

- QUESTIONS ORDER * PART-A **** (1A)-(2A)-(2B)-(2C)-(3A)-(3D)
 • PART-B **** (4B)-(4C)-(4A)-(4C CONTINUATION)

PART A

$$(Q4) \text{ Using } y'(t) = -10(y-t^2) + 2t \quad y(0) = 1
 at t_0 = 0, y_0 = 1$$

$$\frac{dy}{dt} = -10(y-t^2) + 2t$$

Sln

$$\text{at } t_1 = 0.1, y_1 = 0.3778794$$

$$y' = -10[0.3778794 - (0.1)^2] + 2(0.1) = -3.478740$$

$$\text{at } t_2 = 0.2, y_2 = 0.1753352$$

$$y' = -10[0.1753352 - (0.2)^2] + 2(0.2) = -0.9533540$$

$$\text{at } t_3 = 0.3, y_3 = 0.1397871$$

$$y' = -10[0.1397871 - (0.3)^2] + 2(0.3) = 0.1021290$$

We need to calculate the values of y_4 at $t_4 = 0.4$ and y_5 at $t_5 = 0.5$.

Now to find y_4 and y_5 we need to first write the generalized milne predictor and correct formulae as

$$y_{j+1} = y_{j-1} + h \left(2 + \frac{1}{3} \nabla^2 + \frac{1}{3} \nabla^3 + \frac{29}{90} \nabla^4 + \frac{25}{90} \nabla^5 \right) f_j$$

For the values stating $j=n$ and the ∇ value given by the difference in the interval of time t , the equation is simplified as

$$y_{n+1}^P = y_{n+1} = \frac{4}{3}h (2y_{n-2}^I - y_{n-1}^I + 2y_n^I)$$

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and similarly for the correct method

$$y_{n+1}^c = y_{n-1} + \frac{4}{3} (y_{n-1}^c + 4y_n^c + y_n^c)$$

$$\text{So at } n=3 \Rightarrow t=3$$

$$y_4^c = y_2 + \frac{4}{3} h (2y_1^c - y_2^c + 2y_2^c)$$

$$= 1 + \frac{4}{3} (0.1) [2(-3.4787940) - (-0.9533540) + 2(0.1021290)]$$

$$y_4^c (0.4) = 0.2266699$$

$$y_4^c(t) = -10(y-t^2) + 2t$$

$$y_4^c = -10(0.2266699 - (0.4)^2) + 2(0.4) = 0.1333013$$

as for the correct method

$$y_4^c = y_2 + \frac{4}{3} (y_2^c + 4y_3^c + y_4^c)$$

$$= 0.1753353 + \frac{4}{3} (-0.9533540 + 4(0.1021290) + 0.1333013) \\ = 0.2251743$$

$$y_4^c = -10(0.2251743 - (0.4)^2) + 2(0.4) = 0.1482562$$

$$\text{at } t=0.5 \quad y(0.5) = ?$$

$$\text{at } n=4 \Rightarrow t=4$$

$$y_5^c = \frac{4}{3} h (2y_2^c - y_3^c + 2y_4^c)$$

$$0.3718794 + \frac{4}{3} (0.1) [2(0.1753353) - (0.1021290) + 2(0.1333013)] \\ = 0.1455815$$

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$$y_5^1 = -10 \left(0.1455815 - (0.5)^3 \right) + 2(0.5) = 2.6442350$$

as per the
Concord method

$$= 0.1397871 + 0.1/3 \left(0.1021290 + 4(0.1333013) + 2.044144 \right)$$

$$y_5^1 = -10 \left(0.2291044 - (0.5)^3 \right) + 2(0.5) = 1.2089559$$

Table

j	t	y
0	0.0	1.0000000
1	0.1	0.3778794
2	0.2	0.1753353
3	0.3	0.1397871
4	0.4	0.2266699

