예제1) (1, 1), (2, 1), (3, 2), (4, 2), (5, 3) 의 회소 제곱 직선.

sol)

x_t	Z_t
1	1
2	1
3	2
4	2
5	3

$$Z_t = \boxed{\beta_0 + \beta_1 x_t} + \varepsilon_t$$

$$1 = \beta_0 + \beta_1 \cdot 1 + \varepsilon_1$$

$$1 = \beta_0 + \beta_1 \cdot 2 + \varepsilon_2$$

$$2 = \beta_0 + \beta_1 \cdot 3 + \varepsilon_3$$

$$2 = \beta_0 + \beta_1 \cdot 4 + \varepsilon_4$$

$$3 = \beta_0 + \beta_1 \cdot 5 + \varepsilon_5$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}}_{\beta} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}}_{Z}$$

$$M^T M \beta = M^T Z \Rightarrow \beta = (M^T M)^{-1} M^T Z$$

예제2)

x_t	y_t	Z_t
1	1	3.5
1	-1	3.0
2	2	3.0
2	-2	2.5

$$Z_t = \beta_1 x_t + \beta_2 y_t + \varepsilon_t$$

예제2)

x_t	y_t	z_t
1	1	3.5
1	-1	3.0
2	2	3.0
2	-2	2.5

① $z_t = \beta_1 x_t + \beta_2 y_t + \varepsilon_t$

② $z_t = \beta_0 + \beta_1 x_t + \beta_2 y_t + \varepsilon_t$

Sol) ①

$\beta_1 \cdot 1 + \beta_2 \cdot 1 = 3.5$

$\beta_1 \cdot 1 + \beta_2 \cdot (-1) = 3.0$

$\beta_1 \cdot 2 + \beta_2 \cdot 2 = 3.0$

$\beta_1 \cdot 2 + \beta_2 \cdot (-2) = 2.5$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 3.5 \\ 3.0 \\ 3.0 \\ 2.5 \end{pmatrix}$$

$$M^T M \beta = M^T Z$$

②

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \\ 1 & 2 & -2 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}}_{\beta} = Z$$

$$M^T M \beta = M^T Z$$

3x3

X_{t1}	X_{t2}	Z_t
X_{11}	X_{12}	Z_1
X_{21}	X_{22}	Z_2
$\vdots$	$\vdots$	$\vdots$
X_{n1}	X_{n2}	Z_n

$$Z_1 = \beta_0 + \beta_1 X_{11} + \beta_2 X_{12} + \varepsilon_1$$

$$Z_2 = \beta_0 + \beta_1 X_{21} + \beta_2 X_{22} + \varepsilon_2$$

$$\vdots$$

$$Z_n = \beta_0 + \beta_1 X_{n1} + \beta_2 X_{n2} + \varepsilon_n$$

$$\Rightarrow \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 & X_{11} & X_{12} \\ 1 & X_{21} & X_{22} \\ \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\boxed{Z = X\beta + \varepsilon.}$$

X_{t1}	X_{t2}	$\cdots$	X_{tp}	Z_t
X_{11}	X_{12}		X_{1p}	Z_1
X_{21}	X_{22}	$\cdots$	X_{2p}	Z_2
$\vdots$	$\vdots$		$\vdots$	$\vdots$
X_{n1}	X_{n2}		X_{np}	Z_n

$$Z = X\beta + \varepsilon$$

$$X = \begin{pmatrix} 1 & X_{11} & \cdots & X_{1p} \\ 1 & X_{21} & \ddots & X_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{n1} & \cdots & X_{np} \end{pmatrix}$$

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}.$$

$\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$ 현재 가정 $\begin{cases} E(\varepsilon_t) = 0. \\ Var(\varepsilon_t) = \sigma_\varepsilon^2 \\ \sim N(0, \sigma_\varepsilon^2). \end{cases}$

$(t = 1, 2, \cdots, n)$

+ $t_1 \neq t_2$ 이면 $Cov(\varepsilon_{t_1}, \varepsilon_2) = 0.$

※ 공분산 $Cov(\cdot, \cdot)$.

