PART B

1) Using Cholesky decomposition method

let $A = LL^T \Rightarrow LL^T x = b \Rightarrow Lz = b \Rightarrow L^T x = z$

System of linear eqn

$4x_1 - 2x_2 + 4x_3 + 2x_4 = 30$

$-2x_1 + 5x_2 - 2x_3 + 3x_4 = -27$

$4x_1 - 2x_2 + 13x_3 - x_4 = 93$

$2x_1 + 3x_2 - x_3 + 15x_4 = -15$

Soln

In matrix form

$$\underbrace{\begin{pmatrix} 4 & -2 & 4 & 2 \\ -2 & 5 & -2 & 3 \\ 4 & -2 & 13 & -1 \\ 2 & 3 & -1 & 15 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 30 \\ -27 \\ 93 \\ -15 \end{pmatrix}}_{b}$$

$$\underbrace{\begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & 0 & L_{44} \end{pmatrix}}_{L^T} = \begin{pmatrix} A \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} & L_{11}L_{41} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} & L_{21}L_{41} + L_{22}L_{42} \\ L_{11}L_{31} & L_{21}L_{31} + L_{32}L_{22} & L_{31}^2 + L_{32}^2 + L_{33}^2 & L_{31}L_{41} + L_{32}L_{42} + L_{33}L_{43} \\ L_{11}L_{41} & L_{41}L_{21} + L_{42}L_{22} & L_{41}L_{31} + L_{42}L_{32} + L_{43}L_{33} & L_{41}^2 + L_{42}^2 + L_{43}^2 + L_{44}^2 \end{pmatrix} \quad (*)$$

Equating (*) to A and solving for L and L^T we have

$\Rightarrow L_{11}^2 = 4 \Rightarrow L_{11} = 2$

$2L_{21} = -2 \Rightarrow L_{21} = -1$

$2L_{31} = 4 \Rightarrow L_{31} = 2$

$2L_{41} = 2 \Rightarrow L_{41} = 1$

$1 + L_{22}^2 = 5 \Rightarrow L_{22}^2 = 4 \Rightarrow L_{22} = 2$

$\Rightarrow -1.1 + 2 L_{42} = 3$

$2L_{42} = 4 \Rightarrow L_{42} = 2$

$\Rightarrow 4 + 0 + L_{33}^2 = 13 \Rightarrow L_{33} = 3$

$\Rightarrow 2.1 + 0 + 3L_{43} = -1$

$2 + 3L_{43} = -1 \Rightarrow L_{43} = 1$

$\Rightarrow 1 + 4 + 1 + L_{44}^2 = 15$

$\Rightarrow L_{44}^2 = 15 - 6 = 9$

$\Rightarrow L_{44} = 3$

$$\Rightarrow L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix} \qquad L^T = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$\Rightarrow L L^T \underset{\sim}{x} = b$

$$\Rightarrow L\underset{\sim}{z} = b \qquad \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$\Rightarrow 2z_1 = 30 \Rightarrow z_1 = 15$

$\Rightarrow -z_1 + z_2 = -27 \Rightarrow -15 + 2z_2 = -27 \Rightarrow z_2 = -6$

$\Rightarrow 2z_1 + 3z_3 = 93 \Rightarrow 3z_3 = 93 - 30 \Rightarrow z_3 = 21$

$\Rightarrow z_1 + 2z_2 - z_3 + 3z_4 = -18 \Rightarrow 15 - 12 - 21 + 3z_4 = -18 \Rightarrow z_4 = 0$

$$\Rightarrow \underset{\sim}{z} = \begin{pmatrix} 15 \\ -6 \\ 21 \\ 0 \end{pmatrix}$$

$$\Rightarrow L^T x = z \Rightarrow \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -15 \\ -6 \\ 21 \\ 0 \end{pmatrix}$$

$3x_3 = 21 \Rightarrow x_3 = 7$

$3x_4 = 0 \Rightarrow x_4 = 0$, $3x_3 = 21$

$\Rightarrow 2x_2 + 2x_4 = 6 \Rightarrow x_2 = -3$

$\Rightarrow 2x_1 - x_2 + 2x_3 + x_4 = 15 \Rightarrow x_1 = -1$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

