1D Ising Model with Transverse Field

The Hamiltonian of the system is:

$$H = -\sum_{n=1}^{N-1} \sigma_n^z \sigma_{n+1}^z - \sum_{n=1}^{N} g_n \sigma_n^x$$

where:

- N is the number of spin-1/2 particles.
- g is the transverse magnetic field strength.
- σ_n^x, σ_n^z spin-1/2 operators associated with site n in the direction of $\hat{x}$, $\hat{z}$ respectively

The model is in a **paramagnetic phase**, when $g >> 1$. In this case ground state of the system has no degeneracy and it is a state in which all spins are pointing along $\hat{x}$:

$$|GS\rangle = |\rightarrow\rightarrow\rightarrow \ldots \rightarrow\rightarrow\rightarrow\rangle$$

On the other hand, the most excited state in this phase would be the following one:

$$|\leftarrow\leftarrow\leftarrow \ldots \leftarrow\leftarrow\leftarrow\rangle$$

The model is in a **ferromagnetic phase**, when $g << 1$ or $g \rightarrow 0$. A ground state of the system has a two-fold degeneracy and it can be one of the following states (or their superposition):

$$|\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle, \quad |\downarrow\downarrow\downarrow \ldots \downarrow\downarrow\downarrow\rangle$$

On the other hand, the most excited state in this phase would be:

$$|\uparrow\downarrow\uparrow \ldots \downarrow\uparrow\downarrow\rangle \quad \text{or} \quad |\downarrow\uparrow\downarrow \ldots \uparrow\downarrow\uparrow\rangle$$

We can make Jordan-Wigner transformation to spinless non-interacting fermions c_n, $c_n^\dagger$ given by:

$$\sigma_n^x = 1 - 2c_n^\dagger c_n, \quad \sigma_n^z = -(c_n + c_n^\dagger)\prod_{m<n}(1 - 2c_m^\dagger c_m)$$

Our Hamiltonian becomes quadratic in creation and annihilation fermionic operators c_n, $c_n^\dagger$:

$$H = 2\sum_{n=1}^{N} g_n c_n^\dagger c_n - 2\sum_{n=1}^{N-1}(g_n c_n^\dagger c_n + c_{n+1} c_n)$$

This quadratic Hamiltonian can be diagonalized to:

$$H = \sum_{m=1}^{N} \omega_m \left(\gamma_m^\dagger \gamma_m - \frac{1}{2}\right)$$

by the Bogoliubov transformation:

$$c_n = \sum_{m=1}^{N} \left(u_{n,m}\gamma_m + v_{n,m}^* \gamma_m^\dagger\right),$$

where m is enumerating N eigenmodes of stationary Bogoliubov-de-Gennes (BdG) equations:

$$\omega_m u_{n,m}^\pm = 2g_n u_{n,m}^\mp - 2u_{n\mp 1,m}^\mp$$

with $\omega_m \geq 0$. Here $u_{n,m}^\pm = (u_{n,m} \pm v_{n,m})/\sqrt{2}$

By putting $\omega_m \to i\hbar\partial_t$ we obtain time-dependent version of BdG equation:

$$i\hbar\partial_t u^{\pm}_{n,m} = 2g_n(t)u^{\mp}_{n,m} - 2u^{\mp}_{n\mp1,m}$$

With boundary conditions

$$u^{\pm}_{n+1} = u^{\pm}_0 = 0$$

We can write down this equation using two N-component vectors $\vec{u} = (\vec{u}^+, \vec{u}^-)^T$

$$i\hbar\partial_t \begin{pmatrix} \vec{u}^+ \\ \vec{u}^- \end{pmatrix} = H_1(t) \begin{pmatrix} \vec{u}^+ \\ \vec{u}^- \end{pmatrix} + H_2 \begin{pmatrix} \vec{u}^+ \\ \vec{u}^- \end{pmatrix}$$

Formal solution to this equation would be:

$$\begin{pmatrix} \vec{u_t}^+ \\ \vec{u_t}^- \end{pmatrix} = e^{-i\hbar(H_1(t)+H_2)} \begin{pmatrix} \vec{u_{t=0}}^+ \\ \vec{u_{t=0}}^- \end{pmatrix}$$

We can divide our evolution time t into N small time steps dt:

$$U(t) \approx \prod_{n=0}^{N-1} e^{-i\hbar dt(H_1(ndt+dt/2)+H_2)}$$

and then perform second-order Suzuki-Trotter decomposition:

$$e^{-i\hbar dt(H_1(ndt+dt/2)+H_2)} = e^{-i\hbar\frac{dt}{2}H_2} e^{-i\hbar dt H_1(ndt+dt/2)} e^{-i\hbar\frac{dt}{2}H_2} + O(dt^3)$$

