PART B

(4b) Using cholesky decomposition method

let $A = LL^T \Rightarrow LL^T x = b \Rightarrow Lz = b \Rightarrow L^T x = z$

System of linear eqn

$4x_1 - 2x_2 + 4x_3 + 2x_4 = 30$

$-2x_1 + 5x_2 - 2x_3 + 3x_4 = -27$

$4x_1 - 2x_2 + 13x_3 - x_4 = 93$

$2x_1 + 3x_2 - x_3 + 15x_4 = -15$

Soln

In matrix form

$$\overset{A}{\begin{pmatrix} 4 & -2 & 4 & 2 \\ -2 & 5 & -2 & 3 \\ 4 & & & \end{pmatrix}} \overset{x}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \overset{b}{\begin{pmatrix} 30 \\ -27 \end{pmatrix}}$$

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Soln

In matrix form

$$\overset{A}{\begin{pmatrix} 4 & -2 & 4 & 2 \\ -2 & 5 & -2 & 3 \\ 4 & -2 & 13 & -1 \\ 2 & 3 & -1 & 15 \end{pmatrix}} \overset{x}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} = \overset{b}{\begin{pmatrix} 30 \\ -27 \\ 93 \\ -15 \end{pmatrix}}$$

$$\Rightarrow \overset{L}{\begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{41} & L_{42} & L_{44} \end{pmatrix}} \overset{L^T}{\begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & & \end{pmatrix}} = \begin{pmatrix} A \end{pmatrix}$$

System of linear eqn

$$4x_1 - 2x_2 + 4x_3 + 2x_4 = 30$$
$$-2x_1 + 5x_2 - 2x_3 + 3x_4 = -27$$
$$4x_1 - 2x_2 + 13x_3 - x_4 = 93$$
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Soln

In matrix form

$$\underset{A}{\begin{pmatrix} 4 & -2 & 4 & 2 \\ -2 & 5 & -2 & 3 \\ 4 & -2 & 13 & -1 \\ 2 & 3 & -1 & 15 \end{pmatrix}} \underset{x}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} = \underset{b}{\begin{pmatrix} 30 \\ -27 \\ 93 \\ -15 \end{pmatrix}}$$

$$\Rightarrow \underset{L}{\begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}} \underset{L^T}{\begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & 0 & L_{44} \end{pmatrix}} = \begin{pmatrix} A \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} & L_{11}L_{41} \\ L_{11}L_{21} & L_{21}^2+L_{22}^2 & L_{21}L_{31}+L_{22}L_{32} & L_{21}L_{41}+L_{22}L_{42} \\ L_{11}L_{31} & L_{21}L_{31}+L_{32}L_{22} & L_{31}^2+L_{32}^2+L_{33}^2 & L_{31}L_{41}+L_{32}L_{42}+L_{33}L_{43} \\ L_{11}L_{41} & L_{41}L_{21}+L_{42}L_{22} & L_{41}L_{31}+L_{42}L_{32}+L_{43}L_{33} & L_{41}^2+L_{42}^2+L_{43}^2+L \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 4 & -2 & 4 & 2 \\ -2 & 5 & -2 & 3 \\ 4 & -2 & 13 & -1 \\ 2 & 3 & -1 & 15 \end{pmatrix}} \underset{x}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -15 \end{pmatrix}$$

$$\Rightarrow \underset{L}{\begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}} \underset{L^T}{\begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & 0 & L_{44} \end{pmatrix}} = \begin{pmatrix} A \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} & L_{11}L_{41} \\ L_{11}L_{21} & L_{21}^2+L_{22}^2 & L_{21}L_{31}+L_{22}L_{32} & L_{21}L_{41}+L_{22}L_{42} \\ L_{11}L_{31} & L_{21}L_{31}+L_{32}L_{22} & L_{31}^2+L_{32}^2+L_{33}^2 & L_{31}L_{41}+L_{32}L_{42}+L_{33}L_{43} \\ L_{11}L_{41} & L_{41}L_{21}+L_{42}L_{22} & L_{41}L_{31}+L_{42}L_{32}+L_{43}L_{33} & L_{41}^2+L_{42}^2+L_{43}^2+L_{44}^2 \end{pmatrix} \quad (*)$$

...ating (*) to A and solving for L and L^T we
...e

$L_{11}^2 = 4 \Rightarrow L_{11} = 2$	$2L_{41} = 2 \Rightarrow L_{41} = 1$
$2L_{21} = -2 \Rightarrow L_{21} = -1$	$1 + L_{22}^2 = 5 \Rightarrow L_{22}^2 = 4 \Rightarrow L$...
$2L_{31} = 4 \Rightarrow L$...	

$$\begin{pmatrix} 4 & -2 & 13 & -1 \\ 2 & 3 & -1 & 15 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \begin{pmatrix} \\ -15 \end{pmatrix}$$

$L \qquad L^T$

$$\Rightarrow \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & 0 & L_{44} \end{pmatrix} = \begin{pmatrix} A \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} & L_{11}L_{41} \\ L_{11}L_{21} & L_{21}^2+L_{22}^2 & L_{21}L_{31}+L_{22}L_{32} & L_{21}L_{41}+L_{22}L_{42} \\ L_{11}L_{31} & L_{21}L_{31}+L_{32}L_{22} & L_{31}^2+L_{32}^2+L_{33}^2 & L_{31}L_{41}+L_{32}L_{42}+L_{33}L_{43} \\ L_{11}L_{41} & L_{41}L_{21}+L_{42}L_{22} & L_{41}L_{31}+L_{42}L_{32}+L_{43}L_{33} & L_{41}^2+L_{42}^2+L_{43}^2+L_{44}^2 \end{pmatrix} (*)$$

$\Rightarrow$ Equating $(*)$ to A and solving for L and L^T we have

$\Rightarrow L_{11}^2 = 4 \Rightarrow L_{11} = 2$

$2L_{21} = -2 \Rightarrow L_{21} = -1$

$2L_{31} = 4 \Rightarrow L_{31} = 2$

$2L_{41} = 2 \Rightarrow L_{41} = 1$

$1 + L_{22}^2 = 5 \Rightarrow L_{22}^2 = 4 \Rightarrow L_{22} = 2$

$\Rightarrow -1.1 + 2L_{42} = 3$

$2L_{42} = 4 \Rightarrow L_{42} = 2$

$\Rightarrow 4 + 0 + L_{33}^2 = 13 \Rightarrow L_{33} = 3$

$\Rightarrow 2.1 + 0 + 3L_{43} = -1$

$2 + 3L_{43} = -1 \Rightarrow L_{43} = 1$

$\Rightarrow 1 + 4 + 1 + L_{44}^2 = 15$

$\Rightarrow L_{44}^2 = 15 - 6 = 9$

$\Rightarrow L_{44} = 3$

$$\Rightarrow L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix} \qquad L^T = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$\Rightarrow LL^T x = b$

$$Lz = b \quad \begin{pmatrix} 2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \end{pmatrix} \begin{pmatrix} 30 \end{pmatrix}$$

$\Rightarrow -1.1 + 2L_{42} = 3$

$2L_{42} = 4 \Rightarrow L_{42} = 2$

$\Rightarrow 4 + 0 + L_{33}^2 = 13 \Rightarrow L_{33} = 3$

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$\Rightarrow L L^T x = b$

$$\Rightarrow Lz = b \quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$\Rightarrow 2z_1 = 30 \Rightarrow z_1 = 15$

$= -27 \Rightarrow -15 + 2z_2 = -27 \Rightarrow z_2 = -6$

$$\Rightarrow L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix} \quad L^T = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$\Rightarrow L L^T x = b$

$$\Rightarrow Lz = b \quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$\Rightarrow 2z_1 = 30 \Rightarrow z_1 = 15$

