Mandelbrot and Julia Sets

Gleb Sokolovski, s2015488

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1 Introduction

Fractals are mathematical objects that display self-similarity, meaning that they look similar at different levels of magnification. Mandelbrot and Julia sets are two famous examples of fractals, both of which are generated by iterations of complex functions. In this paper, we will explore the mathematics behind these fractals and their connection.

2 Julia Sets

We begin by defining what's known as a filled-in Julia set. A reminder that the Reimann sphere $\mathbb{C}^*$ is the defined as the extended complex plane such that $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$

Definition 2.1. Let $c \in \mathbb{C}^*$ be a complex number on the Reimann sphere. For each point z_0 of the plane, we generate a sequence that satisfies the iteration :

$$z_{n+1} = z_n^2 + c. \tag{2.1}$$

We define a set of z_0's for which the sequence diverges to infinity as the filled-in Julia set. And we denote it as K_C.

Following this, we arrive at the following defintion.

Definition 2.2. Julia set J_C is the boundary of the filled-in set (the set of "exceptional points").

Proposition 2.3. Let z_n be an element of the sequence $z_{n+1} = z_n^2 + c$. If z_n is farther than 2 from the origin, then successive iterates will grow without bound.

Proof. For a complex number $z_n = x_n + i \cdot y_n$, we define the absolute value as $|z_n| = \sqrt{(x_n^2 + y_n^2)}$. So we want to show if some z_n satisfies $|z_n| > \max(2, |c|)$, then the sequence $z_n, z_{n+1}, ...$ will diverge to infinity.

Suppose $|z_n| > \max(2, |c|)$. Then we can write (2.1) as

$$|z_n| = 2 + \epsilon,$$

for some $\epsilon > 0$. And since $|z_n^2| = |z_n^2 + c - c| \le |z_n^2 + c| + |c|$ due to triangle inequality, we can say that

$$\begin{aligned}
|z_n^2 + c| &\ge |z_n^2| - |c| = |z_n|^2 - |c| \\
&> |z_n|^2 - |z_n| (because |z_n| > |c|) \\
&= (|z_n| - 1) \cdot |z_n| = (1 + \epsilon) \cdot |z_n|
\end{aligned}$$

That is, $|z_{n+1}| > (1 + \epsilon) \cdot |z_n|$. Iterating, $|z_{n+k}| > (1 + \epsilon)^k \cdot |z_n|$.

Now observe that if $|c| > 2$, then (noting $|c + 1| > 1$ because $|c| > 2$)

$$z_0 = 0$$
$$z_1 = c$$
$$z_2 = c_2 + c = c \cdot (c + 1)$$
$$\rightarrow |z_2| = |c| \cdot |c + 1| > |c|.$$

$\square$

Example 2.4. If we paint the sequence produced by z_0 at its center that doesn't diverge to infinity in black colour, and otherwise we paint the pixel a colour determined by how quickly sequence gets further than 2 from the origin (hence diverges), we get interesting "fractals", like the one at Figures 1 and 2.
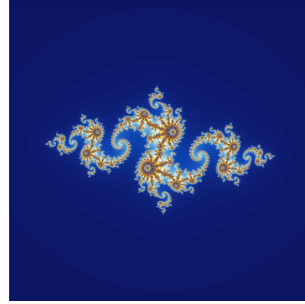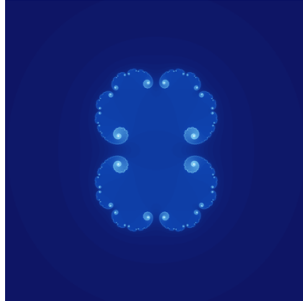

Figure 1: Julia set for f_c, $c = 0.285 + 0i$ [4]   Figure 2: Julia set for f_c, $c = -0.8 + 0.156i$ [4]

Let's now look at the two different types of Julia sets.

Definition 2.5 ([3])**.** There are two types of Julia sets: connected sets - also known as Fatou Sets - and Cantor sets - also known as Fatou Dust. The Fatou set is the set of points in the complex plane where the Julia set is locally connected. More precisely, a point z in the Fatou set of a Julia set J is said to be locally connected if there exists a neighborhood U of z such that the set of all points in U that belong to J is connected. While the Cantor set is a subset C of the Fatou set F of a Julia set J that satisfies the following properties:

1. C is compact and connected.

2. C has no isolated points.

3. C is perfect, meaning that every point of C is a limit point of C.

4. C is nowhere dense, meaning that its closure has empty interior.

5. C has a self-similar structure, meaning that it can be expressed as the union of finitely many copies of itself, each of which is scaled by a fixed factor and translated to a different location.

Example 2.6. Consider the complex quadratic polynomial $f(z) = z^2 - 1$. The Julia set J of this polynomial is the set of all points in the complex plane that have chaotic behavior under iterations of f.

