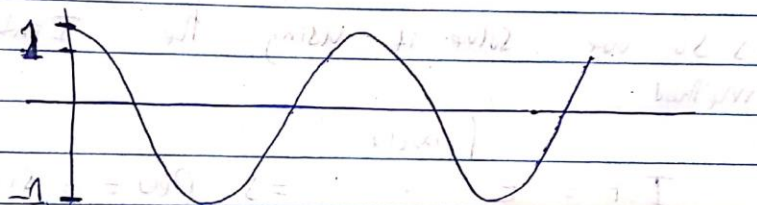q) Consider the IVP

$$\cos x \frac{dy}{dx} - \sin x \, y = 0 \qquad at \quad y(0) = 2$$

(i) Range of x -values

Soln
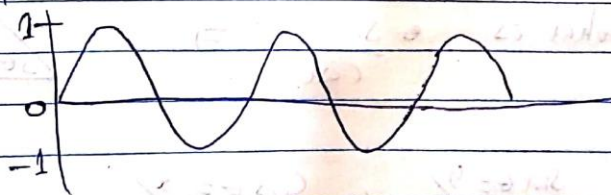
We simply consider the range of sine and Cosine functions and extinguish the valid x values for which the solution will exist

$\Rightarrow$ Sine function takes the form

Sinx ranges from $(-1, 1)$

Cosine function takes the form

Cosine ranges from $(-1, 0, 1)$

Thus, range of x values is given by interval $\Rightarrow$

interval $\boxed{[-1, 1]}$

(ii) Solution

S.m

We first write the equation is standard form

i.e $\cos x \dfrac{dy}{dx} - \sin(x)y = 0$ with $y(o) = 2$

$\Rightarrow \dfrac{dy}{dx} - \dfrac{\sin x}{\cos x} y = 0$

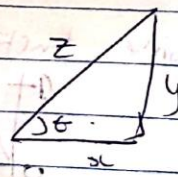$\Rightarrow$ This equation takes the form $\dfrac{dy}{dx} + P(x)y = Q(x)$

$\Rightarrow$ So we solve it using the Integrating factor method

$I.F = e^{\int P(x)dx}$ $\Rightarrow P(x) = -\dfrac{\sin x}{\cos x}$

$\Rightarrow I.F = e^{\int -\frac{\sin x}{\cos x} dx}$

but what is $\dfrac{\sin x}{\cos x}$ $\Rightarrow$

let $\sin \theta = \dfrac{y}{z}$, $\cos \theta = \dfrac{x}{z}$

$\Rightarrow \dfrac{\cos x}{\sin x} = \dfrac{x}{z} \cdot \dfrac{z}{y} = \dfrac{x}{y} = \tan \theta = \dfrac{x}{y}$

$\Rightarrow \dfrac{\sin x}{\cos x} = \dfrac{1}{\tan x}$

$$\Rightarrow I.F = e^{\int -\frac{1}{\tan x} dx} \qquad = e^{\int -\cot x \, dx}$$

$$= e'$$

$$\Rightarrow \text{But} \quad \int \cot x \, dx = \ln \sin x + c$$

$$\int -\cot x \, dx \qquad \ln \cdot \sin x$$

$$\Rightarrow \quad e \qquad = e \qquad = \pm \sin x.$$

$\Rightarrow$ Solution of tho trithe we have,

$$y \cdot I.F = \int Q(x) \cdot I.F \, dx + C_1$$

$$\Rightarrow y \cdot (-\sin x) = \int 0 \cdot (-\sin x) \, dx + C_1$$

$$= -y \sin x = \int 0 \, dx + C_1$$

$$= -y \sin x = C + C_1 \qquad \Rightarrow y = -\frac{(C + C_1)}{\sin x}$$

$$\Rightarrow y(x) = -\cosec [C + C_1]$$

b) Given $y_1(x) = e^{-x}$ $y_2(x) = \dfrac{e^{-x}}{x^2}$ Solutions

to

$$x\dfrac{d^2y}{dx^2} + (2x+3)\dfrac{dy}{dx} + (x+3)y = 0$$

verify that Satisfies Abel's identity

Soln

We need to note what Abel's theorem or identity
entails

    $\Rightarrow$ $W(y_1(x), y_2(x))$ should be either identically zero,
    or never Vanishes

