3) $\left(x_n\right)_{n=1}^{\infty}$ $\qquad$ $x_n = \dfrac{5}{2n-1}$

prove.

Using definition of convergence we have

$\Rightarrow$ This sequence $\{x_n\}$ is said to converge to a real number x if for every $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that

$$\left| x_n - x \right| < \varepsilon \qquad \forall n > N(\varepsilon)$$

$\Rightarrow$ $\displaystyle \lim_{n \to \infty} x_n = ?$

$\Rightarrow$ $\displaystyle \lim_{n \to \infty} \frac{5}{2n-1}$ $\Rightarrow$ $\displaystyle \lim_{n \to \infty} \frac{5}{2n-1} = 0$

$$\left| \frac{5}{2n-1} - 0 \right| < \varepsilon \qquad \forall n > N(\varepsilon)$$

Now $\left| \dfrac{5}{2n-1} \right| < \varepsilon$ $\Rightarrow$ $2n - 1 > \dfrac{5}{\varepsilon}$

$\Rightarrow$ $n > \left(\dfrac{5}{\varepsilon} - 1 \right) \bigg/ 2$ $\Rightarrow$ $N(\varepsilon) > \dfrac{5}{2\varepsilon} - \dfrac{1}{2}$

Thus taking the limit

$$\lim_{n \to \infty} \frac{5}{2n-1} = \frac{5}{2(\infty)-1} = \frac{5}{\infty} = 0$$

So this sequence converges to zero.

$(x_n)_{n=1}^{\infty}$ $\qquad$ $x_n = \dfrac{1}{10} \sin \dfrac{n\pi}{2}$ $\qquad \forall n$

Using the definition of convergence we have

Taking the limit we have

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \dfrac{1}{10} \sin \dfrac{n\pi}{2} = \dfrac{1}{10} \lim\limits_{n\to\infty} \sin \dfrac{n\pi}{2}$

range of sine function is $[-1, 1]$

Thus $\dfrac{1}{10} \lim\limits_{n\to\infty} \sin \dfrac{n\pi}{2} = 0$

Using Convergence definition

$\forall \varepsilon > 0$, $\exists N > 0$ Such that $n > N(\varepsilon)$

$\left| \dfrac{1}{10} \sin \dfrac{n\pi}{2} - 0 \right| < \varepsilon$

$-1 - \left| \dfrac{1}{10} \sin \dfrac{n\pi}{2} \right| < \varepsilon$

We find the $N(\varepsilon)$ such that it minimises the above function, thus we have

$\left| \dfrac{1}{10} \sin \dfrac{n\pi}{2} \right| \leq \dfrac{1}{n^2} < \dfrac{1}{N^2} < \varepsilon \implies N^2 > \dfrac{1}{\varepsilon}$

Take $\implies N(\varepsilon) > \left(\dfrac{1}{\varepsilon} \right)^{\frac{1}{2}}$

$\Rightarrow$ Thus sequence does converge to zero.

(c) $\left(x_n\right)_{n=1}^{\omega}$ $\qquad$ $x_n = \begin{cases} 1 & n \text{ is a power of 2} \\ 0 & \text{otherwise} \end{cases}$

Soln

We use the theorem that a sequence of real numbers converges to x if and only if every subsequence of $\{x_n\}$ converges to x

$\Rightarrow$ Suppose $x_n \to x$, for every

$\qquad \varepsilon > 0$, $\exists\ N(\varepsilon) \in J^{+}$ such that

$$|x_n - x| < \varepsilon \qquad \forall n > N(\varepsilon)$$

Taking $n_k > N(\varepsilon)$, then $|x_{n_k} - x| < \varepsilon$

$$\forall n_k > N(\varepsilon)$$

$\Rightarrow \{x_n\} = \{x_1, x_2, x_3, \dots\}$

$\qquad \{0, 1, 0, 1, 0, 1 \dots\}$

the corresponding subsequence

$\{x_{n_k}\}$ for $k=2$ =

$$\{x_{2n}\}_{n=1}^{\infty} = \{x_2, x_4, x_6 \dots\}$$

$$\{1, 1, 1, \dots\}$$

This subsequence converge to 1

We consider another subsequen :

$$\left(x_{2n-1}\right)_{n=1}^{\omega} = \{x_1, x_3, x_5, \cdots \}$$

$$\{0, 0, 0, \cdots \}$$

This subsequence converge to 0.

From the theorem definition for convergence

we have, torner that

$$\lim_{n \to \omega} x_{2n} = 1 \qquad \text{and} \qquad \lim_{n \to \omega} x_{2n-1} = 0$$

$$|x_{2n-1}| < \varepsilon \qquad \text{and} \qquad |x_{2n} - 0| < \varepsilon$$

$\Rightarrow$ We can not find a $N(\varepsilon)$ such that

$$|x_n - x| < \varepsilon \qquad \text{for every} \qquad \varepsilon > 0$$

Thus, since $0 \neq 1$, the sequence diverge,

d) $\left(x_n\right)_{n=1}^{\infty}$ $\quad x_n = \dfrac{1}{\log_2 n} \quad n \geq 2$

Soln.

Using definition of Convergence

$\forall \varepsilon > 0, \quad \exists N > 0 \quad \text{s.t} \quad n > N(\varepsilon)$

$\left| x_n - x \right| < \varepsilon$

$\Rightarrow \left| \dfrac{1}{\log_2 n} - x^2 \right| < \varepsilon$

2) $\lim\limits_{n \to \infty} \dfrac{1}{\log_2 n} = \lim\limits_{n \to \infty} \dfrac{1}{\log_2 (\infty)}$

$\Rightarrow$ this value is approximately ∞

i.e $\log_2(\infty) = \infty \Rightarrow$

$\lim\limits_{n \to \infty} \dfrac{1}{\infty} = 0$

$\Rightarrow \left| \dfrac{1}{\log_2 n} - 0 \right| < \varepsilon$

We need to choose $N(\varepsilon)$ s.t $\forall \varepsilon > 0 \; \exists N(\varepsilon)$

$n > N(\varepsilon)$

$\Rightarrow \left| \dfrac{1}{\log_2 n} \right| < \varepsilon$

We now find the $N(\epsilon)$ such such that it minimises, the above function.

$$\left|\frac{1}{\log_2 n}\right| \leq \frac{1}{n} < \frac{1}{N} < \epsilon$$

$$\Rightarrow N > \frac{1}{\epsilon}$$

Thus, we that $N(\epsilon) = \frac{1}{\epsilon}$

$$\Rightarrow |x_n - \ell| < \epsilon \qquad \forall n > N(\epsilon) > \frac{1}{\epsilon}$$

We take $N(\epsilon) = \frac{1}{\epsilon}$

Thus the sequence converges to zero

Thanks