

PART B

(4b) Using Cholesky decomposition method

$$\text{let } A = LL^T \Rightarrow LL^T x = b \Rightarrow Lz = b \Rightarrow L^T z = x$$

System of linear eqns

$$4x_1 - 2x_2 + 4x_3 + 2x_4 = 30$$

$$-2x_1 + 5x_2 - 2x_3 + 3x_4 = -27$$

$$4x_1 - 2x_2 + 12x_3 + 2x_4 = 93$$

$$2x_1 + 3x_2 - x_3 + 15x_4 = -15$$

Sln

In matrix form

$$\begin{array}{c|cc|c} & A & x & b \\ \hline & 4 & -2 & 4 & 2 \\ & -2 & 5 & -2 & 3 \\ & 4 & -2 & 13 & -1 \\ & 2 & 3 & -1 & 15 \end{array} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -15 \end{pmatrix}$$

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In matrix form

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$L$        $L^T$

$$\Rightarrow \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & 0 & L_{44} \end{pmatrix} = \begin{pmatrix} A \end{pmatrix}$$

System of linear eqns

$$\begin{aligned} 4x_1 - 2x_2 + 4x_3 + 2x_4 &= 30 \\ -2x_1 - 5x_2 - 2x_3 + 3x_4 &= -27 \\ 4x_1 - 2x_2 + 12x_3 + 2x_4 &= 93 \\ 2x_1 + 3x_2 - 2x_3 + 15x_4 &= -15 \end{aligned}$$

Sln

In matrix form

$$\begin{array}{c|ccccc} A & x & b \\ \hline 4 & -2 & 4 & 2 & x_1 & 30 \\ -2 & 5 & -2 & 3 & x_2 & -27 \\ 4 & -2 & 13 & -1 & x_3 & 93 \\ 2 & 3 & -1 & 15 & x_4 & -15 \end{array}$$

$L$

$L^T$

$$\Rightarrow \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & 0 & L_{44} \end{pmatrix} = \begin{pmatrix} A \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} L_{11}^2 \quad L_{11}L_{21} \quad L_{11}L_{31} \quad L_{11}L_{41} \\ L_{11}L_{21} \quad L_{21}^2 + L_{22}^2 \quad L_{21}L_{31} + L_{22}L_{32} + L_{21}L_{41} + L_{22}L_{42} \\ L_{11}L_{31} \quad L_{21}L_{31} + L_{32}L_{22} \quad L_{31}^2 + L_{32}^2 + L_{33}^2 \quad L_{31}L_{41} + L_{32}L_{42} + L_{33}L_{43} \\ L_{11}L_{41} \quad L_{41}L_{21} + L_{42}L_{22} \quad L_{41}L_{31} + L_{42}L_{32} + L_{43}L_{33} \quad L_{41}^2 + L_{42}^2 + L_{43}^2 + L_{44}^2 \end{array}$$

$$\begin{array}{c|ccccc} A & x & b \\ \hline 4 & -2 & 4 & 2 & x_1 & 30 \\ -2 & 5 & -2 & 3 & x_2 & -27 \\ 4 & -2 & 13 & -1 & x_3 & 93 \\ 2 & 3 & -1 & 15 & x_4 & -15 \end{array}$$

$L$

$L^T$

$$\Rightarrow \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ 0 & L_{22} & L_{32} & L_{42} \\ 0 & 0 & L_{33} & L_{43} \\ 0 & 0 & 0 & L_{44} \end{pmatrix} = \begin{pmatrix} A \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} L_{11}^2 \quad L_{11}L_{21} \quad L_{11}L_{31} \quad L_{11}L_{41} \\ L_{11}L_{21} \quad L_{21}^2 + L_{22}^2 \quad L_{21}L_{31} + L_{22}L_{32} + L_{21}L_{41} + L_{22}L_{42} \\ L_{11}L_{31} \quad L_{21}L_{31} + L_{32}L_{22} \quad L_{31}^2 + L_{32}^2 + L_{33}^2 \quad L_{31}L_{41} + L_{32}L_{42} + L_{33}L_{43} \\ L_{11}L_{41} \quad L_{41}L_{21} + L_{42}L_{22} \quad L_{41}L_{31} + L_{42}L_{32} + L_{43}L_{33} \quad L_{41}^2 + L_{42}^2 + L_{43}^2 + L_{44}^2 \end{array}$$

Putting (\*) to A and using for L and  $L^T$  we

$$\begin{array}{ll} L_{11}^2 = 4 \Rightarrow L_{11} = 2 & 2L_{41} = 2 \Rightarrow L_{41} = 1 \\ L_{21} = -2 \Rightarrow L_{21} = -1 & 1 + L_{22}^2 = 5 \Rightarrow L_{22}^2 = 4 \Rightarrow L_{22} = \pm 2 \end{array}$$

$$\begin{array}{c}
 \left( \begin{array}{ccccc} 4 & -2 & 13 & -1 & \\ 2 & 3 & -1 & 15 & \end{array} \right) \left( \begin{array}{c} 23 \\ 24 \\ L^T \end{array} \right) = \left( \begin{array}{c} 73 \\ -15 \end{array} \right) \\
 \Rightarrow \left( \begin{array}{cccc} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{array} \right) \left( \begin{array}{cccc} L_{11} & L_{12} & L_{13} & L_{14} \\ 0 & L_{22} & L_{23} & L_{24} \\ 0 & 0 & L_{33} & L_{34} \\ 0 & 0 & 0 & L_{44} \end{array} \right) = \left( \begin{array}{c} A \end{array} \right) \\
 \Rightarrow \left( \begin{array}{cccc} L_{11}^2 & L_{11}L_{12} & L_{11}L_{13} & L_{11}L_{14} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} + L_{23}L_{33} + L_{24}L_{34} + L_{22}L_{42} \\ L_{11}L_{31} & L_{21}L_{31} + L_{32}L_{22} & L_{31}^2 + L_{32}^2 + L_{33}^2 & L_{31}L_{41} + L_{32}L_{42} + L_{33}L_{43} \\ L_{11}L_{41} & L_{21}L_{41} + L_{42}L_{22} & L_{31}L_{41} + L_{42}L_{32} + L_{43}L_{33} & L_{41}^2 + L_{42}^2 + L_{43}^2 + L_{44}^2 \end{array} \right) \\
 \Rightarrow \text{Equating } (*) \text{ to } A \text{ and solving for } L \text{ and } L^T \text{ we get} \\
 \end{array}$$

$$\left| \begin{array}{l}
 L_{11}^2 = 4 \Rightarrow L_{11} = 2 \\
 2L_{21} = -2 \Rightarrow L_{21} = -1 \\
 2L_{31} = 4 \Rightarrow L_{31} = 2
 \end{array} \right| \left| \begin{array}{l}
 2L_{41} = 2 \Rightarrow L_{41} = 1 \\
 1 + L_{22}^2 = 5 \Rightarrow L_{22}^2 = 4 \Rightarrow L_{22} = 2
 \end{array} \right|$$

$$\left| \begin{array}{l}
 -1 \cdot 1 + 2L_{42} = 3 \\
 2L_{42} = 4 \Rightarrow L_{42} = 2
 \end{array} \right| \left| \begin{array}{l}
 1 + 4 + 1 + L_{44}^2 = 15 \\
 L_{44}^2 = 15 - 6 = 9
 \end{array} \right| \\
 \Rightarrow \left| \begin{array}{l}
 4 + 0 + L_{33}^2 = 13 \Rightarrow L_{33} = 3 \\
 2 + 3L_{43} = -1 \Rightarrow 3L_{43} = -3 \Rightarrow L_{43} = -1
 \end{array} \right| \left| \begin{array}{l}
 L_{44} = 3
 \end{array} \right|$$

$$\Rightarrow L = \left( \begin{array}{cccc} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{array} \right) \quad L^T = \left( \begin{array}{cccc} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{array} \right)$$

$$\Rightarrow LL^T x = b$$

$$\left( \begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & -1 & 3 \end{array} \right)$$

$$\begin{aligned}
 &\Rightarrow -1 \cdot 1 + 2L_{42} = 3 \\
 &\Rightarrow 2L_{42} = 4 \Rightarrow L_{42} = 2 \\
 &\Rightarrow 4 + 0 + L_{33}^2 = 13 \Rightarrow L_{33} = 3 \\
 &\Rightarrow 2 \cdot 1 + 0 + 3L_{43} = -1 \\
 &\Rightarrow 2 + 3L_{43} = -1 \Rightarrow L_{43} = 1
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow L &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix} \quad L^T = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \\
 \Rightarrow L L^T x &= b
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow L z &= b \quad \left( \begin{array}{cccc|c} 2 & 0 & 0 & 0 & z_1 \\ -1 & 2 & 0 & 0 & z_2 \\ 2 & 0 & 3 & 0 & z_3 \\ 1 & 2 & -1 & 3 & z_4 \end{array} \right) = \left( \begin{array}{c} 30 \\ -27 \\ 93 \\ -18 \end{array} \right)
 \end{aligned}$$

$$2z_1 = 30 \Rightarrow z_1 = 15$$

$$\Rightarrow -27 \Rightarrow -15 + 2z_2 = -27 \Rightarrow z_2 = -6$$

$$\begin{aligned}
 \Rightarrow L &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \end{pmatrix} \quad L^T = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \\
 \Rightarrow L L^T x &= b
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow L z &= b \quad \left( \begin{array}{cccc|c} 2 & 0 & 0 & 0 & z_1 \\ -1 & 2 & 0 & 0 & z_2 \\ 2 & 0 & 3 & 0 & z_3 \\ 1 & 2 & -1 & 3 & z_4 \end{array} \right) = \left( \begin{array}{c} 30 \\ -27 \\ 93 \\ -18 \end{array} \right)
 \end{aligned}$$

$$2z_1 = 30 \Rightarrow z_1 = 15$$

$$\Rightarrow -z_1 + z_2 = -27 \Rightarrow -15 + 2z_2 = -27 \Rightarrow z_2 = -6$$

$$\Rightarrow 2z_1 + 3z_3 = 93 \Rightarrow 3z_3 = 93 - 30 \Rightarrow z_3 = 21$$

$$\Rightarrow z_1 + 2z_2 - z_3 + 3z_4 = -18 \Rightarrow 15 - 12 - 21 + 3 \cdot (-6) = -18 \Rightarrow z_4 = 0$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 15 \\ -6 \\ 21 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} L & b \\ \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & 3 \\ \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \\ \end{pmatrix}$$

$$\Rightarrow z_1 = 30 \Rightarrow z_1 = 15$$

$$\Rightarrow -z_1 + z_2 = -27 \Rightarrow -15 + z_2 = -27 \Rightarrow z_2 = -6$$

$$\Rightarrow 2z_1 + 3z_3 = 93 \Rightarrow 3z_3 = 93 - 30 \Rightarrow z_3 = 21$$

$$\Rightarrow z_1 + 2z_2 - z_3 + 3z_4 = -18 \Rightarrow 15 - 12 - 21 + 3z_4 = -18 \Rightarrow z_4 = 0$$

$$\Rightarrow \begin{pmatrix} z \\ \end{pmatrix} = \begin{pmatrix} 15 \\ -6 \\ 21 \\ 0 \\ \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} L & x \\ \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \\ \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \end{pmatrix} = \begin{pmatrix} -15 \\ -6 \\ 21 \\ 0 \\ \end{pmatrix}$$

$$3x_4 = 0 \Rightarrow x_4 = 0, \quad 3x_3 = 21 \Rightarrow x_3 = 7$$

$$\Rightarrow 2x_2 + 2x_4 = 6 \Rightarrow x_2 = -3$$

$$\Rightarrow 2x_1 - x_2 + 2x_3 + x_4 = 15 \Rightarrow x_1 = -1$$

$$\Rightarrow \begin{pmatrix} x \\ \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \\ \end{pmatrix}$$

