Proportional Dynamics in Exchange Economies

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Abstract

We study the *Proportional Response* dynamic in exchange economies, where each player starts with some amount of money and a good. Every day, players bring one unit of their good and submit bids on goods they like, each good gets allocated in proportion to the bid amounts, and each seller collects the bids received. Then every player updates the bids proportionally to the contribution of each good in their utility.

This dynamic models a process of learning how to bid and has been studied in a series of papers on Fisher and production markets, but not in exchange economies. Our main results are as follows:

1. For linear utilities, the dynamic converges to market equilibrium utilities and allocations, while the bids and prices may cycle. We give a combinatorial characterization of limit cycles for prices and bids.
2. We introduce a lazy version of the dynamic, where players may save money for later, and show this converges in everything: utilities, allocations, and prices.
3. For CES utilities in the substitute range $[0, 1)$, the dynamic converges for all parameters.

This answers an open question about exchange economies with linear utilities, where tatonnement does not converge to market equilibria, and no natural process leading to equilibria was known. We also note that proportional response is a process where the players exchange goods throughout time (in out-of-equilibrium states), while tatonnement only explains how exchange happens in the limit.

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To only look at situations where the Invisible Hand has finished its work cannot lead to a real understanding of how that work is accomplished.

Fisher *[Fis83]*

1 Introduction

Market *dynamics* have been an integral part of general equilibrium theory since its inception. The introduction of general equilibrium theory by Walras [Wal96] was accompanied by the idea of the *tâtonnement* process. Fisher designed in 1891 a device to compute an equilibrium (Brainard and Scarf [BS05]). The most popular interpretation of tâtonnement, however, is as *fictitious* play. An auctioneer, playing the role of *the invisible hand*, calls out prices to which the agents respond with their demand, then the auctioneer adjusts the prices, and the process repeats until the excess demand is zero. At this point exchange actually happens, at equilibrium prices. This, however, is not necessarily how actual markets function in practice. Moreover, general equilibrium theory itself suffers from a lack of a *descriptive model* of out of equilibrium exchange: what happens when the excess demand is positive? Further, the causal linkage between demand and prices is unspecified. Demand is a response to the price as well as the price is a response to the demand. (See [Fis83] for some disequilibrium extensions; there is no widely agreed upon model.) The question is fundamentally both algorithmic and economic: Algorithmically, the question is about an effective and efficient locally controlled network process that computes an equilibrium. Economically, the question is about an incentives-motivated multi-agent process that converges to equilibrium.

Shapley and Shubik [SS77] sought to address these issues via the *trading post* mechanism. This is first of all a descriptive model that specifies concrete outcomes as a result of player strategies, and therefore it can be viewed as a non-cooperative game. Prices are a result of strategic actions; the higher the demand for a good, the higher its price, and vice versa. It requires that all trade be *monetary* and that the players pay *cash in advance*.[1] Each player submits a cash bid on each *good*. The goods are then distributed *in proportion to the bids*, and the per-unit price of a good is set to be the sum of bids on that good (this implies that the bids of a player should add up to at most the cash at hand). The same mechanism has been rediscovered multiple times; in particular it has been proposed for sharing resources in computer networks (e.g. Kelly [Kel97]) and computer systems (e.g. Feldman, Lai, and Zhang [FLZ05]).

This still leaves open the question of dynamics: (how) do players reach a market equilibrium in the trading post mechanism? The predominant answer to this in the last decade or so has been the *Proportional Response* dynamic (see Zhang [Zha11]), where buyers iteratively update their bid on each good *in proportion to the utility* they received from that good in the previous iteration. For the case of linear utilities, this implies that the ratio of bids in successive iterations is proportional to the *bang-per-buck* for that good, in contrast to the *best response* which distributes *all* the cash among the goods with the *highest* bang-per-buck. Such an update is similar in spirit to the multiplicative weights update algorithms used in online learning (also to proportional tâtonnement), except that there are no parameters such as the *step size* to tune carefully! It still magically seems to work.

[1] The assumption is that there is one special commodity used as the means of payment, which is called *cash* or *money*. It may or may not have an intrinsic utility of its own.

1.1 Our results

This brings us to the topic of this paper, the study of proportional response dynamics in pure exchange markets. The players are both buyers and sellers, and the exchange at each step is fueled by the players revenue from the previous round, and a fresh batch of goods of fixed quantities at the hands of the players. We focus primarily on *linear utilities*. We show convergence for the range of CES utilities in the substitutes regime, but the case of linear utilities is the most difficult to handle. Linear utilities represent a particularly interesting special case because the process of tâtonnement is not well defined in markets with such utilities. Tâtonnement adjusts the prices based on the excess demand for a good, but with linear utilities the demand is a set function. Also, the demand is discontinuous: a small change in price can lead to a large change in demand. This makes tâtonnement especially unsuited as a process describing market dynamics for linear utilities. In contrast, linear utilities pose no such problem for proportional response dynamics. For complements, especially in the extreme case of Leontief utilities, there is scant hope for fast convergence since computationally the problem of finding an equilibrium is PPAD-hard (see Codenotti, Saberi, Varadarajan, and Ye [CSVY06]). However this does not rule out the existence of a slowly converging process.

The dearth of convergence results in this setting is not for the lack of trying (see, e.g., gradient descent based algorithms [CLL19]). The difficulty might be attributed to the fact that the dynamics can cycle! Consider the scenario where there are two sets of players such that each set buys all its goods from the other set. Suppose that the sets start with widely unequal amounts of cash. Then in each iteration, the total amount of cash of each set moves to the other side, forever. However there is still hope: the allocations and the utilities of the players could converge to those of a market equilibrium, even if the prices oscillate. (In fact, in the above example the relative prices in each set may converge, even though the price scales alternate between the sides.)

We study the standard model of exchange economy where each player comes to the market with a unit endowment of an exclusive good. For linear utilities, this is without loss of generality. The equilibrium utilities are unique, while the equilibrium allocation may not be. Our main results are as follows:

1. We show that the kind of cycling described above is essentially the only one possible. We characterize the *limit cycles* of the dynamics as follows: there are equivalence classes of players such that within each class the ratio of price to equilibrium price is a constant. Further, the classes form a cycle, where the players in each class only buy goods from the players in the next class in the cycle. The allocations and utilities correspond to a market equilibrium, and remain invariant all along the limit cycle. (Theorem 6)
2. We show that the allocation and hence the utilities converge (Theorem 5). The result in the previous item only shows that the limit set in the allocation space is the set of equilibrium allocations. Convergence to this set is implied.
3. We introduce a *lazy* version [2] of the proportional response dynamic, where players saves a certain fraction of their cash at hand for future rounds (Definition 2). The fraction can be different for each player, as long as it is in $(0, 1)$, but it doesn't change over time. We show that for this version, there is no cycling: prices, allocation and utilities all converge to a market equilibrium (Theorem 4).

[2] The name is motivated by lazy random walks.

1.2 Previous work

Most of the results for the convergence of the proportional response dynamics to date have been for *Fisher* markets, where the players act just as buyers, and each step is fueled by a fixed income of each player and a fresh batch of goods. Zhang [Zha11] shows convergence of proportional response dynamics to the market equilibrium for *Constant Elasticity of Substitution* (CES) utilities in the *substitutes* regime. Birnbaum, Devanur, and Xiao [BDX11] interpret proportional response as *mirror descent* on the convex program of Shmyrev [Shm09] that captures equilibria in Fisher markets with *linear* utilities, and also extends it to some other markets. Cheung, Cole, and Tao [CCT18] extend the approach in [BDX11] to show that proportional response converges for the entire range of CES utilities including *complements*, with linear utilities on one extreme and *Leontief* utilities on the other extreme. Cheung, Hoefer, and Nakhe [CHN19] show that the dynamics stays close to equilibrium even when the market parameters are changing slowly over time, once again for CES utilities.

Wu and Zhang [WZ07] consider a tit-for-tat dynamic in an exchange setting, which they also call proportional response, but this is different from what is generally accepted today as proportional response. In addition, their analysis is for the special case where the value of a good is the same to each player and there is no money. We show that tit-for-tat and proportional response are functionally different, i.e. they have different trajectories in terms of allocations and utilities, even in the special case of utilities analyzed in [WZ07] and given the same starting configurations (see Appendix C). In fact, we find an example with arbitrary linear utilities (i.e. that do not satisfy the special symmetry property under which convergence of tit-for-tat was known), for which both utilities and allocations in the tit-for-tat dynamics cycle (Appendix C).

