Unforeseen Evidence

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Abstract

In this note, I proposes a normative updating rule, *extended Bayesianism*, for the incorporation of probabilistic information arising from the process of becoming more aware. Extended Bayesianism generalizes standard Bayesian updating to allow the posterior to reside on richer probability space than the prior. I then provided an observable criterion on prior and posterior beliefs such that they were consistent with extended Bayesianism.

Key words: extended Bayesianism; reverse Bayesianism; conditional expectations.

Conditioning on Unforeseen Evidence

Decision maker's (DM's) who are *unaware*, cannot conceive of, nor articulate, the decision relevant contingencies they are unaware of. Nonetheless, such agents may hold sophisticated probabilistic beliefs regarding those contingencies they *are* aware of. How then should an agent's probabilistic beliefs respond to the discovery of novel contingencies? This note proposes a normative updating rule for the incorporation of probabilistic information arising from the process of becoming more aware.

Let Ω denote an (at most countable) objective, albeit possibly unobservable, state space.[1] Let Σ_t, a sigma-algebra on Ω, represent the events the DM can conceive of at $t \in \{0, 1\}$. By nature of the problem, we assume that $\Sigma_0 \subseteq \Sigma_1$. The DM's subjective uncertainty, given her current understanding, is taken to be a probability distribution, π_t, on the probability space (Ω, Σ_t). Set $S_t = \{\omega \in \Omega \mid \pi_t(E) > 0 \text{ for the smallest } E \in \Sigma_t \text{ with } E \supseteq \omega\}$ to denote the support of π_t. It is easy to show this is the smallest event in Σ_t with π_t-probability 1.

The tenet of *reverse Bayesianism* (RB), as introduced by Karni and Vierø (2013), states that when the DM becomes more aware, her probabilistic assessments regarding previously understood contingencies do not change. Formally: $\pi_1(E) = \pi_0(E)$ for all $E \in \Sigma_0$, so that π_1 is an extension of π_0 to the richer algebra.[2] Thus, RB essentially posits that becoming more aware is not in and of itself informative—learning how to distinguish between new events does not provide any probabilistic information regarding the likelihood of those events previously understood.

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[1] There are no intrinsic problems in entertaining uncountable state spaces. The conditions stated in this paper are in spirit all that is needed. However, care needs to taken as the support of a measure is no longer well defined (without, e.g., additional topological restrictions) so conditioning events are identified only up to sets of measure zero.

[2] In Karni and Vierø (2013), there are actually two distinct ways the DM can become more aware, *refinement*, which is essentially what is characterized here, and *expansion* where by the underlying state-space gets larger. It seems to me that allowing the state-space to expand is fundamentally at odds with the natural definition of a state space as a representation of *all* decision relevant states of affairs. One can diffuse this tension and represent expansions via refinements by setting an event $E^\star \in \Sigma_0$ to collect "that which is not yet understood." $E^\star$ gets carved up with each new discovery. This latter method has the added benefit of allowing the DM to reason about her own unawareness.

There are many intuitive situations, however, where becoming aware intrinsically does provide information. Incontrovertibly, if the DM becomes aware of an event E, she *must* learn that she used to be unaware of E.[3] But, even without appealing to introspection, it is reasonable that the mere existence of a concept can serve as evidence regarding contingencies the DM was already aware of. This is essentially the "problem of old evidence" (Glymour, 1980).

Example 1. Players i and j are playing a card game. i initially thinks it is highly likely that he fully understands the rules of the game, and further that j's behavior is not rationalizable according to these rules. Hence i believes its is highly likely that j is irrational. i then discovers that there are in fact two variants of the game. Although i does not learn any hard information about the rules of either variant, he now places much less probability on the event that he fully understands the rules (of the game j believes they are playing), and therefore less probability on the event that j is irrational. ∎

Even if becoming aware does not intrinsically change beliefs, it may well be that by the time the DM's beliefs can actually be elicited, she has taken into account some additional probabilistic information. That is to say, despite the DM adhering to RB, the beliefs elicited at time 1 reflect not only the expansion of awareness but also conventional updating.