5/0.5 0.2

Table

j	t	y	y¹	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	0.0	1.0000000	-10.0000000				
				6.5212060			
					-3.9957660		
1	0.1	0.3778794	-3.4787940			2.5258090	
				2.5254400			-2.080144
					-1.4699570		
2	0.2	0.1753353	-0.9533540			0.4456463	
				1.0554830			1.623146
					-1.0243107		
3	0.3	0.1397871	0.1021290			2.0687930	
				0.0311723			
					1.0444823		
4	0.4	0.2266699	0.1333013				
				1.0756546			
5	0.5	0.2291044	1.2089559				

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$\nabla^5 f$

3, 7032911

(2a) For the predictor-mixer predictor method.

$$y_{j+1} = y_{j-1} + h \left(2 + \frac{1}{3} \nabla^2 + \frac{1}{3} \nabla^3 + \frac{29}{90} \nabla^4 + \frac{28}{90} \nabla^5 + \dots \right) f_j$$

For $j=n$

$$\Rightarrow y_{n+1} = y_{n-1} + h \left(2 + \frac{1}{3} y_{n-1} + \frac{1}{3} y_{n-2} + \frac{29}{90} y_{n-3} + \dots \right) f_n$$

$$\Rightarrow E y_{n+1} = y_{n+1}$$

$$E y_{n-1} = y_{n-1}$$

$$E^2 y_{n-2} = y_{n-2}$$

$$\Gamma(E) = \frac{28}{90} E^4 + \frac{29}{90} E^3 + \frac{1}{3} E^2 + \frac{1}{3} E + 2$$

$$P(E) = (E^3 - E)$$

$$(E^3 - E) y_{n-2} - h \left(\frac{28}{90} E^4 + \frac{29}{90} E^3 + \frac{1}{3} E^2 + \frac{1}{3} E + 2 \right) y_{n-2} = 0$$

$$P(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1) = 0 \quad \text{--- 1st characteristic}$$

$$\Gamma(\xi) = \frac{28}{90} \xi^4 + \frac{29}{90} \xi^3 + \frac{1}{3} \xi^2 + \frac{1}{3} \xi + 2 \quad \text{--- 2nd characteristic}$$

Root condition

$$\text{at } P(\xi) = 0$$

$$\Rightarrow P(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1)$$

$$\Rightarrow \xi = 0, \xi_1 = 1, \xi_2 = -1$$

These are the simple roots corresponding to the mixer-predictor method

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Stability

$$P(\xi) = \frac{28}{q_0} \xi^4 + \frac{29}{q_0} \xi^3 + \frac{1}{3} \xi^2 + \frac{1}{3} \xi + 2 = 0$$

$$\Rightarrow |\xi_4| > 0$$

Hence zero stable since this will yield complement

Convergence and stability consistency

\Rightarrow We suppose the method is zero-stable, then assume that $P(\xi)$ has double roots in the unit circle. Further, product of roots of $P(\xi)$ we set as 1, but now ξ_1 and ξ_2 are equal to $\pm i$, they not strictly complex hence they converge and are consistent.

$$\xi = 0, \xi_{1,2} = \pm i, \xi_3 = -1$$

\Rightarrow For the correct method

$$as \quad j=n$$

$$y_{j+1} = y_j = h \left(2 - 2\tau + \frac{1}{3}\tau^2 - \frac{1}{q_0}\tau^4 - \frac{1}{q_0}\tau^5 - \dots \right) f_{j+1}$$

\Rightarrow summing terms into

$$y_{n+1} = y_{n-1} + h \left(2 - 2y_n^1 + \frac{1}{3}y_{n-1}^1 - \frac{1}{q_0}y_{n-2}^1 - \frac{1}{q_0}y_{n-3}^1 - \dots \right) f_{n+1}$$

$$\Rightarrow E y_{n-2}^1 = y_n^1$$

$$E y_{n-2}^1 = y_{n-1}^1$$

$$E^2 y_{n-2}^1 = y_n^1$$

$$\Rightarrow P(E) = -\frac{1}{q_0} E^3 - \frac{1}{q_0} E^2 + \frac{1}{3} E + 2 - 2$$

$$= -\frac{1}{q_0} E^5 - \frac{1}{q_0} E^2 + \frac{1}{3} E$$

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$$P(E) = E^3 - E^2$$

$$\Rightarrow (E^3 - E^2)y_{n-2} - h \left(-\frac{1}{q_0}E^3 - \frac{1}{q_0}E^2 + \frac{1}{3}E \right) y_{n-2} = 0$$

$$P(\xi) = (\xi^3 - \xi^2) - \xi^2(\xi - 1) = \xi^2(\xi - 1) \quad \text{--- 1st characteristic}$$

$$T(\xi) = -\frac{1}{q_0}\xi^3 - \frac{1}{q_0}\xi^2 + \frac{1}{3}\xi \quad \text{--- 2nd characteristic}$$

