a) For each of these functions, find $f'(x)$. Show your working.

i) $f(x) = x^3 + 2x^2 + 4$

Solution

$$f'(x) = 3x^2 + 4x$$

ii) $f(x) = \frac{x^2+5}{x-2}$

Solution

Here we use the quotient rule to differentiate

$$f'(x) = \frac{(x-2)(2x) - (x^2+5)(1)}{(x-2)^2}$$

$$f'(x) = \frac{2x^2 - 4x - x^2 + 5}{(x-2)^2}$$

$$\Rightarrow f'(x) = \frac{x^2 - 4x + 5}{(x-2)^2}$$

iii) $f(x) = \sin(2x)\cos(4x)$

Solution

We use the product rule to differentiate this function

$$\Rightarrow f'(x) = sin(2x) * (-sin(4x))(4) + cos(4x) * cos(2x)(2)$$

$$\Rightarrow f'(x) = -4sin(2x)sin(4x) + 2cos(4x)cos(2x)$$

iv) $f(x) = \cos(e^{4x})$

Solution

differentiate $= e^{4x}$ with respect to x to obtain $\Rightarrow 4e^{4x}$

$$\Rightarrow f'(x) = -4e^{4x} sin(e^{4x})$$

v) $f(x) = \frac{e^{x^2}-1}{\ln x}$

Solution

Using quotient rule to differentiate

$$\Rightarrow f'(x) = \frac{ln(x) * (2x) * e^{x^2} - (e^{x^2} - 1) * (\frac{1}{2})}{ln(x)^2}$$

$$\Rightarrow f'(x) = \frac{2x^2 e^{x^2} lnx - e^{x^2} + 1}{xln^2 x}$$

For each of these functions, find $\int f(x)dx$. Show your working.

vi) $f(x) = x^2 - 4x + 3$

Solution

$$\Rightarrow \int f(x)dx = \int (x^2 - 4x + 3)dx$$

$$\Rightarrow \int f(x)dx = x^3 - \frac{4x^2}{2} + 3x + c$$

$$\Rightarrow \int f(x)dx = x^3 - 2x^2 + 3x + c$$

vii) $f(x) = \sin(4x + \pi)$

Solution

$$\Rightarrow \int f(x)dx = \int sin(4x + \Pi)dx$$

We use integration by substitution

Let u=4x+Π

differentiate the above wrt x we have $\Rightarrow du = 4dx \Rightarrow dx = \frac{du}{4}$

$$\Rightarrow \text{substituting the values we obtain} \Rightarrow \int sin(u)\frac{du}{4}$$

$$\Rightarrow \frac{1}{4}\int sin(u)du \Rightarrow \frac{1}{4}(-cos(u)) + c$$

$$\Rightarrow -\frac{1}{4}cos(4x + \Pi) + c$$

viii) $f(x) = e^{3x+4}$

Solution

$$\Rightarrow \int f(x)dx = \int e^{3x+4}dx$$

We use substitution, let u=3x+4

Then, du=3dx $\Rightarrow dx = \frac{du}{3}$

After substitution we get, $\Rightarrow \int e^u \frac{du}{3} = \frac{1}{3}\int e^u du$

$$\Rightarrow \frac{1}{3}e^u + c \equiv\Rightarrow \frac{1}{3}e^{3x+4} + c$$

ix) $f(x) = 4x\cos(3x)$

Solution

We use integration by parts rule

$$\Rightarrow \int f(x)dx = \int 4xcos(3x)dx$$

Let u=4x and dv=cos(3x)dx

Differentiating u and integrating dv wrt to x we get:

$$\Rightarrow du = 4dx \text{ and } v = \frac{1}{3}sin(3x)$$

$$\Rightarrow 4x * \frac{1}{3}sin(3x) - \int \frac{1}{3}sin(3x) * 4dx$$

$$\Rightarrow \frac{4}{3}xsin(3x) - \frac{4}{3}(-\frac{1}{3})cos(3x) + c$$

$$\Rightarrow \frac{4}{3}xsin(3x) + \frac{4}{9}cos(3x) + c$$

x) $f(x) = x^3 \ln x$

Solution

$$\Rightarrow \int f(x)dx = \int x^3 lnxdx$$

We use integration by parts

$\Rightarrow$ We let u=lnx and $dv = x^3dx$

$\Rightarrow$ differentiating u and integrating dv wrt x we have:

$$\Rightarrow du = \frac{1}{x}dx \text{ and } \Rightarrow v = \frac{x^4}{4}$$

$$\Rightarrow \frac{x^4}{4}lnx - \int \frac{x^4}{4} * \frac{1}{x}dx$$

$\Rightarrow$ Upon simplification we get: $\Rightarrow \frac{x^4}{4}lnx - \frac{1}{4}\int x^3 dx$

$$\Rightarrow \frac{x^4}{4}lnx - \frac{x^4}{16} + c$$

b) In some situations, we would like to differentiate a function more than once. We call this *higher-order differentiation*, and we can define it in the following way:

$$f^{(0)}(x) = f(x)$$
$$f^{(n+1)}(x) = \left(f^{(n)}\right)'(x)$$

i) Let $f(x) = 2x^5 + 3x^4 + 4$. Calculate $f^{(3)}(x)$. Show your working.