In an *exchange* setting, Branzei, Mehta, and Nisan [BMN18] generalize the definition of proportional response from Fisher markets to a *production economy*; this is the definition we use, for the special case of constant amounts. In the production market, players make new goods from the ones they acquire through the trading post mechanism, and the new goods are sold in the next iteration. The dynamic leads to *universal growth* of the market, where the amount of goods produced grow unboundedly over time, but also to growing inequality between the players on the most efficient production cycle and the rest. In particular, the dynamic learns through local interactions a global feature of the exchange graph—the cycle with the highest geometric mean.

1.3 Difficulties and techniques

The strongest convergence results for proportional response dynamics in Fisher markets are achieved via the mirror descent interpretation on suitable convex programs [BDX11, CCT18]. Devanur, Garg, and Végh [DGV16] show a similar (but more complicated) convex program for linear utilities in exchange markets. It is therefore tempting to conjecture that a similar mirror descent interpretation would extend to exchange markets as well, but unfortunately this doesn't seem to be the case. There are many difficulties, but the easiest to explain is the following. In the Fisher case, the proportional response bids are by definition in the feasible region of the convex program, which asks that the price of a good equal the total bids placed on it, and that the budget of a player equal the total of his own bids. In the exchange case, the price of a good is still the total bid placed on it by definition, but the total bid a player issues equals his earnings from the previous iteration, which may differ from the total bid placed on its good in the current iteration. The convex program requires these equality constraints, so the bids don't stay inside the feasible region as in the Fisher case.

We use the KL divergence between equilibrium bids and the current bids as a Lyapunov function; this divergence was also used in [Zha11] when analyzing Fisher markets. The convergence of utilities is the easiest, and follows almost exactly the analysis for Fisher markets. Beyond utilities, the cycling of bids presents more difficulties. We characterize the limit cycles by considering the zero set of a certain set of equations. We argue that their structure is like that of the price ratios in the limit cycles described above. The convergence of allocation follows from showing that the KL divergence between the (equilibrium and current) bids can be decomposed into a positive linear combination of the KL divergence between the prices and the KL divergences between the allocations for each good. This also implies that the KL divergence between equilibrium prices and prices on a limit cycle must be an invariant.

For the lazy version we show that adding a suitably weighted KL divergence between equlibrium prices and current budgets to the Lyapunov function does the trick. It collapses the limit cycles so that the limit set is now just the equilibria. This then gives us that the Lyapunov function must go to zero, which implies convergence of prices as well.

1.4 Other related work

The study of convergence of tâtonnement goes back at least as far as Arrow, Block, and Hurwicz [ABH59], which was soon followed by examples of cycling (see Scarf [Sca60] and Gale [Gal63]). For markets with weak gross substitutes utilities (WGS), a polynomial time convergence of a discrete time process was shown by Codenotti, McCune, and Varadarajan [CMV05], and Cole and Fleischer [CF08] showed fast convergence not just for static markets, but also "ongoing" markets. (See also Fleischer, Garg, Kapoor, Khandekar, and Saberi [FGK+08].) This was followed up by similar analysis for some markets with complementarities (Cheung, Cole, and Rastogi [CCR12]; Cheung and Cole [CC14]; Avigdor, Rabani, and Yadgar [AERY14]), then for all "Eisenberg-Gale" markets in the Fisher model [CCD19]. Some of these analyses apply to ongoing and/or asynchronous settings.

Several approaches have been explored for the (centralized) computation of equilibria in a linear exchange market: the ellipsoid method (Jain [Jai07]; Codenotti, Pemmaraju, and Varadarajan [CPV05]), interior point algorithms (Ye [Ye08]), combinatorial flow based methods (Jain, Mahdian, and Saberi [JMS03]; Devanur and Vazirani [DV03]; Duan and Mehlhorn [DM15]; Duan, Garg, and Mehlhorn [DGM16]). Extending the Fisher market version of [DPSV08]) recently led to a strongly polynomial time algorithm (Garg and Végh [GV19]). Other approaches include auction based algorithms [GK06, BGH19], cell decomposition [DPS03, DK08], complementary pivoting [GMSV15], and computational versions of Sperner's lemma [EW11, Sca77]. On the other hand, computing an equilibrium with even the simplest kind of complementarities is PPAD-hard [CSVY06, CDDT09].

There has been extensive work on understanding dynamics in games and auction settings under various behavioural models of the agents, such as best-response dynamics, multiplicative weight updates, fictitious play (e.g., [FS99, KPT09, DDK15, MPP15, PP16a, RST17, DS16, HKMN11, PP16b, LST16]) and best response processes and other dynamics of learning how to bid in market settings [CD11, NSVZ11, BR11, LB10, BBN17, DK17, CDE+14, BFR19]). In the former the focus has been on convergence to an equilibrium, preferable Nash, and if not then (coarse) correlated equilibria, and the rate of covergence. In the latter the focus has been on either convergence points and their quality (price-of-anarchy), or dynamic mechanisms such as ascending price auctions to reach efficient allocations (e.g. the Ausubel auction [Aus04]).

1.5 Organization of the paper

In Section 2, we give the definitions of the two variants of the proportional response dynamics, and market equilibria and state some useful properties. We also show numeric examples of cycling for the non-lazy version. Section 3 defines a Lyapunov function for the dynamics, which shows convergence of utilities. We also show convergence of allocation and prices for the strictly lazy version. In Section 4, we characterize limit cycles, and show convergence of allocation for the non-lazy version. The deferred proofs are in Appendix A. Sections B and C show some more numerical examples, and comparisons with the tit-for-tat dynamics.

2 Preliminaries

There are n agents, each of which has one unit of an eponymous good. The goods are divisible. The agents have linear utilities given by a matrix $A = \{a_{i,j}\}$, where $a_{i,j} > 0$ is the valuation of agent i for one unit of the good owned by agent j. We assume that every agent has a certain quantity of a numeraire to start with, which we call money. All the prices will be determined in terms of this numeraire. The agents don't have any utility for the money; it just facilitates exchange of goods.

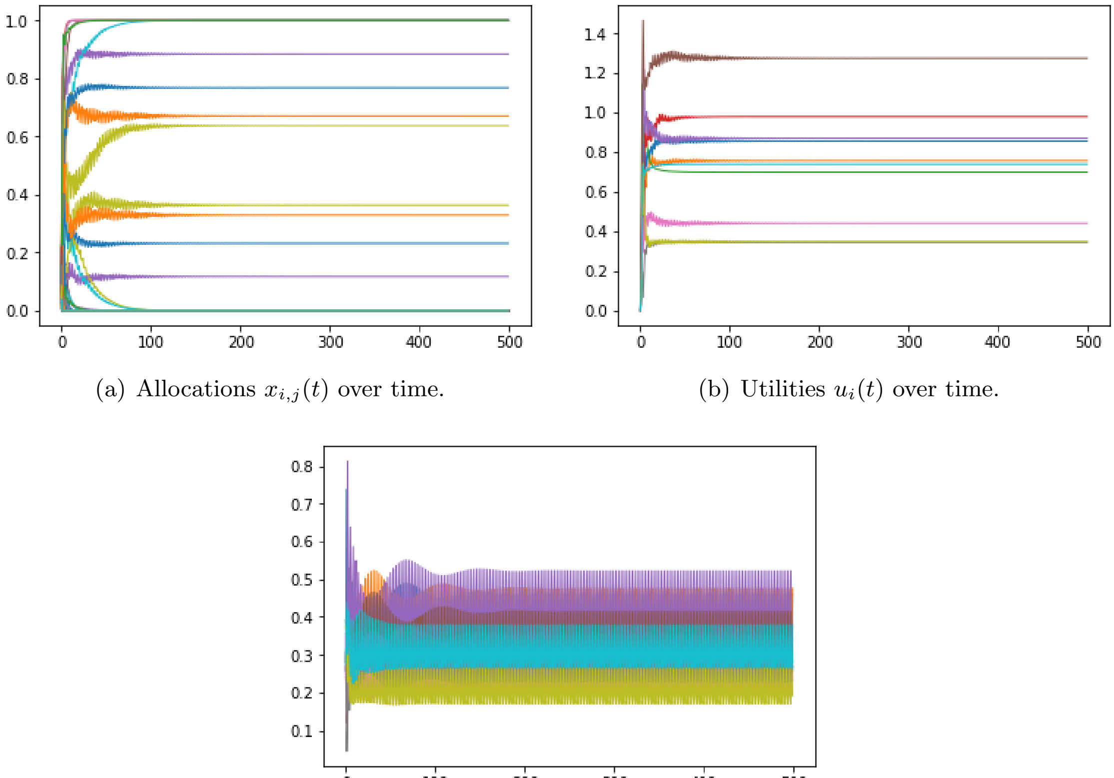
Proportional Response in Exchange Economies: The proportional response dynamic describes a process in which the players come to the market every day with one unit of good and some budget, which is split into bids. The players bid on the goods, then the seller of each good allocates it in proportion to the bid amounts and collects the money from selling, which becomes its budget in the next round. Finally the players update their bids in proportion to the contribution of each good to their utility.

