1.We define the following matrices:

$$A = \begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ -1 & 5 \\ 4 & 6 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 5 & 2 \\ -1 & 4 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 4 & 3 & 2 \\ -1 & 5 & 0 \\ 2 & 4 & 0 \end{pmatrix}.$$

1. Calculate the following additions on these matrices:

a) $A + A$

Solution

$$\Rightarrow \begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} (2+2) & (4+4) \\ (-2-2) & (-5-5) \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ -4 & -10 \end{pmatrix}$$

b) $3D$ (this is equivalent to $D + D + D$).

Solution

$$\begin{pmatrix} (4+4+4) & (3+3+3) & (2+2+2) \\ (-1-1-1) & (5+5+5) & (0+0+0) \\ (2+2+2) & (4+4+4) & (0+0+0) \end{pmatrix} = \begin{pmatrix} 12 & 9 & 6 \\ -3 & 15 & 0 \\ 6 & 12 & 0 \end{pmatrix}$$

2. Calculate the following multiplications on these matrices.

a) AA

Solution

$$\Rightarrow AA \Rightarrow \begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} (2*2)+(4*2) & (2*4)+(4*-5) \\ (-2*2)+(-5*-2) & (-2*4)+(-5*5) \end{pmatrix} = \begin{pmatrix} -4 & -12 \\ 6 & 17 \end{pmatrix}$$

b) AC

c) BA

d) BC

e) CD

f) DB

g) D^2 (this is equivalent to DD)

Solution

$$\Rightarrow AC \Rightarrow \begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} 1 & 5 & 2 \\ -1 & 4 & 0 \end{pmatrix} = \begin{pmatrix} (2*1)+(4*-1) & (2*5)+(4*4) & (2*2)+(0*0) \\ (-2*1)+(-5*-1) & (-2*2)+(-5*4) & (-2*2)+(-5*0) \end{pmatrix}$$

$$\Rightarrow AC \Rightarrow \begin{pmatrix} -2 & 26 & 4 \\ 3 & -24 & -4 \end{pmatrix}$$

$$\Rightarrow BA \Rightarrow \begin{pmatrix} 2 & 1 \\ -1 & 5 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} (2*2)+(1*-2) & (2*4)+(1*-5) \\ (-1*2)+(5*-2) & (-1*4)+(5*-5) \\ (4*2)+(6*-2) & (4*4)+(6*-5) \end{pmatrix}$$

$$\Rightarrow BA \Rightarrow \begin{pmatrix} 2 & 3 \\ -12 & -29 \\ -4 & -14 \end{pmatrix}$$

$$\Rightarrow BC \Rightarrow \begin{pmatrix} 2 & 1 \\ -1 & 5 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 5 & 2 \\ -1 & 4 & 0 \end{pmatrix} = \begin{pmatrix} (2*1)+(1*-1) & (2*5)+(1*4) & (2*2)+(0*0) \\ (-1*1)+(5*-1) & (1*5)+(5*4) & (-1*2)+(5*0) \\ (4*1)+(6*-1) & (4*5)+(6*4) & (4*2)+(6*0) \end{pmatrix}$$

$$\Rightarrow BC \Rightarrow \begin{pmatrix} 1 & 14 & 4 \\ -6 & 15 & -2 \\ -2 & 44 & 8 \end{pmatrix}$$

$$\Rightarrow CD \Rightarrow \begin{pmatrix} 1 & 5 & 2 \\ -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ -1 & 5 & 0 \\ 2 & 4 & 0 \end{pmatrix} = \begin{pmatrix} (1*4)+(5*-1)+(2*2) & (1*3)+(5*5)+(2*4) & (1*2)+(5*0+(2*0)) \\ (-1*4)+(4*-1)+(0*2) & (-1*3)+(4*5)+(0*4) & (-1*2)+(4*0)+(0*0) \end{pmatrix}$$

$$\Rightarrow CD \Rightarrow \begin{pmatrix} 3 & 36 & 2 \\ -8 & 17-2 & \end{pmatrix}$$

$$\Rightarrow DB \Rightarrow \begin{pmatrix} 4 & 3 & 2 \\ -1 & 5 & 0 \\ 2 & 4 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 5 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} (4*2)+(3*-1)+(2*4) & (4*1)+(3*5)+(2*6) \\ (-1*2)+(5*-1)+(0*4) & (-1*1)+(5*5)+(0*6) \\ (2*4)+(4*-1)+(0*4) & (2*1)+(4*5)+(0*0) \end{pmatrix}$$

$$\Rightarrow DB \Rightarrow \begin{pmatrix} 13 & 31 \\ -7 & 24 \\ 0 & 22 \end{pmatrix}$$

$$\Rightarrow D^2 = DD \Rightarrow \begin{pmatrix} 4 & 3 & 2 \\ -1 & 5 & 0 \\ 2 & 4 & 0 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ -1 & 5 & 0 \\ 2 & 4 & 0 \end{pmatrix} = \begin{pmatrix} (4*4)+(3*-1)+(2*2) & (4*3)+(3*5)+(2*4) & (4*2)+(3*0)+(2*0) \\ (-1*4)+(5*-1)+(0*2) & (-1*3)+(5*5)+(0*4) & (-1*2)+(5*0)+(0*0) \\ (2*4)+(4*-1)+(0*2) & (2*3)+(4*5)+(0*4) & (2*2)+(4*0)+(0*0) \end{pmatrix}$$

$$\Rightarrow D^2 = DD \Rightarrow \begin{pmatrix} 17 & 35 & 8 \\ -9 & 22 & -2 \\ 4 & 26 & 4 \end{pmatrix}$$

