PART A Question (1a)

(1a)

Using $y'(t) - 10(y-t^2)+2t$ $\quad y(0) = -1$

at $t_0 = 0$, $y_0 = 0$

Soln

$$dy/_{dt} = -10(y-t^2)+2t$$

at $t_1 = 0.1$, $y_1 = 0.3778794$

$$y' = -10[0.3778794 - (0.1)^2] + 2(0.1) = -3.4787940$$

at $t_2 = 0.2$ $\quad y_2 = 0.1753352$

$$y' = -10[0.1753352 - (0.2)^2] + 2(0.2) = -0.9533540$$

at $t_3 = 0.3$ $\quad y_3 = 0.1397871$

$$y' = -10[0.1397871 - (0.3)^2] + 2(0.3) = 0.1021290$$

We need to calculate the values at y_4 at $t_4 = 0.5$ and y_5 at $t_5 = 0.5$

⇒ Using the generalized milne predictor and corector formulae to find y_4 and y_5

$$y_{j+1} = y_{j-1} + h\left(2 + \tfrac{1}{3}\nabla^2 + \tfrac{1}{3}\nabla^3 + \tfrac{29}{90}\nabla^4 + \tfrac{28}{90}\nabla^5 + \dots\right) f_{i_j}$$

→ For values starting $j=n$ and the ∇ value given by the difference in the interval of time t, this equation is simplified into

$$y^p_{n+1} = y_{n-3} + \tfrac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

and similarly for the corector method

$$y_{n+1}^{c} = y_{n-1} + h/3 \left(y'_{n-1} + 4y_n' + y'_{n+1} \right)$$

So at $n=3 \Rightarrow j=3$

$$y_4^{p} = y_0 + \tfrac{4}{3} h \left(2y_1' - y_2' + 2y_2' \right)$$

$$= 1 + \tfrac{4}{3}(0.1)\left[2(-3.4787940) - (-0.9533540) + 2(0.1021290)\right]$$

$$y_4^{p}(0.4) = 0.2266699$$

$$y_4' = -10\left[0.2266699 - (0.4)^2\right] + 2(0.4) = 0.1333013$$

as for corector method

$$y_4^{c} = y_2 + h/3 \left(y_2' + 4y_3' + y_4' \right)$$

$$= 0.1753353 + 0.1/3 \left[-0.9533540 + 4(0.1021290) + 0.1333013\right.$$

$$= 0.2251743$$

$$y_4' = -10\left[0.2251743 - (0.4)^2\right] + 2(0.4) = 0.1482566$$

at $t = 0.5$ $y(0.5) = ?$

$$y_5^{p} = \tfrac{4}{3} h \left(2y_2' - y_3' + 2y_4' \right) + y_1$$

$$= 0.3778794 + 4/3\,(0.1)\left[2(-0.9533540) - (0.1021290) + 2(0.1333013)\right]$$

$$= 0.1455815$$

$$y_5' = -10\left[0.1455815 - (0.5)^2\right] + 2(0.5) = 2.0442850$$

as for the corector method

$= 0.1397871 + \frac{0.1}{3}\left[0.1021290 + 4(0.1333013) + 2.044185\right] = 0.2291044$

$y_5' = -10\left[0.2291044 - (0.5)^2\right] + 2(0.5) = 1.2089559.$

Table.

j	t	y	y'	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$
0	0.0	1.0000000	-10.000000					
				6.5212060				
					-3.9957660			
1	0.1	0.3778794	-3.4787940			2.5258090		
				2.5254400			-2.080144	
					-1.4699570			~~3.7032911~~
2	0.2	0.1753353	-0.9533540			0.4456463		
				1.0554830			1.6231467	
					-1.0243107			
3	0.3	0.1397871	0.1021290			2.0687930		
				0.0311723				
					1.0444823			
4	0.4	0.2266699	0.1333013					
				1.0756546				
5	0.5	0.2291044	1.2089559					

Question 2(a)

(2a) For the Nystrom-Milne predictor Method

$y_{j+1} = y_{j-1} + h\left(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 + \dots\right)f_j$

for $j = n$

$\Rightarrow y_{n+1} = y_{n-1} + h\left(2 + \frac{1}{3}y_{n-1}' + \frac{1}{3}y_{n-2}' + \frac{29}{90}y_{n-3}' + \dots\right)f_n$

