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-Revision Materials For a PDE Derivation (Proof)

Proof

From the non-linear pdes of the form

$$\begin{cases} \partial_t u + u, \Delta_x u + \frac{1}{2}Tr(\sigma\sigma^T \Delta_x^2 u) = F(\cdot,\cdot,u,\Delta_x u,\Delta_x^2 u) \quad (1) \\ u(T,\cdot) = g \end{cases}$$

on $[0,T] \times \mathbb{R}^d$ & on $\mathbb{R}^d$

=> We first assume that the solution given as u to this general non-linear pde is smooth C^2, thus we denote its parameter by (Y, Z, Γ)

=> Defining the parameters we have

$$Y_t = u(t,X_t), \quad Z_t = \Delta_x u(t,X_t), \quad \Gamma_t = \Delta_x^2 u(t,X_t) \text{ on } t\in$$

=> Writing the $u(t,X_t)$ solution is differential form we

$$dY_t = du(t,X_t) = -F(s,X_s,Y_s,Z_s,\Gamma) - Z_s^T\sigma(s,X_s)d$$

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On interval $t \in [0,T]$

=> Thus the solution satisfying the above differential form of a non-linear pde is given as

$$Y_t = g(X_T) - \int^T F_t(s,X_s,Y_s,Z_s,\Gamma)ds - \int^T Z_s^T\sigma(s,X_s)dW_s$$

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-Revision Materials For a PDE Derivation (Proof)

Proof

From the non-linear pdes of the form

$$\begin{cases}\partial_t u + u_t D_x u + \frac{1}{2} T_r(\sigma\sigma^T D_x^2 u) = F(\cdot,\cdot,u,D_x u, D_x^2 u) & (1)\\ u(T,\cdot) = g\end{cases}$$

on $[0,T] \times \mathbb{R}^d$ & on $\mathbb{R}^d$

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=> From the existence and uniqueness of the above solution under Lipschitz properties, is that there exist a constant C which depends on both f and α such that $\forall t \in [0,T]$

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=> From the existence and uniqueness of the above solution under Lipschitz properties, is that there exist a constant C which depends on both f and α such that $\forall t \in [0,T]$

$$x, z' \in \mathbb{R}^d, \quad y, y' \in \mathbb{R}^d, \quad Z, Z' \in \mathbb{R}^d$$

=> Therefore

-Revision Materials For a PDE Derivation (Proof)

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$$x,x'\in\mathbb{R}^d,\quad y,y'\in\mathbb{R}^d,\quad Z,Z'\in\mathbb{R}^d$$

=> Therefore

$$|F_t(t,x,y,\sigma(t,x)^T,z) - F_t(t,x',y',\sigma(t,x')^T,z') \le C_{f,\alpha}\big(|x-x'|$$
$$(1+|z|_2) + |y-y'| + |z-z'|_2\big)$$

=> Assuming the lipschitz ...

$u(t,X_t)$ solution is differential form we have

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$$|F_t(t,x,y,\sigma(t,x)^T,z) - F_t(t,x',y',\sigma(t,x')^T,z') \le C_{f,\alpha}\big(|x-x'|_2(\cdots$$
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=> Assuming the lipschitz convergence criterion is satisfied, we have that for some α, the solution $U(t,X_t)$ is bounded

=> Note that our generator function F_t does depend on x, y and z, thus by lipschitz proper... the function reduces to ...