$\Rightarrow -z_1 + z_2 = -27 \Rightarrow -15 + 2z_2 = -27 \Rightarrow z_2 = -6$

$\Rightarrow 2z_1 + 3z_3 = 93 \Rightarrow 3z_3 = 93 - 30 \Rightarrow z_3 = 21$

$\Rightarrow z_1 + 2z_2 - z_3 + 3z_4 = -18 \Rightarrow 15 - 12 - 21 + 3z_4 = -18 \Rightarrow z$

$$z = \begin{pmatrix} 15 \\ -6 \end{pmatrix}$$

$\Rightarrow LUx = b$

$$\Rightarrow Lz = b \quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$\Rightarrow 2z_1 = 30 \Rightarrow z_1 = 15$

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$\Rightarrow z_1 + 2z_2 - z_3 + 3z_4 = -18 \Rightarrow 15 - 12 - 21 + 3z_4 = -18 \Rightarrow z_4 = 0$

$$\Rightarrow z = \begin{pmatrix} 15 \\ -6 \\ 21 \\ 0 \end{pmatrix}$$

$$\Rightarrow Ux = z \Rightarrow \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -15 \\ -6 \\ 21 \\ 0 \end{pmatrix}$$

$3x_4 = 0 \Rightarrow x_4 = 0$, $3x_3 = 21$, $3x_3 = 21 \Rightarrow x_3 = 7$

$\Rightarrow 2x_2 + 2x_4 = -6 \Rightarrow x_2 = -3$

$\Rightarrow 2x_1 - x_2 + 2x_3 + x_4 = 15 \Rightarrow x_1 = -1$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

(4c)(i) Explanation

=> Also known as triangularization or factorization to compute the inverse A^{-1} by first co[...] both the upper and lower triangular matrices by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \end{pmatrix} \text{ and } U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \end{pmatrix}$$

$3x_4 = 0 \Rightarrow x_4 = 0$, $3x_3 = 21$ $3x_3 = 21 \Rightarrow x_3 = 7$

$\Rightarrow 2x_2 + 2x_4 = -6$ $\Rightarrow x_2 = -3$

$\Rightarrow 2x_1 - x_2 + 2x_3 + x_4 = 15$ $\Rightarrow x_1 = -1$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

(4c)(i) Explanation

⇒ Also known as triangularization or factorization is used to compute the inverse A^{-1} by first computing both the upper and lower triangular matrices given by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \text{ and } U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

we then get the product of LU so as to be equal to the Matrix A and calculating the coefficients

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

(4c)(i) Explanation

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$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \text{ and } U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

⇒ we then get the product of LU so as to be equal to the Matrix A and calculating the coefficients of matrices L and U which we then obtain the A^{-1}

$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \end{pmatrix}$ $U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \end{pmatrix}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

(4c)(i) Explanation

=> Also known as triangularization or factorization is used to complete the inverse A^{-1} by first computing both the upper and lower triangular matrices given by

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \text{ and } U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

=> We then get the product of LU so as to equate ~~to~~ to the matrix A and calculating the coefficients for matrices L and U which we then obtain matrix A^{-1}

(ii) $$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

let $l_{ii} = 1$

U

=> L = (

=> Also known as triangularization or factorization is used to complete the inverse A^{-1} by first computing both the upper and lower triangular matrices given by

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \text{ and } U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

=> We then get the product of LU so as to equate ~~to~~ to the matrix A and calculating the coefficients for matrices L and U which we then obtain matrix A^{-1}

(ii) $$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

let $l_{ii} = 1$

U

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix} \times \begin{pmatrix} u_{12} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

$\Rightarrow u_{11}=4 \quad , u_{12}=-2 \quad , u_{13}=4 \quad , u_{14}=2$

$\Rightarrow 4L_{21}=-2 \Rightarrow L_{21}=-\frac{1}{2} \quad , -\frac{1}{2}\cdot 2+u_{24}=3 \Rightarrow u_{24}=4$

$\Rightarrow 4 \cdot L_{31}=4 \Rightarrow L_{31}=1 \quad , 1\cdot -2+4L_{32}=-2 \quad , 4L_{32}=-2+2 \quad , L_{32}=0$

$\Rightarrow 4+u_{33}=13 \Rightarrow u_{33}=9 \quad , 4L_{41}=2 \Rightarrow L_{41}=\frac{1}{2}$

$\Rightarrow 2+9L_{43}=-1 \quad , L_{43}=-\frac{1}{3} \quad , 4L_{42}=4 \quad , L_{41}=1$

$\Rightarrow 1+4+1+u_{44}=15 \Rightarrow u_{44}=9$

$$L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad U=\begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$\Rightarrow u_{11}=4 \quad , u_{12}=-2 \quad , u_{13}=4 \quad , u_{14}=2$

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$\Rightarrow 4+u_{33}=13 \Rightarrow u_{33}=9 \quad , 4L_{41}=2 \Rightarrow L_{41}=\frac{1}{2}$

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$$L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad U=\begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$Lz=b$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}=\begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$\Rightarrow z_1=30$

$\Rightarrow \frac{1}{2}z_1+z_2=-27 \quad , z_2=-12$

$\Rightarrow$ [illegible]

$\Rightarrow 4 \cdot L_{31} = 4 \Rightarrow L_{31} = 1$, $1 \cdot 2 + 4L_{32} = -2$, $4L_{32} = -2+2$, $L_{32} = 0$

$\Rightarrow 4 + u_{33} = 13 \Rightarrow u_{33} = 9$, $4L_{41} = 2 \Rightarrow L_{41} = \frac{1}{2}$

$\Rightarrow 2 + 9L_{43} = -1$, $L_{43} = -\frac{1}{3}$, $4L_{42} = 4$, $L_{41} = 1$

$\Rightarrow 1 + 4 + 1 + u_{44} = 15 \Rightarrow u_{44} = 9$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad u = \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$\underline{Lz = b}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$\Rightarrow z_1 = 30$

$\Rightarrow \frac{1}{2} z_1 + z_2 = -27$, $z_2 = -12$

$\Rightarrow z_1 + z_3 = 93$, $z_3 = 63$

$\Rightarrow z_4 + \frac{1}{2} \cdot 30 - 12 - \frac{1}{3} \cdot 63 = -18$

$\Rightarrow z_4 = 0$

$$\Rightarrow \underline{z} = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad u = \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$\underline{Lz = b}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

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$\underline{ux = z}$

$$\Rightarrow \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$\Rightarrow x_4 = 0$

$9x_3 = 63 \Rightarrow x_3 = 7$

$4x_2 = -12 \Rightarrow x_2 = -3$

$$\underline{x} = \begin{pmatrix} -1 \\ -3 \\ 7 \\ \ldots \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$Lz = b$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$\Rightarrow z_1 = 30$

$\Rightarrow \frac{1}{2}z_1 + z_2 = -27$, $z_2 = -12$

$\Rightarrow z_1 + z_3 = 93$, $z_3 = 63$

$\Rightarrow z_4 + \frac{1}{2}\cdot 30 - 12 - \frac{1}{3}\cdot 63 = -18$

$\Rightarrow z_4 = 0$

$$\Rightarrow z = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$$Ux = z \Rightarrow \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$\Rightarrow x_4 = 0$

$9x_3 = 63 \Rightarrow x_3 = 7$

$4x_2 = -12 \Rightarrow x_2 = -3$

$4x_1 + 6 + 28 = 30 \Rightarrow x_1 = -1$

$$x = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

(iii) Why is not the most efficient?