(4c)(i) Explanation

$\Rightarrow$ Also known as triangularization or factorization is used to compute the inverse A^{-1} by first computing both the upper and lower triangular matrices given by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \text{ and } U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

$\Rightarrow$ We then get the product of LU so as to equate ~~it~~ to the matrix A and calculating the coefficients for matrixes L and U which we then obtain inverse A^{-1}

(ii) $$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

let $L_{ii} = 1$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{pmatrix} \times \overset{U}{\begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}}$$

$\Rightarrow u_{11} = 4 \quad , u_{12} = -2 \quad , u_{13} = 4 \quad , u_{14} = 2$

$\Rightarrow 4L_2 = -2 \Rightarrow L_{21} = -\frac{1}{2} \quad , -\frac{1}{2} \cdot 2 + u_{24} = 3 \Rightarrow u_{24} = 4$

$\Rightarrow 4 . L_{31} = 4 \Rightarrow L_{31} = 1 \quad , 1 \cdot 2 + 4L_{32} = -2 \quad , 4L_{32} = -2+2 \ , L_{32} = 0$

$\Rightarrow 4 + u_{33} = 13 \Rightarrow u_{33} = 9 \quad , 4L_{41} = 2 \Rightarrow L_{41} = \frac{1}{2}$

$\Rightarrow 2 + 9L_{43} = -1 \quad , L_{43} = -\frac{1}{3} \quad , 4L_{42} = 4 \quad , L_{41} = 1$

$\Rightarrow 1 + 4 + 1 + u_{44} = 15 \Rightarrow u_{44} = 9$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad u = \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$Lz = b$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

$\Rightarrow z_1 = 30$

$\Rightarrow \frac{1}{2} z_1 + z_2 = -27 \quad , z_2 = -12$

$\Rightarrow z_1 + z_3 = 93 \quad , z_3 = 63$

$\Rightarrow z_4 + \frac{1}{2} . 30 - 12 - \frac{1}{3} . 63 = -18$

$\Rightarrow z_4 = 0$

$$\Rightarrow z = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$$ux = z \Rightarrow \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$\Rightarrow x_4 = 0$

$9x_3 = 63 \Rightarrow x_3 = 7$

$4x_2 = -12 \Rightarrow x_2 = -3$

$4x_1 + 6 + 28 = 30 \Rightarrow x_1 = -1$

$$x = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

(iii) Why is not the most efficient? method FAILS

=> This is because LU decomposition is equal to zero any of the pivot elements

i.e, if $L_{ii} = 1$ and $U_{ii} = 0$ or if

$U_{ii} = 1$ and $L_{ii} = 0$

A better efficient method to solve the above System would be to use the Jacobi Iterative method or the Gauss - Siedel approximation Iterative method

(4a)

$$\begin{array}{c|cc} 3/4 & 3/4 & 0 \\ 1 & 1/4 & 3/4 \\ \hline & 1/2 & 1/2 \end{array} \Rightarrow \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

=> $c_1 = 3/4$, $c_2 = 1$, $a_{11} = 3/4$ $a_{12} = 0$

$a_{21} = 1/4$, $a_{22} = 3/4$, $b_1 = 1/2$, $b_2 = 1/2$

=> For an A-stable function above we have that it

$\bar{C} = \{ z \in \mathbb{R} , \mathbb{R}e(z) \leq 0 \}$

=> so the above Condition is met iff

$\mathbb{R}(iy) \leq 1$ $y \in \mathbb{R}$

$R(z)$ is analytic for $\mathbb{R}e(z) < 0$

=> Thus the stability function for the above Butcher tableau is given by

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{3}{16}z^3 + \frac{1}{48}z^4$$

Note that all non-zero entries are ~~above~~ the below the diagonal elements

ii) let $A = \begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}$ & $b^T = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