3. There is one last possible matrix multiplication involving two of these matrices. What is it? What is the result of the multiplication?

$$\Rightarrow CB \Rightarrow \begin{pmatrix} 1 & 5 & 2 \\ -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 5 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} (1*2)+(5*-1)+(2*4) & (1*1)+(5*5)+(2*6) \\ (-1*2)+(4*-1)+(0*4) & (-1*1)+(4*5)+(0*6) \end{pmatrix} = \begin{pmatrix} 5 & 38 \\ -6 & 19 \end{pmatrix}$$

Calculate $\det(A)$, and using it, calculate A^{-1}.

$$\Rightarrow det(A) \Rightarrow \begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix} = (2*-5)+(-2*4) = -2$$

$$\Rightarrow A^{-1} \Rightarrow -\frac{1}{2}\begin{pmatrix} -5 & -4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2.5 & 2 \\ -1 & -1 \end{pmatrix}$$

Using the result of the previous question, solve the following system of linear equations:

$$\begin{aligned} 2x_1 + 4x_2 &= 22 \\ -2x_1 - 5x_2 &= -26 \end{aligned}.$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2.5 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 22 \\ -26 \end{pmatrix}$$

$$\Rightarrow x_1 = (2.5*22)+(2*-26) = 3$$

$$\Rightarrow x_2 = (-1*22)+(-1*-26) = 4$$

We define the following matrices:

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 4 & 2 \\ 0 & -1 \end{pmatrix}$$

1. Find the eigenvalues for A,C and D.

$$\Rightarrow \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} = \begin{vmatrix} 4-\Lambda & 0 \\ 0 & 4-\Lambda \end{vmatrix} = 0$$

$$\Rightarrow 16 - 8\Lambda + \Lambda^2 = 0 \Rightarrow \Lambda_1 = 0, \Lambda_2 = 4$$

$$\Rightarrow C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} = \begin{vmatrix} 1-\Lambda & 0 \\ 0 & -1-\Lambda \end{vmatrix} = 0$$

$$\Rightarrow C \Rightarrow -1 - \Lambda + \Lambda + \Lambda^2 = 0, \Lambda_1 = 1, \Lambda_2 = -1$$

$$\Rightarrow D \Rightarrow \begin{pmatrix} 4 & 2 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} = \begin{vmatrix} 4-\Lambda & 2 \\ 0 & -1-\Lambda \end{vmatrix} = 0$$

$$\Rightarrow D \Rightarrow \Lambda^2 - 3\Lambda - 4 = 0, \Lambda_1 = 4, \Lambda_2 = -1$$

2. Find the eigenvectors corresponding to the eigenvalues for A,C and D.

$$\Rightarrow A \Rightarrow \Lambda_1 = 0, \Rightarrow \begin{pmatrix} 4-0 & 0 \\ 0 & 4-0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 4v_1 + 0v_2 = 0, \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow A \Rightarrow \Lambda_2 = 4, \Rightarrow \begin{pmatrix} 4-4 & 0 \\ 0 & 4-4 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0v_3 + 0v_4 = 0, \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow C \Rightarrow \Lambda_1 = -1, \Rightarrow \begin{pmatrix} 1--1 & 0 \\ 0 & -1--1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2v_1 + 0v_2 = 0, \Rightarrow \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\Rightarrow C \Rightarrow \Lambda_2 = 1, \Rightarrow \begin{pmatrix} 1-1 & 0 \\ 0 & -1-1 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0v_3 - 2v_4 = 0, \Rightarrow \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow D \Rightarrow \Lambda_2 = 4, \Rightarrow \begin{pmatrix} 4-4 & 2 \\ 0 & -1-4 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0v_3 + 2v_4 = 0, \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow D \Rightarrow \Lambda_1 = -1, \Rightarrow \begin{pmatrix} 4--1 & 2 \\ 0 & -1--1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 5v_1 + 2v_2 = 0, \Rightarrow \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

3. Since A is of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, we know it is a uniform scaling transformation with a scale factor of 4. Using the fact that

$$B = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

predict what transformation B is.

$$\Rightarrow B \text{ has a uniform scaling transformation with a scale factor } \frac{1}{2}$$

4. Using the fact that the eigenvectors of a linear transformation only change in magnitude under that linear transformation, try to predict what linear transformation C represents. Test out your theory using some simple areas.

$$\Rightarrow C \text{ represents a magnification of a scale factor 1,which is rotated in a } 90^o \text{ right in clockwise direction}$$

$\Rightarrow$ For example we have q= $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, we transform it as follows

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (1 * 2) + (0 * 3) \\ (0 * 2) + (-1 * 3) \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$