Part condition at $P(\xi) = 0$

$$\Rightarrow P(\xi) = T^2(\xi - 1) = 0$$

$$\Rightarrow \xi_1 = 1, \xi_{2,3} = 0$$

These are simple roots of the corresponding characteristic equation

Stability

$$T(E) = T(\xi) = -\frac{1}{q_0}\xi^3 - \frac{1}{q_0}\xi^2 + \frac{1}{3}\xi \quad \text{at } P(\xi) = 0$$

$$= -\frac{1}{q_0}\xi^3 - \frac{1}{q_0}\xi^2 + \frac{1}{3}\xi = 0$$

$$\Rightarrow |\xi_3| > 1$$

Hence zero-stable since it yields complex roots

Convergence and consistency

$$P(\xi) = \xi^2(\xi - 1) = 0$$

We suppose the method is zero-stable, then assume that $P(\xi)$ has double roots in the unit circle

Since the roots of $P(\xi)$ is 1 and ξ_3 , then they are not equal and neither equal to ± 1 , then they are strictly complex

$$\Rightarrow \xi \neq \xi_{2,3}$$

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(2b) From the Nijstrom-milnes method,

$$y_{n+1} = y_{n-1} + h \left(2 + \frac{1}{3}T^2 + \frac{1}{3}T^3 + \underbrace{\frac{2}{9}T^4 + \frac{2}{9}T^5}_{T^6} \right) f_1$$

Setting $T=1$, a then simplified Nijstrom-milnes predictor method is given as follows:

$$y_{n+1} = y_{n-3} + \frac{4}{3}h (2y_{n-2}^1 - y_{n-1}^1 + 2y_n^1)$$

$$\Rightarrow y_{n+1} = y_{n-3} - \frac{4}{3}h (2y_{n-2}^1 - y_{n-1}^1 + 2y_n^1)$$

$$(1) \quad E y_{n-2}^1 = y_{n-1}^1$$

$$E y_{n-1}^1 = y_n^1$$

$$E^2 y_{n-2}^1 = y_n^1$$

$$\Rightarrow J(E) = \frac{8}{3}E^2 - \frac{4}{3}E + \frac{6}{3}$$

$$P(E) = (E^3 - 0)$$

$$\Rightarrow (E^3 - 0) y_{n-2} = h \left(\frac{8}{3}E^2 - \frac{4}{3}E + \frac{6}{3} \right) y_{n-2}^1 = 0$$

$$\Rightarrow (i) \quad P(\xi) = \xi^3 = \xi(\xi^2) \quad \text{--- 1st characteristic polynomial}$$

$$(ii) \quad J(\xi) = \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{6}{3} \quad \text{--- 2nd characteristic polynomial}$$

Root Condition

- The explicit method of the Nijstrom-milnes method is said to satisfy the root condition if the roots of the equation $P(\xi) = 0$ lies inside the closed unit disk in the complex plane and are simple if they lie on the circle.

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$$P(\xi) = 0$$

From our equation

$$P(\xi) = \xi^3 \Rightarrow \xi_{1,2,3} = 0$$

Stability

\Rightarrow The linear predictor method is zero stable if given $y^1 = f(x_0)$, $y(x_0) = y_0$, $f(x_0)$ satisfy Lipschitz condition and root condition holds.

