

Solve the homogeneous heat equation for
 $L=2$, $k=10$ and dirichlet homogeneous BC
and $T_C = 0$ given by $f(x) = 5 + 5 \sin^2(\pi x/2)$

Soln

We let the heat equation with the given conditions be given as

$$\frac{\partial u}{\partial t} = 10 \frac{\partial^2 u}{\partial x^2} \quad \text{with } k=10 > 0 \quad (*)$$

$$\text{for } u(0,t) = 0$$

$$u(2,t) = 0$$

and

$$u(x,0) = f(x) = 5 + 5 \sin^2(\pi x/2)$$

We will be using the method of separation of variables to solve the above eqn

\Rightarrow So we let $u(x,t) = X(x)T(t)$

$$u(x,t) = X(x)T(t)$$

\Rightarrow Differentiating the above w.r.t x and t respectively

$$u_t(x,t) = X_{12} T'(t) \quad \text{and} \quad u_{xx}(x,t) = X''(x)T(t)$$

\Rightarrow Substituting into $(*)$ we have

$$XT' = 10XT''$$

\Rightarrow dividing all by $U(x,t) = XT$ we have,

$$\frac{T'}{10T} = \frac{X''}{X} = -\lambda$$

\Rightarrow Separating the equations into 2 independent equations

$$\frac{T'}{10T} = -\lambda \quad \text{--- (1) First and}$$

$$\frac{X''}{X} = -\lambda \quad \text{--- (2) Second}$$

\Rightarrow Solving (1) we have,

$$\frac{dT}{10T} = T' = -\lambda \Rightarrow \frac{dT}{T} = -10\lambda dt$$

Upon integration

$$\ln T = -10\lambda t + C$$

$$\Rightarrow T(t) = e^{-10\lambda t + C} = e^{-10\lambda t} \cdot e^C$$

Let $e^C = A$

$$\Rightarrow T(t) = Ae^{-10\lambda t}$$

and Solving (2) we have,

$$\frac{x''}{x} = -\lambda \Rightarrow x'' + \lambda x = 0 \quad \text{for } x \neq 0.$$

\Rightarrow This is a 2nd order ODE equation, we solve by characteristic method.

$$a^2 + \lambda = 0 \Rightarrow a^2 = -\lambda \Rightarrow a = \pm \sqrt{-\lambda}$$

$$X(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$

\Rightarrow Now that we have the 2 solutions to $T(t)$ and $X(x)$, next we note if the hypothesis that for $\lambda \neq 0$, we will re-write the $X(x)$ solution as

$$X(x) = \bar{c}_1 \sin \sqrt{\lambda} x + \bar{c}_2 \cos \sqrt{\lambda} x$$

\Rightarrow Thus A, c_1, c_2, \bar{c}_2 and \bar{c}_1 are what is called constants of integration

such that $\lim_{t \rightarrow \infty} \Phi(t) = 0$ and not $+\infty$

\Rightarrow Hence we have the solution $U(x,t)$ given as

$$U(x,t) = X(x) T(t)$$

$$U(x,t) = A e^{-\lambda t} \left[\bar{c}_1 \sin \sqrt{\lambda} x + \bar{c}_2 \cos \sqrt{\lambda} x \right]$$

\Rightarrow Thus we have

$$U(x,t) = \bar{e}^{10xt} [B \sin \sqrt{\lambda} x + D \cos \sqrt{\lambda} x]$$

with constants of integration being
B and D.

\Rightarrow Next, since we have the solution
to $U(x,t)$, we now apply the
Boundary conditions,

at $x=0$, we have

$$U(0,t) = \bar{e}^{10xt} [0 + D \cdot 1]$$

$$= \bar{e}^{10xt} (D) = D \bar{e}^{10xt} = 0$$

similarly

for $U(x,t) = 0$ at $x=2$

$$\Rightarrow U(x,t) = \bar{e}^{10xt} [B \sin \sqrt{\lambda} \cdot 2 + D \cos \sqrt{\lambda} \cdot 2] = 0$$