$$Cov(X, Y) := E[(X - \bar{X})(Y - \bar{Y})]$$

※ $E[X+Y] = E[X] + E[Y]$

$$= \frac{1}{n \to n-1} \boxed{\sum_{t=1}^{n} (x_t - \bar{x})(y_t - \bar{y})}$$

$$\begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix} \cdot \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix}$$

$$E[XY - \bar{X}Y - X\bar{Y} + \bar{X}\bar{Y}]$$

$$= \boxed{E[XY] - \bar{X}\bar{Y}} \cancel{- \bar{Y}\bar{X}} \cancel{+ \bar{X}\bar{Y}}$$

$$= \boxed{E[XY] - E[X]E[Y]}.$$

X, Y : 독립 $\Rightarrow$ $E[XY] = E[X]E[Y]$

$\Leftrightarrow$ $\Downarrow$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 0$$

$$\boxed{※\ u \cdot v = 0 \Leftrightarrow u \perp v}$$

$$\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix} \Rightarrow E(\varepsilon) = \begin{pmatrix} E(\varepsilon_1) \\ E(\varepsilon_2) \\ \vdots \\ E(\varepsilon_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

"공분산 행렬"

대칭!

$$Cov(\varepsilon) = E(\varepsilon\varepsilon^T) = E(\varepsilon\varepsilon')$$

$$\underset{n\times 1}{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}} \underset{1\times n}{\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}} = \begin{pmatrix} \varepsilon_1^2 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 \cdots & \varepsilon_1\varepsilon_n \\ \varepsilon_2\varepsilon_1 & \varepsilon_2^2 & \varepsilon_2\varepsilon_3 \cdots & \varepsilon_2\varepsilon_n \\ & & \ddots & \\ & & & \varepsilon_n^2 \end{pmatrix}$$

※ $A^T = A$

A : 대칭행렬

$$E(\varepsilon_t^2) = E\big((\varepsilon_t - \underbrace{0}_{\overline{\varepsilon_t}})^2\big)$$

$$= E\big(|\varepsilon_t - \overline{\varepsilon_t}|^2\big)$$

$$= Var(\varepsilon_t) = \sigma_\varepsilon^2$$

$t_1 \neq t_2$

$$\Rightarrow E(\varepsilon_{t_1}\varepsilon_{t_2}) = E\big((\varepsilon_{t_1} - 0)(\varepsilon_{t_2} - 0)\big)$$

$$= E\big((\varepsilon_{t_1} - \overline{\varepsilon_{t}})(\varepsilon_{t_2} - \overline{\varepsilon_{t_2}})\big)$$

$$= Cov(\varepsilon_{t_1}, \varepsilon_{t_2})$$

$$= 0.$$

$$\therefore Cov(\varepsilon) = \sigma_\varepsilon^2 I$$

$$\Rightarrow \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

다중정규분포

평균: 0 공분산 $\sigma_\varepsilon^2 I$

$$Z = X\beta + \varepsilon$$

$$(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n)_{1\times n} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}_{n\times 1} = \varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_n^2$$

$$\varepsilon^T \varepsilon$$

$$X^T X \beta = X^T Z$$

$$\boxed{\beta = (X^T X)^{-1} X^T Z}$$

$$S(\beta) = \varepsilon^T \varepsilon = \varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_n^2$$

$$\varepsilon' \varepsilon = (Z - X\beta)^T (Z - X\beta)$$

$$\varepsilon = Z - X\beta$$

$$= (Z^T - (X\beta)^T)(Z - X\beta)$$

$$= (Z^T - \beta^T X^T)(Z - X\beta)$$

$$= Z^T Z - \underbrace{Z^T X\beta}_{1\times n \ \ n\times 1} - \underbrace{\beta^T X^T Z}_{1\times n \ \ n\times 1} + \beta^T X^T X \beta$$

$Z : n\times 1$

$X\beta : n\times 1$

$$= \underbrace{Z^T Z}_{\|Z\|^2} - 2\beta^T X^T Z + \underbrace{\beta^T X^T X\beta}_{\|X\beta\|^2}$$

$$\frac{\partial S}{\partial \beta} = -2X^T Z + 2X^T X\beta = 0$$

$$\boxed{X^T X\beta = X^T Z} \Rightarrow \boxed{\beta = (X^T X)^{-1} X^T Z}$$