Definition. *Say that* (π_1, π_0) *satisfies* extended Bayesianism *(EB), if there exists a probability distribution* $\bar{\pi}$ *on* (Ω, Σ_1) *such that*

EC1 $\bar{\pi}(S_1) > 0$,

EC2 $\bar{\pi}(E) = \pi_0(E)$ *for all* $E \in \Sigma_0$, *and*

EC3 $\pi_1(E) = \frac{\bar{\pi}(E \cap S_1)}{\bar{\pi}(S_1)}$ *for all* $E \in \Sigma_1$.

An interpretation is as follows: If (π_0, π_1) satisfies EB it is *as if* π_1 was constructed by conditioning π_0 on the event S_1. We say 'as if' because when $S_1 \notin \Sigma_0$ then the π_0 probability of S_1 is undefined. However, in this case, we make sense of conditioning by first extending π_0 to the richer algebra ($\pi_0 \to \bar{\pi}$) and then constructing π_1 by conditioning this extension ($\bar{\pi} \to \pi_1$). The overall transition ($\pi_0 \to \pi_1$) satisfies reverse Bayesianism if and only if $S_1 = \Omega$ so that the conditioning step is trivial and satisfies canonical Bayesianism if $\Sigma_0 = \Sigma_1$ so that the discovered evidence was expected at time 0.

Example 2. Let $\Sigma_0 = \{\omega_1, \omega_2, \omega_3\}$, Σ_0 be generated by the partition $\{\{\omega_1\}, \{\omega_2, \omega_3\}\}$ and Σ_1 by the discrete partition. Let π_0 be given by $\pi_0(\{\omega_1\}) = \pi_0(\{\omega_2, \omega_3\}) = \frac{1}{2}$. Finally let $\pi_1(\omega_1) = \frac{2}{3}$, $\pi_1(\omega_2) = \frac{1}{3}$ and $\pi_1(\omega_3) = 0$. Then (π_0, π_1) satisfies EB, as witnessed by $\bar{\pi}$ on (Ω, Σ_1) given by $\bar{\pi}(\omega_1) = \frac{1}{2}$, $\bar{\pi}(\omega_2) = \frac{1}{4}$ and $\bar{\pi}(\omega_3) = \frac{1}{4}$. ∎

Example 3. Let $\Sigma_0 = \{\emptyset, \Omega\}$. Then (π_0, π_1) satisfies EB irrespective of π_1. ∎

[3] In purely semantic "state-space" models, introspection is not captured. However, by starting with a first order language with an awareness modality and setting the states as possible worlds, one can make precise sense out of the event "i used to be unaware of the event E." See, for example, Halpern and Rêgo (2009); Halpern and Piermont (2019).

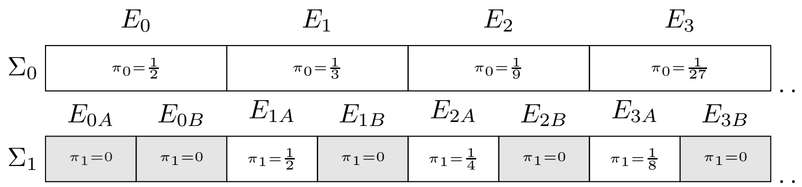

Figure 1: A visual representation of the state space from Example 4.

