Solving the Reswitching Paradox in the Sraffian Theory of Capital

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June 2019

The possibility of the reswitching of techniques in Piero Sraffa's intersectoral model, namely the returning capital-intensive techniques with monotonic changes in the profit rate, is traditionally considered as a paradox putting at stake the viability of the neoclassical theory of production. It is argued here that this phenomenon can be rationalized within the neoclassical paradigm. Sectoral interdependencies can give rise to non-monotonic effects of progressive variations in income distribution on relative prices. The reswitching of techniques is therefore the result of cost-minimizing technical choices facing returning ranks of relative input prices in full consistency with the neoclassical perspective. (*JEL* B12, B13, B51, D33, D57, D61, Q11)

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The use of the concept of marginal productivity in Piketty's (2014) *Capital in the Twenty-First Century* has triggered a critical debate (for example, with different views, Galbraith, 2014 and

Solow, 2014) drawing on the Cambridge controversy of the 1960s and Piero Sraffa's (1960) canonical book. However, this controversy has never reached a definite conclusion regarding its relevance and contents as demonstrated by the recent revival of discussions (Coen and Harcourt, 2003, Pasinetti, 2003, Scazzieri, 2008, Garegnani, 2012, Schefold, 2013, Backhouse, 2014, Gram and Harcourt, 2017). One of the major problems with Sraffa's model of prices is that, as Afriat (1987, p. 189) noted, "there is an obstacle to the application of the theory, *since the arithmetic of it is impossible"* (emphasis added) as there are too many conditions imposed by Sraffa on his system of prices. Afriat's criticism did not come alone. Samuelson (1962, 1966, 1975, 1983, 2000), Hicks (1965), Morishima (1966), Solow (1969, 1975), Stiglitz (1973, 1974), and Sen (1974), among many others, have shown perplexity about certain other mathematical aspects. Samuelson (2000, p. 113), in particular, referred to Sraffa's (1960) book as "a work in mathematical economics by an amateur, an autodidact. It has the properties of such. The book has more in it that the author knows. It is not the better for its imperfections."

Surprisingly, nobody with appropriate skills has so far unveiled the fallacy of the interpretation of reswitching of techniques as violation of the neoclassical theory of production. Indeed, Samuelson (2000, p. 117) himself praised the last seven pages dedicated to the choice of techniques in Sraffa (1960) to "constitute the novelty of the work's contribution." He had previously claimed that, though this 100-page book "presents results that are compatible with marginalist theory or certain modern generalizations of that theory of the linear programming type, we have no right to indict Sraffa for being a marginalist" (Samuelson, 1961, p. 423). In this perspective, it is argued here that the reswitching of techniques is not contradictory but fully consistent with the neoclassical paradigm.

As Keynes observed, old habits die hard, even in the field of scientific thinking. Once flawed principles have been established, it is easier to persuade young students than elderly scholars about old misleading views. As noted by Al-Khalili (2013), true paradoxes in science are statements that lead to circular and self-contradictory arguments or describe logically impossible situations. They are generally due to false assumptions or erroneous linking true assumptions with wrong conclusions or, if starting from true assumptions and using a correct logic, the true conclusions appear contrary to the common sense arising from the narrow interpreter's vision (see also Sorensen, 2005; Sainsbury, 2009). The "reswitching" paradox, while it is presented as a contradiction of the marginalist rationality, is in fact a case of deductive incoherence in misleading interpretations of the obtained results (on the coherence theory of truth, see for example Priest, 2000, 2006a, 2006b).

Much of the confusion and debate have come from the definition of the capital input price. In different occasions, Hicks (1965, p. 140, fn.1; 1979) recalled that it is the net earnings of the proprietors of capital goods rather than the rate of interest that should be considered as the price of capital services in the general case (see, for example, Petri, 2016 among recent discussions). It is well known, at least since Walras (1896), that the unit cost of using durable capital goods in production is the *rental price of capital service* where the rate of interest comes in *simultaneous interrelation* with the acquisition prices of capital goods. This was in line with the earlier studies of Wicksell (1893, 1901), who noted a discrepancy between the marginal productivity of capital and the rate of interest, given that, in the equilibrium of a social system, the former is equal to the real price of capital services, not simply to the rate of interest.[1]

In the Sraffian intersectoral model of production, the rate of interest may indeed interact in various directions with all relative prices. The intersectoral interaction between the rate of interest and relative input prices is generally non-linear. Different techniques yield equilibrium price solutions that may correspond to different levels of the interest rate. As it will be shown in the present paper, the price-taking producers may "reswitch" techniques as a cost-minimizing response to changes in relative input prices. [2]

To be sure, initially, Joan Robinson (1953) was very cautious when discussing the "curious possibility" of the reswitching of techniques pointed out to her by Ruth Cohen. In this view, the "perverse" behavior of the curve defined in the real wage-interest coordinate space, when it occurs at all, can be met only rarely and over a limited range. The reswitching was recognized by Champernowne (1953) in his comment on her article and reaffirmed by Robinson (1956) herself and became more fully explicit in Sraffa's (1960, Ch. XII) last seven pages. Fisher (1907, pp. 352-53; 1930, Ch. XI, especially p. 279), in his classic works on the rate of interest was

[1] Since the reswitching of techniques has been raised as a paradoxical phenomenon without any reference to capital aggregation, it is treated here without fully discussing its implication on aggregation theory.

[2] Gallaway and Shukla (1976) recalled that the most profitable technique is not necessarily the one with the highest rate of profit for a given real wage (see Laibman and Nell, 1977: 883-84 for a discussion). Salvadori (1985) reached the same conclusion in the joint production case, but he claimed that this is due to the existence of joint production. However, none of these authors went as far as to overhaul the interpretation of the "reswitching" of techniques in terms of violation of the marginalist price-quantity behaviour.

aware of the possibility of reversing in the capital *value* intensity in relation with interest (see, for a discussion, Samuelson, 1966 and the historical notes of Velupillai, 1975, 1995).

Many years later, Joan Robinson intervened again on the subject to confirm her views about the unimportance of reswitching relative to other major issues such as those regarding the existence of an aggregate pseudo production function of an economic system or whether there exists an accumulation taking place in a given state of technical knowledge (Robinson and Naqvi, 1967 and Robinson, 1969, 1975a, 1975b). As she described it, "[w]hat "reswitching" showed was that a higher real-wage rate is not necessarily associated with higher net output per head, and a lower rate of profit with a higher value of capital per man employed" (Robinson, 1975b, p. 553). She noted that a good deal of exploration of the possible magnitude and behavior of the interest effect was needed before saying whether this phenomenon is a mere theoretical "rigmarole", or whether there is likely to be anything in reality corresponding to it. However, while the aggregation theory has reached a mature state in the mainstream neoclassical field, the remote possibility of a perverse behavior of the relation between capital intensity and interest rate has become the powerhouse argument of the Sraffian critics (Mas-Colell, 1989). This appeared consistent with Karl Popper's swan example where only one observed black swan is sufficient to falsify the theory stating that all swans are white.

An inspiring *incipit* of the article by Robinson (1975a) is worth quoting:

The story of what is known as the debate over the reswitching of the techniques is a sad example of how controversies arise between contestants who confront the conclusions of their arguments without first examining their respective assumptions. How is it possible to have a controversy over *a purely logical point? When various theorists each set out their assumptions clearly, after eliminating errors, they can agree about what conclusions follow from what assumptions* (*ibid*, p. 32).

In the spirit of these words, the following reflections are proposed.