$\Rightarrow E\,y_{n-2}' = y_{n+1}'$

$E y'_{n-2} = y'_{n-1} \Rightarrow E^2 y'_{n-2} = y'_n$

$\sigma(E) = \frac{28}{90} E^4 + \frac{29}{90} E^3 + \frac{1}{3} E^2 + \frac{1}{3} E + 2$

$\rho(E) = E^3 - E$

$(E^3 - E) y_{n-2} - h\left(\frac{28}{90}E^4 + \frac{29}{90}E^3 + \frac{1}{3}E^2 + \frac{1}{3}E + 2\right) y'_{n-2} = 0$

$\rho(\xi) = \xi^3 - \xi = \xi(\xi^2 - 1) = 0$ — 1st characteristic

$\sigma(\xi) = \frac{28}{90}\xi^4 + \frac{29}{90}\xi^3 + \frac{1}{3}\xi^2 + \frac{1}{3}\xi + 2$ — 2nd characteristic

Root condition

at $\rho(\xi) = 0$

$\Rightarrow \rho(\xi) = \xi^3 - \xi = \xi(\xi^2 - 1)$

$\Rightarrow \xi_1 = 0, \quad \xi_2 = 1, \quad \xi_3 = -1$

These are the simple roots corresponding to the milne-predictor method

Stability

$\sigma(\xi) = \frac{28}{90}\xi^4 + \frac{29}{90}\xi^3 + \frac{1}{3}\xi^2 + \frac{1}{3}\xi + 2 = 0$

$\Rightarrow |\xi_4| > 0$

Hence zero stable since this will yield simple roots

Convergence and consistency

We suppose the method is zero stable, then assume that $\rho(\xi)$ has double roots in the unit circle

Further product of roots of $\rho(\xi)$ we set as 1, but now ξ_1 and ξ_2 are equal to ± 1, thus not strictly complex hence they converge and are consistent

$$\xi_1 = 0 \quad , \quad \xi_2 = +1 , \quad \xi_3 = -1$$

$\Rightarrow$ For the corrector method as $j = n$

$$y_{n+1} = y_{n-1} + h\left(2 - 2y_n' + \tfrac{1}{3}y_{n-1}' - \tfrac{1}{90}y_{n-2}' - \tfrac{1}{90}y_{n-3}' + \dots\right)f$$

$$\Rightarrow E\, y_{n-2}' = y_{n-1}'$$

$$E\, y_{n-2}' = y_n' - 1 \Rightarrow E^2 y_{n-2}' = y_n'$$

$$\Rightarrow \sigma(E) = -\tfrac{1}{90}E^3 - \tfrac{1}{90}E^2 + \tfrac{1}{3}E + 2 - 2 =$$

$$\Rightarrow -\tfrac{1}{90}E^3 - \tfrac{1}{90}E^2 + \tfrac{1}{3}E$$

$$\Rightarrow \rho(E) = E^3 - E^2$$

$$\Rightarrow (E^3 - E^2)\, y_{n-2} - h\left(-\tfrac{1}{90}E^3 - \tfrac{1}{90}E^3 + \tfrac{1}{3}E\right) y_{n-2}' = 0$$

$$\rho(\xi) = (\xi^3 - \xi^2) = \xi^2(\xi - 1) \quad \text{— 1st characteristic}$$

$$\sigma(\xi) = -\tfrac{1}{90}\xi^3 - \tfrac{1}{90}\xi^2 + \tfrac{1}{3}\xi \quad \text{— 2nd characteristic}$$

Root condition at $\rho(\xi) = 0$

$$\Rightarrow \rho(\xi) = \xi^2(\xi - 1) = 0 \Rightarrow \xi_1 = 1 \quad , \xi_{2,3} = 0$$

These are simple roots of the corresponding corrector method

Stability

$\sigma(E) = \sigma(\xi) = -\frac{1}{90}\xi^3 - \frac{1}{90}\xi^2 + \frac{1}{3}\xi = 0$ at $\rho(\xi) = 0$

$= |\xi_3| > 1$

Hence zero stable since it yields complex roots

Convergence and Consistency

$\rho(\xi) = \xi^2(\xi - 1) = 0$

We suppose the method is zero stable, then assume that $\rho(\xi)$ has double roots in the unit circle

Further, product of roots of $\rho(\xi)$ is 1 and since the roots are not equal and neither equal to ± 1, then they are strictly complex

$\Rightarrow \xi = \xi_{1,2,3}$

Question (2b)

$\Rightarrow$ From the Nystrom - predictor milnes method

$$y_{j+1} = y_{j-1} + h\left(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 + \dots\right) f_j$$

Setting $j = n$, a simplified Nystrom - milnes predictor method is given as follows

$$y_{n+1} = y_{n-3} + \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$\Rightarrow y_{n+1} - y_{n-3} - \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

let. $Ey'_{n-2} = y'_{n-1}$, $Ey'_{n-1} = y'_n$.