\Rightarrow Thus we have in our case function

$$P(\xi) = \frac{8}{3}\xi^2 - 4\frac{1}{3}\xi + \frac{8}{3} \text{ which is zero-stable since it yields simple}$$

$$\Rightarrow \frac{8}{3}\xi^2 - 4\frac{1}{3}\xi + \frac{8}{3} = 0 \Rightarrow |\xi_{1,2}| \geq 0$$

Convergence & Consistency

\Rightarrow The linear predictor above is consistent if it has order $p \geq 1$

$$P(\xi) = \xi^2(\xi) = 0$$

(lets suppose the method is zero-stable, then $P(\xi)$ has double roots into unit circle)

\Rightarrow Factor product of roots of $P(\xi)$ is 1, and since the roots are neither equal to ± 1 , hence should be strictly complex.

$$\Rightarrow \xi(\xi^2) = 0 \Rightarrow \xi_1 = 0, \xi_2 = 0, \xi_3 = 0$$

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(2c) Derivation of 4th order Adams Bashforth Suen

Consider the AM general function given by

$$y_{n+1} = y_n + \frac{h}{24} (55y_n - 59y_{n-1} + 37y_{n-2} - 9y_{n-3})$$

$$\text{Let } E y_{n-2}^1 = y_{n-1}^1$$

$$E y_{n-2}^1 = y_{n-1}^1$$

Substituting into our function above we have

$$E^2 y_{n-2}^1 = y_n^1$$

$$\sigma(E) = -\frac{9}{24}E^3 + \frac{37}{24}E^2 - \frac{59}{24}E + \frac{55}{24}$$

$$P(E) = (E^3 - 1)$$

$$(E^3 - 1) y_{n-2} - h \left(-\frac{9}{24}E^3 + \frac{37}{24}E^2 - \frac{59}{24}E + \frac{55}{24} \right) y_{n-2}^1 = 0$$

Setting the root condition

$$P(\xi) = 0 \Rightarrow$$

characteristic polynomial

$$\sigma(\xi) = -\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24} = 1^{\text{st}} \text{ characteristic}$$

$$P(\xi) = (\xi^3 - 1) = 2^{\text{nd}} \text{ characteristic}$$

For the root condition (1) we have

$$P(\xi) = 0$$

$$(\xi^3 - 1) = 0 \Rightarrow \xi^3 = 1$$

$$-\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24} = 0$$

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$$\Rightarrow \frac{g}{24} \left(-9g^2 + \frac{37}{24}g - \frac{57}{24} \right) = -\frac{55}{24}$$

Interval : if from $x=1$ to $x=4$, the abscissa values lie within the margin interval

$$x^2 - 9x + 5 = 0$$

$$(x-1)(x-8)$$

$$x^2 - 9x + 5 = 0 \quad (1) \quad (2)$$

$$(x-1)(x-8)$$

$$x^2 - 9x + 5 = 0 \quad (1) \quad (2)$$

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Derviation of 4th order explicitly using Butterworth

Sln

Let the condition eqns be defined by

$$b_1 + b_2 + b_3 + b_4 = 1 \quad (1)$$

$$b_2 c_2 + b_3 c_3 + b_4 c_4 = \frac{1}{2} \quad (2)$$

$$b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{3} \quad (3)$$

$$b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 = \frac{1}{4} \quad (4)$$

$$c_2 = a_{21}$$

$$c_3 = a_{31} + a_{32}$$

$$c_4 = a_{41} + a_{42} + a_{43}$$

$$b_3 (a_{32} c_2 + b_4 (a_{42} c_2 + a_{43} c_3)) = \frac{1}{8} \quad (5)$$

$$b_3 a_{32} c_2 + b_4 a_{42} = b_2 (1 - c_2) \quad (6)$$

$$b_4 a_{43} = b_3 (1 - c_3) \quad (7)$$

$$0 = b_4 (1 - c_4) \quad (8)$$

so $\Rightarrow b_2 = \frac{1}{4}$ and $c_2 = \frac{1}{2}$ we have

Substituting $b_2 = \frac{1}{4}$ and $c_2 = \frac{1}{2}$ in the equation

(1) - (3) we have

$$b_1 + \frac{1}{2} + b_3 + b_4 = 1 \Rightarrow b_1 + b_3 + b_4 = \frac{1}{2} \quad (9)$$

