3. State whether Sequences Converges to zero or not and prove

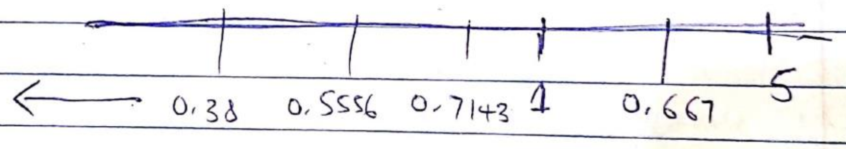
(i)
(a) $(x_n)_{n=1}^{\infty}$ $\quad x_n = \dfrac{5}{2n-1} \quad \forall n$

Solution

So first we generate terms of the sequence as below for $n = 1, 2, 3, ---$

$$x_n = \left\{ 5, \frac{5}{3}, 1, \frac{5}{7}, \frac{5}{9}, \frac{5}{11}, \frac{5}{13}, ---- \right\}$$

lets plot this in a number line



So examining the trend of this sequence is that it is decreasing towards Zero

$\Rightarrow$ This sequence will Converge to Zero

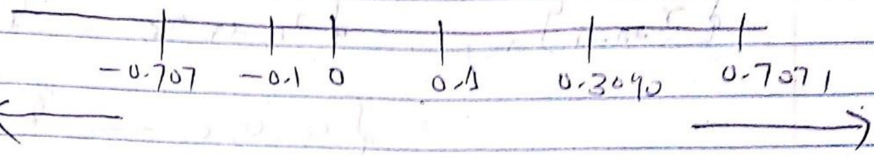
(b) $(x_n)_{n=1}^{\infty}$ $\quad x_n = \dfrac{1}{10} \sin\left(\dfrac{n\pi}{2}\right) \quad \forall n$

Soln

First we gonna obtain some values of the sequence given as below

$x_n = \left\{ \frac{1}{10}, 0, -\frac{1}{10}, -0.7071, 0.7071, 0.3090, \ldots \right\}$

plotting these sample values



Examining these values, simply indicates a sequence which is diverging from zero outwards
$\Rightarrow$ So this sequence does not converge to zero

(c) $(x_n)_{n=1}^{\infty}$ $\quad x_n = \begin{cases} 1 & n \text{ is a power of } 2 \\ 0 & \text{otherwise} \end{cases}$

Solution

The first thing we do to this sequence is that we expand it to obtain its sample values
$\Rightarrow$ So basically for n being a power 2 implies values for n being even and then we can take 0 values when n is odd

$\Rightarrow x_n = \{ x_1, x_2, x_3, x_4, \ldots \}$

$\{ 0, 1, 0, 1, 0, 1, \ldots \}$

lets get 2 subsequences of the above sequence.

$$\{x_{2n}\}_{n=1}^{\infty} = \{x_2, x_4, x_6, ---\}$$

$$= \{1, 1, 1, ---\}$$

This subsequence converges to 1

$$\{x_{2n-1}\}_{n=1}^{\infty} = \{x_1, x_3, x_5, ---\}$$

$$= \{0, 0, 0, ---\}$$

This subsequence converges to 0

Remember that for a convergent sequence then both subsequences must be convergent as well

Thus $0 \neq 1$, Thus this does not converge to zero

(d) $(x_n)_{n=1}^{\infty}$ $\quad x_n = \dfrac{1}{\log_2 n}$ for $n \geq 2$

Solution

We proceed as follows

For $n=2$

$\Rightarrow \log_2 2 = x \Rightarrow 2^x = 2 \Rightarrow x \log 2 = 2$

$x = 6.644$

$\Rightarrow 1/6.644 = 0.1505$

$n=3$

$\Rightarrow \log_2 3 = x \Rightarrow 2^x = 3 \Rightarrow x \log 2 = 3$

$x = 9.966$

$\Rightarrow \frac{1}{9.966} = 0.1003$

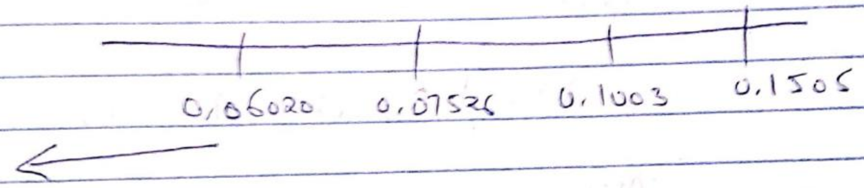
$\Rightarrow$ for $n=4$

$\log_2 4 = x \Rightarrow 2^x = 4 \quad x \log 2 = 4$

$x = 13.29$

$\Rightarrow \frac{1}{13.29} = 0.07526$

plotting these values we have



So easily this sequence is converging to zero

$\Rightarrow$ Increasing the value of n, results to obtaining a value which is decreasing towards zero