(4c)(i) Elimination

- Also known as triangulation or factorization  
to complete the inverse A<sup>-1</sup> by first  
both the upper and lower triangular matrices  
by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ \end{pmatrix}$$

$$\begin{aligned}
 5x_4 &= 0 \Rightarrow x_4 = 0 & 3x_3 &= 3, \quad 3x_3 = 21 \Rightarrow x_3 = 7 \\
 2x_2 + 2x_4 &= 6 \Rightarrow x_2 = -3 \\
 2x_1 - 2x_2 + 2x_3 + x_4 &= 15 \Rightarrow x_1 = -1
 \end{aligned}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

(4c)(i) Elimination

- Also known as triangulation or factorization is used to complete the inverse  $A^{-1}$  by first computing both the upper and lower triangular matrices given by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

Then get the product of LU so as b equals Matrix A and calculating the coefficients

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

(4c)(i) Elimination

- Also known as triangulation or factorization is used to complete the inverse  $A^{-1}$  by first computing both the upper and lower triangular matrices given by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

We then get the product of LU so as b equals the Matrix A and calculating the coefficients matrices L and U which we then obtain  $A^{-1}$

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \end{pmatrix} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

$$x = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

(G.C) (ii) Explanation

Also known as triangularization or factorization is used to complete the matrix A by first computing both the upper and lower triangular matrices given by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}$$

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

$\Rightarrow$  we then get the product of LU so as to equate it to the matrix A and calculating the coefficients for matrices L and U which we then obtain inverse  $A^{-1}$

$$(ii) L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

$$(el) L_{11} = 1$$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}$$

(G.C) (ii)  
Also known as triangularization or factorization is used to complete the matrix A by first computing both the upper and lower triangular matrices given by

$$L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}$$

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

$\Rightarrow$  we then get the product of LU so as to equate it to the matrix A and calculating the coefficients for matrices L and U which we then obtain inverse  $A^{-1}$

$$(ii) L = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

$$(el) L_{11} = 1$$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{pmatrix} \times \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$

$$\Rightarrow u_{11} = 4, \quad u_{12} = -2, \quad u_{13} = 4, \quad u_{14} = 2$$

$$\Rightarrow 4L_2 = -2 \Rightarrow L_{21} = -\frac{1}{2}, \quad -\frac{1}{2} \cdot 2 + u_{24} = 3 \Rightarrow u_{24} = 4$$

$$\Rightarrow 4L_{31} = 4 \Rightarrow L_{31} = 1, \quad 1 \cdot 2 + 4L_{32} = -2, \quad 4L_{32} = -2+2, \quad L_{32} = 0$$

$$\Rightarrow 4 + u_{33} = 13 \Rightarrow u_{33} = 9, \quad 4L_{41} = 2 \Rightarrow L_{41} = \frac{1}{2}$$

$$\Rightarrow 2 + 9 + u_{43} = 11 \Rightarrow L_{43} = -\frac{1}{3}, \quad 4L_{42} = 4, \quad L_{41} = 1$$

$$\Rightarrow 1 + 4 + 3 + u_{44} = 15 \Rightarrow u_{44} = 9$$

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad U = \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$\Rightarrow u_{11} = 4, \quad u_{12} = -2, \quad u_{13} = 4, \quad u_{14} = 2$$

$$\Rightarrow 4L_2 = -2 \Rightarrow L_{21} = -\frac{1}{2}, \quad -\frac{1}{2} \cdot 2 + u_{24} = 3 \Rightarrow u_{24} = 4$$

$$\Rightarrow 4L_{31} = 4 \Rightarrow L_{31} = 1, \quad 1 \cdot 2 + 4L_{32} = -2, \quad 4L_{32} = -2+2, \quad L_{32} = 0$$

$$\Rightarrow 4 + u_{33} = 13 \Rightarrow u_{33} = 9, \quad 4L_{41} = 2 \Rightarrow L_{41} = \frac{1}{2}$$

$$\Rightarrow 2 + 9 + u_{43} = 11 \Rightarrow L_{43} = -\frac{1}{3}, \quad 4L_{42} = 4, \quad L_{41} = 1$$

$$\Rightarrow 1 + 4 + 3 + u_{44} = 15 \Rightarrow u_{44} = 9$$

$$L = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad U = \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$L_{21} = -\frac{1}{2}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix} \Rightarrow z_1 = 30$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix} \Rightarrow \frac{1}{2}z_1 + z_2 = -27, \quad z_2 = -12$$

$$\begin{aligned}
 &\Rightarrow 4L_{31} = 4 \Rightarrow L_{31} = 1, \quad 1 \cdot 2 + 4L_{22} = -2, \quad 4L_{32} = -2 \Rightarrow L_{32} = 0 \\
 &\Rightarrow 4 + U_{33} = 13 \Rightarrow U_{33} = 9, \quad 4L_{41} = 2 \Rightarrow L_{41} = \frac{1}{2} \\
 &\Rightarrow 2 + 9 + U_{33} = 13 \Rightarrow U_{33} = 9, \quad 4L_{42} = 4 \Rightarrow L_{42} = 1 \\
 &\Rightarrow 1 + 4 + 3 + U_{44} = 15 \Rightarrow U_{44} = 9
 \end{aligned}$$

used

on

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad U = \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$L^{-1} = b$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & z_1 \\ -\frac{1}{2} & 1 & 0 & 0 & z_2 \\ 1 & 0 & 0 & 0 & z_3 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 & z_4 \end{array} \right) \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \right) = \left( \begin{array}{c} 30 \\ -27 \\ 93 \\ -18 \end{array} \right) \Rightarrow z_1 = 30$$

$$\Rightarrow \frac{1}{2}z_1 + z_2 = -27, \quad z_2 = -12$$

$$\Rightarrow z_1 + z_3 = 93, \quad z_3 = 63$$

$$\Rightarrow z_4 + \frac{1}{2} \cdot 30 - 12 - \frac{1}{3} \cdot 63 = -18 \Rightarrow z_4 = 0$$

$$\Rightarrow \underline{\underline{z}} = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$$4 \quad -2 \quad 4 \quad 2 \quad | \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \quad | \quad \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix}$$

using  
pivot

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 \end{pmatrix} \quad U = \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$L^{-1} = b$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & z_1 \\ -\frac{1}{2} & 1 & 0 & 0 & z_2 \\ 1 & 0 & 0 & 0 & z_3 \\ \frac{1}{2} & 1 & -\frac{1}{3} & 1 & z_4 \end{array} \right) \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \right) = \left( \begin{array}{c} 30 \\ -27 \\ 93 \\ -18 \end{array} \right) \Rightarrow z_1 = 30$$

$$\Rightarrow \frac{1}{2}z_1 + z_2 = -27, \quad z_2 = -12$$

$$\Rightarrow z_1 + z_3 = 93, \quad z_3 = 63$$

$$\Rightarrow z_4 + \frac{1}{2} \cdot 30 - 12 - \frac{1}{3} \cdot 63 = -18 \Rightarrow z_4 = 0$$

$$\Rightarrow \underline{\underline{z}} = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$$\underline{\underline{Ux}} = \underline{\underline{z}} \Rightarrow \left( \begin{array}{cccc|c} 4 & -2 & 4 & 2 & z_1 \\ 0 & 4 & 0 & 4 & z_2 \\ 0 & 0 & 9 & -3 & z_3 \\ 0 & 0 & 0 & 9 & z_4 \end{array} \right) \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \right) = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_4 = 0$$

$$9x_3 = 63 \Rightarrow x_3 = 7$$

$$4x_2 - 12 - 1x_2 = -3$$

$$\underline{\underline{x}} = \begin{pmatrix} -1 \\ -3 \\ 7 \end{pmatrix}$$

$$\begin{aligned}
 L_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} & 1 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} &= \begin{pmatrix} 30 \\ -27 \\ 93 \\ -18 \end{pmatrix} \Rightarrow z_1 = 30 \\
 &\Rightarrow \frac{1}{2}z_1 + z_2 = -27, z_2 = -12 \\
 &\Rightarrow z_1 + z_2 = 93, z_3 = 63 \\
 &\Rightarrow z_4 + \frac{1}{2}z_3 - \frac{1}{2}z_2 = -12 \\
 &\Rightarrow z_4 = 0 \\
 \Rightarrow \underline{\underline{Z}} &= \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix} \\
 U_{\infty} &= 2 \Rightarrow \begin{pmatrix} 4 & -2 & 4 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 30 \\ -12 \\ 63 \\ 0 \end{pmatrix} \\
 \Rightarrow z_4 &= 0 \\
 9z_3 &= 63 \Rightarrow z_3 = 7 \\
 4z_2 &= -12 \Rightarrow z_2 = -3 \\
 4z_1 + 6z_2 + 28 &= 30 \Rightarrow z_1 = -1 \\
 \underline{\underline{Z}} &= \begin{pmatrix} -1 \\ -3 \\ 7 \\ 0 \end{pmatrix}
 \end{aligned}$$

(iii) why is not the method straght?

$\Rightarrow$  This is because LU decomposition method FAILS if any of the pivot elements is equal to zero

i.e., if  $L_{ii} = 0$  and  $U_{ii} = 0$  or if  
 $U_{ii} = 0$  and  $L_{ii} = 0$

A better efficient method to solve the above system would be to use the Jacobi Iterative method or the Gauss-Siedel approximation Iterative method.

(iii) Why is not the method efficient?  
 $\Rightarrow$  This is because LU decomposition method FAILS if any of the pivot elements is equal to zero.

i.e., if  $L_{ii} = 0$  and  $U_{ii} \neq 0$  or if  
 $U_{ii} = 0$  and  $L_{ii} \neq 0$

A better efficient method to solve the above system would be to use the Jacobi Iterative method or the Gauss-Siedel Approximation Iterative method.

$$(a) \begin{array}{|c|cc|} \hline & \frac{3}{4} & 0 \\ \hline 1 & \frac{1}{4} & \frac{3}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \\ \hline \end{array} \Rightarrow \left( \begin{array}{|c|c|} \hline c & A \\ \hline b^T & \\ \hline \end{array} \right)$$

$$\Rightarrow c_1 = \frac{3}{4}, c_2 = 1, a_{11} = \frac{3}{4}, a_{12} = 0$$

i.e., if  $L_{ii} = 1$  and  $U_{ii} = 0$   
 $U_{ii} = 1$  and  $L_{ii} = 0$

A better efficient method to solve the above system would be to use the Jacobi Iterative method or the Gauss-Siedel Approximation Iterative method.

$$(4a) \begin{array}{|c|cc|} \hline & \frac{3}{4} & 0 \\ \hline 1 & \frac{1}{4} & \frac{3}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \\ \hline \end{array} \Rightarrow \left( \begin{array}{|c|c|} \hline c & A \\ \hline b^T & \\ \hline \end{array} \right)$$

$$\Rightarrow c_1 = \frac{3}{4}, c_2 = 1, a_{11} = \frac{3}{4}, a_{12} = 0, a_{21} = \frac{1}{4}, a_{22} = \frac{3}{4}, b_1 = \frac{1}{2}, b_2 = \frac{1}{2}$$

$\Rightarrow$  For an A-Stable function above we have that it  
 $C = \{z \in \mathbb{R}^+, \operatorname{Re}(z) \leq 0\}$

$\Rightarrow$  So the above condition is met iff

$$\operatorname{Re}(z_1) \leq 1$$

A better efficient method  
System would be to use the  
method of the Crank-Nicolson approximation iteration

$$(1a) \quad \begin{array}{c|cc} \frac{2}{3} & 0 \\ \hline 1 & \frac{1}{3} & \frac{2}{3} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \Rightarrow c \mid \frac{A}{bT}$$

$$\Rightarrow q = \frac{3}{4}, c_2 = 1, a_{11} = \frac{3}{4}, a_{12} = 0 \\ a_{21} = \frac{1}{4}, a_{22} = \frac{3}{4}, b = \frac{1}{2}, b_2 = \frac{1}{2}$$

$\Rightarrow$  For an A-Dmissible function above we have that it.

$$C = \{z \in \mathbb{R}, \operatorname{Re}(z) \leq 0\}$$

$\Rightarrow$  So the above condition is met iff

$$|\operatorname{Re}(iz)| \leq 1 \quad i \in \mathbb{R}$$

$R(z)$  is analytic for  $\operatorname{Re}(z) < 0$

$\Rightarrow$  This is the stability function for the above Butcher tableau.  
Then by