$E^2 y'_{n-2} = y'_n$

$\Rightarrow \sigma(E) = \frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3}$, $\sigma(E) = E^3 - 0$

$\Rightarrow (E^3 - 0) y_{n-2} - h\left(\frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3}\right) y'_{n-2} = 0$

(i) $\rho(\xi) = \xi^3 = \xi(\xi^2)$ — 1st characteristic

(ii) $\sigma(\xi) = \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{8}{3}$ — 2nd characteristic

root condition

$\Rightarrow$ The explicitly method of the Nystrom _ milnes method is said to satisfy the root condition if the roots of the equation $\rho(\xi) = 0$ lies inside the closed unit disk in the complex plane and are simple if they lie on the circle

$\rho(\xi) = 0$

$\Rightarrow$ From our equation

$\rho(\xi) = \xi^3 = 0 \Rightarrow \xi_{1,2,3} = 0$

Stability

$\Rightarrow$ The linear predictor method is zero stable if given $y' = f(x,y)$, $y(x_0) = y_0$, $f(x,y)$ satisfy lipschitz condition and root condition holds

$\Rightarrow$ Thus we have in our case function

=

$\sigma(\xi) = \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{8}{3}$ which is zero-stable since it yields simple roots

$\Rightarrow \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{8}{3} = 0 \quad |\xi_{1,2}| > 0$

Convergence & Consistency

The linear predictor above is Consistent if it has order $p \geq 1$

$$\rho(\xi) = \xi^2(\xi) = 0$$

Further product of roots of $\rho(\xi)$ is 1, and since the roots are neither equal to ± 1, hence said to be strictly complex

$$\Rightarrow \xi(\xi^2) = 0 \Rightarrow \xi_1 = 0, \xi_2 = 0, \xi_3 = 0$$

Question 2C

Derivation of 4^{th} order Adam Bashforth

Soln

We consider the general function given by

$$y_{n+1} = y_n + \frac{h}{24}\left(55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right)$$

let $Ey_{n-2}' = y_{n-1}'$, $Ey_{n-2}' = y_{n-1}'$

$E^2 y_{n-2}' = y_n'$

$\Rightarrow$ substituting these into our function above

$$\sigma(E) - \frac{9}{24}E^3 + \frac{37}{24}E^2 - \frac{59}{24}E + \frac{55}{24}$$

$$\rho(E) = E^3 - 1$$

$$(E^3 - 1)y_{n-2} - h\left(-\frac{9}{24}E^3 + \frac{37}{24}E^2 - \frac{59}{24}E + \frac{55}{24}\right)y_{n-2}' = 0$$

Setting the root condition $\rho(\xi) = 0$ characteristic polynomial

$$\sigma(E) = -\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24}$$ — 1st characteristic

$\rho(\xi) = (\xi^3 - 1)$ ___ 2nd characteristic

=) For the root condition (h.l) we have

$\rho(\xi) = 0$, $(\xi^3 - 1) = 0$ =) $\xi^3 = 1$

$= -\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24} = 0$

$\Rightarrow \xi\left(-\frac{9}{24}\xi^2 + \frac{37}{24}\xi - \frac{59}{24}\right) = -\frac{55}{24}$

Interval is from -1 to 1, the above values lies within this margin.

Question 2d.

