PAGE NO.:
DATE: / /

Q1 Solve the homogeneous heat equation for $L=2$, $k=10$ and dirichlet homogeneous BC and IC given by
$$f(x) = 5 + 5\sin^2(\pi x/2)$$

Soln

We let the heat equation with the given conditions be given as

$$\frac{\partial u}{\partial t} = 10\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad k = 10 > 0 \qquad (*)$$

for $u(0,t) = 0$

$u(2,t) = 0$

and

$$u(x,0) = f(x) = 5 + 5\sin^2(\pi x/2)$$

We will be using the method of separation of variables to solve the above eqn

=> So we Let

$$u(x,t) = X(x)\,T(t)$$

=> Differentiating the above w.r.t x and t respectively

$$u_t(x,t) = X(x)\,T'(t) \quad \text{and} \quad u_{xx}(x,t) = X''(x)\,T(t)$$

=> Substituting into (*) we have

$XT' = 10XT''$

$\Rightarrow$ dividing all by $u(x,t) = XT$ we have

$$\frac{T'}{10T} = \frac{X''}{X} = -\lambda$$

$\Rightarrow$ Separating the equations into 2 independent equations

$$\frac{T'}{10T} = -\lambda \quad \text{— (1)} \qquad \text{and}$$

$$\frac{X''}{X} = -\lambda \quad \text{— (2)}$$

$\Rightarrow$ Solving (1) we have

$$\frac{dT}{10T\,dt} = \frac{T'}{10T} = -\lambda \quad \Rightarrow \quad \frac{dT}{T} = -10\lambda dt$$

Upon integration

$$\ln T = -10\lambda t + c$$

$$\Rightarrow \quad T(t) = e^{-10\lambda t + c} = e^{-10\lambda t} \cdot e^{c}$$

$$\text{Let } e^{c} = A$$

$$\Rightarrow \quad T(t) = Ae^{-10\lambda t}$$

and Solving (2) we have,

$\frac{X''}{X} = -\lambda \implies X'' + \lambda X = 0$

$\Rightarrow$ This is a 2^{nd} order ode equation, we solve by characteristic method

$$a^2 + \lambda = 0 \implies a^2 = -\lambda \implies a = \pm\sqrt{-\lambda}$$

$$X(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$

$\Rightarrow$ Now that we have the 2 solutions to $T(t)$ and $X(x)$, next we note of the hypothesis that for $\lambda \geq 0$, we will re-write the $X_1(x)$ solution as

$$X(x) = \bar{c}_1 \sin\sqrt{\lambda}x + \bar{c}_2 \cos\sqrt{\lambda}x$$

$\Rightarrow$ Thus A, c_1, c_2, $\bar{c}_2$ and $\bar{c}_1$ are what is called constants of integration

Such that $\lim_{t \to \infty} \phi(t) = 0$ and not $+\infty$

$\Rightarrow$ Hence we have the solution $U(x,t)$ given as

$$U(x,t) = X(x)\,T(t)$$

$$U(x,t) = A e^{-10\lambda t}\left[\bar{c}_1 \sin\sqrt{\lambda}x + \bar{c}_2 \cos\sqrt{\lambda}x\right]$$

$\Rightarrow A\bar{C}_1 = B$ and $A\bar{C}_2 = D$

$\Rightarrow$ Thus we have

$$U(x,t) = e^{-10\lambda t}\left[B\sin\sqrt{\lambda}\,x + D\cos\sqrt{\lambda}\,x\right]$$

with constants of integration being B and D

$\Rightarrow$ Next, since we have the solution $U(x,t)$, we now apply the Boundary Conditions.

at $x = 0$

$$U(0,t) = e^{-10\lambda t}\left[0 + D.1\right]$$

$$= e^{-10\lambda t}(D) = De^{-10\lambda t} = 0$$

$$\Rightarrow D = 0$$

similarly

for $U(a,t) = 0$ at $x = a$

$$\Rightarrow U(a,t) = e^{-10\lambda t}\left[B\sin\sqrt{\lambda}.a + D\cos\sqrt{\lambda}.a\right] = 0$$

