PAGE NO.:
DATE: / /

Solve the homogeneous heat equation for $L=2$, $k=10$ and dirichlet homogeneous BC and IC given by
$f(x) = 5 + 5\sin^2(\pi x/2)$

Soln

We let the heat equation with the given conditions be given as

$$\frac{\partial u}{\partial t} = 10\frac{\partial^2 u}{\partial x^2} \quad \text{with } k = 10 > 0 \qquad \text{———} (*)$$

for $u(0,t) = 0$

$u(2,t) = 0$

and

$u(x,0) = f(x) = 5 + 5\sin^2(\pi x/2)$

We will be using the method of separation of variables to solve the above eqn

$\Rightarrow$ So we let

$$u(x,t) = X(x)T(t)$$

$\Rightarrow$ Differentiating the above w.r.t x and t respectively

$$u_t(x,t) = X(x)T'(t) \quad \text{and} \quad u_{xx}(x,t) = X''(x)T(t)$$

$\Rightarrow$ Substituting into $(*)$ we have

$XT' = 10XT''$

$\Rightarrow$ dividing all by $U(x,t) = XT$ we have

$$\frac{T'}{10T} = \frac{X''}{X} = -\lambda$$

$\Rightarrow$ Separating the equations into 2 independent equations

$$\frac{T'}{10T} = -\lambda \quad \text{—— (1)} \quad \text{and}$$

$$\frac{X''}{X} = -\lambda \quad \text{—— (2)}$$

$\Rightarrow$ Solving (1) we have

$$\frac{dT}{10T\,dt} = \frac{T'}{10T} = -\lambda \quad \Rightarrow \quad \frac{dT}{T} = -10\lambda\,dt$$

Upon integration

$$\ln T = -10\lambda t + c$$

$$\Rightarrow \quad T(t) = e^{-10\lambda t + c} = e^{-10\lambda t} \cdot e^{c}$$

let $e^{c} = A$

$$\Rightarrow \quad T(t) = Ae^{-10\lambda t}$$

and Solving (2) we have

PAGE NO.:
DATE: / /

$$\frac{X''}{X} = -\lambda \quad \Rightarrow \quad X'' + \lambda X = 0$$

$\Rightarrow$ This is a 2^{nd} order ode equation, we solve by characteristic method

$$a^2 + \lambda = 0 \quad \Rightarrow \quad a^2 = -\lambda \quad \Rightarrow \quad a = \pm\sqrt{-\lambda}$$

$$X(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$

$\Rightarrow$ Now that we have the 2 solutions to $T(t)$ and $X(x)$, next we note of the hypothesis that for $\lambda \geq 0$, we will re-write the $X(x)$ solution as

$$X(x) = \bar{c}_1 \sin\sqrt{\lambda}x + \bar{c}_2 \cos\sqrt{\lambda}x$$

$\Rightarrow$ Thus A, c_1, c_2, $\bar{c}_2$ and $\bar{c}_1$ are what is called constants of integration

Such that $\lim_{t \to \infty} \phi(t) = 0$ and not $+\infty$

$\Rightarrow$ Hence we have the solution $U(x,t)$ given as

$$U(x,t) = X(x)\,T(t)$$

$$U(x,t) = A e^{-10\lambda t}\left[\bar{c}_1 \sin\sqrt{\lambda}x + \bar{c}_2 \cos\sqrt{\lambda}x\right]$$

=> Thus we have

$$U(x,t) = e^{-10\lambda t}\left[B\sin\sqrt{\lambda}\,x + D\cos\sqrt{\lambda}\,x\right]$$

with constants of integration being B and D.

=> Next, since we have the solution $U(x,t)$, we now apply the to Boundary conditions,

at $x=0$

$$U(0,t) = e^{-10\lambda t}\left[0 + D.1\right]$$

$$= e^{-10\lambda t}(D) = D e^{-10\lambda t} = 0$$

$$\Rightarrow D = 0$$

similarly

for $U(a,t) = 0$ at $x = a$

$$\Rightarrow U(a,t) = e^{-10\lambda t}\left[B\sin\sqrt{\lambda}.a + D\cos\sqrt{\lambda}.a\right] = 0$$