=) Finding region of absolute stability for 4th order Adam Bashforth

=) Pr Sin

=) Thus the local truncation error for m-step Adam-Bashforth is $O(h^{m+2})$ i.e. it is (m+1) - order in accuracy

Consistency $O = \sum_{n=1}^{\infty} a_n = \rho(1) = 1$

has to be a root $\rho(\xi)$

$\Rightarrow \rho(\xi) = \sigma(\xi)\ln(\xi) = c(\xi-1)^{p+1} + O(|\xi-1|^{p+2})$

$\Rightarrow \sigma(\xi) = \frac{\rho(\xi)}{\ln(\xi)} + O\left(\frac{|\xi-1|^{p+1}}{\ln \xi}\right)$

choose $s = 2$ $\rho(\xi) = 7(\xi-1)\xi = \xi^2 - \xi$

$\frac{\rho(\xi)}{\ln(\xi)} = 1 + \frac{3}{2}\xi + \frac{6}{12}\xi + O(\xi^3)$

$$\sigma(\xi) = \xi^2 - \xi \Rightarrow \sigma(\xi) = 1 + \tfrac{3}{2}(\xi - 1) + \tfrac{6}{12}(\xi-1)^2 + O(\xi-1)^3$$

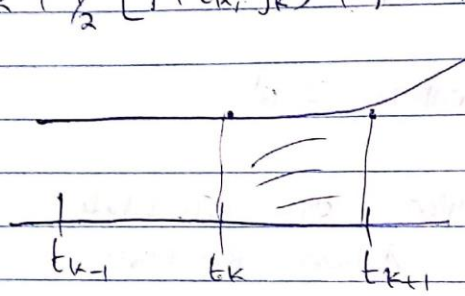
Regim of Stability

$$\Rightarrow \xi^2 - \xi = \xi(\xi - 1) = 0$$

$$\Rightarrow \xi = 0 \quad \text{and} \quad \xi = 1$$

From the 1-step Adam-Bashforth method we have for $m = 1$

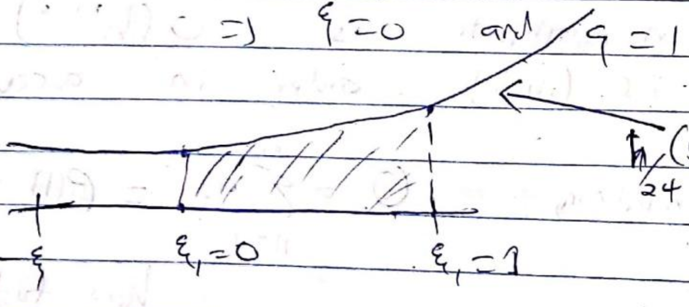
$$y_{k+1} = y_k + \tfrac{h}{2}\left[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right]$$



general representation of Stability area

For $\xi(\xi - 1) = 0 \Rightarrow \xi^2 - \xi$

$$\Rightarrow \xi = 0 \quad \text{and} \quad \xi = 1$$



$\tfrac{h}{24}(55y_n' - 59y_{n-1}' + 37y_{n-2}')$

Question 2e.

For the Nystrom - Milne predictor method Using the 1st characteristic we have

$$\rho(\xi) = \xi^3 - \xi = \xi(\xi^2 - 1) = 0$$

$$\sigma(\xi) = \tfrac{28}{90}\xi^4 + \tfrac{29}{90}\xi^3 + \tfrac{1}{3}\xi^2 + \tfrac{1}{3}\xi + 2$$

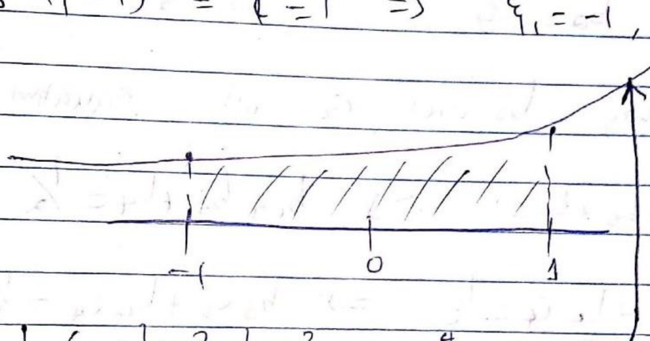
=> By the general boundary locals method

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} q_1(s)\,ds = h\left(2+\tfrac{1}{3}\nabla^2+\tfrac{1}{3}\nabla^3+\tfrac{29}{90}\nabla^4+\tfrac{28}{90}\nabla^5+\ldots\right)f_{i,}$$