Solution

To find $f^{(3)}(x)$ then we need to find the values for both $f^{(1)}(x), f^{(2)}(x)$

$$\Rightarrow f^{(1)}(x) = 10x^4 + 12x^3$$

$$\Rightarrow f^{(2)}(x) = 40x^3 + 36x^2$$

$$\Rightarrow f^{(3)}(x) = 120x^2 + 72x$$

ii) Now let $f(x)$ be a polynomial of order n, i.e. one whose highest non-zero coefficient is c_n. For example, the polynomial above is of order 5, since it has highest non-zero coefficient $c_5 = 2$. Show that $f^{(n+1)}(x) = 0$

Solution

Given: $f(x) = 2x^5 + 3x^4 + 4$,we have that this polynomial is order 5 i.e n=5

Therefore, $f^{(n+1)}(x) = f^{(5+1)}(x) = ?$

Since we have found above that $f^{(3)}(x) = 120x^2 + 72x,$ we find, $f^{(4)}(x), f^{(5)}(x), f^{(6)}(x)$

$\Rightarrow f^{(4)}(x) = 240x + 72$

$\Rightarrow f^{(5)}(x) = 240$

$\Rightarrow f^{(6)}(x) = 0,$ hence the proof

iii) Now let $f(x) = \sin x$. Calculate $f^{(5)}(x)$. Show your working.

Solution

$\Rightarrow f^{(1)}(x) = cos(x)$

$\Rightarrow f^{(2)}(x) = -sin(x)$

$\Rightarrow f^{(3)}(x) = -cos(x)$

$\Rightarrow f^{(4)}(x) = sin(x)$

$\Rightarrow f^{(5)}(x) = cos(x)$

iv) Fix some $a \in \mathbb{R}$. With $f(x) = \sin(x)$ as before, we define a new function $g : \mathbb{N} \to \mathbb{R}$ such that $g(n) = f^{(n)}(a)$. Write an alternative definition of $g(n)$ that does not involve differentiation.

Solution

Since f(x) =sin (x) ,and we know that $f^{(5)}(x) = cos(x)$;

Then, for $a \in \mathbb{R}$, $\Rightarrow a \in \mathbb{N}$, therefore g(n) can be defined as g(n)=cos(a) ,given that $a \in \mathbb{R}$

c) Below is an alternative but equivalent definition of the definite integral of $f: \mathbb{R} \to \mathbb{R}$ between a and b, given $x_k = a + k\left(\frac{b-a}{n}\right)$ for $0 \le k \le n$:

$$I_n(f,a,b) = \left(\frac{b-a}{n}\right)\sum_{k=0}^{n-1} f(x_k)$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} I_n(f,a,b)$$

i) For $f(x) = x^2$, calculate $I_5(f,-2,8)$. Show your working.

Solution

$$I_5(f,-2,8) = (\frac{8--2}{n})\sum_{k=0}^{n-1} f(x_k)$$

$$\Rightarrow \frac{10}{5}\sum_{k=0}^{4}(-2+2k)^2$$

$$\Rightarrow \frac{10}{5}((_-2)^2+0+4+16+36) = 60 * \frac{10}{5} = 120$$

ii) We now define

$$J_n(f,a,b) = \left(\frac{b-a}{n}\right)\sum_{k=1}^{n} f(x_k)$$

For $f(x) = x^2$, calculate $J_5(f,-2,8)$. Show your working.