Therefore $R(z) = N(z)/D(z)$ for polynomial $E(y) = D(iy)D(-iy) - N(iy)N(-iy)$

=) For simplicity we follow the below conditions

if A

(i) All roots $D(z)$ are in the right hand side

(ii) $E(iy) \geq 0$ for all $y \in \mathbb{R}$

then the method is A-stable,

$$R(z) = \frac{\det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}\right]}{\det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z\begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}\right]}$$

$$\Rightarrow R(z) = \frac{\det\begin{bmatrix} 1/4 + 1/2 z & 0 \\ -1/4 & 1/4 + 1/2 z \end{bmatrix}}{\det\begin{bmatrix} 1 - 3/4 z & 0 \\ -1/4 z & 1 - 3/4 z \end{bmatrix}} = \frac{\left(1/4 + 1/2 z\right)^2 + 1/4}{\left(1 - 3/4 z\right)^2 + 1/4 z}$$

$$\Rightarrow N(z) = 1/4 + \left(1/4 + 1/2 z\right)^2 = 1/4 z^2 + 1/4 z + 5/16$$

$$D(z) = 1/4 z + \left(1 - 3/4 z\right)^2 = 9/16 z^2 - \frac{11}{4} z + 1$$

let $a = 9/16$, $b = -11/4$ $c = 1$

$$z = \frac{11/4 \pm \sqrt{\frac{121}{16} - \frac{36}{16}}}{9/8} \Rightarrow z_1 = 4.493$$

$$z_2 = 0.3956$$

$\Rightarrow$ Thus the roots of $D(z)$ are all in the R.H.S so the first condition is met / satisfied

$\Rightarrow E(y) = D(iy)D(-iy) - N(iy)N(-iy)$

$\Rightarrow \cancel{\left(1 - \frac{11}{4}iy\right)}$

$\Rightarrow \left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right)\left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right)$

$\Rightarrow -\left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right)\left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right)$

$\Rightarrow \frac{3}{6} + \frac{1}{4}iy - \frac{1}{4}y^2 - \frac{55}{36}iy - \frac{11}{16}y^2 - \frac{15}{32}y^2 - \frac{9}{64}iy^3 - \frac{9}{64}y^4 \geq 0$

$\Rightarrow$ Thus since $E(y) \geq 0$ and roots $D(z)$ are in the R.H.S plane then we can conclude that the method is A-stable

(4 c(i))

Continuation

Having determined L and U of the system $A\underline{x} = \underline{b}$

$\Rightarrow \underline{L}\,\underline{U}\,\underline{x} = \underline{b}$

$\Rightarrow$ simplified into $\underline{L}\,z = \underline{b}$

whereby $z = L^{-1}\underline{b}$ or $z = \underline{U}\underline{x}$

$\Rightarrow$ The z-values are obtained by inverse L^{-1} of the matrix A^{-1} multiplied by the $\underline{b}$-values

$\Rightarrow L^{-1}U^{-1} = A^{-1}$

$\boxed{X}$

PART A

(1a) Using $y'(t) = -10(y-t^2)+2t$ $y(0)=-1$

at $t_0=0$, $y_0=0$

$$\frac{dy}{dt} = -10(y-t^2)+2t$$

Soln

at $t_1 = 0.1$, $y_1 = 0.3778794$

$$y' = -10\left[0.3778794 - (0.1)^2\right] + 2(0.1) = -3.4787940$$

at $t_2 = 0.2$ $y_2 = 0.1753352$

$$y' = -10\left[0.1753353 - (0.2)^2\right] + 2(0.2) = -0.9533540$$

at $t_3 = 0.3$, $y_3 = 0.1397871$

$$y' = -10\left[0.1397871 - (0.3)^2\right] + 2(0.3) = 0.1021290$$

we need to calculate the values of y_4 at $t_4=0.4$ and y_5 at $t_5=0.5$

Now to find the y_4 and y_5 we need to first write the generalized milne predictor and corrector formulae as

$$y_{j+1} = y_{j-1} + h\left(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 + \dots\right)f_j$$

For the values stating $j=n$ and the ∇ value given by the difference in the interval of time t, this equation is simplified into

$$y^p_{n+1} = y_{n-3} = \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

and similarly for the corrector method

$$y_{n+1}^{c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y_n' + y'_{n+1}\right)$$