$$\frac{1}{4} + b_3 c_2 + b_4 c_4 = \frac{1}{2} \Rightarrow b_3 c_3 + b_4 c_4 = \frac{1}{4} \quad (10)$$

$$\frac{1}{2} \cdot \frac{1}{8} + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{3} \Rightarrow b_3 c_3^3 + b_4 c_4^3 = \frac{1}{24} \quad (11)$$

$$\frac{1}{2} \cdot \frac{1}{8} + b_3 c_3^3 + b_4 c_4^3 = \frac{1}{4} \Rightarrow b_3 c_3^3 + b_4 c_4^3 = \frac{3}{16} \quad (12)$$

$$b_1 = \frac{1}{8} - b_3 - b_4, b_2 = \frac{1}{2}, b_3 = \frac{1}{2} - b_1 - b_4, b_4 = \frac{1}{2} - b_1 - b_3$$

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$$c_2 = \frac{1}{2}, \quad c_3 = \left(\frac{1}{4} - b_4 c_4\right) / b_3, \quad c_4 = \left(\frac{1}{4} - b_3 c_3\right) / b_4$$

$$a_{21} = \frac{1}{2}, \quad a_{31} = \left(\frac{1}{4} - b_4 c_4\right) / b_3 - a_{22}, \quad a_{32} = \left(\frac{1}{4} - b_4 c_4\right) / b_3 - a_{31}$$

$$a_{41} = \left(\frac{1}{4} - b_3 c_3\right) / b_4 - a_{42} - a_{43}, \quad a_{42} = \left(\frac{1}{4} - b_3 c_3\right) / b_4 - a_{41} - a_{43}$$

$$a_{43} = \left(\frac{1}{4} - b_3 c_3\right) / b_4 - a_{41} - a_{42}$$

Re Butcher tableau

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & & \\ \hline 1 & & 1 & \\ \hline 2 & & \frac{1}{2} & \\ \hline \frac{1}{4} - b_4 c_4 / b_3 & \left(\frac{1}{4} - b_4 c_4\right) / b_3 - a_{32} & \left(\frac{1}{4} - b_4 c_4\right) / b_3 - a_{31} & \\ \hline \frac{1}{4} - b_3 c_3 / b_4 & \left(\frac{1}{4} - b_3 c_3\right) / b_4 - a_{42} - a_{43} & \left(\frac{1}{4} - b_3 c_3\right) / b_4 - a_{41} - a_{43} & \left(\frac{1}{4} - b_3 c_3\right) / b_4 - a_{41} - a_{42} \\ \hline & \left(\frac{1}{2} - b_3 - b_4\right) & \left(\frac{1}{2}\right) & \left(\frac{1}{2} - b_1 - b_4\right) & \left(\frac{1}{2} - b_1 - b_3\right) \\ \hline \end{array}$$

(3d) Eqns. derivations

$$\frac{ds}{dt} = \beta \bar{s} / n$$

$$\frac{dI}{dt} = \beta \bar{s} / n - \gamma I - \mu I$$

$$\frac{dR}{dt} = \gamma I$$

$$\frac{dD}{dt} = \mu I$$

Given data

$$n = 5000, 000$$

$$\beta = 0.4$$

$$\gamma = 0.10 / 0.5 = 2.4$$

$$\mu = 0.2$$

See Code

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PART B

Using Cholesky decomposition method

$$\text{Let } A = LL^T \Rightarrow LL^T x = b \Rightarrow Lz = b \Rightarrow L^T z = z$$

System of linear eqn

$$4x_1 - 2x_2 + 4x_3 + 2x_4 = 30$$

$$-2x_1 + 5x_2 - 2x_3 + 3x_4 = -27$$

$$4x_1 - 2x_2 + 13x_3 + 2x_4 = 93$$

$$2x_1 + 3x_2 - 2x_3 + 15x_4 = -15$$

Sum

In matrix form

$$\begin{array}{c|c|c} A & x & b \\ \hline \begin{pmatrix} 4 & -2 & 4 & 2 \\ -2 & 5 & -2 & 3 \\ 4 & -2 & 13 & -1 \\ 2 & 3 & -1 & 15 \end{pmatrix} & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} & \begin{pmatrix} 30 \\ -27 \\ 93 \\ -15 \end{pmatrix} \end{array}$$