$\Rightarrow$ with $D=0$, we simply the above further into

$$e^{-10\lambda t}\left[B\sin\sqrt{\lambda}.a\right] = 0$$

$\Rightarrow$ Thus, for a non-trivial solution

DATE: / /

$\sin\sqrt{\lambda}.2 = 0$

=> We need to find Values for which their sine function equals zero

=> $\sqrt{\lambda}.2 = n\pi$ for $n = 1, 2, 3, ---$

=> Taking square's for both sides we have

$4\lambda = n^2\pi^2 \Rightarrow \lambda = \frac{n^2\pi^2}{4}$

=> $\lambda_n = n^2\pi^2/4$ for $n = 1, 2, 3, ----$

=> We now substitute this solution (λ_n) into $u(x,t)$ solution

=> Remember that $D=0$ and $\lambda_n = n^2\pi^2/4$, $n = 1, 2, -$

=> Thus $u(x,t)$ becomes

$u(x,t) = e^{-10\lambda t}[B\sin\sqrt{\lambda}x]$

$\Rightarrow u(x,t) = Be^{-10\lambda t}\sin\sqrt{\lambda}x$

=> For n number of values $n = 1, 2, 3 ---$

=> $u_n(x,t) = B_n\sin\left(\frac{n^2\pi^2}{4}\right)^{\frac{1}{2}} e^{-10\left(\frac{n^2\pi^2}{4}\right)t}$

DATE: / /

$\Rightarrow U_n(x,t) = B_n \sin \frac{n\pi}{2} x . e^{-10\frac{n^2\pi^2}{4}t}$

$\Rightarrow U_n(x,t) = B_n \sin \frac{n\pi}{2} x . e^{-\frac{5}{2}n^2\pi^2 t}$

$\Rightarrow$ By the principle of superposition we have

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{2} x . e^{-\frac{5}{2}n^2\pi^2 t}$$

$\Rightarrow$ We now apply the I C to the above equation

$$U(x,0) = f(x) = 5 + 5\sin(\pi/2\, x)$$

at $t=0$, we have

$$U(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{2} x = 5 + 5\sin(\pi/2\, x)$$

$\Rightarrow$ We expand the summation to get

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{2} x = B_1 \sin \pi/2 + B_2 \sin 2\pi/2 + \cdots$$
$$= 5 + 5\sin(n\pi/2)$$

$\Rightarrow$ For $B_1 = 5$, $B_2 = B_3 = B_4 = \cdots B_n = 0$

for $n = 1, 2, 3, \cdots$

DATE: / /

For the particular Solution

$$U(x,t) = 5 + 5\sin^2\left(\frac{\pi}{2}x\right) e^{\frac{5}{2}\pi^2 t} + 0 \cdot \sin^2\left(\frac{2\pi}{2}\right)x \cdot e^{\frac{5}{2}\cdot 4\pi^2 t}$$

$$\Rightarrow U(x,t) = 5 + 5\sin^2\left(\frac{\pi}{2}x\right) \cdot e^{\frac{5}{2}\pi^2 t}$$

$\boxed{X}$

Q2 Solve homogeneous heat equation for $L=4$, $k=6$ and Neuman Constant BC. and IC

$\frac{du}{dx}(0,t) = -1$ and $\frac{du}{dx}(L,t) = 4$

with $f(x) = 20 - 5(x-2)^2 = u(x,0)$

Soln

We need to write the heat equation with $L=4$ and $k=6$ as

$$\frac{\partial u}{\partial t} = 6\frac{\partial^2 u}{\partial x^2} \quad \text{for } k=6>0 \quad \text{---}(*)$$