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I. Background concepts: "marginalist" values of capital goods

Before unfolding the full specification and solution of the reference model, let me offer the curious reader my view of the origin of the Sraffian interpretation of reswitching. This clarification may also answer the question of why eminent economists have fallen into an oversight. The lack of reliable disaggregated data during the first half of the twentieth century has contributed to the adoption of aggregate economic models with the notable exceptions of Leontief (1941, 1953) and Afriat (1972, 2014) (on the latter, see also Afriat and Milana, 2009). The use of the aggregate production functions had made the economists familiar with the concept of economy-wide optimal factor demands. In the particular case of a Clark-Ramsey economy producing one single commodity by means of the same commodity and labour in an aggregate economy (where the output can be seen as partly re-usable as an input of production), it is customary to consider two aggregate inputs. Denoting the quantity and producer price levels of gross output of the aggregate economy respectively with y and p_y, under the competitive equilibrium condition and constant returns to scale where supernormal profits are zero, p_y equals the marginal minimum cost of production λ. Then, the optimality first-order conditions yield the marginal factor productivities to equal the factor price ratios:

(1) $$MPL \equiv \frac{\partial f(\mathbf{x})}{\partial x_L} = \frac{w_L}{\lambda} = \frac{w_L}{p_y}$$

(2) $$MPK \equiv \frac{\partial f(\mathbf{x})}{\partial x_K} = \frac{w_K}{\lambda} = \frac{p_y(\delta + r)}{p_y} = \delta + r$$

where x_L and x_K are respectively quantities of labour and capital inputs, δ is the depreciation rate taking values within the interval $0 \leq \delta \leq 1$, and r is the interest rate, whose equilibrium level is equal to the profit rate and is determined in the financial market, and w_L is the labour wage rate assumed here to be paid *post factum*.

The textbook definition of the marginal rate of cost-minimizing input substitution subject to a standard neoclassical production function $f(\mathbf{x})$ is

(3) $$MRS \equiv \left. \frac{dx_L}{dx_K} \right|_{isoquant} = -\frac{MPK}{MPL}$$

Therefore, in view of (2) and (3), in the model with a homogeneous output,

(4) $$\left. -\frac{dx_L}{dx_K} \right|_{isoquant} = \frac{w_K}{w_L}$$

Hence the equality $\frac{w_K}{w_L} = \frac{(\delta + r)}{w_L / p_y}$ has led to the ratio of $(\delta + r)$ to w_L / p_y, as an indicator of the price of capital input relative to the real labour wage. Such a customary formulation became mistaken when it was extended to the interindustry models and the error has leaked into many studies bringing about paradoxical meanings of the results.

In an interindustry model with heterogeneous outputs, the definition of marginal rate of substitution leads to the equilibrium equality

$$(5) \qquad -\frac{dx_L}{dx_K}\bigg|_{isoquant} = \frac{w_K}{w_L} = \frac{p_K(\delta + r)}{w_L}$$

The correct expressions for the real wages of capital and labour inputs are in fact $\frac{p_K(\delta + r)}{p_y}$ and (w_L / p_y) respectively, except of course the real rental of the self-produced capital good, which is equal to $\delta + r$ when $p_K = p_y$. As it will be seen later, an equivalent discrete form of the (negative) marginal rate of input substitution given by (5) is implied by Sraffa's (1960, Ch. XII) model of a cost-minimizing choice over a finite set of alternative techniques of production. However, in an intersectoral model and even in macroeconomic models where the output is produced for both immediate consumption and accumulation purposes, the relative acquisition prices are affected by changes in income distribution, where the output price results as an aggregate of the respective price indexes in consumption and investment activities, that is $p_y = g(p_C, p_K)$ with $p_y \neq p_C \neq p_K$. The model is to be solved by taking account of the interrelation between the interest rate and these relative prices (see, for example, Harcourt, 1970, p. 45). In the next section, this also applies to the intersectoral models where monotonic changes in the interest rate affect relative prices in a non-linear way while playing a key role in equilibrium solutions.

II. Accounting for prices in the Leontief-Sraffa model

The Sraffian model of production of commodities by means of commodities has been often stylized in the simplest form with two sectors using two or three inputs. In most examples of the two-sector two-input model, the focused sector produces the consumption good using labour rewarded *post factum* while the capital goods are acquired *ex ante*, at the start of the current period, from the second sector producing the capital good using labour and a quantity of its output that in turn is also acquired *ex ante* from itself.

The Sraffian solutions are more clearly seen from the accounting system of a generalized Leontief-Sraffa type model. Let us consider a simplified production system with the following characteristics. All commodities can be produced over one certain period of time at constant returns to scale with no joint production, out of themselves and out of one or more primary factors produced in the past periods of time. In a fully competitive equilibrium of the current period, there are no supernormal profits on production activity where the rate of interest and the rate of profits result to be the same in all sectors. In such conditions, the unit value of output equals everywhere the average total cost of production. The general system of price accounting equations can be expressed in matrix form as

$$\mathbf{p} = w_L \mathbf{a}_0 + \mathbf{pB}(\hat{\boldsymbol{\delta}} + \hat{\mathbf{r}}) + \mathbf{pA} \quad (6)$$

where $\mathbf{p}$ is the n-order row vector of output prices; w_0 is the *ante factum* labour wage rate; w_L is the labour wage rate paid *post factum,* which is equivalent to the present value of w_0, the labour wage paid *ante factum*, so that $w_L = w_0(1+r)$ (differently from most numerical examples, the original Sraffa's, 1960 model does not consider the labour wage paid *ante factum*); $\mathbf{a}_0$ is the *n*-order row vector of direct input-output coefficients of labour*;* **I** is the $(n \times n)$-order unit matrix**; r** is the *n*-order vector of internal rates of return or rates of profit; $\boldsymbol{\delta}$ is the *n*-order vector of non-negative depreciation rates of capital goods (the hat ^ indicates the transformation of a vector in a diagonal matrix); $\mathbf{A}$ is the $(n \times n)$-order Leontief matrix of direct input-output coefficients for intermediate circulating goods produced and consumed during the current period of production; $\mathbf{B}$ is the $(n \times n)$-order Leontief matrix of input-output coefficients for the services of capital goods pre-existing at the start of the current period of production.

In Sraffa's model of production of commodities by means of commodities where all $\delta_i = 1$, the accounting equation (6) of an economy in equilibrium condition with all sectors scoring the same profit rate *r* can be fully solved in one single step with the following reduced form if the Hawkins and Simon condition on the viability of the system is satisfied:

(7) $$\mathbf{p} = w_L \mathbf{a}_0 [\mathbf{I} - \mathbf{A} - (1+r)\mathbf{B}]^{-1}$$

To follow Sraffa's reasoning, the system of price accounting equations (6) can be solved alternatively in two steps. The first step computes the price components defined at the level of the Leontievian "vertically integrated sectors" containing all the interindustry transactions occurring in the current period to produce the respective final quantities of commodities. This solution is obtained by taking account of the input-output interactions between the prices of inputs produced during the same current period and considering the prices of pre-existing factor inputs as predetermined variables:

$$
\begin{aligned}
\mathbf{p} &= w_L \mathbf{a}_0 (\mathbf{I}-\mathbf{A})^{-1} + \mathbf{p}(1+r)\mathbf{B}(\mathbf{I}-\mathbf{A})^{-1} \\
&= w_L \mathbf{a}_L + \mathbf{p}(1+r)\mathbf{A}_K \\
&= \mathbf{w}\mathbf{A}_T
\end{aligned} \tag{8}
$$

where $\mathbf{a}_L \equiv \mathbf{a}_0(\mathbf{I}-\mathbf{A})^{-1}$ is the n-order row vector of Leontief's *direct and indirect* input-output requirements of labour inputs; $\mathbf{A}_K \equiv \mathbf{B}(\mathbf{I}-\mathbf{A})^{-1}$ is the $(n \times n)$-order matrix of Leontief's *direct and indirect* input-output coefficients for inputs of capital goods services; $\mathbf{w} \equiv [w_L \ \mathbf{w}_K]$ is the $(n+1)$-order row vector of wage rates for inputs of labour and capital goods services; $\mathbf{w}_K \equiv \mathbf{p}(1+r)$ is the n-order row vector of rental prices or user cost of capital goods; $\mathbf{A}_T \equiv \begin{bmatrix} \mathbf{a}_L \\ \mathbf{A}_K \end{bmatrix}$ is the $[(n+1)\times n]$-order matrix of Leontief's direct and indirect input-output requirements for total labour and capital goods.

In the second step, the reduced form of the Sraffa's price model is derived from the second line of (8):

$$
\mathbf{p} = w_L \mathbf{a}_L [\mathbf{I} - (1+r)\mathbf{A}_K]^{-1} \tag{9}
$$

The prices $\mathbf{p}$ are positive if $0 \leq r \leq R$, where R is the maximum profit rate attainable in the production system, that is $R = (1/\lambda) - 1$, and λ represents the dominant eigenvalue of $\mathbf{A}_K$ (see, for example, Pasinetti, 1977, pp. 95-97).