$$\begin{array}{c|c|c} L & L^T & \\ \hline \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} & \begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & 0 & L_{44} \end{pmatrix} & \begin{pmatrix} A \end{pmatrix} \end{array}$$

$$\Rightarrow \begin{pmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} & L_{11}L_{41} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} + L_{21}L_{41} + L_{22}L_{42} \\ L_{11}L_{31} & L_{21}L_{31} + L_{32}L_{22} & L_{31}^2 + L_{32}^2 + L_{33}^2 & L_{31}L_{41} + L_{32}L_{42} + L_{33}L_{43} \\ L_{11}L_{41} & L_{21}L_{41} + L_{42}L_{22} & L_{31}L_{41} + L_{43}L_{33} & L_{41}^2 + L_{42}^2 + L_{43}^2 + L_{44}^2 \end{pmatrix}$$

Equating (*) to A and solving for L and L^T we have

$$\begin{array}{l} L_{11}^2 = 4 \Rightarrow L_{11} = 2 \\ 2L_{21} = -2 \Rightarrow L_{21} = -1 \\ 2L_{31} = 4 \Rightarrow L_{31} = 2 \end{array} \quad \left| \begin{array}{l} 2L_{41} = 2 \Rightarrow L_{41} = 1 \\ 1 + L_{21}^2 = 5 \Rightarrow L_{22}^2 = 4 \Rightarrow L_{22} = 2 \end{array} \right.$$

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$$\Rightarrow -1 \cdot 1 + 2 \cdot L_{42} = 3$$

$$2L_{42} = 4 \Rightarrow L_{42} = 2$$

$$\Rightarrow 1 + 4 + 1 + L_{44}^2 = 15$$

$$\Rightarrow L_{44}^2 = 15 - 6 = 9$$

$$\Rightarrow 4 + 0 + L_{33}^2 = 13 \Rightarrow L_{33} = 3$$

$$\Rightarrow L_{44} = 3$$

$$\Rightarrow 2 \cdot 1 + 0 + 3L_{43} = -1$$

$$2 + 3L_{43} = -1 \Rightarrow L_{43} = 1$$

$$\Rightarrow L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix}$$

$$L^T = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\Rightarrow L L^T x = b$$

$$\Rightarrow L z = b \quad \left(\begin{array}{cccc|c} 2 & 0 & 0 & 0 & z_1 \\ -1 & 2 & 0 & 0 & z_2 \\ 2 & 0 & 3 & 0 & z_3 \\ 1 & 2 & -1 & 3 & z_4 \end{array} \right) \quad \left(\begin{array}{c} 30 \\ -27 \\ 93 \\ -18 \end{array} \right)$$

$$\Rightarrow 2z_1 = 30 \Rightarrow z_1 = 15$$

$$\Rightarrow -z_1 + z_2 = -27 \Rightarrow -15 + 2z_2 = -27 \Rightarrow z_2 = -6$$

$$\Rightarrow 2z_1 + 3z_3 = 93 \Rightarrow 3z_3 = 93 - 30 \Rightarrow z_3 = 21$$

$$\Rightarrow z_1 + 2z_2 - z_3 + 3z_4 = -18 \Rightarrow 15 - 12 - 21 + 3z_4 = -18 \Rightarrow z_4 = 0$$

$$\Rightarrow z = \begin{pmatrix} 15 \\ -6 \\ 21 \\ 0 \end{pmatrix}$$

$$\Rightarrow L^T z = b \Rightarrow \left(\begin{array}{cccc|c} 2 & -1 & 2 & 1 & z_1 \\ 0 & 2 & 0 & 2 & z_2 \\ 0 & 0 & 3 & -1 & z_3 \\ 0 & 0 & 0 & 3 & z_4 \end{array} \right) \quad \left(\begin{array}{c} 30 \\ -27 \\ 93 \\ -18 \end{array} \right)$$