So at $n=3 \Rightarrow j=3$

$$y_4^{p} = y_0 + \frac{4}{3}h\left(2y_1' - y_2' + 2y_2'\right)$$

$$= 1 + \frac{4}{3}(0.1)\left[2(-3.4787940) - (-0.9533540) + 2(0.1021290)\right]$$

$$y_4^{p}(0.4) = 0.2266699$$

$y'(t) = -10(y-t^2)+2t$

$$y_4' = -10\left(0.2266699 - (0.4)^2\right) + 2(0.4) = 0.1333013$$

as for the corrector method

$$y_4^{c} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$= 0.1753353 + \frac{0.1}{3}\left(-0.9533540 + 4(0.1021290) + 0.1333013\right)$$

$$= 0.2251743$$

$$y_4' = -10\left(0.2251743 - (0.4)^2\right) + 2(0.4) = 0.1482566$$

at $t=0.5$ $\quad y(0.5) = ?$

at $n=4 \Rightarrow j=4$

$$y_5^{p} = \frac{4}{3}h\left(2y_2' - y_3' + 2y_4'\right)$$

$$0.3778794 + \frac{4}{3}(0.1)\left[2(9533540) - (0.1021290) + 2(0.1333013)\right]$$

$$= 0.1455815$$

$y_5' = -10\left(0.1455815 - (0.5)^2\right) + 2(0.5) = 2.0442850$

as for the corrector method

$= 0.1397871 + 0.1/3\left(0.1021290 + 4(0.1333013) + 2.0442850\right)$

$= 0.2291044$

$y_5' = -10\left(0.2291044 - (0.5)^2\right) + 2(0.5) = 1.2089559$

Table

j	t	y
0	0.0	1.0000000
1	0.1	0.3778794
2	0.2	0.1753353
3	0.3	0.1397871
4	0.4	0.2266699
5	0.5	0.2

Table

j	t	y	y'	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
0	0.0	1.0000000	-10.0000000				
				6.5212060			
					-3.9957660		
						2.5258090	
1	0.1	0.3778794	-3.4787940				
				2.5254400			-2.080144
					-1.4699570		
						0.4456463	
2	0.2	0.1753353	-0.9533540				
				1.0554830			1.623146
					-1.0243107		
						2.0687930	
3	0.3	0.1397871	0.1021290				
				0.0311723			
					1.0444823		
4	0.4	0.2266699	0.1333013				
				1.0756546			
5	0.5	0.2291044	1.2089559				

$\nabla^5 f$

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2a) For the Nystrom-Milne predictor method

$$y_{j+1} = y_{j-1} + h\left(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 + \dots\right) f_j$$

For $j = n$

$$\Rightarrow y_{n+1} = y_{n-1} + h\left(2 + \frac{1}{3}y_{n-1}' + \frac{1}{3}y_{n-2}' + \frac{29}{90}y_{n-3}' + \dots\right) f_{j}$$

$$\Rightarrow E y_{n-2}' = y_{n+1}'$$

$$E y_{n-2}' = y_{n-1}'$$

$$E^2 y_{n-2}' = y_n'$$

$$\sigma(E) = \frac{28}{90}E^4 + \frac{29}{90}E^3 + \frac{1}{3}E^2 + \frac{1}{3}E' + 2$$

$$\rho(E) = (E^3 - E)$$

$$(E^3 - E)y_{n=2} - h\left(\frac{28}{90}E^4 + \frac{29}{90}E^3 + \frac{1}{3}E^2 + \frac{1}{3}E + 2\right)y_{n-2}' = 0$$

$$\rho(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1) = 0$$ — 1st characteristic

$$\sigma(\xi) = \frac{28}{90}\xi^4 + \frac{29}{90}\xi^3 + \frac{1}{3}\xi^2 + \frac{1}{3}\xi + 2$$ — 2nd characteristic