$\Rightarrow$ let $U(x,t) = X_{(x)} T_{(t)}$

$\Rightarrow U_t(x,t) = X_{(x)} T'_{(t)}$ and $U_{xx}(x,t) = X''_{(x)} T_{(t)}$

PAGE NO.:
DATE: / /

$\Rightarrow$ Substitute these into $(*)$, we have

$$XT' = 6X''T$$, dividing by $X_{(x)}T_{(t)}$

$$\frac{T'}{6T} = \frac{X''}{X} = -\lambda$$, we normally set it to a constant

$\Rightarrow$ Separating the variables, we have

$$\frac{T'}{6T} = -\lambda \quad \text{—} (1) \qquad \text{and} \qquad \frac{X''}{X} = -\lambda \quad \text{—} (2)$$

$\Rightarrow$ Solving for (1) and (2), we get

$$\frac{T'}{6T} = \frac{dT}{6T\,dt} = -\lambda \Rightarrow \frac{dT}{T} = -6\lambda\,dt$$

$$\Rightarrow \ln T = -6\lambda t + c$$

$\Rightarrow$ Taking exponential on both sides

$$T(t) = e^{-6\lambda t + c} = e^{-6\lambda t} \cdot e^{c}$$

let $e^{c} = A$

$$\Rightarrow T(t) = Ae^{-6\lambda t}$$

$\Rightarrow$ Solving for (2)

$$X''/X = -\lambda \Rightarrow X'' + \lambda X = 0$$

PAGE NO.:
DATE: / /

=> This is a 2^{nd} order ode, we solve by Characteristic approach

$$a^2 + \lambda = 0 \qquad a^2 = -\lambda \quad \Rightarrow \quad a = \pm\sqrt{-\lambda}$$

$$\Rightarrow \quad X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

=> By the hypothesis $\lambda \geq 0$, so that

$\lim_{t \to \infty} \phi(t) = 0$ and not $+\infty$, we have

$$X(x) = \bar{c}_1 \sin\sqrt{\lambda}x + \bar{c}_2 \cos\sqrt{\lambda}x$$

=> $U(x,t)$ is given as

$$U(x,t) = X(x)T(t) = A\bar{e}^{6\lambda t}\left[\bar{c}_1 \sin\sqrt{\lambda}x + \bar{c}_2 \cos\sqrt{\lambda}x\right]$$

$$\Rightarrow \quad B = A\bar{c}_1 \quad \text{and} \quad D = A\bar{c}_2$$

$$\Rightarrow U(x,t) = \bar{e}^{6\lambda t}\left[B \sin\sqrt{\lambda}x + D\cos\sqrt{\lambda}x\right] \quad \text{—— (3)}$$

=> We now apply the B.c,

$$\frac{du(0,t)}{dx} = -1 \quad \text{and} \quad \frac{du}{dx}(4,t) = 4$$

=> We differentiate equation (3) w.r.t x so we can apply those B.Cs

PAGE NO.
DATE / /

$$U_x(x,t) = e^{-6\lambda t}\left[\sqrt{\lambda}\, B \cos\sqrt{\lambda}\, x - \sqrt{\lambda}\, D \sin\sqrt{\lambda}\, x\right]$$

applying B.C, we get

$$U_x(0,t) = e^{-6\lambda t}\left[B\sqrt{\lambda}.1 + 0\right] = -1$$

$$\Rightarrow e^{-6\lambda t} . B\sqrt{\lambda} = -1$$

$$\Rightarrow B = e^{6\lambda t}/\sqrt{\lambda}$$

$$\Rightarrow U_x(4,t) = e^{-6\lambda t}\left[B\sqrt{\lambda} \cos\sqrt{\lambda}.4 - \sqrt{\lambda}.D \sin\sqrt{\lambda}.4\right] = 4$$