Example 4. Let $\Omega = \mathbb{N} \times \{A, B\}$ with Σ_0 generated by $\mathbb{N}$ and Σ_0 by the discrete partition. Set $\pi_0(E_0) = \frac{1}{2}$ and $\pi_0(E_n) = 3^{-n}$ for $n > 0$. Set $\pi_1(E_{0A}) = \pi_1(E_{nB}) = 0$ and $\pi_0(E_{nA}) = 2^{-n}$ for all $n > 0$ (see figure 1). Then (π_0, π_1) does not satisfy EB. ∎

Fixing S_0 and S_1 (and, of course, the state space and sigma algebras) there may be multiple π_0's such that (π_0, π_1) satisfies EB for a fixed π_1 (namely those priors that keep the relative likelihoods of events within S_1 equal). Also, there might be multiple π_1's such that (π_0, π_1) satisfies EB for a fixed π_0 (namely those posteriors that ascribe different probabilities to $E \in \Sigma_1 \setminus \Sigma_0$).

Observability

Bayesian updating is the normative benchmark for how probabilistic judgements should respond to the acquisition of new evidence. Unfortunately, in cases where $S_1 \notin \Sigma_0$, Bayes' rule cannot be directly verified, as there was no prior belief regarding the likelihood of the conditioning event. The notion of commensurability, below, provides a simple resolution, advancing an observable restriction on (π_0, π_1) equivalent extended Bayesianism.

Definition. *Say that π_1 is* commensurate *to π_0 if*

P1 $\inf_{E \in S_0} \frac{\pi_0(E)}{\pi_1(E)} > 0$, *and,*

P2 *for all $E, F \in \Sigma_0$ with $E \subseteq S_1$,*

$$\frac{\pi_0(E)}{\pi_0(F)} \leq \frac{\pi_1(E)}{\pi_1(F)} \tag{1}$$

and where (1) holds with equality whenever $F \subseteq S_1$. (We here associate $\frac{x}{0}$ with $+\infty$ for all $x \in \mathbb{R}$.)

Remark 1. For all $E \in \Sigma_0$ with $E \subseteq S_1$, $\pi_0(E) \leq \pi_1(E)$. This follows by setting F to Ω in (P1).

Remark 2. (P1) implies that π_1 is absolutely continuous with respect to π_0: $S_1 \subseteq S_0$. If this was not the case, then F, the smallest event in Σ_0 containing $S_1 \setminus S_0$, is non-empty. Since $F \cap S_1 \neq \emptyset$ and $F \cap S_0 = \emptyset$, we must have $\pi_1(F) > 0$ but $\pi_0(F) = 0$. So then $\frac{\pi_0(\emptyset)}{\pi_0(F)} = +\infty \not\leq 0 = \frac{\pi_1(\emptyset)}{\pi_1(F)}$, a contradiction.

Remark 3. If there exists a non-empty $E \in \Sigma_0$ with $E \subseteq S_1$, then (P2) is implied by (P1). To see this note that for such E, we have for all $F \in \Sigma_1$: $0 < \frac{\pi_0(E)}{\pi_1(E)} \leq \frac{\pi_0(F)}{\pi_1(F)}$, where Remark 2 establishes that $0 < \pi_0(E)$.

Theorem 1. *π_1 is commensurate to π_0 if and only if (π_0, π_1) satisfies EB.*

$$\begin{tikzcd} (\Sigma_0, \pi_0) \arrow[r, "\bar{\pi}"] \arrow[rr, bend left, dashed, "\bar{\pi}''"] & (\Sigma_1, \pi_1) \arrow[r, "\bar{\pi}'"] & (\Sigma_2, \pi_2) \end{tikzcd}$$

Figure 2: The existence of the extensions $\bar{\pi}$ and $\bar{\pi}'$ ensure the existence of an extension $\bar{\pi}''$.

