a) Consider the IVP

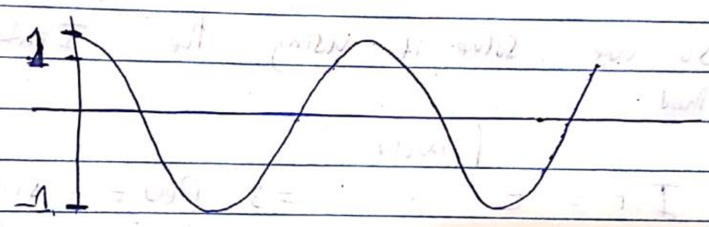
$$\cos x \frac{dy}{dx} - \sin x \, y = 0 \quad \text{at } y(0) = 2$$

(i) Range of x-values

Soln

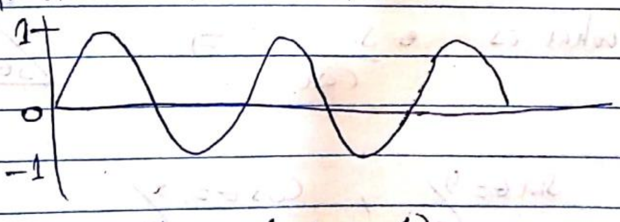
We simply consider the range of sine and Cosine functions and extinguish the valid x-values for which the solution will exist

$\Rightarrow$ Sinx function takes the form



Sinx ranges from $(-1, 1)$

Cosine function takes the form



Cosine ranges from $(-1, 0, 1)$

Thus, range of x-values is given by interval $\Rightarrow$

interval $[-1, 1]$

(ii) Solution

Soln

We first write the equation is standard form

i.e. $\cos x \frac{dy}{dx} - \sin(x) y = 0$ with $y(0) = 2$

$\Rightarrow \frac{dy}{dx} - \frac{\sin x}{\cos x} y = 0$

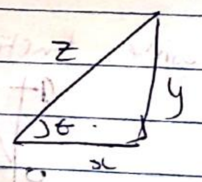
$\Rightarrow$ This equation takes the form $\frac{dy}{dx} + P(x)y = Q(x)$

$\Rightarrow$ So we solve it using the Integrating factor method

$I.F = e^{\int p(x)dx}$ $\Rightarrow$ $p(x) = -\frac{\sin x}{\cos x}$

$\Rightarrow I.F = e^{\int -\frac{\sin x}{\cos x} dx}$

but what is $\frac{\sin x}{\cos x}$ $\Rightarrow$



let $\sin\theta = y/z$, $\cos\theta = x/z$

$\Rightarrow \frac{\cos x}{\sin x} = \frac{x}{z} \cdot \frac{z}{y} = \frac{x}{y} = \tan\theta = \frac{x}{y}$

$\Rightarrow \frac{\sin x}{\cos x} = \frac{1}{\tan\theta}$

$\Rightarrow I.F = e^{\int -\frac{1}{\tan x} dx} = e^{\int -\cot x dx}$

$\Rightarrow$ But $\int \cot x \, dx = \ln \sin x + c$

$\Rightarrow e^{\int -\cot x dx} = e^{\ln - \sin x} = - \sin x.$

$\Rightarrow$ Solution of the further we have,

$y . I.F = \int Q(x) . I.F dx + C_1$

$\Rightarrow y.(- \sin x) = \int 0 . (- \sin x) dx + C_1$

$= - y \sin x = \int 0 \, dx + C_1$

$= - y \sin x = C + C_1 \quad \Rightarrow y = - \frac{(C + C_1)}{\sin x}$

$\Rightarrow \underline{y(x) = - \operatorname{cosec} x [C + C_1]}$

with $y(0) = 2$, we have $x = 0, y = 2$.