$\Rightarrow$ For $B = e^{6\lambda t}/\sqrt{\lambda}$

$$\Rightarrow e^{-6\lambda t}\left[\frac{e^{6\lambda t}}{\sqrt{\lambda}} . \sqrt{\lambda} \cos\sqrt{\lambda}.4 - \sqrt{\lambda}.D \sin\sqrt{\lambda}.4\right] = 4$$

$$\Rightarrow \cos\sqrt{\lambda}.4 - \sqrt{\lambda}.D \sin\sqrt{\lambda}.4 = 4$$

$$\Rightarrow \sqrt{\lambda}.D \sin\sqrt{\lambda}.4 = \cos\sqrt{\lambda}.4 - 4$$

$$D = \frac{\cos\sqrt{\lambda}.4 - 4}{\sqrt{\lambda} \sin\sqrt{\lambda}.4}$$

$$\Rightarrow \frac{1}{\sqrt{\lambda}}\left[\cot\sqrt{\lambda}.4 - \frac{4}{\sin\sqrt{\lambda}.4}\right]$$

$$U(x,t) = e^{-6\lambda t}\left[\frac{e^{6\lambda t}}{\sqrt{\lambda}} . \sqrt{\lambda} \sin\sqrt{\lambda}\, x + \left(\frac{\cos\sqrt{\lambda}.4 - 4}{\sqrt{\lambda} \sin\sqrt{\lambda}.4}\right) \cos\sqrt{\lambda}\, x\right]$$

PAGE NO.:
DATE: / /

$\Rightarrow$ By hypothesis $\lambda \geq 0$, then $\lim_{x \to \infty} \phi(x) = 0$

not $+\infty$

$\Rightarrow$ Now, we clearly note that there will be infinitely many solutions for $\sqrt{\lambda}.x$ and resulting $U(x,t)$, so we choose the one that leads to the easiest problem for $U(x,t)$

$$\Rightarrow \sin^{-1}(\sin\sqrt{\lambda}.x = \pi, 2\pi, 3\pi, ---$$

$$\Rightarrow 16\lambda = n^2\pi^2 \Rightarrow \lambda = n^2\pi^2/16$$

$$\Rightarrow \lambda_n = n^2\pi^2/16 \quad n = 1, 2, 3 ---$$

$$U_{\text{Non-Homogeneous B.C}}(x,t) = \sum_{n=1}^{\infty} B_n \sin\left[\left(n\pi/4\right)^2\right]^{\frac{1}{2}} x . e^{-\frac{n^2\pi^2}{16}.6t}$$

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{4} x . e^{-\frac{6}{16} n^2\pi^2 t}$$

Thus

$$\alpha + U_{\text{Non-homogeneous B.C}}(x,t) = U(x,t) \Rightarrow$$

$$\alpha = \left[\frac{U_x(4,t) - U_x(0,t)}{4}\right] x + U_x(0,t)$$

ϕ

$$U(x,t) = \left(\frac{4-(-1)}{4}\right)x + U_x(0,t) = \frac{5}{4}x - 1$$

=) Now we apply the I.C as below

$$U(x,0) = f(x) = 20 - 5(x-2)^2 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{4} + \frac{5}{4}x - 1$$

=) We expand $\frac{5}{4}x - 1$ in half-interval

with Fourier series as

$$\frac{5}{4}x - 1 = \frac{2}{4}\sum_{n=1}^{\infty}\left[\int_0^4 \left(\frac{5}{4}x - 1\right)s \sin\frac{n\pi s}{4}\,ds\right]\sin\frac{n\pi x}{4}$$

=) Now, plugging in $U(x,0) = f(x)$ and identifying coefficients to field B_n, we will solution given as

$$U(x,t) = \frac{2}{4}\sum_{n=1}^{\infty}\left[\int_0^4 \left(20 - 5(x-2)^2 - \left[\left(\frac{5}{4}x - 1\right)s\right]\sin\frac{n\pi s}{4}\,ds\right] \cdot Q$$

with $Q = \sin\frac{n\pi x}{4} \cdot e^{-\frac{6}{16}n^2\pi^2 t}$

Thus final solution becomes

$$U(x,t) = \frac{1}{2}\sum_{n=1}^{\infty}\left[\int_0^4 \left(21 - 5(x-2)^2 - \frac{5}{4}xs\right)\sin\frac{n\pi s}{4}\,ds\right]\sin\frac{n\pi x}{4}\, e^{-\frac{6}{16}n^2\pi^2 t}$$

PAGE NO.:
DATE: / /

Q3 Solve the non-homogeneous heat equation for $L = 6$, $k = 12$ with time-independent $q(x) = 3e^{-x}$, mixed homogeneous B.C

and I.C $f(x) = 2\sin(\pi x/6)$

Soln.