$\Rightarrow$ This is because LU decomposition method FAILS if any of the pivot elements is equal to zero

i.e, if $L_{ii} = 1$ and $U_{ii} = 0$ or if $U_{ii} = 1$ and $L_{ii} = 0$

A better efficient method to solve the above system would be to use the Jacobi Iterative method or the Gauss - Siedel approximation Iterative method

3/ (3)

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(4 a)

$$\begin{array}{c|cc} \frac{3}{4} & \frac{3}{4} & 0 \\ 1 & \frac{1}{4} & \frac{3}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \Rightarrow \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

$\Rightarrow c_1 = \frac{3}{4}$, $c_2 = 1$, $a_{11} = \frac{3}{4}$, $a_{12} = 0$

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$a_{21} = \frac{1}{4}$, $a_{22} = \frac{3}{4}$, $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{2}$

$\Rightarrow$ For an A-stable function above we have that if

$\bar{C} = \{ z \in \mathbb{R}, \mathbb{R}e(z) \leq 0 \}$

$\Rightarrow$ so the above condition is met iff

$|\mathbb{R}(iy)| \leq 1$ $\forall y \in \mathbb{R}$

A better efficient method for the Jacobi Iteration System would be to use the approximation Iteration method or the Gauss-Siedel method

(4 a)

$$\begin{array}{c|cc} 3/4 & 3/4 & 0 \\ 1 & 1/4 & 3/4 \\ \hline & 1/2 & 1/2 \end{array} \Rightarrow \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

$\Rightarrow c_1 = \frac{3}{4}$, $c_2 = 1$, $a_{11} = \frac{3}{4}$, $a_{12} = 0$

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$R(z)$ is analytic for $\mathbb{R}e(z) < 0$

$\Rightarrow$ Thus the stability function for the above Butcher tableau is given by

method

(4 a)

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$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{3}{16}z^3 + \frac{1}{48}z^4$$

Note that all non-zero entries are ~~above~~ the below the diagonal elements method

(4a) $\begin{array}{c|cc} 3/4 & 3/4 & 0 \\ 1 & 1/4 & 3/4 \\ \hline & 1/2 & 1/2 \end{array}$ $\Rightarrow$ $\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$

$\Rightarrow c_1 = \frac{3}{4}$, $c_2 = 1$, $a_{11} = \frac{3}{4}$, $a_{12} = 0$, $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{2}$

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Let $A = \begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}$ & $b^T = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

Therefore $R(z) = N(z) / D(z)$ for polynomial

$E(y) = D(iy)D(-iy) - N(iy)N(-iy)$

$\Rightarrow$ For simplicity we follow the below conditions

if A

(i) All roots $D(z)$ are in the right hand side

(ii) $E(iy) \geq 0$ for all $y \in \mathbb{R}$

Then the method is A-stable,

$$R(z) = \det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}\right]$$

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$$\Rightarrow R(z) = \det\begin{bmatrix} \frac14 + \frac12 z & 0 \\ -\frac14 & \frac14 + \frac12 z \end{bmatrix} = \frac{\left(\frac14 + \frac12 z\right)^2}{\left(1 - \frac34 z\right)^2}$$

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$$\Rightarrow N(z) = \frac14 + \left(\frac14 + \frac12 z\right)^2 = \frac14 z^2 + \frac14 z + \frac{5}{16}$$

(iii) $E(y) \geq 0$ for all $y \in \mathbb{R}$
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$$R(z) = \frac{\det\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + z\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix} - \begin{pmatrix}\frac{3}{4} & 0\\ \frac{1}{4} & \frac{3}{4}\end{pmatrix}\right]}{\det\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - z\begin{pmatrix}\frac{3}{4} & 0\\ \frac{1}{4} & \frac{3}{4}\end{pmatrix}\right]}$$

$$\Rightarrow R(z) = \frac{\det\begin{bmatrix}\frac{1}{4}+\frac{1}{2}z & 0\\ -\frac{1}{4} & \frac{1}{4}+\frac{1}{2}z\end{bmatrix}}{\det\begin{bmatrix}1-\frac{3}{4}z & 0\\ -\frac{1}{4}z & 1-\frac{3}{4}z\end{bmatrix}} = \frac{\left(\frac{1}{4}+\frac{1}{2}z\right)^2 + \frac{1}{4}}{\left(1-\frac{3}{4}z\right)^2 + \frac{1}{4}z}$$

$$\Rightarrow N(z) = \frac{1}{4} + \left(\frac{1}{4}+\frac{1}{2}z\right)^2 = \frac{1}{4}z^2 + \frac{1}{4}z + \frac{5}{16}$$

$$D(z) = \frac{1}{4}z + \left(1-\frac{3}{4}z\right)^2 = \frac{9}{16}z^2 - \frac{11}{4}z + 1$$

let $a = \frac{9}{16}$, $b = -\frac{11}{4}$ $c = 1$

$$R(z) = \frac{\det\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + z\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix} - \begin{pmatrix}\frac{3}{4} & 0\\ \frac{1}{4} & \frac{3}{4}\end{pmatrix}\right]}{\det\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - z\begin{pmatrix}\frac{3}{4} & 0\\ \frac{1}{4} & \frac{3}{4}\end{pmatrix}\right]}$$

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let $a = \frac{9}{16}$, $b = -\frac{11}{4}$ $c = 1$

$$z = \frac{\frac{11}{4} \pm \sqrt{\frac{121}{16} - \frac{36}{16}}}{\frac{9}{8}} \Rightarrow z_1 = 4.493$$

$$z_2 = 0.3956$$

$\Rightarrow$ Thus the roots of $D(z)$ are all in the R.H.S so the first condition is met / satisfied

$$\Rightarrow E(y) = D(iy)D(-iy) - N(iy)N(-iy)$$

$$\Rightarrow \cancel{\left(1 - \tfrac{11}{4} iy\right)}$$

$$\Rightarrow \left(1 - \frac{11}{4} iy - \frac{9}{16} y^2\right)\left(\frac{5}{6} + \frac{1}{4} iy - \frac{1}{4} y^2\right)$$

$$\Rightarrow -\left(\frac{5}{6} + \frac{1}{4} iy - \frac{1}{4} y^2\right)\left(1 - \frac{11}{4} iy - \frac{9}{16} y^2\right)$$

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$$\Rightarrow \frac{3}{6} + \frac{1}{4} iy - \frac{1}{4} y^2 - \frac{55}{36} iy - \frac{11}{16} y^3 - \frac{15}{32} y^2 - \frac{9}{64} iy^3 - \frac{9}{64} y^4 \geq 0$$

Thus since $E(y) \geq 0$ and roots $D(z)$ are in the R.H.S plane then we can conclude that the method is A-stable

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(4 c(i))

Continuation

Having determined L and U of the system $Ax =$

$\Rightarrow LUx = b$

simplified into $Lz = b$

$\Rightarrow \cancel{\left(1 - \frac{11}{4} iy\right)}$

$\Rightarrow \left(1 - \frac{11}{4} iy - \frac{9}{16} y^2\right)\left(\frac{5}{6} + \frac{1}{4} iy - \frac{1}{4} y^2\right)$

$\Rightarrow -\left(\frac{5}{6} + \frac{1}{4} iy - \frac{1}{4} y^2\right)\left(1 - \frac{11}{4} iy - \frac{9}{16} y^2\right)$

$\Rightarrow \frac{3}{6} + \frac{1}{4} iy - \frac{1}{4} y^2 - \frac{55}{36} iy - \frac{11}{16} y^2 - \frac{15}{32} y^2 - \frac{9}{64} iy^3 - \frac{9}{64} y^4 \geq 0$

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(4 c(i))