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-Revision Materials For a PDE Derivation (Proof)

form of a non-linear pde satisfying the above differential is given as

$$Y_t = g(X_T) - \int_t^T F_t(s, X_s, Y_s, Z_s \Gamma)\,ds - \int_t^T Z_s^T \sigma(s, X_s)\,dW_s$$

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$$x, z' \in \mathbb{R}^d, \quad y, y' \in \mathbb{R}^d, \quad z, z' \in \mathbb{R}^d$$

=> Therefore

$$|F_t(t, x, y, \sigma(t,x)^T, z) - F_t(t, x', y', \sigma(t,x')^T, z')| \leq C_{f,\alpha}\big(|x - x'|_2(\dots(1 + |z|_2) + |y - y'| + |z - z'|_2\big)$$

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$$F_t(x,y,z) = (\delta_t + \alpha_t)x + (\delta_t + \alpha_t)y + (\delta_t + \alpha_t)z + \gamma_t$$

=> By the discounting and Girsanov's change of measure our solution function Y_t becomes

$$Y_t = E^{P^\alpha}\Big[e^{\int_t^T \delta_s ds} g(X_T) + \int_t^T e^{\int_t^u \delta_u du} \gamma_s ds \,\Big|\, (\delta_t + \alpha_t)x + (\delta_t + \alpha_t)y + (\delta_t + \alpha_t)z + \dots + \gamma_t \Big|$$

P^α is a probability measure

=> Using the comparison theorem to our nonlinear pde solution above, we have

$$Y_t = \operatorname*{ess\,sup}_{\alpha \in A} E^{P^\alpha}\Big[e^{\int_t^T \delta_s ds} g(X_T) + \int_t^T e^{\int_t^u \delta_u du} \gamma_s ds \,\Big|\, (\delta_t + \alpha_t)x + \dots$$

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on $t \in [0,T]$

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=> Note that the diffusion process X in R associated with our non-linear pde is given as

$$X_t = X_0 + \int_0^t \mu(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s \qquad t \in [0,T]$$

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=) Note that the diffusion process X in $\mathbb{R}$ associated with our non-linear pde is given as

$$X_t = X_0 + \int_0^t \mu(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s \quad t \in [0,T]$$

=) Taking the total derivatives of the function we have

$$dX_t = dX_0 + \mu(s,X_s)ds + \sigma(s,X_s)dW_s \quad \text{in } \mathbb{R}^d$$

=) Thus under standard lipschitz assumption on the coefficients μ, $\exists$ a unique solution to the ab... condition

$t \in [0,T]$

$$F_t(x,y,z) = (\delta_t + \alpha_t)x + (\delta_t + \alpha_t)y + (\delta_t + \alpha_t)z + \gamma_t$$

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on $t \in [0,T]$

=) Note that the diffusion process X in $\mathbb{R}$ associated with our non-linear pde is given as

$$X_t = X_0 + \int_0^t \mu(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s \quad t \in [0,T]$$

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$$Y_t(x,y,z) = (\delta_t + \alpha_t)x + (\delta_t + \alpha_t)y + (\delta_t + \alpha_t)z + \gamma t$$

⇒ By the discounting and Girsanov's change of measure our solution function Y_t becomes

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on $t \in [0,T]$

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⇒ Thus under standard lipschitz assumption on the coefficients μ, $\exists$ a unique solution to the above function under given initial condition

⇒ Implying that a forward non-linear pde estimate

our solution function Y_t becomes

$$Y_t = E^{P^{\alpha}}\left[e^{\int_t^T \delta_s ds} g(X_T) + \int_t^T e^{\int_t^u \delta_u du} \gamma_s ds \,\middle|\, (\delta_t+\alpha_t)x + (\delta_t+\alpha_t)y + (\delta_t+\alpha_t)z + \dots + \gamma t\right]$$

P^{α} is a probability measure

⇒ Using the comparison theorem to our nonlinear pde solution above, we have

$$Y_t = \operatorname*{ess\,sup}_{\alpha \in A} E^{P^{\alpha}}\left[e^{\int_t^T \delta_s ds} g(X_T) + \int_t^T e^{\int_t^u \delta_u du} \gamma_s ds \,\middle|\, (\delta_t+\alpha_t)x + \dots + (\delta_t+\alpha_t)y + (\delta_t+\alpha_t)z + \gamma t\right]$$

on $t \in [0,T]$

⇒ Note that the diffusion process X in $\mathbb{R}$ associated with our non-linear pde is given as

$$X_t = X_0 + \int_0^t \mu(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s \qquad t \in [0,T]$$