Definition 1 (Proportional Response Dynamic)**.** *The initial bids of player i are $b_{i,j}(0)$, which are non-zero whenever $a_{i,j} > 0$. Then, at each time t, the following steps occur:*

> **Exchange of goods.** *Each player i brings one unit of its good and submits bids $b_{i,j}(t)$. The player receives an amount $x_{i,j}(t)$ of each good j, where*
>
> $$x_{i,j}(t) = \begin{cases} \frac{b_{i,j}(t)}{\sum_{k=1}^{n} b_{k,j}(t)}, & \text{if } b_{i,j}(t) > 0 \\ 0, & \text{otherwise} \end{cases}$$
>
> **Utility.** *Each player i computes the utility for its bundle: $u_i(t) = \sum_{j=1}^{n} a_{i,j} \cdot x_{i,j}(t)$.*
>
> **Bid update.** *Each player i collects the money made from selling: $B_i(t+1) = \sum_{k=1}^{n} b_{k,i}(t)$ and updates his bids proportionally to the contribution of each good in his utility:*
>
> $$b_{i,j}(t+1) = \left(\frac{a_{i,j} \cdot x_{i,j}(t)}{u_i(t)}\right) \cdot B_i(t+1)$$

Note: The sum of bids on a good can be seen as its *price*, so we will write $p_i(t) = \sum_{k=1}^{n} b_{k,i}(t)$.

Figure 1 shows an example trajectory in a ten player economy. In this example it can be seen that the allocations and utilities converge while the prices continue cycling even after several hundred rounds.



(a) Allocations $x_{i,j}(t)$ over time.

(b) Utilities $u_i(t)$ over time.

(c) Prices $p_i(t)$ over time.

Figure 1: Proportional response dynamic in an economy with $n = 10$ players for three components over $T = 500$ rounds: allocations, utilities, and prices. The graph consists of three components (C_1, C_2, C_3), so that players in C_i only consume goods from C_{i+1}, where $C_4 := C_1$.

Lazy Proportional Response in Exchange Economies: Our first contribution is to define a more general framework for proportional response, which can be seen as a lazy version of the dynamic and where a player may choose to only spend some fraction α_i of its total money in each round, while keeping the remaining fraction of $1 - \alpha_i$ in the bank.

Definition 2 (Lazy Proportional Response Dynamic)**.** *Initially each player i has some amount of money $m_i(0)$. The player splits a fraction $\alpha_i \in (0, 1]$ of the money into initial bids $b_{i,j}(0)$, which are non-zero whenever $a_{i,j} > 0$. I.e., the budget for spending at time 0, which is $B_i(0) \triangleq \alpha_i \cdot m_i(0)$, is split into bids $b_{i,j}(0)$ satisfying $\sum_j b_{i,j}(0) = \alpha_i \cdot m_i(0)$, and the player saves the remaining portion of money $(1 - \alpha_i) \cdot m_i(0)$ in the bank. At each time t, the following steps take place:*

Exchange of goods. *Each player i brings one unit of its good and submits bids $b_{i,j}(t)$. The player receives an amount $x_{i,j}(t)$ of each good j, where*

$$x_{i,j}(t) = \begin{cases} \frac{b_{i,j}(t)}{\sum_{k=1}^{n} b_{k,j}(t)}, & \text{if } b_{i,j}(t) > 0 \\ 0, & \text{otherwise} \end{cases}$$

Utility. *Each player i computes the utility for its bundle: $u_i(t) = \sum_{j=1}^{n} a_{i,j} \cdot x_{i,j}(t)$.*

Bid update. *Each player i collects the money made from selling: $p_i(t) = \sum_{k=1}^{n} b_{k,i}(t)$. Now the total money of player i is the price of the good sold and the money saved: $m_i(t) = p_i(t) + (1-\alpha_i)B_i(t)/\alpha_i$. This gets split again into a fraction of $(1-\alpha_i)$ that is saved in the bank and a fraction α_i that becomes the budget for spending in the next round:*

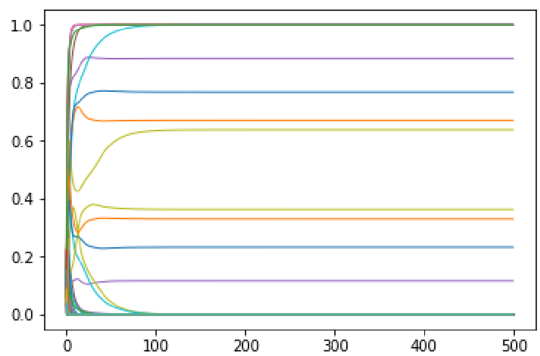
$$B_i(t+1) = \alpha_i \cdot m_i(t+1) = \alpha_i \cdot p_i(t) + (1-\alpha_i)B_i(t).$$

The player updates his bids proportionally to the contribution of each good in his utility:

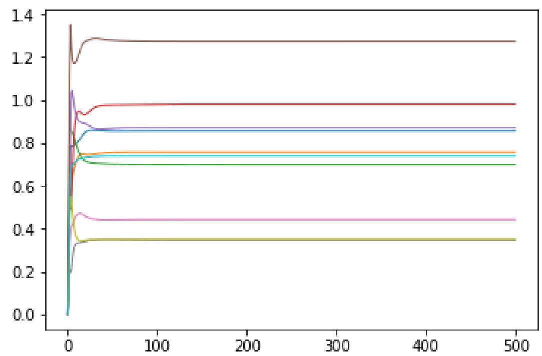
$$b_{i,j}(t+1) = \left(\frac{a_{i,j} \cdot x_{i,j}(t)}{u_i(t)}\right) \cdot B_i(t+1)$$

Notice that the special case of this dynamic where $\alpha_i = 1$ for all agents i is simply the proportional response dynamic of Definition 1.

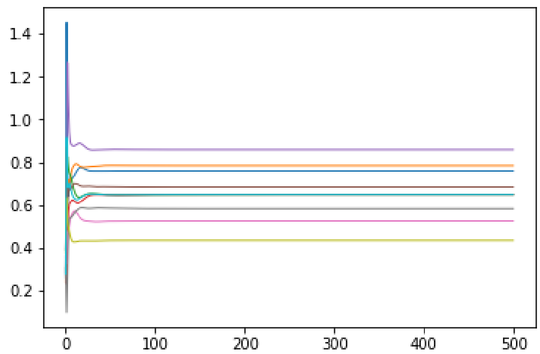
In Figure 2 we show the dynamic for the same valuations and initial bids as in Figure 1 but where the players also have half of their money in the bank at each time unit.



(a) Allocations $x_{i,j}(t)$ over time.



(b) Utilities $u_i(t)$ over time.



(c) Prices $p_i(t)$ over time.

Figure 2: Lazy version of the dynamic where the players have the same valuations and initial budgets and bids as in Figure 1, but they also have a matching amount saved in the bank, and in each round they save 50% of their money.

2.1 Market Equilibria

We review the definition of market equilibria and some useful properties. We assume that for each good j there is at least one player i such that $a_{i,j} > 0$. An equilibrium is given by a set of prices p_i for each $i \in [n]$ and a set of allocations $x_{i,j} \geq 0$ for each pair $i, j \in [n]$ such that

Market clearing: the goods are all sold, i.e., $\forall\ j, \sum_i x_{i,j} = 1$.

Optimal allocation: each buyer gets an optimal bundle of goods, i.e., $\forall\ i, x_{i,j}$s maximize the sum $\sum_j a_{i,j} x_{i,j}$ subject to the budget constraint $\sum_j p_j x_{i,j} \leq p_i$.

It is known that equilibrium utilities are unique. Equilibrium allocations and prices may not be unique, but equilibrium allocations, equilibrium prices, the set of equilibria $(x_{i,j}, \log p_j)$, and the set of equilibria $(b_{i,j}, p_j)$ where $b_{i,j} = x_{i,j} p_j$, all form convex sets [Gal76, Cor89, Mer03, Flo04, DGV16].