=> For the 1 - step Nystrom - milne we have for $m=1$

$$0 = \sum_{n=1} a_n \Rightarrow \rho(\xi) = (\xi^3-1) \Rightarrow \rho(1) = 0$$

$$\Rightarrow (\xi^3-1) = \xi^3 = 1 \Rightarrow \xi_1 = -1,\ \xi_{12} = -1,\ \xi_{13} = 1$$



$$h\left(2+\tfrac{1}{3}\nabla^2+\tfrac{1}{3}\nabla^2+\tfrac{29}{90}\nabla^4+\tfrac{29}{90}\nabla^5+\ldots\right)f_{i,}$$

Question 3 a

Derivation of 4^{th} order explicity Runge - kutta method, $b_2 = \frac{1}{4}$, $c_2 = \frac{1}{2}$

soln

let the condition eqns be defined by

$$b_1 + b_2 + b_3 + b_4 = 1 \quad \text{--- (1)}$$

$$b_2 c_2 + b_3 c_3 + b_4 c_4 = \tfrac{1}{2} \quad \text{--- (2)}$$

$$b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = \tfrac{1}{3} \quad \text{--- (3)}$$

$$b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 = \tfrac{1}{4} \quad \text{--- (4)}$$

$b_3 c_3 a_{32} c_2 + b_4 c_4 (a_{42} c_2 + a_{43} c_3) = 1/8$ — (5)

$b_3 a_{32} + b_4 a_{42} = b_2 (1 - c_2)$ — (6)

$b_4 a_{43} = b_3 (1 - c_3)$ — (7)

$0 = b_4 (1 - c_4)$ — (8)

$c_2 = a_{21}$, $c_3 = a_{31} + a_{32}$, $c_4 = a_{41} + a_{42} + a_{43}$

So for $b_2 = 1/4$, $c_2 = 1/2$ we have

=> Substituting b_2 and c_2 into equation 1 — 3

$b_1 + 1/2 + b_3 + b_4 = 1 \Rightarrow b_1 + b_3 + b_4 = 1/2$ — (9)

$1/4 + b_3 c_3 + b_4 c_4 = 1/2 \Rightarrow b_3 c_3 + b_4 c_4 = 1/4$ — (10)

$1/2 \cdot 1/4 + b_3 c_3^2 + b_4 c_4^2 = 1/3 \Rightarrow b_3 c_3^2 + b_4 c_4^2 = 5/24$ — (11)

$1/2 \cdot 1/8 + b_3 c_3^3 + b_4 c_4^3 = 1/4 \Rightarrow b_3 c_3^3 + b_4 c_4^3 = 3/16$ — (12)

$b_1 = 1/2 - b_3 - b_4$, $b_2 = 1/2$, $b_3 = 1/2 - b_1 - b_4$

$b_4 = 1/2 - b_1 - b_3$

$c_2 = 1/2$, $c_3 = (1/4 - b_4 c_4)/b_3$, $c_4 = (1/4 - b_3 c_3)/b_4$

$a_{21} = 1/2$, $a_{31} = (1/4 - b_4 c_4)/b_3 - a_{32}$, $a_{32} = (1/4 - b_4 c_4)/b_3 - a_{31}$

$a_{41} = (1/4 - b_3 c_3)/b_4 - a_{42} - a_{43}$, $a_{42} = (1/4 - b_3 c_3)/b_4 - a_{41} - a_{43}$

$a_{43} = (1/4 - b_3 c_3)/b_4 - a_{41} - a_{42}$

=) The Butcher tableau

$$X = \left(\tfrac{1}{4} - b_3 c_3\right)/b_4 - a_{41} - a_{42}$$

$$\begin{array}{c|cccc}
0 & 0 & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\left(\frac{1}{4}-b_4c_4\right)/b_3 & \left(\frac{1}{4}-b_4c_4\right)/b_3 - a_{32} & \left(\frac{1}{4}-b_4c_4\right)/b_3 - a_{31} & & \\
\left(\frac{1}{4}-b_3c_3\right)/b_4 & \left(\frac{1}{4}-b_3c_3\right)/b_4 - a_{42} - a_{43} & \left(\frac{1}{4}-b_3c_3\right)/b_4 - a_{41} - a_{43} & X & \\
\hline
 & \frac{1}{2} - b_3 - b_4 & \frac{1}{2} & \frac{1}{2} - b_1 - b_4 & \frac{1}{2} - b_1 - b_3
\end{array}$$