Root condition

at $\rho(\xi) = 0$

$$\Rightarrow \rho(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1)$$

$$\Rightarrow \xi = 0, \quad \xi_2 = 1, \quad \xi_1 = -1$$

These are the simple roots corresponding to the milne-predictor method

Stability

$$\sigma(\xi) = 28/90\,\xi^4 + 29/90\,\xi^3 + \tfrac{1}{3}\xi^2 + \tfrac{1}{3}\xi + 2 = 0$$

$$\Rightarrow |\xi_4| > 0$$

Hence zero stable since this will yield simple roots

Convergence and ~~Stab~~ Consistency

=) We suppose the method is zero-stable, then assume that $\rho(\xi)$ has double roots in the unit circle. Further, product of roots of $\rho(\xi)$ we set as 1, but now ξ_1 and ξ_2 are equal to ± 1, thus not strictly complex hence they converge and are consistent

$$\xi = 0 \quad , \xi_{12} = \pm 1 \quad , \xi_3 = -1$$

=) For the corrector method,

as $j = n$

$$y_{j+1} = y_{j-1} = h\left(2 - 2\nabla + \tfrac{1}{3}\nabla^2 - \tfrac{1}{90}\nabla^4 - \tfrac{1}{90}\nabla^5 - \dots\right) f_{j+1}$$

=) simplifies into

$$y_{n+1} = y_{n-1} + h\left(2 - 2y_n' + \tfrac{1}{3}y_{n-1}' - \tfrac{1}{90}y_{n-2}' - \tfrac{1}{90}y_{n-3}' + \dots\right) f_{j+1}$$

$$\Rightarrow E y_{n-2}' = y_{n-1}'$$

$$E y_{n-2}' = y_{n-1}'$$

$$E^2 y_{n-2}' = y_n'$$

$$\Rightarrow \sigma(E) = -\tfrac{1}{90}E^3 - \tfrac{1}{90}E^2 + \tfrac{1}{3}E + 2 - 2$$

$$= -\tfrac{1}{90}E^3 - \tfrac{1}{90}E^2 + \tfrac{1}{3}E$$

$$P(E) = E^3 - E^2$$

$$\Rightarrow (E^3 - E^2) y_{n-2} - h\left(-\frac{1}{40}E^3 - \frac{1}{40}E^2 + \frac{1}{3}E\right) y'_{n-2} = 0$$

$$\rho(\xi) = (\xi^3 - \xi^2) = \xi^2(\xi - 1)$$ — 1^{st} characteristic

$$\sigma(\xi) = -\frac{1}{40}\xi^3 - \frac{1}{40}\xi^2 + \frac{1}{3}\xi$$ — 2^{nd} characteristic

Root condition at $\rho(\xi) = 0$

$$\Rightarrow \rho(\xi) = \xi^2(\xi - 1) = 0$$

$$\Rightarrow \xi_1 = 1, \quad \xi_{2,3} = 0$$

These are simple roots of the corresponding corrector method

Stability

$$\sigma(E) = \sigma(\xi) = -\frac{1}{40}\xi^3 - \frac{1}{40}\xi^2 + \frac{1}{3}\xi \quad \text{at } \rho(\xi) = 0$$

$$= -\frac{1}{40}\xi^3 - \frac{1}{40}\xi^2 + \frac{1}{3}\xi = 0$$

$$\Rightarrow |\xi_3| > 1$$

Hence Zero-stable since it yields complex roots

Convergence and Consistency

$$\rho(\xi) = \xi^2(\xi - 1) = 0$$

we suppose the method is zero-stable, then assume that $\rho(\xi)$ has double roots in the unit circle

$\Rightarrow$ Further, product of roots of $\rho(\xi)$ is 1 and since the roots are not equal and neither equal to ± 1, then they are strictly complex

$$\Rightarrow \xi \neq \xi_{2,3}$$

(2b) From the Nystrom - predictor Milne method,

$$y_{j+1} = y_{j-1} + h\left(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 f \ldots\right) f_j$$

Setting $j = n$, a simplified Nystrom - Milne's predictor method is given as follows

$$y_{n+1} = y_{n-3} + \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$\Rightarrow y_{n+1} - y_{n-3} - \frac{4}{3}h\left(2y'_{n-2} - y'_{n-1} + 2y_n'\right)$$