\Rightarrow with $D=0$, we simply the above

$$e^{-10xt} [B \sin \sqrt{\lambda} \cdot 2] = 0$$

\Rightarrow Thus, for a non-trivial solution

$$\sin \sqrt{\lambda} z = 0$$

\Rightarrow We need to find values for which their sine function equals zero

$$\Rightarrow \sqrt{\lambda} z = n\pi \quad \text{for } n=1, 2, 3, \dots$$

\Rightarrow Taking square's for both sides we have

$$4\lambda = n^2 \pi^2 \Rightarrow \lambda = \frac{n^2 \pi^2}{4}$$

$$\Rightarrow \lambda_n = \frac{n^2 \pi^2}{4} \quad \text{for } n=1, 2, 3, \dots$$

\Rightarrow We now substitute this solution (λ_n) into $U(x,t)$ solution

$$\Rightarrow \text{Remember that } D=0 \quad \text{and } \lambda_n = \frac{n^2 \pi^2}{4}, n=1, 2, \dots$$

\Rightarrow Thus $U(x,t)$ becomes

$$U(x,t) = e^{i\omega t} [B \sin \sqrt{\lambda} x]$$

$$\Rightarrow U(x,t) = B e^{i\omega t} \sin \sqrt{\lambda} x$$

\Rightarrow For n number of values $n=1, 2, 3, \dots$

$$\Rightarrow U_n(x,t) = B_n \sin \left(\frac{n^2 \pi^2}{4} \right) x e^{-i\omega \left(\frac{n^2 \pi^2}{4} \right) t}$$

$$\Rightarrow u_n(x,t) = B_n \sin \frac{n\pi}{a} x \cdot e^{-\frac{10n^2\pi^2 t}{4}}$$

$$\Rightarrow B_n \sin \frac{n\pi}{a} x \cdot e^{-\frac{5n^2\pi^2 t}{2}}$$

By the principle of superposition we have

$$u(x,t) = u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a} x \cdot e^{-\frac{5n^2\pi^2 t}{2}}$$

\Rightarrow Now apply the IC to the above equation.

\Rightarrow But what is B_n

$$B_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

$$\text{with } f(x) = 5 + 5 \sin^2 \left(\frac{\pi x}{a} \right)$$

$$\Rightarrow B_n = \int_0^a \left[5 + 5 \sin^2 \left(\frac{\pi x}{a} \right) \right] \sin \frac{n\pi x}{a} dx$$

$$\Rightarrow \int_0^a 5 \sin \frac{n\pi x}{a} dx + \int_0^a 5 \sin^2 \left(\frac{\pi x}{a} \right) \sin \left(\frac{n\pi x}{a} \right) dx$$

$$-\frac{5}{n\pi} \left. \cos \frac{n\pi x}{a} \right|_0^a = \frac{10}{n\pi} [(-1)^n - 1] = \left(\frac{-20}{n\pi} \right) - 1$$

$$\Rightarrow \int_0^a 5 \sin^2 \left(\frac{\pi x}{a} \right) \sin \left(\frac{n\pi x}{a} \right) dx$$

$$\Rightarrow \sin^2 x = \frac{1}{2} [1 - \cos 2x]$$

$$\Rightarrow \sin^2 \left(\frac{\pi x}{2}\right) = \frac{1}{2} \left[1 - \cos 2 \cdot \frac{\pi x}{2}\right]$$

$$= \frac{1}{2} [1 - \cos \pi x]$$

$$\Rightarrow \frac{5}{2} \int_0^2 (1 - \cos \pi x) \sin \frac{n\pi x}{2} dx$$

$$\Rightarrow \frac{5}{2} \left[\int_0^2 \sin n\pi x dx - \int_0^2 \cos \pi x \sin \frac{n\pi x}{2} dx \right]$$

$$\Rightarrow -\frac{5}{2} \left[\frac{2}{n\pi} \cos n\pi x \Big|_0^2 \right] - \frac{5}{2} \int_0^2 \cos \pi x \sin \frac{n\pi x}{2} dx$$

$$\Rightarrow -\frac{5}{2} \left[\frac{2}{n\pi} [(-1)^n - 1] \right]$$

$$= -\left(\frac{10}{n\pi}\right) - \frac{5}{2} \int_0^2 \cos \pi x \sin \frac{n\pi x}{2} dx$$

$$\Rightarrow \cos \pi x \sin \frac{n\pi x}{2} = \frac{1}{2} \left[\sin \pi \left(1 + \frac{n}{2}\right)x - \sin \pi \left(1 - \frac{n}{2}\right)x \right]$$