=> with $D=0$, we simply the above further into

$$e^{-10\lambda t}\left[B\sin\sqrt{\lambda}.a\right] = 0$$

=> Thus, for a non-trivial solution

PAGE NO.:
DATE: / /

$\sin\sqrt{\lambda}.2 = 0$

=> We need to find values for which their sine function equals zero

=> $\sqrt{\lambda}.2 = n\pi$ for $n = 1, 2, 3, ----$

=> Taking square's for both sides we have

$$4\lambda = n^2\pi^2 \Rightarrow \lambda = \frac{n^2\pi^2}{4}$$

=> $\lambda_n = n^2\pi^2/4$ for $n = 1, 2, 3, ----$

=> We now substitute this solution (λ_n) into $U(x,t)$ solution

=> Remember that $D = 0$ and $\lambda_n = n^2\pi^2/4$, $n = 1, 2, --$

=> Thus $u(x,t)$ becomes

$$u(x,t) = e^{-10\lambda t}\left[B \sin\sqrt{\lambda}\, x\right]$$

$$\Rightarrow u(x,t) = B e^{-10\lambda t} \sin\sqrt{\lambda}\, x$$

=> For n number of values $n = 1, 2, 3, ---$

$$\Rightarrow u_n(x,t) = B_n \sin\left(\frac{n^2\pi^2}{4}\right)^{1/2} e^{-10\left(\frac{n^2\pi^2}{4}\right)t}$$

PAGE NO.:
DATE: / /

$$\Rightarrow U_n(x,t) = B_n \sin \frac{n\pi}{2}x \cdot e^{-\frac{10n^2\pi^2 t}{4}}$$

$$\Rightarrow B_n \sin \frac{n\pi}{2}x \cdot e^{-\frac{5}{2}n^2\pi^2 t}$$

By the principle of superposition we have

$$U(x,t) = U_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{2}x \cdot e^{-\frac{5}{2}n^2\pi^2 t}$$

$\Rightarrow$ We now apply the IC to the above equation.

$\Rightarrow$ But what is B_n

$$B_n = \frac{2}{2}\int_0^2 f(x) \sin \frac{n\pi x}{2}\, dx$$

with $f(x) = 5 + 5\sin^2\left(\frac{\pi x}{2}\right)$

$$\Rightarrow B_n = \int_0^2 \left[5 + 5\sin^2\left(\frac{\pi x}{2}\right)\right] \sin \frac{n\pi x}{2}\, dx$$

$$\Rightarrow \int_0^2 5 \sin \frac{n\pi x}{2}\, dx + \int_0^2 5\sin^2\left(\frac{\pi x}{2}\right) \sin\left(\frac{n\pi x}{2}\right) dx.$$

$$-\frac{5 \cdot 2}{n\pi} \cos \frac{n\pi x}{2}\Big|_0^2 = \frac{10}{n\pi}\left[(-1) - 1\right] = \left(\frac{-20}{n\pi}\right) - 1$$

$$\Rightarrow \int_0^2 5\sin^2\left(\frac{\pi x}{2}\right) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$\Rightarrow \sin^2 x = \frac{1}{2}\left[1 - \cos 2x\right]$$

$$\Rightarrow \sin^2\left(\frac{\pi x}{2}\right) = \frac{1}{2}\left[1 - \cos 2 \cdot \frac{\pi x}{2}\right]$$

$$= \frac{1}{2}\left[1 - \cos \pi x\right]$$

$$\Rightarrow \frac{5}{2}\int_0^2 (1 - \cos \pi x) \sin \frac{n\pi x}{2}\, dx$$

$$\Rightarrow \frac{5}{2}\left[\int_0^2 \sin \frac{n\pi x}{2}\, dx - \int_0^2 \cos \pi x \sin \frac{n\pi x}{2}\, dx\right]$$

$$\Rightarrow -\frac{5}{2}\left[\frac{2}{n\pi}\cos \frac{n\pi x}{2}\Big|_0^2\right] - \frac{5}{2}\int_0^2 \cos \pi x \sin \frac{n\pi x}{2}\, dx$$

$$\Rightarrow -\frac{5}{2}\left[\frac{2}{n\pi}\left[(-1) - 1\right]\right]$$

$$= -\left(\frac{-10}{n\pi}\right) - \frac{5}{2}\int_0^2 \cos \pi x \sin \frac{n\pi x}{2}\, dx$$

$$\Rightarrow \cos \pi x \sin \frac{n\pi x}{2} = \frac{1}{2}\left[\sin \pi\left(1 + \frac{n}{2}\right)x - \sin \pi\left(1 - \frac{n}{2}\right)x\right]$$

$$\Rightarrow \frac{5}{2}\left[\int_0^2 \sin \pi\left(1 + \frac{n}{2}\right)x - \int_0^2 \sin \pi\left(1 - \frac{n}{2}\right)x\right]$$