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{3}{4}z^3 + \frac{1}{2}z^4$$

Method  
method

$$(1a) \quad \begin{array}{c|cc} \frac{2}{3} & 0 \\ \hline 1 & \frac{1}{3} & \frac{2}{3} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \Rightarrow c \mid \frac{A}{bT}$$

$$\Rightarrow q = \frac{3}{4}, c_2 = 1, a_{11} = \frac{3}{4}, a_{12} = 0 \\ a_{21} = \frac{1}{4}, a_{22} = \frac{3}{4}, b = \frac{1}{2}, b_2 = \frac{1}{2}$$

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$R(z)$  is analytic for  $\operatorname{Re}(z) < 0$

$\Rightarrow$  This is the stability function for the above Butcher tableau.  
Then by

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{3}{16}z^3 + \frac{1}{48}z^4$$

Note that all non-zero entries are above the below the  
diagonal elements

$$(4a) \quad \begin{array}{c|cc} & 3/4 & 0 \\ \hline 1 & 1/4 & 3/4 \\ & 1/2 & 1/2 \end{array} \Rightarrow C \left| \begin{array}{c|c} & A \\ & b^T \end{array} \right.$$

$$\Rightarrow c_1 = \frac{3}{4}, \quad c_2 = 1, \quad a_{11} = \frac{3}{4}, \quad a_{12} = 0 \\ a_{21} = \frac{1}{4}, \quad a_{22} = \frac{3}{4}, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{2}$$

$\Rightarrow$  For an A-stable function above we have that it.

$$C = \{z \in \mathbb{R}, \operatorname{Re}(z) \leq 0\}$$

$\Rightarrow$  So the above condition is met iff

$$\operatorname{Re}(iy) \leq 1 \quad y \in \mathbb{R}$$

$R(z)$  is analytic for  $\operatorname{Re}(z) < 0$

$\Rightarrow$  This is the stability function for the above Butcher's tableaux

True by

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{3}{16}z^3 + \frac{1}{48}z^4$$

Note that all non-zero entries are ~~other~~ the below diagonal elements

$$\Delta = 1 - 8z - 4z^2 - 8z^3 + 2z^4$$

$$\text{Let } A = \begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix} \quad \text{and } b^T = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$$

$$\text{Now our } R(z) = \frac{\Delta(z)}{D(z)} \quad \text{for polynomial} \\ F(y) = D(iy)\Delta(-iy) - N(iy)N(-iy)$$

$\Rightarrow$  For simplicity we follow the below conditions at A

(i) All roots  $D(z)$  are in the right-hand side

(ii)  $E(y) \geq 0$  for all  $y \in \mathbb{R}$

Then the method is A-stable,

$$R(z) = \det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 3/4 & 0 \\ 1/4 & 3/4 \end{pmatrix} \right]$$

*All zero it*

$$\text{Let } A = \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad \text{and } b = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Now write  $R(z) = \frac{N(z)}{D(z)}$  for polynomial

$$E(y) = D(y)D(-iy) - N(y)N(-iy)$$

$\Rightarrow$  For simplicity we follow the below conditions  
it  $A$

(i) All roots  $D(z)$  are in the right hand side

(ii)  $E(y) \geq 0$  for all  $y \in \mathbb{R}$

Then the method is A-Stable,

$$R(z) = \det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right]$$

$$\det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right]$$

$$\Rightarrow R(z) = \det \left[ \begin{array}{cc} \frac{1}{4} + \frac{1}{2}z & 0 \\ -\frac{1}{4} & \frac{1}{4} + \frac{1}{2}z \end{array} \right] = \frac{\left( \frac{1}{4} + \frac{1}{2}z \right)^2}{(1 - \frac{3}{4}z)^2}$$

$\Rightarrow$  For simplicity we follow the below conditions  
it  $A$

(i) All roots  $D(z)$  are in the right hand side

(ii)  $E(y) \geq 0$  for all  $y \in \mathbb{R}$

Then the method is A-Stable,

$$R(z) = \det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right]$$

$$\det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right]$$

$$\Rightarrow R(z) = \det \left[ \begin{array}{cc} \frac{1}{4} + \frac{1}{2}z & 0 \\ -\frac{1}{4} & \frac{1}{4} + \frac{1}{2}z \end{array} \right] = \frac{\left( \frac{1}{4} + \frac{1}{2}z \right)^2 + \frac{1}{4}}{(1 - \frac{3}{4}z)^2 + \frac{1}{4}z}$$

$$\det \left[ \begin{array}{cc} 1 - \frac{3}{4}z & 0 \\ -\frac{1}{4}z & 1 - \frac{3}{4}z \end{array} \right]$$

$$\Rightarrow N(z) = \frac{1}{4} + \left( \frac{1}{4} + \frac{1}{2}z \right)^2 = \frac{1}{4}z^2 + \frac{1}{4}z + \frac{5}{16}$$

Then the method is a better,

$$P(z) = \det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} - \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \right]$$

$$\det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right]$$

$$\Rightarrow P(z) = \det \frac{\begin{bmatrix} \frac{1}{4} + \frac{1}{2}z & 0 \\ -\frac{1}{4} & \frac{1}{4} + \frac{1}{2}z \end{bmatrix}}{\begin{bmatrix} 1 - \frac{3}{4}z & 0 \\ -\frac{1}{4}z & 1 - \frac{3}{4}z \end{bmatrix}} = \frac{\left(\frac{1}{4} + \frac{1}{2}z\right)^2 + \frac{1}{4}}{\left(1 - \frac{3}{4}z\right)^2 + \frac{1}{4}z^2}$$

$$\Rightarrow N(z) = \frac{1}{4} + \left(\frac{1}{4} + \frac{1}{2}z\right)^2 = \frac{1}{4}z^2 + \frac{1}{4}z + \frac{9}{16}$$

$$D(z) = \frac{1}{4}z + \left(1 - \frac{3}{4}z\right)^2 = \frac{9}{16}z^2 - \frac{11}{4}z + \frac{1}{4}$$

$$\text{let } a = \frac{9}{16}, b = -\frac{11}{4}, c = \frac{1}{4}$$

$$z = \frac{-b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}} \Rightarrow z_1 = 4.493$$

$$P(z) = \det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} - \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \right]$$

$$\det \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right]$$

$$\Rightarrow P(z) = \det \frac{\begin{bmatrix} \frac{1}{4} + \frac{1}{2}z & 0 \\ -\frac{1}{4} & \frac{1}{4} + \frac{1}{2}z \end{bmatrix}}{\begin{bmatrix} 1 - \frac{3}{4}z & 0 \\ -\frac{1}{4}z & 1 - \frac{3}{4}z \end{bmatrix}} = \frac{\left(\frac{1}{4} + \frac{1}{2}z\right)^2 + \frac{1}{4}}{\left(1 - \frac{3}{4}z\right)^2 + \frac{1}{4}z^2}$$

$$\Rightarrow N(z) = \frac{1}{4} + \left(\frac{1}{4} + \frac{1}{2}z\right)^2 = \frac{1}{4}z^2 + \frac{1}{4}z + \frac{9}{16}$$

$$D(z) = \frac{1}{4}z + \left(1 - \frac{3}{4}z\right)^2 = \frac{9}{16}z^2 - \frac{11}{4}z + \frac{1}{4}$$

$$\text{let } a = \frac{9}{16}, b = -\frac{11}{4}, c = \frac{1}{4}$$

$$z = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \Rightarrow z_1 = 4.493$$

$$z_2 = 0.3956$$

$\frac{9}{8}$

$\Rightarrow$  Thus the roots of  $A(z)$  are all in the R.H.S so the first condition is met / satisfied

$$\Rightarrow E(y) = D(iy)D(-iy) - N(iy)N(-iy)$$

$$\Rightarrow \left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right)$$

$$\Rightarrow \left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right) \left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right)$$

$$\Rightarrow -\left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right) \left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right)$$

$$\Rightarrow \frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2 - \frac{55}{24}iy - \frac{11}{16}y^2 - \frac{15}{16}y^2 - \frac{9}{16}y^4 > 0$$

$\Rightarrow$  Thus the roots of  $E(y)$  are all in the R.H.S so the first condition is met / satisfied

$$\Rightarrow E(y) = D(iy)D(-iy) - N(iy)N(-iy)$$

$$\Rightarrow \left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right)$$

$$\Rightarrow \left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right) \left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right)$$

$$\Rightarrow -\left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right) \left(1 - \frac{11}{4}iy - \frac{9}{16}y^2\right)$$

$$\Rightarrow \frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2 - \frac{55}{36}iy - \frac{11}{16}y^2 - \frac{15}{16}y^2 - \frac{9}{64}y^4 > 0$$

Thus since  $E(y) > 0$  and roots  $A(z)$  are in the R.H.S plane than we can conclude that the method is  $q$ -stable.

$$\Rightarrow E(y) = b_1(y) b_2(y) - n_1(y) n_2(y)$$

$$\Rightarrow \left(1 - \frac{1}{4}iy\right)$$

$$= \left(1 - \frac{1}{4}iy - \frac{9}{16}y^2\right) \left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right)$$

$$\Rightarrow -\left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right) \left(1 - \frac{1}{4}iy - \frac{9}{16}y^2\right)$$

$$\Rightarrow \frac{3}{6} + \frac{1}{4}iy - \frac{1}{4}y^2 - \frac{5}{36}iy - \frac{11}{16}y^2 - \frac{1}{32}y^3 - \frac{9}{64}y^4 \geq 0$$

$\Rightarrow$  Thus since  $E(y) \geq 0$  and roots  $b_i(z)$  come in the R.H.S plane than we can conclude that the method is A-stable.

(4 c(i))

Continuing

Having determined L and U of the system  $Ax = b$

$$\Rightarrow L \underline{U} z = b$$

$$\text{Simplified into } Lz = b$$

$$\Rightarrow \left(1 - \frac{1}{4}iy\right)$$

$$= \left(1 - \frac{1}{4}iy - \frac{9}{16}y^2\right) \left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right)$$

$$\Rightarrow -\left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right) \left(1 - \frac{1}{4}iy - \frac{9}{16}y^2\right)$$

$$\Rightarrow \frac{3}{6} + \frac{1}{4}iy - \frac{1}{4}y^2 - \frac{5}{36}iy - \frac{11}{16}y^2 - \frac{1}{32}y^3 - \frac{9}{64}y^4 \geq 0$$

$\Rightarrow$  Thus since  $E(y) \geq 0$  and roots  $b_i(z)$  come in the R.H.S plane than we can conclude that the method is A-stable.

(4 c(ii))

Continuing

Having determined L and U of the system  $Ax = b$

$$\Rightarrow L \underline{U} z = b$$

$$\Rightarrow \text{Simplified into } Lz = b$$

$$\text{whence } z = L^{-1}b \quad \text{or} \quad z = \underline{U}^{-1}b$$

$\Rightarrow$  The  $z$ -values are obtained by inverse ( $L^{-1}$ ) of the matrix  $A^{-1}$  multiplied by the  $b$ -values

$$\Rightarrow L^{-1} \underline{U}^{-1} = A^{-1}$$

$$\Rightarrow -\left(\frac{5}{6} + \frac{1}{4}iy - \frac{1}{4}y^2\right) + \frac{1}{4}y^2 = 0$$

$$\Rightarrow \frac{3}{6} + \frac{1}{4}iy - \frac{1}{4}y^2 = \frac{5}{6} - iy - \frac{1}{4}y^2 - \frac{1}{4}y^2 = \frac{9}{6} - \frac{9}{6}y^2 = 0$$

$\Rightarrow$  Thus, since  $E(y) \geq 0$  and now  $A(z)$  come in the R.H.S  
plane then we can conclude that the method is  
 $A$ -stable.