Let $Ey'_{n-2} = y'_{n-1}$

$\therefore$

$Ey'_{n-1} = y_n'$

$E^2 y'_{n-2} = y_n'$

$$\Rightarrow \sigma(E) = \frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3}$$

$$\rho(E) = (E^3 - 0)$$

$$\Rightarrow (E^3 - 0)y_{n-2} - h\left(\frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3}\right)y'_{n-2} = 0$$

$\Rightarrow$ (i) $\rho(\xi) = \xi^3 = \xi(\xi^2)$ — 1st characteristic polynomial

(ii) $\sigma(\xi) = \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{8}{3}$ — 2nd characteristic polynomial

Root Condition

- The explicit method of the Nystrom - Milne's method is said to satisfy the root condition if the roots of the equation $\rho(\xi) = 0$ lies inside the closed unit disk in the complex plane and are simple if they lie on the circle

$\rho(\xi) = 0$

$\Rightarrow$ From our equation

$\rho(\xi) = \xi^3 = 0 \quad \Rightarrow \xi_{1,2,3} = 0$

Stability

$\Rightarrow$ The linear predictor method is zero stable if given $y' = f(x,y)$, $y(x_0) = y_0$, $f(x,y)$ satisfy lipschitz condition and root condition holds

$\Rightarrow$ Thus we have in our case function

$$\sigma(\xi) = \frac{5}{3}\xi^2 - \frac{4}{3}\xi + \frac{5}{3}$$ which is zero-stable since it yields simple roots

$$\Rightarrow \frac{5}{3}\xi^2 - \frac{4}{3}\xi + \frac{5}{3} = 0$$

$$\Rightarrow |\xi_{1,2}| \geq 0$$

Convergence & Consistency

$\Rightarrow$ The linear predictor above is consistent if it has order $p \geq 1$

$$\rho(\xi) = \xi^2(\xi) = 0$$

Lets suppose the method is zero-stable, then $\rho(\xi)$ has double roots in the unit circle

$\Rightarrow$ Further product of roots of $\rho(\xi)$ is 1, and since the roots are neither equal to ± 1, have said to be strictly complex

$$\Rightarrow \xi(\xi^2) = 0 \Rightarrow \xi_1 = 0, \xi_2 = 0 \; \xi_3 = 0$$

Derivation of 4^{th} order explicity Runge Kutta method

$b_2 = 1/4$, $c_2 = 1/2$

Soln

let the Conditions eqns be defined by

$b_1 + b_2 + b_3 + b_4 = 1$ — ①

$b_2c_2 + b_3c_3 + b_4c_4 = \frac{1}{2}$ — ②

$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = \frac{1}{3}$ — ③

$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = \frac{1}{4}$ — ④

$b_3c_3a_{32}c_2 + b_4c_4(a_{42}c_2 + a_{43}c_3) = \frac{1}{8}$ — ⑤

$c_2 = a_{21}$

$c_3 = a_{31} + a_{32}$

$c_4 = a_{41} + a_{42} + a_{43}$

$b_3a_{32} + b_4a_{42} = b_2(1-c_2)$ — ⑥

$b_4a_{43} = b_3(1-c_3)$ — ⑦

$0 = b_4(1-c_4)$ — ⑧

so for $b_2 = 1/4$ and $c_2 = 1/2$ we have

Subsituting $b_2 = 1/4$ and $c_2 = 1/2$ into equation ① — ③ we have

$b_1 + \frac{1}{2} + b_3 + b_4 = 1 \Rightarrow b_1 + b_3 + b_4 = \frac{1}{2}$ — ⑨

$\frac{1}{4} + b_3c_3 + b_4c_4 = \frac{1}{2} \Rightarrow b_3c_3 + b_4c_4 = \frac{1}{4}$ — ⑩

$\frac{1}{2}\cdot\frac{1}{4} + b_3c_3^2 + b_4c_4^2 = \frac{1}{2} \Rightarrow b_3c_3^2 + b_4c_4^2 = 5/24$ — ⑪