Question 3 d

=) Equation derivations

$$dS/dt = BIS/N$$

$$dI/dt = BIS/N - \gamma I - \mu I$$

$$dR/dt = \gamma I$$

$$dD/dt = \mu I$$

Given data

$N = 5004000$

$B = 0.4$

$\gamma = 0.10105 \approx 0.1$

$\mu = 0.2$

$\boxed{\text{See Code}}$

PART B

Question 4a

$$\Rightarrow \begin{array}{c|cc} 3/4 & 3/4 & 0 \\ 1 & 1/4 & 3/4 \\ \hline & 1/2 & 1/2 \end{array} \quad \Rightarrow \quad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

$\Rightarrow c_1 = 3/4$, $c_2 = 1$, $a_{11} = 3/4$, $a_{12} = 0$
$a_{21} = 1/4$; $a_{22} = 3/4$, $b_1 = 1/2$, $b_2 = 1/2$

For an A-stable function above we have that if

$$\bar{C} = \{ z \in \mathbb{R} , \mathbb{R}e(z) \leq 0 \}$$

So this condition is met iff

$$\mathbb{R}(iy) \leq 1$$

$\mathbb{R}(z)$ is analytic for $\mathbb{R}e(z) < 0$

Thus the stability function for the above Butcher tableau is given by

$$\mathbb{R}(z) = 1 + z + \frac{1}{2} z^2 + \frac{3}{16} z^3 + \frac{1}{48} z^4$$

Note $\Rightarrow$ All non-zero entries are below the diagonal elements

Let $A = \begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}$ $\quad 1 b^T = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

Therefore $R(z) = N(z) / D(z)$ for polynomial

$$E(y) = D(iy) D(-iy) - N(iy) N(-iy)$$

=) For simplicity, we follow the below conditions i.e if
(i) All roots $D(z)$ are in the R.H.S
(ii) $E(y) \geq 0 \quad \forall y \in R$
then this method is A- stable

$$R(z) = \frac{\det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}\right]}{\det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z\begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}\right]}$$

$$R(z) = \frac{\det\begin{bmatrix} 1/4 + 1/2 z & 0 \\ -1/4 & 1/4 + 1/2 z \end{bmatrix}}{\det\begin{bmatrix} 1 - 3/4 z & 0 \\ -1/4 z & 1 - 3/4 z \end{bmatrix}} = \frac{\left(\frac{1}{4} + \frac{1}{2}z\right)^2 + \frac{1}{4}}{\left(1 - \frac{3}{4}z\right)^2 + \frac{1}{4}z}$$

$$=) \quad N(z) = 1/4 + \left(1/4 + 1/2 z\right)^2 = 1/4 z^2 + 1/4 z + 5/16$$

$$D(z) = 1/4 z + \left(1 - 3/4 z\right)^2 = 9/16 z^2 - 11/4 z + 1$$

let $a = 9/16 \quad b = -11/4 \quad c = 1$

$$z = \frac{11/4 \pm \sqrt{121/16 - 36/16}}{9/8} \qquad z_1 = 4.493, \quad z_2 = 0.3956$$

=) Thus the roots of $D(z)$ are all in the R.H.S so the 1^{st} condition is met

$$=) \quad E(y) = D(iy)D(-iy) - N(iy)N(-iy)$$

$$=) \left[1 - 11/4 iy - 9/16 y^2\right]\left(5/6 + 1/4 iy - 1/4 y^2\right)$$

$$=) -\left(5/6 + 1/4 iy - 1/4 y^2\right)\left(1 - 11/4 iy - 9/16 y^2\right) \geq 0$$

$\Rightarrow$ Thus since $E(y) \geq 0$ and roots $N(z)$ are in R.H.S we can conclude that the method is A-stable

Question 4b

$\Rightarrow$ Using cholesky decomposition method

let $A = LL^T \Rightarrow LL^T x = b$. $Lz = b \Rightarrow L^T x = z$

$$4x_1 - 2x_2 + 4x_3 + 2x_4 = 30$$
$$-2x_1 + 5x_2 - 2x_3 + 3x_4 = -27$$
$$4x_1 - 2x_2 + 13x_3 - x_4 = 93$$
$$2x_1 + 3x_2 - x_3 + 15x_4 = -15$$