(4c(i))

Continuation

Having determined  $L$  and  $U$  of the system  $Az = b$

$$\Rightarrow Lz = b$$

$$\Rightarrow \text{Simplified into } Lz = b$$

$$\text{whence } z = L^{-1}b \quad \text{or} \quad z = U^{-1}b$$

$\rightarrow$  The  $z$ -values are obtained by inverse ( $= L^{-1}$ ) of  
the matrix  $A^{-1}$  multiplied by the  $b$ -values

$$\Rightarrow L^{-1}U^{-1} = A^{-1}$$



### PART A

$$(B_a) \text{ Using } y'(t) = -10(y-t) + 2t \quad y(0) = -1$$

$$at \quad t_0 = 0, y_0 = 0$$

$$\frac{dy}{dt} = -10(y-t) + 2t$$

Given

$$at \quad t_1 = 0.1, \quad y_1 = 0.3778794$$

$$y' = -10 \left[ 0.3778794 - (0.1)^2 \right] + 2(0.1) = -3.478740$$

$$at \quad t_2 = 0.2 \quad y_2 = 0.1753352$$

$$y' = -10 \left[ 0.1753352 - (0.2)^2 \right] + 2(0.2) = -0.95$$

PART A

$$(D) \text{ Using } y'(t) = -10(y-t^2) + 2t \quad y(0) = -3 \\ \text{at } t_0 = 0, y_0 = 0$$

$$\frac{dy}{dt} = -10(y-t^2) + 2t$$

Solu

$$\text{at } t_1 = 0.1, y_1 = 0.3778794$$

$$y' = -10[0.3778794 - (0.1)^2] + 2(0.1) = -3.4787940$$

$$\text{at } t_2 = 0.2, y_2 = 0.1753352$$

$$y' = -10[0.1753352 - (0.2)^2] + 2(0.2) = -0.9533540$$

$$\text{at } t_3 = 0.3, y_3 = 0.1397871$$

$$y' = -10[0.1397871 - (0.3)^2] + 2(0.3) = 0.1021290$$

$$\frac{dy}{dt} = -10(y-t^2) + 2t$$

$$\text{at } t_1 = 0.1, y_1 = 0.3778794$$

$$y' = -10[0.3778794 - (0.1)^2] + 2(0.1) = -3.4787940$$

$$\text{at } t_2 = 0.2, y_2 = 0.1753352$$

$$y' = -10[0.1753352 - (0.2)^2] + 2(0.2) = -0.9533540$$

$$\text{at } t_3 = 0.3, y_3 = 0.1397871$$

$$y' = -10[0.1397871 - (0.3)^2] + 2(0.3) = 0.1021290$$

We need to calculate the values of  $y_4$  at  $t_4 = 0.5$  and  $y_5$  at  $t_5 = 0.5$ .

Now we have to find the  $(y_4 - \text{and } y_5)$  we need to find using the generalized milne predictor and corrector.

$$t_1 = 0.1, y_1 = 0.3778794$$

$$y' = -10 \left[ 0.3778794 - (0.1)^3 \right] + 2(0.1) = -3.4787940$$

at  $t_2 = 0.2, y_2 = 0.1753352$

$$y' = -10 \left[ 0.1753352 - (0.2)^3 \right] + 2(0.2) = -0.9533540$$

at  $t_3 = 0.3, y_3 = 0.1297871$

$$y' = -10 \left[ 0.1297871 - (0.3)^3 \right] + 2(0.3) = 0.1021290$$

We need to calculate the values of  $y_4$  at  $t_4 = 0.4$

and  $y_5$  at  $t_5 = 0.5$

Now to find  $y_4$  and  $y_5$  we need to first write the generalized milne predictor and correct formulation as

$$y_{j+1} = y_{j-1} + h \left( 2 + \frac{1}{3} \nabla^2 + \frac{1}{3} \nabla^3 + \frac{29}{90} \nabla^4 + \frac{25}{90} \nabla^5 \right) f$$

For the values starting  $j=n$  and the  $\nabla$  value

at  $t_1 = 0.1, y_1 = 0.3778794$

$$y' = -10 \left[ 0.3778794 - (0.1)^3 \right] + 2(0.1) = -3.4787940$$

at  $t_2 = 0.2, y_2 = 0.1753352$

$$y' = -10 \left[ 0.1753352 - (0.2)^3 \right] + 2(0.2) = -0.9533540$$

at  $t_3 = 0.3, y_3 = 0.1297871$

$$y' = -10 \left[ 0.1297871 - (0.3)^3 \right] + 2(0.3) = 0.1021290$$

We need to calculate the values of  $y_4$  at  $t_4 = 0.4$

and  $y_5$  at  $t_5 = 0.5$

Now to find  $y_4$  and  $y_5$  we need to first write the generalized milne predictor and correct formulation as

$$y_{j+1} = y_{j-1} + h \left( 2 + \frac{1}{3} \nabla^2 + \frac{1}{3} \nabla^3 + \frac{29}{90} \nabla^4 + \frac{25}{90} \nabla^5 \right) f$$

$$t_2 = 0.2 \quad y_2 = 0.1753352$$

$$y' = -10 [0.1753353 - (0.2)^2] + 2(0.2) = -0.9533540$$

$$\text{at } t_3 = 0.3, \quad y_3 = 0.1297871$$

$$y' = -10 [0.1297871 - (0.3)^2] + 2(0.3) = 0.1021290$$

We need to calculate the values of  $y_4$  at  $t_4 = 0.4$   
and  $y'_4$  at  $t_4 = 0.5$ .

Now to find  $y_{j+1}$  and  $y'_k$  we need to first  
write the generalized milne predictor and correct  
formulation as

$$y_{j+1} = y_{j-1} + h \left( 2 + \frac{1}{3}T^2 + \frac{1}{3}T^3 + \frac{29}{90}T^4 + \frac{25}{90}T^5 + \right) f_i$$

For the values stating  $j=n$  and the  $\nabla$  value given  
by the difference in the interval of time  $t$ , this  
equation is simplified as

$$y_{n+1}^P = y_{n-3} = \frac{4}{3}h \left( 2y_{n-2} - y_{n-1} + 2y_n \right)$$

$$\text{at } t_2 = 0.2 \quad y_2 = 0.1753352$$

$$y' = -10 [0.1753353 - (0.2)^2] + 2(0.2) = -0.9533540$$

$$\text{at } t_3 = 0.3, \quad y_3 = 0.1297871$$

$$y' = -10 [0.1297871 - (0.3)^2] + 2(0.3) = 0.1021290$$

We need to calculate the values of  $y_4$  at  $t_4 = 0.4$   
and  $y'_4$  at  $t_4 = 0.5$ .

Now to find  $y_{j+1}$  and  $y'_k$  we need to first  
write the generalized milne predictor and correct  
formulation as

$$y_{j+1} = y_{j-1} + h \left( 2 + \frac{1}{3}T^2 + \frac{1}{3}T^3 + \frac{29}{90}T^4 + \frac{25}{90}T^5 + \right) f_i$$

For the values stating  $j=n$  and the  $\nabla$  value given  
by the difference in the interval of time  $t$ , this  
equation is simplified as

$$y_{n+1}^P = y_{n-3} = \frac{4}{3}h \left( 2y_{n-2} - y_{n-1} + 2y_n \right)$$

R-Hd

$$y' = -10 \left[ 0.1753355 - (0-2)^2 \right] + 2(0.2) = -0.9533540$$

at  $t_3 = 0.3$ ,  $y_3 = 0.1297871$

$$y' = -10 \left[ 0.1397871 - (0.3)^2 \right] + 2(0.3) = 0.1021290$$

We need to calculate the values of  $y_j$  at  $t_4 = 0.4$  and  $y'_j$  at  $t_5 = 0.5$

Now to find the  $y_j$  and  $y'_j$  we need to first write the generalized milne predictor and corrector formulae as

$$y_{j+1} = y_{j-1} + h \left( 2 + \frac{1}{3} \Delta^2 + \frac{1}{3} \Delta^3 + \frac{29}{90} \Delta^4 + \frac{2}{90} \Delta^5 \dots \right) f_i$$

For the values stating  $j=n$  and the  $\Delta$  value given by the difference in the interval at time  $t$ , this equation is simplified as

$$y_{n+1}^P = y_{n-3} = \frac{4}{3} h \left( 2y_{n-2}^I - y_{n-1}^I + 2y_n^I \right)$$

and similarly for the corrector method

$$y_{n+1}^C = y_{n-1} + \frac{h}{3} \left( y_{n-1}^I + 4y_n^I + y_{n+1}^I \right)$$

So at  $n=3 \Rightarrow j=3$

$$y_4^P = y_0 + \frac{4}{3} h \left( 2y_1^I - y_2^I + 2y_2^I \right)$$

$$= 1 + \frac{4}{3}(0.1) \left[ 2(-3.4787940) - (-0.9533540) + 2(0.1021290) \right]$$

$$y_4^P(0.4) = 0.2266699$$

$$y'(t) = -10(y-t^2) + 2t$$

and similarly for the corrector method

$$y_{n+1}^c = y_{n-1} + \frac{h}{3} (y_{n-1}' + 4y_n' + y_{n+1}')$$

so at  $n=3 \Rightarrow j=3$

$$y_4' = y_0 + \frac{h}{3} h (2y_1' - y_2' + 2y_2')$$

$$= 1 + \frac{h}{3} (0.1) [2(-3.4787940) - (-0.9533540) + 2(0.1021290)]$$

$$y_4' (0.4) = 0.2266699$$

$$y'(t) = -10(y-t^2) + 2t$$

$$y_4' = -10(0.2266699 - (0.4)^2) + 2(0.4) = 0.1333013$$

as for the corrector method

$$y_{n+1}^c = y_{n-1} + \frac{h}{3} (y_{n-1}' + 4y_n' + y_{n+1}')$$

so at  $n=3 \Rightarrow j=3$

$$y_4' = y_0 + \frac{h}{3} h (2y_1' - y_2' + 2y_2')$$

$$= 1 + \frac{h}{3} (0.1) [2(-3.4787940) - (-0.9533540) + 2(0.1021290)]$$

$$y_4' (0.4) = 0.2266699$$

$$y'(t) = -10(y-t^2) + 2t$$

$$y_4' = -10(0.2266699 - (0.4)^2) + 2(0.4) = 0.1333013$$

as for the corrector method

$$y_4^c = y_2 + \frac{h}{3} (y_2' + 4y_3' + y_4')$$

$$= 0.1753353 + \frac{h}{3} (-0.9533540 + 4(0.1021290) + 0.1333)$$

$$= 0.2251743$$

$$y_4' = -10(0.2251743 - (0.4)^2) + 2(0.4)$$

$$y_4 = y_2 + \frac{h}{3} (y_1' + 2y_2' + y_3') \\ = 1 + \frac{0.1}{3} (0.4) [(-3.4787940) - (-0.953540) + 2(0.1021240)]$$

$$y_4''(0.4) = 0.2266699$$

$$y'(t) = -10(y-t^2) + 2t$$

$$y_4' = -10(0.2266699 - (0.4)^2) + 2(0.4) = 0.1333013$$

as for the correct method

$$y_4^c = y_2 + \frac{h}{3} (y_1' + 4y_2' + y_3')$$

$$= 0.1753353 + 0.1/3 (-0.953540 + 4(0.1021240) + 0.1333013) \\ = 0.2251743$$

$$y_4^l = -10(0.2251743 - (0.4)^2) + 2(0.4) = 0.1482564$$

$$\text{at } t=0.5 \quad y(0.5) = ?$$

$$\text{at } n=24 \Rightarrow \delta=4$$

$$y'(t) = -10(y-t^2) + 2t$$

$$y_4' = -10(0.2266699 - (0.4)^2) + 2(0.4) = 0.1333013$$

as for the correct method

$$y_4^c = y_2 + \frac{h}{3} (y_1' + 4y_2' + y_3')$$

$$= 0.1753353 + 0.1/3 (-0.953540 + 4(0.1021240) + 0.1333013) \\ = 0.2251743$$

$$y_4^l = -10(0.2251743 - (0.4)^2) + 2(0.4) = 0.1482564$$

$$\text{at } t=0.5 \quad y(0.5) = ?$$

$$\text{at } n=24 \Rightarrow \delta=4$$

$$y_5^p = \frac{4}{3} h (2y_2' - y_3' + 2y_4')$$

$$0.3718794 + \frac{4}{3}(0.1) T_2(9533540) - (0.1021240) T_2(1122)$$

$$y(t) = -10(y - t^2) + t$$

$$y_4' = -10(0.2266099 - (0.4)^2) + 2(0.4) = 0.1333013$$

as in the correct method

$$y_4^c = y_2 + \frac{1}{3}(y_2' + 4y_3' + y_4')$$

$$= 0.1753353 + 0.1/3(-0.9533540 + 4(0.1021290) + 0.1333013) \\ = 0.2251743$$

$$y_4' = -10(0.2251743 - (0.4)^2) + 2(0.4) = 0.1482566$$

$$\text{at } t = 0.5 \quad y(0.5) = ?$$

$$\text{at } n=4 \Rightarrow j=4$$

$$y_5^p = \frac{4}{3}h(2y_2' - y_3' + 2y_4') \quad \text{using } h = 0.1$$

$$0.3778794 + \frac{4}{3}(0.1)[2(9533540) - (0.1021290) + 2(0.1333013)] \\ = 0.1455815$$

$$y_5' = -10(0.1455815 - (0.5)^2) + 2(0.5) = 2.0442850$$

as in the correct method

$$= 0.1397871 + 0.1/3(0.1021290 + 4(0.1333013) + 2.0442850) \\ = 0.2291044$$

$$y_5' = -10(0.2291044 - (0.5)^2) + 2(0.5) = 1.2089559$$

Table

y

0.000000

3778794

10  $(0.2291044 - (0.1)^2) + 2(0.1) = 1.208759$

Table

j	t	y
0	0.0	1.000000
1	0.1	0.3778794
2	0.2	0.1752353
3	0.3	0.1397871
4	0.4	0.2266699
5	0.5	0.2

