1. A pack of cards contains 52 cards. For the purposes of this question, assume that each card has a number from 1 to 13, and a suit of Hearts, Clubs, Diamonds or Spades.

a. We draw one card from the pack, replace it, and then draw another. What are the chances that both cards are Hearts? What are the chances that the cards are different suits? Express your answers using a probability function P.

Solution

$$\Rightarrow P(2hearts) = \mathrm{P(heart\ and\ heart)} \Leftrightarrow P(heart) = \frac{13}{52}$$

$$\Rightarrow P(2hearts) = \frac{13}{52} * \frac{13}{52} = \frac{1}{16}$$

$$\Rightarrow \mathrm{P(cards\ from\ different\ suites)} = \mathrm{P(S\ and\ H)} + \mathrm{P(S\ and\ D)} + \mathrm{P(S\ and\ C)} + \mathrm{P(D\ and\ H)} + \mathrm{P(C\ and\ H)} + \mathrm{P(D\ and\ C)}$$

$$\Rightarrow \mathrm{P(cards\ from\ different\ suites)} = \frac{1}{16} * 6 = \frac{3}{8}$$

b. Now we draw one card from the pack, do not replace it, and then draw another. What are the chances that both cards are Hearts? What are the chances that the cards are different suits? Again, express your answers using a probability function P.

Solution

$$\Rightarrow \mathrm{P(2hearts)} = \mathrm{P(heart\ and\ heart)} = \frac{13}{52} * \frac{12}{51} = \frac{1}{17}$$

$$\Rightarrow \mathrm{P(cards\ are\ from\ different\ suites)} = \mathrm{P(S\ and\ H)} + \mathrm{P(S\ and\ D)} + \mathrm{P(S\ and\ C)} + \mathrm{P(D\ and\ H)} + \mathrm{P(C\ and\ H)} + \mathrm{P(D\ and\ C)}$$

$$\Rightarrow \mathrm{P(cards\ are\ from\ different\ suites)} = \frac{1}{17} * 6 = \frac{6}{17}$$

2. If dangerous fires are uncommon (0.5%) but smoke is fairly common (15%) and 90% of dangerous fires make smoke, then use Bayes Theorem to determine the percentage of occasions that smoke means a dangerous fire.

Solution

$$\Rightarrow \mathrm{P(Fire)}=0.5\%\ ,\mathrm{P(Smoke)}=15\%,\ \mathrm{P(Smoke|Fire)}=90\%,$$

$$\Rightarrow \mathrm{P(Fire|Smoke)} = \frac{P(Fire) * P(Smoke|fire)}{P(Smoke)} = \frac{0.5\% \ * 90\ \%}{15\%} = 3\%$$

3. Earlier, we stated in a Theorem three properties of a probability space $(\Omega, \mathcal{F}, P)$:

1. $P(\Omega \setminus A) = 1 - P(A)$.

Proof

$\Rightarrow$ The events $\Omega \backslash A$ are disjoint thus we have that $P(A \cup \Omega \backslash A) = P(A) + P(\Omega \backslash A)$

$\Rightarrow$ note that $P(A \cup \Omega \backslash A) = P(A \cup \Omega \backslash A) = P(\Omega) = 1$

$\Rightarrow P(A) + P(\Omega \backslash A) = 1 \Rightarrow P(\Omega \backslash A) = 1 - P(A)$

2. If $A \subseteq B$ then $P(B) = P(A) + P(B \setminus A) \geq P(A)$

Proof

$\Rightarrow$ we know that if $A \subseteq B, then -> P(B) \geq P(A)$

$\Rightarrow$ Therefore $(B \cap B \backslash A) \cup A = B \Rightarrow P(B) = P(A) + P(B \cap B \backslash A)$

$\Rightarrow$ But ,$P(B \cap B \backslash A) \geq 0 \Leftrightarrow P(B) = P(A) + P(B \cap B \backslash A) \geq P(A)$

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof

$\Rightarrow P((B \cap A^c) \cup A) = P(A) + P(B \cap A^c) = P(B)$

$\Rightarrow P(B) = P(B \cap A^c) \cup P(B \cap A) = P(B \cap A^c) + P(B \cap A)$

$\Rightarrow P(B \cap A^c) = P(B) - P(B \cap A) \Rightarrow$

$\Rightarrow$ Therefore ,$P(A \cup B) = P(A) + P(B) - P(A \cap B)$