PAGE NO.:
DATE: / /

$$\Rightarrow \frac{5}{2}\left[\frac{-2}{(2+n)\pi}\cos\pi\left(1+\frac{n}{2}\right)x\Big|_0^2 + \frac{2}{(2-n)\pi}\cos\pi\left(1-\frac{n}{2}\right)x\Big|_0^2\right]$$

$$\Rightarrow \frac{5}{2}\left[\frac{-2}{(2+n)\pi}(-2) - \frac{2}{(2-n)\pi}(-2)\right]$$

$$\Rightarrow \frac{-10}{(2+n)\pi} - \frac{10}{(2-n)\pi} = \frac{-10}{\pi}\left(\frac{1}{2+n} + \frac{1}{2-n}\right)$$

$$\Rightarrow \frac{-10}{n\pi} - \frac{10}{\pi}\left(\frac{4}{4-n^2}\right) = \frac{-10}{\pi}\left(\frac{1}{n} + \frac{4}{4-n^2}\right)$$

$\Rightarrow$

$$B_n = +\frac{20}{n\pi} + \frac{10}{\pi}\left(\frac{1}{n} \cancel{-} \frac{4}{4-n^2}\right) = \frac{20}{n\pi} + \frac{10}{\pi}\left(\frac{1}{n} - \frac{4}{4-n^2}\right)$$

Particular solution

$$U(x,t) = \sum_{n=1}^{\infty}\left[\frac{+20}{n\pi} + \frac{10}{\pi}\left(\frac{1}{n} \cancel{+} \frac{4}{4-n^2}\right)\right] \sin\frac{n\pi}{2}x \cdot e^{-\frac{5}{2}n^2\pi^2 t}$$

[X.]

Solve homogeneous heat equation for
$L=4$, $k=6$ and
Neuman Constant BC. and IC
$\frac{du}{dx}(0,t) = -1$ and
$\frac{du}{dx}(L,t) = 4$

with $f(x) = 20 - 5(x-2)^2 = u(x,0)$

Soln

We need to write the heat equation with
$L=4$ and $k=6$ as

$$\frac{\partial u}{\partial t} = 6\frac{\partial^2 u}{\partial x^2} \quad \text{for } k=6>0 \quad \text{—}(*)$$

$\Rightarrow$ let $u(x,t) = X_{(x)}T_{(t)}$

$\Rightarrow$ $u_t(x,t) = X_{(x)}T'_{(t)}$ and $u_{xx}(x,t) = X''_{(x)}T_{(t)}$

=> Substitute these into (*), we have

$$XT' = 6X''T \quad , \text{ dividing by } X_{(x)}T_{(t)}$$

$$\frac{T'}{6T} = \frac{X''}{X} = -\lambda \quad , \text{ we normally set it to a constant}$$

=> Separating the variables, we have

$$\frac{T'}{6T} = -\lambda \quad — (1) \qquad \text{and} \qquad \frac{X''}{X} = -\lambda \quad — (2)$$

=> Solving for (1) and (2), we get

$$\frac{T'}{6T} = \frac{dT}{6T\,dt} = -\lambda \Rightarrow \frac{dT}{T} = -6\lambda\,dt$$

$$\Rightarrow \ln T = -6\lambda t + c.$$

=> Taking exponential on both sides

$$T(t) = e^{-6\lambda t + c} = e^{-6\lambda t} \cdot e^{c}$$

$$\text{let } e^{c} = A$$

$$\Rightarrow T(t) = Ae^{-6\lambda t}$$

=> Solving for (2)

$$X''/X = -\lambda \Rightarrow X'' + \lambda X = 0$$

PAGE NO.:
DATE: / /

$\Rightarrow$ This is a 2^{nd} order ode, we solve by characteristic approach

$$a^2 + \lambda = 0 \qquad a^2 = -\lambda \quad \Rightarrow \quad a = \pm\sqrt{-\lambda}$$

$$\Rightarrow \quad X_{(x)} = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$\Rightarrow$ By the hypothesis $\lambda \geq 0$, so that