In this price accounting system of n equations with the $(n + 2)$ price-type variables $\mathbf{p}, w_L, r$, the technology can autonomously determine all relative prices except one, which can be chosen arbitrarily. Let us divide all prices and wage rate through (9) by one arbitrary output price, say p_j

. Moreover, let w_L / p_j, or r, or a normalized output price, say p_i / p_j, be pre-determined; then, the Sraffian system of price equations can be respectively solved to find the n-tuple of either $(\frac{1}{p_j}\mathbf{p}, r)$, or $R = (1/\lambda) - 1$, or the rest of $(n-2)$ normalized output prices along with $(\frac{w_L}{p_j}, r)$. In the Sraffian approach, the output prices and labour wage are usually expressed in real terms as ratios to the output price of one focused commodity while the level of the real wage rate or rate of profits is conjecturally fixed. [3]

The equation system (9) is the further reduced form of (8) giving rise to the accounting expression of the price decomposition of the so-called Sraffian "sub-systems". Some authors, for example Gram (1973), Pertz and Teplitz (1979), considered the second and third lines of (8) as two alternative views of the same equation. In the following discussion of the reswitching paradox, *both* (8) and (9) reduced forms provide complementary information needed to satisfy Sraffa's (1960, Ch. 9) requirements for identifying the switch points between alternative techniques of production. In these switch points, the techniques coexist with all the n-tuples of relative prices plus the real wage and the interest rate being equal. Contrary to the usual

[3] Sraffa defined also a theoretical "standard commodity" to be used as a *numéraire* whose computed price indeed depends on income distribution, but is invariant with respect to changes in the relative prices of other commodities. The price of the standard commodity corresponds to a weighted average of commodity prices where the weights are the elements of the eigenvector of the matrix $\mathbf{A}_K$. However, the dependence of movements of this price on the profit and wage rates, support the abovementioned Afriat's criticism about the arithmetic impossibility of the simultaneous determination of a price "ultimately equal to the labour that has gone into making it" and being also the result of a particular income distribution. More recent discussions on the meaning of Sraffa's "standard commodity", on which there is not yet a commonly accepted view, include those of Bellino (2004), Baldone (2006), and Wright (2014, 2017).

discussions in the reswitching debate, which have been generally centered on the equation of the focused commodity in the reduced form (9), sufficient conditions for the existence of genuine reswitching points need to be checked also on the full range of all relative prices by exploring more directly the structural form (8).

The *j*th sub-system features the following real labour wage as function of the profit rate for a given technology:

$$\frac{w_L}{p_j} = \frac{1}{\mathbf{a}_L[\mathbf{I} - (1+r)\mathbf{A}_K]^{-1}\mathbf{e}_j} \tag{10}$$

where $\mathbf{e}_j$ is a column vector with all its elements equal to zero except the *j*th one, which is equal to unity.

The capital input prices expressed in terms of the *j*th commodity are obtained as

$$\mathbf{w}_K \frac{1}{p_j} = \mathbf{p}\frac{(1+r)}{p_j} = \frac{w_L}{p_j}\mathbf{a}_L[\frac{1}{(1+r)}\mathbf{I} - \mathbf{A}_K]^{-1} \tag{11}$$

Dividing (11) through by $\frac{w_L}{p_j}$ yields

$$\frac{1}{w_L}\mathbf{w}_K = \mathbf{a}_L[\frac{1}{1+r}\mathbf{I} - \mathbf{A}_K]^{-1} \tag{12}$$

which, in view of (8), (11), and (12), is equivalent to

$$\frac{1}{w_L}\mathbf{w}_K = \frac{1+r}{w_L}\mathbf{w}\mathbf{A}_T \tag{13}$$

III. The Sraffian price system from the perspective of linear programming

In order to clarify further the meaning of the rental prices of capital goods used in the system (8) and their relationship with the interest rate, the third line of (8) can be complemented with the objective function specifying explicitly the assumption of cost minimization throughout the economy in the following classical programming problem

$$C(\mathbf{w}) = Min_{\mathbf{w}} \ \mathbf{w} \cdot \mathbf{v}$$

(11) sub to

$$\mathbf{p} = \mathbf{w}\mathbf{A}_T$$

where $\mathbf{v}$ is a given (column) vector of primary inputs.

The foregoing minimization problem has the dual counterpart of the quantity maximization

$$R(\mathbf{f}) = Max_{\mathbf{f}} \ \mathbf{p} \cdot \mathbf{f} \tag{12}$$

sub to

$$\mathbf{A}_T\mathbf{f} = \mathbf{v}$$

where $\mathbf{f}$ is a given (column) vector of final outputs.

The Lagrangian functions of problems (11) and (12) are, respectively,

$$L(\mathbf{w},\ \boldsymbol{\lambda}_C) = C(\mathbf{w}) + (\,\mathbf{p} - \mathbf{w}\mathbf{A}_T\,)\boldsymbol{\lambda}_C \tag{13}$$

$$L(\boldsymbol{\lambda}_R, \mathbf{f}) = R(\mathbf{f}) + \boldsymbol{\lambda}_R(\,\mathbf{v} - \mathbf{A}_T\mathbf{f}\,)$$

The conditions for a stationary point of $L(\mathbf{w},\ \boldsymbol{\lambda}_C)$ are

$$\begin{aligned} &\nabla_{\mathbf{w}^*} L(\mathbf{w}^*, \boldsymbol{\lambda}_C^*) = \nabla_{\mathbf{w}^*} C(\mathbf{w}^*) - \mathbf{A}_T\boldsymbol{\lambda}_C^* = \mathbf{0} \\ &\nabla_{\boldsymbol{\lambda}_C^*} L(\mathbf{w}^*, \boldsymbol{\lambda}_C^*) = \mathbf{p} - \mathbf{w}^*\mathbf{A}_T = \mathbf{0} \end{aligned} \tag{14}$$

and those of $L(\boldsymbol{\lambda}_R, \mathbf{f})$ are

$$\begin{aligned} \nabla_{\mathbf{f}^*} L(\boldsymbol{\lambda}_R^*, \mathbf{f}^*) &= \nabla_{\mathbf{f}^*} R(\mathbf{f}^*) - \boldsymbol{\lambda}_R^* \mathbf{A}_T = \mathbf{0} \\ \nabla_{\boldsymbol{\lambda}_R^*} L(\boldsymbol{\lambda}_R^*, \mathbf{f}^*) &= \mathbf{v} - \mathbf{A}_T \mathbf{f}^* = \mathbf{0} \end{aligned} \tag{15}$$

Solving the $2n + 1$ equations (14) yields solutions for the $2n + 1$ unknowns: $n + 1$ instruments (input prices) $\mathbf{w}^*$ and n Lagrangian multipliers $\boldsymbol{\lambda}_C^*$. Similarly, simultaneously solving the $2n + 1$ equations (15) yields solutions for other $2n + 1$ unknowns: $n + 1$ Lagrangian multipliers $\boldsymbol{\lambda}_R^*$ and n instruments (final outputs) $\mathbf{f}^*$. It is straightforward to note that $\nabla_{\mathbf{w}^*} C(\mathbf{w}^*) = \mathbf{v}$ and $\boldsymbol{\lambda}_C^* = \mathbf{f}^*$ while $\nabla_{\mathbf{f}^*} R(\mathbf{f}^*) = \mathbf{p}$ and $\boldsymbol{\lambda}_R^* = \mathbf{w}^*$. Therefore, under the assumptions made, the equality of optimal total cost expenditure and revenue is attained, that is $R^* = C^*$ following from $(\mathbf{p} - \mathbf{w}^* \mathbf{A}_T) \boldsymbol{\lambda}_C^* = \mathbf{0}$, hence $\mathbf{p}\mathbf{f}^* = \mathbf{w}^* \mathbf{A}_T \mathbf{f}^*$ and $\boldsymbol{\lambda}_R(\mathbf{v} - \mathbf{A}_T \mathbf{f}^*) = \mathbf{0}$, hence $\mathbf{w}^* \mathbf{v} = \mathbf{p}\mathbf{f}^*$.