$y(0) = 2 = - \operatorname{cosec} x \, A \quad \Rightarrow \frac{A}{\sin x} = -2$

$\Rightarrow A = -2 \sin x \Big|_{x=0} \quad \Rightarrow A = 0$

$\Rightarrow \underline{y(x) = 0}$

b) Given $y_1(x) = e^{-x}$ $y_2(x) = e^{-x}/x^2$ Solutions

to

$$x\frac{d^2y}{dx^2} + (2x+3)\frac{dy}{dx} + (x+3)y = 0$$

verify that Satisfies Abel's identity

<u>Soln</u>

We need to note what Abel's theorem or identity entails

$\Rightarrow$ $W(y_1(x), y_2(x))$ should be either identically zero, or never vanishes

$\Rightarrow$ So we going to prove this by showing that

$$W' + pW = 0$$

$\Rightarrow$ Therefore we have

(i) We first compute the wronskian $W(x)$

(ii) Compute wronskian derivative $W'(x)$

(iii) Use Abel's identity $W'(x) + pW = 0$ to show that wronskian is identically zero or it never vanishes

$\Rightarrow$ First we write the equation is standard form

i.e

$$\frac{d^2y}{dx^2} + \left(\frac{2x+3}{x}\right)\frac{dy}{dx} + \frac{(x+3)}{x}y = 0$$

$\Rightarrow$ $p(x) = \frac{2x+3}{x}$ $\qquad q(x) = \frac{x+3}{x}$

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \quad = y_1(x) = e^{-x}, \; y_2(x) = e^{-x}/x^2$$

$=$

$$y_1'(x) = -e^{-x} \quad , \; y_2'(x) = \frac{-x^2 e^{-x} - 2x e^{-x}}{x^4}$$

$$\Rightarrow y_2'(x) = -e^{-x}/x^2 - 2/x^3 \, e^{-x}$$

$$\Rightarrow \begin{vmatrix} e^{-x} & e^{-x}/x^2 \\ -e^{-x} & -e^{-x}/x^2 - 2/x^3 \, e^{-x} \end{vmatrix} = e^{-x}\left[-e^{-x}/x^2 - 2/x^3 \, e^{-x}\right] + e^{-2x}/x^2$$

$$= -e^{-2x}/x^2 - 2e^{-2x}/x^3 + e^{-2x}/x^2 = W(y_1, y_2)$$

For Abel's identity $W' + pW = 0$, we have to find W'

$$W' = -y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2'$$

$$W' = -\left[\frac{-2e^{-2x}x^2 - 2xe^{-2x}}{x^4}\right] - 2\left[\frac{-2e^{-2x}x^3 - 3x^2 e^{-2x}}{x^6}\right] + \left[\frac{-2e^{-2x}x^2 - 2xe^{-2x}}{x^4}\right]$$

$$= \frac{2e^{-2x}}{x^2} + \frac{2e^{-2x}}{x^3} + \frac{4e^{-2x}}{x^3} + \frac{6e^{-2x}}{x^4} - \frac{2e^{-2x}}{x^2} - \frac{2e^{-2x}}{x^3} = W'(y_1, y_2)$$

⇒ Abel's identity

$$w' + pw = 0 \qquad p(x) = \frac{2x+3}{x}$$

$$\Rightarrow \frac{2e^{-2x}}{x^2} + \frac{2e^{-2x}}{x^3} + \frac{4e^{-2x}}{x^3} + \frac{6e^{-2x}}{x^4} - \frac{2e^{-2x}}{x^2} - \frac{2e^{-2x}}{x^3} + \dots$$

$$+ \left(\frac{2x+3}{x}\right)\left[\frac{e^{-2x}}{x^2} - \frac{e^{-2x}}{x^2} - \frac{2e^{-2x}}{x^3}\right] = 0$$

$$\cancel{\frac{2e^{-2x}}{x^2}} + \cancel{\frac{2e^{-2x}}{x^3}} + \cancel{\frac{4e^{-2x}}{x^3}} + \cancel{\frac{6e^{-2x}}{x^4}} - \cancel{\frac{2e^{-2x}}{x^2}} - \cancel{\frac{2e^{-2x}}{x^3}} + \dots$$

$$-\cancel{\frac{4e^{-2x}}{x^3}} - \cancel{\frac{6e^{-2x}}{x^4}} = 0$$

⇒ So this simplifies into Zero hence Satisfying the Abel's identity

END OF PROOF

THANK YOU !!!

|Sojore1| ✓✓