$$\begin{cases} \frac{dT}{dt} = p_T \times A - u_T \times T \\ \frac{dA}{dt} = p_A \times I(t \leq h) - u_A \times A \end{cases} \tag{1}$$

Solution

in matrix form we have

$$\Rightarrow \begin{pmatrix} T \\ A \end{pmatrix}' = \begin{pmatrix} -\mu_T & p_T \\ 0 & -\mu_A \end{pmatrix} \begin{pmatrix} T \\ A \end{pmatrix} + \begin{pmatrix} 0 \\ Ip_A \end{pmatrix}$$

$\Rightarrow$ finding the eigenvalues

$$\Rightarrow \begin{pmatrix} -\mu_T & p_T \\ 0 & -\mu_A \end{pmatrix} - \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} = \begin{vmatrix} -\mu_T - \Lambda & -p_T \\ 0 & -\mu_A - \Lambda \end{vmatrix} = 0$$

$$\Rightarrow -\mu_T(-\mu_A - \Lambda) - \Lambda(-\mu_A - \Lambda) - 0 = 0$$

$\Rightarrow$ solving the above function we get:

$$\Rightarrow \Lambda_1 = -\mu_A, \Lambda_2 = -\mu_T$$

$\Rightarrow$ eigenvector associated with, $\Lambda_1 = -\mu_A$,

$$\Rightarrow \begin{pmatrix} -\mu_T + \mu_A & p_T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_1(t) = \begin{pmatrix} \mu_A \\ -\mu_T + \mu_A \end{pmatrix} e^{-\mu_A t}$$

$\Rightarrow$ eigenvector associated with, $\Lambda_2 = -\mu_T$,

$$\Rightarrow \begin{pmatrix} 0 & p_T \\ 0 & -\mu_A + \mu_T \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_2(t) = \begin{pmatrix} \mu_T \\ -\mu_T + \mu_A \end{pmatrix} e^{-\mu_T t}$$

$$\Rightarrow \begin{pmatrix} T \\ A \end{pmatrix}(t) = \frac{-\mu_T I p_A t}{\mu_T^2 - \mu_A^2} \begin{pmatrix} \mu_A \\ -\mu_T + \mu_A \end{pmatrix} e^{-\mu_A t} + \frac{\mu_T I p_A t}{\mu_T^2 - \mu_A^2} \begin{pmatrix} \mu_T \\ -\mu_T + \mu_A \end{pmatrix} e^{-\mu_T t}$$

$$\begin{cases} \frac{dT}{dt} = p_T \times A - u_T \times T \\ \frac{dA}{dt} = p_A \times A \times I(t \leq h) - u_A \times A \end{cases} \tag{2}$$

Solution

in matrix form we have

$$\Rightarrow \begin{pmatrix} T \\ A \end{pmatrix}' = \begin{pmatrix} -\mu_T & p_T \\ 0 & p_A I - \mu_A \end{pmatrix} \begin{pmatrix} T \\ A \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\mu_T & p_T \\ 0 & p_A I - \mu_A \end{pmatrix} - \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} = \begin{vmatrix} -\mu_T - \Lambda & p_T \\ 0 & p_A I - \mu_A - \Lambda \end{vmatrix} = 0$$

$$\Rightarrow -\mu_T(p_A I - \mu_A - \Lambda) - \Lambda(p_A I - \mu_A - \Lambda) - 0 = 0$$

$$\Rightarrow \text{solving the above function we get:}$$

$$\Rightarrow \Lambda_1 = -\mu_T, \Lambda_2 = p_A I$$

$$\Rightarrow \text{eigenvector associated with, } \Lambda_1 = -\mu_T,$$

$$\Rightarrow \begin{pmatrix} 0 & p_T \\ 0 & p_A I - \mu_A + \mu_T \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_1(t) = \begin{pmatrix} p_A I \\ p_A I - \mu_A + \mu_T \end{pmatrix} e^{-\mu_T t}$$

$$\Rightarrow \text{eigenvector associated with, } \Lambda_2 = p_A I,$$

$$\Rightarrow \begin{pmatrix} \mu_T - p_A I & p_T \\ 0 & p_A I - \mu_A + \mu_T \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_2(t) = \begin{pmatrix} \mu_A \\ p_A I + \mu_T \end{pmatrix} e^{p_A I t}$$

$$\Rightarrow \begin{pmatrix} T \\ A \end{pmatrix}(t) = c_1 \begin{pmatrix} p_A I \\ p_A I - \mu_A + \mu_T \end{pmatrix} e^{-\mu_T t} + c_2 \begin{pmatrix} \mu_A \\ p_A I + \mu_T \end{pmatrix} e^{p_A I t}, t \leq h$$

$$\begin{cases} \frac{dT}{dt} = p_T \times A - u_T \times T \\ \frac{dA}{dt} = -u_A \times A \end{cases} \tag{3}$$