If the resources $\mathbf{v}$ are allowed to vary, then using the resulting modified Lagrangian function derived from problem (12) yields (the demonstration is omitted to save space and can be found, for example, in Intriligator, 1971, pp. 36-38):

$$\nabla_{\mathbf{v}} L(\mathbf{v}, \boldsymbol{\lambda}_R^*, \mathbf{f}^*) = \boldsymbol{\lambda}_R^* = \mathbf{w}^* \tag{16}$$

In view of (8), the optimal input price vector $\mathbf{w}^* \equiv [w_L^* \ \mathbf{w}_K^*]$ with the rental prices $\mathbf{w}_K^* = \mathbf{p}_K^*(1 + r)$ for the inputs of capital goods services is consistent with the definition of capital rental price dating back at least to Walras. As they measure the sensitivity of the objective value to the marginal changes in the respective resource quantities, they are often called “shadow prices”.[4] Similarly, sensitivity analysis can be applied to changes in the input-output coefficients of matrix $\mathbf{A}_T$. The obtained results are discussed in the following section.

IV. The cost-minimizing choice over alternative methods of production

Regarding two different methods of production, Sraffa (1960, p. 98) claimed:

Two different methods of producing the same basic commodity can only co-exist at the points of intersection (that is to say at those rates of profits at which the prices of production by the two methods are equal), since the two economic systems (which are respectively characterized by the two methods, but are alike in every other respect) will at such points necessarily have also the same commodity-wage and the same system of relative prices.

DEFINITION 1: *The Sraffian point(s) of intersections of two methods of production, say method I and method II, are defined as those satisfying the following requirements simultaneously:*

(i) *the interest rate $r = r^*$ where the numerical value(s) of r* is (are) the solution(s) of the following equality of the real wage equation for the focused sector:*

$$\mathbf{a}_L^I[\mathbf{I}-(1+r)\mathbf{A}_K^I]^{-1}\mathbf{e}_j = \mathbf{a}_L^{II}[\mathbf{I}-(1+r)\mathbf{A}_K^{II}]^{-1}\mathbf{e}_j \tag{17}$$

[4] Since Sraffa assumed cost minimization behavior, the explicit introduction of the objective of cost-minimization in his intersectoral price model brings about the neoclassical rental prices of capital goods (earlier statements are given, for example, by Bruno, 1969, p. 47 and later by Salvadori, 1982).

(ii) *the numerical value(s) of r^* satisfying (17) should also satisfy the consistency requirement that the two methods yield the same system of relative prices in a genuine switch point, that is*

$$\mathbf{a}_L^I[\mathbf{I}-(1+r^*)\mathbf{A}_K^{II}]^{-1}\mathbf{e}_i-\mathbf{a}_L^{II}[\mathbf{I}-(1+r^*)\mathbf{A}_K^{II}]^{-1}\mathbf{e}_i=0; \quad \forall i\neq j \tag{18}$$

which, using (8), is equivalent to

$$\mathbf{w}^*(\mathbf{A}_T^{II}-\mathbf{A}_T^I)\mathbf{e}_i=0; \quad \forall i\neq j \tag{19}$$

A problem may arise using only the real wage-profit rate curves in search of the identification of the points of intersection "where the prices of production by the two methods are equal" (Sraffa, 1960, p. 102 and Fig. 8). In general, the intersection points in the 2-dimensional coordinate wage-profit space, if any, do not map into intersection points of the alternative techniques in the *n*-dimensional coordinate space of the real input prices. In other words, equations (17) and (18)-(19) in general do not hold simultaneously.

Figure 1 represents the case of the decomposable model showing two points where the two methods bring about the same labour wage and the same profit rate, but these entail different relative input prices. By contrast, the Sraffian switching point is defined as one where the two techniques achieve the same relative input prices *and* the same profit rate.

[Insert Figure 1 here]

A different situation can now be considered in the real price input frontier of a production system with the two commodities, three inputs, and two techniques. One or multiple intersection points can occur between the two techniques in the 2-dimensional coordinate space of real wage and profit rate. These intersection points may not map into corresponding points of the locus of straight line, which is shown in **Figure 2**, connecting the points A and B. Sraffa himself made clear that only if the number of alternative techniques is *lower* than the number of factor inputs, then the reswitching of techniques is possible. Only in such a case may all relative prices and the profit rate be equal in multiple points across the alternative techniques. By contrast, if the number of alternative techniques is equal to or higher than the number of factor inputs, the reswitching of techniques is not possible.

[Insert Figure 2 here]

He argued in the following terms:

This co-existence [of alternative methods of production] is possible because with k basic equations (representing k methods of production) and $k+1$ *unknowns (representing* $k-1$ *[relative] prices, the wage* w_L *and the rate of profits r [with the total number of inputs being equal to* $k+1$*]) there is room for one more basic equation (or method of production) even though it does not bring with it an additional product and an additional price. With* $k+1$ *methods of production [with the number of methods being equal to the number of unknown real input prices], however, it is no longer possible to vary at will the rate of profits, its level is now fully determined. At any other level of the rate of profits the two methods are incompatible, and the two distinct systems to which they belong have no point of contact.* (Sraffa, 1960, p. 90.)

The last case considered in the Sraffa's foregoing passage can be described mathematically as follows.

A. *The case of the number of techniques being equal to the number of factor inputs*

With the number of the alternative techniques being equal to the number of the factor inputs, the real factor prices are univocally determined provided the coefficient matrix is not singular. Let us consider the general case of n-sector Sraffian model with (n+1) inputs (commodity inputs plus one labour input) and (n+1) different techniques with each sector supplying its own product as input to the other sectors and to itself. In view of the third line of (8), let the system of the real factor-price equations of the focused sector, say now the 1st one, be represented as follows

$$
\begin{aligned}
&a_{L1}^{I}\frac{w_L}{p_1}+a_{11}^{I}\frac{w_{K1}}{p_1}+\ \dots\ +a_{n1}^{I}\frac{w_{Kn}}{p_1}=1\\
&a_{L1}^{II}\frac{w_L}{p_1}+a_{11}^{II}\frac{w_{K1}}{p_1}+\ \dots\ +a_{n1}^{II}\frac{w_{Kn}}{p_1}=1\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_{L1}^{(n+1)}\frac{w_L}{p_1}+a_{11}^{(n+1)}\frac{w_{K1}}{p_1}+\ \dots\ +a_{n1}^{(n+1)}\frac{w_{Kn}}{p_1}=1
\end{aligned}
\tag{20}
$$

In matrix form, the solution is

$$
\begin{bmatrix} \dfrac{w_L}{p_1} & (1+r) & \dots & \dfrac{w_{Kn}}{p_1} \end{bmatrix} = \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}\begin{bmatrix} a_{L1}^{I} & a_{L1}^{II} & \dots & a_{L1}^{(n+1)} \\ a_{11}^{I} & a_{11}^{II} & \dots & a_{11}^{(n+1)} \\ . & . & . & . \\ a_{n1}^{I} & a_{n1}^{II} & \dots & a_{n1}^{(n+1)} \end{bmatrix}^{-1}
\tag{21}
$$

provided the matrix to be inverted is not singular. In this case, there is no degree of freedom for r or $\frac{w_L}{p_1}$ with an income distribution fixed over different technical conditions. As Sraffa pointed out in the above reported text, the solution is unique implying that reswitching is not possible.