We note the following condition, which is guarateed to hold at any equilibrium in an exchange market ($*$s indicate equilibrium quantities):

$$\frac{u_i^*}{p_i^*} = \frac{a_{i,j}}{p_j^*}, \text{for all } i, j\ s.t.\ x_{i,j}^* > 0 \tag{1}$$

Fisher Markets: A variant of this model is the Fisher market, where there is a distinction between buyers and sellers. There are n players and n goods, and the utilities are as before. In addition, each player i comes to the market with a fixed budget B_i. The equilibrium conditions are the same as before, except that the budget constraint of player i is that $\sum_j p_j x_{i,j} \leq B_i$. The following convex program, called the *Eisenberg-Gale* convex program, captures equilibria in the Fisher market (with linear utilities): the set of optimal solutions to this program is equal to the set of equilibrium allocations and utilities [EG59].

$$\max \sum_i B_i \log u_i \text{ s.t.}$$

$$\forall\ i, u_i \leq \sum_j a_{i,j} x_{i,j},$$

$$\forall\ j, \sum_i x_{i,j} \leq 1,$$

$$x_{i,j} \geq 0.$$

Moreover, equilibrium utilities and prices are unique in the Fisher market [EG59].

We conclude this section by noting that the fixed points of the lazy proportional response dynamics are market equilibria. We defer the proof to Appendix A.

Theorem 1. *Any fixed point of lazy proportional response is a market equilibrium.*

3 Convergence of lazy proportional response

In this section we study the dynamic and show convergence of utilities for any combination of the values $\alpha_i \in (0, 1]$.

3.1 Convergence of Utilities

In this section we show that the utilities of the players converge for any valuations, initial configuration of the bids $b_{i,j}(0)$, and savings fractions $\alpha_i \in (0, 1]$ of the players. We let u_i^* denote the (unique) equilibrium utilities.

Theorem 2. *For any initial non-degenerate bids, the utilities of the players in the lazy dynamic converge to the market equilibrium utilities u_i^*; that is, $\lim_{t\to\infty} u_i(t) = u_i^*$.*

The high level idea is to show that a Lyapunov function for the dynamics is the Kullback-Leibler (KL) divergence between the vector with the bids and prices at a market equilibrium, and bids and budgets for the dynamic. We let p_i^* and $x_{i,j}^*$ denote some equilibrium price and allocation resp., and let $b_{i,j}^* = p_j^* x_{i,j}^*$. For each $i, j \in N$, define

$$z_{i,j}(t) = \begin{cases} \left(\frac{b_{i,j}^*}{b_{i,j}(t)}\right)^{b_{i,j}^*} & if\ b_{i,j}^* > 0, b_{i,j}(t) > 0 \\ 1 & otherwise \end{cases} \tag{2}$$

and

$$w_i(t) = \begin{cases} \left(\frac{p_i^*}{B_i(t)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} & if\ p_i^* > 0, B_i(t) > 0 \\ 1 & otherwise \end{cases} \tag{3}$$

Then our Lyapunov function is $f(t) = \prod_{i,j \in [n]} z_{i,j}(t) \cdot \prod_{i \in [n]} w_i(t)$. Assume w.l.o.g. that the sum of all the budgets is 1. Similarly we normalize p_i^* so that they sum up to 1. With this, we get that

$$\log f(t) = d_{KL}(b_{i,j}^*, ||b_{i,j}(t)) + \tfrac{1-\alpha_i}{\alpha_i} d_{KL}(p_i^* || B_i(t)).$$

The key fact we use about $f(t)$ is an iterative formula relating $f(t+1)$ to $f(t)$. We first define the functions $h, g : \mathbb{N} \to \Re$ by

$$g(t) = \prod_{i : p_i^* > 0} \left(\frac{u_i(t)}{u_i^*}\right)^{p_i^*}$$

and

$$h(t) = \left(\prod_{i \in [n]} \frac{p_i(t) \cdot B_i(t)^{\left(\frac{1-\alpha_i}{\alpha_i}\right)}}{(\alpha_i \cdot p_i(t) + (1-\alpha_i) \cdot B_i(t))^{\frac{1}{\alpha_i}}}\right)^{p_i^*}.$$

Then we have the following lemma, the proof of which is deferred to Appendix A.

Lemma 1. *With the lazy proportional response dynamics, we have the identity:*

$$f(t+1) = f(t) \cdot g(t) \cdot h(t).$$

We now show convergence of utilities.

Proof of Theorem 2. Observe that $\log g(t) = \sum_{i \in N} p_i^* \cdot \log\left(\frac{u_i(t)}{u_i^*}\right)$. Consider the Fisher market obtained by setting the budget of each player to the equilibrium price of its own good. Then the expression $\sum_{i \in N} p_i^* \cdot \log u_i$, which is the Eisenberg-Gale objective, is maximized by u_i^*. It follows that $\sum_{i \in N} p_i^* \cdot \log u_i(t) < \sum_{i \in N} p_i^* \cdot \log u_i^*$. Thus $g(t) \le 1$ for all t, and the equality holds if and only if $u_i(t) = u_i^*$ for all i.

Using the weighted arithmetic mean-geometric mean inequality, we have that

$$p_i(t)^{\alpha_i} \cdot B_i(t)^{1-\alpha_i} \le \alpha_i \cdot p_i(t) + (1-\alpha_i) \cdot B_i(t)$$

for all $\alpha_i \in (0, 1]$. Thus $h(t) \le 1$ for all t.

Since $g(t) < 1$ and $h(t) \le 1$, we obtain that $f(t+1) < f(t)$ for all the times t where the utilities are not the equilibrium utilities. It follows that $f(t)$ is monotonically decreasing and bounded from below, so $\lim_{t\to\infty} f(t)$ exists and is non-zero. Since $f(t+1) = f(t) \cdot g(t) \cdot h(t)$, we get that $\lim_{t\to\infty} g(t) = 1$, and so the dynamic reaches the market equilibrium utilities. $\square$

3.2 Convergence of Bids and Allocations in the Lazy Dynamic

In this subsection, we focus on the case when $\alpha_i < 1$ for all i. In this case, the limit points of the sequence must correspond to equilibrium prices.

Theorem 3. *If each $\alpha_i < 1$, then any limit point of the sequence $b_{i,j}(t)$ is an equilibrium.*

Proof. Let $\vec{b}$ be a limit point with a converging subsequence $s = (s_1, s_2, \ldots)$. As before, the utilities converge to the equilibrium utility. Further, we have that $lim_{n\to\infty} h(s_n) = 1$, which implies $lim_{n\to\infty} B_i(s_n) = lim_{n\to\infty} p_i(s_n)$ for all i. Thus for the limit point we must have that $\forall\ i, \sum_j b_{i,j} = p_i$. It is now easy to verify that the allocations and prices satisfy the equilibrium conditions. □

With this in hand, we can now show that the equilibrium bids and allocations converge. Note that this automatically implies that the prices converge too.

Theorem 4. *If each $\alpha_i < 1$, then the bids and allocations of the lazy dynamic converge to a market equilibrium.*

Proof. Suppose towards a contradiction that there exists a starting configuration $b_{i,j}(0)$ for which the sequence of bids does not converge. Then since the set of feasible bid matrices is compact, there exist two subsequences of rounds $s' = (s'_1, s'_2, \ldots)$ and $s'' = (s''_1, s''_2, \ldots)$ so that the bids converge to two limit bid matrices $\vec{b}'$ and $\vec{b}''$ along these subsequences, respectively.

From Theorem 3 we have that both $\vec{b}'$ and $\vec{b}''$ are market equilibrium bids. Define the function f with respect to the limit bids $\vec{b}'$ (i.e. set $\vec{b}^* = \vec{b}'$ in the definition of f). Then we have that $\log f(\vec{b}') = 0$ and $\log f(\vec{b}) > 0$ for all $\vec{b} \neq \vec{b}'$. Let $\vec{p}'$ and $\vec{p}''$ be the price vectors corresponding to bids $\vec{b}'$ and $\vec{b}''$. Since the bids along the sequences s' and s'' converge to different limits, by continuity of the KL divergence there exist $\delta > 0$ and an index $k_0 \in \mathbb{N}$ so that $|\log f(\vec{b}(s'_k)) - \log f(\vec{b}(s''_k))| > \delta$ for all $k \geq k_0$. This implies that $\lim_{k\to\infty} \log f(\vec{b}(s''_k)) > 0$, which is a contradiction. Thus the subsequence s'' cannot exist and the bids converge in the limit as required. □

4 Cycling behavior and convergence of allocations in the non-lazy dynamic

Note the bids may in fact cycle in proportional response (the strictly non-lazy version), thus we do not necessarily obtain in the limit the market equilibrium prices. Consider the following economy.