Continuation

Having determined L and U of the system $Ax = b$

$\Rightarrow LUx = b$

$\Rightarrow$ simplified into $Lz = b$

whereby $z = L^{-1} b$ or $z = Ux$

$\Rightarrow$ The z-values are obtained by inverse L^{-1} of the matrix A^{-1} multiplied by the b-values

$\Rightarrow L^{-1}U^{-1} = A^{-1}$

$\Rightarrow -\left(\frac{5}{6}+\frac{1}{4}iy-\frac{1}{4}y^2\right)\left(1 \quad \frac{}{4} \quad \frac{}{16}\right)$

$\Rightarrow \frac{3}{6}+\frac{1}{4}iy-\frac{1}{4}y^2-\frac{55}{36}iy-\frac{11}{16}y^2-\frac{15}{32}y^2-\frac{9}{64}iy^3-\frac{9}{64}y^4 \geq 0$

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(4 c(i))

Continuation

Having determined L and U of the system $A\underline{x} = \underline{b}$

$\Rightarrow \underline{L}\underline{U}\underline{x} = \underline{b}$

$\Rightarrow$ simplified into $\underline{L}\underline{z} = \underline{b}$

whereby $Z = L^{-1}\underline{b}$ or $Z = \underline{U}\underline{x}$

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$\Rightarrow L^{-1}U^{-1} = A^{-1}$

$\boxed{X}$

PART A

(1a) Using $y'(t) - 10(y-t^2)+2t \quad y(0)=-1$

at $t_0 = 0, y_0 = 0$

$\frac{dy}{dt} = -10(y-t^2)+2t$

Soln

at $t_1 = 0.1, y_1 = 0.377894$

$y' = -10\left[0.377894 - (0.1)^2\right] + 2(0.1) = -3.478940$

at $t_2 = 0.2 \quad y_2 = 0.1753352$

$y' = -10\left[0.1753353 - (0.2)^2\right] + 2(0.2) = -0.95$

PART A

(1a) using $y'(t) = -10(y - t^2) + 2t$ $\quad y(0) = -1$

at $t_0 = 0$, $y_0 = 0$

$$\frac{dy}{dt} = -10(y - t^2) + 2t$$

Soln

at $t_1 = 0.1$, $y_1 = 0.3778794$

$y' = -10[0.3778794 - (0.1)^2] + 2(0.1) = -3.4787940$

at $t_2 = 0.2$ $\quad y_2 = 0.1753352$

$y' = -10[0.1753353 - (0.2)^2] + 2(0.2) = -0.9533540$

at $t_3 = 0.3$, $y_3 = 0.1297871$

$y' = -10[0.1397871 - (0.3)^2] + 2(0.3) = 0.1021290$

at $t_0 = 0$, $y_0 = 0$

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Soln

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We need to calculate the values of y_4 at $t_4 =$

and y_5 at $t_5 = 0.5$

Now to find the y_4 and y_5 we need to find

... the Generalized milne predictor and corr at $t_1 = 0.1$, $y_1 = 0.3778794$

$y' = -10[0.3778794 - (0.1)^2] + 2(0.1) = -3.4787940$

at $t_2 = 0.2$ $y_2 = 0.1753352$

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at $t_3 = 0.3$, $y_3 = 0.1297871$

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We need to calculate the values of y_4 at $t_4 = 0.4$ and y_5 at $t_5 = 0.5$

Now to find the y_4 and y_5 we need to first write the generalized milne predictor and corrector formulae as

$$y_{j+1} = y_{j-1} + h\left(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 + \dots\right) f$$

For the values stating $j = n$ and the ∇ value

... the interval of time t

at $t_1 = 0.1$, $y_1 = 0.3778794$

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For the values stating $j = n$ and the ∇ value given by the difference in the interval of time t this equation is simplified into

$$y^p_{n+1} = y_{n-3} = \tfrac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

at $t_2 = 0.2$ $\quad y_2 = 0.1753352$

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R.H.S is

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For the values stating $j = n$ and the ∇ value given by the difference in the interval of time t this equation is simplified into

$$y^p_{n+1} = y_{n-3} = \tfrac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

and similarly for the corrector method

$$y^c_{n+1} = y_{n-1} + \tfrac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1}\right)$$

So at $n = 3 \Rightarrow j = 3$

$$y^p_4 = y_0 + \tfrac{4}{3}h\left(2y'_1 - y'_2 + 2y'_3\right)$$

$$= 1 + \tfrac{4}{3}(0.1)\left[2(-3.4787940) - (-0.9533540) + 2(0.102\right.$$

$$y^p_4(0.4) = 0.2266699$$

$$y'(t) = -10(y - t^2) + 2t$$

and similarly for the corrector method

$$y^c_{n+1} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y_n' + y'_{n+1}\right)$$

So at $n=3 \Rightarrow j=3$

$$y_4^p = y_0 + \frac{4}{3}h\left(2y_1' - y_2' + 2y_2'\right)$$

$$= 1 + \frac{4}{3}(0.1)\left[2(-3.4787940) - (-0.9533540) + 2(0.1021290)\right]$$

$$y_4^p(0.4) = 0.2266699$$

$$y'(t) = -10(y - t^2) + 2t$$

$$y_4' = -10\left(0.2266699 - (0.4)^2\right) + 2(0.4) = 0.1333013$$

s for the corrector method

$$y^c_{n+1} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y_n' + y'_{n+1}\right)$$

So at $n=3 \Rightarrow j=3$

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$$y_4' = -10\left(0.2266699 - (0.4)^2\right) + 2(0.4) = 0.1333013$$

as for the corrector method

$$y_4^c = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$= 0.1753353 + 0.1/3\left(-0.9533540 + 4(0.1021290) + 0.133\right.$$

$$= 0.2251743$$

$$y_4' = -10\left(0.2251743 - (0.4)^2\right) + 2(\ldots$$

$y_4 = y_0 + \frac{4}{3} h (\ldots$

$= 1 + \frac{4}{3}(0.1)[2(-3.4787940) - (-0.9533540] + 2(0.1021240)$

$y_4^p(0.4) = 0.2266699$

$y'(t) = -10(y - t^2) + 2t$

$y_4' = -10(0.2266099 - (0.4)^2) + 2(0.4) = 0.1333013$

as for the corrector method

$y_4^c = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$

$= 0.1753353 + 0.1/3(-0.9533540 + 4(0.1021240) + 0.1333013)$

$= 0.2251743$

$y_4' = -10(0.2251743 - (0.4^2)) + 2(0.4) = 0.1482566$

at $t = 0.5$ $y(0.5) = ?$

at $n = 4 \Rightarrow 5 = 4$

$y'(t) = -10(y - t^2) + 2t$

$y_4' = -10(0.2266099 - (0.4)^2) + 2(0.4) = 0.1333013$

as for the corrector method

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at $t = 0.5$ $y(0.5) = ?$

at $n = 4 \Rightarrow 5 = 4$

$y_5^p = \frac{4}{3}h(2y_2' - y_3' + 2y_4')$

$0.3778794 = \frac{4}{3}(0.1)[2(9533540) - (0.1021240) + 2(0.1333$

$y'(t) = -10(y - t^2) + 2t$

$y_4' = -10(0.2766099 - (0.4)^2) + 2(0.4) = 0.1333013$

as for the corrector method

$y_4^c = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$

$= 0.1753353 + 0.1/3(-0.9533540 + 4(0.1021290) + 0.1333013)$

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at $n = 4 \Rightarrow j = 4$