In correspondence of the genuine switch point, the equation system (20) can be rearranged in order to relate the relative input prices to the sensitivity analysis of technical coefficients based on the Lagrangian of the linear programming problem (12). Starting with a binary comparison of techniques, say *I* and *II,* which at a switch point coexist with equal relative unit costs and prices, taking the difference of the respective cost equations and rearranging yield the following single equation in the n relative input prices for the focused sector:

$$(a_{L1}^{II} - a_{L1}^{I}) + \frac{1}{w_L^*}\mathbf{w}_K^*(\mathbf{a}_{K1}^{II} - \mathbf{a}_{K1}^{I}) = 0 \tag{22}$$

Applying the transitivity property of index numbers having common relative prices as weights, the whole system of binary comparison of n techniques at the same switch point yields

$$\begin{aligned} &\frac{1}{w_L^*}\mathbf{w}_K^*(\mathbf{a}_{K1}^{II} - \mathbf{a}_{K1}^{I}) = -(a_{L1}^{II} - a_{L1}^{I}) \\ &\frac{1}{w_L^*}\mathbf{w}_K^*(\mathbf{a}_{K1}^{III} - \mathbf{a}_{K1}^{II}) = -(a_{L1}^{III} - a_{L1}^{II}) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\frac{1}{w_L^*}\mathbf{w}_K^*(\mathbf{a}_{K1}^{n^{th}} - \mathbf{a}_{K1}^{n^{th}}) = -(a_{L1}^{n^{th}} - a_{L1}^{n^{th}}) \end{aligned} \tag{23}$$

Solving for the rental rates of capital goods relative to wage in the common switch point yields

$$\frac{1}{w_L^*}\mathbf{w}_K^* = \mathbf{c}_1 \cdot (\mathbf{I} - \mathbf{D})^{-1} \tag{24}$$

where $\mathbf{c}_1$ is a (row) vector with elements $c_{i1} \equiv -\Delta a_{L1}/\Delta a_{Ki1} \quad \forall i$, and $\mathbf{D}$ is a matrix with elements

$d_{ss} = 0 \quad \forall s$ and $d_{st} = -\dfrac{a_{Kt1}^{\beta} - a_{Kt1}^{\alpha}}{a_{Ks1}^{\beta} - a_{Ks1}^{\alpha}}$ where $s \neq t;\ s,t = 1,...,n;\quad \beta \neq \alpha;\ \beta,\alpha = I, II,..,n^{th}$.

With only one capital good in the economy, the solution in terms of relative input prices is $\dfrac{w_K^*}{w_L^*} = -\dfrac{a_{L1}^{II} - a_{L1}^{I}}{a_{K1}^{II} - a_{K1}^{I}}$. This shows that the Sraffian model is the exact counterpart of the (negative) marginal rate of substitution of capital-labour inputs defined in (5) for the case of continuous spectrum of techniques of production.

At the intersection point of the cost budget lines, the relative input prices are in fact the same with the two techniques:

$$\begin{aligned} \mathbf{w}_K^* &= \mathbf{p}^{(I)*}(1+r^*) \\ &= \mathbf{p}^{(II)*}(1+r^*) \end{aligned} \tag{25}$$

As Sraffa (1960, p. 90) himself noted, there is a unique vector of capital input prices and there is no reswitching in the general case where the number of alternative techniques is not lower than the number of commodity inputs and the rate of profit is uniquely determined.

The non-linear relationship between profit rate and relative input prices in terms of labour is at the heart of the apparent paradox of the reswitching of techniques, which may appear along intervals of possible values of the given profit rate or labour wage. The non-linear relationship between r and relative prices implies that the profit rate and the capital input rentals in terms of labour may not change proportionally. The degree of non-linearity of such relation depends strictly on the rank of the Sraffian matrices being equal to the number of the producing sectors. In a two-sector model, the rental price relative to labour costs is a quadratic function of the profit rate. In general, with n sectors, such relationship has at most an n-degree polynomial form.[5]

With the two-sectors, two inputs, two-techniques, the solution (21) becomes

$$\begin{bmatrix} \frac{w_L}{p_1} & \frac{w_{K2}}{p_1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a_{L1}^{I} & a_{L1}^{II} \\ a_{K21}^{I} & a_{K21}^{II} \end{bmatrix}^{-1} \tag{26}$$

where there is no degree of freedom of r provided the relative price p_2 / p_1 has a level satisfying the technological condition (or vice versa). This is the case of the model proposed by Samuelson (1962, p. 205) and later used by Hicks (1965, pp. 139-59), and others including Spaventa (1970, 1973), Garegnani (1970), Harris (1973), Sato (1974), and Gram (1976) to discuss the reswitching of techniques. All these authors overlooked that, if this prototype model have common levels for w_L / p_1 and w_{K2} / p_1, then this is achieved in correspondence with different levels of r using the alternative technique. This implies and is implied by the fact that the two techniques will differ in the relative price p_2 / p_1 in such a solution. Conversely, if the two techniques lead to two common levels of w_L / p_1 and r, then they generally differ in the relative rental price w_{K2} / p_1 and relative price p_2 / p_1. This excludes reswitching as defined by Sraffa. The literature cited above in this section seemed to be unaware that, in the 2-2-2 Sraffian model, reswitching is impossible although

[5] For example, Schefold (1976) remains in the realm of the output prices within the Sraffian framework, whereas Gram (1976), offers a comparison of the two model solutions, but his analysis maintains the Sraffian interpretation of the real wage-profit rate relation as the real factor-price curve, but he does not clarify that this is expressed in reduced form.

Bruno *et al.* (1966) had previously declared reswitching impossible in the presence of "only one capital good in the system". They wrote:

Can we get reswitching if all activities use the same capital good? The answer to this question turns out to be negative, and we have the following theorem:

THEOREM*: In a two-sector economy with many alternative independent techniques for producing the two goods, if there is* only one *capital good in the system, reswitching cannot occur.* (*Ibid*, p. 536. Emphasis in original.)

This contention, however, turns out to be misleading as the impossibility of reswitching also arises in the more general case with more than one capital good. Sraffa (1960, p. 90) himself was aware of the impossibility of reswitching due to the full determinacy (or over-determinacy) that arises when the number of inputs is equal to (or lower than) the number of alternative techniques. As Stiglitz (1973, 1974) recognized, this implies that the impossibility of reswitching occurs when the alternative techniques are infinite in number as in the neoclassical case of a continuous spectrum of input-output coefficients.

Illustration of Case A of equal number of inputs and techniques using the Garegnani's (1970) numerical example. —A selection of seven numerical examples were presented by Garegnani (1970) in a series of bilateral comparisons in the framework of the 2-2-2 Sraffian model, apparently overlooking the impossibility of reswitching in this case. The aim was to show "how far the relation of the rate of interest and the value of capital per worker in the production of a commodity can differ from what traditional theory postulates" (*ibid*, p. 428). The input-output coefficients of labour and capital goods are defined as continuous functions of a parameter u with the exception of the constant labour coefficient (set equal to 1) of the sector producing the capital good. The labour coefficient in the production of consumption goods and the capital coefficient in the production of capital goods are increasing functions of *u*, whereas the labour coefficient in the production of consumption goods is decreasing in *u* as shown in **Table 1**. Taking seven values of *u* in increasing order from 0 to 1.505, Garegnani compared the simultaneous solutions of the model with the seven alternative techniques. The external envelope of the resulting real wage-profit rate curves would suggest the reswitching of techniques along the resulting frontier.

[Insert Table 1 here]

Garegnani stated:

the cheaper system will be the same at both wage rates and price systems. Moreover, the tendency of producers to switch to whichever system is cheaper in the existing price situation will bring them to the system giving the highest wL; while systems giving the same wL for the same r will be indifferent and can co-exist (*ibid*, p. 411).

This contention can be contrasted with the resulting relative levels of total costs of production. In order to save space, we now take only the intermediate cases of $u = 0.75$ and $u = 0.50$, respectively

Technique δ ($u = 0.75$)

$$\mathbf{A}_W^{(\delta)} = \begin{array}{l c} & \begin{array}{cc}\text{Consumption good} & \text{Capital good}\end{array} \\ \begin{array}{l}\text{Labour input} \\ \text{Capital good input}\end{array} & \begin{bmatrix} 4.834 & 1.000 \\ 0.133 & 0.851 \end{bmatrix} \end{array}$$

Technique γ ($u = 0.50$)

$$\mathbf{A}_W^{(\gamma)} = \begin{array}{l c} & \begin{array}{cc}\text{Consumption good} & \text{Capital good}\end{array} \\ \begin{array}{l}\text{Labour input} \\ \text{Capital good input}\end{array} & \begin{bmatrix} 3.930 & 1.000 \\ 0.237 & 0.845 \end{bmatrix} \end{array}$$

The numerical solutions are shown in **Table 2**.