Table

j	t	y*	$\nabla^1 y$	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
0	0.0	1.000000	-10.000000			
1	0.1	0.3778794	-3.4787940		2.5258090	
2	0.2	0.1752353	-0.9533540	1.0554830	0.4456463	1.62314
3	0.3	0.1397871	0.1021290	0.6311723	2.0687930	
4	0.4	0.2266699	0.1333013	1.0756546	1.044823	

6.5212060  
 $\rightarrow 9957660$

-1.4699570

-1.0243107

Table

j	t	y
0	0.0	1.000000
1	0.1	0.3778794
2	0.2	0.1752353
3	0.3	0.1397871
4	0.4	0.2266699
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Table

j	t	y*	$\nabla^1 y$	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
0	0.0	1.000000	-10.000000			
1	0.1	0.3778794	-3.4787940		2.5258090	
2	0.2	0.1752353	-0.9533540	1.0554830	0.4456463	1.62314
3	0.3	0.1397871	0.1021290	0.6311723	2.0687930	
4	0.4	0.2266699	0.1333013	1.0756546	1.044823	

6.5212060  
 $\rightarrow 9957660$

-1.4699570

-1.0243107

j	t	$y^*$	$\nabla^1 y^*$	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
0	0.0	1.000000	-10.000000			
1	0.1	0.3778794	-3.4787940			
2	0.2	0.1753353	-0.9533540			
3	0.3	0.1397871	0.1021290			
4	0.4	0.2266699	0.1333013			
5	0.5	0.2291044	1.2089559			

6.5212060       $\rightarrow 9957660$   
 2.5254400      2.5258090  
 -1.4699570      -2.080144  
 0.4456463      1.6231457  
 -1.0243167  
 2.10687930  
 0.6311723      1.0444823  
 1.0756546

$$\nabla^5 f$$

$$3.7032918 = (2.0) + (2.0 + 2.0 + 2.0 + 1.0) \cdot 0.1 = 2.0$$

(2.29) For the ESSN method + modified predictor method.

$$y_{j+1} = y_j + h(2 + \frac{1}{3}P^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{28}{90}\nabla^5 + \dots)$$

$$\text{For } j = n$$

$$\Rightarrow y_{j+1} = y_j + h(2 + \frac{1}{6}y_1' + \frac{1}{6}y_2' + \frac{29}{90}y_3')$$

$\frac{d^5}{dt^5}$

$$3, 7032911 - (y_{n+1} - 2(2 + \frac{1}{3}E^2 + \frac{1}{3}E^3 + \frac{29}{90}E^4 + \frac{28}{90}E^5 + \dots)h)$$

but now 360 - 210 = 150

(2a) For three system - mince predictor method

$$y_{j+1} = y_{j-1} + h \left( 2 + \frac{1}{3}E^2 + \frac{1}{3}E^3 + \frac{29}{90}E^4 + \frac{28}{90}E^5 + \dots \right) +$$

for  $j=n$

$\sqrt{J+1}$

=)

$$y_{n+1} = y_{n-1} + h \left( 2 + \frac{1}{3}y_{n-1}' + \frac{1}{3}y_{n-2}' + \frac{29}{90}y_{n-3}' + \dots \right) + f_n$$

$$\Rightarrow E y_{n-2}' = y_{n-1}'$$

$$E y_{n-3}' = y_{n-1}'$$

$$E^2 y_{n-2}' = y_n'$$

(2a) For three system - mince predictor method

$$y_{j+1} = y_{j-1} + h \left( 2 + \frac{1}{3}E^2 + \frac{1}{3}E^3 + \frac{29}{90}E^4 + \frac{28}{90}E^5 + \dots \right) +$$

for  $j=n$

$\sqrt{J+1}$

=)

$$y_{n+1} = y_{n-1} + h \left( 2 + \frac{1}{3}y_{n-1}' + \frac{1}{3}y_{n-2}' + \frac{29}{90}y_{n-3}' + \dots \right) + f_n$$

$$\Rightarrow E y_{n-2}' = y_{n-1}'$$

$$E y_{n-3}' = y_{n-1}'$$

$$E^2 y_{n-2}' = y_n'$$

$$\sigma(E) = \frac{28}{90}E^4 + \frac{29}{90}E^3 + \frac{1}{3}E^2 + \frac{1}{3}E + 2$$

$$P(E) = (E^3 - E)$$

$$(E^3 - E)y_{n-2}' - h \left( \frac{28}{90}E^4 + \frac{29}{90}E^3 + \frac{1}{3}E^2 + \frac{1}{3}E + 2 \right) y_{n-2}' = 0$$

$$\Rightarrow y_{n+1} = y_{n-1} + h \left( 2 + \frac{1}{3} y_{n+1} + \frac{1}{3} y_{n-2} + \frac{29}{90} y_{n-3} \right) \quad \dots \quad (i)$$

$$\Rightarrow E y_{n+1} = y_{n-1}$$

$$E y_{n-3} = y_{n-1}$$

$$\sigma(E) = \frac{28}{90} E^4 + \frac{29}{90} E^3 + \frac{1}{3} E^2 + \frac{1}{3} E + 2$$

$$P(E) = (E^3 - E)$$

$$(E^3 - E) y_{n-2} - h \left( \frac{28}{90} E^4 + \frac{29}{90} E^3 + \frac{1}{3} E^2 + \frac{1}{3} E + 2 \right) y_{n-2} = 0$$

$$P(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1) = 0 \quad \text{--- 1st characteristic}$$

$$\sigma(\xi) = \frac{28}{90} \xi^4 + \frac{29}{90} \xi^3 + \frac{1}{3} \xi^2 + \frac{1}{3} \xi + 2 - 2^{\text{nd}} \text{ char}$$

out Condition

$$\text{at } P(\xi) = 0$$

$$\frac{E y_{n-2}}{E^3 y_{n-2}} = \frac{y_{n-2}}{y_n}$$

$$\sigma(E) = \frac{28}{90} E^4 + \frac{29}{90} E^3 + \frac{1}{3} E^2 + \frac{1}{3} E + 2$$

$$P(E) = (E^3 - E)$$

$$(E^3 - E) y_{n-2} - h \left( \frac{28}{90} E^4 + \frac{29}{90} E^3 + \frac{1}{3} E^2 + \frac{1}{3} E + 2 \right) y_{n-2} = 0$$

$$P(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1) = 0 \quad \text{--- 1st characteristic}$$

$$\sigma(\xi) = \frac{28}{90} \xi^4 + \frac{29}{90} \xi^3 + \frac{1}{3} \xi^2 + \frac{1}{3} \xi + 2 - 2^{\text{nd}} \text{ char}$$

out Condition

$$\text{at } P(\xi) = 0$$

$$\Rightarrow P(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1)$$

$$\xi = 0, \xi_1 = 1, \xi_2 = -1$$

are the simple roots corresponding to 4 min

$$\Rightarrow E y_{n+1} = y_n$$

$$E y_{n-1} = y_{n-1}$$

$$E^2 y_{n-2} = y_{n-2}$$

$$O(E) = \frac{28}{90} E^4 + \frac{29}{90} E^3 + \frac{1}{3} E^2 + \frac{1}{3} E + 2$$

$$P(E) = (E^3 - E)$$

$$(E^3 - E) y_{n-2} - h \left( \frac{28}{90} E^4 + \frac{29}{90} E^3 + \frac{1}{3} E^2 + \frac{1}{3} E + 2 \right) y_{n-2} = 0$$

$$P(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1) = 0 \quad \text{--- 1st characteristic}$$

$$P(\xi) = \frac{28}{90} \xi^4 + \frac{29}{90} \xi^3 + \frac{1}{3} \xi^2 + \frac{1}{3} \xi + 2 = 0 \quad \text{--- 2nd characteristic}$$

Root condition

$$\text{at } P(\xi) = 0$$

$$\Rightarrow P(\xi) = (\xi^3 - \xi) = \xi(\xi^2 - 1)$$

$$\Rightarrow \xi = 0, \xi_1 = 1, \xi_2 = -1$$

Those are the simple roots corresponding to the more-practical

### Stability

$$J(\xi) = \frac{28}{90} \xi^4 + \frac{29}{90} \xi^3 + \frac{1}{3} \xi^2 + \frac{1}{3} \xi + 2 = 0$$

$$\Rightarrow |\xi_4| > 0$$

Hence zero stable since the will yield implement

Convergence and consistency

We suppose the method is zero-stable, then assuming that  $P(\xi)$  has double roots in the unit circle

Further, product of roots of  $P(\xi)$  we set as

now  $\xi_1$  and  $\xi_2$  are equal to  $\pm i$ , thus not strictly complex hence they converge and

Stability

$$\Gamma(\xi) = \frac{28}{90} \xi^4 + \frac{29}{60} \xi^3 + \frac{1}{3} \xi^2 + \frac{1}{5} \xi + 2 = 0$$

$$\Rightarrow |\xi_4| > 0$$

Hence zero stable since no will yield complex

Convergence and ~~stability~~ consistency

$\Rightarrow$  We suppose the method is zero-stable, then assume that  $P(\xi)$  has double roots in the unit circle

Further, product of roots of  $P(\xi)$  we set as 1, but now  $\xi_1$  and  $\xi_2$  are equal to  $\pm i$ , thus not strictly complex hence they converge and are

consistent

$$\xi = 0, \xi_{1,2} = \pm i, \xi_3 = -1$$

For the correct method,

as  $J=n$

$$y_{j+1} = y_j = h \left( 2 - 2\tau + \frac{1}{3}\tau^2 - \frac{1}{90}\tau^4 - \frac{1}{90}\tau^5 - \dots \right) f_j$$

$$\Rightarrow |\xi_4| > 0$$

Hence zero stable since no will yield complex

Convergence and ~~stability~~ consistency

$\Rightarrow$  We suppose the method is zero-stable, then assume that  $P(\xi)$  has double roots in the unit circle

Further, product of roots of  $P(\xi)$  we set as 1, but now  $\xi_1$  and  $\xi_2$  are equal to  $\pm i$ , thus not strictly complex hence they converge and are

consistent

$$\xi = 0, \xi_{1,2} = \pm i, \xi_3 = -1$$

$\Rightarrow$  For the correct method,

as  $J=n$

$$y_{j+1} = y_j = h \left( 2 - 2\tau + \frac{1}{3}\tau^2 - \frac{1}{90}\tau^4 - \frac{1}{90}\tau^5 - \dots \right) f_{j+1}$$