8 Give sequence $(x_n)_{n=1}^{\infty}$ based on properties

a) We need to find a sequence x_n such that $x \neq 0$ and $y \neq 0$ for converging sequences x_n and y_n respectively

=> let $x_n = \left(1+\frac{1}{n}\right)^{\frac{1}{n}}$

so we test this sequence to see if it meets the requirements

=> We generate some values

$x_n = \{2, 1.225, 1.1006, ----\}$

This sequence is converging to 1 which is not zero

=> passes the test ✓

=> lets test (y_n) as well

$y_n = 2x_n^n$

$\Rightarrow 2\left[1+\frac{1}{n}\right]^{\frac{1}{n}\cdot n} = 2\left[1+\frac{1}{n}\right]$

$y_n = \{4, 3, 2.21, ----$

This sequence converge to 2 which is not zero

=> looking good ✓

(b) x_n is bounded but (y_n) does not converge

Solution

=) So here we need a sequence x_n which is bounded and a corresponding y_n sequence which doesn't converge

=) We proceed as follows

let $x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

let's test the sequence x_n to see if it meets the requirements

=) $x_n = \{0, 1, 0, 1, ---\}$

So this sequence is bounded because if we chose x, it will lie between 0 and 1

ie, $0 \le x \le 1$ (bounded)

=) lets see (y_n) sequence if it does converge or not

for $n=1$, $x_1 = 0$,

$y_n =$ $n=2$, $x_2 = 1$

$n=3$, $x_3 = 0$

$n=4$, $x_4 = 1$

$y_n = 2x_n^n$ =) $y_1 = 2x_1^1 = 0$

$y_2 = 1$ . . . . . . --

$\{y_n\} : \{0, 2, 0, 2, \ldots\}$

Using 2 subsequence from the above sequence we have

$$(y_{2n})_{n=1}^{\infty} = \{2, 2, 2, \ldots\}$$

Converges to 2.

$$(y_{2n-1})_{n=1}^{\infty} = \{0, 0, 0 \ldots\}$$

Converges to 0

Using the theorem that for a sequence to be convergent then its subsequences must converge as well

$\Rightarrow$ But $0 \neq 2$.

Thus $(y_n)_{n=1}^{\infty}$ does not converge

$\Rightarrow$ Test passed ✓

(c) (x_n) does not converge (to any $x \in \mathbb{R}$) but (y_n) converges to zero

Solution

let $\left(\frac{1}{2n}\right)^{1/n} = x_n$

$\Rightarrow$ Generating some numericals we have

$$x_n = \{0.5, 0.5, 0.55, 0.99, \ldots\}$$

=> So this sequence will be converging to ∞ (infinity) => implying that the sequence does not converge to any $x \in \mathbb{R}$

Test passed ✓✓

let's test the (y_n) sequence as well

$$y_n = 2(x_n)^n = 2\left[\left(\frac{1}{2n}\right)^{\frac{1}{n}}\right]^n$$

$$= \frac{1}{n}$$

=> Generating numerical

$$y_n = \{1, 0.5, 0.3333, 0.25, ---\}$$

This sequence converges to zero

Test passed for (y_n) as well ✓✓

(d) (x_n) does not converge (to any $x \in \mathbb{R}$) but (y_n) converges to $y \neq 0$

Solution

In this one, we can let $(x_n) = n^2$
=> Then we test its convergence

$$x_n = \{1, 4, 9, 16, ---\}$$

This sequence converges towards ∞ (infinity) which is not any $x \in \mathbb{R}$

$\Rightarrow$ We run a similar test to (y_n) as well

$$y_n = 2x_n^n = 2[n^2]^n = 2n^{2n}$$

$$\Rightarrow \{y_n\} = \{2, 32, 1458, \text{---}\}$$

$\Rightarrow$ This sequence similarly converges to ∞ (infinity) which is not ~~any~~ $y = 0$

Thus, $(x_n) = n^2$ is looking great ~~h~~

$n \in \mathbb{N}$

END OF (B)

THANK YOU!!!