$\lim_{t \to \infty} \phi(t) = 0$ and not $+\infty$, we have

$$X(x) = \bar{c}_1 \sin\sqrt{\lambda}\, x + \bar{c}_2 \cos\sqrt{\lambda}\, x$$

$\Rightarrow$ $U(x,t)$ is given as

$$U(x,t) = X_{(x)} T_{(t)} = A e^{-6\lambda t}\left[\bar{c}_1 \sin\sqrt{\lambda}x + \bar{c}_2 \cos\sqrt{\lambda}\, x\right]$$

$$\Rightarrow \quad B = A\bar{c}_1 \quad \text{and} \quad D = A\bar{c}_2$$

$$\Rightarrow U(x,t) = e^{-6\lambda t}\left[B \sin\sqrt{\lambda}\, x + D \cos\sqrt{\lambda}\, x\right] \quad \text{—— (3)}$$

$\Rightarrow$ We now apply the B.C,

$$\frac{dU(0,t)}{dx} = -1 \quad \text{and} \quad \frac{du}{dx}(4,t) = 4$$

$\Rightarrow$ We differentiate equation (3) w.r.t x so we can apply those B.Cs

$$U_x(x,t) = e^{-6\lambda t}\left[\sqrt{\lambda}\, B \cos\sqrt{\lambda}\, x - \sqrt{\lambda}\, D \sin\sqrt{\lambda}\, x\right]$$

applying B.C, we get

$$U_x(0,t) = e^{-6\lambda t}\left[B\sqrt{\lambda}.1 + 0\right] = -1$$

$$\Rightarrow e^{-6\lambda t}. B\sqrt{\lambda} = -1$$

$$\Rightarrow B = e^{6\lambda t}/\sqrt{\lambda}$$

$$\Rightarrow U_x(4,t) = e^{-6\lambda t}\left[B\sqrt{\lambda}\cos\sqrt{\lambda}.4 - \sqrt{\lambda}.D\sin\sqrt{\lambda}.4\right] =$$

$$\Rightarrow \text{For } B = e^{6\lambda t}/\sqrt{\lambda}$$

$$\Rightarrow e^{-6\lambda t}\left[\frac{e^{6\lambda t}}{\sqrt{\lambda}}.\sqrt{\lambda}\cos\sqrt{\lambda}.4 - \sqrt{\lambda}.D\sin\sqrt{\lambda}.4\right] = 4$$

$$\Rightarrow \cos\sqrt{\lambda}.4 - \sqrt{\lambda}.D\sin\sqrt{\lambda}.4 = 4$$

$$\Rightarrow \sqrt{\lambda}.D\sin\sqrt{\lambda}.4 = \cos\sqrt{\lambda}.4 - 4$$

$$D = \frac{\cos\sqrt{\lambda}.4 - 4}{\sqrt{\lambda}\sin\sqrt{\lambda}.4}$$

$$\Rightarrow \frac{1}{\sqrt{\lambda}}\left[\cot\sqrt{\lambda}.4 - \frac{4}{\sin\sqrt{\lambda}.4}\right]$$

$$U(x,t) = e^{-6\lambda t}\left[\frac{e^{6\lambda t}}{\sqrt{\lambda}}.\sqrt{\lambda}\sin\sqrt{\lambda}x + \left(\frac{\cos\sqrt{\lambda}.4 - 4}{\sqrt{\lambda}\sin\sqrt{\lambda}.4}\right)\cos\sqrt{\lambda}x\right]$$

=> By hypothesis $\lambda \geq 0$ for $\lim_{x \to \infty} \phi(x) = 0$ not to

$$U(x,t) = U_{non-homogenous\ B.C}(x,t) + S(x)$$

=> Now we clearly note that there will be infinitely many solutions for $\sqrt{\lambda}.x$ and resulting $U(x,t)$, so we choose the one which leads to the easiest problem solution for $U(x,t)$