$\Rightarrow$ So we going to prove this by showing that

$$W' + PW = 0$$

$\Rightarrow$ Therefore we have

(i) we first compute the wronskian
$$W(x)$$

(ii) Compute wronskian derivative $W'(x)$

(iii) Use Abel's identity $W'(x) + PW = 0$ to show
that wronskian is identically zero or it
never vanishes

$\Rightarrow$ First we write the equation is standard form

i.e. $$\dfrac{d^2y}{dx^2} + \left(\dfrac{2x+3}{x}\right)\dfrac{dy}{dx} + \dfrac{(x+3)}{x}y = 0$$

$\Rightarrow$ $p(x) = \dfrac{2x+3}{x}$ $\qquad$ $q(x) = \dfrac{x+3}{x}$

$$W(y_1,y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \quad - y_1(x) = \bar{e}^x \quad , y_2(x) = \dfrac{\bar{e}^x}{x^2}$$

$=$

$$y_1'(x) = -\bar{e}^x \qquad , y_2'(x) = -\dfrac{x^2 \bar{e}^x - 2x \bar{e}^x}{x^4}$$

$\Rightarrow$ $y_2'(x) = -\dfrac{\bar{e}^x}{x^2} - \dfrac{2}{x^3}\bar{e}^x$

$$\Rightarrow \begin{vmatrix} \bar{e}^x & \dfrac{\bar{e}^x}{x^2} \\ -\bar{e}^x & -\dfrac{\bar{e}^x}{x^2} - \dfrac{2}{x^3}\bar{e}^x \end{vmatrix} = \bar{e}^x\left[-\dfrac{\bar{e}^x}{x^2} - \dfrac{2}{x^3}\bar{e}^x\right] + \dfrac{\bar{e}^{2x}}{x^2}$$

$$= -\dfrac{\bar{e}^{2x}}{x^2} - \dfrac{2\bar{e}^{2x}}{x^3} + \dfrac{\bar{e}^{2x}}{x^2} = W(y_1,y_2)$$

For Abel's identity $W' + PW = 0$, we have to find W'

$$W' = -y_1'y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2'$$

$$W' = -\left[-\dfrac{2\bar{e}^{2x}x^2 - 2x\bar{e}^{2x}}{x^4}\right] - 2\left[-\dfrac{2\bar{e}^{2x}x^3 - 3x^2\bar{e}^{2x}}{x^6}\right] + \left[-\dfrac{2\bar{e}^{2x}x^2 - 2x\bar{e}^{2x}}{x^4}\right]$$

$$= \dfrac{2\bar{e}^{2x}}{x^2} + \dfrac{2\bar{e}^{2x}}{x^3} + \dfrac{4\bar{e}^{2x}}{x^3} + \dfrac{6\bar{e}^{2x}}{x^4} - \dfrac{2\bar{e}^{2x}}{x^2} - \dfrac{2\bar{e}^{2x}}{x^3} = W'(y_1,y_2)$$

$\Rightarrow$ Abel's identity

$$w' + pw = 0 \qquad\qquad p(x) = \frac{2x+3}{x}$$

$$\Rightarrow 2e^{-2x}\!\!\Big/\!x^0 + 2e^{-2x}\!\!\Big/\!x^3 + 4e^{-2x}\!\!\Big/\!x^3 + 6e^{-2x}\!\!\Big/\!x^4 - 2e^{2x}\!\!\Big/\!x^2 - 2e^{2x}\!\!\Big/\!x^3 + \dots$$

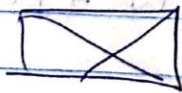$$+ \left(\frac{2x+3}{x}\right)\left[\frac{e^{-2x}}{x^2} - \frac{e^{-2x}}{x^2} - \frac{2e^{-2x}}{x^3}\right] = 0$$

$$2e^{-2x}\!\!\Big/\!x^2 + 2e^{-2x}\!\!\Big/\!x^3 + 4e^{-2x}\!\!\Big/\!x^3 + 6e^{-2x}\!\!\Big/\!x^4 - 2e^{-2x}\!\!\Big/\!x^2 - 2e^{-2x}\!\!\Big/\!x^3 + \dots$$

$$- 4e^{-2x}\!\!\Big/\!x^3 - 6e^{-2x}\!\!\Big/\!x^4 = 0$$

$\Rightarrow$ So this simplifies into Zero hence satisfying the Abel's identity

END OF PROOF

THANK YOU !!!

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$\underline{\text{Sojbre1}}$ ✓✓