$$\Rightarrow \frac{5}{2} \left[\int_0^2 \sin \pi \left(1 + \frac{n}{2}\right)x - \int_0^2 \sin \pi \left(1 - \frac{n}{2}\right)x \right]$$

$$\Rightarrow \frac{5}{2} \left[\frac{-2}{(2+n)\pi} \cos \pi \left(1 + \frac{n}{2}\right)x \Big|_0^2 + \frac{-2}{(2-n)\pi} \cos \pi \left(1 - \frac{n}{2}\right)x \Big|_0^2 \right]$$

$$\Rightarrow \frac{5}{2} \left[\frac{-2}{(2+n)\pi} (-2) - \frac{-2}{(2-n)\pi} (-2) \right]$$

$$\Rightarrow -\frac{10}{(2+n)\pi} - \frac{10}{(2-n)\pi} = -\frac{10}{\pi} \left(\frac{1}{2+n} + \frac{1}{2-n} \right)$$

$$\Rightarrow -\frac{10}{n\pi} - \frac{10}{\pi} \left(\frac{4}{4-n^2} \right) = -\frac{10}{\pi} \left(\frac{1+4}{n \cdot \frac{4}{4-n^2}} \right)$$

\therefore

$$B_n = \frac{+20}{n\pi} + \frac{10}{\pi} \left(\frac{1-4}{n \cdot \frac{4}{4-n^2}} \right) = \frac{20}{n\pi} + \frac{10}{\pi} \left(\frac{1-4}{n \cdot \frac{4}{4-n^2}} \right)$$

Particular solution

Solution:

$$U(x,t) = \sum_{n=1}^{\infty} \left[\frac{+20}{n\pi} + \frac{10}{\pi} \left(\frac{1-4}{n \cdot \frac{4}{4-n^2}} \right) \right] \sin \frac{n\pi x}{2} e^{-\frac{n^2 \pi^2 t}{2}}$$

Solve homogeneous heat equation for
 $L=4$, $K=6$ and initial &
 Neumann Constant B.C. and I.C.
 $\frac{du}{dx}(0,t) = -1$ and $\frac{du}{dx}(L,t) = 4$
 with $f(x) = 20 - 5(x-2)^2 = u(x,0)$

Soln

We need to write the heat equation with
 $L=4$ and $K=6$ as

$$\frac{\partial u}{\partial t} = 6 \frac{\partial^2 u}{\partial x^2} \text{ for } K=6 > 0 \quad (*)$$

$$\Rightarrow \text{let } u(x,t) = X(x)T(t)$$

$$\Rightarrow u_t(x,t) = X(x)T'_t(t) \text{ and } u_{xx}(x,t) = X''(x)T(t)$$

\Rightarrow Substitute these into (*), we have

$$XT' = G X''T \quad , \text{ dividing by } X''T(t)$$

$$\frac{T'}{GT} = \frac{X''}{X} = -\lambda \quad , \text{ we normally set it to a constant}$$

\Rightarrow Separating the variables, we have

$$\frac{T'}{GT} = -\lambda \quad \text{--- (1)} \quad \text{and} \quad \frac{X''}{X} = -\lambda \quad \text{--- (2)}$$

\Rightarrow Solving for (1) and (2), we get

$$\frac{T'}{GT} = \frac{dT}{GTdt} = -\lambda \Rightarrow \frac{dT}{T} = -G\lambda dt$$

$$\Rightarrow \ln T = -G\lambda t + C$$

\Rightarrow Taking exponential on both sides

$$T(t) = e^{-G\lambda t + C} = e^{-G\lambda t} \cdot e^C$$

$$(e^C = A)$$

$$\Rightarrow T(t) = A e^{-G\lambda t}$$

\Rightarrow Solving for (2)

$$\frac{X''}{X} = -\lambda \Rightarrow X'' + \lambda X = 0$$

\Rightarrow This is a 2nd order ODE, we solve by characteristic approach

$$a^2 + \lambda = 0 \quad a^2 = -\lambda \Rightarrow a = \pm \sqrt{-\lambda}$$

$$\Rightarrow X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

\Rightarrow By the hypothesis $\lambda > 0$, so that

$\lim_{t \rightarrow \infty} \phi(t) = 0$ and not $+\infty$, we have

$$X(x) = \bar{c}_1 \sin \sqrt{\lambda} x + \bar{c}_2 \cos \sqrt{\lambda} x$$

$\Rightarrow U(x,t)$ is given as

$$U(x,t) = X(x)T(t) = A \bar{e}^{6xt} [\bar{c}_1 \sin \sqrt{\lambda} x + \bar{c}_2 \cos \sqrt{\lambda} x]$$

