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1. Approximation of functions by polynomials.

(a) Working in radians, obtain the Taylor polynomial $P_5(x)$ of degree 5 about $x_0 = 0$ of $f(x) = \frac{1}{2}\cos(3x)$

Soln

$\Rightarrow f(x) = P_n(x) + R_n(x)$

$\Rightarrow f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \cdots + \frac{f^n(c)(x-c)^n}{n!} + R(x)$

$\Rightarrow$ where $R_n(x) = \frac{f^{(n+1)}(x)}{(n+1)!}(x-c)^{n+1}$

$\Rightarrow$ Now, we calculate the polynomial ~~$P_5(x)$ at~~

$P_5(x)$ at $x_0 = 0$ with $c = 0$

$\Rightarrow$ But first we need to find the derivatives of $f(x)$ to degree 5

$\Rightarrow f(x) = \frac{1}{2}\cos 3x$

$\Rightarrow f'(x) = -\frac{1}{2} \cdot 3 \sin 3x$

$= -\frac{3}{2}\sin 3x$

$\Rightarrow f''(x) = -\frac{3}{2} \cdot 3 \cos 3x$

$= -\frac{9}{2} \cos 3x$

$\Rightarrow f'''(x) = \frac{9 \cdot 3}{2} \sin 3x$

$= \frac{27}{2} \sin 3x$

$\Rightarrow f^{iv}(x) = \frac{27 \cdot 3}{2} \cos 3x$

$= \frac{81}{2} \cos 3x$

$\Rightarrow f^{v}(x) = -\frac{81 \cdot 3}{2} \sin 3x$

$= -\frac{243}{2} \sin 3x$

$\Rightarrow$ Now we evaluate the above derivatives at $x_0 = 0$, Thus we have.

$\Rightarrow f(0) = \frac{1}{2} \cdot \cos 0 = \frac{1}{2}$

$\Rightarrow f'(0) = -\frac{3}{2} \sin 0 = 0$

$\Rightarrow f''(0) = -\frac{9}{2} \cos 0 = -\frac{9}{2}$

$\Rightarrow f'''(0) = \frac{27}{2} \sin 0 = 0$

$\Rightarrow$

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$\Rightarrow f^{iv}(0) = \frac{81}{2} \cos 0 = \frac{81}{2}$

$\Rightarrow f^{v}(0) = -\frac{243}{2} \sin 0 = 0$

$\Rightarrow f(x) \simeq \frac{1}{2} + 0.x - \frac{9}{2} \cdot \frac{x^2}{2!} + \frac{0.x^3}{3!} + \frac{81}{2} \frac{x^4}{4!} + \frac{0.x^5}{5!}$

$\Rightarrow f(x) \simeq \frac{1}{2} - \frac{9}{4} x^2 + \frac{81}{48} x^4$

$\Rightarrow P_5(x) = \frac{1}{2} - \frac{9}{4} x^2 + \frac{81}{48} x^4$

(b) To find polynomial estimate of $f(\frac{1}{69})$

Soln

$\Rightarrow$ ~~$P_5(\frac{1}{69})$~~

$f(x) \simeq \frac{1}{2} - \frac{9}{4} x^2 + \frac{81}{48} x^4$

$\Rightarrow \frac{1}{2} - \frac{9}{4} \left(\frac{1}{69}\right)^2 + \frac{81}{48} \left(\frac{1}{69}\right)^4$

$= 0.499527484$

$\Rightarrow$ To 8 decimal places we have

$= 0.49952748$

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(c) Construct an upper bound $\epsilon$ for error of the approximation Calculated in (b)

Soln

Remember that $R_n(x) = \frac{f^{(n+1)}(x)}{(n+1)!}(x-c)^{n+1}$

Then $R_5(x) = \frac{f^6(x)}{6!}(x-c)^6$

$\Rightarrow$ We now need to find the derivative of

$f^6(x) = ?$

$\Rightarrow$ from $f^5(x) = -\frac{243}{2} \sin 3x$

$\Rightarrow$ we have

$$f^6(x) = -\frac{243 \cdot 3}{2} \cdot \cos 3x.$$

$$= -\frac{729}{2} \cos 3x$$

$$\Rightarrow R_5(x) = \frac{-\frac{729}{2} \cos 3x \, (x-c)^6}{6!}$$

= and $x$ between with $c = 0$ 0 and $1/69$

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$\Rightarrow -\frac{729}{2.6!} \cos 3x \cdot x^6$

$x$ between 0 and $\frac{1}{69}$

$\Rightarrow -\frac{729}{1440} \cdot \cos 3x, x^6$

$\Rightarrow$ Now to get the upper bound we have that

$$|R_5(x)| \leq \left| -\frac{729}{1440} \cos 3x \cdot x^6 \right| = \left| \frac{729}{1440} \cdot x^6 \cdot \cos 3x \right|$$

$$= \left| R_5\left(\frac{1}{69}\right) \right| \leq \left| \frac{729}{1440} \cdot \left(\frac{1}{69}\right)^6 \cdot \cos 0 \right|$$

(d) Find actual error in estimate (b) and magnitude of upper bound (Compare and Comment)

Soln

$$\left| R_5\left(\frac{1}{69}\right) \right| \leq \left| \frac{729}{1440} \cdot \cos 0 \left(\frac{1}{69}\right)^6 \right|$$

$$= 4.691054 \times 10^{-12}$$

$\Rightarrow$ Next we Calculate the actual Value of $f(x)$

$\Rightarrow f(x) = \frac{1}{2} \cos 3x$

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$\Rightarrow \frac{1}{2} \cos(3 \times \frac{1}{64})$

$= 0.499999856$ —(i)

and with approximated value of (from part b)

$= 0.49952748 4$ —(ii)

We calculate the actual error to be

actual error = (i) − (ii) = ?

$= 4.7237204 \times 10^{-4}$

$\Rightarrow$ Thus, we are getting a larger magnitude of the actual error compared to the magnitude of the upper bound $\epsilon$ given by $4.7237204 \times 10^{-4}$ and $4.691054 \times 10^{-12}$ respectively

$\Rightarrow$ This is because the upper bound magnitude value will ensure a sharper error bound values, in other words, one that is as close to the actual error as possible.

$\boxed{\times}$

THANK YOU!!!