Solution

$$J_5(f,-2,8) = (\frac{8--2}{5})\sum_{k=1}^{n} f(x_k)$$

$$f(x_k) = (a+k(\frac{b-a}{n}))^2 = (-2+k(\frac{8--2}{5}))^2$$

$$\Longrightarrow \frac{10}{5}\sum_{k=1}^{5}(-2+2k)^2 \equiv 2\sum_{k=1}^{5}(-2+2k)^2$$

$\Rightarrow 2(0+4+16+36+64) = 2(120) = 240$

iii) Calculate $\int_{-2}^{8} x^2 dx$

Solution

$$\int_{-2}^{8} x^2 dx = \frac{x^3}{3} | \text{ from -2 } \longrightarrow 8$$

$$\Rightarrow (\frac{(8^3)}{3}) - (\frac{(-2)^3}{3}) = 170\frac{2}{3} - (-2\frac{2}{3}) = 173.33$$

iv) We call a function $f: \mathbb{R} \to \mathbb{R}$ *monotone increasing* if $x > y$ implies $f(x) > f(y)$. Show that, if f is monotone increasing, then, for any $a, b \in \mathbb{R}, a < b$, and $n \in \mathbb{N}$

$$I_n(f, a, b) < J_n(f, a, b)$$

Solution

Let $a, b \in \mathbb{R}$, thus since $a <$ b for a,b in $\mathbb{R}$, then :

$$\Rightarrow (\frac{b-a}{n}) \sum_{k=0}^{n-1} f(x_k) \leq (\frac{b-a}{n}) \sum_{k=1}^{n} f(x_k)$$

This implies that $\exists k, \in J_n(f, a, b)$ such that $k_1 > k_0 \in I_n(f, a, b)$

Hence, $J_n(f, a, b) > I_n(f, a, b)$

v) Without formally proving it, explain why:

$$\lim_{n\to\infty} J_n(f, a, b) = \int_a^b f(x)dx = \lim_{n\to\infty} I_n(f, a, b)$$

Solution

This is because as the limit for the both functions approaches infinity the value for (b-a)/n which forms both parts of the two functions will approach a zero value, which further implies of equality between the two function solutions

a) We define the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \quad C = \begin{pmatrix} -3 & 2 \\ -4 & 3 \\ 1 & 4 \end{pmatrix}$$

i) All but one of the following matrix operations are well-defined. Indicate the one which is not, and calculate the rest. Show your working.

Solution

B^2 is not well defined

$$\Rightarrow 4C = 4\begin{pmatrix} -3 & 2 \\ -4 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} (4*-3) & (4*2) \\ (4*-4) & (4*3) \\ (4*1) & (4*4) \end{pmatrix} = \begin{pmatrix} -12 & 8 \\ -16 & 12 \\ 4 & 16 \end{pmatrix}$$

$$\Rightarrow AB = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & 5 & -1 \\ 2 & -1 & -5 \end{pmatrix} = \begin{pmatrix} (2*3)+(1*2) & (2*5)+(1*-1) & (2*-1)+(1*-5) \\ (4*3)+(5*2) & (4*5)+(5*-1) & (4*-1)+(5*-5) \end{pmatrix}$$

$$\Rightarrow AB = \begin{pmatrix} 8 & 9 & -7 \\ 22 & 15 & -29 \end{pmatrix}$$

$$\Rightarrow CA = \begin{pmatrix} -3 & 2 \\ -4 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} (-3*2)+(2*4) & (-3*1)+(2*5) \\ (-4*2)+(3*4) & (-4*1)+(3*5) \\ (1*2)+(4*4) & (1*1)+(4*5) \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 4 & 11 \\ 18 & 21 \end{pmatrix}$$

$$\Rightarrow det(A) = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} = (2*5)-(4*1) = 6$$

$$\Rightarrow A^{-1} = \frac{1}{6}\begin{pmatrix} 5 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{6} \\ -\frac{4}{6} & \frac{2}{6} \end{pmatrix}$$

ii) Find the eigenvalues and corresponding eigenvectors for A. Show your working.

Solution

Eigenvalues

$$|A - \Lambda I| = 0 \Rightarrow |\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\Lambda & 1 \\ 4 & 5-\Lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\Lambda)(5-\Lambda) - 4 = 0 \Rightarrow 2(5-\Lambda) - \Lambda(5-\Lambda) = 0$$

$$\Rightarrow 10 - 2\Lambda - 5\Lambda + \Lambda^2 - 4 = 0 \Rightarrow \Lambda^2 - 7\Lambda + 6 = 0$$

Let $a = 1, b = -7, c = 6$

$$\Rightarrow \Lambda = \frac{7 \pm \sqrt{49-24}}{2} = \frac{7 \pm 5}{2} \Rightarrow \Lambda_1 = 6, \Lambda_2 = 1$$

Eigenvectors

Eigenvector associated with:

$\Lambda_1 = 6$,

$$\Rightarrow \begin{pmatrix} (2-6) & 1 \\ 4 & (5-6) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -4v_1 + v_2 = 0 \Rightarrow v_2 = 4v_1,$$

$$\Rightarrow \text{choose } ,v_1 = 1, v_2 = 4 \Rightarrow \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Eigenvector associated with:

$$\Lambda_2 = 1$$

$$\Rightarrow \begin{pmatrix} (2-1) & 1 \\ 4 & (5-1) \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_3 + v_4 = 0 \Rightarrow v_4 = -v_3$$

$$\Rightarrow \text{choose}, v_3 = 1, v_4 = -1, \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b) Consider the following system of linear equations:

$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= -1 \\ 3x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + 3x_2 - 4x_3 &= 2 \end{aligned}$$

i) Express this system of linear equations with a single function $f : \mathbb{R}^3 \to \mathbb{R}^3$, without using a matrix.

