Nystrom extension

=> Given $K = \begin{pmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{pmatrix}$

prove $K_{BB} = K_{BA} K_{AA}^{-1} K_{AB}$

proof

Now given the matrix $K = \begin{pmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{pmatrix}$, we need to note that there will be a non-zero eigenvalue λ

=> Thus, we apply singular-value decomposition on K to obtain the below function

$$K = U \lambda U^T$$

whereby U stores the eigenvectors and the corresponding eigenvalues arranged in λ

=> Now, picking n elements uniformly at random, we obtain a simplified version of full rank given by matrix K_{AA}

=> Our main task now is to evaluate every entry in matrix K_{AA} which is $n \times n$ and similarly for K_{BA}

=> Inorder to achieve this we need to split up the eigen decomposition on the K matrix as below

=> Since $K = U \lambda U^T$, it follows that the eigenvalues λ stored in U resulting to the K-matrix will be given as

$$K = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \lambda \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^T$$

=> Simplifying this further we get

$$K = \begin{bmatrix} U_1 \lambda U_1^T & U_1 \lambda U_2^T \\ U_2 \lambda U_1^T & U_2 \lambda U_2^T \end{bmatrix} \text{———} (*)$$

where U is overally $n \times n$

=> Notice therefore we have K_{AA} from $(*)$ given by the first entry on the matrix $(*)$

=> $K_{AA} = U_1 \lambda U_1^T$

=> Now, since we know what K_{AA} is, we can easily calculate the value of U_1 and λ by performing a singular-value eigen decomposition on K_{AA}

=> Also, similarly from $(*)$ $K_{BA} = U_2 \lambda U_1^T$

=> Next we find the value of U_2 from K_{BA} by right-multiplying all sides by $U_1 \lambda^{-1}$

=> This results into:

$$U_2 = K_{BA} U_1 \lambda^{-1}$$

=> Next, is to find K_{BB} and prove its equivalence to

=> Note that from matrix representation (*), we have

K_{BB} given by

$$K_{BB} = U_2 \lambda U_2^T$$

=> But since we have U_2, we now just substitute into the above equation to find K_{BB}

$$\Rightarrow K_{BB} = (K_{BA} U_1 \lambda^{-1}) \lambda (K_{BA} U_1 \lambda^{-1})^T$$

=> This results into

$$K_{BB} = K_{BA} U_1 (\lambda^{-1} \lambda) \lambda^{-1} U_1^T K_{BA}^T$$

$$= K_{BA} U_1 \lambda^{-1} U_1^T K_{BA}^T$$

=> Note that $K_{AA} = U_1 \lambda U_1^T$, implying that

$$K_{AA}^{-1} = U_1 \lambda^{-1} U_1^T$$

=> Next, replace K_{AA}^{-1} into K_{BB} equation above

$$\Rightarrow K_{BA} K_{AA}^{-1} K_{BA}^T$$

Note that $K_{BA}^T = K_{AB}$, substitute this above we have

$$K_{BB} = K_{BA} K_{AA}^{-1} K_{AB}$$

$\boxed{X}$

THANK YOU!!!