Proof. The 'if' direction is easy: Assume (π_0, π_1) satisfies EB with $\bar{\pi}$ the mitigating measure. Take some $E, F \in \Sigma_0$ with $E \subseteq S_1$. Then, by the properties of $\bar{\pi}$,

$$\frac{\pi_0(E)}{\pi_0(F)} = \frac{\bar{\pi}(E)}{\bar{\pi}(F)} \leq \frac{\bar{\pi}(E \cap S_1)}{\bar{\pi}(F \cap S_1)} = \frac{\pi_1(E)}{\pi_1(F)}$$

with equality whenever $F \subseteq S_1$, establishing (P1). (P2) holds because

$$\frac{\pi_0(E)}{\pi_1(E)} = \bar{\pi}(S_1)\frac{\pi_0(E)}{\bar{\pi}(E \cap S_1)} \geq \bar{\pi}(S_1)\frac{\pi_0(E)}{\bar{\pi}(E)} = \bar{\pi}(S_1)\frac{\pi_0(E)}{\pi_0(E)} = \bar{\pi}(S_1).$$

for all $E \in \Sigma_0$.

Towards the 'only if' direction, assume that π_1 is commensurate to π_0. We must find a $\bar{\pi}$ on (Ω, Σ_1) such that the conditions of EB hold. Since, Ω is denumerable, Σ_0 and Σ_1 are generated by partitions of Ω—call theses $\mathbb{P}_0$ and $\mathbb{P}_1$, respectively—and it isuffices to specify $\bar{\pi}$ on the cells of $\mathbb{P}_1$.

But, first, we must set a value, β, for $\bar{\pi}(S_1)$. If there exists an $E \in \Sigma_0$ with $E \subseteq S_1$, then set $\beta = \frac{\pi_0(E)}{\pi_1(E)}$. By (P1), the choice of E is irrelevant and by Remark 1, $\beta \leq 1$. Further, Remark 2 indicates that $0 < \beta$ and following the logic of Remark 3 we have $\beta \leq \inf_{S_0} \frac{\pi_0(E)}{\pi_1(E)}$. If no such E exists, take an arbitrary $0 < \beta \leq \inf_{S_0} \frac{\pi_0(E)}{\pi_1(E)} \leq 1$.

Now for all $P \in \mathbb{P}_1$ with $P \subseteq S_1$, set $\bar{\pi}(P) = \beta\pi_1(P)$. For each $Q \in \mathbb{P}_0$, such that $Q \not\subseteq S_1$, choose a representative $P^Q \in \mathbb{P}_1$ with $P^Q \subseteq Q \setminus S_1$. Set $\bar{\pi}(P^Q) = \pi_0(Q) - \beta\pi_1(Q)$. Since $\beta \leq \frac{\pi_0(Q)}{\pi_1(Q)}$, this is a well defined probability. For any remaining $P \in \mathbb{P}_1$, set $\bar{\pi}(P) = 0$.

It is straightforward to verify that $\bar{\pi}$ is a witness to (π_0, π_1) satisfying EB. First, $\bar{\pi}(S_1) = \beta > 0$, so (EC1) is satified. Next, notice for all $Q \in \mathbb{P}_0$, such that $Q \not\subseteq S_1$, $\bar{\pi}(Q) = \pi_0(Q)$ by construction. If there is some $Q \in \mathbb{P}_0$, such that $Q \subseteq S_1$, then $\beta = \frac{\pi_0(Q)}{\pi_1(Q)}$, so that $\bar{\pi}(Q) = \beta\pi_1(Q) = \pi_0(Q)$, and so (EC2) holds for all $Q \in \mathbb{P}_0$. Finally, for $\pi_1(E) = \pi_1(E \cap S_1) = \frac{\bar{\pi}(E \cap S_1)}{\beta} = \frac{\bar{\pi}(E \cap S_1)}{\bar{\pi}(S_1)}$, so (EC3) holds. ∎

Repeated Conditioning

If the DM discovers unforeseen evidence more than once, the observed subjective probabilities will form a finite sequence, $\pi_0 \ldots \pi_N$, over increasingly fine algebras, $\Sigma_0 \ldots \Sigma_N$. For a DM who adheres to Bayesianism to the extent possible under unawareness, each (π_n, π_{n+1}) will satisfy EB. Even if the modeler cannot feasibly observe each π_n, this hypothesis can be falsified, since under this assumption, (π_n, π_m) will satisfy EB for all *all* π_m with $m \geq n$.