$y_5^p = \frac{4}{3}h(2y_2' - y_3' + 2y_4')$

$0.3778794 + \frac{4}{3}(0.1)[2(9533540) - (0.1021290) + 2(0.1333013)]$

$= 0.1455815$

$y_5' = -10(0.1455815 - (0.5)^2) + 2(0.5) = 2.0442850$

as for the corrector method

$= 0.1397871 + 0.1/3(0.1021290 + 4(0.1333013) + 2.044185)$

$= 0.2291044$

$y_5' = -10(0.2291044 - (0.5)^2) + 2(0.5) = 1.2089559$

table

y

.000000

3778794

$(0.2291044 - (0.1)^2) + 2(0.1) = 1.2084569$

Table

j	t	y
0	0.0	1.0000000
1	0.1	0.3778794
2	0.2	0.1753353
3	0.3	0.1397871
4	0.4	0.2266699
5	0.5	0.2

Table

j	t	y^*	y'	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
0	0.0	1.000000	-10.000000				
				6.5212060			
					-3.9957660		
1	0.1	0.3778794	-3.4787940			2.5258090	
				2.5254400			-2.080144
					-1.4699570		
2	0.2	0.1753353	-0.9533540			0.4456463	
				1.0554830			1.62314
					-1.0243107		
						2.0687930	
	0.3	0.1397871	0.1021290				
				0.0311723			

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2	0.2	0.1753353	-0.9533540			0.4456463	
				1.0554830			1.6231467
					-1.0243107		
						2.0687930	
3	0.3	0.1397871	0.1021290				
				0.0311723			
					1.0444823		
4	0.4	0.2266699	0.1333013				
				1.0756546			

~~0~~	~~0.0~~	~~1.000000~~
~~1~~	~~0.1~~	~~0.3778794~~
~~2~~	~~0.2~~	~~0.1753353~~
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4	0.4	0.2266699	0.1333013				
				1.0756546			
5	0.5	0.2291044	1.2089559				

$\nabla^5 f$

3.7032911

2a) For the Adams-Milne predictor method.

$$y_{j+1} = y_{j-1} + h\left(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 + \dots\right.$$

For $j = n$

$\Rightarrow$

$$y_{n+1} = y_{n-1} + h\left(2 + \frac{1}{3}y'_{n} + \frac{1}{3}y'_{n-2} + \frac{29}{\dots}\right.$$

$\nabla^5 f$

3.7032911

2a) For the Adams-Milne predictor method

$$y_{j+1} = y_{j-1} + h\left(2 + \tfrac{1}{3}\nabla^2 + \tfrac{1}{3}\nabla^3 + \tfrac{29}{90}\nabla^4 + \tfrac{28}{90}\nabla^5 + \dots\right) f_j$$

For $j = n$

=>

$$y_{n+1} = y_{n-1} + h\left(2 + \tfrac{1}{3}y'_{n-1} + \tfrac{1}{3}y'_{n-2} + \tfrac{29}{90}y'_{n-3} + \dots\right) f_n$$

=> $E y'_{n-2} = y'_{n+1}$

$E y'_{n-2} = y'_{n-1}$

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2a) For the Adams-Milne predictor method

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$E y'_{n-2} = y'_{n-1}$

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$$\sigma(E) = \tfrac{28}{90}E^4 + \tfrac{29}{90}E^3 + \tfrac{1}{3}E^2 + \tfrac{1}{3}E' + 2$$

$$\rho(E) = (E^3 - E)$$

$$(E^3 - E) y_{n-2} - h\left(\tfrac{28}{90}E^4 + \tfrac{29}{90}E^3 + \tfrac{1}{3}E^2 + \tfrac{1}{3}E + 2\right) y'_{n-2} = 0$$

$$\Rightarrow y_{n+1} = y_{n-1} + h\left(2 + \frac{1}{3}y'_{n-1} + \frac{1}{3}y'_{n-2} + \frac{29}{90}y'_{n-3} + \dots\right) f_n$$

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$\rho(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1) = 0$ — 1st characteris

$\sigma(\xi) = \frac{28}{90}\xi^4 + \frac{29}{90}\xi^3 + \frac{1}{3}\xi^2 + \frac{1}{3}\xi + 2$ — 2nd char

Root Condition

at $\rho(\xi) = 0$

$$E y'_{n-3} = y'_{n-1}$$

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Root Condition

at $\rho(\xi) = 0$

$$\Rightarrow \rho(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1)$$

$$\Rightarrow \xi = 0, \quad \xi_2 = 1, \quad \xi_1 = -1$$

se are the simple roots corresponding to the m

tral

$\Rightarrow E y_{n-2} = y_{n+1}$

$E y'_{n-2} = y'_{n-1}$

$E^2 y_{n-2} = y'_n$

$\sigma(E) = \frac{28}{90}E^4 + \frac{29}{90}E^3 + \frac{1}{3}E^2 + \frac{1}{3}E' + 2$

$\rho(E) = (E^3 - E)$

$(E^3 - E) y_{n-2} - h\left(\frac{28}{90}E^4 + \frac{29}{90}E^3 + \frac{1}{3}E^2 + \frac{1}{3}E + 2\right) y'_{n-2} = 0$

$\rho(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1) = 0$ — 1st characteristic

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Root condition

at $\rho(\xi) = 0$

$\Rightarrow \rho(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1)$

$\Rightarrow \xi = 0, \ \xi_2 = 1, \ \xi_1 = -1$

These are the simple roots corresponding to the milne-p[illegible] method

Stability

$\sigma(\xi) = 28/90\,\xi^4 + 29/90\,\xi^3 + \frac{1}{3}\xi^2 + \frac{1}{3}\xi + 2 = 0$

$\Rightarrow |\xi_4| > 0$

Hence zero stable since this will yield simple roots

Convergence and ~~Stab~~ Consistency

We suppose the method is zero-stable, then assume that $\rho(\xi)$ has double roots in the unit circle of roots of $\rho(\xi)$ we set as [illegible] Further, product of roots of $\rho(\xi)$ are equal to ± 1, thus not now ξ_1 and ξ_2 are [illegible] hence they converge and strictly complex

~~Stability~~

$$\sigma(\xi) = 28/90\,\xi^4 + 29/90\,\xi^3 + \tfrac{1}{3}\xi^2 + \tfrac{1}{3}\xi - 2 = 0$$

$$\Rightarrow |\xi_4| > 0$$

Hence zero stable since this will yield complex roots

Convergence and ~~Stab~~ Consistency

⇒ We suppose the method is zero-stable, then assume that $\rho(\xi)$ has double roots in the unit circle. Further, product of roots of $\rho(\xi)$ we set as 1, but now ξ_1 and ξ_2 are equal to ± 1, thus not strictly complex hence they converge and are consistent

$$\xi = 0, \quad \xi_{12} = \pm 1, \quad \xi_3 = -1$$

For the corrector method

as $j = n$

$$y_{j+1} = y_{j-1} = h\left(2 - 2\nabla + \tfrac{1}{3}\nabla^2 - \tfrac{1}{90}\nabla^4 - \tfrac{1}{90}\nabla^5 \ldots\right) f_{j+1}$$

$$\Rightarrow |\xi_4| > 0$$

Hence zero stable since this will yield complex roots

Convergence and ~~Stab~~ Consistency

⇒ We suppose the method is zero-stable, then assume that $\rho(\xi)$ has double roots in the unit circle. Further, product of roots of $\rho(\xi)$ we set as 1, but now ξ_1 and ξ_2 are equal to ± 1, thus not strictly complex hence they converge and are consistent

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⇒ For the corrector method

as $j = n$

$$y_{j+1} = y_{j-1} = h\left(2 - 2\nabla + \tfrac{1}{3}\nabla^2 - \tfrac{1}{90}\nabla^4 - \tfrac{1}{90}\nabla^5 \ldots\right) f_{j+1}$$