Solution

in matrix form we have

$$\Rightarrow \begin{pmatrix} T \\ A \end{pmatrix}' = \begin{pmatrix} -\mu_T & p_T \\ 0 & -\mu_A \end{pmatrix} \begin{pmatrix} T \\ A \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\mu_T & p_T \\ 0 & -\mu_A \end{pmatrix} - \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} = \begin{vmatrix} -\mu_T - \Lambda & p_T \\ 0 & -\mu_A - \Lambda \end{vmatrix} = 0$$

$$\Rightarrow -\mu_T(-\mu_A - \Lambda) - \Lambda(-\mu_A - \Lambda) - 0 = 0$$

$\Rightarrow$ solving the above function we get:

$$\Rightarrow \Lambda_1 = -\mu_A, \Lambda_2 = -\mu_T$$

$\Rightarrow$ eigenvector associated with, $\Lambda_1 = -\mu_A$,

$$\Rightarrow \begin{pmatrix} -\mu_T + \mu_A & p_T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_1(t) = \begin{pmatrix} p_T \\ \mu_T - \mu_A \end{pmatrix} e^{-\mu_A t}$$

$\Rightarrow$ eigenvector associated with, $\Lambda_2 = -\mu_T$,

$$\Rightarrow \begin{pmatrix} 0 & p_T \\ 0 & -\mu_T + \mu_A \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_2(t) = \begin{pmatrix} \mu_A \\ \mu_T - \mu_A \end{pmatrix} e^{-\mu_T t}$$

$$\Rightarrow \begin{pmatrix} T \\ A \end{pmatrix}(t) = c_1 \begin{pmatrix} p_T \\ -\mu_A + \mu_T \end{pmatrix} e^{-\mu_A t} + c_2 \begin{pmatrix} \mu_A \\ \mu_T - \mu_A \end{pmatrix} e^{-\mu_T t}, t \le h$$

$$\begin{cases} \frac{dT}{dt} = p_{Ts} \times As + p_{Tl} \times Al - u_T \times T \\ \frac{dAs}{dt} = p_{As} \times I(t \le h) - u_{Al} \times As \\ \frac{dAl}{dt} = 0 \end{cases} \tag{4}$$

Solution

in matrix form we have

$$\Rightarrow \begin{pmatrix} T \\ A_s \\ A_l \end{pmatrix}'(t) = \begin{pmatrix} -\mu_T & p_{Ts} & p_{Tl} \\ 0 & -\mu_{Al} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ A_s \\ A_l \end{pmatrix} + \begin{pmatrix} 0 \\ p_{As} I \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{finding the eigenvalues:}$$

$$\Rightarrow \begin{pmatrix} -\mu_T & p_{Ts} & p_{Tl} \\ 0 & -\mu_{Al} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix} = \begin{vmatrix} -\mu_T - \Lambda & p_{Ts} & p_{Tl} \\ 0 & -\mu_{Al} - \Lambda & 0 \\ 0 & 0 & -\Lambda \end{vmatrix} = 0$$

$$\Rightarrow -\mu_T - \Lambda \begin{vmatrix} -\mu_{Al} - \Lambda & 0 \\ 0 & -\Lambda \end{vmatrix} = 0$$

$$\Rightarrow -\mu_T - \Lambda[\Lambda\mu_{Al} + \Lambda^2] - 0 = 0$$

$$\Rightarrow \Lambda_1 = -\mu_T, \Lambda_2 = \frac{-\mu_T}{2\mu_{Al}}, \Lambda_3 = \frac{\mu_T}{2\mu_{Al}}$$

$$\Rightarrow \text{eigenvector associated with, } \Lambda_1 = -\mu_T$$

$$\Rightarrow \begin{pmatrix} 0 & p_{Ts} & p_{Tl} \\ 0 & -\mu_{Al} + \mu_T & 0 \\ 0 & 0 & \mu_T \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_1(t) = \begin{pmatrix} \mu_{Al} \\ p_{Tl} \\ -p_{Ts} \end{pmatrix} e^{-\mu_T t}$$

$$\Rightarrow \text{eigenvector associated with, } \Lambda_2 = \frac{-\mu_T}{2\mu_{Al}}$$

