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$$c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{—— } (*)$$

B.C $\quad u(0,t) = 0$

$\qquad\quad u(L_0,t) = f(t)$

I.C $\quad u(x,0) = 0$

$$\frac{\partial u(x,0)}{\partial t} = 0$$

Soln

=> Note that wave equations are parabolic types of D.Es

=> So, we first need to obtain the Laplace transform with respect to 't' as follows

$$L\{u_{tt}(x,t)\} = s^2 u(x,s) - s\,u(x,0) - \frac{\partial u}{\partial t}(x,0) \quad \text{— (1)}$$

=> Taking the laplace transform for eqn (*) we obtain

$$L\left\{c^2 \frac{\partial^2 u(x,t)}{\partial x^2}\right\} = L\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\}$$

$$\Rightarrow \quad c^2 \frac{\partial^2 u(x,s)}{\partial x^2} = L\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} = L\{u_{tt}(x,t)\} \quad \text{— (2)}$$

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=> Substituting (2) into (1) we have

$$\Rightarrow c^2 \frac{\partial^2 U(x,s)}{\partial x^2} = s^2 U(x,s) - sU(x,0) - \frac{\partial U}{\partial t}(x,0)$$

=> Applying the Initial condition we get

$\Rightarrow U(x,0) = 0$

$\Rightarrow \frac{\partial U}{\partial t}(x,0) = 0$

$$\Rightarrow c^2 \frac{\partial^2 U(x,s)}{\partial x^2} = s^2 U(x,s) \longleftarrow (3)$$

=> We need to transform the B.C.s as well

$\Rightarrow \bar{U}(0,s) = 0$

$\Rightarrow \bar{U}(L,s) = f(s)$

=> We now solve eqn (3) as follows.

=> It is a 2nd order ode so we can solve it using the characteristic approach

$$\Rightarrow c^2 \frac{d^2 U}{dx^2} = s^2 U$$

$$\Rightarrow \frac{d^2 U}{dx^2} = \frac{s^2}{c^2} U$$

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$\Rightarrow \lambda^2 = \frac{s^2}{c^2} \Rightarrow \lambda = \pm \frac{s}{c}$

$\Rightarrow U(x,s) = c_1 e^{\frac{s}{c}x} + c_2 e^{-\frac{s}{c}x}$

$\Rightarrow$ Applying the B.Cs, we have that

$U(0,s) = 0$

$\Rightarrow$ This implies that we have

$\bar{U} \longrightarrow 0$ as $x \longrightarrow \infty$

$\Rightarrow$ Thus we have $c_1 = 0$

$\Rightarrow U(x,s) = c_2 e^{-\frac{s}{c}x}$

$\Rightarrow$ From 2nd B.C. we have

$\Rightarrow \bar{F}(s) = \bar{U}(L_0, s) = c_2$

$\Rightarrow U(x,s) = F(s) e^{-\frac{s}{c}x}$ ——— (4)

$\Rightarrow$ Next, we take the laplace inverse transform of eqn (4) to obtain $U(x,t)$

$\Rightarrow$ Note that

$$L(f(t-a)H(t-a) = \int_{\infty}^{-1} e^{-as} L(f(t))$$

$\Rightarrow$ We obtain the following

$\Rightarrow U(x,t) = L^{-1}\left(e^{-\frac{x}{c}s} L(f(t))\right)$

$\Rightarrow \quad = f(t-\frac{x}{c}).H(t-\frac{x}{c}) = L^{-1}\left(e^{-\frac{x}{c}s} L(f(t))\right)$

$\Rightarrow$ The soln $U(x,t)$ becomes

$$U(x,t) = \begin{cases} f(t-\frac{x}{c}) & \text{with } x < tc \\ 0 & \text{with } t < \frac{x}{c} \end{cases}$$

THANK YOU!!!