⇒ simplifies into

$$y_{n+1} = y_{n-1} + h\left(2 - 2y_n' + \tfrac{1}{3}y_{n-1}' - \tfrac{1}{90}y_{n-2}' - \tfrac{1}{90}y_{n-3}' \ldots\right) f$$

$$\Rightarrow \nabla y_{n-2}' = y_{n-1}'$$

Consistency

Suppose the method is zero-stable. Then assume that $\rho(\xi)$ has double roots in the unit circle. Further, product of roots of $\rho(\xi)$ we set as 1, but now ξ_1 and ξ_2 are equal to ± 1, thus not strictly complex hence they converge and are consistent

$$\xi = 0, \quad \xi_{12} = \pm 1, \quad \xi_3 = -1$$

=) For the corrector method

as $J = n$

$$y_{j+1} = y_{j-1} = h\left(2 - 2\nabla + \tfrac{1}{3}\nabla^2 - \tfrac{1}{90}\nabla^4 - \tfrac{1}{90}\nabla^5 - \dots\right) f_{j+1}$$

=> simplifies into

$$y_{n+1} = y_{n-1} + h\left(2 - 2y_n' + \tfrac{1}{3}y_{n-1}' - \tfrac{1}{90}y_{n-2}' - \tfrac{1}{90}y_{n-3}' \dots\right) f_{j+1}$$

$$\Rightarrow E y_{n-2}' = y_{n-1}'$$

$$E y_{n-2}' = y_{n-1}'$$

$$E^2 y_{n-2}' = y_n'$$

$$\vdots \qquad \vdots$$

$$\sigma(E) = \dots E^3 \dots E^2 + \tfrac{1}{3}E + 2 - 2$$

are equal to ± 1, thus not strictly complex hence they converge and are consistent

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$$\Rightarrow E y_{n-2}' = y_{n-1}'$$

$$E y_{n-2}' = y_{n-1}'$$

$$E^2 y_{n-2}' = y_n'$$

$$\vdots \qquad \vdots$$

$$\Rightarrow \sigma(E) = -\tfrac{1}{90}E^3 - \tfrac{1}{90}E^2 + \tfrac{1}{3}E + 2 - 2$$

$$= -\tfrac{1}{90}E^3 - \tfrac{1}{90}E^2 + \tfrac{1}{3}E$$

$$P(E) = E^3 - E^2$$

$$\Rightarrow (E^3 - E^2)\, y_{n-2} - h\left(-\frac{1}{90}E^3 - \frac{1}{90}E^2 + \frac{1}{3}E\right) y'_{n-2} = 0$$

$$\rho(\xi) = (\xi^3 - \xi^2) = \xi^2(\xi - 1)$$ — 1st characteristic

$$\sigma(\xi) = -\frac{1}{90}\xi^3 - \frac{1}{90}\xi^2 + \frac{1}{3}\xi$$ — 2nd characteristic

Root condition at $\rho(\xi) = 0$

$$\Rightarrow \rho(\xi) = \xi^2(\xi - 1) = 0$$

$$P(E) = E^3 - E^2$$

$$\Rightarrow (E^3 - E^2)\, y_{n-2} - h\left(-\frac{1}{90}E^3 - \frac{1}{90}E^2 + \frac{1}{3}E\right) y'_{n-2} = 0$$

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Root condition at $\rho(\xi) = 0$

$$\Rightarrow \rho(\xi) = \xi^2(\xi - 1) = 0$$

(2b)

$$\Rightarrow \xi_1 = 1, \quad \xi_{2,3} = 0$$

hese are simple roots of the corresponding corrector
ethod

tability

$$\rho(\xi) = (\xi^3 - \xi^2) = \xi^2(\xi - 1) \quad \text{— 1st characteristic}$$

$$\sigma(\xi) = -\frac{1}{90}\xi^3 - \frac{1}{90}\xi^2 + \frac{1}{2}\xi \quad \text{— 2nd characteristic}$$

Root condition at $\rho(\xi) = 0$

$\Rightarrow \rho(\xi) = \xi^2(\xi - 1) = 0$

$\Rightarrow \xi_1 = 1, \ \xi_{2,3} = 0$

These are simple roots of the corresponding collocation method

<u>Stability</u>

$$\sigma(E) = \sigma(\xi) = -\frac{1}{90}\xi^3 - \frac{1}{90}\xi^2 + \frac{1}{3}\xi \quad \text{at } \rho(\xi) = 0$$

$$= -\frac{1}{40}\xi^3 - \frac{1}{90}\xi^2 + \frac{1}{3}\xi = 0$$

$$\Rightarrow |\xi_3| > 1$$

Hence Zero - stable

Root condition at $\rho(\xi) = 0$

$\Rightarrow \rho(\xi) = \xi^2(\xi - 1) = 0$

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$$\Rightarrow |\xi_3| > 1$$

Hence Zero - stable since it yields complex roots

Convergence and Consistency

$$\rho(\xi) = \xi^2(\xi - 1) = 0$$

suppose the method is zero - stable, then assume $\rho(\xi)$ has

These are simple roots of the corresponding collocation method

Stability

$$\sigma(E) = \sigma(\xi) = -\frac{1}{40}\xi^3 - \frac{1}{90}\xi^2 + \frac{1}{3}\xi \quad \text{at } \rho(\xi) = 0$$

$$= -\frac{1}{40}\xi^3 - \frac{1}{90}\xi^2 + \frac{1}{3}\xi = 0$$

$$\Rightarrow |\xi_3| > 1$$

Hence zero-stable since it yields complex roots

Convergence and consistency

$$\rho(\xi) = \xi^2(\xi - 1) = 0$$

we suppose the method is zero-stable, then assume that $\rho(\xi)$ has double ~~not~~ roots in the unit circle

Further, product of roots of $\rho(\xi)$ is 1 and since the roots are not equal and neither equal to ± 1, then they are strictly complex

$$\Rightarrow \xi \neq \xi_{2,3}$$

These are simple roots of the correspo

method

Stability

$$\sigma(E) = \sigma(\xi) = -\frac{1}{40}\xi^3 - \frac{1}{90}\xi^2 + \frac{1}{3}\xi \quad \text{at } \rho(\xi) = 0$$

$$= -\frac{1}{40}\xi^3 - \frac{1}{90}\xi^2 + \frac{1}{3}\xi = 0$$

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Convergence and consistency

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$\Rightarrow$ Further, product of roots of $\rho(\xi)$ is 1 and since the roots are not equal and neither equal to ± 1, then they are strictly complex

$$\Rightarrow \xi \neq \xi_{2,3}$$

(2b) From the Nystrom-predictor Milne's method.

$$y_{j+1} = y_{j-1} + h\left(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 + \dots\right) f_j$$

Setting $j=n$, a ~~then~~ simplified Nystrom-Milne's predictor method is given as follows

$$y_{n+1} = y_{n-3} + \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$\Rightarrow y_{n+1} - y_{n-3} - \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

Let $E y'_{n-2} = y'_{n-1}$

(2b) From the Nystrom-predictor Milne's method.

$$y_{j+1} = y_{j-1} + h\left(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 + \dots\right) f_j$$

Setting $j=n$, a ~~then~~ simplified Nystrom-Milne's predictor method is given as follows

$$y_{n+1} = y_{n-3} + \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$\Rightarrow y_{n+1} - y_{n-3} - \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

Let $E y'_{n-2} = y'_{n-1}$

$$E y'_{n-1} = y'_n$$

$$E^2 y'_{n-2} = y'_n$$

$$\Rightarrow \sigma(E) = \frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3}$$

Setting $j = n$, a simplified Milne-Simpson predictor method is given as follows

$$y_{n+1} = y_{n-3} + \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$\Rightarrow y_{n+1} - y_{n-3} - \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