- QUESTIONS ORDER * PART-A **** (1A)-(2A)-(2B)-(2C)-(3A)-(3D)
 • PART-B **** (4B)-(4C)-(4A)-(4C CONTINUATION)

$$3x_4 = 0 \Rightarrow x_4 = 0, \quad 3x_3 = 3 \quad \Rightarrow x_3 = 1$$

$$\Rightarrow 2x_2 + 2x_4 = 6 \Rightarrow x_2 = -3$$

$$\Rightarrow 2x_1 - x_2 - 2x_3 + x_4 = 15 \Rightarrow x_1 = -1$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

(4c) (i) Explanation

\Rightarrow Also known as triangulation or factorization is used to complete the inverse A^{-1} by first computing both the upper and lower triangular matrices given by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

\Rightarrow we then get the product of LU so as to equate that the matrix A and calculating the coefficients for matrices L and U which we then obtain matrix A^{-1}

$$(ii) \quad L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

$$\text{let } U_{11} = 1$$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{pmatrix} \times \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

- QUESTIONS ORDER * PART-A **** (1A)-(2A)-(2B)-(2C)-(3A)-(3D)
 • PART-B **** (4B)-(4C)-(4A)-(4C CONTINUATION)

$$\Rightarrow u_{11}=4, u_{12}=-2, u_{13}=4, u_{14}=2$$

$$\Rightarrow 4L_2 = -2 \Rightarrow L_{21} = -\frac{1}{2}, -\frac{1}{2} \cdot 2 + u_{22} = 3 \Rightarrow u_{22} = 4$$

$$\Rightarrow 4L_{31} = 4 \Rightarrow L_{31} = 1, 1 - 2 + 4L_{32} = -2, 4L_{32} = -2 + 2, L_{32} = 0$$

$$\Rightarrow 4 + u_{33} = 13 \Rightarrow u_{33} = 9, 4L_{41} = 2 \Rightarrow L_{41} = \frac{1}{2}$$

$$\Rightarrow 2 + 9L_{43} = 11, L_{43} = -\frac{1}{3}, 4L_{42} = 4, L_{42} = 1$$

$$\Rightarrow 1 + 4 + 3 + u_{44} = 15 \Rightarrow u_{44} = 9$$

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad U = \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$Lz = b$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix} \Rightarrow z_1 = 30$$

$$\Rightarrow \frac{1}{2}z_1 + z_2 = -27, z_2 = -12$$

$$\Rightarrow z_1 + z_2 = 93, z_3 = 63$$

$$\Rightarrow z_4 + \frac{1}{2}z_1 - \frac{1}{3}z_2 = -12 \Rightarrow z_4 = 0$$

$$\Rightarrow z = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$$Ux = b \Rightarrow \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_4 = 0$$

$$9x_3 = 63 \Rightarrow x_3 = 7$$

$$4x_2 = -12 \Rightarrow x_2 = -3$$

$$4x_1 + 6x_2 = 30 \Rightarrow x_1 = -1$$

$$z = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

- QUESTIONS ORDER *
- PART-A **** (1A)-(2A)-(2B)-(2C)-(3A)-(3D)
 - PART-B **** (4B)-(4C)-(4A)-(4C CONTINUATION)

(iii) Why is not the method efficient?
 \Rightarrow This is because LU decomposition method FAILS
any of the pivot elements is equal to zero

i.e., if $L_{ii} = 0$ and $U_{ii} = 0$ or if
 $U_{ii} = 1$ and $L_{ii} = 0$

A better efficient method to solve the above system would be to use the Jacobi Iterative method or the Gauss-Siedel approximation Iterative method.

$$(1a) \quad \begin{array}{c|cc} \frac{3}{4} & \frac{3}{4} & 0 \\ 1 & \frac{1}{4} & \frac{3}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \Rightarrow \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

$$\Rightarrow c_1 = \frac{3}{4}, \quad c_2 = 1, \quad a_{11} = \frac{3}{4}, \quad a_{12} = 0 \\ a_{21} = \frac{1}{4}, \quad a_{22} = \frac{3}{4}, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{2}$$

\Rightarrow For an A-stable function above we have that it.

$$C = \{z \in \mathbb{C}, \operatorname{Re}(z) \leq 0\}$$

\Rightarrow So the above condition is met iff

$$\operatorname{Re}(iz) \leq 1 \quad \forall z \in C$$

$R(z)$ is analytic for $\operatorname{Re}(z) < 0$

\Rightarrow This is the stability function for the above Butcher tableau

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{3}{16}z^3 + \frac{1}{48}z^4$$

Note that all non-zero entries are above the diagonal.