⇒ Taking the total derivatives of the function we have

$$dX_t = dX_0 + \mu(s,X_s)ds + \sigma(s,X_s)dW_s \quad \text{in } \mathbb{R}^d$$

⇒ Thus under standard lipschitz assumption on the coefficients μ, $\exists$ a unique solution to the above function under given initial condition

⇒ Implying that a forward non-linear pde estimate solution is given by

$$Y_t = g(X_t) - \int_t^T F(s,X_s,Y_s,Z_s,\Gamma_s)ds - \int_t^T Z_s^T \sigma(s,X_s)dW_s$$

But by the definition of the Markov properties w

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solution ... comparison theorem to our nonlinear pde above, we have

$$Y_t = \operatorname{ess\,sup}_{\alpha \in A} E^{P^\alpha}\Big[e^{\int_t^T \delta_s ds} g(X_T) + \int_t^T e^{\int_t^u \delta_u du} \gamma_s ds \,\Big|\, (\delta_t + \alpha_t)x + \cdots$$

$$+ (\delta_t + \alpha_t)y + (\delta_t + \alpha_t)z + \gamma t \Big|$$

$\Rightarrow$ Note that the diffusion process X on $t \in [0,T]$ in $\mathbb{R}$ associated with our non-linear pde is given as

$$X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \qquad t \in [0,T]$$

$\Rightarrow$ Taking the total derivatives of the function we have

$$dX_t = dX_0 + \mu(s, X_s) ds + \sigma(s, X_s) dW_s \quad \text{in } \mathbb{R}^d$$

$\Rightarrow$ Thus under standard lipschitz assumption on the coefficients μ, $\exists$ a unique solution to the above function under given initial condition

$\Rightarrow$ Implying that a forward non-linear pde estimate solution is given by

$$Y_t = g(X_t) - \int_t^T F(s, X_s, Y_s, Z_s, \Gamma_s) ds - \int_t^T Z_s^T \sigma(s, X_s) dW_s$$

$\Rightarrow$ But by the definition of the Markov properties we define the solution Y ~~restricted~~ restricted on the domain s and t as follows

$$Y_s - Y_t = U(s, X_s) - U(t, X_t) =$$

$$= \int_t^s F(s, X_u, Y_u, Z_u, \Gamma_u) ds - \int_t^s Z_u^T \sigma(s, X_u) dW_u \qquad t \in [0,T]$$

$\Rightarrow$ Thus from the Itô's formula we have that for a non-linear pde in the ~~form~~ x, y, z direction

$$\int_t^s \left(\frac{\partial u}{\partial t} + Lu\right)(u, X_u, Y_u) du + \int_t^s \sigma^T (X_u, Y_u) D_x U(u, X_u, Y_u) dW_u$$

$$= - \int_t^s F(s, X_u, Y_u, Z_u) du + \int_t^s Z_u dW_u$$

with the solution $Y_t = U(t, X_t)$ satisfying the fully non-linear equation (1)

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$$= \int_t^s F(s, X_u, Y_u, Z_u, \Gamma_u)\, ds - \int_t^s Z_u^T \sigma(s, X_u)\, dW_u \qquad t \in [0,T]$$

$\Rightarrow$ Thus from the Ito's formula we have that for a non-linear pde in the x, y, z direction

$$\int_t^s \left(\frac{\partial u}{\partial t} + Lu\right)(u, X_u, Y_u)\, du + \int_t^s \sigma^{+}(X_u, Y_u)\, D_x u\,(u, X_u, Y_u)\, dW_u$$

$$= - \int_t^s F(s, X_u, Y_u, Z_u)\, du + \int_t^s Z_u\, dW_u$$

with the solution $Y_t = u(t, X_t)$ satisfying the fully non-linear equation (1)