a) Translate the following sentences into propositional logic. Your formalizations should be as detailed as possible.

i) Alice will go to the cinema or the theatre.

Solution

P=Alice will go the cinema

Q=Alice will go the theatre

$$P \vee Q$$

ii) Two is a prime number and not an odd number.

Solution

P=Two is a prime number

Q=Two is not an odd number

$$P \wedge Q$$

iii) If the speed limit is 30mph and I am driving at 25mph, then I am not breaking the law.

Solution

P=The speed limit is 30mph

Q=I am driving at 25mph

R=I am not breaking the law

$$(P \wedge Q) \rightarrow R$$

iv) If Bob is not sleeping then he is working, eating, or relaxing.

Solution

P=Bob is not sleeping

Q=Bob is working, eating

R=Bob is relaxing

$$P \rightarrow (Q \vee R)$$

v) Carlos will go to the park only if it is not raining.

Solution

P=Carlos will go to the park

Q=it is not raining

$$P \Leftrightarrow Q$$

Translate the following sentences into predicate logic. Your formalizations should be as detailed as possible.

vi) Everyone is mortal.

Solution

Predicate= mortal (one)

X=one

$$\forall x\,(one(x) \rightarrow mortal(x))$$

vii) Unicorns do not exist.

Solution

Predicate=exist (unicorn)

X=unicorn

$$\forall x\,(unicorn(x) \rightarrow exist(x))$$

viii) Every professional tennis player could beat any amateur tennis player.

Solution

Predicate=beat (x, y)

X=professional tennis player

Y=any amateur tennis player

$$\forall x\,(professionaltennisplayer(x) \rightarrow beat(x, anyamateurtennisplayer))$$

ix) Everyone has either a father or a mother.

Solution

Predicate=either (x, y)

X= one

Y= parent

$$\forall x\,(one(x) \rightarrow either(x, father) \vee either(x, mother))$$

x) Somebody has visited every country that currently exists.

Solution

Predicate=visited (x, y)

X= xbody

Y=every country that currently exists

$$\exists x\,(xbody(x) \wedge visited(x, everycountrythatcurrentlyexists))$$

b) Consider the following formula of propositional logic:

$$P = ((A \wedge B) \vee (\neg A \wedge \neg B)) \rightarrow C$$

i) Suppose A = "I will go to the shops", B = "I will go out for lunch" , C = "My partner will be unhappy". Translate P into an English sentence.

Solution

If I will go to the shops and go out for lunch, or I will neither go to the shops nor go out for lunch, then my partner will be unhappy

ii) Suppose A is true, B is false and C is true. Is P true or false? Briefly explain why.

Solution

P is True

Reason

A=True

B=False

C=True

$$P = ((A \wedge B) \vee (\neg A \wedge \neg B)) \rightarrow C$$

Thus for $\mathrm{A} \wedge B \Longrightarrow T \wedge F \Longrightarrow F$

Similarly

For $\neg A \wedge \neg B \Longrightarrow F \wedge T \Longrightarrow F$

Therefore, we have

For $F \vee F \Longrightarrow F$

We finally have False implying True

With False (F) $\rightarrow$ True(T) we have True(T)

iii) Which truth values for A, B and C result in P being false?

Solution

A=True

B=True

C=False

c) Consider the sentence "There is an animal in the zoo such that, if that animal is sleeping, then every animal in the zoo is sleeping.", formalised in predicate logic by the sentence $Q = \exists x(Px \rightarrow \forall y Py)$. For each of the questions below, explain your answer.

i) Suppose the zoo contains two animals, and both are sleeping. Is Q true or false?

Solution

True. Because that specific animal x which is sleeping is one of the two animals that are sleeping

ii) Now suppose one animal is sleeping, and one is awake. Is Q true or false?

Solution

False. Because of the two animals, animal x is sleeping implying that the rest must be sleeping which is not the case if one animal is awake

iii) Now suppose we still know that the zoo contains two animals, but we do not know how many are sleeping or awake. Can we know if Q is true or false?

Solution

Yes. Because if one of the two animals sleeping, then that particular animal will be x, further implying that the other animal is sleeping or if animal x is not sleeping then the other animal is not sleeping

iv) Now suppose we don't know how many animals the zoo contains, except that there is at least one animal, and we don't know how many are awake or asleep. Can we know if Q is true or false?

Solution

No. Because in this case animal x can not be determined to dictate if the rest of the animals are asleep or not

Question 2

a) Identify which of these relations are reflexive, which are symmetric, which are transitive, and which are equivalence relations. You do not need to show any working.

i) A is the set of all animals, $R_1 = \{(a,b) \mid a \text{ is the same species as } b\} \subseteq A \times A$

Solution

Reflexive

ii) $R_2 = \{(m,n) \mid m^2 \leq n^2\} \subseteq \mathbb{Z} \times \mathbb{Z}$

Solution

Reflexive and Transitive

iii) $R_3 = \{(x,y) \mid x+y<1\} \subseteq \mathbb{R} \times \mathbb{R}$

Solution

Reflexive, symmetric, transitive ➔ equivalent

iv) $B = \{0,1,2\}$, $R_4 = \{(0,0),(0,1),(1,0),(1,1),(2,2)\} \subseteq B \times B$

Solution

Equivalent

v) $B = \{0,1,2\}$, $R_5 = \{(0,1),(1,2),(2,0)\} \subseteq B \times B$

Solution

None

b) We define a function $f : \mathbb{N} \to \mathbb{Z}$ such that:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

i) Prove that f is an injection.