= sum of terms

$$y_{n+1} = y_{n-1} + h \left( 2 + 2y_n' + \frac{1}{3}y_{n-1}' - \frac{1}{90}y_{n-2}' - \frac{1}{90}y_{n-3}' \right) f$$

$$\therefore E y_{n-2}' = y_{n-1}'$$

Suppose the method is zero-stable, then assume  
 that  $P(\xi)$  has double roots in the unit circle.  
 Further, product of roots of  $P(\xi)$  we set as 1, but  
 now  $\xi_1$  and  $\xi_2$  are equal to  $\pm i$ , thus not  
 strictly consistent. Complex hence they converge and are

$$\xi = 0, \xi_{1,2} = \pm i, \xi_3 = -1$$

$\Rightarrow$  For the corrector method,  
 as  $j=n$

$$y_{j+1}^1 = y_j^1 = h \left( 2 - 2\tau + \frac{1}{3}\tau^2 - \frac{1}{90}\tau^4 - \frac{1}{90}\tau^5 - \dots \right) f_{j+1}$$

$\Rightarrow$  Simplifies into

$$y_{n+1}^1 = y_{n-1}^1 + h \left( 2 + 2y_n^1 + \frac{1}{3}y_{n-1}^1 - \frac{1}{90}y_{n-2}^1 - \frac{1}{90}y_{n-3}^1 \right) f_{n+1}$$

$$\Rightarrow E y_{n-2}^1 = y_{n-1}^1$$

$$E y_{n-2}^1 = y_{n-1}^1$$

$$E^2 y_{n-2}^1 = y_{n-1}^1$$

$$-3 \quad E^2 + \sqrt{E} + 2 - 2$$

Consistent Complex equal to  $\pm i$ , thus not  
 hence they converge and are

$$\xi = 0, \xi_{1,2} = \pm i, \xi_3 = -1$$

$\Rightarrow$  For the corrector method,  
 as  $j=n$

$$y_{j+1}^1 = y_j^1 = h \left( 2 - 2\tau + \frac{1}{3}\tau^2 - \frac{1}{90}\tau^4 - \frac{1}{90}\tau^5 - \dots \right) f_{j+1}$$

$\Rightarrow$  Simplifies into

$$y_{n+1}^1 = y_{n-1}^1 + h \left( 2 + 2y_n^1 + \frac{1}{3}y_{n-1}^1 - \frac{1}{90}y_{n-2}^1 - \frac{1}{90}y_{n-3}^1 \right) f_{n+1}$$

$$\Rightarrow E y_{n-2}^1 = y_{n-1}^1$$

$$E y_{n-2}^1 = y_{n-1}^1$$

$$E^2 y_{n-2}^1 = y_{n-1}^1$$

$$\Rightarrow F(E) = -\frac{1}{90}E^3 - \frac{1}{90}E^2 + \frac{1}{3}E + 2 - 2$$

$$= -\frac{1}{90}E^5 - \frac{1}{90}E^2 + \frac{1}{3}E$$

$$P(E) = E^3 - E^2$$

$$\Rightarrow (E^3 - E^2)y_{n-2} - h \left( -\frac{1}{q_0} E^3 - \frac{1}{q_0} E^2 + \frac{1}{3} E \right) y_{n-2} = 0$$

$$P(\xi) = (\xi^3 - \xi^2) = \xi^2(\xi - 1) \quad \text{--- 1st characteristic}$$

$$J(\xi) = -\frac{1}{q_0} \xi^3 - \frac{1}{q_0} \xi^2 + \frac{1}{3} \xi \quad \text{--- 2nd characteristic}$$

Put condition at  $\xi = 0$   $P(\xi) = 0$

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$$\Rightarrow \xi_1 = 1, \xi_{2,3} = 0$$

These are simple roots of the corresponding characteristic equation

ability

Final answer:  $y_n = C_1 + C_2 n + C_3 n^2$

$$P(\xi) = (\xi^3 - \xi^2) - \xi^2(\xi - 1) \text{ since } -\xi^2 \text{ characteristic}$$

$$\sigma(\xi) = -\frac{1}{q_0}\xi^3 - \frac{1}{q_0}\xi^2 + \frac{1}{2}\xi - 2^{\text{st}} \text{ characteristic}$$

Pearl condition at  $P(\xi) = 0$

$$\Rightarrow P(\xi) = T^2(\xi - 1) = 0$$

$$\Rightarrow \xi_1 = 1, \xi_{2,3} = 0$$

These are simple root of the corresponding cutters  
method

### Stability

$$\sigma(E) = \sigma(\xi) = -\frac{1}{q_0}\xi^3 - \frac{1}{q_0}\xi^2 + \frac{1}{2}\xi \text{ at } P(\xi) = 0$$

$$= -\frac{1}{q_0}\xi^3 - \frac{1}{q_0}\xi^2 + \frac{1}{2}\xi = 0$$

$$\Rightarrow |\xi_3| > 1$$

Hence zero-stable

Pearl condition at  $P(\xi) = 0$

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These are simple root of the corresponding cutters  
method

### Stability

$$\sigma(E) = \sigma(\xi) = -\frac{1}{q_0}\xi^3 - \frac{1}{q_0}\xi^2 + \frac{1}{2}\xi \text{ at } P(\xi) = 0$$

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$$\Rightarrow |\xi_3| > 1$$

Hence zero-stable since it yields complex root

convergence and consistency

$$P(\xi) = \xi^2(\xi - 1) = 0$$

Suppose the method is zero-stable, then any  
 $P(\xi)$  has roots at least

These are simple roots of the corresponding characteristic equation

Stability

$$\begin{aligned} J(E) = \sigma(\xi) &= -\frac{1}{40}\xi^3 - \frac{1}{40}\xi^2 + \frac{1}{3}\xi \quad \text{at } \rho(\xi)=0 \\ &= -\frac{1}{40}\xi^3 - \frac{1}{40}\xi^2 + \frac{1}{3}\xi = 0 \\ &\Rightarrow |\xi_3| > 1 \end{aligned}$$

Hence zero-stable since it yields complex roots

Convergence and consistency

$$P(q) = q^2(q-1) = 0$$

We suppose the method is zero-stable, then assuming that  $P(q)$  has double roots in the unit circle

Further, product of roots of  $P(q)$  is 1 and

since the roots are not equal and neither equal to  $\pm 1$ , then they are strictly complex

$$\Rightarrow q \neq q_{2,3}$$

These are simple roots of the characteristic equation

Stability

$$\begin{aligned} J(E) = \sigma(\xi) &= -\frac{1}{40}\xi^3 - \frac{1}{40}\xi^2 + \frac{1}{3}\xi \quad \text{at } \rho(\xi)=0 \\ &= -\frac{1}{40}\xi^3 - \frac{1}{40}\xi^2 + \frac{1}{3}\xi = 0 \\ &\Rightarrow |\xi_3| > 1 \end{aligned}$$

Hence zero-stable since it yields complex roots

Convergence and consistency

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(Qb) From the Nystrom-prediction method,

$$y_{itn} = y_{i-1} + h \left( 2 + \frac{1}{3}T^2 + \frac{1}{3}T^3 + \frac{29}{90}T^4 + \frac{28}{90}T^5 - \right) f_i$$

Setting  $T=1$ , a then simplified Nystrom-prediction method is given as follows

$$y_{n+1} = y_{n-3} + \frac{4}{3}h \left( 2y_{n-2}^1 - y_{n-1}^1 + 2y_n^1 \right)$$

$$\Rightarrow y_{n+1} = y_{n-3} - \frac{4}{3}h \left( 2y_{n-2}^1 - y_{n-1}^1 + 2y_n^1 \right)$$

$$(at E y_{n-2}^1 = y_{n-1}^1)$$

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$$(at E y_{n-2}^1 = y_{n-1}^1)$$

$$E y_{n-1}^1 = y_n^1$$

$$E^2 y_{n-2}^1 = y_n^1$$

$$\Rightarrow \sigma(E) = \frac{8}{3}E^2 - \frac{4}{3}E + \frac{5}{3}$$

Setting  $\xi = n$ , a more simplified algorithm emerges  
predictor method is given as follows

$$y_{n+2} = y_{n-2} + \frac{4}{3}h (2y_{n-1} - y_{n-2} + 2y_n)$$

$$\Rightarrow y_{n+2} - y_{n-2} - \frac{4}{3}h (2y_{n-1} - y_{n-2} + 2y_n) = 0$$

$$(2) \quad E^2 y_{n-2} = y_{n-1}$$

$$E y_{n-1} = y_n$$

$$E^2 y_{n-1} = y_n$$

$$\therefore \sigma(E) = \frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3}$$

$$P(E) = (E^3 - 0)$$

$$\Rightarrow (E^3 - 0) y_{n-2} = h \left( \frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3} \right) y_{n-2} = 0$$

$$\Rightarrow (i) \quad p(\xi) = \xi^3 = \xi(\xi^2) \quad \text{--- 1st characteristic poly}$$

$$(ii) \quad \sigma(E) = \frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3} \quad \text{--- 2nd characteristic poly}$$

$$\Rightarrow y_{n+2} + \frac{4}{3}h (2y_{n-1} - y_{n-2} + 2y_n)$$

$$\Rightarrow y_{n+2} - y_{n-2} - \frac{4}{3}h (2y_{n-1} - y_{n-2} + 2y_n) = 0$$

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$$\Rightarrow (i) \quad p(\xi) = \xi^3 = \xi(\xi^2) \quad \text{--- 1st characteristic polynomial}$$

$$(ii) \quad \sigma(E) = \frac{8}{3}E^2 - \frac{4}{3}E + \frac{8}{3} \quad \text{--- 2nd characteristic polynomial}$$

### Routh Condition

The explicit method of the Milne-Milner method is said to satisfy the Routh condition if the roots of the equation  $p(\xi) = 0$  lie inside the closed unit disk in the complex plane or

Heads

$$\begin{aligned} E^2 y_{n-2} &= y_{n-1} \\ E y_{n-1} &= y_n \\ E^2 y_{n-2} &= y_n \\ \Rightarrow \Gamma(E) &= \frac{8}{3} E^2 - \frac{4}{3} E + \frac{8}{3} \\ P(E) &= (E^2 - 0) \\ \Rightarrow (E^2 - 0) y_{n-2} &= h \left( \frac{8}{3} E^2 - \frac{4}{3} E + \frac{8}{3} \right) y_{n-2} = 0, \\ \Rightarrow i) P(\xi) &= \xi^3 = \xi(\xi^2) \quad \text{--- 1st characteristic polynomial} \\ ii) \Gamma(\xi) &= \frac{8}{3} \xi^2 - \frac{4}{3} \xi + \frac{8}{3} \quad \text{--- 2nd characteristic polynomial} \end{aligned}$$

Root Condition

- The explicit method or the Adams-Moulton method is said to satisfy the root condition if the roots of the equation  $P(\xi) = 0$  lie inside the closed unit disk in the complex plane and are simple if they lie on the circle.

$P(\xi) = 0$

$\Rightarrow$  From our equation

$$P(E) = \xi^3 = 0 \Rightarrow \xi_{1,2,3} = 0$$

Stability

$\Rightarrow$  The linear predictor method is zero-stable if given  $y^1 = f(x_1)$ ,  $y(x_0) = y_0$ ,  $f(x_0)$  satisfy Lipschitz condition and root condition holds.

$\Rightarrow$  Thus we have in our case function

$$\Gamma(\xi) = \frac{8}{3} \xi^2 - \frac{4}{3} \xi + \frac{8}{3} \quad \text{which is zero-stable since it yields simple}$$

$$P(\xi) = 0$$

From our equation

$$P(\xi) = \xi^3 = 0 \Rightarrow \xi_{1,2,3} = 0$$

(3a)

### Stability

$\Rightarrow$  The linear predictor method is zero stable if given  $y^1 = f(z_0)$ ,  $y(x_0) = y_0$ ,  $f(z_{2N})$  satisfy Lipschitz condition and root condition holds.