=> $\sqrt{\lambda}.x = n\pi$ for $n = 1, 2, 3 ...$

$$\lambda_n = n^2\pi^2/16 \quad \text{for } n = 1, 2, 3, ...$$

=> For $U(x,t)$ derivative function the U (non-homogenous B.C) function we can choose to approximate the easiest problem of the superposition principle to be given as

$$U_{non-homogenous}(x,t) = \sum_{n=1}^{N} B_n \cos\left(\frac{n\pi x}{4}\right) e^{-\frac{n^2\pi^2}{k16}t} + B_0/2 \text{ (fourier series constant)}$$

=> now we calculate

B_0 and B_n

$$\Rightarrow B_0 = \frac{2}{4}\int_0^4 f(x) = \frac{1}{2}\int_0^4 20 - 5(x-2)^2\, dx.$$

PAGE NO.:
DATE: / /

$$= \frac{1}{2}\left[\int_0^4 20\,dx - 5\int_0^4 (x-2)^2\,dx\right]$$

$$= \frac{1}{2}\left[20x\Big|_1^4 - 5\int_0^4 (x-2)^2\,dx\right]$$

let $u = x-2$ $du = dx$.

$$= 5\int_0^4 u^2\,du = \frac{u^3}{3}\Big|_0^4$$

$$\Rightarrow \frac{5}{3}(x-2)^3\Big|_0^4 = \frac{1}{2}\left[(60) - \frac{80}{3}\right] = \frac{50}{3}$$

$$B_n = \frac{1}{2}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\int_0^4 \left[20 - 5(x-2)^2\right]\cos\frac{n\pi x}{4}\,dx$$

$$= \frac{1}{2}\left[\int_0^4 20\cos\frac{n\pi x}{4}\,dx - 5\int_0^4 (x-2)^2\cos\frac{n\pi x}{4}\,dx\right]$$

$$= \frac{1}{2}\left[\underbrace{\frac{-20.4}{n\pi}\sin\frac{n\pi x}{4}\Big|_0^4}_{0} - 5\int_0^4 (x-2)^2\cos\frac{n\pi x}{4}\,dx\right]$$

$$= -\frac{5}{2}\int_0^4 (x-2)^2\cos\frac{n\pi x}{4}\,dx.$$

DATE: / /

u	dv
$(x-2)^2$	$\cos\frac{n\pi x}{4}$
$2(x-2)$	$\frac{4}{n\pi}\sin\frac{n\pi x}{4}$
2	$-\frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}$
0	$-\frac{64}{n^3\pi^3}\sin\frac{n\pi x}{4}$

$$= -\frac{5}{2}\left[\underbrace{\frac{4(x-2)^2}{n\pi}\sin\frac{n\pi x}{4}\Big|_0^4}_{0} + \frac{32(x-2)}{n^2\pi^2}\cos\frac{n\pi x}{4}\Big|_0^4 - \underbrace{\frac{128}{n^3\pi^3}\sin\frac{n\pi x}{4}\Big|_0^4}_{0}\right]$$

$$\Rightarrow -\frac{5}{2}\left[\frac{32(x-2)}{n^2\pi^2}\cos\frac{n\pi x}{4}\Big|_0^4\right]$$

$$= -\frac{5}{2}\left[\frac{32(x-2)}{n^2\pi^2}\cos n\pi + \frac{64}{n^2\pi^2}\right]$$

$$= -\frac{5}{2}\left[\frac{64}{n^2\pi^2}\cos n\pi + 1\right]$$

$$B_n = -\frac{160}{n^2\pi^2}\cos n\pi - \frac{5}{2}$$

$$U(x,t) = \frac{50/3}{2} + \sum_{n=1}^{\infty}\left(-\frac{160}{n^2\pi^2}\cos n\pi - \frac{5}{2}\right)\cos\frac{n\pi x}{4}\, e^{-\frac{n^2\pi^2 t}{16.6}} + 5x$$

$$= U(x,t) = \frac{25}{3} + \sum_{n=1}^{\infty}\left(-\frac{160}{n^2\pi^2}\cos n\pi - \frac{5}{2}\right)\cos\frac{n\pi x}{4}\, e^{-\frac{n^2\pi^2 t}{616}} + 5x$$

PAGE NO.:
DATE: / /

Q3 Solve the non-homogeneous heat equation for $L = 6$, $k = 12$, $q(x) = 3e^{-x}$ with time-independent, mixed homogeneous B.C and IC $f(x) = 2\sin(\pi x/6)$

Soln.