Example 1 (Bid cycling). *Let $N = \{1, 2\}$, with utilities $a_{1,1} = a_{2,2} = 0$, $a_{1,2} = a_{2,1} = 1$ and initial bids $b_{1,1}(0) = b_{2,2}(0) = 0$, $b_{1,2}(0) = 1/3$, $b_{2,1}(0) = 2/3$. Then at the market equilibrium the prices of the two goods are equal, so $p^*_1 = p^*_2 = 1/2$, with $b^*_{1,2} = b^*_{2,1} = 1/2$. Thus $f(0) = \left(\frac{1/2}{1/3}\right)^{1/2} \cdot \left(\frac{1/2}{2/3}\right)^{1/2}$. Since the players swap their budgets throughout the dynamic, we get that $f(2k+1) = \left(\frac{1/2}{2/3}\right)^{1/2} \cdot \left(\frac{1/2}{1/3}\right)^{1/2} = f(2k) = f(0)$ for all $k \in \mathbb{N}$.*

More generally, prices do not converge in bipartite graphs when the two sides are unbalanced. For example, consider any economy where the underlying graph is bipartite. Suppose the initial sums of the budgets on the two sides of the graph are different. Then the prices do not converge. This follows from the fact that the two sides will keep swapping their money each iterartion.

This phenomenon is analogous to what happens with periodic Markov chains. We note that cycling can happen even if the valuation matrix has all the edges non-zero and the bids are strictly positive. Rather, what determines cycling are the consumption graph in the equilibrium allocation together with the initial distribution of bids.

We first show that the allocation always converges and later characterize the instances where the bids cycle.

Theorem 5. *The allocation in the (non-lazy) proportional response dynamic converges to a market equilibrium allocation.*

Proof. For any initial bids $b_{i,j}(0)$, there is a subsequence of bids converging to some limit $\vec{b}'$. We note that the limit $\vec{b}'$ may not be market equilibrium bids. However, the allocation corresponding to $\vec{b}'$ must give equilibrium utilities, by Theorem 2. Any such allocation is also an equilibrium. (We are not sure if this property is known before; we could not find any reference. We give a proof this in Appendix A, as Theorem 7.) Let this allocation be $\vec{x}' = \vec{x^*}$, the corresponding equilibrium price be $\vec{p^*}$ and the corresponding bids be $\vec{b^*}$.

Let $\vec{p}'$ be the prices induced by the bids $\vec{b}'$. Since the allocation under the two bid profiles is the same, we get

$$\frac{b'_{i,j}}{b^*_{i,j}} = \frac{\left(\frac{b'_{i,j}}{p'_j}\right)}{\left(\frac{b^*_{i,j}}{p^*_j}\right)} \cdot \frac{p'_j}{p^*_j} = \frac{x'_{i,j}}{x^*_{i,j}} \cdot \frac{p'_j}{p^*_j} = \frac{p'_j}{p^*_j}$$

Let $j \in [n]$ be arbitrary but fixed. Then we obtain

$$\sum_{i\in[n]} b^*_{i,j} \cdot \log \frac{b'_{i,j}}{b^*_{i,j}} = \sum_{i\in[n]} b^*_{i,j} \cdot \log \frac{p'_j}{p^*_j} = p^*_j \cdot \log \frac{p'_j}{p^*_j} \tag{4}$$

Suppose for the sake of contradiction that the sequence of allocations of the dynamic does not converge. Then there exists another subsequence of bids converging to a different limit $\vec{b}'' \neq \vec{b}'$ which has the property that $\vec{x}'' \neq \vec{x}'$, where $\vec{x}''$ is the allocation at the bid profile $\vec{b}''$. We use an identity which we state in Lemma 2 to obtain

$$\sum_{i\in[n]} b^*_{i,j} \cdot \log \frac{b''_{i,j}}{b^*_{i,j}} = p^*_j \cdot \log \frac{p''_j}{p^*_j} + p^*_j \cdot \sum_{i\in[n]} x'_{i,j} \cdot \log \frac{x''_{i,j}}{x'_{i,j}}.$$

From Theorem 2, the KL divergence between the fixed point bids $\vec{b^*}$ and the dynamic bids $\vec{b}(t)$ is decreasing and converges to some constant $\alpha \geq 0$. Since both $\vec{b}', \vec{b}''$ are limit points of $\vec{b}(t)$, this implies that

$$\begin{aligned} &\sum_{j\in[n]} \sum_{i\in[n]} b^*_{i,j} \cdot \log \frac{b'_{i,j}}{b^*_{i,j}} = \sum_{j\in[n]} \sum_{i\in[n]} b^*_{i,j} \cdot \log \frac{b''_{i,j}}{b^*_{i,j}} \iff \\ &\sum_{j\in[n]} p^*_j \cdot \log \frac{p'_j}{p^*_j} = \sum_{j\in[n]} p^*_j \cdot \log \frac{p''_j}{p^*_j} + \sum_{j\in[n]} p^*_j \cdot \sum_{i\in[n]} x'_{i,j} \cdot \log \frac{x''_{i,j}}{x'_{i,j}} \end{aligned} \tag{5}$$

Among all possible choices of limit bid profiles $\vec{b}'$ and market equilibrium bids $\vec{b^*}$ with the same allocation, select the pair $(\vec{b}', \vec{b^*})$ that minimizes the sum $\sum_{j\in[n]} p^*_j \cdot \log \frac{p'_j}{p^*_j}$; this is possible since the bid space is compact so the infimum of a set of accummulation points is itself an accummulation point (See Lemma 4.1 in [Kha02] for example). For this choice of bids, we obtain

$$\sum_{j\in[n]} p_j^* \cdot \log \frac{p''_j}{p_j^*} \geq \sum_{j\in[n]} p_j^* \cdot \log \frac{p'_j}{p_j^*} = \sum_{j\in[n]} p_j^* \cdot \log \frac{p''_j}{p_j^*} + \sum_{j\in[n]} p_j^* \cdot \sum_{i\in[n]} x'_{i,j} \cdot \log \frac{x''_{i,j}}{x'_{i,j}} \tag{6}$$

Note that each term $\sum_{i\in[n]} x'_{i,j} \cdot \log \frac{x''_{i,j}}{x_{i,j'}}$ represents the KL-divergence between the allocation of good i at the limits x' and x''. Since the KL divergence is always non-negative, we get that in fact $\sum_{i\in[n]} x'_{i,j} \cdot \log \frac{x''_{i,j}}{x'_{i,j}} = 0$ for each j, so the allocation at the limit $\vec{b}''$ is the same as at $\vec{b}'$. Thus the assumption that the limit $\vec{b}''$ had a different allocation from $\vec{b}'$ was incorrect, so the allocations must converge. □

Lemma 2. *For any two bids $\vec{b}$ and $\vec{b}'$ and corresponding allocations and prices, we have that*

$$\sum_{i\in[n]} b_{i,j} \cdot \log \frac{b'_{i,j}}{b_{i,j}} = p_j \cdot \log \frac{p'_j}{p_j} + p_j \cdot \sum_{i\in[n]} x_{i,j} \cdot \log \frac{x'_{i,j}}{x_{i,j}}.$$

The proof of this lemma is in Appendix A. We now characterize the limit cycles in the price space. Note that we already have from Theorem 5 that the allocation must remain an invariant along any limit cycle.

Theorem 6. *The limit bids of the proportional response dynamic are either an equilibrium or there exist equivalence classes $C_1, \ldots, C_k$, where $C_i \cap C_j = \emptyset$, $C_i \subseteq N$ for all i, such that there exists $\lambda_i(t) \geq 0$ for each C_i with the property that the price of each good $j \in C_i$ satisfies $p_j(t)/p_j^* = \lambda_i(t)$, for some equilibrium price $\vec{p^*}$.*

Proof. We have from Theorem 5 that the allocation converges to an equilibrium; let $\vec{p^*}$ be the corresponding equilibrium price, and $\vec{u^*}$ the equilibrium utilities. Consider a limit point, and consider taking one step of the dynamics from the limit point. We denote the limit point by $\vec{b}(t)$, just to indicate that the next step after this is $\vec{b}(t+1)$. The point $\vec{b}(t+1)$ is also a limit point of the sequence.[3] By the definition of the update rule, we have that

$$u_i(t+1) = \sum_{j=1}^{n} a_{i,j} \cdot x_{i,j}(t+1) = \sum_{j=1}^{n} a_{i,j} \cdot \frac{b_{i,j}(t+1)}{\sum_{k=1}^{n} b_{k,j}(t+1)}$$

By Theorem 2, the utilities converge in the limit to the market equilibrium utilities, therefore

[3]This is a well known property of continuous time dynamical systems, and the same holds for discrete time systems as well. A quick proof: if s_n is the subsequence whose limit is $\vec{b}(t)$, then by continuity of the update rule, we get that the subsequence $s_n + 1$ must have $\vec{b}(t+1)$ as its limit.