Proof

To prove injection, we show that

$$\forall x1, x2 \in A \ f(x1) = f(x2) \text{ then } \ \mathrm{x1{=}x2}$$

For x being even we let

X1=2P1

X2=2P2

Therefore

$$f(x1) = f(x2) \equiv f(2P1) = f(2P2) \equiv \frac{2P1}{2} = \frac{2P2}{2} \equiv P1 = P2$$

Similarly for x being odd we let

X1=2P1+1

X2=2P2+1

Therefore

$$f(2P1+1) = f(2P2+1) \equiv -\frac{(2P1+1)+1}{2} = -\frac{(2P2+1)+1}{2} \equiv -\frac{2P1}{2} = -\frac{2P2}{2} \equiv P1 = P2$$

Hence f is an injection

ii) Prove that f is a bijection.

Proof

Since we have shown the function is an injection, we now show that the function is a surjection for us to prove that the function is a bijection

We need to show that $\exists s \in \mathbb{Z}$ such that $f(s) = q$ for $q \in \mathbb{N}$

So for x being even we have s even

$$s = q \Longrightarrow f(s) = \frac{2q}{2} = q \quad \Rightarrow s = q$$

Similarly

With x odd, we also have s odd ,thus $q = -\frac{p+1}{2}$

But with s=p

$$\Rightarrow f(s) = -\frac{s+1}{2} \equiv -\frac{p+1}{2} \equiv q$$

➔ f is a surjection, hence a bijection

iii) Given that f is a bijection, find the inverse function $f^{-1}: \mathbb{Z} \to \mathbb{N}$

Solution

$$f^{-1}(x) = \begin{pmatrix} 2x \text{ if x is even} \\ -2x+1 \text{ if x is odd} \end{pmatrix}$$

iv) Given the previous parts, what can we say about the cardinality of $\mathbb{N}$ and $\mathbb{Z}$?

Solution

Since there is a bijection between $\mathbb{N}$ and $\mathbb{Z}$ we can conclude that

the two sets have the same cardinality

c) In this problem sheet, we will call a relation R *geometric* if $\forall x, y, z(xRy \wedge xRz \rightarrow yRz)$

i) Prove that if a relation is geometric and reflexive, then it is also symmetric.

Solution

Since R is geometric, we need to show that R is Reflexive and then is also symmetric

Let there be $x, y, z \in \mathbb{R}$

Thus R= $\{(x,x),(x,y),(y,x),...\}$

Therefore we have a $(a,a) \in \mathbb{R}$ such that R is geometric

Implying that $(x,x),(y,y),(z,z) \in R$,Hence reflexive

Now we have $(x,y) \in R,\ and\ (y,x) \in R$ which are in R, $\forall x,y,z \in R$

Hence R is symmetric

ii) Using the previous part, prove that if a relation is geometric and reflexive, then it is an equivalence relation.

Solution

Since we have shown that the geometric function above is reflexive and thus symmetric, now we need to show that the function R is transitive in order to prove equivalence

Thus $\forall x,y,z \in R,$ the set $R = \{(x,x),(x,y),(y,x),(x,z),...\}$

Therefore $\forall x,y,z \in R$, $\exists x,y \in R$ and $y,z \in R$ such that $x,z \in R$

From the given set R, $x,y \in R$ and $y,z \in R$

Implying that $x,z \in R$ which is geometric from the defination,thus a transitive relation

Hence an Equivalent relation

iii) Let $f: A \to B$ be a function. We define the relation

$$R \subseteq A \times A, \quad R = \{(x,y) \mid f(x) = f(y)\}$$

By showing that it is geometric and reflexive, show that R is an equivalence relation. What are the equivalence classes of R?

Solution

We need to show that the above function is geometric and reflexive first, for geometric

$$\text{We have that } \exists x, y \in R, \text{ such that } x, y \in A * A$$

$$\forall x, y\, (xRf(y) \longrightarrow yRf(x))$$

Now we show that this geometric function is reflexive

$$\text{Therefore } \exists x, y \in R \text{ such that } \mathrm{R}=\{(x,x),(x,y),(y,x),\dots\}$$

$$\text{For reflexivity } \exists(x,x) \text{ and } (y,y) \in R \text{ such that R is geometric}$$

Hence reflexive

Next, we show equivalence by proving symmetric and transitivity properties

$$\text{Thus } \forall x, y \in A * A,\ \exists x, y \in R \text{ and } y, x \in R, \text{ which is true } \forall x, y \in R$$

Hence symmetric

Lastly, we show transitivity, that given

$$\forall x, y \in A * A,\ \exists x, y \in R \text{ and } \exists y, x \in R \text{ such that } x, x \in R \subseteq A * A$$

Thus, the relation is equivalent

Equivalence classes

$$[x] = \{x \in A * A \mid * Rf(x)\}$$

$$[y] = \{x \in A * A \mid * Rf(y)\} \qquad \forall x, y \in R \subseteq A * A$$