3) $\left(x_n\right)_{n=1}^{\infty}$ $x_n = \frac{5}{2n-1}$

proof.

Using definition of Convergence we have

=> This sequence $\{x_n\}$ is said to converge to a real number $x$ if for every $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that

$$\left| x_n - x \right| < \varepsilon \qquad \forall n > N(\varepsilon)$$

$$\Rightarrow \lim_{n \to \infty} x_n = ?$$

$$\Rightarrow \lim_{n \to \infty} \frac{5}{2n-1} \Rightarrow \lim_{n \to \infty} \frac{5}{2n-1} = 0$$

$$\left| \frac{5}{2n-1} - 0 \right| < \varepsilon \qquad \forall n > N(\varepsilon)$$

Now $\left| \frac{5}{2n-1} \right| < \varepsilon \Rightarrow 2n - 1 > 5/\varepsilon$

$$\Rightarrow n > \left( \frac{5}{\varepsilon} - 1 \right) / 2 \Rightarrow N(\varepsilon) > \frac{5}{2\varepsilon} - \frac{1}{2}$$

Thus taking the limit

$$\lim_{n \to \infty} \frac{5}{2n-1} = \frac{5}{2(\infty)-1} = 5/\infty = 0$$

So this sequence converge to zero!

$(x_n)_{n=1}^{\infty}$ $\quad x_n = \frac{1}{10} \sin \frac{n\pi}{2} \quad \forall n.$

Using the definition of convergence we have

Taking the limit we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{10} \sin \frac{n\pi}{2} = \frac{1}{10} \lim_{n\to\infty} \sin \frac{n\pi}{2}$$

range of sine function is $[-1, 1]$

Thus $\frac{1}{10} \lim_{n\to\infty} \sin \frac{n\pi}{2} = 0$

Using convergence definition

$\forall \varepsilon > 0, \exists N > 0$ such that $n > N(\varepsilon)$

$$\left| \frac{1}{10} \sin \frac{n\pi}{2} - 0 \right| < \varepsilon$$

$$= \left| \frac{1}{10} \sin \frac{n\pi}{2} \right| < \varepsilon$$

We find the $N(\varepsilon)$ such that it minimises the above function, thus, we have

$$\left| \frac{1}{10} \sin \frac{n\pi}{2} \right| \leq \frac{1}{n^2} < \frac{1}{N^2} < \varepsilon = N^2 > \frac{1}{\varepsilon}$$

Take $\Rightarrow N(\varepsilon) > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}}$

=) Thus sequence does converge to zero

(c) $\left(x_n\right)_{n=1}^{\infty}$ $\quad x_n = \begin{cases} 1 & n \text{ is a power of } 2 \\ 0 & \text{otherwise} \end{cases}$

Soln

We use the theorem that a sequence of real numbers converge to $x$ if and only if every subsequence of $\{x_n\}$ converges to $x$

=) Suppose $x_n \to x$, for every

$$\varepsilon > 0 \,,\ \exists\ N(\varepsilon) \in J^+ \quad \text{such that}$$

$$|x_n - x| < \varepsilon \quad \forall n > N(\varepsilon)$$

Taking $n_k > N(\varepsilon)$, then $|x_{n_k} - x| < \varepsilon$

$$\forall n_k > N(\varepsilon)$$

$$\Rightarrow \{x_n\} = \{x_1, x_2, x_3, \ldots\}$$

$$\{0, 1, 0, 1, 0, 1 \ldots\}$$

The corresponding subsequence

$\{x_{n_k}\}$ for $k = 2$ =

$$\{x_{2n}\}_{n=1}^{\infty} = \{x_2, x_4, x_6 \ldots\}$$

$$\{1, 1, 1, \ldots\}$$

This subsequence converge to 1

We consider another subsequence :

$$\left(x_{2n-1}\right)_{n=1}^{\infty} = \{x_1, x_3, x_5, \ldots\}$$

$$\{0, 0, 0, \ldots\}$$

This subsequence converges to 0.

From the theorem definition for convergence we have found that

$$\lim_{n\to\infty} x_{2n} = 1 \quad \text{and} \quad \lim_{n\to\infty} x_{2n-1} = 0$$

$$\left|x_{2n}-1\right| < \varepsilon \quad \text{and} \quad \left|x_{2n}-0\right| < \varepsilon$$

$\Rightarrow$ We can not find a $N(\varepsilon)$ such that

$|x_n - x| < \varepsilon$ for every $\varepsilon > 0$

Thus, since $0 \neq 1$, the sequence diverge,

(d) $(x_n)_{n=1}^{\infty}$ $\quad x_n = \dfrac{1}{\log_2 n}$ $\quad n \geq 2$

Soln.

Using definition of Convergence

$\forall \varepsilon > 0, \ \exists N > 0$ s.t. $n > N(\varepsilon)$

$|x_n - x| < \varepsilon$

$$\Rightarrow \left| \frac{1}{\log_2 n} - x \right| < \varepsilon$$

$$\Rightarrow \lim_{n \to \infty} \frac{1}{\log_2 n} = \lim_{n \to \infty} \frac{1}{\log_2(\infty)}$$

$\Rightarrow$ This value is approximately $\infty$

i.e $\log_2(\infty) = \infty \Rightarrow$

$$\lim_{n \to \infty} \frac{1}{\infty} = 0$$

$$\Rightarrow \left| \frac{1}{\log_2 n} - 0 \right| < \varepsilon$$

We need to choose $N(\varepsilon)$ s.t. $\forall \varepsilon > 0 \ \exists N(\varepsilon)$ $\forall \varepsilon$ $s.t.$
$n > N(\varepsilon)$

$$\Rightarrow \left| \frac{1}{\log_2 n} \right| < \varepsilon$$

We now find the $N(\varepsilon)$ ~~such~~ such that it minimises the above function.

$$\left|\frac{1}{\log_2 n}\right| \leq \frac{1}{n} < \frac{1}{N} < \varepsilon$$

$$\Rightarrow N > \frac{1}{\varepsilon}$$

Thus, we that $N(\varepsilon) = \frac{1}{\varepsilon}$

$$\Rightarrow |x_n - e| < \varepsilon \quad \forall n > N(\varepsilon) > \frac{1}{\varepsilon}$$

We take $N(\varepsilon) = 1/\varepsilon$

Thus the sequence converges to zero

Thanks