The Fatou set F of this Julia set consists of all points in the complex plane that have stable behavior under iterations of f.

As for an example of a Cantor set (or Fatou dust) for this Julia set, we could consider the set of points that are attracted to a repelling periodic orbit of f. Specifically, let p be the fixed point of f given by $p = (1 + \sqrt{5})/2$, and let C be the set of all points in the complex plane that converge to p under iterations of f. This set C is a Cantor set.

We just need one more definition to set us up perfectly for the important theorem. It's the definition many have heard of - but I will put in it just in case you haven't.

Definition 2.7. Critical points are the points in the complex plane where the derivative of a function is zero or undefined. In other words:

$$\text{CPs}(f(z)) = \{z : f'(z) = 0 \,||\, f'(z) \text{ is undefined }\}.$$

Theorem 2.8 (Dichotomy Theorem). For any polynomial f, if all the critical points of f belong to K_f, then J_f is connected and if none of the critical points of f belong to K_f, then J_f is a Cantor set.

Proof. To prove these statements, we need to use the theory of complex dynamics, which is a branch of mathematics that studies the iteration of complex functions. In particular, we will need to use the theory of iteration of polynomials, which is a special case of complex dynamics. First, we define the filled-in Julia set K_f of a polynomial f to be the set of points z in the complex plane such that the iterates $f^n(z)_n = 0^\infty$ are bounded. That is,

$$K_f = \{z : f^n(z)_n = 0^\infty \text{ is bounded }\}.$$

The Julia set J_f is then defined as the boundary of K_f, that is, $J_f = \partial K_f$. Now, we can prove the two statements:

1. If all the critical points of f belong to K_f, then J_f is connected.

2. If none of the critical points of f belong to K_f, then J_f is a Cantor set.

I will not get into the proof of them, but you can read about it here [2].

□

3   Mandelbrot Sets

The Mandelbrot set, M, is defined by the same iteration process used to define Julia sets, but applied in a different way. Instead of using the complex plane to represent the different choices of z_0, it represents different values of c.

Definition 3.1. Let $c \in \mathbb{C}^*$ be a complex number on the Reimann sphere. For the point $z_0 = 0$, we generate a sequence that satisfies the iteration of the equation 2.1. We define a set of c's for which the sequence does not diverge to infinity as the Mandelbrot set. And we denote it as M.

Remark 3.2. Starting with $z_0 = 0$ allows us to directly compute the sequence for each value of c without having to calculate a separate starting value for each point in the complex plane. It also ensures that we are always working with the same initial value, making it easier to compare different values of c and generate a consistent image of the Mandelbrot set.
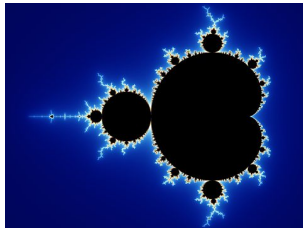

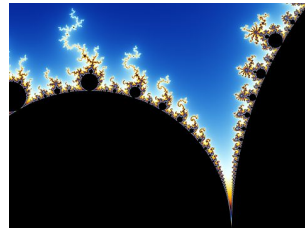Figure 3: Famous Mandelbrot Example [5]

Figure 4: Zoomed in to the gap between the "head" and the "body" [5]

As with Julia sets, we paint the pixel black if the sequence produced by the c at its center belongs to M, and otherwise we paint the pixel a color determined by how quickly the sequence diverges.

Figure 4 is sometimes also called the "seahorse valley".

4 Applications of Julia and Mandelbrot Sets

For a while Julia and Mandelbrot sets didn't have much use. Their main purpose was the amazing shapes they produced in computer graphics. However it has since been used in Chaos theory, cryptography, signal processing, Number theory, and a lot more. There was a very interesting paper released by M. A. Alia and A. B. Samsudin about the new key exchange protocol based on what we have looked at today - definitely recommend a read [1].

References

[1] M. A. Alia and A. B. Samsudin. 'New Key Exchange Protocol Based on Mandelbrot and Julia Fractal Sets'. In: (). URL: http://paper.ijcsns.org/07_book/200702/200702B17.pdf.

[2] V. Blankers et al. 'Julia and Mandelbrot Sets for Dynamics over the Hyperbolic Numbers'. In: (). URL: https://www.mdpi.com/2504-3110/3/1/6.

[3] Eric W. Weisstein. 'Julia Set'. In: (). URL: https://mathworld.wolfram.com/JuliaSet.html.

[4] Wikipedia. 'Julia Sets'. In: (). URL: https://en.wikipedia.org/wiki/Julia_set.

[5] Wikipedia. 'Mandelbrot Sets'. In: (). URL: https://en.wikipedia.org/wiki/Mandelbrot_set.