Solution

$$\Rightarrow A\underset{\rightarrow}{x} = \underset{\rightarrow}{b}$$

ii) Using the fact that

$$\begin{pmatrix} 2 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ -11 & \frac{-7}{2} & \frac{13}{2} \\ -7 & -2 & 4 \end{pmatrix}$$

calculate the solution to the system of linear equations above. Show your working.

Solution

$$\Rightarrow \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 11 & -\frac{7}{2} & \frac{13}{2} \\ -7 & -2 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow x_1 = (1 * -1) + (\frac{1}{2} * 7) + (-\frac{1}{2} * 2) = 3\frac{1}{2} = \frac{7}{2}$$

$$\Rightarrow x_2 = (11 * -1) + (-\frac{7}{2} * 7) + (\frac{13}{2} * 2) = -22\frac{1}{2} = -\frac{45}{2}$$

$$\Rightarrow x_3 = (-7 * -1) + (-\frac{7}{2} * 7) + (4 * 2) = 1$$

$$\Rightarrow (x_1, x_2, x_3) = (\frac{7}{2}, -\frac{45}{2}, 1)$$

iii) Using the fact that

$$\det\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix} = 0$$

explain why we can't use the same method as before to find solutions to the system of linear equations below:

$$x_1 + 2x_2 - 3x_3 = 0$$
$$3x_1 - x_2 + 2x_3 = 0$$
$$5x_1 + 3x_2 - 4x_3 = 0$$

Solution

This is because the determinant being zero implies that there will be no inverse for the system of equations, hence we cannot find its solutions

iv) For the purposes of this question, we will call a system of linear equations *uniform* when all the constants are zero. For example, the system in the previous part is uniform. Explain why we can always find at least one solution for any uniform system of linear equations.

Solution

This is because for any given or said value of x3, the values for x1 and x2 will be a combination of the two. Further implying that we can only have one or more solutions to both x1 and x2 in order to find the value for x3

c) Let A be a 2×2 matrix.

i) Show that if $\det A = 0$, then A has an eigenvalue of 0.

Solution

$$\Rightarrow \text{Let A be given by , } A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\Rightarrow \text{Thus,we need to show that A has eigenvalues zero:}$$

$$\Rightarrow |A - \Lambda I| = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} = \begin{vmatrix} (1-\Lambda) & 2 \\ 1 & (2-\Lambda) \end{vmatrix} = 0$$

$$\Rightarrow 1(2-\Lambda) - \Lambda(2-\Lambda) = 0 \Rightarrow \Lambda^2 - 3\Lambda = 0$$

$$\Rightarrow \Lambda(\Lambda - 3) = 0 \Rightarrow \Lambda_1 = 3, \Lambda_2 = 0$$

$$\text{Thus,we have the matrix with eigenvalue } 0$$

ii) Show that if A has an eigenvalue of 0, then $\det A = 0$.

Solution

Therefore, since we know that A has an eigenvalue zero, then we proceed as follows:

$\Rightarrow$ Let $\begin{pmatrix} 1-\Lambda & 2 \\ 1 & 2-\Lambda \end{pmatrix}$, be the matrix with eigenvalue 0

Therefore, $\Rightarrow \begin{pmatrix} 1-0 & 2 \\ 1 & 2-0 \end{pmatrix} =$; This matrix has a determinant, $(1*2)-(2*1)=0$

iii) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation corresponding to the 2×2 matrix A. Suppose A has an eigenvalue of 0, corresponding to the eigenvector $\binom{a}{b}$. Geometrically describe the effect of this transformation on areas in 2-dimensional Euclidean space.

Solution

Since A has a corresponding eigenvector , $\begin{pmatrix} a \\ b \end{pmatrix}$, then it implies that

on any 2-dimensional Euclidean space any value will be magnified or transformed by

by a vector factor value of , $\begin{pmatrix} a \\ b \end{pmatrix}$ in the new 2-d space

THE END