# Validity of Kernels

We have that for any $x_i, x_j \in \mathbb{R}^p$

(i) $f(x_i, x_j) = (\phi(x_i), \phi(x_j))$

ii) for any set of vector

$\{x_i\}_{i=1}^n$ the matrix $k$ computed by

$k(i,j) = f(x_i, x_j)$

proof

We know that a Kernel is valid or rather well defined if the above said conditions are met.

(i) Symmetric

(ii) positive semi-definite

(i) Now if $f_1(x_i, x_j)$ and $f_2(x_i, x_j)$ are valid Kernels then

$$f_3(x_i, x_j) = f_1(x_i, x_j) + f_2(x_i, x_j) \Rightarrow ?$$

We need to note that

$$f(x_i, x_j) = f_a(x_i, x_j) + f_b(x_i, x_j) \Rightarrow$$

$\Rightarrow \phi(x) = (\phi_1(x) ; \phi_2(x))$ as the feature composition

$\Rightarrow$ Thus we have in our case

$\Rightarrow f_1(x_i, x_j') = (\phi_1(x_i), \phi_1(x_j))$

$f_2(x_i, x_j) = (\phi_2(x_i), \phi_2(x_j))$

$\Rightarrow$ By the second property

$$f_3(x_i, x_j) = f_1(x_i, x_j) + f_2(x_i, x_j)$$

$$\Rightarrow f_3 = f_1 + f_2 \geq 0$$

$$\Rightarrow (\phi_1(x_i), \phi(x_j)) + (\phi_2(x_i), \phi_2(x_j))$$

$$\Rightarrow ([\phi_1(x_i)\ \phi_2(x_i)], [\phi_1(x_j), \phi_2(x_j)]) \geq 0$$

$\Rightarrow$ So we therefore see that $f_3(x_i, x_j)$ can be expressed as an inner product

$f_1 + f_2 \geq 0$ which yields a positive element-wise matrix satisfying the positive semi-definite property

$\Rightarrow$ Hence a valid Kernel

(ii) If $f_1(x_i, x_j)$ and $f_2(x_i, x_j)$ are valid Kernels ~~then~~ prove

$$f_4(x_i, x_j) = f_1(x_i, x_j) f_2(x_i, x_j)$$

Proof

$\Rightarrow$ Thus by construction, the Gram matrix is given as follows

$$f_4 = f_1 \odot f_2$$

where $\odot$ is the denotation of Hadamard entrywise product

=) By the 2nd property that $f_1(x_i, x_j)$ and $f_2(x_i, x_j)$ are symmetric positive semi-definate matrices, then we have the following

$$f_1(x_i, x_j) = \sum_{i=1}^{n} x_i x_i^T \quad \text{and} \quad f_2(x_i, x_j) = \sum_{j=1}^{n} x_j x_j^T$$

=) This results into:

$$f_4(x_i, x_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i x_i^T) \odot (x_j x_j^T)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i \odot x_j)(x_i \odot x_j)^T$$

$$= \sum_{K=1}^{n^2} w_K w_K^T$$

Where $w_k = x_{K/n} \odot x_{K \bmod n}$

=) Simplifying further we have

$$f_4(x_i, x_j) = \sum_{K=1}^{n^2} w_k w_k^T = \sum_{K=1}^{n} (w_K^T)^2 \geq 0$$

=) This is a valid Kernel since we end with a summation of rank 1 with positive coefficients

(iii) If $f_1(x_i,x_j)$ and $f_2(x_i,x_j)$ are valid kernels
prove

$$f_5(x_i,x_j) = \exp(f_1(x_i,x_j))$$

proof

=) So this follows by taking the taylor series and limit of the polynomial case as follows

$$\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{x_i}{i!}$$

=) Taking the limit of the polynomial we have

$$\exp(x) = \lim_{i \to \infty} \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{x_i}{i!}\right)$$

=) Thus we have that

$$f(x_i,x_j) = \lim_{i \to \infty} f_1(x_i,x_j) \geq 0$$

since

$f_1(x_i,x_j) = (\phi_1(x_i), \phi_1(x_j))$ yields a positive pointwise matrix as shown in proof (i) above

=) Therefore $\exp(x)$ is a limit of polynomial which can be approximated with positive coefficients hence a limit of kernel $f_5(x_i,x_j)$

=) Thus since positive semi-definiteness property is also closed under pointwise limits, then kernel

$f_5(x_i, x_j) = \exp(f_1(x_i, x_j))$ is a valid kernel satisfying the $2^{nd}$ property of a valid kernel

END OF PROOF $\boxtimes$