[Insert Table 2 here]

In correspondence of the switch points indicated by Garegnani (1970, p. 429) where the two techniques have the same real wage and interest rate, technique δ turns out to be much cheaper than technique γ, thus contradicting the joint Sraffian requirements (17) and (18)-(19) for a genuine switch point. This example empirically confirms that, at those points, the two techniques yield different relative prices w_K / p_1 and w_K / w_L in contrast with the consistency requirements of equal relative prices in the two systems. Moreover, the double-star point based on the cost ratios is unique implying that there is no genuine reswitching. The two techniques may share the same maximized real rental price for a given real wage, but these can be obtained in correspondence with different profit rates and different relative prices.

B. *The case of the number of techniques being less than to the number of factor inputs*

Confining the discussion of the choice between pairs of techniques and noting that the determinant of the Sraffian inverse matrix implies a polynomial whose possible maximum degree is equal to the number of sectors, the following theorem has been established

THEOREM: The maximum number of genuine switch points (Bruno et al, 1966, p. 542)*: (1) In the general N-sector capital model there may be up to* n *switching points between any two techniques, and thus a technique may recur up to (N -* 1*) times. (2) 'Adjacent' techniques on two sides of a switching point will usually differ from each other only with respect to one activity.*

Techniques in general may differ with respect to M activities ($N \geq M > 1$) only if certain M independent N-th degree polynomials happen to have a common root at that switching point.

This theorem implies that, in the case of two sectors, three inputs and two different techniques, the maximum number of genuine switch points in the coordinate space of real wage and interest rate is equal to 2. These switch points can be mapped into the respective corresponding locus of linear intersection of the two technique planes in the 3-dimensional coordinate space of the real factor prices. Let the system of equations be represented as follows

$$\begin{aligned} a_{L1}^{I}\frac{w_L}{p_1}+a_{K11}^{I}\frac{w_{K1}}{p_1}+a_{K21}^{I}\frac{w_{K2}}{p_1}&=1 \\ a_{L1}^{II}\frac{w_L}{p_1}+a_{K11}^{II}\frac{w_{K1}}{p_1}+a_{K21}^{II}\frac{w_{K2}}{p_1}&=1 \end{aligned} \tag{27}$$

As two (dependent) real input-prices are functions of a third (independent) real input price and, given that $\frac{w_{K1}}{p_1}=\frac{p_1(1+r)}{p_1}=1+r$, the foregoing equation represents the following set of solutions written in matrix form:

$$\begin{bmatrix} \frac{w_L}{p_1} & \frac{w_{K2}}{p_1} \end{bmatrix}\cdot\begin{bmatrix} a_{L1}^{I} & a_{L1}^{II} \\ a_{K21}^{I} & a_{K21}^{II} \end{bmatrix} + (1+r)\begin{bmatrix} a_{K11}^{I} & a_{K11}^{II} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{28}$$

Let us consider two sub-cases regarding the matrix singularity.

Case B1: The matrix $\begin{bmatrix} a_{L1}^{I} & a_{L1}^{II} \\ a_{K21}^{I} & a_{K21}^{II} \end{bmatrix}$ *in (28) is singular.* —The matrix singularity real factor price hyperplane of the focused sector is the same for the two techniques having in common the same real factor price structural equation. the common wage equation is derived from (27):

(29) $$\frac{w_{K2}}{p_1} = 1 - \frac{a_{L1}}{a_{K21}} \cdot \frac{w_L}{p_1} + \frac{a_{K11}}{a_{K21}}(1+r)$$

for $a_{L1} = a_{L1}^{I} = a_{L1}^{II}$, $a_{K11} = a_{K11}^{I} = a_{K11}^{II}$, and $a_{K21} = a_{K21}^{I} = a_{K21}^{II}$.

In this special case, all common levels of the interest rate and real wage rate are sufficient for the two techniques to determine the same real factor price level thus satisfying Sraffa's requirements for the identification of switch points. Almost all the numerical examples proposed by Sraffa's followers were formulated within this special case. The reduced form of the foregoing equation is obtained for the relative price level

(30) $$\frac{p_2^{(T)}}{p_1^{(T)}} = \frac{a_{L2}^{T} + (a_{L1}a_{K21} - a_{L2}^{T}a_{K11})(1+r)}{a_{L1} + (a_{L2}^{T}a_{K12}^{T} - a_{L1}a_{K22}^{T})(1+r)} \quad \text{for } T = I, II$$

and for the real wage assumed to be paid *post factum*

(31) $$\frac{w_L^{(T)}}{p_1^{(T)}} = \frac{1 - (a_{K11} + a_{K22}^{T})(1+r) + (a_{K11}a_{K22}^{T} - a_{K21}a_{K12}^{T})(1+r)^2}{a_{L1} + (a_{L2}^{T)}a_{K12}^{(T} - a_{L1}a_{K22}^{T})(1+r)} \quad \text{for } T = I, II$$

which can be converted to the real wage paid *ante factum* $w_0^{(T)} / p_1$ used in many numerical examples cited below according to the formula $\frac{w_0^{(T)}}{p_1} = \frac{w_L^{(T)}}{p_1}(1+r)^{-1}$.

Genuine reswitching is then possible with common levels r^* in correspondence of the equalities $\frac{w_L^{(I)}}{p_1} = \frac{w_L^{(II)}}{p_1}$ and $\frac{p_2^{(I)}}{p_1} = \frac{p_2^{(II)}}{p_1}$. Reswitching appears as a response to the return to a previous ranking levels of relative input prices as reflected by the ratios

$$\frac{w_L^{(II/I)}}{p_1^{(II/I)}} \equiv \frac{w_L^{(II)}}{p_1^{(II)}} / \frac{w_L^{(I)}}{p_1^{(I)}} \qquad \text{for } i = 1, 2$$

(32) $$\frac{w_{Ki}^{(II/I)}}{p_1^{(II/I)}} \equiv \frac{w_{Ki}^{(II)}}{w_L^{(II)}} / \frac{w_{Ki}^{(I)}}{w_L^{(I)}} \qquad \text{for } i = 1,2$$

$$\frac{w_{Ki}^{(II/I)}}{w_L^{(II/I)}} \equiv \frac{w_{Ki}^{(II)}}{w_L^{(II)}} / \frac{w_{Ki}^{(I)}}{w_L^{(I)}} \qquad \text{for } i = 1,2$$

In view of (10), (11), and (12), these ratios are function of r for the given compared techniques $\mathbf{A}_T^{(I)}$ and $\mathbf{A}_T^{(II)}$.

Case B2: *The matrix* $\begin{bmatrix} a_{L1}^{I} & a_{L1}^{II} \\ a_{K21}^{I} & a_{K21}^{II} \end{bmatrix}$ *is not singular*. —Solving the system (28) yields

(33) $$\begin{bmatrix} \frac{w_L}{p_1} & \frac{w_{K2}}{p_1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a_{L1}^{I} & a_{L1}^{II} \\ a_{K21}^{I} & a_{K21}^{II} \end{bmatrix}^{-1} - (1+r)\begin{bmatrix} a_{K11}^{I} & a_{K11}^{II} \end{bmatrix} \begin{bmatrix} a_{L1}^{I} & a_{L1}^{II} \\ a_{K21}^{I} & a_{K21}^{II} \end{bmatrix}^{-1}$$

This defines the linear locus of points of two intersecting planes in the 3-dimensional coordinate space of real factor prices. This locus of points is the set of solution values corresponding to given levels of the profit rate. Since the two planes are linear, they share *only one* intersecting straight line as shown by the AB line in **Figure 2**. This implies that common levels of the profit rate and real wage are not sufficient for determining the switch points as these should occur only under the *very special condition* of mapping into the intersecting straight line within the coordinate space of relative input prices.

Numerical examples of two-sector model with three inputs and two techniques. —
Various numerical examples of Case $B1$ have been proposed in the literature. **Table 3** contains a synoptic collection of the coefficients of the equations (30) and (31) used in six well known contributions, where the techniques differ only in the capital good producing sector.