We first write the non-homogeneous heat equation with time independent, source term as

$$q(x) = 3e^{-x}$$

$$\Rightarrow \frac{\partial u}{\partial t} = 12\frac{\partial^2 u}{\partial x^2} + 3e^{-x} \quad \text{with Condition}$$

$$U(0,t) = T(t)$$

$$U(6,t) = X(t)$$

$$U(x,0) = f(x) = 2\sin(\pi x/6)$$

$\Rightarrow$ Note that the source term $q(x)$ is both time and space dependent

$\Rightarrow$ Thus, to solve this problem, we first need to solve for a given said function

$\Rightarrow$ let the function be given by $p(x,t)$ in such a way that the B.C are satisfied as below

$p(0,t) = T(t) \Rightarrow p(6,t) = X(t)$

$\Rightarrow$ Thus, our said function $p(x,t)$ will take a simpler form as

$$p(x,t) = T(t) + \frac{x}{6}\left(X(t) - T(t)\right)$$

$\Rightarrow$ Note that we have the function $U(x,t)$ and $p(x,t)$

$\Rightarrow$ Thus, we find its difference and set it to another function as

$$w(x,t) = U(x,t) - p(x,t)$$

$\Rightarrow$ Hence $w(x,t)$ will be the solution to the below given problem

$$\frac{\partial w(x,t)}{\partial t} = 12\frac{\partial^2 w(x,t)}{\partial x^2} + \left[3e^{-x} - \frac{\partial p(x,t)}{\partial t} + 12\frac{\partial^2 p(x,t)}{\partial x^2}\right]$$

with B.C as

$$w(0,t) = 0 \quad \text{and} \quad w(6,t) = 0$$

and I C as

$$w(x,0) = U(x,0) = f(x) - p(x,0) = 2\sin\frac{\pi x}{6} - T(0) - \frac{x}{6}\left(X(0) - T(0)\right)$$

$\Rightarrow$ We can approximate this to $f(x)$

$$\Rightarrow p(x,t) = -2\sin\frac{\pi x}{6} + T(t) + \frac{x}{6}\left(X(t) - T(t)\right) + f(x)$$

with approximated Feu Value

=) Now That we successfully reduced our non-homogenous heat equation, we will be using the method of eigen function expansion to find its solution

=) We proceed as below
=) First we write the homogeneous part of the above reduced problem as

$$\frac{\partial S(x,t)}{\partial t} = 12\frac{\partial^2 S(x,t)}{\partial x^2} \quad \text{———} (*)$$

and B.C

$S(0,t) = 0$ and $S(6,t) = 0$

=) We now apply the method of separation of Variables to obtain an ~~eigenfunction~~ eigenvalue function given by

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with Condition}$$

$\phi(0) = 0$
$\phi(6) = 0$

=) let's solve for the eigenvalue λ_n

=) We let $S(x,t) = X(x)T(t)$
=) $S_t(x,t) = T'X$ and $S_{xx}(x,t) = X''T$

=) substituting into $(*)$ we have

$T'X = 12TX''$, dividing all by $S(x,t)$

$\Rightarrow \frac{T'}{12T} = \frac{X''}{X} = -\lambda$

$\Rightarrow \frac{T'}{12T} = \frac{X''}{X} = -\lambda \Rightarrow \frac{T'}{12T} = -\lambda$ and $\frac{X''}{X} = -\lambda$

$\Rightarrow \frac{dT}{T} = -\lambda 12\, dt \Rightarrow \ln T = -12\lambda t + c$