we have that $u_i(t) = u_i^* = u_i(t+1)$ for all $i \in [n]$. Then we get

$$\begin{aligned}
u_i(t+1) &= \sum_{j=1}^{n} a_{i,j} \cdot \frac{b_{i,j}(t+1)}{\sum_{k=1}^{n} b_{k,j}(t+1)} \\
&= \sum_{j=1}^{n} a_{i,j} \cdot \frac{\frac{a_{i,j}}{u_i^*} \cdot x_{i,j}(t) \cdot B_i(t+1)}{\sum_{k=1}^{n} \frac{a_{k,j}}{u_k^*} \cdot x_{k,j}(t) \cdot B_k(t+1)} \\
&= \sum_{j=1}^{n} a_{i,j} \cdot \frac{\frac{p_j^*}{p_i^*} \cdot x_{i,j}(t) \cdot B_i(t+1)}{\sum_{k=1}^{n} \frac{p_j^*}{p_k^*} \cdot x_{k,j}(t) \cdot B_k(t+1)} \\
&= \sum_{j=1}^{n} \frac{\frac{a_{i,j}}{p_i^*} \cdot x_{i,j}(t) \cdot B_i(t+1)}{\sum_{k=1}^{n} x_{k,j}(t) \cdot \frac{B_k(t+1)}{p_k^*}} \\
&= \sum_{j=1}^{n} \frac{a_{i,j} \cdot x_{i,j}(t) \cdot \left(\frac{p_i(t)}{p_i^*}\right)}{\sum_{k=1}^{n} x_{k,j}(t) \cdot \left(\frac{p_k(t)}{p_k^*}\right)} = u_i(t) = \sum_{j=1}^{n} a_{i,j} \cdot x_{i,j}(t)
\end{aligned} \tag{7}$$

The third equality follows from (1). Let $\lambda_i(t) = p_i(t)/p_i^*$. Then identity (7) is equivalent to the following system of equations, where $a_{i,j}, x_{i,j}(t)$ are given and $\lambda_i(t)$ are variables:

$$\begin{cases} \lambda_i(t) \geq 0, \text{for all } i \in [n] \\ \sum_{j=1}^{n} a_{i,j} \cdot x_{i,j}(t) \cdot \left(\frac{\lambda_i(t)}{\sum_{k=1}^{n} x_{k,j}(t) \cdot \lambda_k(t)} - 1\right) = 0, \quad \text{for all } i \in [n] \end{cases}$$

One solution can be obtained as follows. Define the equivalence relation $\sim$ as follows: $i \sim k$ if and only if there exists $j \in [n]$ such that $x_{i,j}, x_{k,j} > 0$. Then consider the transitive closure of this graph – that is, if player i purchases some other good $\ell \neq j$, then all the players that purchase strictly positive amounts of good ℓ are in the same equivalence class with i and k. Let $C_1, \ldots, C_k$ be equivalence classes with respect to the $\sim$ relation. Then setting $\lambda_i(t) = \lambda_k(t)$ for each $i \sim k$ works. This means that all the goods in the same equivalence class have prices within the same factor away from the market equilibrium price at any point in time.

We show that in fact these are the only solutions. Consider an arbitrary solution to this system and suppose towards a contradiction that there exist two players i, i' in the same equivalence class C_ℓ but with $\lambda_i(t) > \lambda_{i'}(t)$. W.l.o.g., $\lambda_i(t) = \max_{v \in C_\ell} \lambda_v(t)$. Then for all j with $x_{i,j}(t) > 0$ we have

$$\frac{\lambda_i(t)}{\sum_{k=1}^{n} x_{k,j}(t) \cdot \lambda_k(t)} > 1 \iff a_{i,j} \cdot x_{i,j}(t) \cdot \left(\frac{\lambda_i(t)}{\sum_{k=1}^{n} x_{k,j}(t) \cdot \lambda_k(t)} - 1\right) > 0 \tag{8}$$

Summing up inequality 8 over all j we get

$$\sum_{j=1}^{n} a_{i,j} \cdot x_{i,j}(t) \cdot \left(\frac{\lambda_i(t)}{\sum_{k=1}^{n} x_{k,j}(t) \cdot \lambda_k(t)} - 1\right) > 0,$$

which does not satisfy the required system of identities. Thus the assumption must have been false and $\lambda_i(t) = \lambda_{i'}(t)$ for any players i, i' in the same equivalence class. □

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A Missing Proofs

The following theorem states that for a linear exchange market, any equilibrium allocation can be paired with any equilibrium price to get an equilibrium pair of allocation and price.

Theorem 7. *Let $\vec{p^*}$ be any equilibrium price, and $\vec{u^*}$ be equilibrium utilities. Let $\vec{x}$ be any feasible allocation that gives equilibrium utilities to all players, i.e., $\forall\ i$, $\sum_j a_{i,j}x_{i,j} = u_i^*$. Then the pair $(\vec{x}, \vec{p^*})$ is an equilibrium.*

Proof. Let $(\vec{x^*}, \vec{p^*})$ be a pair of equilibrium allocation and price for the exchange market. Consider the Fisher market with budgets $B_i = p_i^*$ for all i, denoted by $\mathsf{Fisher}(\vec{p^*})$.

1. Then the pair $(\vec{x^*}, \vec{p^*})$ is also an equilibrium of this Fisher market, since the equilibrium conditions for the exchange market directly imply the equilibrium conditions for the Fisher market.

2. Since Fisher markets have a unique equilibrium price [EG59], this price must be $\vec{p^*}$.

3. Now suppose $\vec{x}$ be any other allocation as in the hypothesis of the Theorem. This implies that $\vec{x}$ is an optimal solution to the Eisenberg-Gale convex program corresponding to $\mathsf{Fisher}(\vec{p^*})$, and therefore $(\vec{x}, \vec{p^*})$ is also an equilibrium of $\mathsf{Fisher}(\vec{p^*})$.

4. This implies that $(\vec{x}, \vec{p^*})$ is also an equilibrium for the exchange market. Once again, the equilibrium conditions are essentially identical.

□

Proof of Theorem 1. Suppose $\vec{b}^*$ is a fixed point of the proportional dynamic; let $x_{i,j}^*$ be the resulting fixed point allocation and u_i^* the corresponding utilities. For each good j, let $p_j^* = \sum_{i=1}^n b_{i,j}^*$; this quantity can be interpreted as the price of the good at the fixed point. The budget update rule at the fixed point gives:

$$B_i^* = \alpha_i \cdot p_i^* + (1-\alpha_i)B_i^* \iff B_i^* = p_i^*.$$

Using the fact that $B_i^* = p_i^*$ we can write the bid update rule as:

$$b_{i,j}^* = \left(\frac{a_{i,j} \cdot x_{i,j}^*}{u_i^*}\right) \cdot B_i^* = \left(\frac{a_{i,j} \cdot \left(\frac{b_{i,j}^*}{\sum_{k=1}^n b_{k,j}^*}\right)}{u_i^*}\right) \cdot B_i^* = \left(\frac{a_{i,j} \cdot \frac{b_{i,j}^*}{p_j^*}}{u_i^*}\right) \cdot p_i^*$$

If $b_{i,j}^* = 0$ the identity trivially holds. For $b_{i,j}^* > 0$, the identity is equivalent to $u_i^*/p_i = a_{i,j}/p_j$. This condition is the same as the market equilibrium condition for all strictly positive bids in the exchange economy, and so every fixed point is a market equilibrium. □

Proof of Lemma 1. The bid update rule gives

$$b_{i,j}(t+1) = \left(\frac{a_{i,j} \cdot x_{i,j}(t)}{u_i(t)}\right) \cdot B_i(t+1) = \frac{a_{i,j}}{u_i(t)} \cdot \frac{b_{i,j}(t)}{p_j(t)} \cdot B_i(t+1)$$

Expanding $z_{i,j}(t+1)$ we obtain

$$z_{i,j}(t+1) = \left(\frac{b_{i,j}^*}{\left(\frac{a_{i,j} \cdot x_{i,j}(t)}{u_i(t)}\right) \cdot B_i(t+1)}\right)^{b_{i,j}^*} = \left(\frac{b_{i,j}^*}{b_{i,j}(t)} \cdot \frac{u_i(t)}{a_{i,j}} \cdot \frac{p_j(t)}{B_i(t+1)}\right)^{b_{i,j}^*} = z_{i,j}(t) \cdot \left(\frac{u_i(t)}{a_{i,j}} \cdot \frac{p_j(t)}{B_i(t+1)}\right)^{b_{i,j}^*}$$