[Insert Table 3 here]

A numerical example of Case $B2$ has been provided by Pertz (1980), where the techniques differ in both sectors. The input-output coefficients are set as follows:

$$
\begin{array}{ll}
\begin{array}{l}
\qquad \text{sector} \\
\qquad 1 \qquad 2 \\
\mathbf{a}_L^I = [0.8 \quad 1.0] \\
\mathbf{a}_{K1}^I = [0.0 \quad 0.1] \\
\mathbf{a}_{K2}^I = [0.7 \quad 0.0]
\end{array}
&
\qquad
\begin{array}{l}
\qquad \text{sector} \\
\qquad 1 \qquad 2 \\
\mathbf{a}_L^{II} = [1.3 \quad 0.9] \\
\mathbf{a}_{K1}^{II} = [0.0 \quad 0.145] \\
\mathbf{a}_{K2}^{II} = [0.5 \quad 0.0]
\end{array}
\end{array}
$$

Technique I has a lower capital per labour unit than technique II in all the numerical examples considered here, except Bruno *et al.* (1966) where the relative capital intensity is reversed between the two techniques. For given levels of the profit rate, the two techniques are compared in terms of real wages and relative prices of all commodity inputs, as shown respectively in the second and third columns of the **Table 4**, where they are also compared in terms of ratios of the relative rentals of the two capital goods with respect to the wage computed using (12) and shown respectively in the fourth and fifth columns.

All cited authors report the comparisons of the two techniques based only on the real wage values for given levels of the profit rate without checking the full range of the relative price systems. As noted, the cost-minimizing technical choice for given profit rate is affected relative input prices. In all the numerical instances of reswitching, the choice of the more (less) capital-intensive technique is invariably associated with the lower (higher) capital rental price in terms of labour. Therefore, all the well-known counterexamples re-examined here appear perfectly consistent with the expectations of the neoclassical theory of cost-minimizing choices of techniques.

[Insert Table 4 here]

In view of these results, the Sraffians' critical interpretation of the switch points is disproved simply on the ground of the non-monotonic effects of relative input prices resulting from monotonic changes in the rate of profit. For example, Pasinetti's (1966, p. 514) failed to recognize that reswitching happens when facing *non-monotonic* changes (*decreasing* and then *increasing* or vice versa) in relative rental prices of physical capital induced by *monotonic* changes in the profit rate. As illustration, **Figure 3** shows the ratios $w_{Ki}^{(II/I)} / w_{L}^{(II/I)}$ defined in (32) for the rental prices of both capital goods relative to the wage in correspondence of a range of given levels of the profit rate using the coefficients of Garegnani's (1966) numerical example. The results graphically demonstrate how the differential decrease and subsequent increase in relative capital price in terms of labour in correspondence with the monotonic increases in profit rate would bring the cost-minimizing choices of techniques from a point A beyond a point B while passing from System *I* to System *II* and then returning to System *I*. A similar pattern could be observed using all the other numerical counterexamples shown in **Table 4**. Indeed, Sraffa (1960) insisted on the fact that changes in the profit or interest rate may affect the relative prices significantly and in a non-linear way in his intersectoral model. His analysis was a further confirmation of an old discovery that the distribution of income yields non-linear effects on relative prices and the internal structure of production. But the U-turn changes in such effects and their implications for the interpretation of the technical choices was misled and remained hidden in the reswitching debate. In the light of the present solution to the paradox, the return to previously chosen techniques cannot be deemed as a contradiction of the neoclassical paradigm.

It should be noted, however, that the alternative techniques that are close enough to co-exist in genuine switch points tend to differ very slightly in terms of costs of production so that the consequent adjustment in the choice of techniques appears to be a minute phenomenon. On the other hand, these small effects on the costs of production are part of the more general non-linear changes in relative input prices and in the capital structure of production systems triggered by variations in the rate of profit and the rate of interest. From the viewpoint of practical utility of our results, empirical investigations of the actual intersectoral systems of production remain to be fully exploited. Han and Schefold (2006) offered one of the first pioneering examples in this direction although with a Sraffian slant.

[Insert Figure 3 here]

V. Reverse deepening in capital value

The reversing of the relative intensity of *financial* capital or *capital value* in the overall economy under stationary equilibrium with monotonic changes of the interest rate was already noted by Fisher (1907) (see Samuelson, 1966, and Velupillai, 1975, 1995 on Fisher's "discovery") and independently by Robinson (1953) and Champernowne (1953) (Harcourt, 1972, p. 124-76, and Scazzieri, 2008 provide further discussions).

The relative producer cost, say in terms of the jth commodity, of K capital goods per unit of labour in our model is given by (assuming for simplicity a null depreciation rate):

(34)
$$r \cdot \frac{p_{Ki}}{p_j} \cdot \frac{a_{Kij}}{a_{Lj}} = \frac{w_{Ki}}{p_j} \cdot \frac{a_{Kij}}{a_{Lj}}$$

In the right-hand side of the foregoing equality, the "price" component r (the rate of interest or rate of profit) multiplies the "deflated" component $\frac{p_{Ki}}{p_j} \cdot \frac{a_{Kij}}{a_{Lj}}$, which can be interpreted as the value of the ith capital goods per unit of labour in terms of the jth commodity. This value is obtained by evaluating the physical capital goods per unit of labour a_{Ki} with the relative price $\frac{p_{Ki}}{p_j}$. The cost of the ith capital good is decomposed in the second-hand side of (34) in terms of the relative rental price of capital goods $\frac{w_{Ki}}{p_j}$ and the ratio of capital to labour technical coefficients a_{Kij} / a_{Lj}. In the reference intersectoral model, the effects of r on these two variables are called respectively "price" and "real" Wicksell effects. The reverse deepening in capital value can occur when the "price"

Wicksell effect overcome, when positive, the non-positive "real" Wicksell effect (see for example, Burmeister, 2008, and Kurz, 2008 for further discussion).

VI. Conclusion

The reswitching of techniques in the Sraffian intersectoral model of a cost-minimizing economy in stationary equilibrium turns out to be misinterpreted as paradoxical violation of the neoclassical regularity of the producers' choices. The Sraffian analysis has however stressed two important points: 1) monotonic changes in the profit rate (or real wage) affect relative prices non-linearly; 2) in a genuine switch point, two alternative techniques face the same system of relative prices as well as the same real wage and profit rate. This phenomenon gives rise to an apparent paradoxical return to previous factor intensities as the real wage or the profit rate moves monotonically. Drawing on these results, the present article has solved the paradox by showing that the reswitching of techniques can be rationalized as a response to a U-shaped turn in the ranking of the relative factor prices over an interval of the admitted levels of the profit rate. In this view, the reswitching phenomenon turns out to conform consistently with the neoclassical expectation of negative price-quantity correlation in the producer's demand behavior.

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TABLE 1—GAREGNANI'S (1970: 429) FAMILY OF COEFFICIENTS DEFINED PARAMETRICALLY

Parameter u	a_{L1}	a_{K1}	a_{L2}	a_{K2}
0.000	0.500	0.750	1	0.833
0.250	2.504	0.424	1	0.839
0.500	3.930	0.237	1	0.845
0.750	4.834	0.133	1	0.851
1.000	5.478	0.075	1	0.857
1.250	5.974	0.042	1	0.863
1.505	6.391	0.023	1	0.868

TABLE 2—SOLUTIONS WITH TECHNIQUES δ AND γ USING GAREGNANI'S (1970, P. 429) NUMERICAL EXAMPLE

$\frac{w_L}{p_1}$	r		w_K / p_1		w_K / w_L		$C^{(\delta)} / C^{(\gamma)}$	
	δ	γ	δ	γ	δ	γ	$\frac{w_K^{(\delta)}}{w_L^{(\delta)}}$	$\frac{w_K^{(\gamma)}}{w_L^{(\gamma)}}$
0.050*	0.163	0.163	5.690	3.386	113.053	67.362	0.693	0.670
0.162*	0.051	0.051	1.616	1.517	9.952	9.391	0.988	0.984
0.165**	0.041	0.039	1.425	1.425	8.640	8.640	1.000	1.000

* Switch point indicated by Garegnani (1970: 429).

