$$= \sum_{t=1}^{4} (\beta_0 + \beta_1 x_t - Z_t)^2$$

$$(\beta_0 \quad - 1)^2 + (\beta_0 + \beta_1 - 3)^2 \quad -6\beta_0 \; -6\beta_1$$

$$-2\beta_0$$

$$+ (\beta_0 + 2\beta_1 - 4)^2 + (\beta_0 + 3\beta_1 - 4)^2$$

$$-8\beta_0 \; -16\beta_1 \qquad -8\beta_0 \; -24\beta_1$$

$$= 4\beta_0^2 + 14\beta_1^2 - 24\beta_0 - 46\beta_1 + 12\beta_0\beta_1 + C$$

$$= S(\beta_0, \beta_1) \Rightarrow \frac{\partial S}{\partial \beta_0} = 8\beta_0 - 24 + 12\beta_1$$

$$= 0.$$

$$\frac{\partial S}{\partial \beta_1} = 28\beta_1 - 46 + 12\beta_0$$

$$8\beta_0 + 12\beta_1 = 24$$

$$2\beta_0 + 3\beta_1 = 6$$

$$12\beta_0 + 28\beta_1 = 46$$

$$6\beta_0 + 14\beta_1 = 23$$

$$8\beta_0 + 12\beta_1 = 24$$

$$2\beta_0 + 3\beta_1 = 6$$

$$12\beta_0 + 28\beta_1 = 46$$

$$6\beta_0 + 14\beta_1 = 23$$

$$6\beta_0 = 9$$

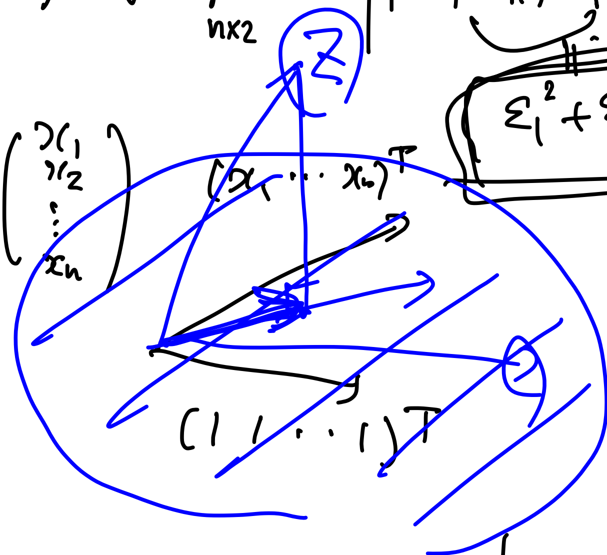
$$6\beta_0 + 9\beta_1 = 18$$

$$5\beta_1 = 5 \Rightarrow \boxed{\begin{aligned}\hat{\beta}_1 &= 1\\ \Rightarrow \hat{\beta}_0 &= 1.5\end{aligned}}$$

$$\left.\begin{aligned}
Z_1 &= \beta_0 + \beta_1 x_1 + \varepsilon_1\\
Z_2 &= \beta_0 + \beta_1 x_2 + \varepsilon_2\\
&\vdots\\
Z_n &= \beta_0 + \beta_1 x_n + \varepsilon_n.
\end{aligned}\right\} \rightarrow$$

$$\left\| \begin{pmatrix} z_1\\ \vdots \\ \vdots \\ z_n\end{pmatrix} - \begin{pmatrix} 1 & x_1\\ 1 & x_2 \\ & \vdots \\ 1 & x_n\end{pmatrix}_{n\times 2} \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}_{2\times 1}\right\|^2 = \left\| \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_n\end{pmatrix}\right\|^2 = \boxed{\varepsilon_1^2+\varepsilon_2^2+\cdots+\varepsilon_n^2}$$

$$\begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix} \quad \begin{pmatrix}x_1\\x_2\\ \vdots\\ x_n\end{pmatrix}$$



$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\end{pmatrix}_{3\times 4} \begin{pmatrix} b_1\\ b_2\\ b_3\\ b_4\end{pmatrix}_{4\times 1} = \begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix}_{3\times 1}$$

$$= b_1\begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix} + b_2\begin{pmatrix}a_{12}\\a_{22}\\a_{32}\end{pmatrix} + b_3\begin{pmatrix}a_{13}\\a_{23}\\a_{33}\end{pmatrix} + b_4\begin{pmatrix}a_{14}\\a_{24}\\a_{34}\\a_{44}\end{pmatrix}$$