Let $E y'_{n-2} = y'_{n-1}$

$$E y'_{n-1} = y'_n$$

$$E^2 y'_{n-2} = y'_n$$

$$\Rightarrow \sigma(E) = \frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3}$$

$$\rho(E) = (E^3 - 0)$$

$$\Rightarrow (E^3 - 0)y_{n-2} - h\left(\frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3}\right)y'_{n-2} = 0$$

$\Rightarrow$ (i) $\rho(\xi) = \xi^3 = \xi(\xi^2)$ — 1st characteristic polynomial

(ii) $\sigma(\xi) = \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{8}{3}$ $\longrightarrow$ 2nd characteristic polynomial

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Root Condition

The explicit method of the Milne-Simpson method is said to satisfy the root condition if the roots of the equation $\rho(\xi) = 0$ lies inside the closed unit disk in the complex plane a

$E y_{n-2}' = y_{n-1}'$

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__Root Condition__

- The explicity method of the Milne method is said to satisfy the root condition if the roots of the equation $\rho(\xi) = 0$ lies inside the closed unit disk in the complex plane and are simple if they lie on the circle

$\rho(\xi) = 0$

$\Rightarrow$ From our equation

$\rho(\xi) = \xi^3 = 0 \Rightarrow \xi_{1,2,3} = 0$

__Stability__

$\Rightarrow$ The linear predictor method is zero stable if given $y' = f(x,y)$, $y(x_0) = y_0$, $f(x,y)$ satisfy lipschitz condition and root condition holds.

$\Rightarrow$ Thus we have in our case function

$\sigma(\xi) = \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{8}{3}$ which is zero-stable since it yields simpl

(3a)

$\rho(\xi)=0$

→ From our equation

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$\Rightarrow \frac{5}{3}\xi^2 - \frac{4}{3}\xi + \frac{5}{3} = 0 \Rightarrow |\xi_{1,2}| \geq 0$

Convergence & Consistency

linear predictor above is Consistent if it h

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Convergence & Consistency

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$\rho(\xi) = \xi^2(\xi) = 0$

Lets suppose the method is zero-stable, then $\rho(\xi)$ h
able roots in the unit circle

The product of roots of $\rho(\xi)$ is A

if given $y' = f(x,y)$ ... and root ...
lip schitz condition and root

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Further product of roots of $\rho(\xi)$ is 1, and since the roots are neither equal to ± 1, have said to be strictly complex

$$\Rightarrow \xi(\xi^2) = 0 \Rightarrow \xi_1 = 0, \xi_2 = 0, \xi_3 = 0$$

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(3a) Derivation of 4^{th} order explicity Runge Kutta method
$b_2 = 1/4$, $c_2 = 1/2$

Soln

Let the Condition eqns be defined by

$$b_1 + b_2 + b_3 + b_4 = 1 \quad \text{—(1)}$$
$$b_2c_2 + b_3c_3 + b_4c_4 = \tfrac{1}{2} \quad \text{—(2)}$$
$$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = \tfrac{1}{3} \quad \text{—(3)}$$
$$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = \tfrac{1}{4} \quad \text{—(4)}$$
$$b_3c_3a_{32}c_2 + b_4c_4(a_{42}c_2 + a_{43}c_3) = \tfrac{1}{8} \quad \text{—(5)}$$

$c_2 = a_{21}$

$c_3 = a_{31} + a_{32}$

$c_4 = a_{41} + a_{42} + a_{43}$

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$$b_3a_{32} + b_4a_{42} = b_2(1 - c_2) \quad \text{—(6)}$$

$$b_4a_{43} = b_3(1 - c_3) \quad \text{—(7)}$$

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Substituting $b_2 = \frac{1}{4}$ and $c_2 = \frac{1}{2}$ into equation (1) — (3) we have

$$b_1 + \tfrac{1}{2} + b_3 + b_4 = 1 \Rightarrow b_1 + b_3 + b_4 = \tfrac{1}{2} \quad — (9)$$

$$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = \tfrac{1}{3} \quad — (3)$$
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$$\tfrac{1}{4} + b_3c_3 + b_4c_4 = \tfrac{1}{2} \Rightarrow b_3c_3 + b_4c_4 = \tfrac{1}{4} \quad — (10)$$

$$\tfrac{1}{2}\cdot\tfrac{1}{4} + b_3c_3^2 + b_4c_4^2 = \tfrac{1}{2} \Rightarrow b_3c_3^2 + b_4c_4^2 = \tfrac{5}{24} \quad — (11)$$

$$\tfrac{1}{2}\cdot\tfrac{1}{8} + b_3c_3^3 + b_4c_4^3 = \tfrac{1}{4} \Rightarrow b_3c_3^3 + b_4c_4^3 = \tfrac{3}{16} \quad — (12)$$

simple notes

$b_3 c_2 a_{32} c_2 + b_4 c_4 (a_{42} c_2 + a_{43} c_3) = \frac{1}{8}$ — (5)

$b_3 a_{32} + b_4 a_{42} = b_2 (1 - c_2)$ — (6)

$b_4 a_{43} = b_3 (1 - c_3)$ — (7)

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$\frac{1}{2} \cdot \frac{1}{4} + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{2} \Rightarrow b_3 c_3^2 + b_4 c_4^2 = \frac{5}{24}$ — (11)

$\frac{1}{2} \cdot \frac{1}{8} + b_3 c_3^3 + b_4 c_4^3 = \frac{1}{4} \Rightarrow b_3 c_3^3 + b_4 c_4^3 = \frac{3}{16}$ — (12)

$b_1 = \frac{1}{2} - b_3 - b_4$, $b_2 = \frac{1}{2}$, $b_3 = \frac{1}{2} - b_1 - b_4$, $b_4 = \frac{1}{2} - b_1 - b_3$

$c_2 = \frac{1}{2}$, $c_3 = (\frac{1}{4} - b_4 c_4)/b_3$, $c_4 = (\frac{1}{4} - b_3 c_3)/b_4$

$a_{21} = \frac{1}{2}$, $a_{31} = (\frac{1}{4} - b_4 c_4)/b_3 - a_{32}$, $a_{32} = (\frac{1}{4} - b_4 c_4)/b_3 - a_{31}$

$a_{41} = (\frac{1}{4} - b_3 c_3)/b_4 - a_{42} - b_{43}$, $a_{42} = (\frac{1}{4} - b_3 c_3)/b_4 - a_{41} - a_{43}$

$a_{43} = (\frac{1}{4} - b_3 c_3)/b_4 - a_{41} - a_{42}$

Butcher tableau

0

1/

$c_2 = \frac{1}{2}$, $c_3 = (\frac{1}{4} - b_4 c_4)/b_3$ $c_4 = (\frac{1}{4} - b_3 c_3)/b_4$

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The Butcher tableau:

$$\begin{array}{c|ccc}
0 & 0 & & \\
\frac{1}{2} & \frac{1}{2} & & \\
\ldots/b_3 & (\frac{1}{4} - b_4 c_4)/b_3 - a_{32} & (\frac{1}{4} - b_4 c_4)/b_3 - a_{31} & \\
\ldots/b_4 & (\frac{1}{4} - b_3 c_3)/b_4 - a_{42} - a_{\ldots} & \ldots &
\end{array}$$

$c_2 = \frac{1}{2}$, $c_3 = (\frac{1}{4} - b_4 c_4)/b_3$ $c_4 = (\frac{1}{4} - b_3 c_3)/b_4$