Theorem 2. *The diagram in Figure 2 commutes.*

Proof. This can be seen easily by appealing to Theorem 1: let π_1 be commensurate to π_0 and π_2 commensurate to π_1. Appealing to (P1), we have

$$\inf_{E \in S_0} \frac{\pi_0(E)}{\pi_2(E)} \geq \inf_{E \in S_0} \frac{\pi_0(E)}{\pi_1(E)}\frac{\pi_1(E)}{\pi_2(E)} \geq \inf_{E \in S_0} \frac{\pi_0(E)}{\pi_1(E)} \inf_{E \in S_1} \frac{\pi_1(E)}{\pi_2(E)} > 0.$$

Similarly, appealing to (P2): for all $E, F \in \Sigma_0 \subseteq \Sigma_1$ with $E \subseteq S_2 \subseteq S_1$,

$$\frac{\pi_0(E)}{\pi_0(F)} \leq \frac{\pi_1(E)}{\pi_1(F)} \leq \frac{\pi_2(E)}{\pi_2(F)}$$

which holds with equality whenever $F \subseteq S_2 \subseteq S_1$. Thus, π_2 is commensurate to π_0; (π_0, π_2) satisfies EB. ∎

A Few Notes on Related Literature

Fagin and Halpern (1991) introduced the notion of outer and inner conditional probability as the upper and lower envelopes of the conditional probabilities of all possible extensions to a richer algebra. In the language of this paper, the outer conditional probability of π_0 on $E \in \Sigma_1$ is

$$\pi_0^*(\cdot|E) = \sup\{\bar{\pi}(\cdot|E) \mid \bar{\pi} \in \Delta(\Omega, \mathcal{S}_1), \bar{\pi} \text{ extends } \pi_0\}$$

and the inner conditional probability is defined by replacing the sup and an inf. Thus it must be that (π_0, π_1) satisfy EB exactly when π_1 lies inside of the outer and inner conditional probabilities (where the conditioning event is S_1) of π_0. As such, filtering through inner and outer probability provides another, indirect, characterization of unforeseen posteriors.

As discussed in Footnote 2, Karni and Vierø (2013) consider two different ways to expand awareness. If we insist on entertaining expansions of the state-space itself so that Σ_0 is defined on Ω and Σ_1 on $\Omega \cup \Omega'$, then we can appropriately generalize the definition of extended Bayesianism to allow π_1 to entertain probability on newly discovered states: Setting $\pi_1 \in \Delta(\Omega \cup \Omega', \Sigma_1)$, say (π_0, π_1) satisfy *generalized extended Bayesianism* (GEB) if $\pi_1(\Omega) > 0$ and $(\pi_0, \pi_1(\cdot \mid \Omega))$ satisfy EB. In this case, we have that the overall transition $(\pi_0 \to \pi_1)$ satisfies reverse Bayesianism if and only if $S_1 = \Omega \cup \Omega'$.

Karni et al. (2018) consider the case where a DM, in the process of becoming more aware, might simultaneously condition her beliefs with respect to some event, E. The only consider expansions of the state space and not refinements of previously describable events (i.e., Ω expands to $\Omega \cup \Omega'$ but $\Sigma_0 = \{E \cap \Omega \mid E \in \Sigma_1\}$). They introduce *generalized reverse Bayesianism*, whereby the relative probabilities of events must remain the same only for events in $S_0 \cap S_1$ (rather than all of S_0 as is the case for RB). This case is clearly captured by GEB. The overall transition $(\pi_0 \to \pi_1)$, where π_1 is defined on $(\Sigma_1, \Omega \cup \Omega')$, satisfies generalized reverse Bayesianism if and only if (π_0, π_1) satisfy GEB and $\Sigma_0 = \{E \cap \Omega \mid E \in \Sigma_1\}$.

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