We first write the non-homogeneous heat equation with time independent source term as

$$q(x) = 3e^{-x}$$

$$\Rightarrow \frac{\partial u}{\partial t} = 12\frac{\partial^2 u}{\partial x^2} + 3e^{-x} \quad \text{with Condition}$$

$$u(0,t) = T(t)$$

$$u(6,t) = X(t)$$

$$u(x,0) = f(x) = 2\sin(\pi x/6)$$

$\Rightarrow$ Note that the source term $q(x)$ is both time and space dependent

$\Rightarrow$ Thus, to solve this problem, we first need to solve for a given said function

$\Rightarrow$ Let the function be given by $p(x,t)$ in such a way that the B.C are satisfied as below

DATE / /

$p(0,t) = T(t) \Rightarrow p(6,t) = X(t)$

$\Rightarrow$ Thus, our said function $p(x,t)$ will take a simpler form as

$$p(x,t) = T(t) + \frac{x}{6}\left(X(t) - T(t)\right)$$

$\Rightarrow$ Note that we have the function $U(x,t)$ and $p(x,t)$

$\Rightarrow$ Thus, we find its difference and set it to another function as

$$w(x,t) = U(x,t) - p(x,t)$$

$\Rightarrow$ Hence $w(x,t)$ will be the solution to the below given problem

$$\frac{\partial w(x,t)}{\partial t} = 12\frac{\partial^2 w(x,t)}{\partial x^2} + \left[3e^{-x} - \frac{\partial p(x,t)}{\partial t} + 12\frac{\partial^2 p(x,t)}{\partial x^2}\right]$$

with B.C as

$$w(0,t) = 0 \quad \text{and} \quad w(6,t) = 0$$

and IC as

$$w(x,0) = U(x,0) = f(x) - p(x,0) = 2\sin\frac{\pi x}{6} - T(0) - \frac{x}{6}\left(X(0) - T(0)\right)$$

$\Rightarrow$ We can approximate this to $f(x)$

$$\Rightarrow p(x,t) = -2\sin\frac{\pi x}{6} + T(t) + \frac{x}{6}\left(X(t) - T(t)\right) + f(x)$$

PAGE NO.:
DATE: / /

with approximated Fou Value

⇒ Now that we successfully reduced our non-homogenous heat equation, we will be using the method of eigen function expansion to find its solution

⇒ We proceed as below
⇒ First we write the homogeous part of the above reduced problem as

$$\frac{\partial S(x,t)}{\partial t} = 12\frac{\partial^2 S(x,t)}{\partial x^2} \quad \text{———} (*)$$

and B.C
$S(0,t) = 0$ and $S(6,t) = 0$

⇒ We now apply the method of separation of variables to obtain an ~~eigenfunction~~ eigenvalue function given by

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \quad \text{with condition}$$

$\phi(0) = 0$
$\phi(6) = 0$

⇒ let's solve for the eigenvalue λ_n

⇒ We let $S(x,t) = X(x)T(t)$
⇒ $S_t(x,t) = T'X$ and $S_{xx}(x,t) = X''T$

⇒ Substituting into $(*)$ we have

$T'X = 12TX''$, dividing all by $S(x,t)$

$\Rightarrow \frac{T'}{12T} = \frac{X''}{X} = -\lambda$

$\Rightarrow \frac{T'}{12T} = \frac{X''}{X} = -\lambda \Rightarrow \frac{T'}{12T} = -\lambda$ and $\frac{X''}{X} = -\lambda$

$\Rightarrow \frac{dT}{T} = -\lambda 12\, dt \Rightarrow \ln T = -12\lambda t + C$