$\Rightarrow T(t) = A e^{-12\lambda t}$

and $X'' + \lambda X = 0 \Rightarrow a^2 + \lambda = 0$

$\Rightarrow a^2 = -\lambda \Rightarrow a = \pm\sqrt{-\lambda}$

$\Rightarrow X(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$

$\Rightarrow$ But with $\lambda > 0$ hypothesis we re-write $X(x)$ as

$X(x) = \bar{c}_1 \sin\sqrt{\lambda}x + \bar{c}_2 \cos\sqrt{\lambda}x$

for the B.C, we have

$X(0) = 0 + \bar{c}_2 = 0 \Rightarrow \bar{c}_2 = 0$

$\Rightarrow X(x) = \bar{c}_1 \sin\sqrt{\lambda}x$

$X(6) = \bar{c}_1 \sin\sqrt{\lambda}.6 = 0$

$\Rightarrow$ For non-trivial solution

$c_1 \neq 0$

$\Rightarrow \sqrt{\lambda}.6 = (n\pi)^2$

$\Rightarrow \lambda.36 = n^2\pi^2$

$\Rightarrow \lambda_n = \frac{n^2\pi^2}{36}$ for $n = 1, 2, 3, ---$

=) The corresponding eigenfunction is given by

$$\phi_n(x) = \sin \frac{n\pi}{6} x$$

=) We now express the unknown solution $w(x,t)$ as a generalized Furrier series of eigenfunction with the time dependent coefficient given by

$$w(x,t) = \sum_{n=1}^{\infty} B_n(t)\, \phi_n(x)$$

=) since we already have $\phi_n(x)$ solution, we now need to calculate the value function for $B_n(t)$. constants coefficient

=) We assume that the term-by-term differentiation is satisfied ; thus substituting $w(x,t)$ value into

$$\frac{\partial w(x,t)}{\partial t} = 12 \frac{\partial^2 (x,t)}{\partial x^2} + \left[3e^{-t} - \frac{\partial \rho(x,t)}{\partial t} + 12 \frac{\partial^2 \rho(x,t)}{\partial x^2} \right] \text{——} (x)'$$

=) simplifying into

DATE: / /

$$\Rightarrow \sum_{n=1}^{\infty} \frac{d B_n(t)}{dt} \phi_n(x) = 12 \sum_{n=1}^{\infty} B_n(t) \frac{d^2 \phi_n(x)}{dx^2} + 3e^{-x}$$

$$= -12 \sum_{n=1}^{\infty} B_n(t) \left(\frac{n^2\pi^2}{36}\right) \phi_n(x)\, dx + 3e^{-x}$$

$\Rightarrow$ Next, we need to expand the term $q(x)$ using generalized Fourier series of eigenfunction

$\Rightarrow$ Thus we have

$$3e^{-x} = \sum_{n=1}^{\infty} q_n(t)\, \phi_n(x) \quad , \text{where}$$

$q_n(t)$ is given by

$$q_n(t) = \frac{\int_0^6 3e^{-x} \phi_n(x)\, dx}{\int_0^6 \phi_n^2(x)\, dx}$$

$\Rightarrow$ Subsituting the above into equation $(*)'$ and identifying the constant coefficient we obtain the differentiation eqn for coefficients given by

$$\frac{d B_n(t)}{dt} + \frac{n^2\pi^2}{36} k B_n(t) = \frac{\int_0^6 3e^{-x} \phi_n(x)\, dx}{\int_0^6 \phi_n^2(x)\, dx}$$

PAGE NO.:
DATE: / /

⇒ We now need to obtain the IC for the above d.e using

$$\omega(x,0) = U(x,0) - P(x,0)$$

⇒ Thus, we end up with

$$B_n(t) = B_n(0)\, e^{-\frac{n^2\pi^2}{36}kt} + e^{-\frac{n^2\pi^2}{36}kt}\int_0^t q_n(\tau)\, e^{\frac{n^2\pi^2}{36}k\tau}\, d\tau$$

⇒ The solution for our problem $\omega(x,t)$ is given by

$$\omega(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{6}$$

⇒ $U(x,t) = \omega(x,t) + P(x,t)$

⇒ Final General Solution is given by

$$U(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{6} + T(t) - 2\sin\frac{\pi x}{6} + \frac{x}{6}\left(X(t) - T(t)\right)$$

$\boxed{\times}$

THANK YOU!!!