$$\Rightarrow \begin{pmatrix} -\mu_T + \frac{\mu_T}{2\mu_{Al}} & p_{Ts} & p_{Tl} \\ 0 & -\mu_{Al} + \frac{\mu_T}{2\mu_{Al}} & 0 \\ 0 & 0 & \frac{\mu_T}{2\mu_{Al}} \end{pmatrix} \begin{pmatrix} v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_2(t) = \begin{pmatrix} \mu_T \\ -p_{Tl} \\ \mu_{Al} \end{pmatrix} e^{-\frac{\mu_T}{2\mu_{Al}} t}$$

$$\Rightarrow \text{eigenvector associated with, } \Lambda_3 = \frac{\mu_T}{2\mu_{Al}}$$

$$\Rightarrow \begin{pmatrix} -\mu_T - \frac{\mu_T}{2\mu_{Al}} & p_{Ts} & p_{Tl} \\ 0 & -\mu_{Al} - \frac{\mu_T}{2\mu_{Al}} & 0 \\ 0 & 0 & -\frac{\mu_T}{2\mu_{Al}} \end{pmatrix} \begin{pmatrix} v_7 \\ v_8 \\ v_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_3(t) = \begin{pmatrix} -\mu_T \\ -\mu_{Al} \\ p_{Ts} \end{pmatrix} e^{\frac{\mu_T}{2\mu_{Al}}t}$$

$$\Rightarrow \begin{pmatrix} T \\ A_s \\ A_l \end{pmatrix}(t) = c_1 \begin{pmatrix} \mu_{Al} \\ p_{Tl} \\ -p_{Ts} \end{pmatrix} e^{-\mu_T t} + c_2 \begin{pmatrix} \mu_T \\ -p_{Tl} \\ p_{Ts} \end{pmatrix} e^{-\frac{\mu_T}{2\mu_{Al}}t} + c_3 \begin{pmatrix} -\mu_T \\ -\mu_{Al} \\ p_{Ts} \end{pmatrix} e^{\frac{\mu_T}{2\mu_{Al}}t} + \begin{pmatrix} \mu_{Al} \\ p_{Tl} - \mu_T \\ p_{Ts} \end{pmatrix} e^{\frac{-\mu_T}{1+4\mu_{Al}}t}, t \leq h$$

$$\begin{cases} \frac{dT}{dt} = p_{Ts} \times As + p_{Tl} \times Al - u_T \times T \\ \frac{dAs}{dt} = p_{As} \times AS \times I(t \leq h) - u_{As} \times As \\ \frac{dAl}{dt} = 0 \end{cases} \tag{5}$$

Solution

in matrix form we have

$$\Rightarrow \begin{pmatrix} T \\ A_s \\ A_l \end{pmatrix}'(t) = \begin{pmatrix} -\mu_T & p_{Ts} & p_{Tl} \\ 0 & p_{As}I - \mu_{As} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ A_s \\ A_l \end{pmatrix}$$

$$\Rightarrow \text{finding the eigenvalues:}$$

$$\Rightarrow \begin{pmatrix} -\mu_T & p_{Ts} & p_{Tl} \\ 0 & p_{As}I - \mu_{As} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix} = \begin{vmatrix} -\mu_T - \Lambda & p_{Ts} & p_{Tl} \\ 0 & p_{As}I - \mu_{As} - \Lambda & 0 \\ 0 & 0 & -\Lambda \end{vmatrix} = 0$$

$$\Rightarrow -\mu_T - \Lambda \begin{vmatrix} p_{As}I - \mu_{As} - \Lambda & 0 \\ 0 & -\Lambda \end{vmatrix} = 0$$

$$\Rightarrow -\mu_T - \Lambda[(p_{As}I - \mu_{As} - \Lambda) - \Lambda] - 0 = 0$$

$$\Rightarrow \Lambda_1 = -\mu_T, \Lambda_2 = \mu_{As}, \Lambda_3 = p_{As}I$$

$$\Rightarrow \text{eigenvector associated with, } \Lambda_1 = -\mu_T$$

$$\Rightarrow \begin{pmatrix} 0 & p_{Ts} & p_{Tl} \\ 0 & p_{As}I - \mu_{As} + \mu_T & 0 \\ 0 & 0 & \mu_T \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_1(t) = \begin{pmatrix} \mu_T \\ p_{As}I - \mu_T \\ -\mu_{As} \end{pmatrix} e^{-\mu_T t}$$

$$\Rightarrow \text{eigenvector associated with, } \Lambda_2 = \mu_{As}$$

$$\Rightarrow \begin{pmatrix} -\mu_T - \mu_{As} & p_{Ts} & p_{Tl} \\ 0 & p_{As}I - \mu_{As} - \mu_{As} & 0 \\ 0 & 0 & -\mu_{As} \end{pmatrix} \begin{pmatrix} v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_2(t) = \begin{pmatrix} -\mu_T \\ \mu_{As} - \mu_T \\ pAsI \end{pmatrix} e^{\mu_{As} t}$$