$$\Rightarrow B = A\bar{c}_1 \text{ and } D = A\bar{c}_2$$

$$\Rightarrow U(x,t) = \bar{e}^{6xt} [B \sin \sqrt{\lambda} x + D \cos \sqrt{\lambda} x] \quad \text{--- (3)}$$

\Rightarrow We now apply the B.C.,

$$\frac{dU(0,t)}{dx} = -1 \quad \text{and} \quad \frac{dU(4,t)}{dx} = 4$$

\Rightarrow We differentiate equation (3) w.r.t x
 So we can apply those B.C.,

$$U_x(x,t) = \bar{e}^{6At} [\sqrt{\lambda} B \cos \sqrt{\lambda} x - \sqrt{\lambda} D \sin \sqrt{\lambda} x]$$

Applying B.C, we get

$$U_x(0,t) = \bar{e}^{6At} [B\sqrt{\lambda}, 1 + 0] = -1$$

$$\Rightarrow \bar{e}^{6At} \cdot B\sqrt{\lambda} = -1$$

$$\Rightarrow B = e^{6At} / \sqrt{\lambda}$$

$$\Rightarrow U_x(4,t) = \bar{e}^{6At} [B\sqrt{\lambda} \cos \sqrt{\lambda} \cdot 4 - \sqrt{\lambda} \cdot D \sin \sqrt{\lambda} \cdot 4] =$$

$$\Rightarrow \text{For } B = e^{6At} / \sqrt{\lambda}$$

$$\Rightarrow \bar{e}^{-6At} \left[\frac{e^{-6At}}{\sqrt{\lambda}} \cdot \sqrt{\lambda} \cos \sqrt{\lambda} \cdot 4 - \sqrt{\lambda} \cdot D \sin \sqrt{\lambda} \cdot 4 \right] = 4$$

$$\Rightarrow \cos \sqrt{\lambda} \cdot 4 - \sqrt{\lambda} \cdot D \sin \sqrt{\lambda} \cdot 4 = 4$$

$$\Rightarrow \sqrt{\lambda} \cdot D \sin \sqrt{\lambda} \cdot 4 = \cos \sqrt{\lambda} \cdot 4 - 4$$

$$D = \frac{\cos \sqrt{\lambda} \cdot 4 - 4}{\sqrt{\lambda} \sin \sqrt{\lambda} \cdot 4}$$

$$\Rightarrow \frac{1}{\sqrt{\lambda}} \left[\frac{\cos \sqrt{\lambda} \cdot 4 - 4}{\sin \sqrt{\lambda} \cdot 4} \right] = 1$$

$$U(x,t) = \bar{e}^{6At} \left[\frac{e^{6At}}{\sqrt{\lambda}} \sin \sqrt{\lambda} x + \left(\frac{\cos \sqrt{\lambda} \cdot 4 - 4}{\sqrt{\lambda} \sin \sqrt{\lambda} \cdot 4} \right) \cos \sqrt{\lambda} x \right]$$

\Rightarrow By hypothesis $\lambda > 0$ for $\lim_{x \rightarrow \infty} \phi_n = 0$ not +

$$U(x,t) = U_{\text{non-homogeneous B.C.}}(x,t) + S(x)$$

\Rightarrow Now we clearly note that there will be infinitely many solutions for $\sqrt{\lambda}, x$ and resulting $U(x,t)$, so we choose the one which leads to the easiest problem solution for $U(x,t)$

$$\Rightarrow \sqrt{\lambda}, x = n\pi \quad \text{for } n = 1, 2, 3, \dots$$

$$\lambda_n = n^2\pi^2 / 16 \quad + \text{ for } n = 1, 2, 3, \dots$$

\Rightarrow For $U(x,t)$ derivative function the $U_{\text{(non-homogeneous B.C.)}}$ function we can choose to approximate the easiest problem of the superposition principle to be given as

$$\text{Unhomogeneous}(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{4}\right) e^{-\frac{n^2\pi^2 t}{16}} + B_0 / \begin{matrix} \text{(Fourier Series)} \\ \text{constant} \end{matrix}$$