$\frac{1}{2}\cdot\frac{1}{8} + b_3c_3^3 + b_4c_4^3 = \frac{1}{4} \Rightarrow b_3c_3^3 + b_4c_4^3 = \frac{3}{16}$ — ⑫

$b_1 = \frac{1}{2} - b_3 - b_4$, $b_2 = \frac{1}{2}$ $b_3 = \frac{1}{2} - b_1 - b_4$, $b_4 = \frac{1}{2} - b_1 - b_3$

$c_2 = 1/2$, $c_3 = (1/4 - b_4 c_4)/b_3$ $c_4 = (1/4 - b_3 c_3)/b_4$

$a_{21} = 1/2$, $a_{31} = (1/4 - b_4 c_4)/b_3 - a_{32}$, $a_{32} = (1/4 - b_4 c_4)/b_3 - a_{31}$

$a_{41} = (1/4 - b_3 c_3)/b_4 - a_{42} - b_{43}$, $a_{42} = (1/4 - b_3 c_3)/b_4 - a_{41} - a_{43}$

$a_{43} = (1/4 - b_3 c_3)/b_4 - a_{41} - a_{42}$

The Butcher tableau:

$$
\begin{array}{c|cccc}
0 & 0 & & & \\
1/2 & 1/2 & & & \\
1/4 - b_4 c_4 / b_3 & (1/4 - b_4 c_4)/b_3 - a_{32} & (1/4 - b_4 c_4)/b_3 - a_{31} & & \\
1/4 - b_3 c_3 / b_4 & (1/4 - b_3 c_3)/b_4 - a_{42} - a_{43} & (1/4 - b_3 c_3)/b_4 - a_{41} - a_{43} & (1/4 - b_3 c_3)/b_4 - a_{41} - a_{42} & \\
\hline
 & (1/2 - b_3 - b_4) & (1/2) & (1/2 - b_1 - b_4) & (1/2 - b_1 - b_3)
\end{array}
$$

(3d) Eqns derivations

$$\frac{dS}{dt} = \beta I S / N$$

$$\frac{dI}{dt} = \beta I S / N - \gamma I - \mu I$$

$$\frac{dR}{dt} = \gamma I$$

$$\frac{dD}{dt} = \mu I$$

Given data
$N = 5\,000,000$
$\beta = 0.4$
$\gamma = 0.1 / 0.05 = 2.1$
$\mu = 0.2$

See code

(2c) Derivation of 4^{th} order Adam Bashforth

Sun

Consider the AM general function given by

$$y_{n+1} = y_n + \frac{h}{24}\left(55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right)$$

Let $Ey_{n-2}' = y_{n-1}'$

$Ey_{n-2}' = y_{n-1}'$

substituting into our function above we have

$E^2y_{n-2}' = y_n'$

$$\sigma(E) - 9/24\,E^3 + 37/24\,E^2 - 59/24\,E + 55/24$$

$$\rho(E) = (E^3 - 1)$$

$$(E^3-1)\,y_{n-2} - h\left(-\tfrac{9}{24}E^3 + \tfrac{37}{24}E^2 + \tfrac{59}{24}E + \tfrac{55}{24}\right)y_{n-2}' = 0$$

Setting the root condition

$\rho(\xi) = 0 \Rightarrow$

Characteristic polynomial

$$\sigma(\xi) = -\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24} \quad \text{— } 1^{st} \text{ characteristic}$$

$$\rho(\xi) = (\xi^3 - 1) \quad \text{— } 2^{nd} \text{ characteristic}$$

For the ~~root~~ root condition (h=1) we have

$\rho(\xi) = 0$

$(\xi^3 - 1) = 0 \Rightarrow \xi^3 = 1$

$$-\frac{9}{24}\xi^3 + \frac{37}{24}\xi^2 - \frac{59}{24}\xi + \frac{55}{24} = 0$$

$\Rightarrow \xi\left(-\frac{9}{24}\xi^2 + \frac{37}{24}\xi - \frac{57}{24}\right) = -\frac{55}{24}$

Interval is from -1 to 1, the above values lies within the margin interval