Using the equilibrium property (1) of the exchange economy, we get

$$z_{i,j}(t+1) = \begin{cases} z_{i,j}(t) \cdot \left(\frac{u_i(t)}{u_i^*} \cdot \frac{p_j(t)}{p_j^*} \cdot \frac{p_i^*}{B_i(t+1)}\right)^{b_{i,j}^*} & if\ b_{i,j}^* > 0 \\ 0 & otherwise \end{cases}$$

Expanding $w_i(t+1)$ gives

$$w_i(t+1) = \left(\frac{p_i^*}{B_i(t+1)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} = \left(\frac{p_i^*}{B_i(t)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} \cdot \left(\frac{B_i(t)}{B_i(t+1)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)}$$

Expanding $f(t+1)$ yields

$$\begin{aligned} f(t+1) &= \prod_{i,j\in[n]} z_{i,j}(t+1) \cdot \prod_{i\in[n]} w_i(t+1) \\ &= \prod_{i,j\in N:b_{i,j}^*>0} z_{i,j}(t) \cdot \left(\frac{u_i(t)}{u_i^*} \cdot \frac{p_j(t)}{p_j^*} \cdot \frac{p_i^*}{B_i(t+1)}\right)^{b_{i,j}^*} \cdot \prod_{i\in[n]} \left(\frac{p_i^*}{B_i(t)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} \cdot \left(\frac{B_i(t)}{B_i(t+1)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} \\ &= f(t) \cdot \left(\prod_{i,j\in N:b_{i,j}^*>0} \left(\frac{u_i(t)}{u_i^*} \cdot \frac{p_j(t)}{p_j^*} \cdot \frac{p_i^*}{B_i(t+1)}\right)^{b_{i,j}^*}\right) \cdot \prod_{i\in[n]} \left(\frac{B_i(t)}{B_i(t+1)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} \end{aligned}$$

Separating the utility terms from the price terms gives

$$\begin{aligned} f(t+1) &= f(t) \cdot \prod_{i:\exists j\ b_{i,j}^*>0} \left(\frac{u_i(t)}{u_i^*}\right)^{\sum_j b_{i,j}^*} \prod_{i:\exists j\ b_{i,j}^*>0} \left(\frac{p_i^*}{B_i(t+1)}\right)^{\sum_j b_{i,j}^*} \prod_{j:\exists i\ b_{i,j}^*>0} \left(\frac{p_j(t)}{p_j^*}\right)^{\sum_i b_{i,j}^*} \prod_{i\in[n]} \left(\frac{B_i(t)}{B_i(t+1)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} \\ &= f(t) \cdot \prod_{i:\exists j\ b_{i,j}^*>0} \left(\frac{u_i(t)}{u_i^*}\right)^{p_i^*} \cdot \prod_{i:\exists j\ b_{i,j}^*>0} \left(\frac{p_i^*}{B_i(t+1)}\right)^{p_i^*} \cdot \prod_{j:\exists i\ b_{i,j}^*>0} \left(\frac{p_j(t)}{p_j^*}\right)^{p_j^*} \cdot \prod_{i\in[n]} \left(\frac{B_i(t)}{B_i(t+1)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} \end{aligned}$$

Note the condition on i that there exists j for which $b_{i,j}^* > 0$ is equivalent to $p_i^* > 0$, which at the market equilibrium is met for every index $i \in [n]$. Similarly, the condition on j that there exists i such that $b_{i,j}^* > 0$ is equivalent to $p_j^* > 0$. Then we can rewrite $f(t+1)$ as follows:

$$\begin{aligned} f(t+1) &= f(t) \cdot \prod_{i\in[n]} \left(\frac{u_i(t)}{u_i^*}\right)^{p_i^*} \cdot \prod_{i\in[n]} \left(\frac{p_i^*}{B_i(t+1)}\right)^{p_i^*} \cdot \prod_{j\in[n]} \left(\frac{p_j(t)}{p_j^*}\right)^{p_j^*} \prod_{i\in[n]} \left(\frac{B_i(t)}{B_i(t+1)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} \\ &= f(t) \cdot \prod_{i\in[n]} \left(\frac{u_i(t)}{u_i^*}\right)^{p_i^*} \cdot \prod_{i\in[n]} \left(\frac{p_i(t)}{B_i(t+1)}\right)^{p_i^*} \prod_{i\in[n]} \left(\frac{B_i(t)}{B_i(t+1)}\right)^{p_i^* \cdot \left(\frac{1-\alpha_i}{\alpha_i}\right)} \end{aligned} \tag{9}$$

We also analyze the term

$$\begin{aligned} &\prod_{i\in[n]} \left(\frac{p_i(t)}{B_i(t+1)}\right) \cdot \prod_{i\in[n]} \left(\frac{B_i(t)}{B_i(t+1)}\right)^{\frac{1-\alpha_i}{\alpha_i}} \\ &= \prod_{i\in[n]} \frac{p_i(t) \cdot B_i(t)^{\left(\frac{1-\alpha_i}{\alpha_i}\right)}}{B_i(t+1)^{\frac{1}{\alpha_i}}} \\ &= \prod_{i\in[n]} \frac{p_i(t) \cdot B_i(t)^{\left(\frac{1-\alpha_i}{\alpha_i}\right)}}{(\alpha_i \cdot p_i(t) + (1-\alpha_i) \cdot B_i(t))^{\frac{1}{\alpha_i}}} \qquad (10) \\ &= h(t)^{\frac{1}{p_i^*}}. \qquad (11) \end{aligned}$$

Recall that $g(t) = \prod_{i:p_i^*>0} \left(\frac{u_i(t)}{u_i^*}\right)^{p_i^*}$. Then it follows that $f(t+1) = f(t) \cdot g(t) \cdot h(t)$ as required.

□

Proof of Lemma 2.

$$\begin{aligned}
\sum_{i\in[n]} b_{i,j} \cdot \log \frac{b'_{i,j}}{b_{i,j}} &= \sum_{i\in[n]} b_{i,j} \cdot \log \frac{x'_{i,j} \cdot p'_j}{x_{i,j} \cdot p_j} \\
&= \sum_{i\in[n]} b_{i,j} \cdot \log \frac{p'_j}{p_j} + \sum_{i\in[n]} b_{i,j} \cdot \log \frac{x'_{i,j}}{x_{i,j}} \\
&= p_j \cdot \log \frac{p'_j}{p_j} + \sum_{i\in[n]} p_j \cdot x_{i,j} \cdot \log \frac{x'_{i,j}}{x_{i,j}} \\
&= p_j \cdot \log \frac{p'_j}{p_j} + p_j \cdot \sum_{i\in[n]} x_{i,j} \cdot \log \frac{x'_{i,j}}{x_{i,j}} \qquad (12)
\end{aligned}$$

□

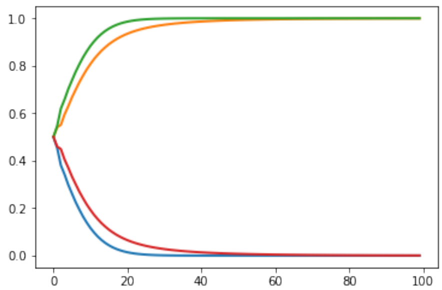
Theorem 8. *Suppose that there is an equilibrium allocation such that the support graph on the set of nodes is connected. Then the equilibrium prices are unique up to a scaling factor*

Proof. Suppose that i and j are such that $x_{i,j} > 0$. Then from the condition (1), we get that the ratio of their equilibrium prices must equal $a_{i,j}/u_i^*$, which is independent of the choice of the equilibrium, since equilibrium utilities are unique. Thus if the support graph is connected, the ratio of any two equilibrium prices remains the same, which means the equilibrium prices form a ray.

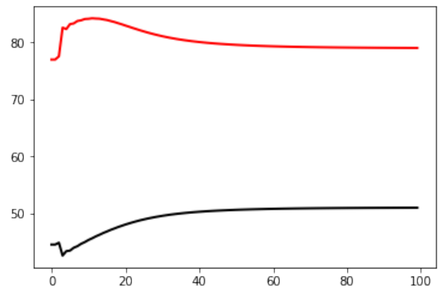
□

B More Examples of Proportional Response Trajectories

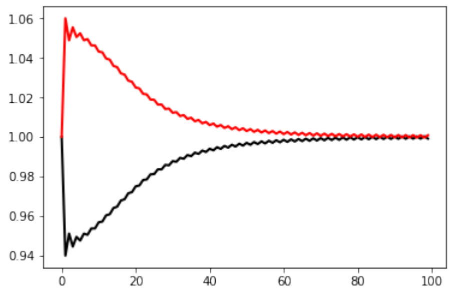
Figure 3 shows a two player economy under the non-lazy proportional response dynamic, where the bids of the players cycle in the limit. This economy has the property that the valuation matrix has all entries non-zero, but the consumption graph market equilibrium is bipartite. For some initial bids, this structure leads to cycling of the pricing.