\Rightarrow Now we calculate

B_0 and B_n

$$\Rightarrow B_0 = \frac{2}{4} \int_0^4 f(x) = \frac{1}{2} \int_0^4 2_0 - 5(x-2)^2 dx.$$

$$= \frac{1}{2} \left[\int_0^4 2x dx - 5 \int_0^4 (x-2)^3 dx \right]$$

$$= \frac{1}{2} \left[2x^2 \Big|_0^4 - 5 \int_0^4 (x-2)^2 dx \right]$$

let $u = x-2 \quad du = dx$

$$= 5 \int_0^4 u^3 du = u^4 \Big|_0^4$$

$$\Rightarrow 5 \int_0^4 (x-2)^3 dx = \frac{1}{2} \left[(64) - \frac{80}{3} \right] = \frac{50}{3}$$

$$B_n = \frac{1}{2} \int_0^4 f(x) \cos\left(\frac{n\pi x}{4}\right) dx$$

$$= \frac{1}{2} \int_0^4 [2x - 5(x-2)^2] \cos\left(\frac{n\pi x}{4}\right) dx$$

$$= \frac{1}{2} \left[\int_0^4 2x \cos\left(\frac{n\pi x}{4}\right) dx - 5 \int_0^4 (x-2)^2 \cos\left(\frac{n\pi x}{4}\right) dx \right]$$

$$= \frac{1}{2} \left[\frac{-20 \cdot 4}{n\pi} \sin(n\pi x) \Big|_0^4 - 5 \int_0^4 (x-2)^2 \cos\left(\frac{n\pi x}{4}\right) dx \right]$$

↓

$$= -\frac{5}{2} \int_0^4 (x-2)^2 \cos\left(\frac{n\pi x}{4}\right) dx.$$

U	$\frac{dv}{dx}$
$(x-2)^2$	$\frac{\cos n\pi x}{4}$
$2(x-2)$	$\frac{4}{n\pi} \frac{\sin n\pi x}{4}$
2	$\frac{-16}{n^2\pi^2} \frac{\cos n\pi x}{4}$
0	$\frac{-64}{n^3\pi^3} \frac{\sin n\pi x}{4}$

$$= -\frac{5}{2} \left[\frac{4(x-2)^2 \sin n\pi x}{n\pi} \Big|_0^4 + \frac{32(x-2) \cos n\pi x}{n^2\pi^2} \Big|_0^4 - \frac{128 \sin n\pi x}{n^3\pi^3} \Big|_0^4 \right]$$

↓ ↓ ↓

0 0 0

$$\Rightarrow -\frac{5}{2} \left[\frac{32(x-2) \cos n\pi x}{n^2\pi^2} \Big|_0^4 \right]$$

$$= -\frac{5}{2} \left[\frac{32(x-2) \cos n\pi + 64}{n^2\pi^2} \right]$$

$$= -\frac{5}{2} \left[\frac{64 \cos n\pi + 1}{n^2\pi^2} \right]$$

$$B_n = -\frac{160}{n^2\pi^2} \cos n\pi - \frac{5}{2}$$

$$U(x,t) = \frac{5}{3} + \sum_{n=1}^{\infty} \left(-\frac{160}{n^2\pi^2} \cos n\pi - \frac{5}{2} \right) \cos n\pi x e^{-\frac{n^2\pi^2 t}{16.6 + 56}}$$

$$= U(x,t) = \frac{5}{3} + \sum_{n=1}^{\infty} \left(-\frac{160}{n^2\pi^2} \cos n\pi - \frac{5}{2} \right) \cos n\pi x e^{-\frac{n^2\pi^2 t}{61.6 + 56}}$$

Q3

Solve the non-homogeneous heat equation
for $L = 6$, $k = 12$ with time-independent
 $q(x) = 3e^{-x}$, mixed homogeneous B.C.

$$\text{and } f(x) = 2 \sin(\pi x/6)$$

Solu.