- QUESTIONS ORDER * PART-A **** (1A)-(2A)-(2B)-(2C)-(3A)-(3D)
 • PART-B **** (4B)-(4C)-(4A)-(4C CONTINUATION)

$$\text{Let } A = \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad \text{and } B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

~~Therefore $P(z) = \det(A - zB)$~~ for polynomial
 $E(y_j) = D(y_j)B(-y_j) - N(y_j)N(-y_j)$

For simplicity we follow the below conditions
 if A

(i) All roots $D(z)$ are in the right hand side

(ii) $E(y_j) \geq 0$ for all $y_j \neq 0$

Then the method is $A - (Au)_+$,

$$P(z) = \det \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right]$$

$$\det \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right]$$

$$\Rightarrow P(z) = \det \frac{\begin{pmatrix} \frac{1}{4} + \frac{1}{2}z & 0 \\ -\frac{1}{4} & \frac{1}{4} + \frac{1}{2}z \end{pmatrix}}{\begin{pmatrix} 1 - \frac{3}{4}z & 0 \\ -\frac{1}{4}z & 1 - \frac{3}{4}z \end{pmatrix}} = \frac{\left(\frac{1}{4} + \frac{1}{2}z\right)^2 + \frac{1}{4}}{(1 - \frac{3}{4}z)^2 + \frac{1}{4}z}$$

$$\Rightarrow N(z) = \frac{1}{4} + \left(\frac{1}{4} + \frac{1}{2}z\right)^2 = \frac{1}{4}z^2 + \frac{1}{4}z + \frac{5}{16}$$

$$D(z) = \frac{1}{4}z + (1 - \frac{3}{4}z)^2 = \frac{1}{16}z^2 - \frac{11}{16}z + 1$$

$$\text{Let } a = \frac{9}{16}, b = -\frac{11}{4}, c = 1$$

$$z = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4c}{4a^2}} \Rightarrow z_1 = 4.493 \\ z_2 = 0.3956$$

- QUESTIONS ORDER * PART-A **** (1A)-(2A)-(2B)-(2C)-(3A)-(3D)
 • PART-B **** (4B)-(4C)-(4A)-(4C CONTINUATION)

\Rightarrow Thus the root of $D(z)$ one all in the R.H.s if
 first condition is met / satisfied

$$\Rightarrow E(y) = D(iy) D(-iy) - N(iy) N(-iy)$$

$$\Rightarrow \left(1 - \frac{11}{4}iy\right)$$

$$= \left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right) \left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right)$$

$$= -\left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right) \left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right)$$

$$= \frac{35}{36} + \frac{1}{4}iy - \frac{1}{4}y^2 - \frac{55}{36}iy - \frac{11}{18}y^2 - \frac{15}{32}y^2 - \frac{9}{64}y^3 - \frac{9}{64}y^4 \geq 0$$

\Rightarrow Thus since $E(y) \geq 0$ and root $D(z)$ one in the R.H.s plane then we can conclude that the method is A-stable.

(4c(ii))

Continuation

Having determined L and U of the system $A\vec{x} = \vec{b}$

$$\Rightarrow L\vec{U}\vec{x} = \vec{b}$$

$$\Rightarrow \text{Simplifying into } L\vec{z} = \vec{b}$$

$$\text{whereby } \vec{z} = L^{-1}\vec{b} \quad \text{or} \quad \vec{z} = U\vec{x}$$

\Rightarrow The \vec{z} -values are obtained by inverse L^{-1} of the matrix A^{-1} multiplied by the \vec{b} -values

$$\Rightarrow L^{-1}U^{-1} = A^{-1}$$