(a) Allocations $x_{i,j}(t)$ over time.



(b) Utilities $u_i(t)$ over time.



(c) Prices $p_i(t)$ over time.

Figure 3: Proportional response dynamic in an economy with $n = 2$ players over $T = 100$ rounds: allocations, utilities, and prices. Player 1 is shown in black and player 2 in red. The initial valuation matrix is non-zero everywhere: $A = [[38.0, 51.0], [79.0, 75.0]]$ and the initial bids are $b_{i,j} = 1/2$ for all i, j. The market equilibrium allocation is $x_{1,2} = x_{2,1} = 1$ and $x_{1,1} = x_{2,2} = 0$. The prices are cycling in the limit.

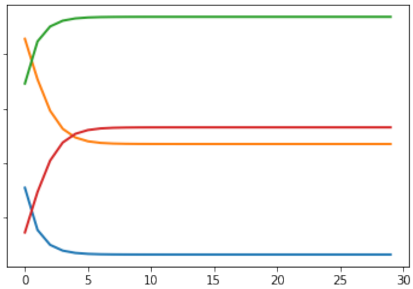
C Comparison between Tit-for-Tat and Proportional Response

In this section we show an execution of the tit-for-tat and proportional response dynamics given the starting configurations. The two dynamics have different trajectories, even in the special case where there exists a vector $\vec{w} = (w_1, \ldots, w_n)$ so that $a_{i,j} = w_j$ for each player i and each good j, which is the case where Wu and Zhang [WZ07] established convergence to market equilibria for the tit-for-tat dynamic.

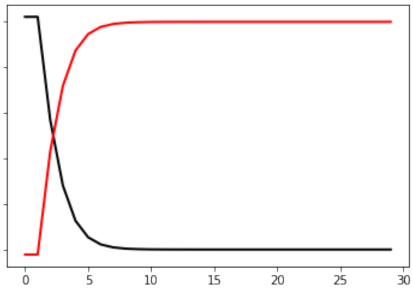
Recall that for general valuations $A = (a_{i,j})$, the tit-for-tat dynamic is

$$y_{i,j}(t+1) = \tfrac{y_{j,i}(t) \cdot a_{i,j}}{u_i(t)}$$

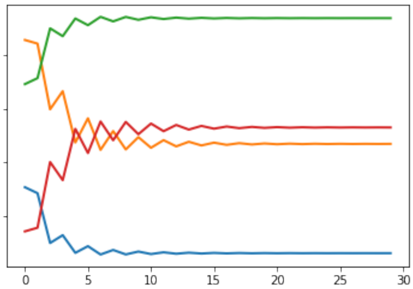
where $y_{j,i}(t)$ is the fraction received by player i from good j in round t and $u_i(t) = \sum_{k=1}^{n} y_{k,i}(t) \cdot a_{i,k}$ is the utility of player i in round t. (The order of the subscripts here is good, player, which is the convention used in [WZ07], as opposed to our notation where the order is player, good.) The tit-for-tat dynamic cycles for this input.



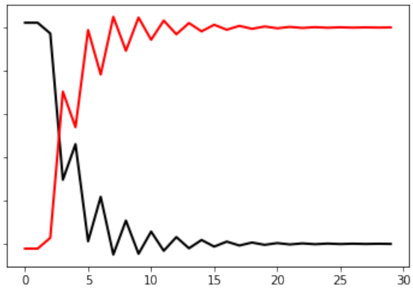
(a) Allocations under tit-for-tat.



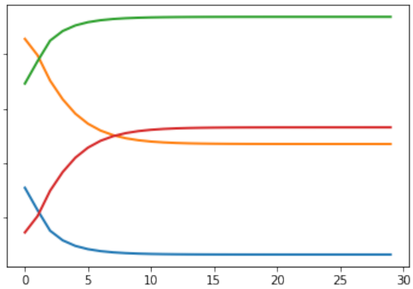
(b) Utilities under tit-for-tat.



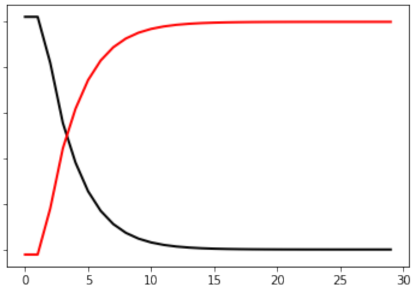
(c) Allocations under non-lazy proportional response.



(d) Utilities under non-lazy proportional response.



(e) Allocations under lazy proportional response with $\alpha_i = 1/2$ for each player i.



(f) Utilities under lazy proportional response with $\alpha_i = 1/2$ for each player i.

Figure 4: Comparison between tit-for-tat, non-lazy proportional response, and lazy proportional response, initialized with the same initial fractions of splitting the goods. The valuation matrix is $\vec{a} = [[1,2],[1,2]]$ and the initial bids for the proportional response executions are $\vec{b} = [[0.4, 0.6],[0.9, 0.1]]$. The initial fractions for tit-for-tat are given by $y_{j,i}(0) = b_{i,j}(0)/p_j(0)$, where $y_{j,i}(0)$ is the fraction of good j that player i receives in round 0.

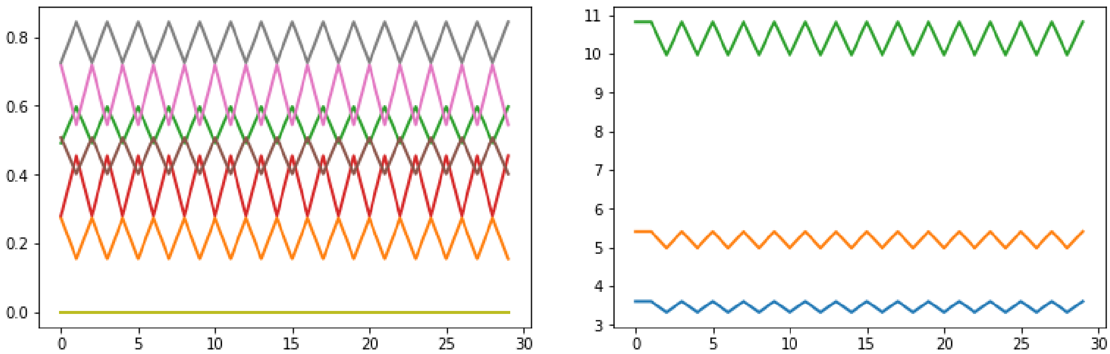
Note: In the figures (including Figure 4), the X axis shows the round number, while the Y axis shows each utility $u_i(t)$ over time and each allocation over time, where an allocation means the fraction received by each player i from good j, for each i, j, for every time unit $t = 0, 1, 2, \infty$.

In Figure 5, we show a three player economy with valuations

$$\mathbf{a} = \begin{vmatrix} 0 & 6 & 4 \\ 3 & 0 & 9 \\ 9 & 6 & 0 \end{vmatrix}$$

and initial fractions

$$\mathbf{y}(0) = \begin{vmatrix} 0.0 & 0.2805339037254016 & 0.7194660962745985 \\ 0.273923422472049 & 0.0 & 0.726076577527951 \\ 0.491752727261851 & 0.5082472727381491 & 0.0 \end{vmatrix}$$



(a) Allocations under tit-for-tat for $T = 30$ rounds.

(b) Utilities under tit-for-tat for $T = 30$ rounds; same initial conditions as in Figure (a)

Figure 5: Tit-for-tat dynamic cycling for both utilities and allocations with period two. The instance is a three player economy where the matrix does not satisfy the symmetry property under which tit-for-tat is known to converge to market equilibria [WZ07] (Recall this property requires that there exist values w_j so that $a_{i,j} = w_j$ for each player i and good j.).

The fractions oscillate between the values at time $t = 0$, equal to $\mathbf{y}(0)$, and those from time $t = 1$, which are equal to:

$$\mathbf{y}(1) = \begin{vmatrix} 0.0 & 0.4552048517736218 & 0.5447951482263783 \\ 0.1553967077250424 & 0.0 & 0.8446032922749576 \\ 0.5978029457196989 & 0.402197054280301 & 0.0 \end{vmatrix}$$

We note that market equilibria exist on such graphs and proportional response converges on such instances for any non-degenerate initial bids.