PAGE NO:
DATE:

$$c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{———} \circledast$$

B.C $\quad u(0,t) = 0$

$\quad\quad\quad u(L_0,t) = f(t)$

I.C $\quad u(x,0) = 0$

$\quad\quad\quad \frac{\partial u(x,0)}{\partial t} = 0$

<u>Soln</u>

=> Note that wave equations are parabolic types of D.Es

=> So, we first need to obtain the Laplace transform with respect to 't' as follows

$$L\{u_{tt}(x,t)\} = s^2 u(x,s) - s u(x,0) - \frac{\partial u}{\partial t}(x,0) \quad — ①$$

=> Taking the laplace transform for eqn ⊛ we obtain

$$L\left\{c^2 \frac{\partial^2 u(x,t)}{\partial x^2}\right\} = L\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\}$$

$$\Rightarrow \quad c^2 \frac{\partial^2 u(x,s)}{\partial x^2} = L\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} = L\{u_{tt}(x,t)\} \quad — ②$$

PAGE NO.:
DATE: / /

=> Substituting ② into ① we have

$$\Rightarrow c^2 \frac{\partial^2 U(x,s)}{\partial x^2} = s^2 U(x,s) - s U(x,0) - \frac{\partial U}{\partial t}(x,0)$$

=> Applying the Initial condition we get

$\Rightarrow U(x,0) = 0$

$\Rightarrow \frac{\partial U}{\partial t}(x,0) = 0$

$$\Rightarrow c^2 \frac{\partial^2 U(x,s)}{\partial x^2} = s^2 U(x,s) \quad \longleftarrow ③$$

=> We need to transform the B.C.s as well

$\Rightarrow \bar{U}(0,s) = 0$

$\Rightarrow \bar{U}(L,s) = f(s)$

=> We now solve eqn ③ as follows

=> It is a 2nd order ode so we can solve it using the characteristic approach

$$\Rightarrow c^2 \frac{d^2 U}{dx^2} = s^2 U$$

$$\Rightarrow \frac{d^2 U}{dx^2} = \frac{s^2}{c^2} U$$

PAGE NO.:
DATE: / /

$\Rightarrow \lambda^2 = \frac{s^2}{c^2} \Rightarrow \lambda = \pm \frac{s}{c}$

$\Rightarrow U(x,s) = c_1 e^{\frac{s}{c}x} + c_2 e^{-\frac{s}{c}x}$

$\Rightarrow$ Applying B.C.,

$U(0,s) = 0$

$\Rightarrow U(0,s) = c_1 + c_2 = 0$

$\Rightarrow c_2 = -c_1$

$\Rightarrow U(x,s) = c_1 e^{\frac{s}{c}x} - c_1 e^{-\frac{s}{c}x} = c_1 \left[e^{\frac{s}{c}x} - e^{-\frac{s}{c}x} \right]$

$\Rightarrow$ From

$U(L_0,s) = c_1 \left[e^{\frac{s}{c}L_0} - e^{-\frac{s}{c}L_0} \right] = \bar{f}(s)$

$\Rightarrow \bar{f}(s) - \bar{U}(L_0,s) = c_1(s)$

$\Rightarrow U(x,s) = c_1(s) \left[e^{\frac{s}{c}L_0} - e^{-\frac{s}{c}L_0} \right]$

$\Rightarrow$ Taking inverse laplace on both sides;

$L^{-1}\{U(x,s)\} = L^{-1}\left\{ c_1(s) \left[e^{\frac{s}{c}L_0} - e^{-\frac{s}{c}L_0} \right] \right\}$

$\Rightarrow U(x,t) = L^{-1}\left\{ c_1(s) e^{\frac{s}{c}L_0} - \bar{c}_1(s) e^{-\frac{s}{c}L_0} \right\}$

$\Rightarrow c_1\left(t + \frac{L_0}{c}\right) - \bar{c}_1\left(t - \frac{L_0}{c}\right)$

PAGE NO.
DATE: / /

$$\Rightarrow u(x,t) = \begin{cases} 0 & x \leq 0 \text{ as } t \to 0 \text{ for } x \geq tc \\ c_1(t + L_0/c) - \bar{c}_1(t - l_0/c) & \frac{x}{c} < t \end{cases}$$

Verification

at

$u(0,s) = 0$

$\Rightarrow$ from $u(x,t)$ we have $x = 0$ with t at s

$\Rightarrow u(0,s) = 0$ since x is a value less or equal to zero from $u(x,t)$

and

$u(L_0,s) \Rightarrow f(s) \Rightarrow u(l_0,t) = f(t)$, we have that

$u(x,t) \Rightarrow u(L_0,t) = c_1(t + l_0/c) - \bar{c}_1(t - l_0/c) = f(t)$

$\Rightarrow u(x,t) = \checkmark$