** Point with equal relative input prices, but different levels of *r* with the two techniques.

TABLE 3—INPUT-OUTPUT COEFFICIENTS IN NUMERICAL EXAMPLES

	Bruno *et al.* (1966, p.537)	Garegnani (1966, p.566, fn.1)	Garegnani (1976, pp. 425-26)	Sato (1976, p.428)	Laibman&Nell (1977, p.881)
δ	1.00	1.00	1.00	1.00	1.00

Common technique in the focused industry					
a_{L1}	1.00	8.90	1.00	1.00	1.00
a_{K11}	0.00	0.00	1/12	0.20	0.10
a_{K21}	0.10	379/423	1/3	0.40	1.00
Other techniques					
System I					
a_{L2}	0.66	9/50	1.0	1.5	0.558720
a_{K21}	0.02	1/2	1/6	0.4	0.135872
a_{K22}	0.30	0.1	1/6	0.2	0.358720
System II					
a_{L2}	0.01	3/2	92/91	1.55	0.567120
a_{K21}	0.71	1/4	137/546	0.5205	0.261712
a_{K22}	0.00	5/12	19/273	0.08	0.117120

(*) The notation indicated for Bruno *et al*. (1966, p. 537) is reversed here to make it consistent with the other cases where the focused industry is the first one.

TABLE 4—"RESWITCHING" AS A RESPONSE TO THE EFFECT OF INCOME DISTRIBUTION ON RELATIVE PRICES

r	$\frac{w_L^{(II/I)}}{p_1^{(II/I)}}$	$\frac{w_{K2}^{(II/I)}}{p_1^{(II/I)}}$	$\frac{w_{K1}^{(II/I)}}{w_L^{(II/I)}}$	$\frac{w_{K2}^{(II/I)}}{w_L^{(II/I)}}$	System in use
Bruno *et al.* (1966, p. 537)					
0.250000	1.010214	0.925823	0.989890	0.916462	*II*
0.465809*	1.000000 *	1.000000*	1.000000 *	1.000000*	*I-II*
1.000000	0.969773	1.084631	1.031168	1.118436	*I*
1.668760*	1.000000 *	1.000000*	1.000000 *	1.000000*	*I-II*
1.857000	1.097942	0.939496	0.910794	0.855688	*II*
Garegnani (1966, p. 566)					
0.010000	0.994627	1.005000	1.005402	1.010430	*I*
0.100000*	1.000000 *	1.000000*	1.000000 *	1.000000*	*I-II*
0.150068	1.001467	0.999320	0.998575	0.997896	*II*
0.200000*	1.000000 *	1.000000*	1.000000 *	1.000000*	*I-II*
0.250643	0.991743	1.001989	1.008326	1.010462	*I*

Garegnani (1976, pp. 425-26)					
0.100000	0.999785	1.000363	1.000215	1.000579	*I*
0.333333*	1.000000 *	1.000000*	1.000000 *	1.000000*	*I-II*
0.450000	1.000020	0.999979	0.999980	0.999959	*II*
0.500000*	1.000000 *	1.000000*	1.000000 *	1.000000*	*I-II*
0.900000	0.998604	1.000713	1.001398	1.002112	*I*
Sato (1976, pp. 428-30)					
0.000000	0.999621	1.000284	1.000379	1.000664	*I*
0.038000*	1.000000 *	1.000000*	1.000000*	1.000000*	*I-II*
0.250000	1.002088	0.999197	0.997916	0.997114	*II*
0.595000*	1.000000*	1.000000*	1.000000*	1.000000*	*I-II*
0.639000	0.986951	1.000263	1.013222	1.013489	*I*
Laibman and Nell (1977, p. 881)					
0.100000	0.999703	1.000156	1.000298	1.000454	*I*
0.200000*	1.000000 *	1.000000*	1.000000*	1.000000*	*I-II*
0.300000	1.000175	0.999952	0.999826	0.999777	*II*
0.400000*	1.000000 *	1.000000*	1.000000*	1.000000*	*I-II*
0.500000	0.998454	1.000132	1.001548	1.001680	*I*
Pertz (1980, p. 1016)					
0.500000	0.930455	1.074743	1.074742	1.155072	*I*
1.170000*	1.000000 *	1.000000*	1.000000*	1.000000*	*I-II*
1.750000	1.030747	0.970167	0.970170	0.941227	*I*
2.250000*	1.000000*	1.000000*	1.000000*	1.000000*	*I- II*
2.500000	0.887452	1.126773	1.126821	1.269672	*I*

* Sraffian switch point; *Note*: The ratio of real capital rental price $w_{K1}^{(II/I)} / p_1^{(II/I)} = (1+r)/(1+r) = 1$ (with the same levels of *r* being conjecturally predetermined for both techniques) is omitted in this table.

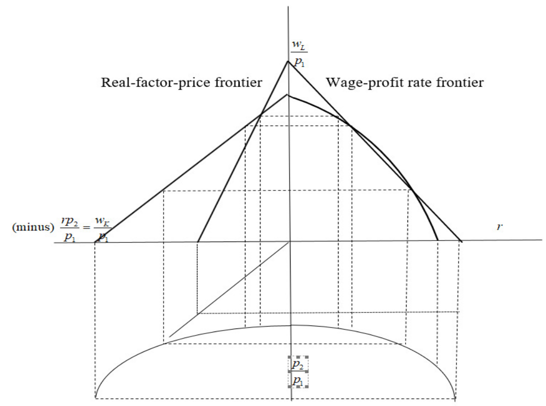

Figure 1. Representation of the two-sector model of an intersectoral system with two inputs and two alternative techniques

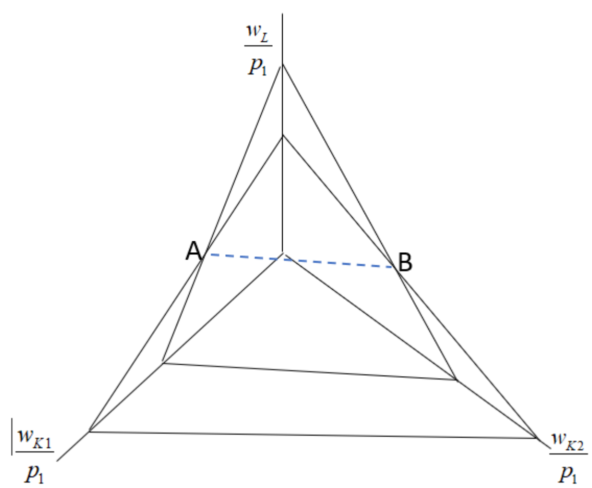

FIGURE 2. REAL FACTOR-PRICE FRONTIER IN A MODEL

WITH 3 INPUTS AND TWO ALTERNATIVE TECHNIQUES

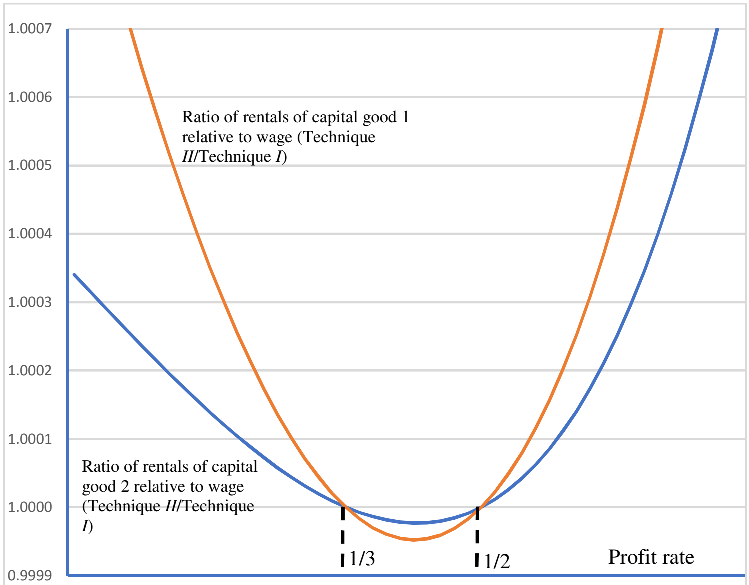

FIGURE 3. NON-LINEAR EFFECTS OF THE PROFIT RATE ON RELATIVE PRICES (USING GAREGNANI'S 1976 NUMERICAL EXAMPLE)