Soln

In matrix form

$$A = \begin{pmatrix} 4 & -2 & 4 & 2 \\ -2 & 5 & -2 & 3 \\ 4 & -2 & 13 & -1 \\ 2 & 3 & -1 & 15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -15 \end{pmatrix}$$

$$\overset{L}{\begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}} \overset{L^T}{\begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & 0 & L_{44} \end{pmatrix}} = \begin{pmatrix} A \end{pmatrix}$$

$$= \begin{pmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} & L_{11}L_{41} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} & L_{21}L_{41} + L_{22}L_{42} \\ L_{11}L_{31} & L_{21}L_{31} + L_{32}L_{22} & L_{31}^2 + L_{32}^2 + L_{33}^2 & L_{31}L_{41} + L_{32}L_{42} + L_{33}L_{43} \\ L_{11}L_{41} & L_{41}L_{21} + L_{42}L_{22} & L_{41}L_{31} + L_{42}L_{32} + L_{43}L_{33} & L_{41}^2 + L_{42}^2 + L_{43}^2 + L_{44}^2 \end{pmatrix}$$

$(*)$

=> Equating (*) to A and solving for L and L^T we have

$L_{11}^2 = 4 \Rightarrow L_{11} = 2$, $2L_{41} = 2 \Rightarrow L_{41} = 1$

$2L_{21} = -2 \Rightarrow L_{21} = -1$, $1 + L_{22}^2 = 5 \Rightarrow L_{22} = 2$

$2L_{31} = 4 \Rightarrow L_{31} = 2$

$-1.1 + 2L_{42} = 3 \Rightarrow L_{42} = 2$, $1 + 4 + 1 + L_{44}^2 = 15$

$\Rightarrow L_{44} = 3$

$4 + 0 + L_{33}^2 = 13 \Rightarrow L_{33} = 3$

$2 + 3L_{43} = -1 \Rightarrow L_{43} = 1$

$Lz = b$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$2z_1 = 30 \Rightarrow z_1 = 15$

$-z_1 + 2z_2 = -27$

$z_2 = -6$

$2z_1 + 3z_3 = 93 \Rightarrow z_3 = 21$

$\Rightarrow z_1 + 2z_2 - z_3 + 3z_4 = -18 \Rightarrow z_4 = 0$

$$z = (15, -6, 21, 0)$$

$L^T x = z$

$$\begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 15 \\ -6 \\ 21 \\ 0 \end{pmatrix}$$

$3x_4 = 0$

$\Rightarrow x_4 = 0$

$3x_3 = 21 \Rightarrow x_3 = 7$

$2x_2 + 2x_4 = -6 \Rightarrow x_2 = -3$

$2x_1 - x_2 + 2x_3 + x_4 = 15 \Rightarrow x_1 = -1$

$$x = (-1, -3, 7, 0)$$

Question 4C

(i) Explanation

=> Also known as triangularization or factorization is used to compute the inverse A^{-1} by computing 1st the upper and lower triangular matrices given by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{23} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

=> We then get the product of LU: so as to equate to matrix A and calculating the coefficients for L and U which we then obtain inverse A^{-1}

=> Having determined L and U of the system $Ax = b$

$\Rightarrow LUx = b$

=> Simplified into $Lz = b$ whereby

$$z = L^{-1}b \quad \text{or} \quad z = Ux$$

=> The z-values are obtained by inverse L^{-1} of matrix A^{-1} multiplied by the b-values

$$\Rightarrow L^{-1}U^{-1} = A^{-1}$$

(ii)

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

let $L_{ii} = 1$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{pmatrix} \times \overset{U}{\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}}$$

$\Rightarrow u_{11} = 4$, $u_{12} = -2$, $u_{13} = 4$, $u_{14} = 2$

$4L_{21} = -2 \Rightarrow L_{21} = -\frac{1}{2}$, $u_{24} = 4$, $L_{31} = 1$, $L_{32} = 0$

$\Rightarrow u_{33} = 9$, $L_{41} = \frac{1}{2}$; $u_{44} = 9$

$Lz = b$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$\Rightarrow z_1 = 30$

$\Rightarrow z_2 = -12$

$\Rightarrow z_3 = 63$

$z_4 = 0$

$z = [30, -12, 63, 0]$

$Ux = z$

$$\begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$\Rightarrow x_4 = 0$, $x_3 = 7$, $x_2 = -3$, $x_1 = -1$

$x = [-1, -3, 7, 0]$

$U_{ii} = 1$ and $L_{ii} = 0$

=) A better efficient method to solve the above system would be to use the Jacobi Iterative method or the Gauss-Siedel approximation Iterative method

THANK YOU!!!