We first write the non-homogeneous heat equation with time independent source term as

$$q(x) = 3e^{-x}$$

$$\Rightarrow \frac{\partial U}{\partial t} = 12 \frac{\partial^2 U}{\partial x^2} + 3e^{-x} \quad \text{with Condition}$$

$$U(0,t) = T(t)$$

$$U(6,t) = X(t)$$

$$U(x,0) = f(x) = 2 \sin(\pi x/6)$$

\Rightarrow Note that the source term $q(x)$ is both time and space dependent

\Rightarrow Thus, to solve this problem, we first need to solve for a given source function.

\Rightarrow let the function be given by $p(x,t)$ in such a way that the B.C. are satisfied as below

$$p(0,t) = T(t) \Rightarrow p(G,t) = X(t)$$

\Rightarrow Thus, our said function $p(x,t)$ will take a simpler form as

$$p(x,t) = T(t) + \frac{x}{G} (X(t) - T(t))$$

\Rightarrow Note that we have the function

$$U(x,t) \text{ and } p(x,t)$$

\Rightarrow Thus, we find its difference and set it to another function as

$$w(x,t) = U(x,t) - p(x,t)$$

\Rightarrow Hence $w(x,t)$ will be the solution to the below given problem

$$\frac{\partial w(x,t)}{\partial t} = 12 \frac{\partial^2 w(x,t)}{\partial x^2} + \left[3e^x - 2 \frac{\partial p(x,t)}{\partial t} + 12 \frac{\partial^2 p(x,t)}{\partial x^2} \right]$$

with B.C as

$$w(0,t) = 0 \text{ and } w(G,t) = 0$$

and I.C as

$$w(x,0) = U(x,0) = f(x) - p(x,0) = 2 \sin \frac{\pi x}{6} - T(0) - \frac{3}{6} (X(0) - T(0))$$

\Rightarrow We can approximate this to $f(x)$

$$\Rightarrow p(x,t) = -2 \sin \frac{\pi x}{6} + T(t) + \frac{x}{6} (X(t) - T(t)) + f(x)$$

with approximated
for Value

\Rightarrow Now That we successfully reduced our non-homogeneous heat equation, we will be using the method of eigenfunction expansion to find its solution.

\Rightarrow We proceed as below

\Rightarrow First we write the homogeneous part. of the above reduced problem as

$$\frac{\partial S(x,t)}{\partial t} = \lambda \frac{\partial^2 S(x,t)}{\partial x^2} \quad (*)$$

and B.C

$$S(0,t) = 0 \quad \text{and} \quad S(6,t) = 0$$

\Rightarrow We now apply the method of separation of variables to obtain an eigenfunction eigenvalue function given by

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \quad \text{with condition}$$

$$\phi(0) = 0$$

$$\phi(6) = 0$$

\Rightarrow let's solve for the eigenvalue λ

$$\Rightarrow \text{We let } S(x,t) = X(x) T(t)$$

$$\Rightarrow S_t(x,t) = T'(t) \quad \text{and} \quad S_{xx}(x,t) = X''(x)$$

\Rightarrow Substituting into $(*)$ we have

$$T'X = 12T X'' \quad , \text{ dividing all by } S(x,t) \text{ but}$$

$$\Rightarrow \frac{T'}{12T} = \frac{X''}{X} = -\lambda \quad \text{divide both sides by } X$$

$$\Rightarrow \frac{T'}{12T} = \frac{X''}{X} = -\lambda \Rightarrow \frac{T'}{12T} = -\lambda \quad \text{and} \frac{X''}{X} = -\lambda$$

$$\Rightarrow \frac{dT}{T} = -\lambda \frac{12dt}{t} \Rightarrow \ln T = -12dt + C$$