$$\Rightarrow \text{eigenvector associated with}, \Lambda_3 = p_{As} I$$

$$\Rightarrow \begin{pmatrix} -\mu_T - p_{As} I & p_{Ts} & p_{Tl} \\ 0 & p_{As} I - \mu_{As} - p_{As} I & 0 \\ 0 & 0 & -p_{As} I \end{pmatrix} \begin{pmatrix} v_7 \\ v_8 \\ v_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_3(t) = \begin{pmatrix} -\mu_T \\ p_{As} I - \mu_T \\ p_{Tl} \end{pmatrix} e^{p_{As} I t}$$

$$\Rightarrow \begin{pmatrix} T \\ A_s \\ A_l \end{pmatrix}(t) = c_1 \begin{pmatrix} \mu_T \\ p_{As} I - \mu_T \\ \mu_{As} \end{pmatrix} e^{-\mu_T t} + c_2 \begin{pmatrix} -\mu_T \\ \mu_{As} - \mu_T \\ p_{As} I \end{pmatrix} e^{\mu_{As} t} + c_3 \begin{pmatrix} -\mu_T \\ p_{Ts} - \mu_T \\ p_{Tl} - \mu_T \end{pmatrix} e^{p_{As} I t}, t \leq h$$

$$\begin{cases} \frac{dT}{dt} = p_{Ts} \times As + p_{Tl} \times Al - u_T \times T \\ \frac{dAs}{dt} = -u_{As} \times As \\ \frac{dAl}{dt} = 0 \end{cases} \tag{6}$$

Solution

in matrix form we have

$$\Rightarrow \begin{pmatrix} T \\ A_s \\ A_l \end{pmatrix}'(t) = \begin{pmatrix} -\mu_t & p_{Ts} & p_{Tl} \\ 0 & -\mu_{As} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ A_s \\ A_l \end{pmatrix}$$

$$\Rightarrow \text{finding eigenvalues:}$$

$$\Rightarrow \begin{pmatrix} -\mu_t & p_{Ts} & p_{Tl} \\ 0 & -\mu_{As} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix} = \begin{vmatrix} -\mu_T - \Lambda & p_{Ts} & p_{Tl} \\ 0 & -\mu_{As} - \Lambda & 0 \\ 0 & 0 & -\Lambda \end{vmatrix} = 0$$

$$\Rightarrow \Lambda_1 = \mu_T, \Lambda_2 = \mu_{As}, \Lambda_3 = -\mu_T$$

$$\text{eigenvector associated with, } \Lambda_1 = \mu_T$$

$$\Rightarrow \begin{pmatrix} -2\mu_T & p_{Ts} & p_{Tl} \\ 0 & -\mu_{As} - \mu_T & 0 \\ 0 & 0 & -\mu_T \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_1(t) = \begin{pmatrix} -\mu_T \\ p_{Ts} \\ \mu_{As} \end{pmatrix} e^{\mu_T t}$$

$$\text{eigenvector associated with, } \Lambda_2 = \mu_{As}$$

$$\Rightarrow \begin{pmatrix} -\mu_T - \mu_{As} & p_{Ts} & p_{Tl} \\ 0 & -2\mu_{As} & 0 \\ 0 & 0 & -\mu_{As} \end{pmatrix} \begin{pmatrix} v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_2(t) = \begin{pmatrix} -\mu_{As} \\ p_{Tl} \\ \mu_T \end{pmatrix} e^{\mu_{As} t}$$

$$\text{eigenvector associated with, } \Lambda_3 = -\mu_T$$

$$\Rightarrow \begin{pmatrix} 0 & p_{Ts} & p_{Tl} \\ 0 & \mu_T - \mu_{As} & 0 \\ 0 & 0 & \mu_T \end{pmatrix} \begin{pmatrix} v_7 \\ v_8 \\ v_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow h_3(t) = \begin{pmatrix} \mu_T \\ p_{Tl} \\ \mu_{As} \end{pmatrix} e^{-\mu_T t}$$

$$\Rightarrow \begin{pmatrix} T \\ A_s \\ A_l \end{pmatrix}(t) = c_1 \begin{pmatrix} -\mu_T \\ p_{Ts} \\ \mu_{As} \end{pmatrix} e^{\mu_T t} + c_2 \begin{pmatrix} -\mu_{As} \\ p_{Tl} \\ \mu_T \end{pmatrix} e^{\mu_{As} t} + c_3 \begin{pmatrix} \mu_T \\ p_{Tl} \\ \mu_{As} \end{pmatrix} e^{-\mu_T t}, t \leq h$$

THE END