$\Rightarrow$  Thus we have in our case function

$$\Gamma(\xi) = \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{8}{3} \text{ which is zero-stable since it yields simple roots}$$

$$\Rightarrow \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{8}{3} = 0 \Rightarrow |\xi_{1,2}| \geq 0$$

### Convergence & Consistency

Linear predictor above is consistent if it has

From our equation

$$P(\xi) = \xi^3 = 0 \Rightarrow \xi_{1,2,3} = 0$$

### Stability

$\Rightarrow$  The linear predictor method is zero stable if given  $y^1 = f(z_0)$ ,  $y(x_0) = y_0$ ,  $f(z_{2N})$  satisfy Lipschitz condition and root condition holds.

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$$\Rightarrow \frac{8}{3}\xi^2 - \frac{4}{3}\xi + \frac{8}{3} = 0 \Rightarrow |\xi_{1,2}| \geq 0$$

### Convergence & Consistency

The linear predictor above is consistent if it has order  $P \geq 1$

$$P(\xi) = \xi^3(\xi) = 0 \Rightarrow P(0, \omega) \in$$

Let's suppose the method is zero-stable, then  $P(\xi)$  has no roots in the unit circle. The product of roots at  $P(\xi) = 0$  is  $1$ .

If given  $y' = f(z)$   
Lipchitz condition and root condition

$\Rightarrow$  Thus we have in our case function

$$\Omega(q) = \frac{8}{3}q^2 - \frac{4}{3}q + \frac{8}{3} \quad \text{which is zero-stable}$$

since it yields simple roots

$$\Rightarrow \frac{8}{3}q^2 - \frac{4}{3}q + \frac{8}{3} = 0 \Rightarrow |q_{1,2}| \geq 0$$

Convergence & consistency

$\Rightarrow$  The linear predictor above is constant if it has order  $P \geq 1$

Let's suppose the method is zero-stable, then  $P(q)$  has double roots into unit circle.  
Further product of roots of  $P(q)$  is 1, and since the roots are neither equal to  $\pm 1$ , have to be strictly complex.

$$\Rightarrow q(q^2) = 0 \Rightarrow q_1 = 0, q_2 = 0, q_3 = 0$$

If given  
Lipchitz condition and root condition

$\Rightarrow$  Thus we have in our case function

$$\Omega(q) = \frac{8}{3}q^2 - \frac{4}{3}q + \frac{8}{3} \quad \text{which is zero-stable}$$

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$$\Rightarrow \frac{8}{3}q^2 - \frac{4}{3}q + \frac{8}{3} = 0 \Rightarrow |q_{1,2}| \geq 0$$

Convergence & consistency

$\Rightarrow$  The linear predictor above is constant if it has order  $P \geq 1$

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$$\Rightarrow q(q^2) = 0 \Rightarrow q_1 = 0, q_2 = 0, q_3 = 0$$

(3a) Derivation of 4<sup>th</sup> order explicitly Runge Kutta method

$$b_2 = \frac{1}{4}, \quad c_2 = \frac{1}{2}$$

Solv

(if the condition eqns be defined by)

$$b_1 + b_2 + b_3 + b_4 = 1 - (1)$$

$$c_2 = a_{21}$$

$$b_2 c_2 + b_3 b_3 + b_4 c_4 = \frac{1}{2} - (2)$$

$$c_3 = a_{31} + a_{32}$$

$$b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{3} - (3)$$

$$b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 = \frac{1}{4} - (4)$$

$$(4) \vdash a_{41} + a_{42} + a_{43}$$

$$b_3 (a_{32} c_2 + b_4 a_4 (a_{42} c_2 + a_{43} c_3)) = \frac{1}{6} - (5)$$

$$b_3 a_{32} c_2 + b_4 a_4 (a_{42} c_2 + a_{43} c_3) = \frac{1}{6} - (5)$$

(3a) Derivation at 4<sup>th</sup> order explicitly Runge Kutta method

$$b_2 = \frac{1}{4}, \quad c_2 = \frac{1}{2}$$

Solv

(if the condition eqns be defined by)

$$b_1 + b_2 + b_3 + b_4 = 1 - (1)$$

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$$(4) \vdash a_{41} + a_{42} + a_{43}$$

$$b_3 (a_{32} c_2 + b_4 a_4 (a_{42} c_2 + a_{43} c_3)) = \frac{1}{6} - (5)$$

$$b_3 a_{32} c_2 + b_4 a_4 (a_{42} c_2 + a_{43} c_3) = \frac{1}{6} - (5)$$

$$b_4 a_{43} = b_3 (1 - c_3) - (7)$$

$$0 = b_4 (1 - c_4) - (8)$$

so for  $b_2 = \frac{1}{4}$  and  $c_2 = \frac{1}{2}$  we have

Let the condition eqns be denoted by

$$b_1 + b_2 + b_3 + b_4 = 1 \quad (1)$$

$$b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{2} \quad (2)$$

$$b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 = \frac{1}{3} \quad (3)$$

$$b_2 c_2^4 + b_3 c_3^4 + b_4 c_4^4 = \frac{1}{4} \quad (4)$$

$$c_2 = a_{21}$$

$$c_3 = a_{31} + a_{32}$$

$$(4) \Rightarrow a_{41} + a_{42} + a_{43}$$

$$b_3 (a_{32} c_2 + b_4 a_{42} (a_{42} c_2 + a_{42} c_3)) = \frac{1}{8} \quad (5)$$

$$b_3 a_{32} c_2 + b_4 a_{42} = b_2 (1 - c_2) \quad (6)$$

$$b_4 a_{43} = b_3 (1 - c_3) \quad (7)$$

$$0 = b_4 (1 - c_4) \quad (8)$$

so for  $b_2 = \frac{1}{4}$  and  $c_2 = \frac{1}{2}$  we have

Substituting  $b_2 = \frac{1}{4}$  and  $c_2 = \frac{1}{2}$  in the equation

(1) - (3) we have

$$b_1 + \frac{1}{2} + b_3 + b_4 = 1 \Rightarrow b_1 + b_3 + b_4 = \frac{1}{2} \quad (9)$$

$$b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{3} \quad (3)$$

$$b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 = \frac{1}{4} \quad (4)$$

$$b_3 (a_{32} c_2 + b_4 a_{42} (a_{42} c_2 + a_{42} c_3)) = \frac{1}{8} \quad (5)$$

$$c_3 = a_{31} + a_{32}$$

$$(4) \Rightarrow a_{41} + a_{42} + a_{43}$$

$$b_3 a_{32} c_2 + b_4 a_{42} = b_2 (1 - c_2) \quad (6)$$

$$b_4 a_{43} = b_3 (1 - c_3) \quad (7)$$

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so for  $b_2 = \frac{1}{4}$  and  $c_2 = \frac{1}{2}$  we have

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(1) - (3) we have

$$b_1 + \frac{1}{2} + b_3 + b_4 = 1 \Rightarrow b_1 + b_3 + b_4 = \frac{1}{2} \quad (9)$$

$$\frac{1}{4} + b_3 c_3 + b_4 c_4 = \frac{1}{2} \Rightarrow b_3 c_3 + b_4 c_4 = \frac{1}{4} \quad (10)$$

$$\frac{1}{2} \cdot \frac{1}{4} + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{3} \Rightarrow b_3 c_3^2 + b_4 c_4^2 = \frac{5}{24} \quad (11)$$

$$\frac{1}{2} \cdot \frac{1}{8} + b_3 c_3^3 + b_4 c_4^3 = \frac{1}{4} \Rightarrow b_3 c_3^3 + b_4 c_4^3 = \frac{3}{16} \quad (12)$$

$$b_3 c_2 c_2 + b_4 c_4 (a_{42} c_2 + a_{43} c_3) = \gamma_4 - (5)$$

$$b_3 a_{42} c_2 + b_4 a_{42} c_3 = b_3 (c_2 - c_3) - (6)$$

$$b_4 a_{43} = b_3 (c_1 - c_2) - (7)$$

$$0 = b_4 (c_1 - c_2) - (8)$$

so we have  $b_3 = \frac{1}{4}$  and  $c_2 = \frac{1}{2}$  we have

Substituting  $b_3 = \frac{1}{4}$  and  $c_2 = \frac{1}{2}$  in the equation  
(1) - (3) we have

$$b_1 + \frac{1}{2} + b_3 + b_4 = 1 \Rightarrow b_1 + b_3 + b_4 = \frac{1}{2} - (9)$$

$$\frac{1}{4} + b_3 c_3 + b_4 c_4 = \frac{1}{2} \Rightarrow b_3 c_3 + b_4 c_4 = \frac{1}{4} - (10)$$

$$\frac{1}{4} + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{2} \Rightarrow b_3 c_3^2 + b_4 c_4^2 = \frac{1}{4} - (11)$$

$$\frac{1}{2} \cdot \frac{1}{8} + b_3 c_3^3 + b_4 c_4^3 = \frac{1}{4} \Rightarrow b_3 c_3^3 + b_4 c_4^3 = \frac{3}{16} - (12)$$

$$b_1 = \frac{1}{2} - b_3 - b_4, \quad b_2 = \frac{1}{2}, \quad b_3 = \frac{1}{2} - b_1 - b_4, \quad b_4 = \frac{1}{2} - b_1 - b_3$$

$$c_2 = \frac{1}{2}, \quad c_3 = (\frac{1}{4} - b_4 c_4) / b_3, \quad c_4 = (\frac{1}{4} - b_3 c_3) / b_4$$

$$a_{21} = \frac{1}{2}, \quad a_{31} = (\frac{1}{4} - b_4 c_4) / b_3 - a_{32}, \quad a_{32} = (\frac{1}{4} - b_4 c_4) / b_3 - a_{31}$$

$$a_{41} = (\frac{1}{4} - b_3 c_3) / b_4 - a_{42} - a_{43}, \quad a_{42} = (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{43}$$

$$a_{43} = (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{42}$$

But then fail again

$$c_2 = \frac{1}{2}, \quad c_3 = (\frac{1}{4} - b_4 c_4) / b_3, \quad c_4 = (\frac{1}{4} - b_3 c_3) / b_4$$

$$a_{21} = \frac{1}{2}, \quad a_{31} = (\frac{1}{4} - b_4 c_4) / b_3 - a_{32}, \quad a_{32} = (\frac{1}{4} - b_4 c_4) / b_3 - a_{31}$$

$$a_{41} = (\frac{1}{4} - b_3 c_3) / b_4 - a_{42} - a_{43}, \quad a_{42} = (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{43}$$

$$a_{43} = (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{42}$$

Butcher tableau

0	0
1	
2	2

$$\frac{1}{b_3} \left( \frac{1}{4} - b_4 c_4 \right) / b_3 - a_{32} \quad \left( \frac{1}{4} - b_4 c_4 \right) / b_3 - a_{31}$$

$$\frac{1}{b_4} \left( \frac{1}{4} - b_3 c_3 \right) / b_4 - a_{43} \quad \left( \frac{1}{4} - b_3 c_3 \right) / b_4 - a_{41}$$

$$c_2 = \frac{1}{2}, \quad c_3 = (\frac{1}{4} - b_4 c_4) / b_3, \quad c_4 = (\frac{1}{4} - b_3 c_3) / b_4$$

$$a_{21} = \frac{1}{2}, \quad a_{31} = (\frac{1}{4} - b_4 c_4) / b_3 - a_{32}, \quad a_{32} = (\frac{1}{4} - b_4 c_4) / b_3 - a_{31}$$

$$a_{41} = (\frac{1}{4} - b_3 c_3) / b_4 - a_{42} - a_{43}, \quad a_{42} = (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{43}$$

$$a_{43} = (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{42}$$

Butcher tableau

0	0
1	
2	2

$$\frac{1}{4} - b_4 c_4 / b_3 \quad \left( \frac{1}{4} - b_4 c_4 \right) / b_3 - a_{32} \quad \left( \frac{1}{4} - b_4 c_4 \right) / b_3 - a_{31}$$

$$\frac{1}{4} - b_3 c_3 / b_4 \quad \left( \frac{1}{4} - b_3 c_3 \right) / b_4 - a_{42} - a_{43} \quad \left( \frac{1}{4} - b_3 c_3 \right) / b_4 - a_{41} - a_{43} \quad \left( \frac{1}{4} - b_3 c_3 \right) / b_4 - a_{41} - a_{42}$$

