Welcome!!!

In this project, we are to proof that

$$\text{Ratio Cut}(S) = y^T L_G y / n \quad \text{given that}$$

$$\text{Ratio Cut}(S) = \left(\frac{1}{|S|} + \frac{1}{|S^c|}\right) \sum_{i \in S, j \in S^c} w_{i,j} \quad \text{and}$$

letting y be the indicator vector of S and S^c with elements equal to

$$y_i = \begin{cases} \left(\frac{|S^c|}{|S|}\right)^{1/2}, & i \in S \\ -\left(\frac{|S|}{|S^c|}\right)^{1/2}, & i \in S^c \end{cases}$$

<u>proof</u>

=) So inorder to proof the ratiocut (S) we first need to focus on what is called the binary clustering defined by cut(S) given by

$$\text{Cut}(S) = \sum_{i \in S, j \in S^c} w_{i,j}$$

=) We then notice that since the Cut(S) clustering function only captures a small number of clusters, we need to extend its functionality into a ratiocut (S) clustering approach

=) This will be given by a polynomial time via the Cut(S) function given by

$\text{RatioCut}(S) = \text{Cut}(S) \times Z$, where Z is defined by

$$Z = \left(\frac{1}{|S|} + \frac{1}{|S^c|}\right)$$

=) Thus we have.

$$\text{RatioCut}(S) = \text{Cut}(S) \times \frac{1}{2}\left(\frac{1}{|S|} + \frac{1}{|S^c|}\right) \quad \text{———} (*)$$

which its indicator Vectorization will be defined by (B) the given definition of the problem

=) Next, for a non-normalized graph laplacian matrix we have

$$L = D - W$$

=) But for every $y_i \in \mathbb{R}^n$, we have $y_i^T L_G y$

=) Expanding this further we have

$$y_i^T L_G y = y_i^T D y_i - y_i^T W y_i$$

=) Implying that $\Longrightarrow \sum_{i=1}^{n} k_i y_i(i)^2 - \sum_{i,j=1}^{n} y_i(i) y_i(j) W_{i,j}$

Where k_i denotes the corresponding eigenvectors

=) This simplifies into

$$= \frac{1}{2}\left[\sum_{i=1}^{n} k_i y_i(i)^2 - 2\sum_{i,j=1}^{n} y_i(i) y_i(j) W_{i,j} + \sum_{j=1}^{n} k_j y_i(j)^2\right]$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} l_{ij}\left(y_i(i) - y_i(j)\right)^2 \quad \text{———} (1)$$

=) Now using the indicator vector of S and S^c as per the definition

$$y_i = \begin{cases} \left(\frac{|S^c|}{|S|}\right)^{\frac{1}{2}}, & i \in S \\ -\left(\frac{|S|}{|S^c|}\right)^{\frac{1}{2}}, & i \in S^c \end{cases}$$

=) Substituting this y_i function into (1) accordingly we have

$$= \frac{1}{2}\sum_{i\in S, j\in S^c} i_{i,j}\left[\left(\frac{|S^c|}{|S|}\right)^{\frac{1}{2}} + \left(\frac{|S|}{|S^c|}\right)^{\frac{1}{2}}\right]^2 + \frac{1}{2}\sum_{i\in S, j\in S} i_{i,j}\left[-\left(\frac{|S^c|}{|S|}\right)^{\frac{1}{2}} - \left(\frac{|S|}{|S^c|}\right)^{\frac{1}{2}}\right]^2$$

=) Applying the cut (S) function we obtained ~~earli~~ earlier to the above equation, we get that

$$= \text{Cut}(S)\left(\frac{|S|+|S^c|}{|S|} + \frac{|S|+|S^c|}{|S^c|}\right)$$

=) This simplifies further into

$$= \text{Cut}(S)\left(\frac{|S^c|}{|S|} + \frac{|S|}{|S^c|} + 2\right)$$ which equals to n-times the ratio cut(S) as defined in (*) for n-number of values

=) Thus we have $\text{Cut}(S)\left(\frac{|S^c|}{|S|} + \frac{|S|}{|S^c|} + 2\right) = \text{Ratiocut}(S) \times n$

=) Since we have shown that $\text{Cut}(S)\left(\frac{|S^c|}{|S|} + \frac{|S|}{|S^c|} + 2\right) = y^T L_G y$

we therefore conclude that

$$n \times \text{Ratiocut}(S) = y^T L_G y \implies \underline{\text{Ratio cut}(S) = y^T L_G y / n}$$

THANK YOU!!!