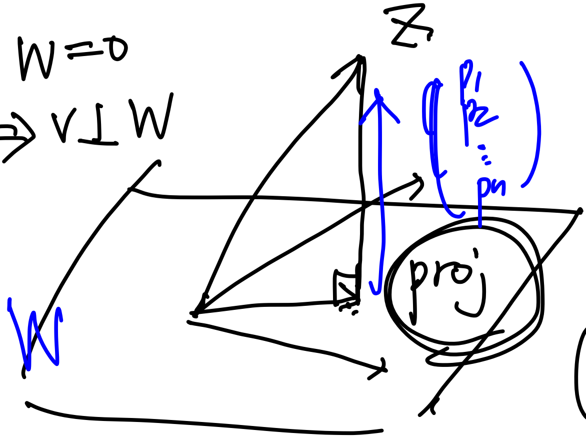
$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \beta_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \beta_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

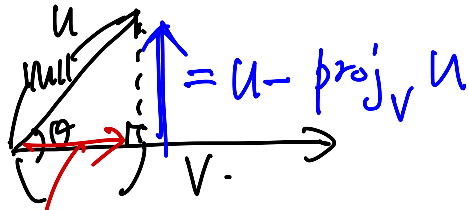
$$\|$$

$$\begin{pmatrix} \beta_0 + \beta_1 x_1 \\ \beta_0 + \beta_1 x_2 \\ \vdots \\ \beta_0 + \beta_1 x_n \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_0 \\ \vdots \\ \beta_0 \end{pmatrix} + \begin{pmatrix} \beta_1 x_1 \\ \beta_1 x_2 \\ \vdots \\ \beta_1 x_n \end{pmatrix}$$

$$= \beta_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \beta_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

※ $v \cdot w = 0 \iff v \perp w$





$\|u\| \cos\theta$.

$$\left(\|u\| \cos\theta \frac{1}{\|v\|}\right) v$$

$$\Big\| Z - \underbrace{\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}}_{\beta} \Big\|^2 \qquad = \left(\frac{\|u\| \|v\| \cos\theta}{\|v\|^2} \right) v$$

$$= \left(\frac{u \cdot v}{\|v\|^2} \right) v = \text{proj}_v u$$

$$Z - M\beta = Z - \text{proj}_W Z$$

$$※ \; M^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

$$M^T (Z - M\beta) = M^T (Z - \text{proj}_W Z) = 0.$$

$$M^T (Z - M\beta) = M^T Z - M^T M \beta = 0$$

$$\Rightarrow M^T M \beta = M^T Z \Rightarrow \boxed{\beta = (M^T M)^{-1} M^T Z}$$

$$\left(M^TM \;\middle|\; \begin{matrix}1&&&\\&1&&\\&&\ddots&\\&&&1\end{matrix}\right) \longrightarrow \left(\begin{matrix}1&&&\\&1&&\\&&\ddots&\\&&&I\end{matrix} \;\middle|\; (M^TM)^{-1}\right)$$

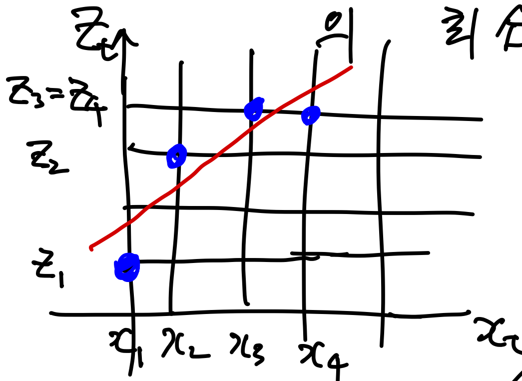
$$\underset{2\times n}{\begin{pmatrix}1&1&\cdots&1\\x_1&x_2&\cdots&x_n\end{pmatrix}}\underset{n\times 2}{\begin{pmatrix}1&x_1\\1&\vdots\\\vdots&\vdots\\1&x_n\end{pmatrix}} = (\qquad)_{2\times 2}.$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$$

$$ad-bc \to \det$$

$$\beta = (M^TM)^{-1}M^TZ.$$

예제3) $(0,1)$, $(1,3)$, $(2,4)$, $(3,4)$ 의 최소제곱 직선.



$$\Bigg\|\underbrace{\begin{pmatrix}1\\3\\4\\4\end{pmatrix}}_{Z} - \underbrace{\begin{pmatrix}1&0\\1&1\\1&2\\1&3\end{pmatrix}}_{M}\underbrace{\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix}}_{\beta.}\Bigg\|^2$$

$$Z_t = \beta_0 + \beta_1 x_t + \varepsilon_t$$

$$M^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 12 \\ 23 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}$$

$$\Rightarrow \frac{1}{56-36} \begin{pmatrix} 14 & -6 \\ -6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 14 & -6 \\ -6 & 4 \end{pmatrix} \begin{pmatrix} 12 \\ 23 \end{pmatrix}$$