$$\left( \frac{1}{2} - b_3 - b_4 \right) \cdot \left( \frac{1}{2} \right) \quad \left( \frac{1}{2} - b_1 - b_4 \right) \quad \left( \frac{1}{2} - b_1 - b_3 \right)$$

$$\begin{aligned}
 a_{21} &= \frac{1}{2}, \quad a_{31} = (\frac{1}{4} - b_4 c_4) b_3 - a_{22}, \\
 a_{41} &= (\frac{1}{4} - b_3 c_3) / b_4 - a_{42} - a_{43}, \quad a_{42} = (\frac{1}{4} - b_2 c_3) / b_4 - a_{41} - a_{43} \\
 a_{43} &= (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{42} \\
 \text{Butcher tableau} & \quad \begin{array}{c|cc|c} 0 & & & \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \\
 & \rightarrow \begin{array}{c|cc|c} 0 & & & \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} + \begin{array}{c|cc|c} 0 & & & \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} + \begin{array}{c|cc|c} 0 & & & \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \\
 -b_4 c_4 / b_3 - a_{32} & \rightarrow (\frac{1}{4} - b_4 c_4) / b_3 - a_{31} \\
 b_3 c_3 / b_4 - a_{42} - a_{43} & (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{43} \quad (\frac{1}{4} - b_3 c_3) b_4 - a_{41} - a_{43} \\
 b_4) & \cdot \left( \frac{1}{2} \right) \quad \left( \frac{1}{2} - b_1 - b_4 \right) \quad \left( \frac{1}{2} - b_1 - b_3 \right)
 \end{aligned}$$

$$\begin{aligned}
 a_{21} &= \frac{1}{2}, \quad a_{31} = (\frac{1}{4} - b_4 c_4) b_3 - a_{22}, \quad a_{32} = (\frac{1}{4} - b_4 c_4) b_3 - a_{31} \\
 a_{41} &= (\frac{1}{4} - b_3 c_3) / b_4 - a_{42} - b_{43}, \quad a_{42} = (\frac{1}{4} - b_2 c_3) / b_4 - a_{41} - a_{43} \\
 a_{43} &= (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{42} \\
 \text{Butcher tableau} & \quad \begin{array}{c|cc|c} 0 & & & \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \\
 & \rightarrow \begin{array}{c|cc|c} 0 & & & \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} + \begin{array}{c|cc|c} 0 & & & \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} + \begin{array}{c|cc|c} 0 & & & \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array} \\
 \frac{1}{4} - b_4 c_4 / b_3 & \rightarrow (\frac{1}{4} - b_4 c_4) / b_3 - a_{32} \rightarrow (\frac{1}{4} - b_4 c_4) / b_3 - a_{31} \\
 \frac{1}{4} - b_3 c_3 / b_4 & (\frac{1}{4} - b_3 c_3) / b_4 - a_{42} - a_{43} \quad (\frac{1}{4} - b_2 c_3) / b_4 - a_{41} - a_{43} \quad (\frac{1}{4} - b_3 c_3) b_4 - a_{41} - a_{43} \\
 & \cdot \left( \frac{1}{2} - b_3 - b_4 \right) \quad \left( \frac{1}{2} \right) \quad \left( \frac{1}{2} - b_1 - b_4 \right) \quad \left( \frac{1}{2} - b_1 - b_3 \right)
 \end{aligned}$$

(P) Equations of derivatives,  $d = \frac{d}{dt}$

$ds_1 = B \bar{F} S$	$\begin{array}{l} d + s_1 d + s_2 d \\ + s_3 d \end{array}$
	Given data

$$\begin{array}{c|ccc} \frac{1}{4} - b_4 c_4 / b_3 & (\frac{1}{4} - b_4 c_4) / b_3 - a_{32} & (\frac{1}{4} - b_4 c_4) / b_3 - a_{31} \\ \hline \frac{1}{4} - b_3 c_3 / b_4 & (\frac{1}{4} - b_3 c_3) / b_4 - a_{42} - a_{43} & (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{43} & (\frac{1}{4} - b_3 c_3) / b_4 - a_{41} - a_{42} \\ \hline & (\frac{1}{2} - b_3 - b_4) & (\frac{1}{2}) & (\frac{1}{2} - b_1 - b_4) & (\frac{1}{2} - b_1 - b_3) \end{array}$$

d) Equations of derivatives, i.e.  $\frac{dS}{dt} = \beta S I / N - \gamma S$ ,  $\frac{dI}{dt} = \beta S I / N - \gamma I - \mu I$ ,  $\frac{dR}{dt} = \gamma I - \mu R$

Given data  
 $N = 5000, 000$   
 $\beta = 0.4$   
 $\gamma = 0.10105 = 2.1$   
 $\mu = 0.2$

See Code

$$\begin{array}{c} \cancel{\frac{1}{2} - b_2 - b_3} / b_2 \quad (\frac{1}{2} - b_2 - b_3) / b_3 = a_{23} \quad (\frac{1}{2} - b_2 - b_3) / b_2 = a_{32} \\ \cancel{\frac{1}{2} - b_2 - b_3} / b_3 \quad (\frac{1}{2} - b_2 - b_3) / b_2 = a_{42} - a_{43} \quad (\frac{1}{2} - b_2 - b_3) / b_3 = a_{34} \\ \hline (\frac{1}{2} - b_2 - b_3) \quad \left(\frac{1}{2}\right) \quad (\frac{1}{2} - b_1 - b_2) \quad (\frac{1}{2} - b_1 - b_3) \end{array}$$

(3d) Equations of motion,	
$\frac{dS}{dt} = \beta FS/N - \gamma I - \mu I$	Given data $N = 5,000,000$
$\frac{dI}{dt} = \beta FS/N - \gamma I - \mu I$	$\beta = 0.4$ $\gamma = 0.10105 = 2.4$
$\frac{dR}{dt} = \gamma I$	$\mu = 0.2$
$\frac{dD}{dt} = \mu I$	
See Code	

(2c) Derivation of 4<sup>th</sup> order Adams Bashforth  
rule

Consider the AM general function given by

$$y_{n+1} = y_n + \frac{h}{24} (55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3})$$

$$\text{let } E y'_{n-2} = y'_{n-1}$$

$$E y'_{n-2} = y'_{n-1}$$

Substituting into our function  
above we have

$$E^2 y'_{n-2} = y'_n$$

(2c) Derivation of 4<sup>th</sup> order Adams Bashforth  
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Substituting into our function  
above we have

$$E^2 y'_{n-2} = y'_n$$

$$T(E) = \frac{9}{24}E^3 + \frac{37}{24}E^2 - \frac{59}{24}E + \frac{55}{24}$$

$$P(E) = (E^3 - 1)$$

$$(E^3 - 1) y_{n+2} - h (-\frac{9}{4}E^3 + \frac{37}{4}E^2 + \frac{59}{4}E + \frac{55}{24}) y'_n$$

(2c) Derivation of 4<sup>th</sup> order Adams-Basforth method

Consider the AM of general function given by

$$y_{n+1} = y_n + \frac{h}{24} (55y_n^1 - 59y_{n-1}^1 + 37y_{n-2}^1 - 9y_{n-3}^1)$$

Let  $Ey_{n-2}^1 = y_{n-1}^1$

$$Ey_{n-2}^1 = y_{n-1}^1$$

Substituting into our function above we have

$$E^2y_{n-2}^1 = y_n^1$$

$$\bar{\sigma}(E) = \frac{1}{24}E^3 + \frac{3}{24}E^2 - \frac{59}{24}E + \frac{55}{24}$$

$$\rho(E) = E^3 - 1$$

$$(E^3 - 1) y_{n-2} - h \left( -\frac{9}{24}E^3 + \frac{37}{24}E^2 + \frac{59}{24}E + \frac{55}{24} \right) y_{n-2}^1$$

Consider the AM of 4<sup>th</sup> order Adams-Basforth method

$$y_{n+1} = y_n + \frac{h}{24} (55y_n^1 - 59y_{n-1}^1 + 37y_{n-2}^1 - 9y_{n-3}^1)$$

Let  $Ey_{n-2}^1 = y_{n-1}^1$

$$Ey_{n-2}^1 = y_{n-1}^1$$

Substituting into our function above we have

$$E^2y_{n-2}^1 = y_n^1$$

$$\bar{\sigma}(E) = \frac{1}{24}E^3 + \frac{3}{24}E^2 - \frac{59}{24}E + \frac{55}{24}$$

$$\rho(E) = E^3 - 1$$

$$(E^3 - 1) y_{n-2} - h \left( -\frac{9}{24}E^3 + \frac{37}{24}E^2 + \frac{59}{24}E + \frac{55}{24} \right) y_{n-2}^1 = 0$$

Setting the root condition

$$\rho(\xi) = 0 \Rightarrow$$

characteristic polynomial

$$\xi^3 - \xi^2 - 59\xi + 55 = 0$$

$$\begin{aligned} \text{Let } E y_{n-2}^1 &= y_{n-1}^1 \\ E y_{n-2}^1 &= y_{n-1}^1 \quad \left| \begin{array}{l} \text{Substituting into our function} \\ \text{above we have} \end{array} \right. \\ E^2 y_{n-2}^1 &= y_n^1 \end{aligned}$$

$$D(E) = -\frac{9}{24}E^3 + \frac{37}{24}E^2 - \frac{59}{24}E + \frac{55}{24}$$

$$P(E) = (E^3 - 1)$$

$$(E^3 - 1) y_{n-2}^1 - h \left( -\frac{9}{24}E^3 + \frac{37}{24}E^2 + \frac{59}{24}E + \frac{55}{24} \right) y_{n-2}^1 = 0$$

Setting the root condition

$$P(q) = 0 \Rightarrow$$

Characteristic polynomial

$$D(q) = -\frac{9}{24}q^3 + \frac{37}{24}q^2 - \frac{59}{24}q + \frac{55}{24} \quad (1^{\text{st}} \text{ characteristic})$$

$$P(q) = (q^3 - 1) \quad (2^{\text{nd}} \text{ characteristic})$$

For the root root condition (H1) we have

$$P(E) = (E^3 - 1)$$

$$(E^3 - 1) y_{n-2}^1 - h \left( -\frac{9}{24}E^3 + \frac{37}{24}E^2 + \frac{59}{24}E + \frac{55}{24} \right) y_{n-2}^1 = 0$$

Setting the root condition

$$P(q) = 0 \Rightarrow$$

Characteristic polynomial

$$D(q) = -\frac{9}{24}q^3 + \frac{37}{24}q^2 - \frac{59}{24}q + \frac{55}{24} \quad (1^{\text{st}} \text{ characteristic})$$

$$P(q) = (q^3 - 1) \quad (2^{\text{nd}} \text{ characteristic})$$

For the root root condition (H1) we have

$$P(q) = 0$$

$$(q^3 - 1) = 0 \Rightarrow q^3 = 1$$

$$-\frac{9}{24}q^3 + \frac{37}{24}q^2 - \frac{59}{24}q + \frac{55}{24} = 0$$

$$P(E) = (E^3 - 1)$$

$$(E^3 - 1) y_{n-2} - h \left( -\frac{q}{24} E^3 + \frac{37}{24} E^2 + \frac{59}{24} E + \frac{55}{24} \right) y_{n-1} = 0$$

Setting the root condition

$$P(q) = 0 \Rightarrow$$

Characteristic polynomial

$$\sigma(q) = -\frac{q}{24} q^3 + \frac{37}{24} q^2 - \frac{59}{24} q + \frac{55}{24} = 0 \text{ characteristic}$$

$$P(q) = (q^3 - 1) \text{ } \underline{\text{3rd}} \text{ characteristic}$$

For the root condition (H) we have

$$P(q) = 0$$

$$(q^3 - 1) = 0 \Rightarrow q^3 = 1$$

$$-\frac{q}{24} q^3 + \frac{37}{24} q^2 - \frac{59}{24} q + \frac{55}{24} = 0$$

$$\Rightarrow q \left( -\frac{q}{24} q^2 + \frac{37}{24} q - \frac{59}{24} \right) = -\frac{55}{24}$$

Interval is from -1 to 1, the above value lies within the margin interval