$\Rightarrow T(t) = A e^{-12\lambda t}$

and $X'' + \lambda X = 0 \Rightarrow a^2 + \lambda = 0$

$\Rightarrow a^2 = -\lambda \Rightarrow a = \pm\sqrt{-\lambda}$

$\Rightarrow X(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$

$\Rightarrow$ But with $\lambda > 0$ hypothesis we re-write $X(x)$ as

$X(x) = \bar{c}_1 \sin\sqrt{\lambda}x + \bar{c}_2 \cos\sqrt{\lambda}x$

for the B.C , we have

$X(0) = 0 + \bar{c}_2 = 0 \Rightarrow \bar{c}_2 = 0$

$\Rightarrow X(x) = \bar{c}_1 \sin\sqrt{\lambda}x$

$X(6) = \bar{c}_1 \sin\sqrt{\lambda}.6 = 0$

$\Rightarrow$ For non-trivial solution

PAGE NO.:
DATE: / /

$c_1 \neq 0$

$\Rightarrow \sqrt{\lambda}.6 = (n\pi)^2$

$\Rightarrow \lambda.36 = n^2\pi^2$

$\Rightarrow \lambda_n = \frac{n^2\pi^2}{36}$ for $n = 1, 2, 3, ---$

$\Rightarrow$ The corresponding eigenfunction is given by

$$\phi_n(x) = \sin \frac{n\pi}{6} x$$

$\Rightarrow$ We now express the unknown solution $w(x,t)$ as a generalized Fourier series of eigenfunction with the time dependent coefficients given by

$$w(x,t) = \sum_{n=1}^{\infty} B_n(t)\, \phi_n(x)$$

$\Rightarrow$ Since we already have $\phi_n(x)$ solution, we now need to calculate the value function for $B_n(t)$ constants coefficient

$\Rightarrow$ We assume that the term-by-term differentiation is satisfied, thus substituting $w(x,t)$ value into

$$\frac{\partial w(x,t)}{\partial t} = 12 \frac{\partial^2 (x,t)}{\partial x^2} + \left[3e^{-x} - \frac{\partial p(x,t)}{\partial t} + 12 \frac{\partial^2 p(x,t)}{\partial x^2} \right] \quad (*)'$$

$\Rightarrow$ simplifying into

PAGE NO.:
DATE: / /

$$\Rightarrow \sum_{n=1}^{\omega} \frac{d B_n(t)}{dt} \phi_n(x) = 12 \sum_{n=1}^{\omega} B_n(t) \frac{d^2 \phi_n(x)}{dx^2} + 3\bar{e}^{x}$$

$$= -12 \sum_{n=1}^{\omega} B_n(t) \left(\frac{n^2\pi^2}{36}\right) \phi_n(x)\, dx + 3\bar{e}^{x}$$

$\Rightarrow$ Next, we need to expand the term $q(x)$ using generalized Fourier Series of eigenfunction

$\Rightarrow$ Thus we have,

$$3\bar{e}^{x} = \sum_{n=1}^{\omega} q_n(t) \phi_n(x) \quad , \text{where}$$

$q_n(t)$ is given by

$$q_n(t) = \frac{\int_0^6 3\bar{e}^{x} \phi_n(x)\, dx}{\int_0^6 \phi_n^2(x)\, dx}$$

$\Rightarrow$ Subtituting the above into equation $(*)'$ and identifying the constant coefficient we obtain the differentiation eqn for coefficients given by

$$\frac{d B_n(t)}{dt} + \frac{n^2\pi^2}{36} k B_n(t) = \frac{\int_0^6 3\bar{e}^{x} \phi_n(x)\, dx}{\int_0^6 \phi_n^2(x)\, dx}$$

PAGE NO.:
DATE: / /

=) We now need to obtain the IC for the d.e using

$$w(x,0) = u(x,0) - p(x,0)$$

=) Thus, we end up with

$$B_n(t) = B_n(0)\, e^{-\frac{n^2\pi^2}{36}kt} + e^{-\frac{n^2\pi^2}{36}kt} \int_0^t q_n(\tau)\, e^{\frac{n^2\pi^2}{36}k\tau}\, d\tau$$

$$\Rightarrow w(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{6}$$

=)

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{6} + T(t) - 2\sin\frac{\pi x}{6} + \frac{x}{6}\left(X(t) - T(t)\right)$$

=)

$$u(x,t) = \sum_{n=1}^{\infty}\left[B_n(0)\, e^{-\frac{n^2\pi^2}{36}kt} + e^{-\frac{n^2\pi^2}{36}kt}\int_0^t q_n(\tau)\, e^{\frac{n^2\pi^2}{36}k\tau}\, d\tau\right]\sin\frac{n\pi x}{6} + \frac{x}{6}\left(X(t) - T(t)\right) - 2\sin\frac{\pi x}{6} + T(t)$$

$\boxed{\times}$