Due to the simple form of H_1 and H_2, each matrix can be diagonalized analytically. H_1 is diagonal in the u, v basis:

$$\begin{aligned} i\hbar\partial_t u_{n,m} &= 2g_n(t)u_{n,m} + ... \\ i\hbar\partial_t v_{n,m} &= -2g_n(t)v_{n,m} + ... \end{aligned}$$

and H_2 is diagonal in $\tilde{u}^{\pm}_{n,m}$ basis:

$$\begin{aligned} i\hbar\partial_t \tilde{u}^+_{n,m} &= ... - 2\tilde{u}^+_{n,m} \\ i\hbar\partial_t \tilde{u}^-_{n,m} &= ... + 2\tilde{u}^-_{n,m} \end{aligned}$$

where:

$$\tilde{u}^+_n = \frac{u^+_n + u^-_{n-1}}{\sqrt{2}}, \quad \tilde{u}^-_n = \frac{u^+_{n+1} - u^-_n}{\sqrt{2}}$$

Density of excitations can be found as an expectation value of the number of Bogoliubov quasiparticles in the final state:

$$p = \frac{1}{N} \sum_m \langle \psi(t_f) | \gamma^\dagger_m \gamma_m | \psi(t_f) \rangle ,$$

where Bogoliubov quasiparticles are defined by the eigenmodes of stationary BdG eqs. at the final point

$$\gamma_m = \sum_{n=1}^{N} \left(u^*_{n,m} c_n + v^*_{n,m} c^\dagger_n \right) ,$$

and fermionic operators are defined by evolved modes:

$$c_n = \sum_{m=1}^{N} \left(u_{n,m}(t_f)\tilde{\gamma}_m + v^*_{n,m}(t_f)\tilde{\gamma}^\dagger_m \right),$$

with a constraint that $\tilde{\gamma}_m \left|\psi(t_f)\right\rangle = 0$.

This was a general idea on how to efficiently simulate dynamics of Ising model with transverse magnetic field.

Kibble-Zurek mechanism

Main idea: Spontaneous symmetry breaking leads to the formation of topological defects.

In Ising model we have the following SSB:

$$\mathbb{Z}_2 \to \mathbb{1}$$

Now, suppose that we change our control parameter over some time scale τ_Q, and near the critical point it behaves as:

$$g \sim \left(\frac{t}{\tau_Q}\right)^r$$

The question could be for example what is the size of domains, or what is the density of these topological defects? Answer:

$$d \sim \tau_Q^{-\frac{r\nu}{rz\nu+1}},$$

where z, ν are so-called critical exponents, and r depends on the rate of change at which we approach the critical point. For Ising model $z = \nu = 1$. For sinusoidal ramp, which approaches the critical point as quadratic function with $r = 2$ we have:

$$d \sim \tau_Q^{-2/3}$$

Example 1

We start at $g = 2$ and go down to the critical point $g_c = 1$. We choose sine protocol:

$$g(t) = \begin{cases} 2, & \text{if } \frac{t}{\tau_Q} < -\frac{\pi}{2} \\ 3/2 - 1/2\sin\left(\frac{t}{\tau_Q}\right), & \text{if } -\frac{\pi}{2} \leq \frac{t}{\tau_Q} \leq \frac{\pi}{2} \\ 1, & \text{if } \frac{t}{\tau_Q} > \frac{\pi}{2} \end{cases}$$

Data for $L = 100$, 200, 300, 400, 500, 1000 confirm the power-law exponent.

Example 2

Dispersion relation for infinite 1D Ising model has the form:

$$\omega_k = 2\sqrt{(g - \cos k)^2 + \sin k^2}$$

one can expand it near the Fermi point at $k = 0$ for small k:

$$\omega_k \approx 2\sqrt{(g_c - 1 + k^2/2)^2 + k^2} \;\approx\; 2k \quad\longrightarrow\quad c = \frac{\mathrm{d}w}{\mathrm{d}k} = 2$$

we see that there is a characteristic velocity in the system. We can think of it as a maximal group velocity at which information can propagate.

Idea: If a speed of ramp $g(t, n)$ moves slower than c, then those parts of the system that cross the critical point earlier may be able to communicate their choice of orientation of the order parameter to the parts that cross the transition later and bias them to make the same choice. Hence, smaller excitation density, and a better approximation of a final GS.