$$\Rightarrow T(t) = A e^{-12\lambda t}$$

$$\text{and } X'' + \lambda X = 0 \Rightarrow a^2 + \lambda = 0$$

$$\Rightarrow a^2 = -\lambda \Rightarrow a = \pm \sqrt{-\lambda}$$

$$\Rightarrow X(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$

\Rightarrow But with $\lambda > 0$ hypothesis we re-write $X(x)$

$$X(x) = \bar{c}_1 \sin \sqrt{\lambda} x + \bar{c}_2 \cos \sqrt{\lambda} x$$

for the B.C., we have

$$X(0) = 0 + \bar{c}_2 = 0 \Rightarrow \bar{c}_2 = 0$$

$$\Rightarrow X(x) = \bar{c}_1 \sin \sqrt{\lambda} x$$

$$X(6) = \bar{c}_1 \sin \sqrt{\lambda} \cdot 6 = 0$$

\Rightarrow For non-trivial solution

$$c_1 \neq 0$$

$$\Rightarrow \sqrt{\lambda} \cdot 6 = (n\pi)^2$$

$$\Rightarrow 1.36 = n^2 \pi^2$$

$$\Rightarrow \lambda_n = \frac{n^2 \pi^2}{36} \quad \text{for } n = 1, 2, 3, \dots$$

\Rightarrow The corresponding eigenfunction is given by

$$\phi_n(x) = \sin \frac{n\pi}{6} x$$

\Rightarrow We now express the unknown solution $w(x,t)$ as a generalized Fourier series of eigenfunctions with the time dependent coefficient given by

$$w(x,t) = \sum_{n=1}^{\infty} B_n(t) \phi_n(x)$$

\Rightarrow Since we already have $\phi_n(x)$ - solution, we now need to calculate the value function for $B_n(t)$. constants (coefficient)

\Rightarrow We assume that the term-by-term differentiation is satisfied, thus substituting $w(x,t)$ value into

$$\frac{\partial w(x,t)}{\partial t} = 12 \frac{\partial^2 w(x,t)}{\partial x^2} + \left[3 \bar{e}^t - \frac{\partial p(x,t)}{\partial t} + 12 \frac{\partial^2 p(x,t)}{\partial x^2} \right] \quad (x)$$

\Rightarrow Simplifying into

$$\Rightarrow \sum_{n=1}^{\infty} \frac{d B_n(t)}{dt} \phi_n(x) = 12 \sum_{n=1}^{\infty} B_n(t) \frac{d^2 \phi_n(x)}{dx^2} + 3 \bar{e}^x$$

$$= -12 \sum_{n=1}^{\infty} B_n(t) \left(\frac{n^2 \pi^2}{36} \right) \phi_n(x) + 3 \bar{e}^x$$

\Rightarrow Next, we need to expand the term $q(t)$ using generalized Fourier Series of eigenfunctions.

\Rightarrow Thus we have,

$$3 \bar{e}^x = \sum_{n=1}^{\infty} q_n(t) \phi_n(x) \quad \text{where}$$

that $q_n(t)$ is given by

$$q_n(t) = \int_0^6 3 \bar{e}^x \phi_n(x) dx$$

\Rightarrow Substituting the above into equation (*) and identifying the constant coefficient, we obtain the differentiation eqn for coefficients given by

$$\frac{d B_n(t)}{dt} + \frac{n^2 \pi^2}{36} k B_n(t) = \int_0^6 3 \bar{e}^x \phi_n(x) dx$$

$$\int_0^6 3 \bar{e}^x \phi_n(x) dx$$

\Rightarrow we now need to obtain the IC for the d.e using

$$\omega(x, 0) = u(x, 0) - p(x, 0)$$

$$\Rightarrow \text{Thus, we end up with } t \frac{n^2 \pi^2 k t}{36} \\ B_{n(t)} = B_n(\omega) e^{-\frac{n^2 \pi^2 k t}{36}} + e^{-\frac{n^2 \pi^2 k t}{36}} \int q_n(t) e^{\frac{n^2 \pi^2 k t}{36}} dt$$

$$\Rightarrow \omega(x, t) = \sum_{n=1}^{\infty} B_{n(t)} \sin \frac{n \pi x}{6}$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin \frac{n \pi x}{6} + T_{(t)} - 2 \sin \frac{\pi x}{6} + \chi \left(x_0 - T_{(t)} \right)$$

$$\Rightarrow U(x, t) = \sum_{n=1}^{\infty} \left[B_n(0) e^{-\frac{n^2 \pi^2 k t}{36}} + e^{-\frac{n^2 \pi^2 k t}{36}} \int_0^t q_n(\tau) e^{\frac{n^2 \pi^2 k \tau}{36}} d\tau \right] \sin \frac{n \pi x}{6} + \\ + \chi \left(x_0 - T_{(t)} \right) - 2 \sin \frac{\pi x}{6} + T_{(t)}$$