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The Butcher tableau:

$$\begin{array}{c|cccc}
0 & 0 & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{4} - b_4 c_4/b_3 & (\frac{1}{4} - b_4 c_4)/b_3 - a_{32} & (\frac{1}{4} - b_4 c_4)/b_3 - a_{31} & & \\
\frac{1}{4} - b_3 c_3/b_4 & (\frac{1}{4} - b_3 c_3)/b_4 - a_{42} - a_{43} & (\frac{1}{4} - b_3 c_3)/b_4 - a_{41} - a_{43} & (\frac{1}{4} - b\ldots & \\
\hline
 & (\frac{1}{2} - b_3 - b_4) & (\frac{1}{2}) & (\frac{1}{2} - b_1 - b_4) & (\frac{1}{6} - \ldots
\end{array}$$

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The Butcher tableau

$$
\begin{array}{c|cccc}
0 & 0 & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
(\frac{1}{4} - b_4 c_4)/b_3 & (\frac{1}{4} - b_4 c_4)/b_3 - a_{32} & (\frac{1}{4} - b_4 c_4)/b_3 - a_{31} & & \\
(\frac{1}{4} - b_3 c_3)/b_4 & (\frac{1}{4} - b_3 c_3)/b_4 - a_{42} - a_{43} & (\frac{1}{4} - b_3 c_3)/b_4 - a_{41} - a_{43} & (\frac{1}{4} - b_3 c_3)/b_4 - a_{41} - a_{42} & \\
\hline
 & (\frac{1}{2} - b_3 - b_4) & (\frac{1}{2}) & (\frac{1}{2} - b_1 - b_4) & (\frac{1}{2} - b_1 - b_3)
\end{array}
$$

1) i) Eqns derivations

Given data

a) $dS/dt = BIS/N$

$\frac{1}{4} - b_4 c_4 / b_3$	$(\frac{1}{4} - b_4 c_4)/b_3 - a_{32}$	$(\frac{1}{4} - b_4 c_4)/b_3 - a_{31}$	
$\frac{1}{4} - b_3 c_3 / b_4$	$(\frac{1}{4} - b_3 c_3)/b_4 - a_{42} - a_{43}$	$(\frac{1}{4} - b_3 c_3)/b_4 - a_{41} - a_{43}$	$(\frac{1}{4} - b_3 c_3)/b_4 - a_{41} \ldots$
	$(\frac{1}{2} - b_3 - b_4)$	$(\frac{1}{2})$	$(\frac{1}{2} - b_1 - b_4)$	$(\frac{1}{2} - b_1 - b_3)$

3d) Eqns derivations

$$dS/dt = \beta I S / N$$

$$dI/dt = \beta I S / N - \gamma I - \mu I$$

$$dR/dt = \gamma I$$

$$dD/dt = \mu I$$

Given data
$N = 5{,}000{,}000$
$\beta = 0.4$
$\gamma = 0.10105 = 2.1$
$\mu = 0.2$

$\boxed{\text{See Code}}$

$\frac{1}{4} - b_4 c_4 / b_3$	$(\frac{1}{4} - b_4 c_4)/b_3 - a_{32}$	$(\frac{1}{4} - b_4 c_4)/b_3 - a_{31}$	
$\frac{1}{4} - b_3 c_3 / b_4$	$(\frac{1}{4} - b_3 c_3)/b_4 - a_{42} - a_{43}$	$(\frac{1}{4} - b_3 c_3)/b_4 - a_{41} - a_{43}$	$(\frac{1}{4} - b_3 c_3)/b_4 - a_{41} \ldots$
	$(\frac{1}{2} - b_3 - b_4)$	$(\frac{1}{2})$	$(\frac{1}{2} - b_1 - b_4)$	$(\frac{1}{2} - b_1 - b_3)$

3d) Eqns derivations

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(2c) Derivation of 4^{th} order Adam Bashforth

Soln

Consider the AM general function given by

$$y_{n+1} = y_n + \frac{h}{24}\left(55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right)$$

Let $Ey_{n-2}' = y_{n-1}'$

$Ey_{n-2}' = y_{n-1}'$

substituting into our function above we have

$E^2y_{n-2}' = y_n'$

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$$\sigma(E) - \frac{9}{24}E^3 + \frac{37}{24}E^2 - \frac{59}{24}E + \frac{55}{24}$$

$$\rho(E) = \left(E^3 - 1\right)$$

$$\left(E^3-1\right)y_{n-2} - h\left(-\frac{9}{24}E^3 + \frac{37}{24}E^2 + \frac{59}{24}E + \frac{55}{24}\right)y_{n-2}'$$

(2c) Derivation of 4^{th} order Adam Bashforth sum

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$Ey_{n-2}' = y_{n-1}'$

$E^2y_{n-2}' = y_n'$

$$\sigma(E) - \tfrac{9}{24}E^3 + \tfrac{37}{24}E^2 - \tfrac{59}{24}E + \tfrac{55}{24}$$

$$\rho(E) = (E^3 - 1)$$

$$(E^3 - 1)y_{n-2} - h\left(-\tfrac{9}{24}E^3 + \tfrac{37}{24}E^2 + \tfrac{59}{24}E + \tfrac{55}{24}\right)y_{n-2}' = 0$$

Setting the root condition

$$\rho(\xi) = 0 \Rightarrow$$

characteristic polynomial

$\sigma(\xi)$ ... 9 ... ^{3} ... 37 ... ξ^2 ... $59\xi + 55/$ — 1^{st} charac

$\frac{h}{24}(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3})$

Let $Ey'_{n-2} = y'_{n-1}$

$Ey'_{n-2} = y'_{n-1}$

$E^2 y'_{n-2} = y'_n$

substituting into our function above we have

$\sigma(E) - \frac{9}{24}E^3 + \frac{37}{24}E^2 - \frac{59}{24}E + \frac{55}{24}$

$\rho(E) = (E^3 - 1)$

$(E^3-1)y_{n-2} - h\left(-\frac{9}{24}E^3 + \frac{37}{24}E^2 + \frac{59}{24}E + \frac{55}{24}\right)y'_{n-2} = 0$

Setting the root condition

$\rho(\xi) = 0 \Rightarrow$

Characteristic polynomial

$\sigma(\xi) = -\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24}$ — 1st characteristic

$\rho(\xi) = (\xi^3 - 1)$ — 2nd characteristic

For the ~~root~~ root condition $(h=0)$ we have

$\rho(E) = (E^3 - 1)$

$(E^3-1)y_{n-2} - h\left(-\frac{9}{24}E^3 + \frac{37}{24}E^2 + \frac{59}{24}E + \frac{55}{24}\right)y'_{n-2} = 0$

Setting the root condition

$\rho(\xi) = 0 \Rightarrow$

Characteristic polynomial

$\sigma(\xi) = -\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24}$ — 1st characteristic

$\rho(\xi) = (\xi^3 - 1)$ — 2nd characteristic

For the ~~root~~ root condition $(h=0)$ we have

$\rho(\xi) = 0$

$(\xi^3 - 1) = 0 \Rightarrow \xi^3 = 1$

$-\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24} = 0$

$\rho(E) = (E^3 - 1)$

$(E^3-1)y_{n-2} - h\left(-\frac{9}{24}E^3 + \frac{37}{24}E^2 + \frac{59}{24}E + \frac{55}{24}\right)y'_{n-2} = 0$

Setting the ~~root~~ condition

$\rho(\xi) = 0 \implies$

Characteristic polynomial

$\sigma(\xi) = -\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24}$ — 1st characteristic

$\rho(\xi) = (\xi^3 - 1)$ ——— 2nd characteristic

For the ~~root~~ root condition (hd) we have.

$\rho(\xi) = 0$

$(\xi^3 - 1) = 0 \implies \xi^3 = 1$

$-\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24} = 0$

$\implies \xi\left(-\frac{9}{24}\xi^2 + \frac{37}{24}\xi - \frac{59}{24}\right) = -\frac{55}{24}$

Interval is from -1 to 1, the above values lies within this margin interval